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ON SUCCESSORS OF SINGULAR CARDINALS

Saharon Shelah

Institute of Mathematics,
The Hebrew University,
Jerusalem, Israel.

Introduction :

We will clarify the situation for the successor of a strong limit singular cardinal λ. We find a special subset $S^*(\lambda^+)$, from which we can find which stationary subsets of λ^+ can be stopped from being stationary by μ-complete forcing (Baumgartner has done this for successor λ^+ of regular $\lambda = \lambda^{<\lambda}$).
For $\lambda \blacksquare \aleph_{\omega+1}$ we succeed in continuing an induction construction done for a λ^+-free not λ^{++} (abelian) group, and similar things for transversals; on those problems see history and references in [Sh 2].
A solution of a related problem - which stationary subsets of λ^+ can be "killed" by a forcing not adding bounded subsets of λ^+-will appear in a paper by U. Avraham, J. Stavi and the author.
We also prove a result related to the title but not to the rest of the paper, improving a result of Gregory [Gr] : assuming G.C.H., for $\lambda \neq \aleph_0$, $\diamond^*_S$ holds, where $S = \{\delta < \lambda^+;\ cf\delta \neq cf\lambda\}$; hence $\diamond_{S_1}$ holds for any stationary $S_1 \subseteq S$.

For a reader interested only with the GCH, he can simplify for himself the part up to section 13. A reader interested in more general cases than those discussed in the main part has to go to the end. There we also show that the special set $S^*(\aleph_{\omega+1})$ can be stationary (even with the GCH).

The main results were announced in the AMS Notices [Sh 3].

Notation: We shall denote infinite cardinals by λ,μ,κ,χ, ordinals by $i,j,\alpha,\beta,\gamma,\xi,\zeta$ limit ordinals by δ, natural numbers by m,n,r,p,q.

Let $\overline{N}$ denote a sequence $<N_i : i<\lambda>$ where for some $\mu,\chi \leqslant \mu$, $N_i \prec (H(\mu),\in)$; $i \subseteq N_i$, $\|N_i\| < \lambda$, $i<j \Rightarrow N_i \prec N_j$, and for limit $\delta, N_\delta = \bigcup_{i<\delta} N_i$. We call this a λ-approximating sequence (for μ).

We denote by d a two-place function from one cardinal to another; $cf\delta$ is the cofinality of δ; $cf^*\delta$ is $cf\delta$ if $cf\delta < \delta$ and is ∞ otherwise. D_δ is the filter over δ generated by the closed unbounded subsets of δ (so we assume $cf\delta > \aleph_0$). If D is a filter over I, $A \subseteq B$ mod D means $I-(A-B) \in D$; similarly $A \equiv B$ mod D means $I-(A-B)\cup(B-A) \in D$. If $A \not\equiv \phi$ mod D, D + A is the filter $\{B : B \cup (I-A) \in D\}$.

Let $CF(\delta,\kappa) = \{i<\delta : cfi = \kappa\}$, similarly $CF(\delta,<\kappa) = \bigcup_{\mu<\kappa} CF(\delta,\mu)$ $CF(\delta,\leqslant\kappa) = \bigcup_{\mu\leqslant\kappa} CF(\delta,\mu)$ $D_{\delta,\kappa} = D_\delta + CF(\delta,\kappa)$ etc.

1. Definition : 1) We say κ is good for λ if $\lambda=\lambda^{<\lambda}, \kappa=\infty$ or there is a family $\underline{P}^0_{\lambda,\kappa}$ such that

a) $|\underline{P}^0_{\lambda,\kappa}| = \lambda$

b) every member of $\underline{P}^0_{\lambda,\kappa}$ is a subset of λ of cardinality κ

c) every subset of λ of cardinality κ contains a member of $\underline{P}^0_{\lambda,\kappa}$

2) We call κ a good cofinality for λ if $\lambda=\lambda^{<\lambda}, \kappa$ is ∞ or if λ and κ are regular and there is a family $\underline{P}_{\lambda,\kappa}$ such that

a) $|\underline{P}_{\lambda,\kappa}| = \lambda$

b) every member of $\underline{P}_{\lambda,\kappa}$ is a subset of λ of cardinality $<\kappa$

c) every subset of λ of cardinality κ has a subset $\{\alpha_i : i<\kappa\}$ such that α_i is increasing and for every $j<\kappa$, $\{\alpha_i : i<j\} \in \underline{P}_{\lambda,\kappa}$

d) $\lambda = \lambda^{<\kappa}$ or $2^\mu < \lambda$ for every $\mu < \kappa$

2. Definition : 1) $\mathrm{Gcf}(\lambda) = \{\kappa : \kappa$ is a good cofinality for $\lambda\}$

$G(\lambda) = \{\kappa : \kappa$ is good for $\lambda\}$

2) $\mathrm{gcf}(\lambda) = \{i < \lambda : \mathrm{cf}^* i \in \mathrm{Gcf}(\lambda)\}$ (note that we use cf^* not cf)

3) $D^g_\lambda = D_\lambda + \mathrm{gcf}(\lambda)$

3. Claim : 1) If $\lambda^\kappa = \lambda$ then κ is good for λ

2) If $\kappa < \infty$ is good for λ then κ is good for λ^+

3) If $\lambda = \sum_{i<\mu} \lambda_i$, $\mathrm{cf}\mu \neq \mathrm{cf}\kappa$, $\lambda_i (i < \mu)$ increasing and $\kappa < \infty$ is good for each λ_i then κ is good for λ

4) If $(\forall \mu < \aleph_\alpha)\mu^\kappa < \aleph_\alpha, \beta < \mathrm{cf}\kappa$, $\mathrm{cf}\aleph_\alpha \neq \mathrm{cf}\kappa$ then κ is good for $\aleph_{\alpha+\beta}$ [in fact $(\forall \mu < \aleph_\alpha)\mu^\kappa \leqslant \aleph_{\alpha+\beta}$ suffice]

5) if λ,κ are regular, κ good for λ then κ is a good cofinality

$2<\kappa$

for λ, provided that $2^{<\kappa} \leqslant \lambda$

6) If λ,κ are regular $\lambda^{<\kappa} = \lambda$ then κ is a good cofinality for λ

7) If $\kappa < \infty$ is a good cofinality for λ then κ is a good cofinality for λ^+

8) If $\lambda = \sum_{i<\mu} \lambda_i$, $\mathrm{cf}\mu \neq \mathrm{cf}\kappa$, $\kappa \in \mathrm{Gcf}(\lambda_i)$ for every $i < \mu$, λ_i increasing, and $\kappa < \infty$ then $\kappa \in \mathrm{Gcf}(\lambda)$

9) If $(\forall \mu < \aleph_\alpha)\mu^{<\kappa} < \aleph_\alpha$, $\mathrm{cf}\aleph_\alpha \neq \kappa$, κ regular, $\beta < \kappa$ then $\kappa \in \mathrm{Gcf}(\aleph_{\alpha+\beta+1})$ [in fact, $(\forall \mu < \aleph_\alpha)\mu^{<\kappa} \leqslant \aleph_{\alpha+\beta+1}$ suffice].

4. Definition : For d a two-place function from δ into $\kappa(\mathrm{cf}\delta > \aleph_0)$ we let $S_1(d) = \{\xi : \xi < \delta$, ξ a limit ordinal such that there is an unbounded $A \subseteq \xi$ on which d is constant$\}$

$S_0(d) = \{\xi : \xi < \delta$, ξ a limit ordinal such that there are unbounded subsets A,B of ξ, such that $(\forall b \in B)(\exists \alpha < \kappa)(\forall a \in A)[a < b \rightarrow d(a,b) \leqslant \alpha]\}$

Remark : Note that d determines δ (as Dom d) but not κ (as d is into κ, not necessarily onto κ), so if the value of κ is not clear we shall write $S_0(d,\kappa)$. In the definition of $S_1(d)$, κ has no role.

5. Claim : For d a two-place function from δ to κ:

1) $S_1(d) \subseteq S_o(d)$,

2) in the definition of $S_\ell(d)$ ($\ell = 0,1$) we can assume A,B have order type cfξ (and generally replace them by unbounded subsets),

3) $CF(\delta, \leqslant \kappa) \subseteq S_o(d)$,

4) If $\ell = 0,1$, $\xi \in S_\ell(d)$, cf$\xi > \aleph_o$, then there is $C \in D_\xi$ such that $C \subseteq S_\ell(d)$.

