HOMOGENEOUS FORCING
1257

SAHARON SHELAH

ABSTRACT. Assume $\kappa = \aleph_0$ or $\kappa = \kappa^{<\kappa} > \aleph_0$, usually an inaccessible.

We shall deal with iterated forcings preserving ${}^{\kappa>}\mathrm{Ord}$ and not collapsing cardinals along a linear order. The aim is to have homogeneous ones, so that for some natural ideals on ${}^{\kappa}2$, we get a model of $\mathsf{ZF} + \mathsf{DC}_\kappa$ + "modulo this ideal, every set is equivalent to a κ-Borel one."

The main application is improving the consistency result of Kellner and Shelah [KS11], and Horowitz and Shelah [HS] on saccharinity. But presently, the homogeneity is only forcing $(Q_t, \mathbf{q} \restriction L_{\mathbf{q},t})$.

§ 0. INTRODUCTION

§ 0(A). **Aim.** Fix $\kappa = \kappa^{<\kappa}$ (maybe $\aleph_0$) and we consider homogeneous iteration of $(<\kappa)$-complete forcing notions, with a version of κ^+-cc, preserving those properties.

To get homogeneity we intend to iterate along a linear order which is quite homogeneous (and so not well-ordered).

Ever since Solovay's celebrated work [Sol70], we know about the connection between the following two issues:

- $\bullet_1$ Forcing notions $\mathbb{P}$ with lots of automorphisms. E.g. for small $\mathbb{P}' \lessdot \mathbb{P}$ and two relevant $\mathbb{P}$-names $\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2$, generic for the same relevant forcing $\mathbb{Q}$ over $\mathbf{V}^{\mathbb{P}'}$, there is an automorphism of $\mathbb{P}$ over $\mathbb{P}'$ mapping $\underset{\sim}{\eta}_1$ to $\underset{\sim}{\eta}_2$.
- $\bullet_2$ Models of $\mathsf{ZF} + \mathsf{DC}+$ "every set of reals is equivalent to a Borel set modulo the null ideal (or other reasonable ideal)". (The relevant forcing $\mathbb{Q}$ was Random Real forcing for the null ideal — and e.g. for the meagre ideal, Cohen forcing.)

Concerning the classical case of Lebesgue measurability, another formulation is "no non-measurable set is easily definable," formulated[1] in $\mathbf{L}[\mathbb{R}]$. See the history and more in [RS04], [RS06].

Date: November 1, 2024.

2020 *Mathematics Subject Classification.* Primary 03E35; Secondary 03E25, 03E15.

Key words and phrases. set theory, forcing, iterated forcing, homogeneity, definability, axiom of choice, ZF+DC.

First typed 2022-03-25. The author thanks an individual who wishes to remain anonymous for generously funding typing services, and thanks Matt Grimes for the careful and beautiful typing. The author would like to thank the Israel Science Foundation for partial support of this research by grant 2320/23 (2023-2027).

References like e.g. [Sh:950, Th0.2$_{=\mathsf{Ly5}}$] mean that the internal label of Theorem 0.2 in Sh:950 is '$\mathsf{y5}$.' The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number 1257 on Saharon Shelah's list.

[1]That is, $\bullet_2$ holds for an inner model $\mathbf{L}[\mathcal{P}(\kappa)]^{\mathbf{V}}$ with $\mathbf{V} \models \mathsf{ZFC}$, so in $\mathbf{V}$ all 'reasonable' sets are 'measurable' for this ideal.

This applies to other ideals $\mathrm{id}(\mathbb{Q}, \underset{\sim}{\eta})$ for a definable forcing notion $\mathbb{Q}$ (mainly a ccc one) and a $\mathbb{Q}$-name $\underset{\sim}{\eta}$ of a real. Generally, it was not so easy to build such forcing notions: it required one to prove the existence of amalgamation in the relevant class of forcings. In Kellner-Shelah [KS11] it was suggested to look at so-called saccharine pairs $(\mathbb{Q}, \underset{\sim}{\eta})$, where $\mathbb{Q}$ is very non-homogeneous. (E.g. forcing with $\mathbb{Q}$ adds just one $(\mathbb{Q}, \underset{\sim}{\eta})$-generic, so we have few cases we need to build automorphisms for.)

Notation 0.1. 1) $\mathrm{id}_\partial(\mathbb{Q}, \underset{\sim}{\eta}) = \mathrm{id}_{<\partial}(\mathbb{Q}, \underset{\sim}{\eta})$ is the ideal consisting of the union of $< \partial$ Borel sets $\mathbf{B}$ such that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin \mathbf{B}$".

2) Let $\mathrm{id}_{\leq\partial}(\mathbb{Q}, \underset{\sim}{\eta})$ be $\mathrm{id}_{<\partial^+}(\mathbb{Q}, \underset{\sim}{\eta})$.

3) $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ will denote ordinals; δ will be a limit ordinal if not stated otherwise.

4) $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$

5) Recall that $\mathbb{L}_{\sigma,\sigma}$ is defined like first-order logic, but allowing $\bigwedge\limits_{i<\alpha} \varphi_i$ for $\alpha < \lambda$ and $(\exists \ldots x_i \ldots)_{i \in I} \varphi$ with I of cardinality $< \sigma$.

Comparing [KS11] to the older results:

- $\bullet_{1.1}$ The forcing $\mathbb{Q}$ collapsed no cardinal, but was not ccc; this[2] we consider a drawback.
- $\bullet_{1.2}$ The model, as in those older results, does satisfy $\mathsf{ZF} + \mathsf{DC}$.
- $\bullet_{1.3}$ The iteration was along a homogeneous linear order.
- $\bullet_{1.4}$ We get only a weak version of measurability, the ideal being $\mathrm{id}_{\leq\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$ instead of $\mathrm{id}_{<\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$.

Alternatively,

- $\bullet'_{1.4}$ Use $\mathrm{id}_{<\aleph_1}(\underset{\sim}{\eta}, \mathbb{Q}) + X$, where X is the set $\{\underset{\sim}{\eta}[\mathbf{G}] : \mathbf{G} \subseteq \mathbb{Q}^{\mathbf{L}}$ is generic over $\mathbf{L}\}$.

The next step was Horowitz-Shelah [HS], where:

- $\bullet_{2.1}$ The forcing is ccc, which is a plus.
- $\bullet_{2.2}$ The model only satisfies ZF; we do not get DC or even $\mathsf{AC}_{\aleph_0}$ — not so good.
- $\bullet_{2.3}$ Again, the iteration is along a homogeneous linear order.
- $\bullet_{2.4}$ This ideal is again $\mathrm{id}_{\leq\aleph_1}(\underset{\sim}{\eta}, \mathbb{Q})$ (or as in $\bullet'_{1.4}$ above).

Here (in 4.1) we regain both ccc (as in $\bullet_{2.1}$) as well as DC (as in $\bullet_{1.2}$). Moreover, we can demand $\mathsf{DC}_{\aleph_1}$ (or more — see §1) which is a significant plus.

We continue [She04b], [She], but do not rely on them. Instead of defining iterations we introduce them axiomatically and allow $\kappa > \aleph_0$ (in the support), but it suffices here to demand that the memory is a set, not an ideal. Unlike [She04b], the present paper does not address forcing $\mathfrak{a} > \mathfrak{d}$. Earlier continuations of [She04b], [She] were the parallels [S⁺a] and [S⁺b] (and later, their descendants [S⁺c], [S⁺d] — all in preparation). There, as in [She04b], we sometimes replace the set $I^{\mathfrak{s}}_s$ (see 1.1) by an ideal (sometimes the whole) and use more general definable forcing notions.

In our iteration we are allowed to replace $\aleph_0$ by some $\kappa = \kappa^{<\kappa}$, so the forcing notions are $(<\kappa)$-complete κ^+-cc. But we need a forcing notion analogous to the one in [HS]: this will hopefully be done in [S⁺e].

[2]Note that Solovay uses Levy collapse of an inaccessible, but the later versions use ccc ones.

§ 0(B). Preliminaries.

Hypothesis 0.2. 1) $\kappa = \kappa^{<\kappa}$ (mainly $\aleph_0$ or an inaccessible).

2) ∂ is a regular cardinal $> \kappa$.

3) D a normal filter on κ^+ such that $S^{\kappa^+}_\kappa := \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\} \in D$.

Definition 0.3. Let $\mathbb{Q}$ be a forcing notion.

1) We say $\mathbb{Q}$ is a *strong κ-forcing* (or '$(\kappa, 1)$-forcing') <u>when</u>:

- (A) If $\kappa = \aleph_0$, then $\mathbb{Q}$ is Knaster (and hence ccc).
- (B) When $\kappa > \aleph_0$:
  - $\bullet_1$ $\mathbb{Q}$ satisfies $*^1_{\kappa,D}$ (which means a strong version of the κ^+-cc; see below in 0.3(4) and more in [She22, 0.2(B)(2)$_{a\text{=Lx2}}$]).
  - $\bullet_2$ $\mathbb{Q}$ is $(<\kappa)$-complete.
  - $\bullet_3$ Any increasing sequence of length $< \kappa$ has a lub.[3]

2) $\mathbb{Q}$ is a *weak* κ-forcing (or '$(\kappa, 2)$-forcing') <u>when</u>:

- (A) If $\kappa = \aleph_0$, then $\mathbb{Q}$ is a ccc forcing.
- (B) As in (1)(B).

3) Whenever we write 'a κ-forcing,' we mean the strong version.

4) For D a normal filter on κ^+ containing $S^{\kappa^+}_{\mathrm{cf}(\kappa)}$, we say the forcing notion $\mathbb{Q}$ satisfies $*^1_{\kappa,D}$ <u>when</u>:

$\kappa = \aleph_0$ and $\mathbb{Q}$ is ccc, <u>or</u> $\kappa > \aleph_0$ and

- $*_a$ Given a sequence $\langle p_i : i < \kappa^+\rangle$ of members of $\mathbb{P}$, there is a set $C \in D$ and a regressive function $\mathbf{h}$ on C such that

$$\alpha, \beta \in C \wedge \mathbf{h}(\alpha) = \mathbf{h}(\beta) \Rightarrow \text{`}p_\alpha \text{ and } p_\beta \text{ have a lub.'}$$

Notation 0.4. 1) Here $\mathfrak{s}$ will denote a combinatorial template (that is, a member of $\mathbf{T}$ — see Definition 1.1).

2) Here $\mathbf{q}, \mathbf{r}, \mathbf{p}$ will denote ATIs (*abstract template iterations*); i.e. members of $\mathbf{Q}_{\mathrm{pre}}$ (the weakest version — see Definition 1.4).

3) L is a linear order (usually $L \subseteq L_\mathfrak{s}$) and $r, s, t \in L$.

L_+ is derived from L, with $\infty, t, t(+) \in L_+$ for $t \in L$. (See below in 1.1(2).)

4) $L_\mathfrak{s}$ or $L_\mathbf{q}$ will be the relevant linear order for $\mathfrak{s}$ or $\mathbf{q}$, etc.

5) $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ denote forcing notions as in Definition 0.3 (which means quasi-orders).

---

[3] It seems sufficient to just demand

- $\bullet_1'$ Instead of clause $(2)_a$ of [She22, 0.2(B)$_{\text{=Lx2}}$], we use the game of length ε of [She00] (with ε a limit ordinal $< \kappa$; the natural choice is $\varepsilon = \partial$).
- $\bullet_2'$ $\mathbb{Q}$ strategically ζ-complete for every $\zeta < \kappa$.
- $\bullet_3'$ Any increasing ∂-sequence has a lub, for one $\partial = \mathrm{cf}(\partial)$.

§ 1. The frame

Definition 1.1. 0) Let $\mathbf{T}$ be the class of (∂, κ)-combinatorial templates (defined below), assuming $\partial = \mathrm{cf}(\partial) > \kappa$. If $\partial = \infty$ we may omit it.

1) A (κ, ∂)-CT (a (κ, ∂)-*combinatorial template*) $\mathfrak{s}$ consists of:

(a) A linear order L (we could have used 'partial'; it does not really matter for our purposes).
We may write $x \in \mathfrak{s}$ instead of $x \in L$, or $x <_{\mathfrak{s}} y$ instead of $x <_L y$.
(b) A sequence $\langle I_t : t \in L\rangle = \langle I^{\mathfrak{s}}_t : t \in L_{\mathfrak{s}}\rangle = \langle I_t[\mathfrak{s}] : t \in L[\mathfrak{s}]\rangle$, where $I_t = I^{\mathfrak{s}}_t \subseteq \{s \in L : s <_L t\} \subseteq L_{\mathfrak{s}}$ has cardinality $< \partial$.
(c) A set $S_t = S^{\mathfrak{s}}_t$ (say, of ordinals) for $t \in L$.

2) We define $t(+)$, L_x, and so forth as follows:

(a) For $x = t \in L$, let $L_x = \{s \in L : s <_L t\}$.
(b) For $t \in L$ and $x = t(+)$, let $L_x := \{s \in L : s \leq_L t\}$.
(c) Naturally, $\langle t : t \in L\rangle\hat{\ }\langle t(+) : t \in L\rangle\hat{\ }\langle\infty\rangle$ is without repetition.
(d) $L_+ = L^+_{\mathfrak{s}} := \{t, t(+) : t \in L\} \cup \{\infty\}$
(e) $<_{L_+}$ is the closure, to a linear order, of the set
$$\big\{t < t(+) : t \in L\big\} \cup \big\{s(+) < t : s <_L t\big\} \cup \big\{t(+) < \infty : t \in L\big\}.$$
(f) Let $L_{\mathfrak{s},\infty} := L_{\mathfrak{s}}$.

3) For $L \subseteq L_{\mathfrak{s}}$, we define $\mathfrak{s} \restriction L \in \mathbf{T}$ as follows.

- $\bullet_1$ $L_{\mathfrak{s}\restriction L} := L$
- $\bullet_2$ $I^{\mathfrak{s}\restriction L}_t := I^{\mathfrak{s}}_t \cap L_{\mathfrak{s}}$.

4) For $s \in L_{\mathfrak{s}}$, let $\mathfrak{s} \restriction s := \mathfrak{s} \restriction L_{\mathfrak{s},s}$.

