SUPERATOMIC BOOLEAN ALGEBRAS: MAXIMAL RIGIDITY 704

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Abstract. We prove that for any superatomic Boolean Algebra of cardinality $> \beth_4$ there is an automorphism moving uncountably many atoms. Similarly for larger cardinals. Any of those results are essentially the best possible.

Annotated Content

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§ 0. Introduction

We show that any superatomic Boolean Algebra has an automorphism moving uncountably many atoms if it is large enough (really, $> \beth_4$); similarly replacing $\aleph_0$ by θ;

[Replacing $\aleph_0$ *where*? The only cardinals in the previous sentence were $\beth_4$ and 'uncountable.']

(an automorphism moves an atom if its image is not itself). We then show that those results are essentially the best possible. Recall that many other natural classes of Boolean Algebras behave differently; there are arbitrarily large members with few automorphisms (and even endomorphisms). Of course, we can express those results in topological terms. (See [Mon] and [Mon90] on Boolean Algebras.)

Rubin and Koppleberg [RK01] have proved the following: if $\diamondsuit_{\lambda^+} + 2^{\lambda^+} = \lambda^{++}$ then there is a superatomic Boolean Algebra $\mathbb{B}$ of cardinality λ^{++} with λ atoms and exactly λ^+ automorphisms answering Question 80 of Monk [Mon96] (i.e. in a preliminary version asking for a consistent example).

By [She01, §1], provably in ZFC, there is a superatomic Boolean Algebra $\mathbb{B}$ such that $|\mathrm{Aut}(\mathbb{B})| < |\mathrm{End}(\mathbb{B})|$ answering Question 96 of Monk [Mon96, p.291].

By [She01, §2], provably in ZFC, there is a superatomic Boolean Algebra $\mathbb{B}$ such that $|\mathrm{Aut}(\mathbb{B})| < |\mathbb{B}|$, answering Problem 80 of [Mon96, p.291].

[So both this *and* [RK01] answer Problem 80? (For reference, #80 in Monk reads "Is $\mathrm{Length}_{H^+}(A) = t(A) \cdot \mathrm{Length}(A)$ for every infinite Boolean Algebra A?")]

In fact, if μ is strong limit, $\mu > \mathrm{cf}(\mu) = \aleph_0$ and $\lambda = \min\{\lambda : 2^\lambda > 2^\mu\}$, then there is a Boolean Algebra $\mathbb{B}$ with 2^μ atoms, 2^λ elements and every automorphism of $\mathbb{B}$ moves $< \mu$ atoms (so $|\mathrm{Aut}(\mathbb{B})| \leq 2^\mu < 2^\lambda$).

Notation

Definition 0.1. 1) For a Boolean Algebra $\mathbb{B}$, its operations are denoted by $x \cap y$, $x \cup y$, $x - y$, and $-x$, and $0_{\mathbb{B}}$ is its zero. Let us define the ideal $\mathrm{id}_\alpha(\mathbb{B})$ by induction:

- $\mathrm{id}_0(\mathbb{B}) := \{0\}$
- $\mathrm{id}_\beta(\mathbb{B}) :=$

$$\big\{x_1 \cup \ldots \cup x_n : n < \omega \text{ and for each } \ell = 1, \ldots, n,\ x_\ell \in \mathbb{B} \text{ and for some } \alpha < \beta, \\ \underline{\text{either}}\ x_\ell \in \mathrm{id}_\alpha(B) \text{ or } x_\ell/\mathrm{id}_\alpha(B) \text{ is an atom of } \mathbb{B}/\mathrm{id}_\alpha(B)\big\}.$$

Hence for limit δ we have

- $\mathrm{id}_\delta(\mathbb{B}) = \bigcup_{\beta<\delta} \mathrm{id}_\beta(\mathbb{B})$.

Let $\mathrm{id}_\infty(\mathbb{B}) := \bigcup_\alpha \mathrm{id}_\alpha(\mathbb{B})$.

2) For $x \in \mathrm{id}_\infty(\mathbb{B})$ let $\mathrm{rk}(x, \mathbb{B}) := \min\{\alpha : x \in \mathrm{id}_{\alpha+1}(\mathbb{B})\}$.

3) $\mathbb{B}$ is *superatomic* if $\mathbb{B} = \mathrm{id}_\infty(\mathbb{B})$.

[The *rank*] $\mathrm{rk}(\mathbb{B})$ is the ordinal α such that $\mathbb{B}/\mathrm{id}_\alpha(\mathbb{B})$ is a finite Boolean Algebra (so $\mathbb{B} = \mathrm{id}_{\alpha+1}(\mathbb{B})$).

4) For a Boolean Algebra $\mathbb{B}$ and $x \in \mathbb{B}$, let

$$\mathbb{B} \restriction x := \mathbb{B} \restriction \{y \in \mathbb{B} : y \leq_{\mathbb{B}} x\}.$$

It is a Boolean Algebra.

5) Define the following by induction on $n = 1, 2, \ldots$:

[So, on $n < \omega$?]

$\beth_1(<\theta) := 2^{<\theta} = \sum\limits_{\kappa<\theta} 2^\kappa$

$\beth_{n+1}(<\theta) := 2^{\beth_n(<\theta)}$.

Observation 0.2. *If $\mathbb{B}$ is a superatomic and D_n is an ultrafilter of $\mathbb{B}$ for $n < \omega$, then for some infinite $u \subseteq \omega$ the sequence $\langle D_n : n \in u\rangle$ converges to some ultrafilter of D of $\mathbb{B}$. (I.e. for every $x \in \mathbb{B}$, for all but finitely many $n \in u$, we have $x \in D_n \Leftrightarrow x \in D$.)*

Proof. Among the pairs

$$\big\{(x, \alpha) : x \in \mathbb{B},\ \mathrm{rk}(x, \mathbb{B}) = \alpha \text{ and } (\exists^\infty n)[x \in D_n]\big\},$$

choose one (x, α) with x minimal. Without loss of generality $x/\mathrm{id}_\alpha(\mathbb{B})$ is an atom. Let $u := \{n < \omega : x \in D_n\}$ and check that $D = \{y \in \mathbb{B} : \mathrm{rk}(y \cap x, \mathbb{B}) = \alpha\}$ is as required. $\square_{0.2}$

§ 1. Superatomic Boolean Algebras have nontrivial automorphisms

Theorem 1.1 (Main Theorem). *Assume*

(a) $\mathbb{B}$ *is a superatomic Boolean Algebra with no automorphism moving* $\geq \theta$ *atoms; that is, if* π *is an automorphism of* $\mathbb{B}$ *then*

$$\big|\{x \in \text{atom}(\mathbb{B}) : \pi(x) \neq x\}\big| < \theta.$$

(b) θ *is regular uncountable.*

Then $|\mathbb{B}| \leq \beth_4(<\theta)$, *so if* $\theta = \sigma^+$ *then* $|\mathbb{B}| \leq \beth_4(\sigma)$.

Remark 1.2. If $|\mathbb{B}|$ is close to $\beth_4(<\theta)$, then the proof says much on the structure of $\mathbb{B}$.

Proof. Let $\mathbb{B}$ be the Boolean algebra satisfying clause (a) and let μ be the number of atoms of $\mathbb{B}$. Without loss of generality

$\boxtimes_1$ $\mathbb{B}$ is a Boolean Algebra of subsets of μ and its atoms are the singletons $\{\alpha\}$ for $\alpha < \mu$. (So e.g. $\mathbb{B} \models$ '$a - b = c$' iff $a \setminus b = c$.)

Let $I := [\mu]^{<\theta} \cap \mathbb{B} = \{x \in \mathbb{B} : |x| < \theta\}$; clearly I is an ideal of $\mathbb{B}$. Let

$$Y := \{x \in \mathbb{B} : x/I \text{ is an atom of } \mathbb{B}/I\}.$$

We shall prove (after some preliminary matters) that:

$\boxtimes_2$ If $x \in Y$ then $|x| \leq \beth_2(<\theta)$; i.e. $2^{(2^{<\theta})}$.

We shall say that a set $a \subseteq \mu$ is $\mathbb{B}$-*autonomous* if $(\forall y \in I)[y \cap a \in \mathbb{B}]$. In this case we let $\mathbb{B} \restriction a := \mathbb{B} \cap \mathcal{P}(a)$; this notation is compatible with 0.1(4).

Clearly

$\oplus_1$ The family of $\mathbb{B}$-autonomous subsets of μ is a Boolean ring and even a Boolean algebra of subsets of μ (i.e. closed under $a \cap b, a \cup b, a \setminus b$), and includes I and even $\mathbb{B}$.

$\oplus_2$ For a $\mathbb{B}$-autonomous set a, $\mathbb{B} \restriction a := \{x \in \mathbb{B} : x \subseteq a\}$ is a Boolean ring of subsets of a which include $\big\{\{\alpha\} : \alpha \in a\big\}$.

Also,

$\oplus_3$ If a_0, a_1 are $\mathbb{B}$-autonomous subsets of μ, $x \in Y$, $a_0 \subseteq x$, $a_1 \subseteq x$, and $\mathbb{B} \restriction a_0 \cong \mathbb{B} \restriction a_1$ over $\mathbb{B} \restriction (a_1 \cap a_2) := \mathbb{B} \cap \mathcal{P}(a_1 \cap a_2)$, then there is an automorphism h of $\mathbb{B}$ such that h maps a_0 to a_1, a_1 to a_0 and

$$\alpha \in (\mu \setminus a_0) \setminus a_1 \Rightarrow h(\{\alpha\}) = \{\alpha\}.$$

[Why? Let g be an isomorphism from $\mathbb{B} \restriction a_0$ onto $\mathbb{B} \restriction a_1$ over $\mathbb{B} \restriction (a_0 \cap a_1)$. Now we define a permutation h of $\text{atom}(\mathbb{B}) := \{\{\alpha\} : \alpha < \mu\}$; let

$$\alpha \in a_0 \Rightarrow h(\{\alpha\}) = g(\{\alpha\}) \wedge h(g(\{\alpha\})) = \{\alpha\}$$

and $\alpha \in (\mu \backslash a_0) \backslash a_1 \Rightarrow h(\{\alpha\}) = \{\alpha\}$. By the demands on g clearly h is a well defined permutation of $\text{atom}(\mathbb{B})$. Now h can be naturally extended to an automorphism $\hat{h}$ of $\mathcal{P}(\mu)$ as a Boolean Algebra: it is of order two. We have to check that $\hat{h}$ maps $\mathbb{B}$ onto itself (even *into* itself will suffice, because of "order two"). Clearly $\hat{h}(x) = x$ and $\hat{h} \restriction (\mathbb{B} \restriction (\mu \setminus x))$ is the identity, so it is enough to check that $\hat{h} \restriction (\mathbb{B} \restriction x)$ is an automorphism of $\mathbb{B} \restriction x$. But $I \cap (\mathbb{B} \restriction x)$ is a maximal ideal of the Boolean Algebra $\mathbb{B} \restriction x$ (as $x \in Y$), hence it is enough to check that $\hat{h}$ maps $I \cap (\mathbb{B} \restriction x)$ into itself. As

$$b \in I \cap (\mathbb{B} \restriction x) \Rightarrow b = (b \setminus a_0 \setminus a_1) \cup (b \cap a_0 \cap a_1) \cup (b \cap a_0 \setminus a_1) \cup (b \cap a_1 \setminus a_0)$$

and all four [subsets] are in I, clearly it is enough to check the following statements:

$$b \in I \wedge b \subseteq x \setminus a_0 \setminus a_1 \Rightarrow h(b) \in I,$$

$$\ell < 2 \wedge b \in I \wedge b \subseteq x \cap a_\ell \setminus a_{1-\ell} \Rightarrow \hat{h}(b) \in I,$$

and lastly, $b \in I \wedge b \subseteq a_0 \cap a_1 \Rightarrow \hat{h}(b) \in I$.

The second implication holds by the choice of g, the first as $\hat{h}(b) = b$ in this case, and the last one as $h \restriction \big\{\{\alpha\} : \alpha \in a_0 \cap a_1\big\}$ is the identity (so again $\hat{h}(b) = b$).]

$\oplus_4$ If $b \subseteq \mu$ with $|b| \leq 2^{<\theta}$, then for some $\mathbb{B}$-autonomous set c we have $b \subseteq c \subseteq \mu$ and $|c| \leq 2^{<\theta}$.

[Why? Find c satisfying $b \subseteq c \subseteq \mu$ and $|c| \leq 2^{<\theta}$ such that

$$(\forall y \in [c]^{<\theta})\big[(\exists z \in I)[y \subseteq z] \Rightarrow (\exists z \subseteq c)[y \subseteq z \in I]\big].$$

(Just close θ times, recalling θ is regular.) Now if $y \in I$ then $|y| < \theta$ hence $y \cap c \in [c]^{<\theta}$, so there is z such that $y \cap c \subseteq z \in I \wedge z \subseteq c$; hence $y \cap c = y \cap z \in I$. This proves that c is $\mathbb{B}$-autonomous, as required.]

Now we return to the promised $\boxtimes_2$.

Proof. **Proof of $\boxtimes_2$:**

Toward contradiction, assume $x \in Y$ and $|x| > \beth_2(<\theta)$. Let

$$\langle \alpha_i : i < \beth_2(<\theta)^+ \rangle$$

be a sequence of elements of x without repetition. Let a_i be a $\mathbb{B}$-autonomous set of cardinality $\leq 2^{<\theta}$ such that $\{\alpha_{i+\varepsilon} : \varepsilon < 2^{<\theta}\} \subseteq a_i$ (this exists[1] by $\oplus_4$), and without loss of generality $a_i \subseteq x$. (Just use $a_i \cap x$; it is as required by $\oplus_1$.)

For some club C of $\beth_2(<\theta)^+$ we have

$$i < j \in C \Rightarrow a_i \cap \{\alpha_{j+\varepsilon} : \varepsilon < 2^{<\theta}\} = \varnothing,$$

hence $i < j \in C \Rightarrow |a_j \setminus a_i| \geq 2^{<\theta}$. Now $I \cap \mathcal{P}(a_i)$ has cardinality $\leq |a_i|^{<\theta} \leq 2^{<\theta}$ (as θ is regular) hence $\mathbb{B} \restriction a_i$ has cardinality $\leq 2^{<\theta}$. It follows that there are a stationary $S \subseteq \big\{\delta < \beth_2(<\theta)^+ : \mathrm{cf}(\delta) = (2^{<\theta})^+\big\}$ and a^* such that

$$i, j \in S \wedge i \neq j \Rightarrow a_i \cap a_j = a^*$$

(the Δ-system lemma). Also, as $a_j \subseteq X \in Y$ and $|a_j| = 2^{<\theta}$ and $|\mathbb{B}_i \restriction a_i| = 2^{<\theta}$, **[Are these supposed to be i-s or j-s? Also, $\mathbb{B}_i$ hasn't been defined anywhere.]**

the number of isomorphism types of $(\mathbb{B} \restriction a_i, \{\alpha\})_{\alpha \in a^*}$ is at most $\leq \beth_2(<\theta)$. Hence for some $i < j$ from $C \cap S$, we have $\mathbb{B} \restriction a_i \cong B \restriction a_j$ over $\mathbb{B} \restriction a^*$ but $|a_j \setminus a_i| \geq 2^{<\theta} \geq \theta$. Hence by $\oplus_3$ there is an automorphism h of $\mathbb{B}$ which moves $\geq 2^{<\theta}$ atoms, a contradiction. $\square_{\boxtimes_2}$

Next,

$\boxtimes_3$ $|Y/I| \leq \beth_3(<\theta)$.