6. Definition : For a λ-approximating sequence $\overline{N}$ (see notation) let $S_2(\overline{N}) = \{\xi : \xi < \lambda$, ξ a limit such that there is an unbounded $A \subseteq \xi$ of order type cfξ such that $(\forall i < \xi)\,[A \cap i \in N_\xi]$ and $N_\xi \cap \lambda = \xi\}$

7. Claim : 1) If λ is regular, $\overline{N}^o$, $\overline{N}^1$ are λ-approximating sequences for μ_o, μ_1 respectively, and $\mu_\ell > \lambda$, then $S_2(\overline{N}^1) = S_2(\overline{N}^o)$ mod D^g_λ.

Proof : Let $\overline{N}^\ell = \langle N^\ell_i : i < \lambda \rangle$, where $N^\ell_i \prec (H(\mu_\ell),\in)$, and let $C = \{\alpha < \lambda : N^o_\alpha \cap (\bigcup_{j<\lambda} N^1_j) = (\bigcup_{j<\lambda} N^o_j) \cap N^1_\alpha = N^o_\alpha \cap N^1_\alpha$ and $N^\ell_\alpha \cap \lambda = \alpha\}$ (we do not distinguish strictly between a model N and its universe). It is easy to check that C is a closed unbounded subset of λ. By transitivity of equality we can assume $N^o_\alpha \prec N^1_\alpha$. Now suppose $\xi \in C$, and $cf^*\xi \in Gcf(\lambda)$. We shall prove $\xi \in S_2(\overline{N}^o)$ iff $\xi \in S_2(\overline{N}^1)$, thus completing the proof. The "only if" part is now trivial, so we concentrate on the "if" part. Also the case $cf^*\xi = \infty$ is easy, so we assume $cf^*\xi = cf\xi < \xi$.

Let $\kappa = cf\xi < \xi$. We have just assumed $\kappa \in Gcf(\lambda)$, so the appropriate $\underline{P}_{\lambda,\kappa}$ (as in Definition 1.2) exists, hence belongs to $H(\mu_1)$, hence w.l.o.g it belongs to N^o_o, and hence, by assumption, to N^1_o.

If $\xi \in S_2(\overline{N}^1)$, then (by definition) there is an unbounded $A \subseteq \xi$ of order-type cfξ, such that for every $\zeta < \xi$, $A \cap \zeta \in N^1_\xi$.

If $\lambda = \lambda^{<\kappa}$, we can assume $\underline{P}_{\lambda,\kappa} = \{B \subseteq \lambda : |B| < \kappa\} = \{B_i : i < \lambda\}$ (since $|\underline{P}_{\lambda,\kappa}| = \lambda$), and so $\underline{P}_{\lambda,\kappa} \cap N^0_\xi = \underline{P}_{\lambda,\kappa} \cap N^1_\xi = \{B_i : i < \xi\}$, hence $\zeta < \xi \Rightarrow A \cap \zeta \in N^0_\xi$, hence A witnesses that $\xi \in S_2(\overline{N}^0)$. Thus finishing.

So we are left with the case $\lambda < \lambda^{<\kappa}$. Then, by d) of Definition 1.2, $(\forall \mu < \kappa) 2^\mu < \lambda$. So, as $N^\ell_\xi \cap \lambda = \xi$, and A has order-type κ, every subset of A of power $< \kappa$ is included in a set from N^1_ξ of cardinality $<\kappa$, hence it belongs to N^1_ξ. So we can replace A by any subset of it which is unbounded in ξ. In particular, by the choice of $\underline{P}_{\lambda,\kappa}$ (see Definition 2), we can assume $A = \{\alpha_i : i < \kappa\}$, and for $j < \kappa$, $\{\alpha_i : i < j\} \in \underline{P}_{\lambda,\kappa}$ and, as mentioned above, $\{\alpha_i : i < j\} \in N^1_\xi$. But as $|\underline{P}_{\lambda,\kappa}| = \lambda$, $\underline{P}_{\lambda,\kappa} \in N^0_0$, clearly $\underline{P}_{\lambda,\kappa} \subseteq \bigcup_{i<\lambda} N^0_i$, hence (as $\xi \in C$) $\underline{P}_{\lambda,\kappa} \cap N^0_\xi = \underline{P}_{\lambda,\kappa} \cap N^1_\xi$, hence for every $j < i$, $\{\alpha_i : i < j\} \in N^0_i$. So $\{\alpha_i : i < \kappa\}$ witnesses that $\xi \in S_2(\overline{N}^0)$, and this finishes the proof of the theorem.

<u>8. Definition</u> : $S^*(\lambda) \subseteq \lambda$ is defined as $(\lambda - S_2(\overline{N})) \cap \text{gcf}(\lambda)$ for $\overline{N}$ any λ-approximating sequence for λ^+, where λ is regular. (so S^* is uniquely defined mod D_λ only).

<u>9. Definition</u> : For λ singular, a two-place function d from λ^+ to $\kappa = \text{cf}\lambda$ is called <u>normal</u> if for every $i < \kappa, \alpha < \lambda^+$, the set $\{\beta < \alpha : d(\beta,\alpha) \leq i\}$ has cardinality $<\lambda$. It is called subadditive if for $\gamma < \beta < \alpha < \lambda^+$, $d(\gamma,\alpha) \leq \max\{d(\gamma,\beta), d(\beta,\alpha)\}$.

<u>10. Claim</u> : For every singular λ, there is a normal subadditive two-place function d from λ^+ to $\text{cf}\lambda$; moreover, if $\lambda = \sum_{i<\text{cf}\lambda} \lambda_i$ (λ_i increasing), then $|\{\beta < \alpha : d(\beta,\alpha) \leq i\}| \leq \lambda_i$.

<u>Proof</u> : Easy.

11. Claim : 1) Suppose λ is singular, $\kappa = cf\lambda$, $(\forall \mu < \lambda)(\mu^{<\chi} \leq \lambda)$, and d is a normal two-place function from λ^+ to κ. Then for some λ^+-approximating sequence $\bar{N}$ for λ^{++},

$$CF(\lambda^+, \leq \chi) \cap S_o(d) \subseteq S_2(\bar{N}) \bmod D_\lambda.$$

2) Suppose λ is singular, $\kappa = cf\lambda$, χ is regular and is a good cofinality for λ^+, and d is a normal two-place function from λ^+ to κ. Then for some λ^+-approximating sequence $\bar{N}$ for λ^{++}, $CF(\lambda^+, \chi) \cap S_o(d) \subseteq S_2(\bar{N})$.

Proof : 1) Choose a λ^+-approximate sequence $\bar{N}$ for λ^{++} such that $d \in N_o$, $N_i \in N_{i+1}$. Clearly $= \{\delta < \lambda^+ : N_\delta \cap \lambda = \delta\}$ is closed and unbounded. So for every $\alpha < \lambda^+$, $i < \kappa$, the set $A^* = \{\beta < \alpha : d(\beta,\alpha) \leq i\}$ belongs to N_{i+1} and has cardinality $< \lambda$. Hence $P_i^\alpha = \{A : B \subseteq A^*, |B| < \chi\}$ belongs to N_{i+1} and has cardinality $< \lambda$, hence $P_i^\alpha \subseteq N_{i+1}$. So suppose $\delta \in S_o(d)$, and $A,B \subseteq \delta$ are witness to it (i.e. they are unbounded in δ and have order-type $cf\delta$, and for every $b \in B$, for some $i(b) < \kappa$, $(\forall a \in A)(a < b \rightarrow d(a,b) \leq i(b))$). Suppose further $\delta \in C$, $cf\delta \leq \chi$. Then $A,B \subseteq N_\delta$ (as $\delta \subseteq N_\delta$) and for every $b \in B$, $\{a : a \in A, a < b\}$ belongs to $\underline{P}_{i(b)}^b$, hence to N_{i+1}, hence to N_δ. So A witnesses that $\delta \in S_2(\bar{N})$. We have just proved $\delta \in CF(\lambda^+, \leq \chi) \cap S_o(d) \Rightarrow \delta \in S_2(\bar{N})$, thus finishing the proof of the claim.

2) A similar proof.