5) We call $L \subseteq L_{\mathfrak{s}}$ *closed* (really, '$\mathfrak{s}$-closed') <u>when</u> $t \in L \Rightarrow I^{\mathfrak{s}}_t \subseteq L$ (e.g. $L \trianglelefteq L_{\mathfrak{s}}$).

6) We say $\mathfrak{s}$ is *closed* <u>when</u> $I^{\mathfrak{s}}_t$ is $\mathfrak{s}$-closed for every $t \in L_{\mathfrak{s}}$.

7) Let $\sigma(\mathfrak{s}) := \min\big\{\partial > \kappa^+ : \partial = \mathrm{cf}(\partial) \text{ and } s \in L_{\mathfrak{s}} \Rightarrow |I^{\mathfrak{s}}_s| < \partial\big\}$.

8) We say π is an *isomorphism from* $\mathfrak{s}_1$ *onto* $\mathfrak{s}_2$ (for $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbf{T}$) <u>when</u>
$$\pi : L_{\mathfrak{s}_1} \to L_{\mathfrak{s}_2}$$
is an order-preserving function mapping $I^{\mathfrak{s}_1}_t$ onto $I^{\mathfrak{s}_2}_{\pi(t)}$ for each $t \in L_{\mathfrak{s}_1}$.

Definition 1.2. We define a two-place relation $\leq_{\mathbf{T}}$ (obviously a partial order) on the class of combinatorial templates by:
$$\mathfrak{s}_1 \leq_{\mathbf{T}} \mathfrak{s}_2 \text{ \underline{iff}}$$

(a) $L_{\mathfrak{s}_1} \subseteq L_{\mathfrak{s}_2}$ as linear orders.
(b) If $s \in L_{\mathfrak{s}_1}$ then $I^{\mathfrak{s}_1}_s = I^{\mathfrak{s}_2}_s$.

Claim 1.3. 1) *$\leq_{\mathbf{T}}$ is indeed a partial order on $\mathbf{T}$.*

2) *If $\langle \mathfrak{s}_\varepsilon : \varepsilon < \delta\rangle$ is $\leq_{\mathbf{T}}$-increasing <u>then</u> $\bigcup_{\varepsilon<\delta} \mathfrak{s}_\varepsilon$ (naturally defined) exists, is a $\leq_{\mathbf{T}}$-lub, and is unique.*

Proof. Easy. $\square_{1.3}$

Definition 1.4. 1) $\mathbf{Q}^{\rm wk}_{\mathfrak{s}}$ is the class of weak $\mathfrak{s}$-ATIs (see below), and

$$\mathbf{Q}_{\rm wk} := \bigcup_{\mathfrak{s}\in\mathbf{T}} \mathbf{Q}^{\rm wk}_{\mathfrak{s}}.$$

(ATI stands for *abstract template iterations.*)

2) For $\mathfrak{s}$ a combinatorial template, we say $\mathbf{q}$ is a *weak* $\mathfrak{s}$*-ATI* when it consists of:[4]

(A) $\mathfrak{s}\in\mathbf{T}$ (We may write $L_{\mathbf{q}}$ for $L_{\mathfrak{s}}$, etc.)

(B) (a) A weak κ-forcing notion $\mathbb{P}=\mathbb{P}_{\mathbf{q}}$ (as in Definition 0.3(2)).

 (b) For $t\in L$, $\mathbb{P}_t \lessdot \mathbb{P}_{t(+)} \lessdot \mathbb{P}$ are weak κ-forcing notions. (This includes $t=\infty$, in which case $\mathbb{P}_t=\mathbb{P}$.)

 (c) For $t\in L$, $\underset{\sim}{\mathbb{Q}}_t$ is a $\mathbb{P}_t$-name of a weak κ-forcing with set of elements $S_t=S(t)$.

 (d) (See 0.3(1)(B)$\bullet_3$.) If $\kappa>\aleph_0$ and $t\in L$, then there is $\mathbf{H}_t : {}^{\kappa>}(S_t)\to S_t$ such that:

 - $\bullet_1$ $\Vdash_{\mathbb{P}_t}$ "if $\eta\in {}^{\kappa>}(S_t)$ is $\le_{\underset{\sim}{\mathbb{Q}}_t}$-increasing then $\mathbf{H}_t(\eta)$ is a lub of $\{\eta(i) : i<\ell g(\eta)\}$".
 - $\bullet_2$ If $\eta\in {}^2S_t$ and $\{\eta(0),\eta(1)\}$ has a $\le_{\mathbb{Q}_t}$-lub <u>then</u> $\mathbf{H}_t(\eta)$ is that lub.

 (e) If $p\in\mathbb{P}$ then p is a function with domain $\mathrm{dom}(p)\in[L_{\mathfrak{s}}]^{<\kappa}$ and support $\mathrm{supp}(p)\in[L_{\mathfrak{s}}]^{\le\kappa}$, with $\mathrm{supp}(p)\supseteq\mathrm{dom}(p)$. (See more in clause (E)(c).)

(C) (a) **[Notation:]** If $L\subseteq L_{\mathfrak{s}}$ then $\mathbb{P}_L := \mathbb{P}\restriction\{p : \mathrm{supp}(p)\subseteq L\}$.

 (b) If L is $\mathfrak{s}$-closed then $\mathbb{P}_L$ is a weak κ-forcing and $\mathbb{P}_L\lessdot\mathbb{P}$.

 (c) For $t\in L^+_{\mathbf{q}}$, let $\mathbb{P}_t := \mathbb{P}_{L_{\mathbf{q},t}}$.

(D) $\bar{\underset{\sim}{\eta}} := \langle \underset{\sim}{\eta}_t : t\in L\rangle$ with $\underset{\sim}{\eta}_t$ a $\mathbb{P}_{t(+)}$-name of a member of ${}^{S(t)}2$, but we identify $\underset{\sim}{\eta}_t\in {}^{S(t)}2$ with $\{\alpha : \underset{\sim}{\eta}_t(\alpha)=1\}$ such that:

 (a) $\underset{\sim}{\eta}_t(a)=1 \Leftrightarrow a\in\underset{\sim}{\mathbf{G}}_{\mathbb{P}}$, where $\underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ is a $\mathbb{P}_{t(+)}$-generic over $\mathbf{V}$.

 (b) For $\mathfrak{s}$-closed L, $\bar{\underset{\sim}{\eta}}\restriction L$ is a generic of $\mathbb{P}_L$.

(E) (a) $p\in\mathbb{P}$ <u>iff</u>

 (α) p is a function.

 (β) $\mathrm{dom}(p)\in[L_{\mathfrak{s}}]^{<\kappa}$

 (γ) For $s\in\mathrm{dom}(p)$, $p(s)$ is a $\mathbb{P}_s$-name of a member of $\underset{\sim}{\mathbb{Q}}_s$. More specifically, it is of the form $\mathbf{B}(\ldots,\underset{\sim}{\eta}_{t_j}(\varepsilon_j),\ldots)_{j<j_{p(s)}}$, where

 - $\bullet_1$ $t_j\in I_s$
 - $\bullet_2$ $\varepsilon_j\in S_{t_j}$
 - $\bullet_3$ $j_{p(s)}\le\kappa$
 - $\bullet_4$ $\mathbf{B}$ is a κ-Borel function[5] from ${}^{(j_{p(s)})}2$ into some $\mathcal{U}_{p(s)}\in[S_s]^{\le\kappa}$.

 (b) The truth value of $p\le_{\mathbb{P}} q$ is computed in $\mathbf{V}[\bar{\underset{\sim}{\eta}}\restriction A]$, where

$$A=\mathrm{dom}(q)\cup\bigcup\big\{I_s : s\in\mathrm{dom}(q)\big\}.$$

 (c) $\mathrm{supp}(p) := \mathrm{dom}(p)\cup\{\gamma_{p(s),j} : s\in\mathrm{dom}(p),\ j<j_{p(s)}\}$

 (d) $\underset{\sim}{\eta}_s :=$

$$\big\{p(s)(\ldots,\underset{\sim}{\eta}_{t_{p(s),j}}(\varepsilon_{p(s),j},\ldots)_{j<j_{p(s)}}[\mathbf{G}],\ldots) : p\in\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{t(+)}},\ t\in\mathrm{dom}(p)\big\}$$

 exists and is well-founded, noting that $p(s)\in S_s$ is computed from $\big\langle \underset{\sim}{\eta}_t[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{L(s)}}] : t\in I_s\big\rangle$.

---

[4]So $\mathbb{P}=\mathbb{P}_{\mathbf{q}}$, etc. We may omit $\mathfrak{s}$ or $\mathbf{q}$ when it is clear from context.

[5]The point is absoluteness.

(e) For $x \in L_+$, $\mathbb{P}_x \models$ '$p \le q$' iff
 - $\bullet_1$ $\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \subseteq L_x$
 - $\bullet_2$ If $s \in \mathrm{dom}(p)$ then $p \restriction L_s \Vdash_{\mathbb{P}_s}$ '$p(s) \le_{\underset{\sim}{\mathbb{Q}}_s} q(s)$'.

(f) Similar to clause (e), but for $\mathbb{P}$. (This actually follows by setting $x = \infty$.)

Definition 1.5. 1) We define $\mathbf{Q}^{\mathrm{st}}_{\mathfrak{s}}$, $\mathbf{Q}_{\mathrm{st}}$, and say '*strong* ATI' when we replace "weak κ-forcing" by "strong κ-forcing" in 1.4, clauses (B)(a), (C)(a).

2) We define $\mathbf{Q}_{\mathrm{pre}}$, $\mathbf{Q}^{\mathrm{pre}}_{\mathfrak{s}}$ as in Definition 1.4, replacing "weak κ-forcing" by "forcing" in clauses (B)(a), (C)(a).

3) Let $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$ be shorthand for $\mathbf{Q}_{\mathrm{pre}}, \mathbf{Q}_{\mathrm{wk}}$, and $\mathbf{Q}_{\mathrm{st}}$, respectively.

4) When we omit the subscripts, we mean 'pre.' (But not in 1.8(2) below, however.)

5) If $\mathbf{q} \in \mathbf{Q}_{\mathrm{pre}}$ and $L \subseteq L_{\mathbf{q}}$, then $\mathbf{p} = \mathbf{q} \restriction L$ is defined by $\mathfrak{s}_{\mathbf{p}} := \mathfrak{s}_{\mathbf{q}} \restriction L$ and $\mathbb{P}_{\mathbf{p}} := \mathbb{P}_{\mathbf{q},L}$.

6) We define "π is an isomorphism from $\mathbf{q}$ onto $\mathbf{p}$" naturally.

Remark 1.6. 1) Recall that $L_{\mathbf{q}}$ is just a linear order and not necessarily a well-ordering.

2) As a consequence, for a given $\mathbf{q}$, $\langle \underset{\sim}{\mathbb{Q}}_s : s \in L_{\mathbf{q}} \rangle$ does not necessarily determine $\mathbb{P}_{\mathbf{q}}$, but if $\mathfrak{s}$ is as in [She04b, §2] then it is unique.

Observation 1.7. *Let* $\mathbf{q} \in \mathbf{Q}_{\mathrm{pre}}$.

1) *If* $L \subseteq L_{\mathbf{q}}$ *is* $\mathbf{q}$*-closed,* $p \in \mathbb{P}_{\mathbf{q}}$*, and* $p \restriction L \le_{\mathbb{P}_{\mathbf{q}}} q \in \mathbb{P}_{\mathbf{q},L}$*, then*

$$r := \big(p \restriction (\mathrm{dom}(p) \setminus L)\big) \cup q$$

is a lub of p *and* q.

2) *For* $\mathbf{q}$*-closed* L*, we have* $\mathbb{P}_{\mathbf{q},L} \models$ "$p \le q$" *iff*

- $\bullet_1$ $\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \subseteq L$
- $\bullet_2$ *If* $s \in \mathrm{dom}(p)$ *then for some* $\mathbf{q}$*-closed* L_1 *satisfying* $I^{\mathbf{q}}_s \subseteq L_1 \subseteq L \cap L_{\mathbf{q},s}$*, we have* $q \restriction L_1 \Vdash_{\mathbb{P}_{L_1}}$ "$p(s) \le_{\underset{\sim}{\mathbb{Q}}_s} q(s)$".

3) *Like* (2)$\bullet_2$*, replacing "for some" with "for every."*

4) *If* $\mathbf{q}$ *is closed, then in* (2)$\bullet_2$ *we can choose* $L_1 = I^{\mathbf{q}}_s$.

Proof. 1) Note

$(*)_1$ $r \in \mathbb{P}_{\mathbf{q}}$

[Why? First, r is a well-defined function. Second, $\mathrm{dom}(r) \in [L_{\mathbf{q}}]^{<\kappa}$, and third $s \in \mathrm{dom}(r) \Rightarrow$ '$r(s)$ is as required in 1.4(2)(E)(a)(γ).' So by 1.4(2)(E)(a) we are done.]

$(*)_2$ $\mathbb{P}_{\mathbf{q}} \models$ '$p \le r$'

We have to check 1.4(2)(E)(e). Now $\bullet_1$ is trivial, as $\mathrm{dom}(p \restriction L) \subseteq \mathrm{dom}(q) \subseteq L$; as for $\bullet_2$, let $s \in \mathrm{dom}(r)$ and exactly one of the following cases will occur.

Case 1: $s \in \mathrm{dom}(p) \setminus L$.

In this case, $r(s) = p(s)$, so

$$r \restriction L_s \Vdash_{\mathbb{P}_{L_s}} \text{“}p(s) \le_{\underset{\sim}{\mathbb{Q}}_{\mathfrak{s}}} r(s)\text{”}$$

holds trivially.

Case 2: $s \in \mathrm{dom}(p) \cap L$.

Recalling $\mathbb{P}_L \models$ “$(p \restriction L) \le q$” and $\mathbb{P}_L \lessdot \mathbb{P}$ (by 1.4(2)(C)(b)), we have

$$q \restriction I_s \Vdash_{\mathbb{P}_{I_s}} \text{“}p(s) \le_{\underset{\sim}{\mathbb{Q}}_{\mathfrak{s}}} r(s)\text{”},$$

so as $r(s) = q(s)$ we are done.