[Why? If not, we can find $x_i \in Y$ for $i < \beth_3(<\theta)^+$ such that

$$i \neq j \Rightarrow x_i/I \neq x_j/I.$$

As $|x_i| \leq \beth_2(<\theta)$ by $\boxtimes_2$, by the Δ-system lemma, for some unbounded $A \subseteq \beth_3(<\theta)^+$ the set $\langle x_i : i \in A\rangle$ is a Δ-system, hence without loss of generality $\langle x_i : i \in A\rangle$ are pairwise disjoint (by substruction — not really needed, just clearer).

As $\mathbb{B} \restriction x_i$ is a Boolean Algebra of cardinality $\leq \beth_2(<\theta)$ (as $I \cap \mathcal{P}(x_i)$ is a maximal ideal of $\mathbb{B} \restriction x_i$ and $I \cap \mathcal{P}(x_i) \subseteq [x_i]^{<\theta}$, and $|x_i| \leq \beth_2(<\theta)$ by $\boxtimes_2$) there are at most $\beth_3(<\theta)$ isomorphism types of $\mathbb{B} \restriction x_i$. So for some $i \neq j$ in A we have $\mathbb{B} \restriction x_i \cong \mathbb{B} \restriction x_j$, so as in the proof of $\oplus_3$ there is an automorphism h of $\mathbb{B}$ mapping x_i to x_j, x_j to x_i, and $h \restriction (\mathbb{B} \restriction (1_{\mathbb{B}} - x_i - x_j))$ is the identity. Hence h moves $\geq |x_i \setminus x_j| \geq \theta$ atoms, because $x_i \neq x_j \mod I$.]

[1] We can also use $\{a_{i+\varepsilon} : \varepsilon < \theta\}$.

Choose a set $\{x_\alpha : \alpha < \alpha^* \leq \beth_3(<\theta)\}$ of representatives of Y/I, and let $x^* := \bigcup_{\alpha<\alpha^*} x_\alpha$ (so $x^* \subseteq \mu$ and $|x^*| \leq \beth_3(<\theta)$).

Define $J := \{a \in \mathbb{B} : a \cap x^* = \varnothing\}$.

$\boxtimes_4$ $J \subseteq I$.

[Why? If not, there is $x \in J \setminus I$ such that x/I is an atom of $\mathbb{B}/I$, so

$$x/I \in \{x_\alpha/I : \alpha < \alpha^*\}.$$

So $x/I = x_\alpha/I$ for some α, hence $|x \setminus x_\alpha| < \theta$ hence $|x \cap x_\alpha| \geq \theta$ hence $x \cap x^* \neq \varnothing$ hence $x \notin J$, a contradiction.]

Define an equivalence relation $\mathcal{E}$ on $\mathbb{B}$: $y_1 \mathrel{\mathcal{E}} y_2$ iff $y_1 \cap x^* = y_2 \cap x^*$.

Clearly $\mathcal{E}$ has $\leq 2^{|x^*|}$ equivalence classes and $2^{|x^*|} \leq \beth_4(<\theta)$. Also,

$$y_1 \mathrel{\mathcal{E}} y_2 \Rightarrow y_1 \setminus y_2 \in J;$$

in fact, $y_1 \mathrel{\mathcal{E}} y_2 \Leftrightarrow (y_1 \triangle y_2 \in J)$ (see the definition of J).

Choose a set of representatives $\{y_\gamma : \gamma < \gamma^*\}$ for $\mathcal{E}$ (so $|\gamma^*| \leq \beth_4(<\theta)$) and let $\mathbb{B}^*$ be the subalgebra of $\mathbb{B}$ generated by $\{y_\gamma : \gamma < \gamma^*\}$. So $|\mathbb{B}^*| \leq \beth_4(<\theta)$, and (being superatomic) the number of ultrafilters of $\mathbb{B}^*$ is also $\leq \beth_4(<\theta)$. Next, $\mathbb{B}$ is generated by $J \cup \mathbb{B}^*$ because for each $y \in \mathbb{B}$ there is γ such that $y \mathrel{\mathcal{E}} y_\gamma$, $y_\gamma \in \mathbb{B}^*$, $y - y_\gamma \in J$, and $y_\gamma - y \in J$. Hence $y \in \langle J \cup \mathbb{B}^* \rangle$. For D an ultrafilter of $\mathbb{B}^*$, let

$$Z_D := \big\{\alpha < \mu : (\forall y \in \mathbb{B}^*)[\alpha \in y \Leftrightarrow y \in D]\big\}.$$

Clearly,

$\boxtimes_5$ For every $\alpha \in \mu \setminus x^*$ there is a unique ultrafilter $D = D[\alpha]$ on $\mathbb{B}^*$ such that $\alpha \in Z_D$ (and the number of such ultrafilters is $\leq \beth_4(<\theta)$).

Now

$\boxtimes_6$ $\mu \leq \beth_4(<\theta)$.

[Why? Assume not. By $\oplus_4$, for each $i < \mu$ we can find a $\mathbb{B}$-autonomous a_i such that $|a_i| \leq 2^{<\theta}$ and $[i, i + 2^{<\theta}) \subseteq a_i$. Let $\{\beta_{i,\varepsilon} : \varepsilon < \varepsilon_i\}$ enumerate a_i in increasing order. Clearly for some unbounded $A \subseteq \beth_4(<\theta)^+$, for all $i \in A$, the following does not depend on i: ε_i and $D[\beta_{i,\varepsilon}]$ for $\varepsilon < \varepsilon_i$ (use $\boxtimes_5$),

[How is it that both ε_i *and* $\langle D[\beta_{i,\varepsilon}] : \varepsilon < \varepsilon_i \rangle$ don't depend on i? I could be mistaken, but this absolutely doesn't look right to me.]

and $\big\{u \in [\varepsilon_i]^{<\theta} : \{\beta_{i,\varepsilon} : \varepsilon \in u\} \in I\big\}$. And for $\zeta < 2^{<\theta}$, $\varepsilon = \varepsilon(i, \zeta)$ will denote the unique ε such that $\beta_{i,\varepsilon} = i + \zeta$, and without loss of generality $a_j \cap [i, i + 2^{<\theta}) = \varnothing$ for $j < i$ in A.

[Isn't this covered by '$a_i \cap a_j = \varnothing$' below?]

By the Δ-system lemma, without loss of generality for some a^* we have $a_i \cap a_j = a^*$ for $i < j$ in A. So by $\oplus_1$, the set a^* is $\mathbb{B}$-autonomous as well as $a_i \setminus a^*$, so we can use $a_i \setminus a^*$. So without loss of generality $a_i \cap a_j = \varnothing$ for $i \neq j$ in A, and as $|x^*| \leq \beth_4(<\theta)$ clearly without loss of generality $i \in A \Rightarrow a_i \cap x^* = \varnothing$.

[Isn't this the *method* by which you get '$a_i \cap a_j = \varnothing$?']

So for $i \neq j$ in A there is a permutation g of order two of μ interchanging a_i and a_j. That is, $g(\beta_{i,\varepsilon}) = \beta_{j,\varepsilon}$, $g(\beta_{j,\varepsilon}) = \beta_{i,\varepsilon}$, and $g(\{\beta\}) = \beta$ for $\beta \in (\mu \setminus a_i) \setminus a_j$. Clearly g can be extended to an automorphism $\hat{g}$ of $\mathcal{P}(\mu)$, and $\hat{g} \restriction \mathbb{B}^*$ is the identity. (The proof is like that proof of $\oplus_3$, using "$\mathbb{B}$ is generated by $J \cup \mathbb{B}^*$" and "$D[\beta_{i,\varepsilon}]$ does not depend on i.") So we get a contradiction.]

So as $|J| \leq \big|[\mu]^{<\theta}\big| = \mu^{<\theta} \leq \beth_4(<\theta)^{<\theta} = \beth_4(<\theta)$ and $|\mathbb{B}^*| \leq |\mathbb{B}/\mathcal{E}| \leq \beth_4(<\theta)$ and $\mathbb{B}$ is generated by $J \cup \mathbb{B}^*$, together we get the desired conclusion.

This completes the proof of 1.1. $\square_{1.1}$

Discussion 1.3. 0) We can strengthen the conclusion of 1.1 to:

- $\oplus$ One of the following occurs (where I is as in the proof):
  - (a) There is $a \in \mathbb{B} \setminus I$ such that $a \cap a = 0$.
    [Is that even possible in a Boolean Algebra? Doesn't it just imply $a = 0_{\mathbb{B}}$?]
  - (b) There is an ideal $J \subseteq I$ containing $2^{<\theta}$ pairwise disjoint elements such that $a, b \in J \Rightarrow a \cap f(b) = 0_{\mathbb{B}}$.

[What's f? I don't see it defined anywhere. Do you mean 'for all/for some automorphism of $\mathbb{B}$?']

1) We can weaken the assumption "$\mathbb{B}$ is superatomic" to "$\mathbb{B}/I^1_{<\theta}[\mathbb{B}]$ is superatomic," where:

$(*)_1$ For a Boolean Algebra $\mathbb{B}$ and infinite cardinal θ, we define

$$I^1_{<\theta}[\mathbb{B}] := \{x \in \mathbb{B} : \mathbb{B} \restriction x \text{ has (algebraic) density } < \theta\}$$

(see [She92, §1] for a little bit more about this). For $\mathbb{B}$ superatomic, this is the I in the proof of 1.1 on such Boolean Algebras.

[We can choose a maximal set Z of pairwise disjoint elements of

$$\big\{x \in \mathbb{B} \setminus \{0_{\mathbb{B}}\} : \pi(\mathbb{B} \restriction x) < \theta\big\}.$$

Now without loss of generality $\mathbb{B}$ is a Boolean subalgebra of $\mathcal{P}(\mu)$ such that $x \in Z \Rightarrow x \in [\mu]^{<\theta}$, and continue as in the proof of 1.1.]

2) What if we just assume "$\mathbb{B}/I_{<\theta}[\mathbb{B}]$ is atomic?" One point in the proof may fail: the number of ultrafilters of $\mathbb{B}^*$ is not necessarily $\leq |\mathbb{B}^*| \leq \beth_4(<\theta)$ but is $\leq 2^{|\mathbb{B}^*|} \leq 2^{2^{|Y|}} \leq \beth_5(<\theta)$, so we should replace $\beth_4(<\theta)$ by $\beth_5(<\theta)$ in the conclusion in parts (1),(2).

[This notation hasn't been defined, and conflicts with earlier usage. Up to now π has denoted an arbitrary automorphism of $\mathbb{B}$.]

3) In parts (1) and (2) we may replace "$\pi(\mathbb{B} \restriction x)$, the *algebraic density*, is $< \theta$" by "$d(\mathbb{B} \restriction x)$; i.e. $\mathbb{B} \restriction x$ has topological density $< \theta$" (recalling that any Boolean Algebra $\mathbb{B}'$ can be embedded into a Boolean subalgebra of $\mathcal{P}(d(\mathbb{B}'))$. However, the bound is seemingly bigger.

So we use $I^2_{<\theta}[\mathbb{B}] := \{x \in B : d(\mathbb{B} \restriction x) < \theta\}$. Note $I^1_{<\theta}[\mathbb{B}] \subseteq I^2_{<\theta}[\mathbb{B}]$.

4) In both parts (1)-(3) we have to make easy changes to adapt the proof of 1.1. Let $k = 1$ and $\mu_1 = 2^{<\theta}$ for parts (1),(2), and $\mu_1 =$ [?], $\beth_3(<\theta)$, $k = 2$ for part (3). We try to indicate some changes and we redefine I as $I^k_{<\theta}[\mathbb{B}]$.

$\boxtimes'_1$ Without loss of generality $\mathbb{B} \subseteq \mathcal{P}(\mu)$ and $a \in I = I^k_{<\theta}[\mathbb{B}] \Leftrightarrow |a| < \theta$.

[Why? Let Z be a maximal set of pairwise $\mathbb{B}$-disjoint members of $I^k_{<\theta}[\mathbb{B}] \setminus \{0_{\mathbb{B}}\}$. For each $z \in I^k_{<\theta}[\mathbb{B}]$, let $\mathcal{D}_z$ be a dense subset of ultrafilters $(z, \mathbb{B}) = \{D : D$ an ultrafilter of $\mathbb{B}$ be such that $z \in D\}$ of cardinality $< \theta$. Let $\mu = \bigcup_{z \in Z} \mathcal{D}_z$ and let $\overline{D} = \langle D_\alpha : \alpha < \mu\rangle$ list this set. There is a natural mapping $h = h_{\overline{D}}$ from $\mathbb{B}$ to $\mathcal{P}(\mu)$ defined by $h(a) = \{\alpha < \mu : a \in D_\alpha\}$.]

Easily,

$(*)_1$ h embeds $\mathbb{B}$ into $\mathcal{P}(\mu)$.

[Why? h is trivially a homomorphism. If $c \in \mathbb{B} \setminus \{0\}$, then for some $a \in Z$ we have $a \cap c > 0_{\mathbb{B}}$ hence for some $\alpha < D_\alpha \in \mathcal{D}_a$ we have $a \in D$. Let $D = D_\alpha$ for $\alpha < \mu$ so $\alpha \in a$. So the kernel of α is $\{0_{\mathbb{B}}\}$, so we are done.]

[Very little in the last few paragraphs made sense to me. D is an ultrafilter and $\mathcal{D}_z$ is a collection of ultrafilters, but so is $(z, \mathbb{B})$?]

$(*)_2$ h maps $I^k_{<\theta}[\mathbb{B}]$ into $[\mu]^{<\theta}$.

[Why? Let $b \in I^k_{<\theta}[\mathbb{B}]$, and define $Z_b := \{a \in Z : b \cap a > 0\}$; it is a subset of Z. Now, for each $D \in \mathcal{D}_b$ we have $|Z_b \cap D| \leq 1$; in fact, $|Z \cap D| \leq 1$. So if $|Z_b| \geq \theta$ then $|Z_b| > |\mathcal{D}_b|$, so for some $c \in Z_b$ we have $(\forall D \in \mathcal{D}_b)[c \notin D]$, but this contradicts the choice of Z_b. So $|Z_b| < \theta$, so $\bigcup\{D : b \in D$ and $D \in \bigcup\limits_{c \in Z_b} \mathcal{D}_c\}$ has cardinality $\sum\limits_{c \in Z_b} |\mathcal{D}_c| < \theta$ and is a subset of $h(b)$.]

$(*)_3$ h maps $I^k_{<\theta}[\mathbb{B}]$ onto $[\mu]^{<\theta} \cap \mathrm{rang}(B)$ [(?)].

[Why? **[Doubtful.]**]

So without loss of generality,

$\oplus$ h is the identity.

So the rest is easier.

Now

$\oplus$ if we assume $\mathbb{B}/I^k_{<\theta}[\mathbb{B}]$ is superatomic **[. . . then what?]**
Otherwise we have just

$\oplus$ $\mathbb{B} \subseteq \mathcal{P}(\mu)$, I is an ideal of $\mathbb{B} \subseteq [\mu]^{<\theta}$, and $\mathbb{B}/I$ is atomic.

So the assumption toward contradiction is

$\oplus$ $|\mathbb{B}| > \beth_5(<\theta)$ and $\neg(a), \neg(b)$ where

(a) There is an automorphism f of $\mathbb{B}$ such that for some $c \in \mathbb{B} \backslash I$, $f(c0 \cap c = 0_{\mathbb{B}}$.

