# VII [Sh g7]

## STRONG COVERING LEMMA AND CH IN $V[R]$

### §0 Introduction

We prove a strengthening of the covering lemma, not using the fine structure theory (only some well known consequences, see Theorem 0.2). We prove it essentially in all cases in which the covering lemma holds.

This, essentially, is Chapter XIII, sections 1-4 of "Proper Forcing" [Sh-b] (the other sections, 5, 6, are superseded by the other material in this book). My interest in the subject stems from Abraham's (see below), and the last spark were discussions with Harrington and Woodin; and Harrington's willingness to hear the proof while being done. When revising [Sh-b], I was told it does not fit there (though see remark below on connection with properness), not to say that the proof of $\aleph_\omega^{\aleph_0} < \aleph_{[2^{\aleph_0}]^+}$ in [Sh-b, XIII,§5,§6] was misplaced. As the proofs here inspire the proof of $\aleph_\omega^{\aleph_0} < \aleph_{[2^{\aleph_0}]^+}$ (i.e. reconstructing a submodel $M$ by the characteristic function) and are combinatorial in character, we hope it will be more welcomed here. Note that the main problem here is very close to

$$\min\{|S| : S \subseteq \mathcal{S}_{<\kappa}(\lambda) \text{ is stationary}\},$$

which plays an important role in the rest of the book, but is not identical. The characteristic function of a model which has a major role here is used, also for example in [Sh371,§1], a difference being that here we use squares, in other places in the book we use weaker principles which holds in more general circumstances.
The changes compared with [Sh-b, XIII,§1-4] are minor — local improvement in presentation (hopefully) and adding 0.5, 4.18.

The neatest case of the strong covering lemma is

**Theorem 0.1** *Assuming $0^\#$ does not exist (in $V$), $A \subseteq \mathrm{Ord}^V$. If $\aleph_2^V = \aleph_2^{L[A]}$, $M$ a model in $V$ with countably many finitary functions whose set of elements is an ordinal $\alpha$ <u>then</u> for every $b \subseteq \alpha$ there is a set $a \subseteq \alpha$, which belongs to $L[A]$ and is closed under the functions of $M$, $b \subseteq a \subseteq \alpha$, and in $V$, $|a| \leq |b|$.*

The theorem is really much more general, it speaks on a pair of universes $W \subseteq V$, and uses three hypotheses which are known to hold in the case above: the usual covering lemma, the existence of squares and the existence of scales (for successor of singular cardinals, see §1 for the definitions; follows from GCH in the smaller universe). Also the conclusion is stronger: for regular $\kappa < \lambda < \lambda^*$, ($\kappa > \aleph_0$ for simplicity) and ordinal $\alpha$ player I has a winning strategy in the following game of length $\lambda$ :

in the $i$th move, player I chooses $a_i \subseteq \alpha$, $|a_i|^V < \lambda^*$, $\bigcup_{j<i} b_j \subseteq a_i$ and player II chooses $b_i \subseteq \alpha$, $|b_i|^V < \lambda^*$, $\bigcup_{j\leq i} a_j \subseteq b_i$.

In the end player I wins the play if for some closed unbounded $C \subseteq \lambda$ we have: $\delta \in C$ & $\mathrm{cf}\delta = \kappa \Rightarrow \bigcup_{i<\delta} a_i \in W$.

We can conclude that for example, if $0^\# \notin V$, then any forcing notion satisfies quite strong properness condition. I.e. let $G \subseteq P$ be generic over $V$; we know that, for given cardinal $\chi$ and $x \in H^{V[G]}(\chi)$, there are (quite many) $N \prec (H^{V[G]}(\chi)), \in, <^*_\chi, H^V(\chi))$ such that $x \in N$, $N \cap H^V(\chi) \in V$, so there is $q \in G$ which forces this and forces a value to $N \cap H^V(\chi)$; hence in $V$, $q$ is $(N \cap H^V(\chi), P)$-generic. (Of course, this does not say that for any $N' \prec (H(\chi)^V, \in, <^*_\chi)$ we can find such a condition $q$). For example there is such an $N$ which in $V[G]$ has cardinality $\aleph_2^{V[G]}$. This was the rationale for putting this in [Sh- b].