Case 3: $s \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$.

Also in this case, $r(s) = q(s)$ is well-defined (and there is no demand on $q(s)$) so we are done.

$(*)_3$ $\mathbb{P}_{\mathbf{q}} \models$ ‘$q \le r$’

As $r \restriction \mathrm{dom}(q) = q$, this is trivial.

$(*)_4$ If $\mathbb{P}_{\mathbf{q}} \models$ “$p \le r' \wedge q \le r'$” then $\mathbb{P}_{\mathbf{q}} \models r \le r'$.

Easy as well.

2,3,4) Also straightforward. $\square_{1.7}$

Definition 1.8. 1) Let $\mathbf{q}_1 \le_{\mathbf{Q}} \mathbf{q}_2$ (or $\mathbf{q}_1 \le^{\mathrm{wk}}_{\mathbf{Q}} \mathbf{q}_2$) mean:

(a) $\mathbf{q}_\ell$ is a weak $\mathfrak{s}_\ell$-ATI for $\ell = 1, 2$ (where $\mathfrak{s}_\ell = \mathfrak{s}_{\mathbf{q}_\ell}$; recall that $\mathbf{q}_\ell$ determines $\mathfrak{s}_\ell$).
(b) $\mathfrak{s}_1 \le_{\mathbf{T}} \mathfrak{s}_2$
(c) $\mathbb{P}_{\mathbf{q}_1} \lessdot \mathbb{P}_{\mathbf{q}_2}$
(d) $\underset{\sim}{\mathbb{Q}}^{\mathbf{q}_1}_t = \underset{\sim}{\mathbb{Q}}^{\mathbf{q}_2}_t$ for $t \in L_{\mathfrak{s}_1}$.
(e) $\Vdash_{\mathbb{P}_{\mathbf{q}_2}}$ “$\underset{\sim}{\eta}^{\mathbf{q}_1}_t = \underset{\sim}{\eta}^{\mathbf{q}_2}_t$” (and so $S_{\mathbf{q}_1}(t) = S_{\mathbf{q}_2}(t)$) for $t \in L_{\mathfrak{s}_1}$.

2) We define $\le^{\mathrm{pre}}_{\mathbf{Q}}$ as above, changing clause (a) to ‘$\mathbf{q}_\ell \in \mathbf{Q}_{\mathrm{pre}}$’ and omitting clause (c). (I.e. we do not require $\mathbb{P}_{\mathbf{q}_1} \lessdot \mathbb{P}_{\mathbf{q}_2}$.)

We define $\le_{\mathbf{Q}_2} := \le_{\mathbf{Q}} \restriction \mathbf{Q}_2$.

2A) If $\mathbf{r} \le^{\mathrm{pre}}_{\mathbf{Q}} \mathbf{q}$ and $p \in \mathbb{P}_{\mathbf{q}}$, then we define $q := p \restriction \mathbf{r}$ as follows:

- $\bullet_1$ $\mathrm{dom}(q) = \mathrm{dom}(p) \cap L_{\mathbf{r}}$
- $\bullet_2$ If $s \in \mathrm{dom}(q)$ then $q(s) = p(s)$ (recalling 1.2(b)).

3) If $\langle \mathbf{q}_\alpha : \alpha < \delta \rangle$ is $\le_{\mathbf{Q}}$-increasing then “$\mathbf{q} := \bigcup\limits_{\alpha<\delta} \mathbf{q}_\alpha$” will mean the following:

(a) $\mathbf{q} \in \mathbf{Q}$
(b) $\mathfrak{s}_{\mathbf{q}} := \bigcup\limits_{\alpha<\delta} \mathfrak{s}_{\mathbf{q}_\alpha}$
(c) $\mathbf{q}_\alpha \le_{\mathbf{Q}} \mathbf{q}$ for all $\alpha < \delta$.
(d) [Follows] If $s \in L_{\mathbf{q}_\alpha}$ <u>then</u> $\underset{\sim}{\mathbb{Q}}^{\mathbf{q}}_s = \underset{\sim}{\mathbb{Q}}^{\mathbf{q}_\alpha}_s$ and $\underset{\sim}{\eta}^{\mathbf{q}}_s = \underset{\sim}{\eta}^{\mathbf{q}_\alpha}_s$.

4) We say $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha < \alpha_* \rangle$ is $\le_{\mathbf{Q}}$-*increasing continuous* <u>if</u> it is $\le_{\mathbf{Q}}$-increasing and $\mathbf{q}_\delta = \bigcup\limits_{\alpha<\delta} \mathbf{q}_\alpha$ for every limit $\delta < \alpha_*$.

Remark 1.9. 1) Note that in parts (3),(4) of Definition 1.8, for a given $\langle \mathbf{q}_\alpha : \alpha < \delta \rangle$, it is not *a priori* clear that such $\mathbf{q}$ exists — and even if it does, whether it is unique.

2) Regarding 1.8(1)(c), does "$\mathbb{P}_{\mathbf{q}_1} \lessdot \mathbb{P}_{\mathbf{q}_2}$" follow by 1.4(2)(C)(a), as $L_{\mathfrak{s}_1}$ is $\mathbf{q}_2$-closed by Definition 1.2? This is not clear. (See 1.6(2).)

We can only show that given $\mathbf{q}_2$ and a $\mathbf{q}_2$-closed $L \subseteq L_{\mathbf{q}}$, we have $(\mathbf{q}_2 \restriction L) \leq_{\mathbf{Q}} \mathbf{q}_2$.

Observation 1.10. 1) *Assume* $\mathbf{q}_1 \leq^{\text{pre}}_{\mathbf{Q}} \mathbf{q}_2$.

(*A*) *If* $p \in \mathbb{P}_{\mathbf{q}_1}$ *and* $q \in \mathbb{P}_{\mathbf{q}_2}$, *then we have* $(a) \Leftrightarrow (b)$, *where:*
  (*a*) $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \leq q$"
  (*b*) *If* $s \in \operatorname{dom}(p)$ *then* $s \in \operatorname{dom}(q) \wedge q \restriction L_{\mathbf{q}_1,s} \Vdash_{\mathbb{P}_{\mathbf{q}_1,s}}$ "$p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)$".

(*B*) *If* $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \not\geq q$" *and* $s \in \operatorname{dom}(p) \cap L_{\mathbf{q}_1}$, *then*

$$q \restriction L_{\mathbf{q}_1,s} \Vdash_{\mathbb{P}_{\mathbf{q}_1,s}} \text{"}p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)\text{"}.$$

(*C*) *Assume*
  (*a*) $L^2_1 \lhd L^2_2 \trianglelefteq L_{\mathbf{q}_2}$
  (*b*) $\bigwedge\limits_{\ell=1}^{2} [L^1_\ell = L^2_\ell \cap L_{\mathbf{q}_1}]$
  (*c*) $p \in \mathbb{P}_{\mathbf{q}_2 \restriction L^2_1}$ *and* $q \in \mathbb{P}_{\mathbf{q}_1 \restriction L^1_2}$.
  (*d*) $\mathbb{P}_{\mathbf{q}_2, L^2_1} \models q \restriction L^1_1 \leq p^+$.

If in addition, $p^+ \in \mathbb{P}_{\mathbf{q}_2 \restriction L^1_1}$ *is* $\leq_{\mathbb{P}_{\mathbf{q}_2}}$*-above* $q \restriction L_{\mathbf{q}_1 \restriction L^1_1}$ *and* $p \restriction L_{\mathbf{q}_1 \restriction L^1_1}$, *then* $\{p, p^+, q\}$ *have a common upper bound in* $\mathbb{P}_{\mathbf{q}_2 \restriction L^2_2}$.

2) *If* $x \in L^+_{\mathfrak{s}}$ *then* $\mathfrak{s} \restriction L_x \in \mathbf{T}$ *and*

$$\mathbf{q} \in \mathbf{Q}_{\mathfrak{s}} \Rightarrow \mathbf{q} \restriction L_x \in \mathbf{Q}_{\mathfrak{s}_{\mathbf{q}} \restriction x}. \text{ (See 1.1(4) and 1.4(3).)}$$

3) *Assume* $\mathbf{q}_1 \leq_{\mathbf{Q}} \mathbf{q}_2$.

Then

(*a*) *If* $L \subseteq L_{\mathbf{q}_1}$ *then* L *is* $\mathbf{q}_1$*-closed iff* L *is* $\mathbf{q}_2$*-closed.*
(*b*) *If* $L_1 \subseteq L_2$, L_1 *is* $\mathbf{q}_1$*-closed, and* L_2 *is* $\mathbf{q}_2$*-closed (so* $L_\iota \subseteq L_{\mathbf{q}_\iota}$ *for* $\iota = 1, 2$*) then*
  $\bullet_1$ $\mathbb{P}_{\mathbf{q}_1, L_1} \lessdot \mathbb{P}_{\mathbf{q}_2, L_2}$
  $\bullet_2$ *If* $p_\iota \in \mathbb{P}_{\mathbf{q}_\iota, L_\iota}$ *for* $\iota = 1, 2$ *and* $p_1 = p_2 \restriction L_1$ *then*

$$\mathbb{P}_{\mathbf{q}_1, L_1} \models \text{"}p_1 \leq q\text{"} \Rightarrow p_2 \text{ and } q \text{ are compatible in } \mathbb{P}_{\mathbf{q}_2, L_2}.$$

Proof. 1A) First assume $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \leq q$" (i.e. clause (A)(a)). Then for every $s \in \operatorname{dom}(p)$, we have $s \in \operatorname{dom}(q)$ (by 1.4(2)(E)(a) and 1.2) and

$$\Vdash_{\mathbb{P}_{\mathbf{q}_1,s}} \text{"}q \restriction L_{\mathbf{q}_1,s} \Vdash \text{'}p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)\text{'"}$$

by 1.7(3). Together we get clause (A)(b).

[No clue why this is in red. Just say 'ok' and I'll revert it.]

Now assume clause (A)(b). So $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$, and by 1.7(2) we get $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \leq q$".

1B) Similar proof.

1C) Use the proof of 1.7(1).

2),3) Easy. $\square_{1.10}$

Claim 1.11. *If* $\langle \mathbf{q}_\alpha : \alpha < \delta \rangle$ *is* $\le_{\mathbf{Q}}$*-increasing continuous (Note: when* $\kappa > \aleph_0$ *this does* <u>*NOT*</u> *mean that* $\langle \mathbb{P}_{\mathbf{q}_\alpha} : \alpha < \delta \rangle$ *is* $\subseteq$*-increasing continuous!) and* $\mathrm{cf}(\delta) \ge \kappa$*,* <u>*then*</u> $\bigcup\limits_{\alpha<\delta} \mathbf{q}_\alpha$ *exists and is unique.*

Proof. Straightforward — anyhow, we shall use 2.1. $\square_{1.11}$

Claim 1.12. [*Assume* $\kappa = \aleph_0$*.*]

1) *In the definition of* $\mathbf{Q}_{\mathrm{wk}}$ *(*1.4(2)*), we may omit clause* $(B)(b)$*.*

2) *Similarly in* 1.5(1)*, replacing 'weak' by 'strong.'*

Remark 1.13. See more in the proof of 2.6; in particular, proving 1.12(2) for $\kappa > \aleph_0$.

Proof. 1) The $\Leftarrow$ direction is obvious. For '$\Rightarrow$,' let $\langle p_\alpha : \alpha < \kappa^+ \rangle \in {}^{\kappa^+}\mathbb{P}_{\mathbf{q}}$.

Without loss of generality, $\langle \mathrm{dom}(p_\alpha) : \alpha < \kappa^+ \rangle$ is a Δ-system with heart $u \in [L_{\mathbf{q}}]^{<\aleph_0}$. Let $t_0 <_{L_{\mathbf{q}}} \ldots <_{L_{\mathbf{q}}} t_{n-1}$ list u, and let $t_n := \infty$.

We choose $p_\ell \in \mathbb{P}_{\mathbf{q},t_\ell}$ increasing with ℓ such that

$$p_\ell \Vdash_{\mathbb{P}_{\mathbf{q},t_\ell}} \big(\exists^{\kappa^+} \alpha < \kappa^+\big)\big[p_\alpha \restriction L_{\mathbf{q},t_\ell} \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{q},t_\ell}}\big].$$

2) For the strong case, recall 0.3(1)(B)$\bullet_3$. $\square_{1.12}$

§ 2. Unions

Claim 2.1. 1) *If* $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha < \delta \rangle$ *is* $\leq_{\mathbf{Q}_{\rm wk}}$*-increasing continuous (see* 1.8(4)*)* *then* $\mathbf{q}_\delta := \bigcup_{\alpha<\delta} \mathbf{q}_\alpha$ *exists and is unique, belongs to* $\mathbf{Q}_{\rm wk}$*, and* $\overline{\mathbf{q}}\,\hat{}\,\langle \mathbf{q}_\delta \rangle$ *is* $\leq_{\mathbf{Q}}$*-increasing continuous.*

2) *Similarly for* $\leq_{\mathbf{Q}_{\rm st}}$.

Remark 2.2. Note that this is not a repeat of 1.11, as we have dropped the assumption on $\mathrm{cf}(\delta)$.

Proof. 1) Let $\mathfrak{s}_\alpha := \mathfrak{s}_{\mathbf{q}_\alpha}$ and $L_\alpha := L_{\mathfrak{s}_\alpha}$ for $\alpha < \delta$.

Note that $\mathfrak{s} = \mathfrak{s}_{\mathbf{q}} := \bigcup_{\alpha<\delta} \mathfrak{s}_\alpha$ is well defined, but when $\mathrm{cf}(\delta) < \kappa$ we cannot choose $\mathbb{P}_{\mathbf{q}} := \bigcup_{\alpha<\delta} \mathbb{P}_{\mathbf{q}_\alpha}$. We have to choose $\mathbf{q} = \mathbf{q}_\delta$ as follows:

$(*)_1$ (a) $\mathfrak{s}_{\mathbf{q}} = \mathfrak{s}_\delta := \bigcup_{\alpha<\delta} \mathfrak{s}_\alpha$, and let $L_\delta := L_{\mathfrak{s},\delta}$.