(b) There is a permutation π of μ inducing an automorphism of $\mathbb{B}$ such that for some $X \subseteq \mu$ of cardinality $\leq 2^\theta$, the union of $I \cap \mathcal{P}(x)$ such that $\pi(X) \cap X = \varnothing$.

We add:

$\otimes_0'$ If $b \in I$ then $|\mathbb{B} \restriction b| \leq 2$ for some $\sigma < \theta$
[Nothing here depends on σ.]

$\otimes_0''$ If $x \in [\mu] \leq 2^{<\theta}$ then $|X| \leq \theta_k$ (so for $k = 2$ let θ_k be the bound **[End of Line]**

$\otimes_1'$ We say that X is $\mathbb{B}$-*autonomous* when X is a sub-Boolean ring of I and $(\forall a \in I)(\exists b \in X)\big[b \leq_{\mathbb{B}} a \wedge (\forall c \in X)[a \cap c \leq b]\big]$.

$\otimes_3'$ if $X_1, X_2 \subseteq I$ are $\mathbb{B}$-autonomous, $x \in Y$, $X_1 \cup X_2 \subseteq \mathbb{B} \restriction x$, and X_1, X_2 are isomorphic over $X_1 \cap X_2$, then there is an automorphism of $\mathbb{B}$ over $X_1 \cap X_2$ mapping X_1 onto X_2.

$\otimes_4'$ if $X \subseteq I$, $|X| \leq \mu_1$ then there is a $\mathbb{B}$-autonomous $X' \subseteq I$ of cardinality $\leq \mu_1$ such that $X \subseteq X'$.

[Why? If $k = 1$ we can find X' of cardinality $\leq 2^{<\theta}$, if there is $b' \in I$ above ever member of $\mathcal{U}$, then there is such $b' \in X'$; now check as there. **[FILL.]**]

Theorem 1.4. *The pair $\mathbb{B}, I)$ satisfies $\odot$ if the Boolean Algebra $\mathbb{B}$ and ideal I satisfies: if $\boxtimes$ below holds when:*
[Unreadable. I note there are two $\odot$s and zero $\boxtimes$s below.]

$\odot$ (a) *$\mathbb{B}$ has cardinality $\leq \beth_5(<\theta), |I| \leq \beth_4(<\theta)$*

(a)* *if in ? is strengthened to $\mathbb{B} \restriction b$ has algebraic density $< \theta$ then $|\mathbb{B}| \leq \beth_4(<\theta), |I| \leq ?$*

(b) *add on s a $(\lhd\theta)$, . . . see end of §3*

$\odot$ (a) *$\mathbb{B}$ is a Boolean algebra.*

(b) *I is an ideal of $\mathbb{B}$.*

(c) *if $b \in I \setminus \{0_{\mathbb{B}}\}$ then $d(\mathbb{B} \restriction b)$ (the topological density) is $< \theta$.*

(d) *$\mathbb{B}/I$ is an atomic Boolean algebra.*

(e) *For all $b \in \mathbb{B} \setminus I$ and all automorphisms π of $\mathbb{B}$, we have $\pi(b) \cap b \neq 0_{\mathbb{B}}$.*

(f) *For no ideal $J \subseteq I$ of cardinality $2^{<\theta}$ with $2^{<\theta}$ pairwise disjoint non-zero members does $\mathbb{B}$ has an automorphism π such that*

$$b, c \in J \Rightarrow b \cap \pi(c) = 0_{\mathbb{B}}.$$

Discussion 1.5. 1) We can adapt 2.1 from §2 below to the case of 1.3(2); i.e. show that $\beth_5(<\theta)$ cannot be improved in general. E.g. let $\langle d_\zeta : \zeta < \lambda = 2^\mu \rangle$ be an independent family of subsets of μ (so any finite Boolean combination of them is infinite) and let $\mathbb{B}^*$ be the Boolean subalgebra of $\mathcal{P}(\mu)$ generated by

$$\{d_\alpha : \alpha < \lambda = 2^\mu\} \cup \big\{\{i\} : i < \mu\big\}.$$

We let $\{c^*_\gamma : \gamma < 2^\lambda\}$ be an independent family of subsets of λ, and let $X^* := \bigcup_{\alpha<\mu} X_\alpha \cup \{x^*_\gamma : \gamma < 2^\lambda\}$. We ignore $\mathcal{A}'$ (and omit clause (k) of the assumption) and among the generators of $\mathbb{B}$, clause (i), (ii) remains and

(iii)′ $c_\zeta = \{x \in X : \text{for some } \alpha \in d_\zeta \text{ we have } x \in X_\alpha\} \cup \big\{x^*_\gamma : \zeta \in c^*_\gamma, \gamma \in [\mu, 2^\lambda)\big\}$.

[Isn't that first part just a long way of saying ' $\bigcup_{\alpha \in d_\zeta} X_\alpha$?']

2) We may consider replacing automorphism by monomorphisms. The problem is only in the proof of 2.1, "f maps J_1 into J_1" does not seem to follow.

§ 2. Constructing counterexamples

We would like to show that the bound $\beth_4(<\theta)$ from 1.1 is essentially the best possible. The construction (in 2.1) is closely related to the proof in §1, but we need various assumptions. So in particular, κ here corresponds to

$$\sup\{|\mathbb{B} \restriction a| : a \in Y\} \leq \beth_2(<\theta)$$

there, μ here corresponds to $|Y| \leq \beth_3(<\theta)$ there, and λ' here corresponds to $|\mathrm{atom}(\mathbb{B})| \leq \beth_4(<\theta)$ there. We shall deal with them later.

Lemma 2.1. *Assume*

(a) $\aleph_1 \leq \theta = \mathrm{cf}(\theta) \leq \kappa \leq \mu \leq \lambda' \leq \lambda$

(b) *There is an* $\mathcal{A} \subseteq [\mu]^{\aleph_0}$ *of cardinality* μ*, almost disjoint (i.e.* $A \neq B \in \mathcal{A} \Rightarrow |A \cap B| < \aleph_0$*), such that* $(\forall A \in [\mu]^{\theta})(\exists B \in \mathcal{A})[B \subseteq^* A]$.

(c) $\overline{\mathbb{B}} = \langle \mathbb{B}_\alpha : \alpha < \mu \rangle$

(d) $\mathbb{B}_\alpha$ *is a superatomic Boolean Algebra with* $\leq \kappa$ *atoms, such that* $|\mathbb{B}_\alpha| \leq \lambda$ *and any automorphism of* $\mathbb{B}_\alpha$ *moves* $< \theta$ *atoms; moreover, if* $c_1, c_2 \in I_\alpha$ *(see below) and* f *is an isomorphism from* $\mathbb{B}_\alpha \restriction (1 - c_1)$ *onto* $\mathbb{B}_\alpha \restriction (1 - c_2)$*, then*

$$\big|\{x \in \mathrm{atom}(\mathbb{B}_\alpha) : x \leq_{\mathbb{B}_\alpha} c_1 \text{ or } f(x) \neq x\}\big| < \theta.$$

(e) $I_\alpha := \big\{b \in \mathbb{B}_\alpha : \big|\{x \in \mathrm{atom}(\mathbb{B}_\alpha) : x \leq b\}\big| < \theta\big\}$ *is a maximal ideal of* $\mathbb{B}_\alpha$.

(f) *There is an infinite set* $\{a^\alpha_n : n < \omega\}$ *of distinct atoms of* $\mathbb{B}_\alpha$ *such that for every* $a \in I_\alpha$*, the set* $\{n < \omega : a^\alpha_n \leq a\}$ *is finite.*

(g) *If* $\alpha \neq \beta$ *<u>then</u> for no* $a_\alpha \in I_\alpha$ *and* $a_\beta \in I_\beta$ *do we have*

$$\mathbb{B}_\alpha \restriction (1_{\mathbb{B}_\alpha} - a_\alpha) \cong \mathbb{B}_\beta \restriction (1_{\mathbb{B}_\beta} - a_\beta).$$

(h) $\mathbb{B}^*$ *is a superatomic Boolean Algebra.*

(i) $\mathbb{B}^*$ *has* μ *atoms.*

(j) $\mathbb{B}^*$ *has* λ *elements.*[2]

(k) *If* $\lambda' > \mu$ *<u>then</u> we have* $\chi, \mathcal{A}', I^*$ *satisfying:*

  (α) $\mathcal{A}' \subseteq [\lambda']^{\aleph_0}$ *is a MAD family of cardinality* χ.

  (β) I^* *is an ideal of* $\mathbb{B}^*$ *containing* $\mathrm{id}_1(\mathbb{B}^*)$*, included in* $\mathrm{id}_{\mathrm{rk}(\mathbb{B})}(\mathbb{B}^*)$*, such that the Boolean algebra* $\mathbb{B}^*/I^*$ *is isomorphic to*

$$\big\{a \subseteq \chi : |a| < \aleph_0 \vee |\chi \setminus a| < \aleph_0\big\}$$

  (so $\chi \leq |\mathbb{B}^*| = \lambda$ *follows).*

  (γ) *If* π *is a partial*[3] *permutation of* λ'*,* $Z_1 := \lambda' \setminus \mathrm{dom}(\pi)$*,* $Z_2 := \lambda' \setminus \mathrm{rang}(\pi)$*, and* $Z := Z_1 \cup Z_2 \in [\lambda']^{<\theta}$ *satisfies*

$$A \in \mathcal{A}' \Rightarrow \big|\big(A \mathbin{\triangle} \pi''(A)\big) \setminus Z\big| < \aleph_0,$$

  <u>then</u> the support of π *has cardinality* $< \theta$ *(where the support of a permutation is* $\{\alpha < \lambda' : \pi(\alpha) \neq \alpha\}$*).* ‘

<u>Then</u> we can find $\mathbf{B}$ *such that:*

(α) $\mathbf{B}$ *is a superatomic Boolean Algebra.*

(β) $\mathbf{B}$ *has* λ' *atoms and* λ *elements.*

(γ) *every automorphism* g *of* $\mathbf{B}$ *moves* $< \theta$ *atoms; i.e.*

$$|\{x \in \mathrm{atom}(\mathbf{B}) : g(x) \neq x\}| < \theta.$$

---

[2] If there is a tree $\mathcal{T}$ with $\leq \mu$ nodes and $\geq \lambda$ branches (= maximal linearly ordered subsets) then such $\mathbb{B}^*$ exists.

[3] I.e. π is one-to-one such that $\mathrm{dom}(\pi) \subseteq \lambda'$ and $\mathrm{rang}(\pi) \subseteq \lambda'$.

Proof. Without loss of generality $\mathbb{B}^*$ is a Boolean Algebra of subsets of

$$\{w_1, \alpha : \alpha < \mu\} \text{ with } \{\omega_1\alpha\} : \alpha < \mu\}$$

being the atoms of $\mathbb{B}^*$.

[I have no idea what's going on there. There are more right braces than left.]

If $\lambda' = \mu$ let $\mathcal{A}' := \varnothing$, $\chi = 0$, and $I^* = \mathbb{B}^*$.

Without loss of generality, $\mathbb{B}_\alpha$ is a subalgebra of $\mathcal{P}(X_\alpha)$ and the set of atoms of $\mathbb{B}_\alpha$ is $\big\{\{x\} : x \in X_\alpha\big\}$. Also without loss of generality, $\alpha \neq \beta \Rightarrow X_\alpha \cap X_\beta = \varnothing$, and we define $X := \bigcup_{\alpha<\mu} X_\alpha$.

If $\lambda' = \mu$ let $Y^* := \varnothing$; if $\lambda' > \mu$, let $Y^* \subseteq \mathbb{B}^*$ be such that $|Y^*| = \chi$ and $\{y/I^* : y \in Y^*\}$ is the set of atoms of $\mathbb{B}^*/I^*$ with no repetitions. Without loss of generality:

$\boxtimes_0$ For every $y \in Y^*$, for some α, $y/\mathrm{id}_\alpha(\mathbb{B}^*)$ is an atom of $\mathbb{B}^*/\mathrm{id}_\alpha(\mathbb{B}^*)$ and

$$(\forall z \leq_{\mathbb{B}^*} y)[z \in \mathrm{id}_\alpha(\mathbb{B}^*) \Leftrightarrow z \in I^*].$$

[Why is this possible? For each $y \in \mathbb{B}^* \setminus I^*$, let

$$\alpha = \alpha(y) := \min\{\mathrm{rk}_{B^*}(y - x) : x \in I^*\}$$

and choose x^0_y exemplifying it, so $(y - x^0_y)/\mathrm{id}_\alpha(\mathbb{B}^*)$ is the union of finitely many atoms of $\mathbb{B}^*/\mathrm{id}_\alpha(\mathbb{B}^*)$ — say, $y_1/\mathrm{id}_\alpha(\mathbb{B}^*), \ldots, y_n/\mathrm{id}_\alpha(\mathbb{B}^*)$, where $n \geq 1$ and (without loss of generality) $y_\ell \leq_{\mathbb{B}^*} y$. So $\{y_1, \ldots, y_n\}$ cannot be all in I^* and there cannot be two $y_\ell \in \mathbb{B}^* \setminus I^*$, so there is a unique $\ell = \ell_*$ such that $y_\ell \notin I^*$. Let $x^*_y := (1 - y_{\ell_*}) \cup x^0_y$; now $\{y - x^*_y : y \in Y^*\}$ is as required.]

Let Y^+ be such that $Y^+ \subseteq \mathbb{B}^*$, $\langle y/\mathrm{id}_{\mathrm{rk}(y,\mathbb{B}^*)}(\mathbb{B}^*) : y \in Y^+\rangle$ list

$$\big\{y/\mathrm{id}_{\mathrm{rk}(y,\mathbb{B}^*)} : y/\mathrm{id}_{\mathrm{rk}(y,\mathbb{B}^*)}(\mathbb{B}^*) \text{ is an atom of } \mathbb{B}^*/\mathrm{id}_{\mathrm{rk}(y,\mathbb{B}^*)}(\mathbb{B}^*)\big\}$$

[That doesn't look right. I see $y/\mathrm{id}_{blah}(\mathbb{B}^*)$ everywhere else, but never y/id_{blah}. If that's a typo, then why not write $\mathrm{atom}(\mathbb{B}^*/\mathrm{id}_{\mathrm{rk}(y,\mathbb{B}^*)}(\mathbb{B}^*))$?]
with no repetitions, and $Y := \{y \in Y^+ : \mathrm{rk}(\mathbb{B}^*) > \mathrm{rk}(y, \mathbb{B}^*) > 0\}$.

Without loss of generality

$$Y^{\max} := \big\{y \in Y^+ : \mathrm{rk}(y, \mathbb{B}) = \mathrm{rk}(\mathbb{B})\big\}$$

is a partition of $1_{\mathbb{B}}$. For $y \in Y^+$, let D_y be the ultrafilter on $\mathbb{B}^*$ generated by

$$\{y\} \cup \big\{1 - x : x \in \mathbb{B}^*, \mathrm{rk}(x, \mathbb{B}^*) < \mathrm{rk}(y, \mathbb{B}^*)\big\}$$

for each $y \in Y$.

[Which is it? $y \in Y^+$, or $y \in Y$?]