The problem arises as follows: Jensen and Solovay [JS] asked how adding a real can affect a universe.

Now adding $0^\#$ to $L$ causes the collapsing of many cardinals, and they knew that adding some real by forcing may collapse many cardinals; (later in Beller, Jensen and Weltch [BJW] much more radical results were proved: if $V$ satisfies GCH, then there is a generic extension of $V$ (by a class forcing) which preserves cardinalities and has the form $L[a]$) (first it was assumed $0^\# \notin L$). See more on this in Shelah Stanley [ShSt340]. Still $L[a]$ always satisfies GCH. So it was natural to ask, which Jensen and Solovay [JS] do:

**Problem 0.2** If $W$ satisfy $GCH$, $V = W[r]$, $r$ a real, $V$ and $W$ have the same cardinals, does $V$ satisfy CH?

There are also several other variants; for example,

**Problem 0.3** (1) If $W$ satisfies CH, $V = W[r]$, $r$ a real and $\aleph_1^V = \aleph_1^W$ then does $V$ satisfy CH?

(2) Ask in addition that $V$, $W$ have the same cardinals $< 2^{\aleph_0}$, <u>and/or</u> $W$ satisfies GCH

Abraham [A] was interested in this problem, he proved that the conclusion of 0.1 implies a positive answer to the question 0.2, and the author notes 0.1 if $\alpha < \aleph_\omega$. Harrington and Van Liere have similar results, parallely. Abraham [A] have conjectured 0.1 when $V$ and $L$ have the same cardinals. He also gave another application:

If $L[A]$, $L[B]$ have no <u>non-constructible</u> reals then $L[A, B]$ have no non-constructible reals provided that $\aleph_1^{L[A,B]} = \aleph_1^V$.

Just before the present work was done Shelah and Woodin [ShWo159] proved the consistency of negatives answer of problems 0.2, 0.3. For example adding a real to a universe $V$ satisfying GCH may blow up the continuum while not collapsing cardinals, starting with a universe $W$ with enough measurable cardinals; hence, answering 0.2 negatively; the other extreme variant is from the consistency of ZFC we can get $V = W[r]$, with $W$ satisfying CH, $\aleph_1^V = \aleph_1^W$ and $\left(2^{\aleph_0}\right)^V$ arbitrarily large, (i.e, answering problem 0.3(1)). Here, using the strong covering lemma we get several complimentary results, so we know which large cardinals are necessary for which variant; for some variants we know exactly, and for some reasonable lower and upper bound. This is done in section 4, and one of the cases (see 4.11) involve proving somewhat more than the strong covering lemma.

The cases in which we do not have exact results are:

(A) For the first result, (for 0.2) a measurable cardinal is necessary, but Shelah and Woodin [ShWo159] use $\left(2^{\aleph_0}\right)^V$ many; we need a suitable inner model so maybe Mitchell [Mi] work can help to close the case.

(B) The existence of $V = W[r]$, $\aleph_1^V = \aleph_1^W$, $W$ satisfies GCH, but in $V$, CH fails. We need an inaccessible, and a 2-Mahlo cardinal suffices.

(C) For problem 0.2 when $W$ satisfies GCH, $2^{\aleph_0} = \aleph_n$ in $V$, $2 < n < \omega$, $0^\#$ is necessary but $\aleph_n$ measurables suffices.

The obvious approach to the strong covering lemma seemed to be to redo the covering lemma more carefully (and so it was thought); however, this is not our solution. We rather prove by induction on $\alpha$ the statement described above, using only some principles which follows and holds in many other situations.