(b) $p \in \mathbb{P}_{\mathbf{q}}$ iff
 - $\bullet_1$ $\mathrm{dom}(p) \in [L_{\mathfrak{s},\delta}]^{<\kappa}$
 - $\bullet_2$ If $s \in \mathrm{dom}(p)$ then $p \restriction \{s\} \in \bigcup_{\alpha<\delta} \mathbb{P}_{\mathbf{q}_\alpha}$.

(c) '$p \leq_{\mathbb{P}_{\mathbf{q}}} q$' is defined by 1.7(2); that is,
$$\big(\forall s \in \mathrm{dom}(p)\big)\big[q \restriction L_{\mathbf{q}_\beta} \Vdash_{\mathbb{P}_{\mathbf{q}_\beta}} \text{“}p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)\text{”}\big],$$
where $\beta = \beta(s) := \min\{\alpha < \delta : s \in L_\alpha\}$.

Let $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha \leq \delta \rangle$. Easily,

$(*)_2$ (a) $\alpha < \delta \Rightarrow \mathbb{P}_{\mathbf{q}_\alpha} \subseteq \mathbb{P}_{\mathbf{q}}$ (As partial orders, of course.)

(b) If $\beta < \delta$ and $L \subseteq L_\beta$ is $\mathfrak{s}_\delta$-closed, then $\mathbb{P}_{\mathbf{q},L} = \mathbb{P}_{\mathbf{q}_\beta,L}$.

(c) $L \subseteq L_\delta$ is $\mathbf{q}$-closed iff $L \cap L_\alpha$ is $\mathbf{q}_\alpha$-closed for every $\alpha < \delta$.

(d) If L is $\mathfrak{s}_\delta$-closed then $\mathbb{P}_{\mathbf{q},L} = \bigcup_{\alpha<\delta} \mathbb{P}_{\mathbf{q}_\alpha, L\cap L_\alpha}$ (defined as above).

Why? Obvious, but we will elaborate.

Clause (a): Let $\alpha < \delta$.

First, if $p \in \mathbb{P}_\alpha$, then by $(*)_{2.1}$+$(*)_{2.2}$ below we have $p \in \mathbb{P}_\delta$.

$(*)_{2.1}$ $\mathrm{dom}(p) \subseteq L_{\mathbf{q}_\alpha}$ is of cardinality $< \kappa$, by 1.4(2)(E)(a)$(\alpha),(\beta)$. $L_\alpha \subseteq L_{\mathbf{q}_\delta}$ by $(*)_1$(a), so p satisfies $(*)_1$(b)$\bullet_1$.

$(*)_{2.2}$ If $s \in \mathrm{dom}(p)$ then $p \restriction \{s\} \in \mathbb{P}_\alpha$ by 1.4(2)(E)(a), hence $p \restriction \{s\} \in \mathbb{P}_\delta$.

Second, assume $p, q \in \mathbb{P}_\alpha$. Then
$$\mathbb{P}_\alpha \models \text{“}p \leq q\text{”} \Rightarrow \mathbb{P}_\delta \models \text{“}p \leq q\text{”}$$
by $(*)_2$(b) and 1.10(1)(B).

Clauses (b)-(d): Similarly.

$(*)_3$ (a) $\alpha < \delta \Rightarrow \mathbb{P}_{\mathbf{q}_\alpha} \lessdot \mathbb{P}_{\mathbf{q}}$

(b) If $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed then $\mathbb{P}_{\mathbf{q},L} \lessdot \mathbb{P}_{\mathbf{q}}$.

(c) $\langle \underset{\sim}{\eta}_s : s \in L_\delta \rangle$ is a generic for $\mathbb{P}_\delta$.

(d) If $L \subseteq L_\delta$ is $\mathfrak{s}$-closed then $\langle \underset{\sim}{\eta}_s : s \in L\rangle$ is a generic for $\mathbb{P}_{\mathbf{q}_\delta \restriction L}$.

To prove clause (a), let $p \in \mathbb{P}_{\mathbf{q}}$. Now by the assumptions $\langle \mathfrak{s}_{\mathbf{q}_\beta} : \beta < \delta\rangle$ is increasing. So by the choice of $\mathfrak{s}_{\mathbf{q}}$, if $s \in \mathrm{dom}(p)$ then there is an $\alpha_s < \delta$ such that $s \in L_{\alpha_s} \setminus \bigcup\limits_{\beta<\alpha_s} L_\beta$. So easily, recalling $(*)_1$(c), $p_\alpha := p \restriction (\mathrm{dom}(p) \cap L_\alpha)$ satisfies

$$\mathbb{P}_{\mathbf{q}_\alpha} \models \text{“}p_\alpha \le q\text{”} \Rightarrow p \text{ and } q \text{ are compatible in } \mathbb{P}_{\mathbf{q}}.$$

(See 1.7(1). Even their union, as defined as in 1.7(1), is okay.)

So clause (a) holds. The proof of clause (b) is similar.

As for (c), let $\mathbf{G}_\delta \subseteq \mathbb{P}_\delta$ be generic over $\mathbf{V}$. By clause (a), $\mathbf{G}_\alpha := \mathbf{G}_\delta \cap \mathbb{P}_\alpha$ is a generic subset of $\mathbb{P}_\alpha$ for $\alpha < \delta$. So $p \in \mathbf{G}_\delta \Rightarrow p \restriction L_\alpha \in \mathbf{G}_\alpha$, recalling $p \in \mathbb{P}_\delta \Rightarrow p \restriction L_\delta \le_{\mathbb{P}_\delta} p$.

Also,

$$p \in \mathbb{P}_\delta \wedge \bigwedge_{\alpha<\delta} [p \restriction L_\alpha \in \mathbf{G}_\alpha] \Rightarrow p \in \mathbf{G}_\delta$$

because $\mathbb{P}_\delta$ is $(<\kappa)$-complete, and $\mathbb{P}_\delta \models$ “ $\bigwedge\limits_{\alpha<\delta} [p \restriction L_\alpha \le q]$” implies $\mathbb{P}_\delta \models$ “$p \le q$”.

So clause (c) holds. Clause (d) is proved similarly.

Next,

$(*)_4$ If L is $\mathfrak{s}_\delta$-closed then $\mathbb{P}_{\mathbf{q}_{\delta,L}}$ is a weak κ-forcing.

Why? If $\kappa = \aleph_0$ then $\langle \mathbb{P}_{\mathbf{q}_\alpha, L\cap L_\alpha} : \alpha < \delta\rangle$ is a $\lessdot$-increasing continuous sequence of ccc forcing notions with union $\mathbb{P}_{\mathbf{q}_\delta, L}$, and so this is known. Therefore assume $\kappa > \aleph_0$ and then prove that $\mathbb{P}_{\mathbf{q}_{\delta,L}}$ satisfies $*^1_{\kappa,D}$ for D and κ as in 0.3(4).

Let $\langle p_i : i < \kappa^+\rangle \in {}^{\kappa^+}(\mathbb{P}_L)$ be given. First, let $u_i := \mathrm{dom}(p_i)$, so $u_i \in [L]^{<\kappa}$. As $\kappa = \kappa^{<\kappa}$, there are C and $\mathbf{h}$ such that:

$(*)_{4.1}$ (a) $C \in D$ and $\alpha \in C \Rightarrow \mathrm{cf}(\alpha) = \kappa$.

(b) $\mathbf{h}$ is a regressive function on C.

(c) If $\zeta \in \mathrm{rang}(\mathbf{h})$, then for some $v_\zeta \subseteq L$ we have

$$i \ne j \in C \wedge \mathbf{h}(i) = \mathbf{h}(j) = \zeta \Rightarrow u_i \cap u_j = v_\zeta.$$

$(*)_{4.2}$ (a) Without loss of generality $\zeta \in \mathrm{rang}(\mathbf{h}) \Rightarrow C_\zeta := \mathbf{h}^{-1}(\{\zeta\}) \in D^+$.

(b) For $s \in L_{\mathbf{q}_\delta}$ let $\alpha(s) := \min\{\alpha : s \in L_{\mathbf{q}_\alpha}\}$.

[Why? For clause (a) recall that D is a <u>normal</u> filter on κ^+.]

The proof splits into cases.

<u>Case 1</u>: $\mathrm{cf}(\partial) \le \kappa$.

Without loss of generality $\delta \le \kappa$, hence there is a function $\mathbf{g} : \kappa^+ \to \kappa \cap (\delta + 1)$ such that $i < \kappa^+ \Rightarrow p_i \in \mathbb{P}_{\mathbf{q}_{\mathbf{g}(i)}}$. Without loss of generality, $\mathrm{dom}(p_i) = \mathbf{g}(i)$ and $\mathbf{g}(i)$ is a limit ordinal (recalling $\kappa = \mathrm{cf}(\kappa) > \aleph_0$).

Now, using $\mathbf{q}_\alpha \in \mathbf{Q}_{\mathrm{wk}}$ for $\alpha < \delta$, consider $\langle p_i \restriction L_{\mathbf{q}_\alpha} : i < \kappa^+\rangle$. There are $C_\alpha \in D$ and $\mathbf{h}_\alpha$ (a regressive function on C_α) as follows from ‘$\mathbb{P}_{\mathbf{q}_\alpha}$ satisfies $*^1_{\kappa,D}$.’

Now, recalling $\kappa = \kappa^{<\kappa}$ and $(\forall \gamma \in C)[\mathrm{cf}(\gamma) = \kappa]$, we can find C_* and $\mathbf{h}_*$ such that

$(*)_{4.3}$ (a) $C_* \in D$ and

$C_* \subseteq \{j \in C : i < j \wedge s \in u_i \Rightarrow j \in C_{\alpha(s)} \wedge (\exists k \in C \cap j)[\mathbf{h}(j) = \mathbf{h}(k)]\}$.

(b) $\mathbf{h}_*$ is a regressive function on C_*.

(c) If $j \in C_*$ and $\zeta \leq \mathbf{g}(j)$, then $\mathbf{h}_*(j)$ codes $\mathbf{h}_\zeta(j)$.

(d) If $j_1, j_2 \in C_*$, $\mathbf{h}_*(j_1) = \mathbf{h}_*(j_2)$, and $\mathbf{g}(j_1) = \zeta$ then $\mathbf{g}(j_2) = \zeta$ and $\mathbf{h}_\zeta(j_1) = \mathbf{h}_\zeta(j_2)$.

[Why? Easy, but we elaborate.

Let $C_1^* := \{\delta < \kappa^+ : \delta$ a limit ordinal, $\alpha < \delta \Rightarrow \delta \in C_\alpha\}$. So $C_1^* \in D$, as D_α is a normal filter on κ^+ and every C_α belongs to D by our choices. As C_1^* and C belong to the filter D, clearly $C_2^* := C_1^* \cap C$ does as well.

As $\kappa = \kappa^{<\kappa}$, there is a one-to-one function from ${}^{\kappa>}(\kappa^+) \cup \bigcup\limits_{\alpha<\kappa} {}^{\alpha}(\kappa^+)$ into κ^+ such that

$$\beta < \kappa^+ \wedge \eta \in {}^{\kappa>}(\beta + \kappa) \Rightarrow \mathrm{cd}(\eta) < \beta + \kappa.$$

[No idea what 'cd' is; it hasn't been defined anywhere]

Let $C_3^* := \{\delta < \kappa^+ : \alpha < \delta \wedge \eta \in {}^{2}\beta \Rightarrow \mathbf{h}(\eta) < \delta\}$; it is a club of κ^+, hence $C_* := C_2^* \cap C_2^* \in D$.

Lastly, define the function h_* with domain C_* by $\delta \mapsto \mathrm{pr}(\langle \mathbf{h}_*(p_\delta \restriction \varepsilon) : \varepsilon < \mathbf{g}(\delta)\rangle)$. It is easy to check that C_* and h_* are as desired.]

$(*)_{4.4}$ If $p, q \in \mathbb{P}_\delta$, $\alpha_1 < \alpha_2 < \delta$, $\alpha_2 \subseteq \mathrm{dom}(p) \cap \mathrm{dom}(q)$ (for transparency), and for $\ell = 1, 2$, $\{p \restriction \alpha_\ell, q \restriction \alpha_\ell\}$ has a $\leq_{\mathbb{P}_{\alpha_\ell}}$-lub r_ℓ, <u>then</u> r_1 and $r_2 \restriction \alpha_1$ are not equivalent.

(That is, $\gamma < \alpha_1 \Rightarrow r_1(\gamma) \leq_{\underset{\sim}{\mathbb{Q}}_\partial} r_2(\gamma) \leq_{\underset{\sim}{\mathbb{Q}}_\partial} r_2(\gamma)$.)

[$r_2(\gamma) \leq_{\underset{\sim}{\mathbb{Q}}_\partial} r_2(\gamma)$ is true, but uninteresting. I don't see anything else this could have been referring to, and can probably be deleted.]

[Why? Easy.]

$(*)_{4.5}$ If $i, j \in C_*$ with $\mathbf{g}_*(i) = \mathbf{g}_*(j)$, then

$$(\forall \alpha < \delta)[p_i \restriction \alpha, p_j \restriction \delta \text{ has a } \leq_{\mathbb{P}_\alpha}\text{-lub}],$$

hence p_i, p_j have a $\leq_{\mathbb{P}_\delta}$-lub.

[Why? Easy.]

Together we are done. That is, C_* and $\mathbf{h}_*$ are as required.

<u>**Case 2:**</u> $\mathrm{cf}(\delta) > \kappa^+$.

For some $\alpha < \delta$, $\{p_i : i < \kappa^+\} \subseteq \mathbb{P}_{\mathbf{q}_\alpha}$ so the conclusion is obvious.

<u>**Case 3:**</u> $\mathrm{cf}(\delta) = \kappa^+$.

Without loss of generality $\delta = \kappa^+$; hence

$(*)_{4.5}$ In clause $(*)_{4.1}$, without loss of generality, for each $\zeta \in \mathrm{rang}(\mathbf{h})$ and $i \in C$ satisfying $\mathbf{h}(i) = \zeta$, we have

- $v_\zeta \subseteq L_{\mathbf{q}_i}$ and $i < j \in C \Rightarrow p_i \in \mathbb{P}_{\mathbf{q}_j}$.
- C_* and $\mathbf{h}_*$ are as in $(*)_{4.3}$.