Without loss of generality, $Y^* \subseteq Y$ and we have $(\forall y \in Y^+)[y < y^{\max}]$ for some $y^{\max} \in Y^{\max}$. Also, as $\mathbb{B}^*/I^*$ is isomorphic to the Boolean Algebra of finite and co-finite subsets of χ, $y \in Y \Rightarrow \mathrm{rk}(y, \mathbb{B}) < \mathrm{rk}(\mathbb{B})$, and clause (k)($\beta$) of the assumption of 2.1, clearly

$$y \in Y \setminus Y^* \Rightarrow \big\{z \in Y^* : z - y \in \mathrm{id}_{\mathrm{rk}(y',\mathbb{B}^*)}(\mathbb{B}^*)\big\} \text{ is finite.}$$

So without loss of generality, those sets are empty for $y \in Y \setminus Y^*$ (and are singletons for $y \in Y^*$, of course). Note that if $\lambda' > \mu$ then Y^* is of cardinality $|\mathcal{A}'|$, and without loss of generality $|Y \setminus Y^*| = \lambda$.

Let $g : \mu \to X$ be one-to-one and onto, and for $A \in \mathcal{A}$ (from clause (b)), let $\{\gamma_{A,k} : k < \omega\}$ list A without repetition. Let $g^* : \mu \to \mu$ map an ordinal γ to the unique $\alpha < \mu$ such that $g(\gamma) \in X_\alpha$.

For $\beta < \mu$, let $\mathbf{i}(\beta)$ be the unique $i < \omega_1$ such that $(\exists\alpha)[\omega_1\alpha \leq \beta = \omega_1\alpha + i]$. For $A \in \mathcal{A}$ we define $\mathbf{i}(A) := \min\{i < \omega_1 : \mathbf{i}(g^*(\gamma)) < i \text{ for every } \gamma \in u_A\}$.

$\boxtimes_1$ We have $\langle \alpha_A : A \in \mathcal{A},\ u_A \text{ well-defined}\rangle$ such that:

[Hasn't been defined yet. Reading ahead, u_A should be defined whenever $\mathbf{i}(A) < \omega_1$.]

(i) $\alpha_A < \mu$

(ii) $\alpha_A \in \{w - 1\beta + \mathbf{i} : \beta < \mu \text{ and } \mathbf{i}(A) \leq i < \omega_1\}$.

[No idea what that is.]

(iii) $\alpha_{A_1} = \alpha_{A_2} \Rightarrow A_1 = A_2$

(iv) $\{\alpha_A : A \in \mathcal{A},\ \mathrm{rang}(g^* \restriction A) \text{ infinite}\} = \{\omega_1\alpha + i : \alpha < \mu,\ i < \omega_1\}$.

Now, by induction on $i < \omega_1$, we choose y_α when $\mathbf{i}(\alpha) < i$ and u_A, y_A when $\mathbf{i}(A) \leq i$ such that:

$\boxtimes_{1.1}$ (a) $y_\alpha \in Y$ or z_α is an atom of $\mathbb{B}^*$.

(b) If $\mathrm{rang}(g^* \restriction A)$ is finite then $u_A := \varnothing$.

(c) If $\mathrm{rang}(g^* \restriction A)$ is infinite <u>then</u>

(α) u_A is an infinite subset of A.

(β) $g^* \restriction u_A$ is one-to-one.

(γ) $z_A \in Y$

[What are z_α and z_A supposed to be?]

(δ) $\langle D_{g_\beta} : \beta \in u_A\rangle$ converges to $D_{g_{\alpha[A]}}$. By this we mean that for every $x \in \mathbb{B}$, for all but finitely many $\beta \in u_A$, we have

$$x \in D_{z_{\alpha[A]}} \Leftrightarrow x \in D_{y_\beta}.$$

This is easy by Observation 0.2.

For $\alpha < \mu$, let a_α be $\{g(\gamma) : \gamma \in u_A\}$ if $\alpha = \alpha_A$ for some $A \in \mathcal{A}$, and $\varnothing$ if there is no such A. Note that if $u_A = \varnothing$ (i.e. $\mathrm{rang}(g \restriction A)$ is finite) then $a_\alpha = \varnothing$.

Toward defining our Boolean Algebra, let $\{x^*_\gamma : \gamma \in [\mu, \lambda')\}$ be pairwise distinct elements not in X. Let

$$\mathcal{A}'' := \{\{\mu + i : i \in A\} : A \in \mathcal{A}'\};$$

it is a maximal almost disjoint family of countable subsets of $[\mu, \lambda')$, as in clause (k) of the assumption. So if $\mu = \lambda'$ then $\mathcal{A}'' = \mathcal{A}' = \varnothing$ and $|Y^*| = (\lambda' - \mu)^{\aleph_0} = 0$, and if $\lambda' > \mu$ then $|\mathcal{A}''| = |\mathcal{A}'| = |Y^*| = \chi$. Let $\langle d_y : y \in Y^*\rangle$ list $\mathcal{A}''$ with no repetitions.

Now we define our Boolean Algebra $\mathbf{B}$. It is the Boolean Algebra of subsets of $X^* := X \cup \{x^*_\gamma : \gamma \in [\mu, \lambda')\}$ generated by the following sets:[4]

$\boxtimes_2$ (i) The sets $\{a \in \mathbb{B}_\alpha : |a| < \theta\} \cup \{a \cup a_\alpha : a \in \mathbb{B}_\alpha, |a| \geq \theta\}$ for $\alpha < \mu$.

(ii) $\{x^*_\gamma\}$ for $\gamma \in [\mu, \lambda')$.

(iii) The sets c_y (for $y \in Y$), where

$$c_y := \{x \in X : (\exists \alpha < \mu)[x \in X_\alpha \wedge y \in D_{z_\alpha}]\} \cup \{x^*_\gamma : \gamma \in [\mu, \lambda') \cap \bigcup_{y \in Y^*} d_y\}.$$

[Aren't these guys already included by clause (ii)? Also, checking the previous paragraph, I see that "$\gamma \in d_y$ for some $y \in Y^*$" is just a circuitous way of saying $\gamma \in \bigcup \mathcal{A}''$.]

Clearly,

$\otimes_{2.0}$ $\mathbf{B}$ is a subalgebra of $\mathcal{P}(X^*)$ which includes all the singletons (and hence is atomic). It has λ' atoms and λ elements.

[4] Recall that a_α may be empty, and that $X := \bigcup_{\alpha < \mu} X_\alpha$.

[Why? The least trivial is $x \in X \Rightarrow \{x\} \in \mathbf{B}$, but if $x \in X_\alpha$ <u>then</u> $\{x\}$ is an atom of $\mathbb{B}_\alpha$, hence it belongs to $\mathbf{B}$.]

Note that

$\otimes_{2.1}$ (i) For $\alpha, \beta < \mu$, $X_\alpha \cap a_\beta$ has at most one element.

(ii) $X_\alpha \cap X_\beta$ is [____] (except when $\alpha = \beta$).

(iii) $a_\alpha \cap a_\beta$ is finite (when $\alpha \neq \beta$), as $\mathcal{A}$ is MAD.

(iv) $(X_\alpha \cup a_\alpha) \cap (X_\beta \cup a_\beta)$ is finite for $\alpha \neq \beta < \mu$. (This follows from clauses (i)-(iii).)

(v) If $\alpha < \mu$ and $y \in Y$, <u>then</u> either the set $(X_\alpha \cup a_\alpha) \setminus c_y$ is finite or $(X_\alpha \cup a_\alpha) \cap c_y$ is finite.

[Why? Recalling $\mathbb{B}^*$ is a subalgebra of $\mathcal{P}(\mu)$ and the definition of c_y, clearly

$$c_y \cap X_\alpha \in \{X_\alpha, \varnothing\}.$$

Also, $X_\alpha \subseteq c_y$, so if $a_\alpha = \varnothing$ we are done. Assume $\alpha = \alpha_A$ (so u_A is infinite) and it suffices to prove that for all but finitely many $\beta \in a_\alpha$, we have

$$\beta \in c_y \Leftrightarrow X_\alpha \subseteq c_y.$$

But $a_\alpha := \{g(\gamma) : \gamma \in u_A\}$, so this means "for all but finitely many $\gamma \in u_A$ we have $g(\gamma) \in c_y \Leftrightarrow X_\alpha \subseteq c_y$." But the definition of c_y and g^* this means: for all but finitely many $\gamma \in u_A$ we have $g^*(\gamma) \in y \Leftrightarrow y \in D_{z_{g^*(\gamma)}} \Leftrightarrow y \in D_{z_\alpha}$. But $z_\alpha = z_A$ and $\langle D_{z_\gamma} : \gamma \in u_A \rangle$ converges to $D_{z_{\alpha[A]}}$, so we are done.]

[Again, no idea what the z-s were intended to be.]

$\otimes_{2.2}$ For $\alpha < \mu$, we have

$$a \in \mathbb{B}_\alpha \wedge |a| < \theta \Rightarrow a \in \mathbf{B} \wedge \mathbf{B} \restriction a = \mathbb{B}_\alpha \restriction a.$$

But $\mathbb{B}_\alpha \restriction a$ is superatomic for all $a \in \mathbb{B}_\alpha$, so $\{a \in \mathbb{B}_\alpha : |a| < \theta\} \subseteq \mathrm{id}_\infty(\mathbf{B})$.

[Why? For the first implication we should check that for every one of the generators of $\mathbf{B}$ listed in $\boxtimes_2(i)$-(iii) above, its intersection with a belongs to $\mathbb{B}_\alpha \restriction a$. For $\boxtimes_2(ii)$ this is trivial, for $\boxtimes_2(i)$ use $\otimes_{2.1}(i)$-(iv), and for $\boxtimes_2(iii)$ use $\otimes_{2.1}(v)$. The rest follows.]

$\otimes_{2.3}$ For $\alpha < \mu$, let

$$I^+_\alpha := \{a \in \mathbf{B} : a \subseteq X_\alpha \cup a_\alpha \text{ and } |a| < \theta\}.$$

<u>Then</u>

(i) $I^+_\alpha = \{a \cup b : a \in \mathbb{B}_\alpha,\ |a| < \theta, \text{ and } b \subseteq a_\alpha \text{ is finite}\}$

(ii) I^+_α is a maximal ideal of $\mathbf{B} \restriction (X_\alpha \cup a_\alpha)$.

[Why? Easy. The main point concerns $(X_\alpha \cup a_\alpha) \cap (X_\beta \cup a_\beta)$ satisfying clause (i)

[Clause (i) of what?]

when it has cardinality $< \theta$; this holds by $\otimes_{2.1}(iv)$. **[The second point is]** $(X_\alpha \cup a_\alpha) \cap c_y$ has cardinality $< \theta$ or $(X_\alpha \cup a_\alpha) \setminus c_y$ has cardinality $< \theta$, which holds by $\otimes_{2.1}(v)$.]

$\otimes_{2.4}$ $\alpha < \mu \Rightarrow X_\alpha \cup a_\alpha \in \mathrm{id}_\infty(\mathbf{B})$

[Why? First, $X_\alpha \cup a_\alpha \in \mathbf{B}$ by clause (i) of $\boxtimes_2$ above; second, if $X_\alpha \cup a_\alpha \notin \mathrm{id}_\infty(\mathbf{B})$ then by $\otimes_{2.2}$

$$(\exists \zeta)\big[a \in \mathbb{B}_\alpha \wedge |a| < \theta \Rightarrow a \in \mathrm{id}_\zeta(\mathbf{B})\big].$$

Hence by $\otimes_{2.3}$ above, $(X_\alpha \cup a_\alpha)$ is an atom of $\mathbf{B}/\mathrm{id}_\zeta(\mathbf{B})$ for ζ large enough, hence $X_\alpha \cup a_\alpha$ belongs to $\mathrm{id}_{\zeta+1}(\mathbf{B})$: a contradiction.]

$\otimes_{2.5}$ $\mathbf{B} \restriction (X_\alpha \cup a_\alpha) \cong \mathbb{B}_\alpha$ for $\alpha < \mu$. Hence, if $\alpha < \beta < \mu$ then for no $c_\alpha \in \mathbf{B}_\alpha$ such that $c_\alpha \leq_{\mathbf{B}} X_\alpha \cup a_\alpha$ and $|c_\alpha| < \theta$ and $c_\beta \in \mathbf{B}$ with $c_\beta \leq_{\mathbf{B}} X_\beta \cup a_\beta$, $|c_\beta| < \theta$ do we have

$$\mathbf{B} \restriction (X_\alpha \cup a_\alpha \setminus c_\alpha) \cong \mathbf{B} \restriction (X_\beta \cup a_\beta \setminus c_\beta).$$

[Why? By clauses (f)+(e) of the assumption, the first phrase holds. The "hence" follows by clause (g) of the assumption.]

Let J_1 be the ideal of $\mathbf{B}$ generated by

$$\bigcup_{\alpha<\mu} I^+_\alpha \cup \{x^*_\gamma : \gamma \in [\mu, \lambda')\}.$$

(We will see that J_1 is I from the proof of 1.1, positive part, i.e. $J_1 = [\lambda']^{<\theta} \cap B$).

Let J_2 be the ideal of $\mathbf{B}$ generated by $J_1 \cup \{X_\alpha \cup a_\alpha : \alpha < \mu\}$. Let J^+_ℓ be the ideal of the Boolean Algebra $\mathcal{P}(\mu)$ generated by J_ℓ.

$\otimes_{2.6}$ (*i*) $J_1 \subseteq \mathrm{id}_\infty(\mathbf{B})$

(*ii*) $J_1 \subseteq [X^*]^{<\theta}$ is a (proper) ideal.

(*iii*) $J_1 \subseteq J_2 \subseteq \mathrm{id}_\infty(\mathbb{B})$ and J_2/J_1 is the ideal of $\mathbf{B}/J_1$ generated by its atoms; i.e. $\mathrm{id}_1(\mathbf{B}/J_1)$, where the atoms are $(X_\alpha \cup a_\alpha)/J_1$.

[Why? For clause (i), note that $\mathrm{id}_\infty(\mathbf{B})$ is an ideal of $\mathbf{B}$, which contains the generators of J_1 by $\otimes_{2.4}$[5] and the atomicity of the $\{x^*_\gamma\}$. Clause (ii) is obvious. Clause (iii) follows by $J_1 \subseteq J_2 \subseteq \mathbb{B}$[, **which**] holds by the choice of J. By $\otimes_{2.3}$, each $(X_\alpha \cup a_\alpha)/J_1$ is an atom of $\mathbf{B}/J_1$.

But are there more atoms? If not, then by the definition of a $\mathbf{B}$ as generated by ..., we can find $n_1 \leq n_2 < \omega$ and $y_0, \dots, y_{n_2-1} \in Y \cup Y^{\max}$ such that $c = \bigcap_{\ell=0}^{n_1-1} c_{y_\ell} - \bigcup_{\ell=n_1}^{n_k-1} c_{y_\ell}$ satisfies c_y/J_1 is an atom of $\mathbf{B}/J_1$. Let $y = \bigcap_{\ell=0}^{m-1} c_\ell - \bigcup_{\ell=n_1}^{n_2-1} c_{y_\ell} \in \mathbb{B}$.

<u>Case 1</u>: $y \in \mathrm{id}_1(\mathbb{B})$.

Say $y = \{\alpha_i : \ell < n\} \in [\mu]^{<\aleph_0}$ is such that $\mathbf{i}(\alpha_\ell) = 0$ for $\ell < n$. Let

$$\beta \in \mu \setminus \{\alpha_\ell : \ell < \mu\};$$

what is $c \cap X_\beta$? We can prove that it is empty by induction on $\mathbf{i}(\beta)$. Similarly, $c \cap S = \varnothing$, so necessarily $c \in J_2$ as required.