After this work, two beautiful related covering theorems were proved. Carlson proved a stronger theorem from a stronger assumption: if $0^\# \notin V$, any increasing sequence of uncountable regular length of sets of ordinals from $L$ belongs to L. Magidor [Mg3] proved that any somewhat closed submodel of $(L_\alpha, \in)$ is the union of $\leq \aleph_0$ sets from $L$ if $0^\# \notin V$ or at least the core model, $K$, has no Erdös cardinal.

* *

Another question is due to Mathias [M2].

**Question 0.4** Can $V$ satisfy GCH, $A \subseteq \aleph_{\omega_1}$, $V[A]$ has the same cardinals as $V$ and in $V[A]$, $2^{\aleph_0} > \aleph_{\omega_1}$, $\aleph_1^{V[A]} = \aleph_1^V$?

Note that if we replace $\aleph_{\omega_1}$ by a regular cardinal, the answer is negative, and if we replace it by a singular cardinality of cofinality $\aleph_0$, such as $\aleph_\omega$, the answer is positive. By the strong covering lemma if $0^\# \notin V$, or even if $V$ has no inner model with a measurable the answer is no. In fact even if $0^\# \notin L[A]$, $V \models$"$(\forall \alpha < \omega_1)\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$", $\aleph_\alpha^V = \aleph_\alpha^{V[A]}$ for $\alpha = \omega_1$, and for arbitrarily large $\alpha < \aleph_{\omega_1}$, then $V[A] \models$"$2^{\aleph_0} \leq \aleph_{\omega_1}$". It seemed very persuasive that using the inner models for hyper-measurable (see Mitchell [Mi]) we can get stronger inner models for that question (and get the relevant exact equi-consistency result for the question of violating CH by adding a real mentioned above).

Recently by [Sh400], if we replace $\omega_1$ by $\omega_4$, the answer is no. Really a negative answer of 0.4 follows if we can prove in ZFC:

$$(\forall \delta < \omega_1)(\delta \text{ limit } \Rightarrow \mathrm{pp}\aleph_\delta < \aleph_{\omega_1}].$$

Both follows, by the next theorem (see more in [Sh400,§3]).

**Theorem 0.5** *(1) Assume $V$ a model of set theory satisfying the* GCH, $\lambda$ *a strong limit cardinal, $A \subseteq \lambda$ (not in $V$) and $V[A]$ a model of set theory with the same cardinals $\leq \lambda^+$ and*

*$(*)$ In $V[A]$, there is a stationary $S \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$ such that $|S| \leq \lambda$.*

*Then in $V[A]$, $2^{\aleph_0} \leq \lambda$.*

*(2) Assume $V$ a model of set theory, $\lambda$ a strong limit cardinal, $\kappa < \lambda$, $A \subseteq \lambda$ (not in $V$), $V[A]$ is a model of set theory and $(\kappa^+)^V$, $\lambda$, $(\lambda^+)^V$ are cardinals also in $V[A]$ and*

*$(*)$ in $V[A]$, there is a stationary subset of $\mathcal{S}_{\leq \kappa}(\lambda)$ of cardinality $\leq \lambda$.*

*Then in $V[A]$, $\lambda^\kappa \leq \lambda$.*

**Remark 0.5A** The assumption $(*)$ holds for example $\lambda = \aleph_{\omega_4}$ by [Sh400, 4.4 + 3.7]. The proof is similar to that of 4.10.

**Proof:** 1) Let $\mathfrak{A} = \Big(H(\lambda^+)^{V[A]}, H(\lambda^+)^V, A, \in, <^*_\lambda\Big)$ (where $<^*_{\lambda^+} \in V$ is a well ordering of $H(\lambda^+)^V$).