12. Claim : Suppose λ is regular, $\kappa < \chi$, $\kappa < \lambda$, χ is a good cofinality for λ and $(\forall \mu < \chi) 2^\mu < \lambda$ or $\chi = \infty$. Then for every two-place function d from λ to κ and for some λ-approximate sequence $\bar{N}$ for λ^+,

$$S_2(\bar{N}) \cap CF(\lambda, \chi) \subseteq S_1(d).$$

Proof : Choose $\bar{N}$ as λ-approximate sequence for λ^+ such that $d \in N_o$. Suppose $\delta \in S_2(\bar{N}) \cap CF(\lambda, \chi)$. We shall prove $\delta \in S_1(d)$. The case

$\chi = \infty$ is easy, so assume $\chi < \infty$.

As $\delta \in S_2(\bar{N})$, there is a set $\{\alpha_i : i < \chi\} \subseteq \delta$, unbounded in δ, such that for every $j < \chi$, $\{\alpha_i : i < j\} \in N_\delta$. Let h be the function with domain χ, $h(i) = \alpha_i$. Clearly for $j < \chi$, $h|j \in N_\delta$.

Now we define by induction on $i < \chi$ an element x_i and function f_i as follows :

$f_i(j) = d(x_j, \delta)$ for $j < i$ (so Dom $f_i = i$)

x_i is the first ordinal which is bigger than α_i and $x_j (j < i)$ and is such that $(\forall j < i)\,[d(x_j, x_i) = f_i(j)]$.

This can be carried out in $H(\lambda^+)$. But now as $\mu < \chi \Rightarrow 2^\mu < \chi$, and $\mu < \chi = cf\delta \leqslant \delta$, clearly each f_i is in N_δ.

Note also that x_i depends only on f_i and $\{\alpha_j : j \leqslant i\}$ (as for $j < i$, $f_j = f_i|j$). So $x_i \in N_\delta$ for each $i < \chi$.

Now there is an unbounded $S \subseteq \chi$ and $i_o < \kappa$ such that $j \in S \Rightarrow d(x_j, \delta) = i_o$. It is easy to check that $\{x_j : j \in S\}$ witnesses that $\delta \in S_1(d)$.

From now on we concentrate on successors of strong limit singular cardinals. We can conclude e.g.

<u>13. Conclusion</u> : Suppose λ is a singular strong limit. Then for every normal two place function d from λ^+ to $\kappa = cf\lambda$, the following holds :

$$S_o(d) \equiv S_1(d) \cup CF(\lambda^+, \leqslant \kappa) \equiv \lambda^+ - S^*(\lambda^+) \bmod_{D_{\lambda^+}} .$$

(So in particular $S_o(d)$ does not depend on d (when d is normal) up to equivalence $\bmod_{D_{\lambda^+}}$).

<u>Proof</u> : Trivial by 5.1, 5.3, 11 and 12.

<u>14. Claim</u> : If λ is regular, $\kappa < \lambda$ and $(\forall \mu < \lambda)\ \mu^{<\kappa} < \lambda)$, then $CF(\lambda, \leqslant \kappa) \leqslant \lambda - S^*(\lambda) \bmod D_{\lambda^+}$.

<u>Proof</u> : We can find a λ-approximating sequence $\langle N_i : i < \lambda \rangle$ to λ^+ such that every subset of N_i of cardinality $< \kappa$ belongs to N_{i+1}. So $CF(\lambda, \leqslant\kappa) \subseteq S_2(\bar{N})$.

15. Claim : If $\delta \in \lambda - S_1(d)$, d a two-place function from λ to $\kappa < cf\delta$, then $cf\delta$ is not weakly compact.

Proof : If $cf\delta$ is weakly compact then $cf\delta \to (cf\delta)^2_\kappa$.

16. Definition : 1) For a set $S \subseteq \lambda$ let

$$F(S) \equiv \{\delta < \lambda : S \cap \delta \text{ is a stationary subset of } \delta\}$$

2) Define $F^n(S)$ by induction on n:

$$F^0(S) = S,\ F^{n+1}(S) = F(F^n(S)).$$

17. Claim : 1) $FF(S) \subseteq F(S)$.

2) $F(S^*(\lambda)) \subseteq S^*(\lambda)$, hence $F^n(S^*(\lambda)) \subseteq F^m(S^*(\lambda))$ if $n > m \geq 0$.

3) $\delta \in F^n(S)$ implies $cf\delta \geq \aleph_n$; moreover, if $\aleph_\alpha = \min\{cf\delta : \delta \in S\}$, then $\delta \in F^n(S)$ implies $cf\delta \geq \aleph_{\alpha+n}$.

4) If $\alpha \leq \min\{cf\delta : \delta \in \bigcup_{i<\alpha} S_i\}$, $S_i \subseteq \lambda$ then

$$F(\bigcup_{i<\alpha} S_i) = \bigcup_{i<\alpha} F(S_i) \bmod D_\lambda.$$

Proof : 1) Easy

2) By 5.4 (and second part-by induction)

3), 4) Easy.

18. Lemma : Suppose λ is a singular strong limit of cofinality κ. Then for some $C \in D_{\lambda^+}$, for every $\delta \in C$, letting $\langle \alpha_i : i < cf\delta \rangle$ be increasing, continuous and converging to δ, the following holds :

$$\{i : \alpha_i \in S^*(\lambda)\} \supseteq S^*(cf\delta) \bmod D_{cf\delta}$$

Proof : Let d be as in 10. Then by 13, for some $C \in D_{\lambda^+}$, $S^*(\lambda^+) \cap C = S_0(d) \cap C$, so we need only deal with $S_0(d)$.

Now define a two-place function d^* from $cf\delta$ to κ by : $d^*(i,j) = d(\alpha_i, \alpha_j)$. It is easy to check that

$$\{\alpha_i : i \in S_0(d^*)\} \subseteq S_0(d).$$

But by 10, $S_0(d^*) \subseteq cf\delta - S^*(cf\delta)$ (remember $\kappa < cf\delta$), so we are finished.

19. Conclusion : 1) Suppose λ is a singular strong limit, χ,μ regular, $\chi\mu < \lambda$ and $(\forall\mu_1 < \mu)\mu_1^{\chi} < \mu$. Then $F[S^*(\lambda^+) \cap CF(\lambda^+,\chi)] \cap CF(\lambda^+,\mu)$ is not stationary.

2) If $n < \omega$ and $2^{\aleph_k} \leqslant \aleph_{k+n}$ for every $k < \omega$, then $F^n(S^*(\aleph_{\omega+1})) \equiv \phi \bmod D_{\aleph_{\omega+1}}$.

3) If $\aleph_\omega$ is a strong limit and $S^*(\aleph_{\omega+1})$ is stationary, then for some stationary $S \subseteq \aleph_{\omega+1}$, $F(S) = \phi$

Proof : 1) By 14 and 18.

2) Suppose $F^n(S^*(\aleph_{\omega+1}))$ is stationary. Then by 17.4 for some $k < \omega$, $F^n[S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)]$ is stationary. Hence for some $\ell < \omega$, $F^n[S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)] \cap CF(\aleph_{\omega+1},\aleph_\ell)$ is stationary. If $\ell \leqslant k+n$, this contradicts 19.3. But if $\ell > k+n$, then $(\forall\mu < \aleph_\ell)\mu^{\aleph_k} < \aleph_\ell$ (since $2^{\aleph_k} \leqslant \aleph_{k+n}$), hence we get a contradiction by 19.1. So in all cases we get a contradiction; hence $F^n(S^*(\aleph_{\omega+1}))$ is not stationary.

3) Since $S^*(\aleph_{\omega+1})$ is stationary, for some $k < \omega$, $S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)$ is stationary. Let $2^{\aleph_k} = \aleph_{k+n}$ ($n < \omega$ since $\aleph_\omega$ is a strong limit). So $k+n < \ell < \omega$ implies $(\forall\mu < \aleph_\ell)\mu^{\aleph_k} < \aleph_\ell$; hence, by 19.1, $F(S) \subseteq CF(\aleph_{\omega+1}, \leqslant \aleph_{k+n})$, where $S = S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)$. But by 17.1, $F^{n+1}(S) \subseteq F(S)$, hence $\delta \in F^{n+1}(S)$ implies $cf\delta \leqslant \aleph_{k+n}$, and by 17.2 $\delta \in F^{n+1}(S)$ implies $cf\delta \geqslant \aleph_{k+n+1}$ (since $\delta \in S \Rightarrow cf\delta = \aleph_k$), so we get that there is no $\delta \in F^{n+1}(S)$, i.e. $F^{n+1}(S) = \phi$. Since $F^0(S) = S$ is stationary, for some ℓ, $F^\ell(S)$ is stationary but $F(F^\ell(S)) = F^{\ell+1}(S)$ is not; $F^\ell(S)$ is as required.