<u>Case 2</u>: $y \in \mathrm{id}_1(\mathbb{B})$. **[Presumably one of these is a '$\notin$.']**

Then we can find distinct $\beta_n < \mu$ for $n < \omega$ such that

$$n < \omega \Rightarrow \beta_n \in y \wedge \mathbf{i}(\beta_n) = 0.$$

Then $\bigcup_{n<\omega} X_{\beta_n} \subseteq c$, hence $c \notin J_2$. So we are done.]

We shall prove

$\otimes_{2.7}$ $\mathbf{B}/J_2$ is isomorphic to a homomorphic image of $\mathbb{B}^*$.

Toward proving $\otimes_{2.7}$, let $S := \big\{x^*_\gamma : \gamma \in [\mu, \lambda')\big\}$ and define a function h as follows: its domain is $\{c_y : y \in Y \cup Y^{\max}\}$ and $h(c_y) = y$ for $y \in Y \cup Y^{\max}$, so h is injective into $\mathbb{B}^*$.

Now,

$(*)$ If $n_1 \leq n < \omega$, $m_1 \leq m < \omega$, and $y_0, \dots, y_{n-1} \in Y \cup Y^{\max}$ is without repetitions, <u>then</u>:[6]

$$\tau_1 := \bigcap_{\ell<n_1} c_{y_\ell} - \bigcup_{\ell=n_1}^{n-1} c_{y_\ell} \text{ belongs to } J_2 \text{ in } \mathbf{B} \text{ \underline{if}}$$

[5] For $X_\alpha \cup a_\alpha$; that is, for the members of I^+_α.

[6] Really, we get "iff;" but no need.

$$\tau_2 := \bigcap_{\ell<n_1} y_\ell - \bigcup_{\ell=n_1}^{n-1} y_\ell \in \mathrm{id}_1(\mathbb{B}) \text{ in } \mathbb{B}^*.$$

[Why? First, assume that the second statement holds (so $\tau_2 \subseteq \{\alpha_\ell : \ell < m\} \in [\mu]^{<\aleph_0}$). Then, by the choice of the c_y-s, trivially

$$\tau_1' := \bigcap_{\ell<n_1} (c_{y_\ell} \setminus S) - \bigcup_{\ell=n_1}^{n-1} (c_{y_\ell} \setminus S) =$$

$$\bigcup\{X_\beta : y_0, \ldots, y_{n_1-1} \in D_{z_\beta},\, y_{n_1}, \ldots, y_{n_2-1} \notin D_{z_\beta}\} =$$

$$\bigcup\{X_\beta : \varnothing = \tau_2 \in D_{z_\beta}\} \subseteq \bigcup_{\ell<m} X_{\alpha_\ell} \cup a_{\alpha_\ell}.$$

But $(\tau_1' \mathbin{\triangle} \tau_1) \subseteq S \cup \bigcup_{\ell<m} a_{\alpha_\ell}$, so $\tau_1 \subseteq S \mod J_2^+$.

Now assume $\tau_1 \cap S$ is infinite, hence $\lambda' > \mu$. Recall $\mathcal{A}'' = \{d_z : z \in Y^*\}$ is a MAD family of subsets of $\lambda' \setminus \mu$. Hence $\big\{\{x^*_\gamma : \gamma \in d_z\} : z \in Y^*\big\}$ is a MAD family of subsets of $S = \big\{x^*_\gamma : \gamma \in [\mu, \lambda')\big\}$. So necessarily, for some $z \in Y^*$, the set $\tau_1 \cap S \cap \{x^*_\gamma : \gamma \in d_z\}$ is infinite. As $\tau_1 \cap S \cap \{x^*_\gamma : \gamma \in d_z\} \subseteq c_{y_\ell}$ for $\ell < n_1$ and $\mathrm{id}_1(\mathbb{B}^*/J_1)$ is a maximal ideal, and by the choice of Y and Y^*, necessarily $y_\ell = z$; hence $y_0 = z$ and $n_1 = 1$. Similarly $\ell \in [n_1, n_2) \Rightarrow y_\ell \neq z$, hence

$$\ell \in [n_1, n) \Rightarrow y_\ell \cap y_0 = y_\ell \cap z \in \mathrm{id}_{\mathrm{rk}(z,\mathbb{B}^*)}(\mathbb{B}^*) \Rightarrow \big|\{x^*_\gamma : \underset{\sim}{\gamma} \in d_z\} \cap c_{y_\ell}\big| < \aleph_0.$$

Hence clearly $\ell \in [n_1, n) \Rightarrow y_\ell \notin D_z$, but $y_0 \in D_z$ and $\alpha < \mu \Rightarrow \{\alpha\} \notin D_z$ (as $z \in Y$!) hence $\mathbb{B}^* \notin \mathrm{id}_1(\mathbb{B}^*)$ (in contradiction to our present assumption), so necessarily $\tau_1 \cap S$ is finite. Therefore $\tau_1 \cap S \in J_1^+$. Together with the previous paragraph, $\tau_1 \in J_2^+$, but $\tau_1 \in \mathbf{B}$ hence $\tau_1 \in J_2$ as required. That is, $\tau_2 \in \mathrm{id}_1(\mathbb{B}) \Rightarrow \tau_1 \in J_2$. So we have proved $(*)_2$.]

As $\mathbb{B}^*$ is superatomic and by the choice of $Y \cup Y^{\max}$, $\otimes_{2.7}$ clearly follows from $(*)$; in fact, h induces an isomorphism $\hat{h}$ from $\mathbf{B}/J_2$ onto $\mathbb{B}^*$. But $\mathbb{B}^*$ is superatomic and $J_2 \subseteq \mathrm{id}_\infty(\mathbb{B})$ by $\otimes_{2.6}(i)$, hence

$\otimes_{2.8}$ $\mathbf{B}$ is superatomic.

Now as $\big\{\{\alpha\} : \alpha < \mu\big\}$ are the atoms of $\mathbb{B}^*$ – and recall $\{X_\alpha \cup a_\alpha/J_1 : \alpha < \mu\}$ are the atoms of $\mathbf{B}/J_1$ by $\otimes_6(iii)$ – and $J_1 \subseteq [X^*]^{<\theta}$ while $|X_\alpha \cup a_\alpha| \geq \theta$, clearly

$\otimes_{2.9}$ $J_1 = \mathbf{B} \cap [X^*]^{<\theta}$.

For the rest of the proof, let $f \in \mathrm{Aut}(\mathbf{B})$, and toward contradiction we assume $\mathrm{supp}(f) := \{x \in \mathrm{atom}(\mathbb{B}) : f(x) \neq x\}$ has cardinality $\geq \theta$.

Recall that $J_1 = \{a \in \mathbf{B} : |a| < \theta\}$ and $\big\{\{x\} : x \in X^*\big\}$ are the atoms of $\mathbf{B}$, so necessarily f maps J_1 onto itself. Note that $\{(X_\alpha \cup a_\alpha)/J_1 : \alpha < \mu\}$ lists the atoms of $\mathbf{B}/J_1$ by $\otimes_{2.6} + \otimes_{2.7}$. Assume $f(X_\alpha \cup a_\alpha)/J_1 = (X_\beta \cup a_\beta)/J_1$ and $\alpha \neq \beta$; let

$$c_1 := (X_\alpha \cup a_\alpha) - f^{-1}(X_\beta \cup a_\beta) \text{ and } c_2 := (X_\beta \cup a_\beta) - f(X_\alpha \cup a_\alpha).$$

Both (being the difference of two members of $\mathbf{B}$) are in $\mathbf{B}$ and $c_1 \leq X_\alpha \cup a_\alpha$, $c_2 \leq X_\beta \cup a_\beta$, and by the present assumption, of course $c_1, c_2 \in J_1$, hence $|c_1|, |c_2| < \theta$. Now $c_1 \leq X_\alpha \cup a_\alpha$ and $|c_1| < \theta$ implies $c_1 \in I_\alpha^+$, so $c_1 \cap X_\alpha \in I_\alpha$ and $c_1 \setminus X_\alpha$ is finite. Similarly, $c_2 \cap X_\beta \in I_\beta$ and $c_2 \setminus X_\beta$ is finite. Clearly

$$f \restriction \big(\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c_1)\big)$$

is an isomorphism from $\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c_1)$ onto $\mathbf{B} \restriction (X_\beta \cup a_\beta - c_2)$, contradicting $\otimes_{2.5}$ by the "moreover" in clause (d) of the assumption of Lemma 2.1. Hence the automorphism which f induced on $\mathbb{B}^*/J_1$ maps each atom to itself, hence it is the identity. Also, for $\alpha < \mu$ we have $(X_\alpha \cup a_\alpha) \mathbin{\triangle} f(X_\alpha \cup a_\alpha) \in J_1$: that is, it has cardinality $< \theta$. So

$\boxtimes_3$ For each $\alpha < \mu$, letting

$$c^1_\alpha := (X_\alpha \cup a_\alpha) - f^{-1}(X_\alpha \cup a_\alpha) \in J_1$$

and $c^2_\alpha := (X_\alpha \cup a_\alpha) - f(X_\alpha \cup a_\alpha) \in J$, we have that $f \restriction \big(\mathbb{B}_\alpha \restriction (1 - c^1_\alpha)\big)$ is an isomorphism from $\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c^1_\alpha)$ onto $\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c^2_\alpha)$.

Hence

$\boxtimes_4$ $Z_\alpha := \{x \in \operatorname{atom}(\mathbb{B}_\alpha) : x \leq_{\mathbb{B}_\alpha} c^1_\alpha \vee f(x) \neq x\}$ has cardinality $< \theta$.

(Why? By clause (d) of the assumptions on $\mathbb{B}_\alpha$.) Let

$$v := \big\{\alpha < \mu : (\exists x \in X_\alpha)\big[f(\{x\}) \neq \{x\}\big]\big\}.$$

Assume, toward contradiction, that

$\boxtimes_5$ $|v| \geq \operatorname{cf}(\theta)$.

For $\alpha \in v$, choose $x_\alpha \in X_\alpha$ such that $f(\{x_\alpha\}) \neq \{x_\alpha\}$, and (possibly shrinking v) without loss of generality $\alpha, \beta \in v \Rightarrow \{x_\alpha\} \neq f(\{x_\beta\})$. Let $g' : v \to \mu + 1$ be such that $f(\{x_\alpha\}) \subseteq X_{g'(\alpha)}$, where we stipulate $X_\mu := S$. Applying the above to f^{-1}, we could have chosen $(x_i, \alpha_i, \gamma_i)$ by induction on $i < \operatorname{cf}(\theta)$ such that $\alpha_i \in v$, $f(\{x_i\}) \neq \{x_i\}$, $x_i \in X_{\alpha_i}$, $f(\{x_i\}) \subseteq X_{\gamma_i}$, and $\alpha_i, \gamma_i \notin \{\alpha_j, \gamma_j : j < i\} \setminus \{\mu\}$. Let $v = \{\alpha_i : i < \operatorname{cf}(\theta)\}$.

Without loss of generality, either g' is one-to-one into μ or g' is constantly μ. Now by clause (b) of the assumption, without loss of generality, for some $A \in \mathcal{A}$ we have $\{x_\alpha : \alpha \in v\} \supseteq A$. So $\alpha_A < \mu$ is well defined and

$$\big\{x \in X_{\alpha[A]} \cup a_{\alpha[A]} : f(\{x\}) \leq_{\mathbf{B}} X_{\alpha[A]} \cup a_{\alpha[A]}\big\}$$

does not belong to $I^+_{\alpha[A]}$; so by $\boxtimes_3$ (applied to $\alpha = \alpha_A$ and the properties of $c^1_{\alpha[A]}, c^2_{\alpha[A]}$) we obtain an easy contradiction.

We can conclude $\neg\boxtimes_5$, hence v has cardinality $< \operatorname{cf}(\theta)$, hence

$$\big|\{x \in X : f(x) \neq x\}\big| < \theta.$$

If $\mu = \lambda'$ then we are done, so assume $\mu < \lambda'$. Now

$$S := \{x^*_\gamma : \gamma \in [\mu, \lambda')\} = X^* \setminus X \subseteq X^*$$

satisfies:

$\boxtimes_6$ (α) $(\forall b \in \mathbf{B})\big[b \cap S \text{ infinite} \wedge \bigwedge_{\alpha \in v} [b \cap X_\alpha = \varnothing] \Rightarrow \operatorname{rk}(b/J_1, \mathbf{B}/J_1) \geq 1\big]$

(β) If S' satisfies the property of S in clause (α), then $|S' \setminus S| < \theta$.

[Why? Clause (α) is proved by inspecting the definition of $\mathbf{B}$. As for clause (β), if $|S' \setminus S| \geq \theta$, as $S' \setminus S \subseteq X$, clearly then there is $A \in \mathcal{A}$ such that

$$\{g(i) : i \in A\} \subseteq^* S' \setminus S.$$

First, if $\alpha := \alpha_A$ is well-defined then $X_\alpha \cup a_\alpha \in \mathbf{B}$ and

$$\operatorname{rk}((X_\alpha \cup a_\alpha)/J_1, \mathbf{B}/J_1) = 0 < 1$$

but $(X_\alpha \cup a_\alpha) \cap S' \supseteq a_\alpha$ is infinite; a contradiction. Second, if α_A is not well defined, then for some $\alpha < \mu$ we have $\{g(i) : i \in A\} \cap X_\alpha$ is infinite, and we get a similar contradiction.]

Hence for $\iota = 1, -1$ the set

$$S^\iota_f := \big\{x^*_\gamma : \gamma \in [\mu, \lambda'),\ f^\iota(\{x^*_\gamma\}) \subseteq X\big\}$$

has cardinality $< \theta$. Let $S^*_f := S^{-1}_f \cup S^1_f$.

Also for every $y \in Y^*$, letting $\gamma := \operatorname{rk}(y, \mathbb{B}^*)$, we have $c_y \mathbin{\triangle} f(c_y) \in J_1$.

(Just recall that the automorphism that f induced on $\mathbf{B}/J_1$ is the identity, and recall that

$$d \subseteq S \wedge d \in J_1 \Rightarrow d \text{ is finite}$$

by $\otimes_6$, hence the symmetric difference of $\big\{\{x^*_\gamma\} : \gamma \in d_y\big\} \setminus S^*_f$ and $\big\{f(\{x^*_\gamma\}) : \gamma \in d_y\big\} \setminus S^*_f$ is finite.

As $\mathcal{A}'' := \{d_y : y \in Y^*\}$ is a MAD family of subsets of $\lambda' \setminus \mu$ as in clause (k)(α) of the assumption, the set

$$\big\{\gamma \in [\mu, \lambda') : f(\{x^*_\gamma\}) \neq \{x^*_\gamma\}\big\}$$

is of cardinality $< \theta$, so we are done . . .

Not exactly: we have assumed $\boxtimes_1$!

To eliminate this extra assumption we make some minor changes. First, without loss of generality $\mathbb{B}^*$ is a Boolean Algebra of subsets of $\{\alpha : \alpha < \mu \text{ even}\}$ with the singletons being its atoms. Second, for $A \in \mathcal{A}$, we choose $u = u_A$ as follows (if possible). As we can replace u_A by any infinite subset, without loss of generality:[7]

- (A) Either (α) or (β), where
  - (α) $g^*(\gamma_{A,k})$ is odd for every $k \in u$.
  - (β) $g^*(\gamma_{A,k})$ is even for every $k \in u$.
- (B) If case (α) occurs then $\langle g^*(\gamma_{A,k}) : k \in u\rangle$ is without repetitions.
- (C) If case (β) occurs in clause (A), <u>then</u> there is a unique $y = y_A \in Y$ such that $\big\langle \{g^*(\gamma_{A,k})\} : k \in u\big\rangle$ converges to D_{y_A}.