Now easily $i, j \in C_* \wedge \mathbf{h}_*(i) = \mathbf{h}_*(j) \Rightarrow$ "p_i and p_j are comparable."

So clearly we have proved $(*)_4$.

$(*)_5$ $\mathbf{q} \in \mathbf{Q}_{\rm wk}$

[Why? We have to check all clauses of Definition 1.4; this is straightforward by $(*)_1$–$(*)_4$.]

$(*)_6$ $\mathbf{q}_\alpha \leq_{\mathbf{Q}} \mathbf{q}_\delta$ for $\alpha < \delta$.

[Why? We should check Definition 1.8(1). Clause (a) holds by $(*)_5$. Clause (b) holds by $(*)_1$(a) (recalling $\mathbf{p} \leq_{\mathbf{Q}} \mathbf{q} \Rightarrow \mathbf{s}_{\mathbf{p}} \leq_{\mathbf{T}} \mathbf{s}_{\mathbf{q}}$ and 1.3(2)). Clause (c) is covered by $(*)_3$(a), and clauses (d) and (e) are obvious.]

$(*)_7$ $\mathbf{q}_\delta = \bigcup\limits_{\alpha<\delta} \mathbf{q}_\alpha$

[Why? We should check Definition 1.8(3):

Clause (a): $(\mathbf{q} \in \mathbf{Q})$

Holds by $(*)_5$.

Clause (b): $(\mathfrak{s}_{\mathbf{q}_\delta} = \bigcup\limits_{\alpha<\delta} \mathfrak{s}_{\mathbf{q}_\alpha})$

Holds by $(*)_1$(a), recalling $\mathbf{q}_\alpha \leq_{\mathbf{Q}} \mathbf{q}_\beta \Rightarrow \mathfrak{s}_\alpha \leq_{\mathbf{T}} \mathfrak{s}_\beta$ and Claim 1.3(2).

Clause (c): $(\mathbf{q}_\alpha \leq_{\mathbf{Q}} \mathbf{q})$

Holds by $(*)_6$.]

2) Similarly, as the Knaster condition is preserved by the union of $\lessdot$-increasing continuous chains.

So we are done proving 2.1. $\square_{2.1}$

Claim 2.3. 1) *We have '(A) implies (B),' where:*

- $(A)(a)$ $\mathbf{r} \in \mathbf{Q}_{\rm st}$
  - (b) $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{r}}$*-name of a strong* κ*-forcing.*
  - $(b)^+$ *Moreover, it is a* $\mathbb{P}_{\mathbf{r}\restriction L_0}$*-name, where* $L_0 \subseteq L \trianglelefteq L_{\mathbf{r}}$ *is* $\mathbf{r}$*-closed.*
- (B) *There are* $\mathbf{q} \in \mathbf{Q}_{\rm st}$ *and* $t_* \in L_{\mathbf{q}} \setminus L_{\mathbf{r}}$ *such that*
  - (a) $\mathbf{r} \leq_{\mathbf{Q}} \mathbf{q}$
  - (b) $L_{\mathbf{q}} = L + \{t_*\} + (L_{\mathbf{r}} \setminus L)$ *as linear orders.*
  - (c) $\underset{\sim}{\mathbb{Q}}_{\mathbf{q},t_*} = \underset{\sim}{\mathbb{Q}}$ *and* $I^{\mathbf{q}}_{t_*} = L_0$.

2) *Identical to part (1), but replacing 'strong' by 'weak' everywhere (so of interest only when* $\kappa = \aleph_0$*) and adding to the antecedent:*

$(A)(c)$ L_0 *is* $\mathbf{q}$*-closed and* $\mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}$*, where* $\sigma = (2^\kappa)^+$. *(See 0.1(5).)*

3) *In part (2) we can weaken (A)(c) to*

$(A)(c)'$ *If* $\kappa = \aleph_0$ *then* $\Vdash_{\mathbb{P}_{\mathbf{q},L_0}}$ "$\mathsf{MA}_{\aleph_1}$".

Proof. Easy, recalling 1.12. $\square_{2.3}$

Claim 2.4. 1) *For every* $\mathbf{r} \in \mathbf{Q}_{\rm st}$ *and*[6] $\partial = \mathrm{cf}(\partial) \geq \sigma(\mathbf{r})$ *(see* 1.1(7)*) satisfying* $(\forall \alpha < \partial)\big[|\alpha|^{2^\kappa} < \partial\big]$, *there is a* $\mathbf{q} \in \mathbf{Q}_{\rm st}$ *such that:*

$(A)^1_\partial$ (a) $\mathbf{r} \leq_{\mathbf{Q}_2} \mathbf{q}$

(b) $\|\mathbb{P}_{\mathbf{q}}\| = \big\|\mathbb{P}_{\mathbf{r}}\big\|^{<\partial}$

$(B)^1_\partial$ (a) $\mathbf{q}$ *satisfies* $\mathrm{cf}(L_{\mathbf{q}}) \geq \partial$.

(b) *If* $t \in L_{\mathbf{q}}$ *then* $\mathrm{cf}(L_{\mathbf{q},t}) \geq \partial$.

(c) *If* $L \lhd L_{\mathbf{q}}$ *is of cofinality* $\geq \partial$, $L_0 \subseteq L$ *is* $\mathbf{q}$*-closed,* $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{q},L_0}$*-name of a weak* κ*-forcing of cardinality* $< \partial$, *and*

[As I said, the clause that** $L_0 \subseteq L$ **is q-closed had already been added. It needs to be mentioned** before **you start talking about** $\mathbb{P}_{\mathbf{q},L_0}$**-names.]

$$\kappa = \aleph_0 \Rightarrow \mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}$$

(where $\sigma := (2^\kappa)^+$*)* *then*

- *For some* $s \in L$, $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{q},s}$*-name and*

$$\Vdash_{\mathbb{P}_{\mathbf{q},s}} \text{“}\underset{\sim}{\mathbb{Q}}_{\mathbf{q},s} \text{ and } \underset{\sim}{\mathbb{Q}} \text{ are isomorphic”}.$$

2) *Similar to part* (1), *but* $\mathbf{r}, \mathbf{q} \in \mathbf{Q}_{\rm wk}$, $(\forall \alpha < \partial)\big[|\alpha|^\kappa < \partial\big]$, *and*

$(A)^2_\partial$ (a) $\mathbf{r} \leq_{\mathbf{Q}} \mathbf{q}$

(b) *As above.*

$(B)^2_\partial$ (a) *As above.*

(b) *As above.*

(c) *Like* $(B)^1_\partial(c)$, *but replacing 'weak* κ*-forcing' by 'strong* κ*-forcing' and omitting* $\mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}$.

3) *Like part* (1), *but replacing*

$$\text{“}\kappa = \aleph_0 \Rightarrow \mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}\text{”}$$

by $\Vdash_{\mathbb{P}_{\mathbf{r},L_0}}$ “$\mathsf{MA}_{\aleph_1}$”.

(We shall call the resulting clauses $(A)^{0.5}_\partial$ *and* $(B)^{0.5}_\partial$.*)*

Proof. 1) We shall prove more. Let $\mathbf{Q}_*$ be the class of $\mathbf{q} \in \mathbf{Q}_2$ satisfying $(A)^1_\partial$. Consider the statement

⊞ If $\mathbf{p} \in \mathbf{Q}_*$ then there exists $\mathbf{q} \in \mathbf{Q}_*$ such that:

(a) $\mathbf{p} \leq_{\mathbf{Q}_2} \mathbf{q}$

(b) There is $t \in L_{\mathbf{q}}$ such that $s \in L_{\mathbf{p}} \Rightarrow s <_{L_{\mathbf{q}}} t$.

(c) If $t \in L_{\mathbf{p}}$, $L_0 \subseteq L$ is $\mathbf{q}$-closed, and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{q},L_0}$-name of a weak κ-forcing of cardinality $< \partial$, then $\bullet_1$ or $\bullet_2$ holds, where

$\bullet_1$ For some $s \in L_{\mathbf{q},t}$ we have

$$\Vdash_{\mathbb{P}_{\mathbf{q}}} \text{“}\underset{\sim}{\mathbb{Q}}_{\mathbf{q},s} \text{ and } \underset{\sim}{\mathbb{Q}} \text{ are not isomorphic”}.$$

$\bullet_2$ $\Vdash_{\mathbb{P}_{\mathbf{q}}}$ “$\underset{\sim}{\mathbb{Q}}$ is not ccc”.

---

[6] If we omit “$\partial = \mathrm{cf}(\partial) \geq \sigma(\mathbf{r})$,” then in 2.3 we need to expand by $S'_s \subseteq S_{\mathbf{q},s}$ of cardinality $< \partial$ for $s \in L$, and make further changes.

We shall prove that $\boxplus$ is both true and sufficient, together proving part (1).

Why $\boxplus$ is true:

Let

$$\mathcal{Y} := \big\{(t, L, \underset{\sim}{\mathbb{Q}}) : t \in L \cup \{\infty\},\ L \text{ a } \mathbf{p}\text{-closed subset of } L_{\mathbf{p},t} \text{ of cardinality } < \partial, \text{ and } \underset{\sim}{\mathbb{Q}} \text{ a } \mathbb{P}_{\mathbf{q},L}\text{-name of a forcing notion with set of elements an ordinal } < \partial\big\}.$$

Easily, $|\mathcal{Y}| \leq \|\mathbb{P}_{\mathbf{p}}\|^{<\partial}$, hence we can find a sequence $\big\langle (t_\alpha, L_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha) : \alpha < |\mathcal{Y}|\big\rangle$ listing $\mathcal{Y}$.

Now we choose $\mathbf{p}_\alpha$ by induction on $\alpha \leq |\mathcal{Y}|$ such that

$\oplus^1_\alpha$ (a) $\mathbf{p}_\alpha \in \mathbf{Q}_*$
(b) $\mathbf{p}_0 := \mathbf{p}$
(c) $\langle \mathbf{p}_\beta : \beta \leq \alpha\rangle$ is $\leq_{\mathbf{Q}}$-increasing continuous.
(d) If $\alpha = \beta + 1$, then one of the following hold:
- $\bullet_1$ $\Vdash_{\mathbb{P}_{\mathbf{p}_\beta}}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is not ccc" and $\mathbf{p}_\alpha = \mathbf{p}_\beta$.
- $\bullet_2$ For some s_β, $L_{\mathbf{p}_\alpha} \setminus L_{\mathbf{p}_\beta} = \{s_\beta\}$, $L_{\mathbf{p}_\beta, t_\beta} < s_\beta <_{L_{\mathbf{p}_\alpha}} t_\beta$, and $\underset{\sim}{\mathbb{Q}}_{\mathbf{p}_\alpha, s_\beta} = \underset{\sim}{\mathbb{Q}}$.

Why can we carry the induction? The base case is covered by clause (b), and for α a limit ordinal we use Definition 2.1. For $\alpha \leq |\mathcal{Y}|$ successor let $\alpha = \beta + 1$.

So $\boxplus$ does indeed hold.

Why $\boxplus$ is sufficient:

We choose $\mathbf{q}_\alpha$ by induction on $\alpha \leq \partial$ such that

$\oplus^2_\alpha$ (a) $\mathbf{q}_\alpha \in \mathbf{Q}_*$
(b) $\mathbf{q}_0 := \mathbf{p}$
(c) $\langle \mathbf{q}_\beta : \beta \leq \alpha\rangle$ is $\leq_{\mathbf{Q}}$-increasing continuous.
(d) If $\alpha = \beta + 1$ then $\boxplus$ is satisfied, with $(\mathbf{q}_\beta, \mathbf{q}_\alpha)$ standing in for $(\mathbf{p}, \mathbf{q})$.

We can carry the induction, using $\boxplus$ for α a successor. Now,

$\oplus_3$ $\mathbf{q}_\partial$ is as required.

Why? We shall check 2.4(1)(A),(B).

Clauses (A)(a),(b): This means $\mathbf{q}_\partial \in \mathbf{Q}_*$, which holds by $\oplus^2_\partial$.

Clause (B)(a): This says $\mathrm{cf}(L_{\mathbf{q}}) \geq \partial$.

It holds because $\langle L_{\mathbf{q}_\alpha} : \alpha < \partial\rangle$ is increasing continuous and $L_{\mathbf{q}_\beta}$ is bounded in $L_{\mathbf{q}_{\beta+1}}$, by $\boxplus$(b) and $\oplus^2_\alpha$(d).

Clause (B)(b):

Similarly, using $\boxplus$(c) we can find $L_0 \subseteq L_{\mathbf{q}_\partial, t}$ as required, because

$$\kappa = \aleph_0 \Rightarrow (\forall \alpha < \partial)\big[|\alpha|^{\aleph_1} < \partial\big],$$

because necessarily $L_0 \subseteq L_{\mathbf{q}_\beta}$ for some $\beta < \partial$, and by our choice of $\mathbf{q}_{\beta+1}$.

Clause (B)(b): Similarly to (B)(b).

So we are done proving part (1).

2) Repeat the proof of part (1) using $\mathbf{Q}_2$.

3) Straightforward. $\square_{2.4}$

Definition 2.5. We say $\mathbf{q}$ is *strongly* $(<\partial)$*-homogeneous* when

- If $L_\ell \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed for $\ell = 1,2$ and π_1 is an isomorphism from L_1 onto L_2 mapping $\mathbf{q} \restriction L_1$ to $\mathbf{q} \restriction L_2$, then there is an automorphism π_2 of $L_{\mathbf{q}}$ extending π_1 and mapping $\mathbf{q}$ to itself. Hence it induces an automorphism $\hat{\pi}_2$ of $\mathbb{P}_{\mathbf{q}}$ (e.g. mapping $\underset{\sim}{\eta}_t$ to $\underset{\sim}{\eta}_{\pi_2(t)}$).