Theorem 20 : Suppose $S \subseteq \lambda$ is stationary, and $S \subseteq gcf(\lambda) - S^*(\lambda)$, $S \subseteq CF(\lambda,\mu)$. If P is a μ^+-complete forcing (i.e. if $\langle p_i : i < \mu \rangle$ is an increasing sequence of elements of P then some $p \in P$ is $\geqslant p_i$ for every i), then S is stationary even in the universe V^P.

Remark : Remember that λ-complete forcing forces the stationariness of any $S \subseteq \lambda$.

Proof : Let $\bar{N}$ be a λ'-approximate sequence for some $\lambda' > \lambda$, such that a P-name $\underset{\sim}{C}$ of a closed unbounded subset of λ, a $p \in P$, are in N_0. So trivially there is $\delta \in S$, $A \subseteq \delta$ such that $\delta = N_\delta \cap \lambda$ and A has order type cfδ, and for every $\zeta < \delta$, $A \cap \zeta \in N_\delta$. Let $f : \mathrm{cf}\delta \to A$ enumerate A, hence $\zeta < \mathrm{cf}\delta$ implies $f|\zeta \in N_\delta$.
We want to prove that not : $p \Vdash$ "$\underset{\sim}{C}$ is disjoint from S". For this it suffices to find $q \in P$ such that $p \leq q$ and $q \Vdash$ "$\delta \in \underset{\sim}{C}$" (since $\delta \in S$). We can assume that a well-ordering $<^*$ of $P \cup P \times \lambda$ belongs to N_0. Now we define by induction on $i < \mathrm{cf}\delta$, $p_i \in N_\delta$.
We let $p_0 = p$, and for i a limit, p_i is the $<^*$-first p' which is $\geq p_j$ for every j (which exists since P is μ^+-complete).
We let p_{i+1}, β_i be such that (p_{i+1}, β_i) is the $<^*$-first pair (p', β') such that $p' \geq p_i$, $\beta' \geq f(i)$ and $p' \Vdash \beta' \in \underset{\sim}{C}$. There is such (p', β') since $\underset{\sim}{C}$ was a P-name of an unbounded subset of λ. It is easy to check that $p_i, \beta_i \in P \cap N_\delta$, so $\beta_i < \delta$. Hence $\delta = \sup\{\beta_i : i < \mathrm{cf}\delta\}$. Since P is μ^+-complete, there is $q \in P$, $p_i \leq q$ for every $i < \mathrm{cf}\delta$. So q force $\underset{\sim}{C} \cap \delta$ to be unbounded below δ. But $\underset{\sim}{C}$ was a P-name of a closed subset of δ. Hence $q \Vdash$ "$\delta \in \underset{\sim}{C}$". So we are finished.

21. Theorem : Suppose $\mu < \lambda$, μ regular. Then there is a μ-complete forcing P, such that in V^P $S^*(\lambda)$ is not stationary.

Proof : First assume $\lambda = \lambda^{<\lambda}$, so $\underset{-}{P} = \{B \subseteq \lambda : |B| < \lambda\} = \{B_i : i < \lambda\}$, each $B \in \underset{-}{P}$ appearing in $\{B_i : i < \lambda\}$ λ times, and let $\bar{B} = \langle B_i : i < \lambda \rangle$. Clearly there is a λ-approximating sequence $\bar{N}$ of λ^+, with $\bar{B} \in N_0$; and then $\underset{-}{P} \cap N_\delta = \{B_i : i < \delta\}$ for a closed unbounded set of δ's.

So (w.l.o.g.) $S^*(\lambda) \subseteq \{\delta < \lambda : N_\delta \cap \underset{-}{P} = \{B_i : i < \delta\}\}$.

$P = \{\eta = \langle \alpha_i : i \leqslant \zeta \rangle$, an increasing, continuous sequence, where $B_{\alpha_{i+1}} = \{\alpha_j : j \leqslant i\}\}$. The order on P is : $\eta_1 < \eta_2$ iff η_1 is an initial segment of η_2.

It is obvious that P is μ-complete; and if $G \subseteq P$ is generic, let $C[G] = \{\alpha_\delta : \delta$ limit, and $\langle \alpha_i : i \leqslant \xi \rangle \in G, \zeta \geqslant \delta\}$. Clearly in $V[G]$, $C[G]$ is a closed unbounded subset of λ. Now we have to prove only : $C[G] \cap S^* = \phi$, where $S^* = S^*(\lambda)^V$. Suppose, in V, for some $p \in P$, $p \Vdash$ "$\delta \in C[\underset{\sim}{G}]$" where $\delta \in S^*$. Let $p = \langle \alpha_j : j \leqslant \zeta \rangle$, so clearly for some limit $i \leqslant \zeta$, $\delta = \alpha_i$. Since $\delta \in S^*$, $N_\delta \cap \{B_i : i < \lambda\} = \{B_i : i < \delta\}$, and there is no unbounded $A \subseteq \delta$ of order type $cf\delta$, such that $\xi < \delta \Rightarrow A \cap \xi \in N_\delta$. But there is such an A namely $\{\alpha_j : j < i\}$ ($\{\alpha_j : j < j_o < i\}$ belongs to N_δ since it is $B_{j_o+1} - \{j_o\}$), contradiction. So we are finished when $\lambda = \lambda^{<\lambda}$.

If $\lambda < \lambda^{<\lambda}$, let Q be the collapsing of 2^λ to λ, i.e. $P = \{f : \text{Dom } f = \xi < \lambda, \text{Range } f \subseteq 2^\lambda\}$. Note that V^P may have a different $gcf(\lambda)$, but $S^*(\lambda)^{V^Q} \cap gcf(\lambda)^V = S^*(\lambda)^V$. Now in V^Q define P as before, and $Q * P$ (the composition) is as required.

<u>22. Conclusion</u> : Suppose λ is regular, $\mu < \lambda$ regular, $S \subseteq gcf(\lambda)$. There is a μ-complete forcing P such that in V^P, S is not stationary iff $(S - S^*(\lambda)) \cap CF(\lambda, <\mu)$ is stationary.

<u>23. Lemma</u> : Suppose λ is regular, $S \subseteq \lambda$ stationary, but $F(S) = \phi$ and for every $\alpha \in S$, A_α is an unbounded subset of α of order-type $cf\alpha$.

Then for every $S' \subseteq S$ with $|S'| < \lambda$, the family $\{A_\alpha : \alpha \in S'\}$ has a transversal (=one-to-one choice function). Moreover we can find $A'_\alpha \subseteq A_\alpha$ ($\alpha \in S'$), $|A'_\alpha| < cf\alpha$, such that the sets $A_\alpha - A'_\alpha$ ($\alpha \in S'$) are pairwise disjoint.

However $\{A_\alpha : \alpha \in S\}$ does not have a transversal.

Proof : See [Sh 1].

24. Lemma : Suppose λ is singular strong limit, $\kappa = cf\lambda$, $S^*(\lambda^+) = \phi \bmod D_{\lambda^+}$, and let

$$S = \{\delta < \lambda^+ : cf\delta \neq \kappa, \aleph_0 \text{ , and } \lambda\omega \text{ divides } \delta\}$$

Then we can define $A_\alpha \subseteq \alpha$ ($\alpha \in S$), A_α unbounded in α and with order-type $\kappa(cf\alpha)$ (ordinal multiplication), such that

A) $\{A_\alpha : \alpha \in S\}$ has no transversal

B) For every $S' \subseteq S$ with $|S'| < \lambda^+$, $\{A_\alpha : \alpha \in S'\}$ has a transversal. Moreover

B') For every $S' \subseteq S$ with $|S'| < \lambda^+$, there are $A'_\alpha \subseteq A_\alpha (\alpha \in S')$ such that :

(i) they are pairwise disjoint,

(ii) A'_α is a big [and even very big] subset of A_α, which means that there is a closed (in A_α) unbounded [resp. cobounded] $C \subseteq A^*_\alpha$ so that

$$(\forall\delta \in C)\ (\exists\zeta < \kappa)\ (\forall\xi)\ (\delta + \zeta \leqslant \xi < \delta + \kappa \rightarrow \xi \in A'_\alpha).$$

Proof : Stage A :

There is a normal $d : \lambda^+ \rightarrow \kappa$, $\lambda = \sum_{i<\kappa} \lambda_i, \lambda_i < \lambda$, $|\{\beta < \alpha : d(\alpha,\beta) \leqslant i\}| \leqslant \lambda_i$, such that for every $\delta < \lambda^+$, $cf\delta \neq \kappa$, there is $A \subseteq \delta$, $\sup A \square \delta$, $d|A$ bounded, and each $i \in A$ is a successor.