We can represent $\mathfrak{A}$ (in $V[A]$) as an increasing continuous chain $\mathfrak{A}_i$ (for $i < \lambda^+$), $\|\mathfrak{A}_i\|^{V[A]} < \lambda^+$, (because $V[A] \models 2^\lambda \leq \lambda^+$). Similarly in $V$, $H(\lambda^+) = \bigcup_{i<\lambda^+} W_i$, $W_i$ increasing continuous, $|W_i| = \lambda < \lambda^+$,

$$\langle W_i : i < \lambda^+ \rangle \in V.$$

In $V[A]$ the set $\{i < \lambda^+ : H(\lambda^+)^V \cap \mathfrak{A}_i = W_i\}$ is a club of $\lambda^+$, so for some club $E \in V[A]$ of $\lambda^+$ for every $i \in E$, $\mathfrak{A}_i \prec \mathfrak{A}$ and $H(\lambda^+)^V \cap \mathfrak{A}_i = W_i$. Let $\bar{f} = \langle f_i : i < \lambda^+ \rangle \in V$ be such that $f_i$ is a one to one function from $\lambda$ onto $W_i$.

Now for every $r \in ({}^\omega 2)^{V[A]}$ we can find $i_r \in E$ such that $r \in \mathfrak{A}_{i_r}$, and a countable elementary submodel $(N_r, f^r)$ of $(\mathfrak{A}_{i_r}, f_{i_r})$ to which $r$ belongs, and $N_r \cap \lambda \in S$. Let $\mu_r < \lambda$ be such that $N_r \cap H(\lambda)^{V[A]} \subseteq H(\mu_r)^{V[A]}$, let $M_r$ be the elementary sub-model of $\big(H(\lambda^+)^V, f_{i_r}, \in, <^*_{\lambda^+}\big)$ with universe the Skolem Hull of $H(\mu_r)^V \cup \{f_{i_r}\}$ (note: $<^*_{\lambda^+} \in V$ is a well ordering of $H(\lambda^+)^V$). Clearly $M_r \in V$, and $\|M_r\| \leq |H(\mu_r)^V| < \lambda$; as in $V$, $\lambda$ is strong limit the number of isomorphism types of possible $M_r$ is $\leq \lambda$. Also the number of possible $N_r \cap \lambda$ is $\leq |S| \leq \lambda$ (and the number of possible $\mu_r$'s is $\leq \lambda$) so if the conclusion fails for some real $r$ the following set has cardinality $\lambda^+$ (in $V[A]$):

$$R =: \left\{ s \in ({}^\omega 2)^{V[A]} : M_s \cong M_r, \mu_s = \mu_r, N_s \cap \lambda = N_r \cap \lambda \right\}$$

So it is enough to prove (remember $N_r$ is countable):

$(*)$ if $s \in R$, then $s \in N_r$.

As $s \in R$ there is an isomorphism $g_s$ from $M_r$ onto $M_s$, it is unique as $M_r$ satisfies extensionality $\left(\text{being} \prec \left(H(\lambda^+)^V, f_{i_r}, \in, <^*_{\lambda^+}\right)\right)$, and belongs to $V$ as $M_r$, $M_s$ belong to $V$. Clearly $g_s$ is necessarily the identity on $H\left(\mu_r\right)^V$ (as it is a transitive subset of $M_r \cap M_s$). Also as

$(\alpha)$ $N_r \cap \lambda = N_s \cap \lambda \subseteq H(\mu_r)^V$ (an assumption) and

$(\beta)$ $N_r \cap H(\lambda^+)^V = \{f_{i_r}(\alpha) : \alpha \in N_r \cap \lambda\}$ (as $(N_r, f_i^r) \prec (\mathfrak{A}_{i_r}, f_{i_r})$ and choice of $f_{i_r}$)

clearly $g_s$ maps $H\left(\lambda^+\right)^V \cap N_r$ onto $H\left(\lambda^+\right)^V \cap N_s$.
Also $g_s\left(A^{N_r}\right) = A^{N_s}$ as $A \subseteq \lambda$. Now $N_r$ being

$$\prec \mathfrak{A} = \left(H(\lambda^+)^{V[A]}, H(\lambda^+)^V, A, \in, <^*_{\lambda^+}\right),$$