Claim 2.6. 1) *If $\mathbf{q} \in \mathbf{Q}_\ell$ for $\ell \in \{1,2\}$ and $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed, then $\mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\mathbf{q},L}$ is a (κ,ℓ)-forcing. (See 0.3.)*

2) *$(\mathbf{Q}_{\rm st}, \leq_{\mathbf{Q}_{\rm st}})$ satisfies amalgamation.*

3) *For $\kappa = \aleph_0$, $\mathbf{Q}_1$ satisfies a weak version of amalgamation:*[7]

> $(*)$ *If $\mathbf{q}_0 \in \mathbf{Q}_1$, $\mathbf{q}_0 \leq_{\mathbf{Q}} \mathbf{q}_\ell$ for $\ell = 1,2$, $L_{\mathbf{q}_1} \cap L_{\mathbf{q}_2} = L_{\mathbf{q}_0}$, and $\Vdash_{\mathbb{P}_{\mathbf{q}_0}}$ "$\mathsf{MA}_{\aleph_1}$" then there is a $\mathbf{q}_3 \in \mathbf{Q}_1$ such that $\mathbf{q}_\ell \leq \mathbf{q}_3$ for $\ell = 0,1,2$.*

4) *In (3)$(*)$ above, we may replace $\Vdash_{\mathbb{P}_{\mathbf{q}_0}}$ "$\mathsf{MA}_{\aleph_1}$" with the demand "$\mathbf{q}_0 \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbf{q}_1$," where $\sigma := (2^{\aleph_0})^+$.*

Proof. 1) **Case 1:** $\kappa > \aleph_0$ (so the choice of ℓ is immaterial).

Proving "$\mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\mathbf{q},L}$ is $(<\kappa)$-complete" is easy, when $\kappa > \aleph_0$. So it suffices to do the following:

$\boxplus$ (a) Assume $p_* \Vdash_{\mathbb{P}_{\mathbf{q},L}}$ "$\underset{\sim}{q}_\alpha \in \mathbb{P}_{\mathbf{q}}/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{q},L}}$ for $\alpha < \kappa^+$".
(b) Now find $p_{**} \in \mathbb{P}_{\mathbf{q},L}$ above p_* and $\mathbb{P}_{\mathbf{q},L}$-names $\underset{\sim}{C}$, $\underset{\sim}{h}$ as required in $*_{\kappa,D}$.

Now

$(*)_1$ For each $\alpha < \kappa^+$, we can choose $\langle p_{\alpha,\iota}, q_{\alpha,\iota} : \iota < \iota(\alpha) \leq \kappa\rangle$ such that
(a) For $\iota < \iota(\alpha)$, $p_{\alpha,\iota} \in \mathbb{P}_{\mathbf{q},L}$ is above p_*, and
$$p_{\alpha,\iota} \Vdash_{\mathbb{P}_{\mathbf{q},L}} \text{“}\underset{\sim}{q}_\alpha = q^*_{\alpha,\iota}\text{”}.$$
(b) Without loss of generality, $\mathbb{Q}_{\mathbf{q},L} \models$ '$(q^*_{\alpha,\iota} \restriction L) \leq p_{\alpha,\iota}$' for $\iota < \iota(\alpha)$.
(c) Therefore, $r_{\alpha,\iota} := p_{\alpha,\iota} \cup (q^*_{\alpha,\iota} \restriction (L_{\mathbf{q}} \setminus L))$ is a $\leq_{\mathbb{P}_{\mathbf{q}}}$-lub of p_α and q^*_α.
(d) $\langle p_{\alpha,\iota} : \iota < \kappa\rangle$ is a maximal antichain of $\mathbb{P}_{\mathbf{q},L}$.

Next,

$(*)_2$ There are C, h, and $\bar{u}$ such that
(a) $C \in D$
(b) h is a pressing-down function on C
(c) $\bar{u} = \langle u_\zeta : \zeta \in \mathrm{rang}(h)\rangle$
(d) If $\zeta \in \mathrm{rang}(h)$ then
$\bullet_1$ The set $S_\zeta := h^{-1}(\{\zeta\})$ belongs to D^+, and $\iota(\alpha) = j(\zeta)$ for $\alpha \in S_\zeta$.
$\bullet_2$ $\langle \mathrm{dom}(r_\alpha) : \alpha \in S_\zeta\rangle$ is a Δ-system with heart u_ζ.

[7] For $\kappa > \aleph_0$ this is not interesting, and is already covered by 2.10(1).

Next,

$(*)_3$ For each $\zeta \in \text{rang}(h)$, $\iota < j(\zeta)$, and $t \in u_\zeta$, recalling $\Vdash_{\mathbb{P}_{\mathbf{q},t}}$ "$\underset{\sim}{\mathbb{Q}}_t$ satisfies $*_{\kappa,D}$", there are $\mathbb{P}_{\mathbf{q},t}$-names $\underset{\sim}{C}_{\zeta,t,\iota}$ and $\underset{\sim}{h}_{\zeta,t}$ witnessing $*_{\kappa,D}$.

Let (e.g.) $\varepsilon := \omega$. We repeat the process ε times, and then we use $\underset{\sim}{\mathbf{H}}_{\mathbf{q},t}$ from 1.4(2)(B)(d) and '$\kappa^{<\kappa} = \kappa$,' and we get

$(*)_4$ There are $C_*, h_*, \bar{u}^*$, and $\overline{S}^* = \langle S^*_\zeta : \zeta \in \text{rang}(h_*)\rangle$ as in $(*)_2$, but for $\langle r^*_{\alpha,\iota} : \alpha \in S^*_\zeta,\ \iota < j(\zeta) \leq \kappa\rangle$ such that (repeating ourselves a bit)
- (a) $r^*_{\alpha,\iota} \in \mathbb{P}_{\mathbf{q}}$, and $r^*_{\alpha,\iota} \restriction L \Vdash_{\mathbb{P}_{\mathbf{q},L}}$ "$\underset{\sim}{q}_\alpha \leq r^*_{\alpha,\iota}$ in $\mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\mathbf{q},L}$".
- (b) For $\alpha \in S^*_\zeta$, the sequence $\langle r^*_{\alpha,\iota} : \iota < j(\zeta)\rangle$ is a maximal antichain of $\mathbb{P}_{\mathbf{q}}$ above p_*.
- (c) If $\zeta \in \text{rang}(h_*)$, $t \in u^*_\zeta$, and $\alpha_1, \alpha_2 \in S^*_\zeta$, then
$$\Vdash_{\mathbb{P}_{\mathbf{q},t}} \text{"}r^*_{\alpha_1}(t), r^*_{\alpha_1}(t) \text{ have a lub in } \underset{\sim}{\mathbb{Q}}_{\mathbf{q},t}\text{"}.$$

The rest of the proof of part (1) for $\kappa > \aleph_0$ should be clear.

Case 2: $\kappa = \aleph_0$ and $\ell = 1$.

Well known.

Case 3: $\kappa = \aleph_0$ and $\ell = 2$.

Like Case 1, but simpler.

2) So assume

$(*)_0$ for $\ell = 0, 1, 2$,
- (a) $\mathbf{q}_\ell \in \mathbf{Q}_2$
- (b) $\mathbf{q}_0 \leq_{\mathbf{Q}_2} \mathbf{q}_\ell$
- (c) $L_{\mathbf{q}_1} \cap L_{\mathbf{q}_2} = L_{\mathbf{q}_0}$ for transparency.

$(*)_1$ Let L be a linear order with set of elements $L_{\mathbf{q}_1} \cup L_{\mathbf{q}_2}$, and $L_{\mathbf{q}_\ell} \subseteq L$ as linear orders.

$(*)_2$ We define $\mathfrak{s} \in \mathbf{T}$ such that $L_{\mathfrak{s}} = L$ and $I_{\mathfrak{s},t} = I_{\mathfrak{s}_{\mathbf{q}_0},t}$ for $t \in L_{\mathbf{q}_\ell}$.

$(*)_3$ We define $\mathbf{q} \in \mathbf{Q}^2_{\mathfrak{s}}$ above $\mathbf{q}_\ell$ (for $\ell \leq 2$) naturally.

We have to prove that $\mathbf{q} \in \mathbf{Q}_2$; being $(<\kappa)$-complete (with $\kappa > \aleph_0$) is easy, satisfying $*_{\kappa,D}$ is a consequence of 2.6(1), and being closed under finite products and composition.

3) Like part (1), but easier.

4) The point here is proving the implication '(A) $\Rightarrow$ (B),' where

(A)
- (a) $\mathbb{P}_0 \lessdot \mathbb{P}_\ell$ (for $\ell = 1, 2$) are ccc forcing notions.
- (b) $\mathbb{P}_0 \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_1$

(B) $\mathbb{P} := \mathbb{P}_1 *_{\mathbb{P}_0} \mathbb{P}_2$ is ccc.

Why does this hold?

Assume $(p_{\alpha,1}, p_{\alpha,2}) \in \mathbb{P}_1 *_{\mathbb{P}_0} \mathbb{P}_2$ for $\alpha < \omega_1$, and let $\langle q_{\alpha,i} : i < \iota_\alpha \leq \omega\rangle$ be a maximal antichain of $\mathbb{P}_0$ such that each $q_{\alpha,i}$ forces a truth value to '$p_{\alpha,1} \in \mathbb{P}_1/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_0}$' and to '$p_{\alpha,2} \in \mathbb{P}_2/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_0}$.' Similarly, for $\alpha, \beta < \omega_1$, let $\langle q_{\alpha,\beta,i} : i < \iota(\alpha,\beta) \leq \omega\rangle$ be a maximal antichain of $\mathbb{P}_0$ such that each $q_{\alpha,\beta,i}$ forces a truth value to "$p_{\alpha,i}$ and $q_{\beta,i}$ are compatible in $\mathbb{P}_\ell/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_0}$ for $\ell = 1, 2$."

Now, finding a sequence $\langle p'_{\alpha,1} : \alpha < \omega_1 \rangle \in {}^{\omega_1}\mathbb{P}_0$ similar enough to $\langle p_{\alpha,1} : \alpha < \omega_1 \rangle$ over

$$\big\{ q_{\alpha,\iota} : \alpha < \omega_1,\ \iota < \iota(\alpha) \big\} \cup \big\{ q_{\alpha,\beta,i} : \alpha, \beta < \omega_1,\ i < \iota(\alpha,\beta) \big\}$$

will contradict "$\mathbb{P}_2$ satisfies the ccc." $\square_{2.6}$

Claim 2.7. 1) *Assume* $\mathbf{p} \in \mathbf{Q}_2$, L_ℓ *is a* $\mathbf{p}$*-closed subset of* $L_{\mathbf{p}}$ *(for* $\ell = 1, 2$*), and* $\pi : L_1 \to L_2$ *is an isomorphism which induces an isomorphism* $\hat{\pi} : \mathbb{P}_{\mathbf{p},L_1} \to \mathbb{P}_{\mathbf{p},L_2}$.

Then we can find $\mathbf{q}$, π_1, L_1^+, L_2^+ *such that*

- (a) $\mathbf{p} \le_{\mathbf{Q}_2} \mathbf{q}$
- (b) *For* $\ell = 1, 2$, $L_\ell \subseteq L_\ell^+ \subseteq L_{\mathbf{q}}$, L_ℓ^+ *is* $\mathbf{q}$*-closed, and* $L_{\mathbf{p}} \subseteq L_1^+$.
- (c) $\pi_1 \supseteq \pi$ *is an isomorphism from* L_1^+ *onto* L_2^+ *which induces an isomorphism* $\hat{\pi}_1 : \mathbb{P}_{\mathbf{p},L_1^+} \to \mathbb{P}_{\mathbf{p},L_2^+}$.

2) *'If* (A) *then* (B)*,' where*

- (A)
  - (a) $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha \le \delta_* \rangle$ *is* $\le_{\mathbf{Q}}$*-increasing continuous.*
  - (b) $\langle \alpha_\varepsilon = \alpha(\varepsilon) : \varepsilon < \zeta \rangle$ *is an increasing continuous sequence of ordinals with limit* δ_*.
  - (c) $L^1_{\alpha(\varepsilon)}$ *and* $L^2_{\alpha(\varepsilon)}$ *are* $\mathbf{q}_{\alpha(\varepsilon)}$*-closed subsets of* $L_{\alpha(\varepsilon)}$.
  - (d) $\pi_\varepsilon : L^1_{\alpha(\varepsilon)} \to L^2_{\alpha(\varepsilon)}$ *is order-preserving and onto.*
  - (e) π_ε *is an isomorphism from* $\mathbf{q}_{\alpha(\varepsilon)} \restriction L^1_{\alpha(\varepsilon)}$ *onto* $\mathbf{q}_{\alpha(\varepsilon)} \restriction L^2_{\alpha(\varepsilon)}$.
  - (f) $L^1_{\alpha(\varepsilon)}, L^2_{\alpha(\varepsilon)}, \pi_\varepsilon$ *are increasing continuously with* ε.
  - (g) *For* $\ell = 1, 2$, *if* $L_{\mathbf{q}_{\alpha(\varepsilon)}} \not\subseteq L^\ell_{\alpha(\varepsilon)+1}$ *then* $L_{\mathbf{q}_{\alpha(\varepsilon)+1}} \subseteq L^\ell_{\alpha(\varepsilon)+2}$.
- (B) $\pi := \bigcup_{\varepsilon<\zeta} \pi_\varepsilon$ *is an automorphism of* $\mathbf{q}_{\delta_*}$.

Proof. 1) By 2.6(2).

2) Easy. $\square_{2.7}$

Definition 2.8. 1) For $\iota = \frac{1}{2}, 2$, we say $\mathbf{q}$ is (∂, ι)*-saturated* when it satisfies $2.4(\iota)(\mathrm{B})^\iota_\partial$.

2) We say $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha < \alpha_* \rangle$ is (∂, ι)-saturated <u>when</u>:

- (a) $\overline{\mathbf{q}}$ is $\le_{\mathbf{Q}_\iota}$-increasing continuous, recalling 1.5(3) and 1.8(2).
- (b) $\mathbf{q}_\alpha$ is (∂, ι)-saturated for $\alpha < \alpha_*$ non-limit.

Remark 2.9. Recall 1.5(3), so e.g. we denote $\mathbf{Q}_{\mathrm{st}}$ and $\mathbf{Q}_{\mathrm{wk}}$ by $\mathbf{Q}_1, \mathbf{Q}_2$, respectively. We may replace them by other classes.