Pf : Let d be from 10, then $S_1(d) \equiv \phi \bmod D_{\lambda^+}$, hence there is a closed unbounded $C \subseteq \lambda^+$, $C \cap S_0(d) = \phi$. Let $C = \{\alpha_i : i < \lambda^+\}$, α_i increasing and continuous, $\alpha_0 = 0$. For each $i < \lambda^+$, we can find $A^i_\zeta \subseteq (\alpha_i, \alpha_{i+1})(\zeta < \kappa)$ such that : $|A^i_\zeta| \square \lambda_\zeta$, A^i_ζ is closed (in the interval), if $\delta \in A^i_\zeta$ is a limit then $\delta = \sup(\delta \cap A^i_\zeta)$, $\alpha_{i+1} = \sup A^i_\zeta$, for some ζ.

A^i_ζ increases with ζ and $(\alpha_i, \alpha_{i+1}) = \bigcup_{\zeta<\kappa} A^i_\zeta$. Now we define d' by :

if $\alpha < \beta$ then $d'(\beta,\alpha) = d(\beta,\alpha)$ if $(\exists i)(\beta \geq \alpha_i > \alpha)$, and otherwise $d'(\beta,\alpha) = \min \{d(\beta,\alpha), \min\{\zeta : \alpha,\beta \in A^i_\zeta\}\}$. It is easy to check that d' is as required. For showing that every $i \in A$ is a successor, use subadditivity.

Stage B :

For any $\alpha < \lambda^+$ the family

$$\underline{P}_\alpha = \{A \subsetneq \alpha : |A| < \lambda,\ d|A \text{ is bounded, } \mathrm{cf}(\sup A) \neq \kappa\}$$

has cardinality $\leq \lambda$.

Pf : Let $\alpha = \bigcup_{i<\kappa} B_i$, $|B_i| < \lambda$, B_i increasing, and let, for $i < \kappa$, $\zeta < \kappa$, $\underline{P}^\zeta_{\alpha,i} = \{A \in \underline{P}_\alpha : A \cap B_i$ unbounded in A, $d|A$ bounded by $\zeta\}$. Since $A \in P_\alpha \Rightarrow [\mathrm{cf}(\sup A) \neq \kappa$ and $d|A$ bounded$]$, and by the choice of the B_i's, $\underline{P}_\alpha = \bigcup_{\zeta,i<\kappa} \underline{P}^\zeta_{\alpha,i}$, it suffices to prove $|\underline{P}^\zeta_{\alpha,i}| \leq \lambda$ (for given $i,\zeta < \lambda$). Let $B^\zeta_i = B_i \cup \bigcup_{\beta\in B_i} \{\gamma : \gamma < \beta,\ d(\beta,\gamma) \leq \zeta\}$. Clearly $|B^\zeta_i| \leq |B_i| + \lambda_\zeta < \lambda$, and $A \in \underline{P}^\zeta_{\alpha,i}$ implies $A \subseteq B^\zeta_i$. So $|\underline{P}^\zeta_{\alpha,i}| \leq 2^{|B_i| + \lambda_i} < \lambda$, so we have proved stage B.

Stage C :

If P is a family of subsets of A each of cardinality $< \lambda$, but $|\underline{P}| \leq |A| = \lambda$, then there is a set $C \subsetneq A$ such that

(i) $|C| = \kappa$,

(ii) $(\forall A \in P)\ |A \cap C| < \kappa$.

This is trivial.

Stage D :

We define the A_α's by induction on α for $\alpha \in S$. Suppose we arrive at α. Let $\langle \gamma_i : i < \mathrm{cf}\alpha \rangle$ be increasing with limit α, $\gamma_i + \lambda \leq \gamma_{i+1}$.

For a set A of ordinals, let $\mathrm{acc}(A) = \{\delta : \delta$ a limit, $\delta = \sup(A \cap \delta)\}$ (= the set of accumulation points of A). By stage B, $|\underline{P}_\alpha| \leq \lambda$, so by stage C we can find $c^i_\alpha \subsetneq (\gamma_i, \gamma_i + \lambda)$, of power κ such that :

(*) for every $A \in P_\alpha \cup \{\cup\{A_\gamma : \gamma < \alpha, \gamma \in acc(A)\} : A \in P_\alpha\}$, its intersection with c_α^i has power $<\kappa$.

In fact we have to check that $|\cup\{A_\gamma : \gamma < \alpha, \gamma \in acc(A)\}| < \lambda$ (for $A \in \underline{P}_\alpha$), but this is easy : $\lambda \in acc(A) \Rightarrow cf\lambda \leqslant |A| \Rightarrow |A_\gamma| \leqslant \kappa + cf\gamma = \kappa + |A|$, hence the set has power $\leqslant (\kappa + A)\ |A| < \lambda$. We let $A_\alpha = \bigcup_{i<cf\alpha} c_\alpha^i$.

Stage E :

$\{A_\alpha : \alpha \in S\}$ has no transversal.

Because $A_\alpha \subseteq \alpha$, by Fodor's theorem.

Stage E :

We prove (A*) from the lemma. We prove by induction on α that there are big $A'_\beta \subseteq A_\beta$ $(\beta \leqslant \alpha, \beta \in S)$, pairwise disjoint. This will clearly suffice.

Case 1 : For α a successor ordinal, it follows from the induction hypothesis on α-1.

Case 2 : For α such that $(\exists\beta < \alpha)\ \beta + \lambda\omega > \alpha$: proof as in the first case.

Case 3 : For α a limit, $cf\alpha = \aleph_0$. Choose ordinals $\alpha_n < \alpha$, $\alpha_n < \alpha_{n+1}$, $\alpha = \cup\, \alpha_n$, $\alpha_0 = 0$. For each n, by the induction hypothesis there are big $A_\beta^n \subseteq A_\beta$ $(\beta \leqslant \alpha_n)$, pairwise disjoint.

Define A'_β , for $\beta \leqslant \alpha$, $\beta \in S$ (hence $\beta \neq 0$), by :

$$A'_\beta = A_\beta^{n+1} - (\alpha_n + \lambda), \text{ where } \alpha_n < \beta \leqslant \alpha_{n+1}$$

It is easy to check that $A'_\beta \subseteq A_\beta$ is still big, and obviously the A'_β are pairwise disjoint. Note that $\alpha \in S$, so we do not have to define A'_α.

Case 4 : For α limit, not case 2, $cf\alpha > \aleph_0$. There is $E \subseteq \alpha$, unbounded, of order type $cf\alpha$ (hence $<\lambda$) and $E = \{\beta_{i+1} : i < cf\alpha\}$ (the β_i increasing), such that $d|E_i$ is unbounded for $i < cf\alpha$, where

$E_i = \{\beta_{j+1} : j < i\}$, and each β_{i+1} is a successor ordinal. (For $cf\alpha \leqslant \kappa$, any unbounded A of order type $cf\alpha$ is as required). (Remember d is from stage A).

We can define for limit $\delta \leqslant cf\alpha$, $\beta_\delta = \sup\{\beta_{i+1} : i < \delta\}$.

Since $\beta_i + \lambda < \alpha$, we can assume w.l.o.g. $\beta_i + \lambda < \beta_{i+1}$ (by making deletions if necessary). Let $A^i_\beta \subseteq A_\beta$ be big, pairwise disjoint, for $\beta \leqslant \beta_i$ (possible by the induction hypothesis).

We now define A'_β, if $\beta \notin \bigcup_{i<cf\alpha} [\beta_i, \beta_i + \lambda) \cup \{\alpha\}$, by :

$A'_\beta = A^i_\beta - (\beta_i + \lambda)$, where $\beta_i + \lambda < \beta \leqslant \beta_{i+1}$.