"think" that "$H(\lambda^+)^{V[A]}$ is $H(\lambda^+)^V[A]$". But constructing $H(\lambda^+)^{V[A]}$ as $H(\lambda^+)^V$ extended by A is a unique process, so $g_s$ can be extended to an isomorphism from $N_r$ onto $N_s$, but necessarily $s = g_s^{-1}(s)$, so $s = g_s^{-1}(s) \in N_r$ as required.
2) Similarly (note that w.l.o.g. $V = L[B]$ for some $B \subseteq \lambda^+$, hence $V$, $V[A]$ satisfy $2^\lambda = \lambda^+$). $\square_{0.5}$
See more in [Sh410].

## §1 The Strong Covering Lemma: Definition and implications

This section defines our central notions and gives the easy relevant facts.

**Context 1.1** Let $V$ be a universe (of set theory), $W$ a transitive class of $V$ which is a model of ZFC (with the same ordinals) so that $W \subseteq V$. Writing for example, $W_0 \subseteq W_1$ we implicitly assume the corresponding hypothesis.

**Definition 1.2** The pair $(W, V)$ satisfies the $\lambda$-covering lemma ($\lambda$ a cardinal of $V$) if for every set $a \in V$, $a \subseteq \lambda$ (or $a \subseteq W$) of power $< \lambda$ (in $V$), <u>there is</u> a set $b \in W$ such that $a \subseteq b$ and $V \models$"$|b| < \lambda$".
If we omit $\lambda$ this means "for every $\lambda \geq \aleph_2^V$". Without loss of generality $a$, $b$ are sets of ordinals.

**Definition 1.3** The pair $(W, V)$ satisfies the strong $(\lambda, \alpha)$-covering lemma ($\lambda$ regular cardinal of $V$, $\alpha$ an ordinal) if for every model $M$ in $V$ with universe $\alpha$ (always with countably many finitary functions and relations) and $a \subseteq \alpha$, $|a| < \lambda$ (in $V$), there is $b \in W$, $a \subseteq b \subseteq \alpha$, $b$ an elementary submodel of $M$ (i.e., the set of elements of such a submodel) and

$$V \models \text{“}|b| < \lambda\text{”}.$$

Instead of saying for every $\alpha$, we write $\infty$ instead of $\alpha$, or write “the strong $\lambda$-covering”.

Of course, we can replace $\alpha$ by any set in $W$ of the same power, so w.l.o.g. $\alpha$ is a cardinal of $W$; and assume $M$ has Skolem functions so it is enough that $b$ is a submodel.

**Definition 1.4**
(1) The pair $(W, V)$ satisfies the strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma (where $\kappa \leq \lambda \leq \lambda^*$ are regular cardinals in $V$, $\alpha$ an ordinal) if player I wins the following game (in $V$, i.e., has a winning strategy) which we call
“the $(\lambda^*, \lambda, \kappa, \alpha)$-covering game”:
The play last $\lambda$ moves, in the ith move, player I chooses $a_i \in V$, a subset of $\alpha$ of power $< \lambda^*$ (in $V$), which includes $\bigcup_{j<i} b_j$, and player II chooses $b_i$, a subset of $\alpha$ of power $< \lambda^*$ which include $\bigcup_{j \leq i} a_j$.
Player I wins if there is a a closed unbounded subset $C \subseteq \lambda$ such that for every $i \in C \cup \{\lambda\}$, $\mathrm{cf}(i) = \kappa \Rightarrow \bigcup_{j<i} a_j \in W$ (if $\kappa = \lambda$, only $i = \lambda$ count).
We omit $\alpha$ if we mean “for every $\alpha$”.
(2) Let $D$ be a filter on $\{i : i \leq \lambda\}$ i.e., on $\lambda + 1$ and $\lambda^*$, $\lambda$, $\alpha$ are as before. The pair $(W, V)$ satisfies the strong $(\lambda^*, \lambda, D, \alpha)$-covering lemma if player I wins in the following game (i.e. I has a winning strategy in $V$) which we call
“the $(\lambda^*, \lambda, D, \alpha)$-covering game”:
The play last $\lambda$ moves; in the ith move player I chooses $a_i \in V$ a subset of $\alpha$ of power $\lambda^*$ (in $V$) which includes $\bigcup_{j<i} b_j$ and then player II chooses $b_i$ a subset of $\alpha$ of power $< \lambda^*$ which include $\bigcup_{j \leq i} a_j$. Player I wins the game if $\left\{ i \leq \lambda : \bigcup_{j<i} a_j \in W \right\} \in D$.