Claim 2.10. 1) *If* $\lambda = \lambda^{<\partial}$ *and* $\partial = \mathrm{cf}(\partial) > \kappa$ *(recalling* $\mathbf{Q}_{\mathrm{st}} = \mathbf{Q}^{\mathrm{st}}_{\kappa,\partial}$*) then there is a* $\mathbf{q} \in \mathbf{Q}^{\mathrm{st}}_{\kappa,\partial}$ *such that*

- (a) $L_{\mathbf{q}}$ *and* $\mathbb{P}_{\mathbf{q}}$ *have cardinality* λ.
- (b) $\mathbf{q}$ *is strongly homogeneous.*
- (c) $\mathbf{q}$ *is* $(\partial, 1)$*-saturated.*

2) *We can combine part* (1) *with* 2.6(3)*; that is, if* $\partial = \mathrm{cf}(\partial) > \kappa = \aleph_0$ *and* $\lambda = \lambda^{<\partial}$, *then there exists a* $\mathbf{q} \in \mathbf{Q}^{\mathrm{wk}}_{\kappa,\partial}$ *such that*

- (a) $L_{\mathbf{q}}$ *has cardinality* λ.

(b) $\mathbf{q}$ *is strongly homogeneous, when we restrict ourselves to an* $L \subseteq L_{\mathbf{q}}$ *such that* $\Vdash_{\mathbb{P}_{\mathbf{q},L}}$ "$\mathsf{MA}_{\aleph_1}$".

(c) $\mathbf{q}$ *is* $(\partial, \frac{1}{2})$*-saturated.*

3) *Similarly for the* $\prec_{\mathbb{L}_{\sigma,\sigma}}$*-version.*

Proof. 1) By 2.7.

2,3) Easy as well. $\square_{2.10}$

§ 3. More on the iteration

Definition 3.1. 1) For $\iota \le 5$, we say $\mathbb{Q}$ is a (κ, ι)-forcing <u>when</u>

- (A) (a) If $\iota = 0$ it is a forcing.
  - (b) If $\iota = 1$ it is a weak κ-forcing.
  - (c) If $\iota = 2$ then it is a strong κ-forcing.
- (B) If $\iota = 3$ then $\mathbb{Q} = (Q, \le, \mathrm{tr}) = (\mathbb{Q}, \le_{\mathbb{Q}}, \mathrm{tr}_{\mathbb{Q}})$ satisfies the following.
  - (a) It is a strong κ-forcing. (Of course, clauses (b),(c) restrict it even further.)
  - (b) $\mathrm{tr}_{\mathbb{Q}}$ is a function $\mathbb{Q} \to \mathcal{H}(\kappa)$.
  - (c) For each $x \in \mathcal{H}(\kappa)$, for some $\partial(x) = \partial_{\mathbb{Q}}(x) \in [2, \kappa]$, any $< 1 + \partial(x)$ members of $\{p \in \mathbb{Q} : \mathrm{tr}(p) = x\}$ have a common upper bound.
- (C) If $\iota = 4$ then as in (B), but we add
  - (d) If $\sigma < \kappa$ then $\{p \in \mathbb{Q} : \partial(\mathrm{tr}(p)) \ge \sigma\}$ is dense.
- (D) If $\iota = 5$ then as in (B), but $\partial(x) = \kappa$ for every $x \in \mathbb{Q}$.

2) For $\iota \le 5$, let $\mathbf{Q}_\iota$ be the class of $\mathbf{q}$ such that[8]

- (A) $\mathbf{q} \in \mathbf{Q}_{\mathrm{pre}}$
- (B) If $t \in L_{\mathbf{q}}$ then $\Vdash_{\mathbb{P}_{\mathbf{q},t}}$ "$\underset{\sim}{\mathbb{Q}}_t$ is an ι-forcing", and if $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed then $\mathbb{P}_{\mathbf{q},L}$ is a (κ, ι)-forcing.
- (C) If $\iota = 3, 4, 5$ then
  - $\bullet_1$ If $p \in \mathbb{P}_{\mathbf{q}}$ and $s \in \mathrm{dom}(p)$, then $\mathrm{tr}_{\mathbb{Q}_s}(p(s))$ is an object, not just a name.
  - $\bullet_2$ If $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed then $\mathbb{P}_{\mathbf{q},L}$ is a $(\kappa, 2)$-forcing.
- (D) If $\iota = 4$ then in addition to $\bullet_1$ and $\bullet_2$,
  - $\bullet_3$ If $\partial < \kappa$ and $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed then
  $$\big\{p \in \mathbb{P}_{\mathbf{q}} : \big(\forall s \in \mathrm{dom}(p)\big)\big[\partial_{\mathbb{Q}_s}(p(s)) \ge \partial\big]\big\}$$
  is dense in $\mathbb{P}_{\mathbf{q},L}$.

3) For $\iota \le 5$, let $\mathbb{Q}^\iota_{\partial,\kappa}$ be the class of $\mathbf{q} \in \mathbf{Q}_\iota$ such that $t \in L_{\mathbf{q}} \Rightarrow |I_{\mathbf{q},t}| < \kappa$ and $\mathbf{q}$ is strongly $(<\partial)$-homogeneous.

Claim 3.2. *For $\iota = 3, 4, 5$, we can repeat the work done for $\iota = 2$ (i.e. $\mathbf{Q}_2$) in §1-2.*

Proof. Repeating previous proofs, using Definition 3.1. $\square_{3.2}$

Definition 3.3. If clause (A) holds, then we define $\mathbb{P}_{\bar{s}}$ as in clause (B), where:

- (A) (a) $\mathbf{q} \in \mathbf{Q}_1$ and $\kappa = \aleph_0$.
  - (b) $\bar{s} \in {}^\alpha(L_{\mathbf{q}})$ and $u_i \subseteq \alpha$ for $i < \alpha$.
  - (c) $L_{\mathbf{q}} \models$ "$s_i < s_j$" for $i < j < \alpha$.
  - (d) $u_i := \{j < i : s_j \in I_{\mathbf{q},s_i}\}$

[8]We may just demand that for $\mathbf{q}$-closed L, we have that
$$\{p \in \mathbb{P}_{\mathbf{q},L} : s \in \mathrm{dom}(p) \Rightarrow \mathrm{tr}_{\mathbb{Q}_s}(p(s)) \text{ is an object}\}$$
is dense. In this case, if $\kappa > \aleph_0$ then this follows.

(e) $\mathbb{Q}_{\mathbf{q},s_i}$ is definable from $\bar{\underset{\sim}{\eta}}_i = \langle \underset{\sim}{\eta}_{s_j} : j \in u_i \rangle$ (say we have a definition $\bar{\varphi}_{i,\bar{\eta}}$ for any $\bar{\eta} \in X_i := \prod\limits_{\varepsilon \in u_i} {}^{S_\varepsilon}2$, where $S_\varepsilon := S_{\mathbf{q},s_\varepsilon}$).

(B) $\mathbb{P}_{\bar{s}} := \mathbb{P}_{\mathbf{q}} \restriction L$, where

$$L := \big\{p \in \mathbb{P}_{\mathbf{q}} : \mathrm{dom}(p) \subseteq \{s_i : i < \alpha\}, \text{ and if } s_i \in \mathrm{dom}(p) \text{ then } \mathrm{supp}(p(s_i)) \subseteq \{s_j : j \in u_i\}\big\}.$$

Claim 3.4. 1) *For $\kappa = \aleph_0$ and $\mathbf{q}, n, \bar{s}$, X_i (for $i < \alpha$) as in* 3.3(A)(e), *we have $\mathbb{P}_{\mathbf{q},\bar{s}} \lessdot \mathbb{P}_{\mathbf{q}}$ <u>when</u>*

- $\boxplus_1$ *If $i < \alpha$ <u>then</u> the demand on $\mathbb{Q}_{\bar{\varphi}_i, \bar{\underset{\sim}{\eta}}}$ holds absolutely (i.e. even after forcing by any κ-forcing).*
- $\boxplus_2$ *Assuming $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$ is generic over $\mathbf{V}$ and $\bar{\eta} = \langle \underset{\sim}{\eta}_t[\mathbf{G}] : t \in L_{\mathbf{q}} \rangle$, we have: <u>if</u> $\mathbf{V}[\langle \eta_{s_j} : j \in u_i \rangle] \models$ "$\mathcal{J}$ is a maximal antichain of $\mathbb{Q}[\langle \eta_{s_j} : j \in u_i \rangle]$" <u>then</u> $\mathbf{V}[\bar{\eta} \restriction L_{\mathbf{q},s_i}] \models$ "$\mathcal{J}$ is a maximal antichain of $\mathbb{Q}[\bar{\eta} \restriction L_{\mathbf{q},s_i}]$" for $i < \alpha$.*

2) *$\mathbb{Q}^2_{\mathbf{n}}$ from* [HS, Defs. 2,4,5] *satisfies the criteria above. Moreover, so does any Suslin ccc forcing.*

3) *Similarly to parts* (1), (2) *for $\bar{s} = \langle s_\alpha : \alpha < \alpha_* \rangle$, where $s_\alpha \in L_{\mathbf{q}}$ is $<_{\mathbf{q}}$-increasing.*

Proof. 1,2) By (3).

3) Straightforward by induction on α_*. $\square_{3.4}$

§ 4. A consequence

We prove the result promised in the introduction, continuing Kellner-Shelah [KS11] and Horowitz-Shelah [HS].

Theorem 4.1. *Let $\kappa = \aleph_0$, $\partial = (2^{\aleph_0})^+$ (or just $\partial = \partial^{\aleph_0} = \mathrm{cf}(\partial)$, $\partial > 2^{\aleph_0}$ for simplicity), and $\lambda = \lambda^{<\partial}$.*

Let $\mathbf{n} \in \mathbf{N}$ be special, *in the sense of* [HS, Definitions 2,4] *(and so $T_{\mathbf{n}}$ is a finite-branching subtree of ${}^{\omega>}\omega$ as defined there). Let $(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$ be as in* [HS, Definition 5], *except that we restrict ourselves to the (dense) subset of $p \in \mathbb{Q}^2_{\mathbf{n}}$ such that for some $m \ll \ell g(\mathrm{tr}_{p(\alpha)})$,*

$$\nu \in p(\alpha) \Rightarrow \mathrm{nor}(\mathrm{suc}_{p_{\bar{w}}}(\nu)) \geq 1 + \frac{1}{m}$$

(as done in the proof of [HS, Claim 21]*).*

Then there is a $\mathbf{q} \in \mathbf{Q}^2_{\kappa,\partial}$ such that:

- (a) *$L_{\mathbf{q}}$ has cardinality λ, $\mathrm{cf}(L_{\mathbf{q}}) = \mathrm{cf}(\lambda)$, and $t \in L_{\mathbf{q}} \Rightarrow |I_{\mathbf{q},t}| < \lambda$.*
- (b) *For every $t \in L_{\mathbf{q}}$, $\mathbb{Q}_{\mathbf{q},t} = \mathbb{Q}^2_{\mathbf{n}}[\mathbf{V}^{\underset{\sim}{\bar{\eta}} \restriction I_t}]$, so $\underset{\sim}{\eta}_t \in \lim T_{\mathbf{n}}$ is $\underset{\sim}{\eta}^2_{\mathbf{n}}$ (recalling* [HS] *— that is,* 3.4(2)*).*
- (c) *$\mathbf{q}$ is strongly $(<\partial)$-homogeneous (see* 2.5*).*
- (d) *Letting $\mathbf{V}_0 = \mathbf{V}$, $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$, and $\mathbf{V}_1 = \mathrm{HOD}\big(\{\underset{\sim}{\bar{\eta}} \restriction u : u \in [L_{\mathbf{q}}]^{<\partial}\}\big)$:*
  - (α) *$\mathbf{V}_1 \models \mathsf{ZF} + \mathsf{DC}_{<\partial}$*
  - (β) *In $\mathbf{V}_1$, modulo the ideal*

    $$J = J_{\mathbf{n},<\partial} := \mathrm{id}_{<\partial}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}}),$$

    we have:
    - $\bullet_1$ $\lim(T_{\mathbf{n}}) \equiv \{\underset{\sim}{\eta}_t : t \in L_{\mathbf{q}}\} \mod J$
    - $\bullet_2$ *Every subset of $\lim(T_{\mathbf{n}})$ is equivalent to a Borel set modulo J.*

Remark 4.2. 1) The difference with the results in [HS] is that there we do not have "$\mathbf{V}_1$ satisfies $\mathsf{AC}_{\aleph_0}$" (to say nothing of DC), whereas here we have DC (even $\mathsf{DC}_{<\partial}$, with $\partial > \aleph_1$).[9]

2) In $\mathrm{id}_{<\partial}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$, is the '$<\partial$' necessary? ([HS, Definition 18] uses $\mathrm{id}_{\leq\aleph_1}$, in our notation.) That is, can we use $\mathrm{id}_{\leq\aleph_0}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$?

For this we have to use "amoeba for $\mathbb{Q}_{\mathbf{n}}$," hence we have to prove stronger amalgamation (which is far from clear). But see 4.5 below.

Proof. Let $\mathbf{Q}_{\mathbf{n}}$ be the set of $\mathbf{q} \in \mathbf{Q}$ which satisfy 4.1(b). Now we can replace $\mathbf{Q}$ by $\mathbf{Q}_{\mathbf{n}}$ in 2.6, and we rely on 4.3, 4.4, and 4.5 below. $\square_{4.1}$

Claim 4.3. *For $\mathbf{q}$ as in* 4.1*,*

$$\Vdash_{\mathbb{P}_{\mathbf{q}}} \text{“if } \eta \in \lim(T_{\mathbf{n}}) \text{ is } (\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})\text{-generic over } \mathbf{V} \text{ then } \eta \in \{\underset{\sim}{\eta}_s : s \in L_{\mathbf{q}}\}\text{”}.$$

Proof. We continue [HS, p.15, Claim 21] (but there it sufficed to consider iterations of finite length).

So assume

$(*)_1$ $p_* \Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$\underset{\sim}{\eta} \in \lim(T_{\mathbf{n}})$".

[9] As wrongly stated in [JS93], for the ideal of meagre sets.