Clearly, the $A'_\beta \subseteq A_\beta$ are big, pairwise disjoint and disjoint from $D =^{df} \bigcup_{i<cf\alpha} [\beta_i, \beta_{i+1} + \lambda)$. For which β's have we still not defined A'_β ? For $\beta = \beta_i$ ($i \leqslant cf\delta$) i.e., $\beta = \beta_j$, for which $\beta \in S$, hence $cfj \neq \aleph_0, \kappa, 1$. Checking definitions we can see that for each such β, $A_\beta \cap D \subseteq A_\beta$ is big. So it suffices to find pairwise disjoint big $A'_{\beta_j} \subseteq A_{\beta_j}$ ($j \leqslant cf\delta$, j a limit). This we do by induction on j. Suppose we have defined these for every $j' < j$. For j a successor among $\{i \leqslant cf\delta : i \text{ a limit}\}$ or $\beta_j \notin S$, there is no problem. (Remember for j a successor, β_j is a successor, hence $\notin S$). Otherwise, note that $cfj \neq \kappa$, hence $cf(\sup(E_j)) \neq \kappa$, hence $E_j \in P_\alpha$ (see stage B). Now look at Stage D, for β_j. We chose there an increasing continuous sequence of ordinals $\langle \gamma_i : i < cf\,\beta_j \rangle$ converging to β_j. Since $cf\,\beta_j \neq \aleph_0$, there is a closed unbounded $C \subseteq cf\,\beta_j$, such that $i \in C \Rightarrow \gamma_i \in \{\beta_\xi : \xi < j\}$. We then defined $A_{\beta_j} = \bigcup_{i<cf\beta_j} c^i_{\beta_j}$, where $c^i_{\beta_j} \subseteq (\gamma_i, \gamma_i + \lambda)$, has order type κ, and in particular

$$[\cup\{A_\zeta : \zeta \in \delta, \zeta \in acc(E_j)\}] \cap c^i_{\beta_j} \quad \text{has power} < \kappa.$$

But what is $acc(E_j)$? It is just $\{\beta_{j(o)} : j(o) < j, j(o) \text{ a limit}\}$. So $c^i_{\beta_j} \cap [\cup\{A_{j(o)} : j(o) < j, j(o) \text{ a limit}, A_{j(o)} \text{ defined}\}]$ has power $< \kappa$.

Let $A'_{\beta_j} = \cup\{c^i_{\beta_j} - \cup\{A_\zeta : \zeta \in S, \zeta \in acc(E_j)\} : i \in C\}$.

It is easy to check that it is a big subset of A_{β_j}, and obviously it is disjoint from $A_{\beta_{j(o)}}$, where $j(o) < j$ is a limit. So we have finished the proof.

Stage E : Suppose λ singular strong limit, $cf\lambda = \kappa$, S a stationary subset of λ^+, and every member of S divisible by $\lambda\omega$. Suppose further $A_\alpha \subseteq \alpha$, $|A_\alpha| \leq \kappa cf\alpha$ for $\alpha \in S$, and for any $\alpha_o < \lambda^+$, $\{A_\alpha : \alpha < \alpha_o\}$ has a transversal. Then we can find $A^*_\alpha \subseteq \alpha$ for $\alpha \in S$, so that $A^*_\alpha = \{\gamma(\alpha,i) : i < \kappa(cf\alpha)\}$, where $\gamma(\alpha,i)$ increase with i, (hence $|A^*_\alpha| \leq cf\alpha + \kappa$ $(< \lambda)$) and for every $\alpha_o < \lambda^+$ there are pairwise disjoint $A'_\alpha \subseteq A_\alpha$ (for $\alpha < \alpha_o$, $\alpha \in S$), such that for each α for some $i_o < cf\alpha$

$$(\forall i < cf\alpha)(\exists \zeta < \kappa)(\forall \xi)(\zeta \leq \xi < \kappa \;\&\; i_o < i \rightarrow \gamma(\alpha,\kappa i+\xi) \in A'_\alpha).$$

Proof : For every α, choose $B^\xi_\alpha \subseteq \alpha$, B^ξ_α increase with ξ, $\alpha = \bigcup_{\xi<\kappa} B^\xi_\alpha$ and $|B^\xi_\alpha| < \lambda$. We can define functions h_o, h_1, Dom $h_\ell = \lambda^+$, so that for any β_o, $\beta_1 \leq \beta < \lambda^+$, $\xi < \kappa$, $A \subseteq B^\xi_{\beta_o}$, there are λ β^*'s, $\beta \leq \beta^* < \beta + \lambda$, such that $h_1(\beta^*) = \beta_1$, $h_2(\beta^*) = A$. (We define $h_\ell \restriction [\lambda i, \lambda(i+1))$ for each i; the number of possible tuples $< \beta_1, A, \beta, \xi, \beta_o >$ is $\leq \lambda$, so there is no problem).

For each $\alpha \in S$ choose an increasing sequence $\beta(\alpha,i)$ $(i < cf\alpha)$ converging to it.

First note that $(\forall \alpha_o < \alpha)$ $\alpha_o + \lambda < \alpha$ (since $\alpha \in S$) hence w.l.o.g. $\beta(\alpha,i) + \lambda < \beta(\alpha,i+1)$, and $\beta(\alpha,i)$ is divisible by λ.

Now we define by induction on $j = i\kappa + \xi$ $(i < cf\alpha$, $\xi < \kappa)$ an ordinal $\gamma(\alpha,j)$, increasing with j, such that

(i) $\beta(\alpha,i) < \gamma(\alpha,j) < \beta(\alpha,i) + \lambda$,

(ii) $h_1(\gamma(\alpha,j)) = cf\alpha$,

(iii) $h_2(\gamma(\alpha,j)) = A_\alpha \cap B^\xi_{\beta(\alpha,i)}$, and

(iv) $\gamma(\alpha,j) \notin \{A^*_{\alpha(o)} : \alpha(o) \in B^\xi_\alpha\}$.

The last condition excludes $< \lambda$ γ's, and the conditions (ii), (iii)

are satisfied by λ γ's, $\beta(\alpha,i) < \gamma < \beta(\alpha,i) + \lambda$.
So we can define $A^*_\alpha = \{\gamma(\alpha,i) : i < \kappa(cf\alpha)\}$, and $\gamma(\alpha,i)$ increase with i and converge to α.

Now we are given $\alpha(o) < \lambda^+$ and have to find $A'_\alpha \subseteq A^*_\alpha$ as required. By hypothesis, there is a transversal f of $\{A_\alpha : \alpha < \alpha(o)\}$.
Define $A^1_\alpha = \{\gamma(\alpha,\kappa i + \xi) : i < cf\alpha,\ f(A_\alpha) \in A_\alpha \cap B^\xi_{\beta(\alpha,i)}\}$.

Clearly it is a very big subset of A_α.

On $S \cap \alpha(o)$ we define a graph : (α_1,α_2) is an edge iff $A^1_{\alpha_1} \cap A^1_{\alpha_2} \neq \phi$.

Note :

(a) If (α_1,α_2) is an edge then $cf\alpha_1 = cf\alpha_2$ (because $\gamma \in A_{\alpha_\ell}$ implies $h_1(\gamma) = cf\alpha_\ell$).

(b) The valency of any α_1 (= $|\{\alpha_2 : (\alpha_1,\alpha_2)$ is an edge $\}|$) is $\leq |A^*_\alpha|$.

As f is one-to-one, it suffices to prove that $f(A_{\alpha_2}) \in A_{\alpha_1}$ whenever $A_{\alpha_2} \cap A_{\alpha_1} \neq \phi$. If $\gamma = \gamma(\alpha_1,\kappa i_1 + \xi_1) = \gamma(\alpha_2,\kappa i_2 + \xi_2) \in A^1_{\alpha_1} \cap A^1_{\alpha_2}$, then $\beta = \beta(\alpha_1,i_1) = \beta(\alpha_2,i_2)$ (it is the biggest ordinal $\leq \gamma$ divisible by λ), so $A_{\alpha_1} \cap B^{\xi_1}_{\beta(\alpha_1,i_1)} = h_2(\gamma) = A_{\alpha_2} \cap B^{\xi_2}_{\beta(\alpha_2,i_2)}$, but $f(A_{\alpha_2}) \in A_{\alpha_2} \cap B^{\xi_2}_{\beta(\alpha_2,i_2)}$ (since $\gamma \in A^1_{\alpha_2}$) hence $f(A_{\alpha_2}) \in A_{\alpha_1} \cap B^{\xi_1}_{\beta(\alpha_1,i_1)} \subseteq A_{\alpha_1}$, as required.