Note

- ($*$) If u_A is not well defined, then for some finite $W \subseteq \mu$ we have
$$\big\{g(\gamma_{A,k}) : k < \omega\big\} \subseteq \bigcup_{\alpha \in W} X_\alpha.$$

Now we choose $\langle \alpha_A : A \in \mathcal{A},\ u_A \text{ well defined}\rangle$ such that:

- ($**$) $\langle \alpha_A : A \in \mathcal{A}, u_A \text{ well defined}\rangle$ is with no repetitions, each α_A is an odd ordinal $< \mu$ and if possible it lists all of them.

Clearly without loss of generality $\mathbb{B}^*/\mathrm{id}_1(\mathbb{B}^*)$ is nontrivial hence $Y \neq \varnothing$ so choose $y^* \in Y$. Now we define a function g from $\mathbb{B}^*$ into $\mathcal{P}(\mu)$ as follows:

$$\begin{aligned} g(x) := \{&\alpha \in \mu \cap x : \alpha \text{ is even}\} \\ &\cup \big\{\alpha < \mu : \alpha = \alpha_A \text{ for some } A \in \mathcal{A},\ u_A, y_A \text{ are well-defined}, \\ &\qquad\qquad \text{and } x \cap y_A \notin \mathrm{id}_{\mathrm{rk}(y_A, \mathbb{B}^*)}(\mathbb{B}^*)\big\} \\ &\cup \big\{\alpha < \mu : \alpha \text{ is odd, but } \alpha \notin \{\alpha_A : A \in \mathcal{A},\ u_A, y_A \text{ well-defined}\} \\ &\qquad\qquad \text{and } x \cap y^* \notin \mathrm{id}_{\mathrm{rk}(y^*, \mathbb{B})}(\mathbb{B}^*)\big\}. \end{aligned}$$

Easily, g is a homomorphism from $\mathbb{B}^*$ into $\mathcal{P}(\mu)$ as $\mathbb{B}^*$ is superatomic. Let $\mathbb{B}^{**}$ be the Boolean Algebra of subsets of μ generated by $\mathrm{rang}(g) \cup \{(\alpha) : \alpha < \mu\}$. Now we just replace $\mathbb{B}^*$ by $\mathbb{B}^{**} \subseteq \mathcal{P}(\mu)$. $\square_{2.1}$

Discussion 2.2. Why do we use MAD families $\mathcal{A} \subseteq [\mu]^{\aleph_0}$ and not $\subseteq [\mu]^{\aleph_1}$? If we use the latter, we have to take more care with superatomicity, as the intersections of such members may otherwise contradict superatomicity.

[7] Clause (C) is possible as in the justification of $\boxtimes_0$ above.

§ 3. Sufficient conditions for the construction's assumptions

Here we shall show that the assumptions of 2.1 are reasonable. Now in 3.2 we shall reduce clause 2.1(k) to $\mathrm{Pr}(\lambda', \theta)$, where Pr formalizes clause (b) there. In 3.3, 3.5 we give sufficient conditions for $\mathrm{Pr}(\mu, \sigma)$. In fact, it is clear that ([for μ, σ large] enough) it is not easy to fail it. In 3.10 we give a sufficient condition for a strong version of clauses (e)-(f) of 2.1 (and earlier deal with the conditions appearing in it). So at least for some cardinals θ, the statement "not having the assumptions of 2.1" (with $\theta := \sigma^+$ for simplicity, $\kappa := \beth_2(\sigma)$, $\mu := \beth_3(\sigma)$, and λ such that (h)+(i)+(j) of 2.1 holds) has large consistency strength.

Definition 3.1. 1) $\mathrm{Pr}(\chi, \mu, \theta)$ means that $\mu \geq \theta$ and for some $\mathcal{A}$ we have:

- (a) $\mathcal{A} \subseteq [\mu]^{\aleph_0}$
- (b) $\mathcal{A}$ is almost disjoint; i.e. $A \neq B \in \mathcal{A} \Rightarrow |A \cap B| < \aleph_0$.
- (c) $|\mathcal{A}| = \chi$
- (d) $(\forall B \in [\mu]^\theta)(\exists A \in \mathcal{A})[A \subseteq^* B]$.

2) If we omit χ we mean "for some χ."
3) We call $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$ *saturated* if every $A \in [\lambda]^{\aleph_0}$ is either almost contained[8] in a finite union of members of $\mathcal{A}$ or almost contains a member of $\mathcal{A}$.

Fact 3.2. 1) Clause (b) of the assumption of 2.1 is equivalent to $\mathrm{Pr}(\mu, \mu, \mathrm{cf}(\theta))$.
2) Clauses (k)$(\alpha), (\gamma)$ of the assumption of 2.1 follow from

$$\mathrm{Pr}(\chi', \lambda', \theta) \wedge \chi = \chi' + 2^{\aleph_0}.$$

3) If $\mathcal{A} \subseteq [\mu]^{\aleph_0}$ is almost disjoint and saturated, then $\mathrm{Pr}(|\mathcal{A}|, \mu, \aleph_1)$.
4) If $\mu = \mu^{\aleph_0} \geq \theta$ then $\mathrm{Pr}(\mu, \theta) \Leftrightarrow \mathrm{Pr}(\mu, \mu, \theta)$ and $\chi \neq \mu \Rightarrow \neg\,\mathrm{Pr}(\chi, \mu, \theta)$.
5) If $\theta < \mu_1 \leq \mu_2$ and $\mathrm{Pr}(\mu_2, \theta)$ then $\mathrm{Pr}(\mu_1, \theta)$.

Proof. 1) Read the two statements.
2) Let $\mathcal{A} \subseteq [\lambda']^{\aleph_0}$ exemplify $\mathrm{Pr}(\chi', \lambda', \theta)$. For each $A \in \mathcal{A}$, we can find

$$\langle B_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$$

such that:

- $(*)$ (i) $B_{A,\zeta} \in [A]^{\aleph_0}$
  - (ii) $\zeta \neq \varepsilon \Rightarrow B_{A,\zeta} \cap B_{A,\varepsilon}$ is finite.
  - (iii) If π is a partial one-to-one function from A to A such that
$$x \in \mathrm{dom}(\pi) \Rightarrow x \neq \pi(x),$$
then for some $\zeta < 2^{\aleph_0}$ we have
$$\alpha \in B_{A,\zeta} \Rightarrow \alpha \notin \mathrm{dom}(\pi) \vee \pi(\alpha) \notin B_{A,\zeta}.$$

[Why? First find $\langle B'_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$ satisfying (i),(ii). Let $\langle \pi_\zeta : \zeta < 2^{\aleph_0} \rangle$ list the π-s from (iii), and choose $B_{A,\zeta} \in [B'_{A,\zeta}]^{\aleph_0}$ to satisfy clause (iii) for π_ζ. Lastly, let $\mathcal{A}'$ be any MAD family of subsets of A extending $\{B_{A,\zeta} : A \in \mathcal{A},\ \zeta < 2^{\aleph_0}\}$.]

Having found $\langle B_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$, we let $\mathcal{A}' := \{B_{A,\zeta} : A \in \mathcal{A},\ \zeta < 2^{\aleph_0}\}$. It has cardinality $|\mathcal{A}| + 2^{\aleph_0} = \chi' + 2^{\aleph_0}$ and is as required in clauses (k)$(\alpha), (\gamma)$ of 2.1.
3-5) Easy. $\square_{3.2}$

[8] $A \subseteq^* B$ (i.e. "A is almost contained in B") means that $A \setminus B$ is finite.

Claim 3.3. 1) *Assume*

(a) $\kappa_n < \kappa_{n+1} < \kappa < \mu_n < \mu_{n+1} < \mu$ *for* $n < \omega$.

(b) $\kappa := \sum \kappa_n$, $\mu := \sum \mu_n$, *and* $\max \mathrm{pcf}\{\kappa_n : n < \omega\} > \mu$.

(c) κ *is strong limit and* $2^\kappa \geq \mu^+$.

(d) $\langle \mu_n : n < \omega \rangle$ *satisfies the requirements from* [She02, §1], *or at least the conclusion — i.e.*

$\odot$ *For every* $\lambda \geq \mu$, *for some* n, *if* $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \setminus \mu$ *and* $|\mathfrak{a}| < \mu$ *then* $\sup \mathrm{pcf}_{\mu_n\text{-complete}}(\mathfrak{a}) \leq \lambda$.

Then for every $\lambda \geq \kappa$:

$\otimes_{\lambda,\kappa}$ *We can find* $\{\bar{A}_\alpha : \alpha < \alpha^*\}$ *such that*

(α) *Each* $\bar{A}_\alpha$ *has the form* $\langle A_{\alpha,n} : n < \omega \rangle$, *it belongs to* $\prod_{n<\omega} [\lambda]^{\kappa_n}$, *and for each* α *the members of* $\bar{A}_\alpha$ *are pairwise disjoint.*

(β) *If* $\alpha \neq \beta$, *then* $\bar{A}_\alpha$ *and* $\bar{A}_\beta$ *are* almost disjoint*; by this we mean that*

$$f \in \prod_{n<\omega} A_{\alpha,n} \wedge f' \in \prod_{n<\omega} A_{\beta,n} \Rightarrow \left|\mathrm{rang}(f) \cap \mathrm{rang}(f')\right| < \aleph_0.$$

(γ) *[If* $\bar{A} \in \prod_{n<\omega} [\lambda]^{\kappa_n}$, *then] for some* $\alpha < \alpha^*$ *and one-to-one functions* $h_1, h_2 \in {}^\omega\omega$, *we have* $\lim_{n\to\infty} |A_{\alpha,h_1(n)} \cap A_{\alpha,h_2(n)}| = \kappa$.

[$\bar{A}$ doesn't depend on α here. I think the bracketed phrase should be deleted.]

2) *If* $\kappa = \aleph_0$, $\kappa_n = 1$, $\mu_n < \mu_{n+1} < \mu = \sum_{n<\omega} \mu_n < 2^{\aleph_0}$ *and we have* $\odot$ *of* (1)(d), *then the conclusion of* (1) *holds.*

3) *We can conclude in* (1) *that there is* $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$, *an almost disjoint family, such that* $(\forall B \in [\lambda]^\kappa)(\exists A \in \mathcal{A})[A \subseteq B]$.

Proof. By [She00], [She04, §3] (even more). $\square_{3.3}$

Remark 3.4. 1) Are the hypotheses of 3.3(1) reasonable?

1a) Assume that κ is strong limit of cofinality $\aleph_0 < \kappa$ and $2^\kappa > \kappa^{+\omega}$. We let $\mu_n := \kappa^{+1+n}$. There is a sequence $\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$ as in clauses (a)-(c) of 3.3(1); such $\bar{\kappa}$ exists (by [She94, Ch.IX,§5]) and it is hard not to satisfy clause (d) (see [She02]).

1b) Clause (c) (i.e. 'κ is strong limit') is just needed to start the induction.

2) Similarly for 3.3(2).

We quote Goldstern, Judah, and Shelah [GJS91], which implies 3.5(1),(2).

Claim 3.5. *Assume*

$$\mathsf{CH} + \mathsf{SCH} + (\forall \mu > 2^{\aleph_0})[\mathrm{cf}(\mu) = \aleph_0 \Rightarrow \square_{\mu^+}].$$

Then there is a saturated MAD family $\mathcal{A}_\lambda \subseteq [\lambda]^{\aleph_0}$ *(of cardinality* $\lambda^{\aleph_0}$*) for every uncountable* λ.

Proof. This is the main result of Goldstern, Judah, and Shelah [GJS91]. $\square_{3.5}$

Definition 3.6. Let $\mu \geq \theta$.
1) Let $\mathcal{S}_\theta$ be the class of $\bar{a} = \langle a_n : n < \omega \rangle$ such that $|a_n| \leq \theta$, $a_n \subseteq a_{n+1}$,

$$\mathrm{cf}(\theta) = \aleph_0 \Rightarrow |a_n| < \theta,$$

and $\limsup_n |a_{n+1} \setminus a_n| = \theta$. Let $\mathcal{S}_{\theta,\mu} := \{\bar{a} \in \mathcal{S}_\theta : a_n \in [\mu]^{\leq\theta}\}$.
2) For $\bar{a} \in \mathcal{S}_\theta$, let

$$\mathrm{set}(\bar{a}) := \Big\{W \in \big[\bigcup_{n<\omega} a_n\big]^{\aleph_0} : n < \omega \Rightarrow \big|W \cap a_n \setminus \bigcup_{\ell<n} a_\ell\big| < \aleph_0\Big\}.$$

3) For $\bar{a}, \bar{b} \in \mathcal{S}_\theta$, let $\bar{a} \leq^* \bar{b}$ mean $\mathrm{set}(\bar{a}) \supseteq \mathrm{set}(\bar{b})$.
4) We say $\bar{a}, \bar{b} \in \mathcal{S}_\theta$ are *compatible* <u>if</u>

$$(\exists \bar{c} \in \mathcal{S}_\theta)\big[\bar{a} \leq^* \bar{c} \wedge \bar{b} \leq^* \bar{c} \wedge \bigcup_n c_n \subseteq \bigcup_n a_n \cap \bigcup_n b_n\big].$$

(If $\mathrm{cf}(\theta) = \aleph_0 < \theta$, this is equivalent to "$\bigcup_{n<\omega} a_n \cap \bigcup_{n<\omega} b_n$ has cardinality θ.")

Definition 3.7. For $\theta \leq \mu$:
1) Let $\boxtimes_{\theta,\mu}$ be the following.

$\boxtimes_{\theta,\mu}$ There is $\mathcal{S}^* \subseteq \mathcal{S}_{\theta,\mu}$ such that:
- (a) For every $\bar{a} \in \mathcal{S}_{\theta,\mu}$, there is $\bar{b} \in \mathcal{S}^*$ compatible with $\bar{a}$.
- (b) If $\bar{a} \neq \bar{b} \in \mathcal{S}^*$ then $\mathrm{set}(\bar{a}) \cap \mathrm{set}(\bar{b}) = \varnothing$.

2) Let $\boxtimes'_{\theta,\mu}$ mean the following.

$\boxtimes'_{\theta,\mu}$ If $\mathcal{S} \subseteq \mathcal{S}_{\theta,\mu}$ has cardinality $\leq \mu$ then we can find $\mathcal{S}^* \subseteq \mathcal{S}_{\theta,\mu}$ such that:
- (a) For every $\bar{a} \in \mathcal{S}$ there is $\bar{b} \in \mathcal{S}^*$ such that $\bar{b} \leq^* \bar{a}$.
- (b) For every $\bar{b} \in \mathcal{S}^*$ there is $\bar{a} \in \mathcal{S}$ such that $\bar{b} \leq^* \bar{a}$.
- (c) $\langle \mathrm{set}(\bar{b}) : \bar{b} \in \mathcal{S}^* \rangle$ are pairwise disjoint.

3) We may replace μ by a set A (but obviously $\boxtimes_{\theta,A}$ is equivalent to $\boxtimes_{\theta,|A|}$ and $\boxtimes'_{\theta,A}$ to $\boxtimes'_{\theta,|A|}$).