$(*)_2$ For $n < \omega$, let $\bar{p}_n := \langle p_{n,\ell} : \ell < \omega \rangle$ be a maximal antichain of $\mathbb{P}_{\mathbf{q}}$ such that $p_{n,\ell} \Vdash \underset{\sim}{\eta} \restriction n = \nu_{n,\ell}$.

Let $L_* := \bigcup\limits_{n,\ell<\omega} \mathrm{supp}(p_{n,\ell}) \cup \mathrm{supp}(p_*)$; it is a countable subset of $L_{\mathbf{q}}$.

$(*)_3$ (a) For $\eta \in T_{\mathbf{n}}$, define:

$$W_{\mathbf{n},\eta} := \big\{ w \subseteq \mathrm{suc}_{T_{\mathbf{n}}}(\eta) : \mathrm{nor}^{\mathbf{n}}_{\eta}(w) \geq 2 \big\}.$$

(b) For $n < \omega$ define $\Lambda_n := \{\eta \in T_{\mathbf{n}} : \ell g(\eta) < n\}$, so $T_{\mathbf{n}} = \bigcup\limits_{n<\omega} \Lambda_n$.

(c) Define

- $\bullet_1$ $S_n := \{\overline{w} = \langle w_\eta : \eta \in \Lambda_n \rangle : w_\eta \in W_{\mathbf{n},\eta}\}$ for $n < \omega$.
- $\bullet_2$ $S := \bigcup\limits_{n<\omega} S_n$
- $\bullet_3$ $(S, \trianglelefteq)$ is a tree with ω levels such that each level is finite.
- $\bullet_4$ $\lim(S) = \{\overline{w} = \langle w_\eta : \eta \in T_{\mathbf{n}} \rangle : \overline{w} \restriction \Lambda_n \in S_n$ for every $n\}$.

(d) For $\overline{w} \in \lim(S)$ let

$$\mathbf{B}_{\overline{w}} := \{\rho \in \lim(T_{\mathbf{n}}) : \text{for every } n \text{ large enough, } \rho \restriction (n+1) \in w_{\rho \restriction n}\}.$$

$(*)_4$ So $\mathbf{B}_{\overline{w}} = \bigcup\limits_{m<\omega} \mathbf{B}_{\overline{w},m}$, where

$$\mathbf{B}_{\overline{w},m} := \big\{\rho \in \lim(T_{\mathbf{n}}) : (\forall n \geq m)[\rho \restriction (n+1) \in w_{\rho\restriction n}]\big\}$$

is a closed subset of $\lim(T_{\mathbf{n}})$.

As proved there,

$(*)_5$ For $\iota = 1, 2$, $\Vdash_{\mathbb{Q}^{\iota}_{\mathbf{n}}}$ "$\underset{\sim}{\eta}^{\iota}_{\mathbf{n}} \in B_{\overline{w}}$" for every $\overline{w} \in \lim(S)^{\mathbf{V}}$".

Hence as in [HS],

$\boxplus$ By $(*)_1$, it suffices to prove $p_* \nVdash_{\mathbb{P}_{\mathbf{q}}}$ "$\underset{\sim}{\eta} \in \mathbf{B}_{\overline{w}}$ for some $\overline{w} \in \lim(S)^{\mathbf{V}}$".

Toward contradiction, assume

$$\Vdash_{\mathbb{P}_{\mathbf{q}}} \text{"}\underset{\sim}{\eta} \text{ is generic for } (\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}}) \text{ over } \mathbf{V}\text{"},$$

<u>or</u> we just choose $\langle p_{\overline{w}} : \overline{w} \in \lim(S) \rangle$ such that $p_* \leq p_{\overline{w}}$ and $p_{\overline{w}} \Vdash \underset{\sim}{\eta} \in \mathbf{B}_{\overline{w}}$. Note that for $r \in \mathrm{dom}(p_{\overline{w}})$, $\mathrm{tr}(p_{\overline{w}}(r))$ is an object (not just a $\mathbb{P}_{\mathbf{q},s}$-name) because $\mathbf{q} \in \mathbf{Q}^2_{\partial,\kappa}$. We continue as there. $\square_{4.3}$

Claim 4.4. 1) *Forcing with $\mathbb{Q}^2_{\mathbf{n}}$ adds a Cohen real.*

2) *If $\mathbb{Q}$ adds a Cohen real then $\Vdash_{\mathbb{Q}}$ "$(\lim T_{\mathbf{n}})^{\mathbf{V}} \in \mathrm{id}_{\leq\aleph_0}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$".*

Proof. See [HS, Claim 19]. $\square_{4.4}$

Claim 4.5. *In the conclusion of Claim 4.1, we can replace* $\mathrm{id}_{<\partial}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$ *by the ideal* $J' := \mathrm{id}_{\leq\aleph_0}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}}) + Y$*, where in* $\mathbf{V}_1$ *we define*

$$Y := \bigcup \big\{\mathbf{B}^{\mathbf{V}_1} : \mathbf{B} \text{ is a Borel subset of } \mathbf{T}_{\mathbf{n}} \text{ defined in } \mathbf{V}_0 \text{ such that } \Vdash_{\mathbb{Q}^2_{\mathbf{n}}} \text{"}\underset{\sim}{\eta}^2_{\mathbf{n}} \notin \mathbf{B}\text{"}\big\}.$$

Proof. The same proof as in 4.1; that is, in clause (d)(β) we use the ideal J' above instead of $J_{\mathbf{n},<\partial}$. $\square_{4.5}$

* * *

Definition 4.6. 1) Let Φ_κ be the set of pairs $(\bar\varphi, \underset{\sim}{\nu})$ such that

(a) $\bar\varphi$ is a definition of a κ^+-cc forcing notion $\mathbb{Q}_i = \mathbb{Q}_{\bar\varphi,i}$ in $\mathcal{H}(\kappa^+)$ from a parameter $i \in {}^\kappa\mathcal{H}(\kappa)$.

(b) $\Vdash_{\mathbb{Q}_{\bar\varphi,i}}$ "$\underset{\sim}{\nu} \in {}^\kappa\mathcal{H}(\kappa)$"; naturally the generic, but this is not necessary.

(c) Moreover, any κ-forcing preserves the properties of (a) and (b), and

$$\text{“}p \in \mathbb{Q}_{\bar\varphi,i},, p \leq_{\mathbb{Q}_{\bar\varphi,i}} q,\ \langle p_\varepsilon : \varepsilon < \varepsilon_* \rangle \text{ is a } \mathbb{Q}_{\bar\varphi,i}\text{-MAC”}$$

will be absolutely between $\mathbf{V}^{\mathbb{P}_1}$ and $\mathbf{V}^{\mathbb{P}_2}$, where $\mathbb{P}_\ell := \mathbb{P}_{\mathbf{q}_\ell}$, $\mathbf{q}_1 \leq_{\mathbf{Q}} \mathbf{q}_2$, and $c_i \in \mathbf{V}[\mathbb{P}_{\mathbf{q}_1}]$.

(A $\mathbb{Q}$-*MAC* is a maximal antichain of the the forcing notion $\mathbb{Q}$.)

2) For $(\bar\varphi, \underset{\sim}{\nu}) \in \Phi_\kappa$ and $\partial > \kappa$, we define the ideal $\mathrm{id}(\bar\varphi, \underset{\sim}{\nu})$ on $\mathcal{P}({}^\kappa\mathcal{H}(\kappa))$ as usual.

Claim 4.7. *Assume $\lambda = \lambda^{<\partial}$ and $\partial = \mathrm{cf}(\partial) > 2^\kappa$. Then there is $\mathbf{q}$ such that*

(A) $\mathbf{q} \in \mathbf{Q}_{\kappa,\partial}$, $L_{\mathbf{q}}$ has cardinality λ, and $\mathrm{cf}(L_{\mathbf{q}}) = \mathrm{cf}(\lambda)$.

(B) For every $t \in L_{\mathbf{q}}$ there are $(\bar\varphi_t, \underset{\sim}{\nu}) \in \Phi_\kappa$ and $\underset{\sim}{c}_t$ (a $\mathbb{P}_{\mathbf{q},I_t}$-name of a member of ${}^\kappa\mathcal{H}(\kappa)$) such that $\mathbb{Q}_{\mathbf{q},t} = (\underset{\sim}{\mathbb{Q}}_{\bar\varphi_t,\underset{\sim}{c}_t})^{\mathbf{V}[\eta]}$, and let $\underset{\sim}{\nu}_t$ be chosen naturally.

(C) For every $\underset{\sim}{c}$ (a $\mathbb{P}_{\mathbf{q}}$-name of a member of ${}^\kappa\mathcal{H}(\kappa)$), letting $X := \{t \in L_{\mathbf{q}} : (\bar\varphi_t, \underset{\sim}{c}_t) = (\bar\varphi, \underset{\sim}{c})\}$ and $Y := \{\underset{\sim}{\nu}_t : t \in X\}$, we have

(a) $\Vdash_{\mathbb{P}_t} Y \notin \mathrm{id}_{<\partial}(\mathbf{Q}_{\varphi_c}, \underset{\sim}{\nu})$

[Don't recall coloring in this subscript, but it's probably because t isn't defined in clause (C), and only appears as a bound variable in the definitions of X and Y. If you meant this as a continuation of 'for all $t \in L_{\mathbf{q}}$,' I can just repeat that phrase again and change the indices to something else.]

(b) Letting $\mathbf{V}_0 = \mathbf{V}$, $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$, and

$$\mathbf{V}_1 = \mathrm{HOD}^{\mathbf{V}_2}(\{\bar\eta \restriction L : L \in [L_t]^{<\partial}\}, \{Y\}, \mathbf{V})$$

then $\mathbf{V}_1$ is a model of $\mathsf{ZF} + \mathsf{DC}_{<\partial}$ + "every $Z \subseteq Y \subseteq {}^\kappa\mathcal{H}(\kappa)$ is equal to a κ-Borel set modulo the ideal generated by

$$\mathrm{id}_{<\partial}(\underset{\sim}{\mathbb{Q}}_{\bar\varphi,\underset{\sim}{c}}, \underset{\sim}{\nu}) \cup \{{}^\kappa\mathcal{H}(\kappa) \setminus Y\} \cup \{{}^\kappa\mathcal{H}(\kappa)^{\mathbf{V}[\bar\eta \restriction L_t]} : t \in L_{\mathbf{q}}\}\text{”}.$$

(c) If $(\underset{\sim}{\mathbb{Q}}_{\bar\varphi,\underset{\sim}{c}}, \underset{\sim}{\nu})$ does not commute with itself (see below) then we can use the ideal $\mathrm{id}_{<\partial}(\underset{\sim}{\mathbb{Q}}_{\bar\varphi,\underset{\sim}{c}}, \underset{\sim}{\nu}) \cup \{{}^\kappa\mathcal{H}(\kappa) \setminus Y\}$.

(d) If we restrict the parameter $\underset{\sim}{c}_t$ to be from $\mathbf{V}$, we can use $\mathbf{V}_1$ for all $(\bar\varphi, c)$.

Remark 4.8. In 4.7(C)(c) the assumption is very weak. It fails for Cohen reals and Random reals. By [She94], [She04a], among ccc Suslin forcings $\mathbb{Q}$ (see [JS88]) if $\mathbb{Q}$ is not bounding then only Cohen forcings do not commute with themselves.

Probably among the bounding ones, 'Random real' is the only one.

Proof. Straightforward. $\square_{4.7}$

References

[HS] Haim Horowitz and Saharon Shelah, *Saccharinity with ccc*, arXiv: 1610.02706.

[JS88] Haim I. Judah and Saharon Shelah, *Souslin forcing*, J. Symbolic Logic **53** (1988), no. 4, 1188–1207. MR 973109

[JS93] ———, *Retracted: Baire property and axiom of choice*, Israel J. Math. **84** (1993), no. 3, 435–450, arXiv: math/9211213. MR 1244679

[KS11] Jakob Kellner and Saharon Shelah, *Saccharinity*, J. Symbolic Logic **76** (2011), no. 4, 1153–1183, arXiv: math/0511330. MR 2895391

[RS04] Andrzej Rosłanowski and Saharon Shelah, *Sweet & sour and other flavours of ccc forcing notions*, Arch. Math. Logic **43** (2004), no. 5, 583–663, arXiv: math/9909115. MR 2076408

[RS06] ———, *How much sweetness is there in the universe?*, MLQ Math. Log. Q. **52** (2006), no. 1, 71–86, arXiv: math/0406612. MR 2195002

[S+a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F2009.

[S+b] ———, *Tba*, In preparation. Preliminary number: Sh:F2029.

[S+c] ———, *Tba*, In preparation. Preliminary number: Sh:F2330.

[S+d] ———, *Tba*, In preparation. Preliminary number: Sh:F2329.

[S+e] ———, *Tba*, In preparation. Preliminary number: Sh:F2261.

[She] Saharon Shelah, *Are $\mathfrak{a}$ and $\mathfrak{d}$ your cup of tea? Revisited.*, arXiv: 2108.03666 Revised version of [Sh:700].

[She94] ———, *How special are Cohen and random forcings, i.e. Boolean algebras of the family of subsets of reals modulo meagre or null*, Israel J. Math. **88** (1994), no. 1-3, 159–174, arXiv: math/9303208. MR 1303493

[She00] ———, *Was Sierpiński right? IV*, J. Symbolic Logic **65** (2000), no. 3, 1031–1054, arXiv: math/9712282. MR 1791363

[She04a] ———, *Properness without elementaricity*, J. Appl. Anal. **10** (2004), no. 2, 169–289, arXiv: math/9712283. MR 2115943

[She04b] ———, *Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing*, Acta Math. **192** (2004), no. 2, 187–223, Previous title "Are $\mathfrak{a}$ and $\mathfrak{d}$ your cup of tea?" arXiv: math/0012170. MR 2096454

[She22] ———, *Forcing axioms for λ-complete μ^+-c.c*, MLQ Math. Log. Q. **68** (2022), no. 1, 6–26, arXiv: 1310.4042. MR 4413641

[Sol70] Robert M. Solovay, *A model of set theory in which every set of reals is lebesgue measurable*, Annals of Math. **92** (1970), 1–56.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 9190401, Jerusalem, Israel; and, Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

URL: https://shelah.logic.at/