Now we deal with each component C of the graph separately. By (a), all $\alpha \in C$ have the same cofinality, say μ, and by b), $|C| \leq \kappa + \mu$. If $\mu > \kappa$ note that each A^1_α has order type μ and is unbounded below α, hence $\alpha_1 \neq \alpha_2 = C \Rightarrow |A^1_{\alpha_1} \cap A^1_{\alpha_2}| < \mu$.
So let $C = \{\alpha_\zeta : \zeta < \mu\}$, and we can define $A^*_{\alpha_\zeta} = A^1_{\alpha_\zeta} - \bigcup_{\xi<\zeta} A^1_{\alpha_\xi}$, which are as required. If $\mu \leq \kappa$, we give a similar treatment to each $\{\gamma(\alpha,\kappa i + \xi) : \xi < \kappa\}$ for $i < \mu$, $\alpha \in C$.

<u>25. Conclusion</u> :

1) Suppose $\aleph_\omega$ is a strong limit.

a) There is a family of $\aleph_{\omega+1}$ countable subsets of $\aleph_{\omega+1}$ which does not have a transversal, but every subfamily of cardinality $< \aleph_{\omega+1}$ has a transversal.

b) There is an abelian group [group] of power $\aleph_{\omega+1}$, which is not free, but every subgroup of cardinality $< \aleph_{\omega+1}$ is.

2) Suppose $\aleph_{\omega\ell}$ is strong limit for $\ell \leqslant n$. Then a), b) hold for $\aleph_{\omega n+1}$.

Proof : 1 a), 2 a). It is easy to see this after reading Milner and Shelah [MS].

1 b), 2 b) are easy to see.

26. Claim : Suppose λ is strong limit, $cf\lambda = \aleph_0$, $\mu < \kappa$, μ regular and : P is μ-complete or among any μ members of P there are μ which are pairwise compatible.

If in V^P λ is still a strong limit cardinal, then

$$S^*(\lambda^+)^V \cap CF(\lambda,\mu)^V, \; S^*(\lambda^+)^{V^P} \cap CF(\lambda,\mu)^{V^P}$$

are equal (i.e., for some representation they are equal).

Proof : Let $d : \lambda^+ \to \kappa$ be normal. Clearly it is still normal in V^P. By 13 it suffices to prove that the truth value of "$\alpha \in S_1(d)$" is not changed, which is quite easy.

27. Claim : If χ is supercompact, $\lambda > \chi$, $cf\lambda < \chi$, then $S^*(\lambda^+)$ is stationary.

Proof : Let $d : \lambda^+ \to cf\lambda$ be normal and subadditive, and suppose $C \subseteq \lambda^+$ is closed and unbounded.

Suppose $N \prec (H(\lambda^{++}), \in)$, $cf\lambda + 1 \subsetneq N$, $C,d \in N$, $\|N\| < \chi$ and every subset of $N \cap \lambda^+$ belongs to N (this is possible as χ is supercompact). Let $\delta^* = \sup(N \cap \lambda^+)$. Clearly $cf\delta^*$ is the successor of a singular cardinal of cofinality $cf\lambda$ so $cf\delta^* > cf\lambda$. Clearly $C \cap N$ is unbounded, hence $\delta^* \in C$; so it suffices to prove $\delta^* \notin S_0(d)$.

So suppose $A \subseteq \delta^*$ is unbounded, and $d|A$ is bounded by ζ. Let $A = \{\beta_i : i < \delta^*\}$, β_i increasing. We may assume, w.l.o.g., for each i there is γ_i, $\beta_i < \gamma_i < \beta_{i+1}$, $\gamma_i \in N$. Let $\zeta_i = \text{Max}\{\zeta, d(\beta_{i+1},\gamma_i), d(\gamma_i,\beta_i)\} < cf\lambda < cf\delta^*$. So (w.l.o.g.) $\zeta_i = \zeta^*$ for every i. Now if $i < j$, then by the subadditivity :

$$d(\gamma_i,\gamma_j) \leq \max\{d(\gamma_j,\beta_{j+1}), d(\beta_{j+1},\beta_{i+1}), d(\beta_{i+1},\gamma_i)\} \leq \zeta^*$$

So $d|\{\gamma_i : i < cf\delta^*\}$ is bounded, but the set necessarily belongs to N, and, as $N \prec (H(\lambda^{++}), \in)$, there is an unbounded $B \subseteq \lambda^+$ on which d is bounded, giving an easy contradiction to normality.

28. Remark : We in fact prove that if d is a subadditive function, with domain α^*, $\alpha \leq \alpha^*$, and d is bounded on some unbounded $A \subseteq \alpha$, then every unbounded $A' \subseteq \alpha$ has an unbounded subset $A'' \subseteq A' \subseteq \alpha$ such that $d|A''$ is bounded.

29. Conclusion : If ZFC + "$\exists$ a supercompact" is consistent then the following is consistent :

$$\text{ZFC} + \text{GCH} + \text{"}S^*(\aleph_{\omega+1}) \text{ is stationary"}.$$

Proof : Suppose χ is supercompact, and also (w.l.o.g.) GCH holds. Let λ be the first singular cardinal $> \chi$. By 27 we can choose a regular $\mu < \chi$ such that $S^*(\lambda^+) \cap CF(\lambda^+,\mu)$ is stationary. We use Levy collapsing P to collapse every $\mu' < \mu$ to $\aleph_0$ (by finite conditions). So now, in V^P, μ is $\aleph_1$. By 26, in V^P, $S^*(\lambda^+)^{V^P} \supseteq S^*(\lambda^+)^V \cap CF(\lambda^+,\mu)^V$, and the latter obviously remains stationary. Now collapse χ to $\aleph_1$ by a Q which is $\aleph_1$-complete. Again $S^*(\lambda^+)^V \cap CF(\lambda^+,\mu)^V$ remains stationary and is still included in $S^*(\lambda^+)^{P*Q}$.

$\diamond_\lambda$ is not a strong requirement

30. Definition : Let λ be a regular cardinal and $E \subseteq \lambda$ a stationary

set in it.

(1) $\Diamond^*_\lambda(E)$. There is $< W_\alpha : \alpha \in E >$ such that for every α, W_α is a family of subsets of α with $|W_\alpha| \leqslant |\alpha|$, and for every $X \subseteq \lambda$ there is a closed and unbounded $C \subseteq \lambda$ such that $X \cap \alpha \in W_\alpha$ for all $\alpha \in C \cap E$.

(2) $\Diamond_\lambda(E)$. There is $< S_\alpha : \alpha \in E >$ such that $S_\alpha \subseteq \alpha$, and for every $X \subseteq \lambda$, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary in λ.

<u>31. Theorem</u> : (Kunen) : (1) For stationary $E \subseteq \lambda$, $\Diamond^*_\lambda(E)$ implies $\Diamond_\lambda(E)$.

(2) For $E_1 \subseteq E_2 \subseteq \lambda$, $\Diamond_\lambda(E_1)$ implies $\Diamond_\lambda(E_2)$ and $\Diamond^*_\lambda(E_2)$ implies $\Diamond^*_\lambda(E_1)$.

<u>32. Theorem</u> : Suppose $\lambda = 2^\mu = \mu^+$ and for some regular $\kappa < \mu$, either

(i) $\mu^\kappa = \mu$, or

(ii) μ is singular $\kappa \neq cf\mu$ and for every $\delta < \mu, |\delta|^\kappa < \mu$

<u>Then</u> $\Diamond^*_\lambda(E(\kappa))$ where $E(\kappa)$ is the stationary subset $\{\alpha < \lambda : cf\alpha = \kappa\}$.

<u>Remark</u> : Case (i) is due to Gregory [Gr].

<u>Proof</u> : Let $<A_\alpha : \alpha < \lambda >$ be a list of all bounded subsets of λ each appearing λ times (there are λ such subsets as $\lambda = 2^\mu = \mu^+$)

<u>Case (i)</u> : For $\alpha \in E(\kappa)$ let W_α be the set of all unions of no more than κ subsets of α belonging to $< A_\beta : \beta < \alpha>$.

$(W_\alpha = \{\cup Y : |Y| \leqslant \kappa, x \in Y \to x \subseteq \alpha, x \in \{A_\beta : \beta < \alpha\}\})$.

Given $X \subseteq \lambda$, let C be $\{\alpha_i \mid i < \lambda\}$ where α_0 is any successor less than $\lambda, \alpha_\delta = \bigcup_{\beta<\delta} \alpha_\beta$ for limit δ, and α_{i+1} is the least $\alpha > \alpha_i$ such that for some $\gamma < \alpha$, $A_\gamma = X \cap \alpha_i$.