Fact 3.8. 1) Assume $\theta > \mathrm{cf}(\theta) = \aleph_0$ is strong limit, $\theta = \sum_{n<\omega} \theta_n$ with $\theta_n < \theta_{n+1}$, and $\bar{b} \in \mathcal{S}_{\theta,\mu}$. <u>Then</u> we can find $\mathcal{A} \subseteq \mathcal{S}_\theta$ such that:
- (a) If $\bar{a} \in \mathcal{A}$ then $(\forall n)(\exists m)[a_n \subseteq b_m]$ (so $\bar{a} \leq \bar{b}$).
- (b) If $\bar{a} \in \mathcal{A}$ then $|a_n| = \theta_n$; moreover, $\mathrm{otp}(a_n) = \theta_n$ and a_{n+1} is an end extension of a_n.
- (c) If $\bar{a} \in \mathcal{A}$ then $\langle a_n : n < \omega \rangle$ is [strictly?] $\subset$-increasing. **[If we don't say *strictly* increasing, this is redundant by the definition.]**
- (d) If $\bar{a}^1 \neq \bar{a}^2$ then $\mathrm{set}(\bar{a}^1) \cap \mathrm{set}(\bar{a}^2) = \varnothing$.
- (e) If $\bar{c} \in \mathcal{S}_\theta$ is compatible with $\bar{b}$ then it is compatible with some $\bar{a} \in \mathcal{A}$.

2) If $(\forall \alpha < \theta_n)\big[|\alpha|^\sigma < \theta_n = \mathrm{cf}(\theta_n)\big]$ and $<_\alpha$ is a well ordering of $\bigcup_{n<\omega} b_n$ for $\alpha < \sigma$, <u>then</u> we can strengthen (b) to

(b)$^+$ For $\alpha < \sigma$, $\bar{a} \in \mathcal{A}$ and $n < \omega$, $\mathrm{otp}(b_n, <_\alpha \restriction b_n) = \theta_n$; and if $\sigma < \aleph_0$ then b_{n+1} is a $<_\alpha$-end extension of b_n.

3) $\bar a, \bar b \in \mathcal{S}_{\theta,A}$ are incompatible iff $\bigcup_{n<\omega} a_n \cap \bigcup_{n<\omega} b_n$ has cardinality $< \theta$ ($\mathrm{cf}(\theta) = \aleph_0 < \theta$ will suffice).

4) (a) $\boxtimes_{\theta,\mu}$ implies $\boxtimes'_{\theta,\mu}$.

(b) $\boxtimes'_{\theta,\mu}$ is equivalent to $\boxtimes_{\theta,\mu}$ if $\mu = \mu^\theta$.

Proof. As in 3.9 below. $\square_{3.8}$

Claim 3.9. *Assume θ is strong limit, $\theta > \mathrm{cf}(\theta) = \aleph_0$.*

1) *If $\mu \in (\theta, 2^\theta]$ then $\boxtimes'_{\theta,\mu}$ from 3.7 holds.*

2) *Also, if $\theta < \mu < (2^\theta)^{+2^\theta}$ then $\boxtimes'_{\theta,\mu}$.*

3) *If $2^\theta < \mu$ and $\big(\forall \lambda \in (2^\theta, \mu)\big)\big[\mathrm{cf}(\lambda) = \aleph_0 \Rightarrow \lambda^{\aleph_0} = \lambda^+ + \square_\lambda\big]$ then $\boxtimes'_{\theta,\mu}$.*

[Is that '+' denoting cardinal addition, or a conjunction?]

Proof. 1) Straight, as $|\mathcal{S}_{\theta,\mu}| = \mu^\theta = 2^\theta$ we can find $\langle \bar a^\alpha : \alpha < 2^\theta\rangle$ listing $\mathcal{S}_{\theta,\mu}$. Now we choose $\gamma(\alpha)$ and $\bar b^\alpha$ by induction on $\alpha < 2^\theta$ such that

(a) $\bar b^\alpha \in \mathcal{S}_{\theta,\mu}$

(b) $\beta < \alpha \Rightarrow \mathrm{set}(\bar b^\beta) \cap \mathrm{set}(\bar b^\alpha) = \varnothing$

(c) $\bar c^\alpha \le a^{\gamma(\alpha)}$

(d) $\gamma(\alpha) = \min\{\gamma : \bar a^\gamma$ is incompatible with $\bar b^\beta$ for every $\beta < \alpha\}$.

Arriving to α in the induction, choose $\gamma(\alpha)$ by clause (d). We note that

$$\beta < \gamma(\alpha) \Rightarrow c^\alpha_\beta := \bigcup_n a^{\gamma(\alpha)}_n \cap \bigcup_n b^\beta_n \text{ has cardinality } < \theta,$$

hence we can find $\bar b_{\alpha,\varepsilon} \le \bar a^{\gamma(\alpha)}$ for $\varepsilon < 2^\theta$ with $\langle \mathrm{set}(\bar b_{\alpha,\varepsilon}) : \varepsilon < 2^\theta\rangle$ pairwise disjoint. So for all but $\le \theta + |\alpha|$ of the $\varepsilon < 2^\theta$, $\bar b_\alpha = \bar b_{\alpha,\varepsilon}$ is as needed.

2) After reading [She00] this is easy: and anyhow, in subsequent works we give fuller answers.

3) As in [GJS91]. $\square_{3.9}$

Claim 3.10. 1) *Assume*

$\boxtimes_{\theta,\kappa,\mu}$ *θ is strong limit, $\aleph_0 = \mathrm{cf}(\theta) < \theta \le \kappa \le 2^{2^\theta}$, $\mu = 2^\kappa$, and $\boxtimes_{\theta,\kappa}$ (from 3.7) holds (so $\mu = \mu^{\aleph_0}$).*

Then some $\bar{\mathbb{B}} = \langle \mathbb{B}_\alpha : \alpha < \mu\rangle$ satisfies clauses (c)-(g) of 2.1; in fact, $\mathbb{B}_\alpha$ is a subalgebra of $\mathcal{P}(\kappa)$ with two levels and $\mathrm{id}_{<\infty}(\mathbb{B}_\alpha)$ is included in $[\kappa]^{<\aleph_1}$, hence

$$\mathbb{B}_\alpha \subseteq \{a \subseteq \kappa : a \text{ countable or co-countable}\}.$$

2) *As above, except that instead of "θ strong limit, $\mathrm{cf}(\theta) = \aleph_0 < \theta$" we demand $2^\theta = \theta^{\aleph_0} > 2^{\aleph_0} \wedge \theta > \mathrm{cf}(\theta) = \aleph_0$ or $\theta = \aleph_0 \wedge$ "there is no infinite MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ of cardinality $<$ the continuum".*

Proof. 1) Let $\theta = \sum_{n<\omega} \theta_n$, $\theta_n < \theta_{n+1} < \theta$.

Fact 3.11. Letting $\bar a^* = \langle \theta_n : n < \omega\rangle$ (i.e. $a^*_n = \theta_n$) we can find

$$\bar t^{\bar a} = \langle t_{\ell,\alpha} : \ell < 3, \alpha < 2^\theta\rangle$$

such that:

(i) $t_{\ell,\alpha} \in \mathrm{set}(\bar a^*)$ has order type ω.

(*ii*) We will fix a bijection $\pi : 2^\theta \times 2^\theta \to 2^\theta$, and write $t_{2,\alpha,\beta}$ for $t_{2,\pi(\alpha,\beta)}$.

(*iii*) If $(\ell_1, \alpha_1) \neq (\ell_2, \alpha_2)$ then $t_{\ell_1,\alpha_1} \cap t_{\ell_2,\alpha_2}$ is finite.

(*iv*) If $\bar a \in \mathcal{S}_{\theta,\kappa}$ and $\bigcup_{n<\omega} a_n \subseteq \theta$, then for some $\alpha < 2^\theta$ we have

$$\beta < 2^\theta \Rightarrow t_{2,\alpha,\beta} \in \mathrm{set}(\bar a).$$

(*v*) If $\bar a, \bar b \in \mathcal{S}_{\theta,\kappa}$, $\bigcup_{n<\omega} a_n \cup \bigcup_{n<\omega} b_n \subseteq \theta$, $\mathrm{set}(\bar a) \cap \mathrm{set}(\bar b) = \varnothing$, and

$$h : \bigcup_{n<\omega} a_n \to \bigcup_{n<\omega} b_n$$

is one-to-one and maps a_n onto b_n, then for some α, $t_{0,\alpha} \in \mathrm{set}(\bar a)$ and $t_{1,\alpha} \in \mathrm{set}(\bar b)$ and h maps $t_{0,\alpha}$ into a co-infinite subset of $t_{1,\alpha}$.

Proof. **Proof of the fact**: Straightforward. $\square_{3.11}$

Construction: Let $\mathcal{S}^* := \{\bar a^\gamma : \gamma < \gamma^*\}$ exemplify $\boxtimes_{\theta,\kappa}$ (so $|\gamma^*| \leq \kappa^\theta$). Without loss of generality

$$\bar a \in \mathcal{S}^* \wedge n < \omega \Rightarrow \mathrm{otp}(a_n) \wedge a_{n+1} \text{ is an end-extension of } a_n.$$

[What *about* $\mathrm{otp}(a_n)$?]

[Why? By 3.8; i.e. by replacing $\bar a^\gamma$ by a suitable family $\subseteq \{\bar b : \bar b \leq \bar a^\gamma\}$.]

Let $\{X_\gamma : \gamma < \kappa\}$ be a sequence of subsets of 2^θ such that

$$\gamma_1 \neq \gamma_2 \Rightarrow |X_{\gamma_1} \setminus X_{\gamma_2}| = 2^\theta;$$

let $\langle Y_j : j < \mu\rangle$ be a sequence of subsets of κ such that $j_1 \neq j_2 \Rightarrow |Y_{j_1} \setminus Y_{j_2}| = \kappa$; let g_γ be a one-to-one mapping from θ into $\bigcup_{n<\omega} a^\gamma_n$ mapping θ_n onto a^γ_n; and lastly, let $t^\gamma_{\ell,\alpha} := g''_\gamma(t_{\ell,\alpha}) = \{g_\gamma(\zeta) : \zeta \in t_{\ell,\alpha}\}$ for $\ell < 3$ and $\alpha < \gamma^+$

[What's that double-prime doing? From what you wrote, that should be the image of $t_{\ell,\alpha}$ under g_γ — if you're worried about it getting confused for something else, you could write it $g_\gamma[t_{\ell,\alpha}]$.]

(hence $t^\gamma_{2,\alpha,\beta} = g''_\gamma(t^\gamma_{2,\alpha,\beta})$). Let

$$t^\gamma_{3,\alpha,\beta} := \big\{g_\gamma(\varepsilon) : \varepsilon \in t_{2,\alpha,\beta} \text{ and } |t_{2,\alpha,\beta} \cap \varepsilon| \text{ is even}\big\}.$$

For $j < \mu$, let $\mathcal{A}_j$ be the following family of subsets of κ:

$$\begin{gathered}\big\{t^\gamma_{0,\alpha}, t^\gamma_{1,\alpha} : \gamma < \gamma^*,\ \alpha < 2^\theta\big\} \cup \\ \big\{t^\gamma_{2,\alpha,1+\beta} : \gamma < \gamma^*,\ \beta \notin X_\gamma,\ \alpha < 2^\theta\big\} \cup \\ \big\{t^\gamma_{3,\alpha,1+\beta} : \gamma < \gamma^*,\ \beta \in X_\gamma,\ \alpha < 2^\theta\big\} \cup \\ \big\{t^\gamma_{2,\alpha,0} : \gamma < \gamma^*,\ \alpha < 2^\theta,\ \gamma \notin Y_j\big\} \cup \big\{t^\gamma_{3,\alpha,0} : \gamma \in Y_j\big\}.\end{gathered}$$

Clearly,

$\odot_1$ $s \neq t \in \mathcal{A}_j \Rightarrow |s \cap t| < \aleph_0 = |s|$.

Let $\mathcal{A}^+_j$ be a maximal almost disjoint family of countable subsets of κ extending $\mathcal{A}_j$. Let I_j be the Boolean ring of subsets of κ generated by $\mathcal{A}^+_j \cup \big\{\{\varepsilon\} : \varepsilon < \kappa\big\}$ and $\mathbb{B}_j$ be the Boolean algebra of subsets of κ generated by I_j. Now,

$\odot_2$ If $i_0, i_1 < \mu$, $b_0, b_1 \in [\kappa]^\theta$, and $h : b_0 \to b_1$ is bijection such that

$$\big(\forall \alpha \in \mathrm{dom}(h)\big)\big[h(\alpha) \neq \alpha\big],$$

then for some $t^0 \in \mathcal{A}^+_{i_0}$ and $t^1 \in \mathcal{A}^+_{i_1}$, we have $t^0 \subseteq^* b_0$, $t^1 \subseteq^* b_1$, and h maps t^0 into a co-infinite subset of t^1.

[Why? For some $\gamma_0 < \kappa$, the set $b_0 \cap \bigcup\limits_{n<\omega} a_n^{\gamma_0}$ has cardinality θ, so without loss of generality $b_0 \subseteq \bigcup\limits_{n<\omega} a_n^{\gamma_0}$; and similarly, for some $\gamma_1 < \kappa$, without loss of generality, $b_1 \subseteq \bigcup\limits_{n<\omega} a_n^{\gamma_1}$. For $\ell = 0, 1$, let $b_\ell^- \in [\theta]^\theta$ be such that g_{γ_ℓ} maps b_ℓ^- onto b_ℓ. Now without loss of generality $b_0^- \cap b_1^- = \varnothing$ or $b_0^- = b_1^-$. (Recall that we have to preserve "h maps b_0 onto b_1," as well!) If $b_0^- \cap b_1^- = \varnothing$ then by clause (v) of Fact 3.11, some $t_{0,\alpha_0}^{\gamma_0} \in \mathcal{A}_{i_0} \subseteq \mathcal{A}_{i^0}^+$ and $t_{0,\alpha_1}^{\gamma_1} \in \mathcal{A}_{i_1} \subseteq \mathcal{A}_{i_1}^+$ will be as required in the conclusion of $\odot_2$.

So assume $b_0^- = b_1^-$ and let $b_0^* := \{\alpha \in b_0^- : h \circ g_{\gamma_0}(\alpha) \neq g_{\gamma_1}(\alpha)\}$. If b_0^* has cardinality θ, we get the desired conclusion (in $\odot_2$) as above, so assume $|b_0^*| < \theta$; hence without loss of generality $b_0^* = \varnothing$. Also, if $\gamma_0 \neq \gamma_1$ then $|X_{\gamma_0} \setminus X_{\gamma_1}| = 2^\theta$, hence we can find a non-zero ordinal $\beta \in X_{\gamma_0} \setminus X_{\gamma_1}$. By clause (ii) of the fact we can find an ordinal $\alpha < 2^\theta$ such that

$$(\forall \beta < 2^\theta)\big[t_{2,\alpha,\beta}^\gamma \subseteq b_0^-\big]$$

hence we can use $t_{3,\alpha,\beta}^\gamma, t_{2,\alpha,\beta}^\gamma$. So we have to assume $\gamma_0 = \gamma_1$; but then $g_{\gamma_0} = g_{\gamma_1}$ so $h \restriction (b_0 \setminus b_0^*)$ is the identity, a contradiction.]

$\odot_3$ If $i_0 \neq i_1 < \mu$ and[9] $Z \in [\kappa]^{<\kappa}$ and $h : \kappa \setminus Z \to \kappa \setminus Z$ is a bijection, <u>then</u> for some $t^0 \in \mathcal{A}_{i_0}^+$ satisfying $t^0 \subseteq^* \mathrm{dom}(h)$ and $t^1 \in \mathcal{A}_{i_1}^+$, we have: $h''(t^0) \subseteq^* t^1$ and $t^1 \setminus h''(t^0)$ is infinite.

[Why? Let $Z_1 := \{\alpha \in \mathrm{dom}(h) : h(\alpha) \neq \alpha\}$; by $\odot_2$ we know $|Z_1| < \theta$. We know that $Y_{i_0} \setminus Y_{i_1}$ has cardinality μ, hence for some $\gamma \in Y_{i_0} \setminus Y_{i_1}$ we have

$$\mathrm{set}(\bar{a}_\gamma) \cap [Z \cup Z_1]^{\aleph_0} = \varnothing.$$

So $t_{3,\alpha,0}^\gamma \in \mathcal{A}_{i_0} \subseteq \mathcal{A}_j^+$ and $t_{2,\alpha,0}^\gamma \in \mathcal{A}_{i_1} \subseteq \mathcal{A}_{i_1}^+$, so $t_{3,\alpha,0}^\gamma$ is a co-infinite subset of $t_{2,\alpha,0}^\gamma, t_{2,\alpha,0}^\gamma \subseteq^* \kappa \setminus Z \setminus Z_0$ and h maps $t_{3,\alpha,0}^\gamma \setminus Z \setminus Z_0$ to itself, a co-infinite subset of $t_{2,\alpha,0}^\gamma$.]

Clearly $\langle \mathbb{B}_j : j < \mu \rangle$ is as required, so we are done.

2) Similar proof. $\square_{3.10}$

Conclusion 3.12. 1) *Under the assumption* $\boxtimes_{\theta,\kappa,\mu}$ *of* 3.10*, let*

$$\lambda^* = \mathrm{Ded}^+(\mu) := \min\{\lambda : \textit{there is no tree with} \leq \mu \textit{ nodes and} \geq \lambda \textit{ branches}\}$$

(equivalently, no linear order of cardinality λ and density $\leq \mu$). <u>Then</u> for any $\lambda \in [\mu, \lambda^*)$ there is a superatomic Boolean Algebra of cardinality λ and μ atoms with no automorphism moving $\geq \theta$ atoms.

2) *Assume θ is uncountable strong limit of cofinality $\aleph_0$, $\mathrm{pp}_{J_\omega^{\mathrm{bd}}}(\theta) = 2^\theta$ (see* [She94, Ch.IX,§5] *for why this is reasonable), $\kappa = (2^\theta)^{+\alpha} \leq 2^{2^\theta}$, $\alpha < (2^\theta)^+$, $\mu = 2^\kappa$, and $\mu < \lambda < \mathrm{Ded}^+(\mu)$ (e.g. $\lambda = 2^\chi$ for $\chi := \min\{\chi' : 2^{\chi'} > \mu\}$). <u>Then</u> there is a superatomic Boolean Algebra of cardinality λ and μ atoms, with no automorphism moving $\geq \theta$ atoms.*

3) *In part* (2) *we can replace $\kappa = (2^\theta)^{+\alpha}$ by $\kappa = 2^{2^\theta}$, if we are granted a very weak pcf hypothesis (whose negation is not known to be consistent and also of §4). E.g.*

(∗) *If $\mathfrak{a}$ is a countable set of regular cardinals then $\mathrm{pcf}(\mathfrak{a})$ is countable (or just $\leq \aleph_{n(*)}$).*

[$n(*)$ isn't defined anywhere. Do you just want 'for some $n < \omega$?']

[9] By a little more care in indexing, $Z \in [\mu]^{<\mu}$ is okay, and we can choose γ such that $\bigcup\limits_n a_{\gamma,n} \subseteq \kappa \setminus Z \setminus Z_0$.

Proof. 1) We, of course, use Lemma 2.1 with θ^+ here standing for θ there, so we have to show that the assumptions there holds.

Clause (a) of 2.1 holds trivially.

Clause (b) of 2.1 follows from $\boxtimes_{\theta,\kappa}$ (every $(\forall A \in [\mu]^\theta)(\exists B \in \mathcal{A})[B \subseteq A]$ rather than just '$(\forall A \in [\mu]^{\theta^+})$.' There is a sequence $\langle \mathbb{B}_\alpha : \alpha < \mu \rangle$ satisfying clauses (c)-(g) of 2.1 by 3.10. There is a Boolean Algebra $\mathbb{B}^*$ satisfying clauses (h)-(j) of 2.1 because $\lambda < \lambda^*$, so there is a tree $\mathcal{T}$ with μ nodes and $\geq \lambda$ branches, let $\mathcal{Y}$ be a set of λ branches of $\mathcal{T}$ and let $\mathbb{B}$ be the Boolean Algebra of subsets of $\mathcal{T}$ generated by $\{a \subseteq T : a \text{ is linearly ordered by } <_T \text{ and } x \in a \wedge y <_T x \Rightarrow y \in a \text{ and } a \text{ is bounded on } a \in \mathcal{Y}\}$.

[What does it mean for a to be bounded on a?]

Lastly, clause (k) of 2.1 holds vacuously, as we chose $\lambda' = \mu$. $\square_{3.12}$

Claim 3.13. *Assume*

(a) $\mathrm{Pr}(\beth_3, \aleph_1)$

(b) $\lambda^* := \min\{\lambda' : \textit{there is a tree with } \beth_3 \textit{ models of} \geq \lambda' \textit{ branches}\}$

(c) $\lambda \in [\beth_3, \lambda^*)$.

Then there is a superatomic Boolean Algebra with λ elements, $\beth_3$ atoms, and no automorphisms moving uncountably many atoms.

Proof. The main new point is that we can prove a parallel of 3.10 noting that as $\mathrm{Pr}(\beth_3, \aleph_1)$ holds also $\mathrm{Pr}(\beth_2, \aleph_1)$ holds. $\square_{3.13}$

Remark 3.14. 1) So clearly, in many models of ZFC we get that the bound in 1.1 cannot be improved.

2) The question is whether inductively we can get for many θ-s the parallel of 3.10.

3) We can (under weak assumptions) add λ' with $\mu \leq \lambda' \leq (\lambda')^{\aleph_0} \leq \lambda$, and demand that the Boolean algebra has μ' atoms.

[What's μ'? On a perhaps related note, where is λ' used?]

For this we need to check condition (k)(α). We probably can omit the demand "$(\lambda')^{\aleph_0} \leq \lambda$" in the generalization of 3.12 indicated above: for this we just need to weaken "$\mathcal{A}$ is MAD" in 2.1.

Claim 3.15. 1) *Let $\lambda > \aleph_0$. A sufficient condition for the existence of a saturated* MAD *family $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$ is the following.*

$\boxplus_{\lambda,\theta}$ *If $\theta := \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^{\aleph_0}$ is an infinite* MAD *family$\}$, then for every $\mu \in (2^{\aleph_0}, \lambda^{\aleph_0}]$ we have $\neg(a)_{\mu,\theta}$ and*

$$\aleph_0 < \sigma = \mathrm{cf}(\sigma) \leq \theta \Rightarrow \neg(b)_{\mu,\theta},$$

where

$(a)_{\mu,\theta}$ *There is a set $\mathfrak{b} \subseteq \mathrm{Reg} \cap \mu \setminus 2^{\aleph_0}$ of cardinality $\leq \theta$ such that $\Pi\mathfrak{b}/[\mathfrak{b}]^{<\aleph_0}$ is μ-directed. Moreover, for no sequence $\bar{\mathfrak{b}} = \langle \mathfrak{b}_i : i < \theta \rangle$ with each $\mathfrak{b}_i \subseteq \mathrm{Reg} \cap \mu \setminus 2^{\aleph_0}$ finite* ***[do we have]***

$$\mathfrak{c} \subseteq \bigcup_{i<\theta} \mathfrak{b}_i \wedge \max \mathrm{pcf}(\mathfrak{c}) < \mu \Rightarrow \left|\{i < \theta : \mathfrak{b}_i \subseteq \mathfrak{c}\}\right| < \aleph_1.$$

$(b)_{\mu,\theta}$ *μ is regular, $S \subseteq \{\delta < \mu : \mathrm{cf}(\delta) = \mathrm{cf}(\theta)\}$ is stationary, $\bar{A} = \langle A_\delta : \delta \in S \rangle$, $A_\delta \subseteq \delta$, $\mathrm{otp}(A_\delta) = \theta$, and*

$$\delta_2 \neq \delta_2 \Rightarrow A_{\delta_1} \cap A_{\delta_2} \textit{ finite.}$$

2) *Similarly, concerning* $\boxtimes'_{\theta,\mu}$.

Proof. As in [She04]. $\square$

§ 4. On independence

In the bound $\beth_4(\sigma)$, the last 'exponentiation' was really the operation $\mathrm{sa}(\mu)$, where

Definition 4.1. 1) $\mathrm{sa}(\mu) :=$

$$\sup\{|\mathbb{B}| : \mathbb{B} \text{ is a superatomic Boolean Algebra with } \mu \text{ atoms}\}.$$

2) $\mathrm{sa}^+(\mu) := \sup\{|\mathbb{B}|^+ : \mathbb{B}$ is a superatomic Boolean Algebra with μ atoms$\}$.
3) $\mathrm{sa}(\mu, \theta) :=$

$$\sup\Big\{|\mathbb{B}| : \mathbb{B} \text{ is a superatomic Boolean subalgebra of } \mathcal{P}(\mu) \text{ extending } \{a \subseteq \mu : a \text{ finite or cofinite}\} \text{ such that } a \in \mathbb{B} \Rightarrow |a| < \theta \vee |\mu \setminus a| < \theta\Big\}.$$

4) $\mathrm{sa}^+(\mu, \theta) := \sup\big\{|\mathbb{B}|^+ : \mathbb{B}$ is as in (3)$\big\}$.
5) $\mathrm{sa}^*(\theta) := \min\{\lambda : \mathrm{cf}(\lambda) \geq \theta$, and $\mu < \lambda \Rightarrow \mathrm{sa}^+(\mu, \theta) \leq \lambda\}$.

That is, by the proof of Theorem 1.1:

Claim 4.2. *If $\theta = \mathrm{cf}(\theta) > \aleph_0$ and $\mathbb{B}$ is a superatomic Boolean Algebra with no automorphism moving $\geq \theta$ atoms, then $|\mathbb{B}| < \mathrm{sa}^+(\beth_3(<\theta))$; moreover,*

$$|\mathbb{B}| < \mathrm{sa}^+(\beth_2(\mathrm{sa}^*(\theta))).$$

Discussion 4.3. 1) Now consistently $\mathrm{sa}(\aleph_1) < 2^{\aleph_1}$. Why? Because [She99, 8.1] shows the consistency of a considerably stronger statement. It proves that (e.g.) if we start with $\mathbf{V} \models \mathsf{GCH}$ and $\mathbb{P}$ is adding $\aleph_{\omega_1}$ Cohen Reals, then in $\mathbf{V}^{\mathbb{P}}$, ($2^{\aleph_0} = \aleph_{\omega_1} < 2^{\aleph_1} = \aleph_{\omega_1+1}$ and) among any $\aleph_{\omega_1+1}$ members of $\mathcal{P}(\omega_1)$, there are $\aleph_{\omega_1+1}$ which form an independent family. (I.e. any finite nontrivial Boolean combination of them is nonempty; in other words, "$\mathcal{P}(\omega_1)$ has $\aleph_{\omega_1+1}$-free pre-caliber" in Monk's question definition.) Not surprising; this is the same model for "no tree with $\aleph_1$ nodes has $2^{\aleph_1}$ branches" in [Bau70].
2) So the bound $\beth_4(\theta)$ is not always the right one, though this needs the use of more complicated functions.
3) We have not looked at the question: does the use of $\mathrm{sa}^*(\theta)$ in claim 4.2 really help?

Claim 4.4. *Assume*
- (a) $\Upsilon = \Upsilon^{<\Upsilon} < \mu = \mathrm{cf}(\mu) < \chi$
- (b) $\mathrm{cf}(\chi) = \mu$, $(\forall \alpha < \chi)\big[|\alpha|^\mu < \chi\big]$, *and* $(\forall \alpha < \mu)\big[|\alpha|^{<\Upsilon} < \mu\big]$.
- (c) $\mathbb{Q}$ *is a forcing notion of cardinality* $< \chi$ *such that in* $\mathbf{V}^{\mathbb{Q}}$, μ *is a regular cardinal and* $(\forall a \in [\chi]^{<\mu})(\exists b)\big[a \subseteq b \in ([\chi]^{<\mu})^{\mathbf{V}}\big]$.
- (d) $\mathbb{P} := \{f : f$ *a partial function from* χ *to* $\{0,1\}$ *of cardinality* $< \Upsilon\}$, *ordered by inclusion (that is, adding a* χ Υ*-Cohen).*

Then in $\mathbf{V}^{\mathbb{Q}\times\mathbb{P}}$ *we have* ($2^\Upsilon = 2^{<\mu} = \chi$, $2^\mu = \chi^\mu = (\chi^\mu)^{\mathbf{V}} m$ *and)* $\mathrm{sa}(\mu) = \chi < 2^\mu$. *Moreover, the Boolean Algebra* $\mathcal{P}(\mu)$ *has* χ^+*-free pre-caliber.*

Proof. Work in $\mathbf{V}^{\mathbb{Q}}$ like [She99, 8.1], not using "$\mathbb{P}$ is σ-complete" (as it may fail in $\mathbf{V}^{\mathbb{Q}}$). $\square_{4.4}$

On the other hand,

Claim 4.5. *Assume* $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ *satisfies* $\lambda_{n+1} = \min\{\lambda : 2^\lambda > 2^{\lambda_n}\}$. *Then for infinitely many n-s, for some* $\mu_n \in [\lambda_n, \lambda_{n+1})$, *we have* $\mathrm{sa}(\mu_n) = 2^{\mu_n} = 2^{\lambda_n}$. *(In fact,* $\mathrm{sa}^+(\mu_n) = (2^{\mu_n})^+ = (2^{\lambda_n})^+$ *except possibly when* $\mathrm{cf}(2^{\lambda_n}) \leq 2^{\lambda_{n-1}}$.*)*

Proof. By [She96, 3.4] we have $\mu_n \in [\lambda_n, \lambda_{n+1})$ for infinitely many n-s, and for every regular $\chi \leq 2^{\lambda_n} = 2^{\mu_n}$, a tree with $\leq \mu_{n+1}$ nodes, λ_n levels and $\geq \chi$-many λ_n-branches. $\square_{4.5}$

Conclusion 4.6. 1) *Assume θ is strong limit,* $\theta > \mathrm{cf}(\theta) = \aleph_0$ *and* $\mathrm{Pr}(2^{2^\theta}, \theta)$ *and* $\lambda < \mathrm{sa}^+(\beth_3(\theta))$. *Then*

> *There is a superatomic Boolean Algebra without any automorphism moving* $\geq \theta$ *atoms such that* $\mathbb{B}$ *has cardinality* λ *(and has* $\beth_3(\theta)$ *atoms*[10]*).*

2) *Assume* $\mathrm{Pr}(\beth_2, \aleph_1)$ *and* $\lambda < \mathrm{sa}^+(\beth_3)$. *Then* $(*)_{\theta,\lambda}$ *holds.*

Proof. 1) Use 3.10 and 2.1.

2) Similarly, only replace 3.10 by a parallel claim. $\square_{4.6}$

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[10] We can allow fewer atoms and fewer elements.