Now $C' = \{\delta : \delta = \cup \{\alpha_i : \alpha_i < \delta\}\}$ is closed unbounded, and for $\delta \in C \cap E(\kappa)$ there are $i(j)$ and $\gamma_j < \delta$ $(j < \kappa)$ such that

$\bigcup_{j<\kappa} \alpha_{i(j)} = \delta$, $X \cap \alpha_{i(j)} = A_{\gamma_j}$ .So $X \cap \delta = \bigcup_{j<\kappa} A_{\gamma_j} \in W_\delta$.

Case (ii) : For δ such that $cf\delta = \kappa$, let $\delta = \bigcup_{j<\mu} V_j^\delta$, where $\langle V_j^\delta : j < \mu \rangle$ is increasing and for $j < \mu$, $|V_j^\delta| < \mu$.

Let W_δ be $\{\bigcup_{\alpha\in Q} A_\alpha : (\exists j < \mu)\ Q \subseteq V_j^\delta,\ |Q| \leqslant \kappa\}$.

Given $X \subseteq \lambda$ let $f : \lambda \to \lambda$ be such that $X \cap \alpha = A_{f(\alpha)}$ $f(\alpha) > \sup f(\beta)$.

There exists a closed unbounded $C \subseteq \lambda$ such that for $\alpha \in C$, $\beta < \alpha$ implies $f(\beta) < \alpha$.

Let $\delta \in C \cap E(\kappa)$, and for increasing $\langle \delta_i : i < \kappa \rangle$ $\delta = \bigcup_{i<\kappa} \delta_i$.

There exists j such that

$$\kappa = |V_j^\delta \cap \{f(\delta_i) : i < \kappa\}| \text{ hence } X \cap \delta = \cup\{X \cap \delta_i : i < \kappa, f(\delta_i) \in V_j^\delta\} \in W_\delta.$$

33. Conclusion : (GCH) If $\lambda > \aleph_0$, then $\diamondsuit^*_{\lambda^+}(E(\kappa))$ holds, whenever $\kappa \neq cf\lambda$. In particular $\diamondsuit_\lambda$ holds.

Final comments

1) The restriction "λ strong limit" in most cases can be weakened at the expense of complicating the results : assuming $(\forall\mu < \lambda)$ $\mu^{<\chi} < \lambda$, and restricting ourselves to $CF(\lambda^+, <\chi)$ or $CF(\lambda^+, \leqslant \chi)$.

2) A more serious question is whether we can, in 7, replace D_λ^g by D_λ. This remains open.

Note that the natural notion is $S_2(\bar{N})$, and that for regular λ, $I^+(\lambda) = \{A \subseteq \lambda :$ for some λ-approximating sequence $\bar{N}$, $A \subseteq S_2(\bar{N})\}$ is always a normal ideal. Similarly

$I^-(\lambda) = \{A \subseteq \lambda : A \cap B \equiv \phi \bmod D_\lambda$ for every $B \in I^+(\lambda)\}$

is a normal ideal. The meaning of claim 7 is that $I^+(\lambda)$ is $\{A : A \subseteq A_0 \bmod D_\lambda\}$ for some A_0, when $gcf(\lambda) = \lambda$. Another formulation of our question is whether this always holds.

However, we can meanwhile just formulate the later theorems in terms of $I^+(\lambda)$ instead of $S^*(\lambda)$ (and the changes in the proofs are minor). By the way it may be more natural to use $S_3(\bar{N}) = \{\delta :$ there is a function h, Dom h = cfδ, Range h an unbounded subset of δ, $(\forall i < \mathrm{cf}\delta)$ $h|i \in N_\delta$, and $N_\delta \cap \lambda = \delta\}$ (in gcf(λ) it does not matter).

3) Why were we interested mainly in $\aleph_{\omega+1}$ and not in e.g. $\aleph_{\omega+2}$? The answer is that several inductive proofs work for successors of regular cardinal, and it was not clear whether they fail at successors of singulars. (But see remarks 5 and 6 below).

4) It may be of interest to mention our original line of thought, which is not so transparent from the present paper.

We want to prove that $S_2(\bar{N})$ is quite "big", where $\bar{N}$ is an $\aleph_{\omega+1}$-approximating sequence for $\aleph_{\omega+1}$, assuming GCH. So we let $d : \aleph_{\omega+1} \to \aleph_0$ be normal, and using the Erdös-Rado theorem $(2^{\aleph_n})^+ \to (\aleph_{n+1})^2_{\aleph_0}$, prove that if $C \subseteq \aleph_{\omega+1}$ is closed of order type $(2^{\aleph_n})^+$ then it contains C_1 of order type $\aleph_{n+1}$, with d constant on C_1. C_1' (the set of accumulation points of C_1) is $\subseteq S_2(\bar{N})$ and is a closed subset of C of order type $\aleph_{n+1}$. This proves that $S_2(\bar{N})$ is in some sense big.

5) We can try to generalize 4) to other cardinals.

Let $\kappa = \mathrm{cf}\, \aleph_\alpha < \aleph_\alpha$.

<u>Definition</u> : Call an (n+1)-place function d from $\aleph_{\alpha+n}$ to κ <u>normal</u> if for every $\alpha_0 < \ldots < \alpha_n < \aleph_{\alpha+n}$ there is $k < n$ such that

$$\{\alpha < \aleph_{\alpha+n} : d(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \ldots, \alpha_n) = d(\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n)\}$$

has cardinality $< \aleph_\alpha$.

<u>Claim</u> : There is a normal function $d : \aleph_{\alpha+n} \to \kappa$.

<u>Proof</u> : By induction on n.

<u>Lemma</u> : Let $\bar{N}$ be an $\aleph_{\alpha+n}$-approximating sequence for $\aleph_{\alpha+n+1}$, C a closed subset of $\aleph_{\alpha+n}$ of order type $\beth_{n+1}(\kappa + \mu)^+$, where $\mu < \lambda$.

Then C has a closed subset of order type μ^+ which is included in $S_2(\overline{N})$.

Proof : Let $d \in N_o$, $d : \aleph_{\alpha+n} \to \kappa$, d normal. By the Erdös-Rado theorem ($\beth_{n+1}(\kappa + \mu)^+ \to (\mu^+)^{n+1}_\kappa$) there is $C_1 \subseteq C$ of order type μ^+ on which d is constant. If $\delta \in C_1'$, then $C_1 \cap \delta$ witnesses that $\delta \in S_2(\overline{N})$.

6) Suppose $\aleph_\alpha$ is strong limit, $\kappa \blacksquare \mathrm{cf}\, \aleph_\alpha$, γ a successor ordinal, $\kappa \leqslant \mu < \aleph_\alpha$ and $\beth_\gamma(\mu) < \aleph_\alpha$. If $\overline{N}$ is a $\aleph_{\alpha+\gamma}$-approximating sequence for $\aleph_{\alpha+\gamma+1}$, and $C \subseteq \aleph_{\alpha+\gamma}$ has order type $\beth_\gamma(\mu)^+$, then C has a closed subset C_1 of order type μ^+ which is included in $S_2(\overline{N})$.

Proof : We prove a somewhat stronger statement :
If $C \subseteq \aleph_{\alpha+\beta}$, $\beta \leqslant \gamma$ a successor ordinal, and C has order type $\geqslant \beth_\beta(\mu)^+$, then there is $C_1 \subseteq C \cap S_2(\overline{N})$ of order type μ^+, such that for some $\ell < n$, if $\alpha_o < \ldots < \alpha_n \in C_1$ then
$(H(\aleph_{\alpha+\gamma+1}), \in) \models \varphi(\alpha_o, \ldots, \alpha_n)$ & $|\{x : \varphi(\alpha_o, \ldots, \alpha_{\ell-1}, x, \alpha_\ell, \ldots, \alpha_n)\}| < \aleph_\alpha$.
(This imples $C_1' \subseteq S_2(\overline{N})$).

We prove this by induction on β. For finite β this was done above, and the induction step is easy.

REFERENCES

[E] Eklof.

[F] L. Fuchs. Infinite abelian groups, Academic Press, N.Y. & London, Vol. I 1970, Vol. II 1973.

[Gr] Gregory, J. Symb. Logic, Sept. 1976.

[MS] E. Milner and S. Shelah, Two theorems on transversals, Proc. Symp. in Honour of Erdös 60th Birthday, Hungary 1973.

[Sh 1] S. Shelah, Notes in partition calculus, Proc. of Symp. in Honour of Erdös 60th Birthday, Hungary 1973.

[Sh 2] ———, A compactness theorem for singular cardinals. Free Algebras, Whitehead problem and transversals, Israel J. Math. 21 (1975), 319-349.