**Remark 1.4A** The two popular cases are

$$D = \{A \subseteq \lambda + 1 : \lambda \in A\}$$

(then we get the $(\lambda^*, \lambda, \lambda, \alpha)$-covering game) and

$$D = \left\{ A \subseteq \lambda + 1 : \text{ there is a club } C \subseteq \lambda \text{ such that } \{\delta \in C : \mathrm{cf}\delta = \kappa\} \subseteq A \right\}$$

(then we get the $(\lambda^*, \lambda, \kappa, \alpha)$-covering game).

**Claim 1.5** (1) The strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma implies the strong $(\lambda^*, \alpha)$- covering lemma when $[\lambda^* > \lambda \ \text{ or } \ \lambda > \kappa]$ and it implies the strong $((\lambda^*)^+, \alpha)$-covering lemma when $\lambda^* = \lambda = \kappa$ ($\lambda^+$-in $V$'s sense).

(2) The strong $(\lambda^*, \lambda, \kappa, \alpha_0)$-covering lemma implies the strong $(\lambda^*, \lambda, \kappa, \alpha_1)$-covering lemma when $\alpha_0 \geq \alpha_1$.

(3) If $W_1 \subseteq W \subseteq V \subseteq V_1$ are universes of set theory with the same ordinals then:

(a) The strong $(\lambda, \alpha)$-covering lemma for $(W_1, V_1)$ implies the strong $(\lambda, \alpha)$-covering lemma for $(W, V)$.

(b) The strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma for $(W_1, V)$ implies the strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma for $(W, V)$ (see 1.5A).

(4) In the $(\lambda^*, \lambda, \kappa, \alpha)$-covering game, it does not hurt any player to choose bigger sets as long as they are subsets of $\alpha$ of power $< \lambda^*$ (i.e., if he has a winning strategy, increasing the sets he still wins).

(5) If $\lambda_1 \leq \lambda_2 \leq \lambda_3$, and $(W, V)$ satisfies the [strong] $(\lambda_1, \lambda)$-covering lemma for every $\lambda < \lambda_2$, and also the [strong] $(\lambda_2, \lambda_3)$-covering lemma then $(W, V)$ satisfies the [strong] $(\lambda_1, \lambda_3)$-covering lemma.

(6) If $(W_1, W_2)$ satisfies the $(\lambda_1, \lambda_3)$-covering lemma, and $(W_2, W_3)$ satisfies the $(\lambda_1, \lambda_3)$-covering lemma, then $(W_1, W_3)$ satisfies the $(\lambda_1, \lambda_3)$-covering lemma. (Where $W_1 \subseteq W_2 \subseteq W_3$, $\lambda_1 \leq \lambda_3$).

(7) We can replace $\kappa$ by a filter $D$ on $\lambda + 1$ in parts 1), 2), 3), 4).

**Proof:** Left to the reader being trivial.

**Remark 1.5A** Why in 1.5(3)(b) we speak on $(W_1, V)$ and not $(W_1, V_1)$? The winning strategy may be missing from $V_1$ (also the club $C$).

**Definition 1.6** We say $W$ has a square if for any cardinal $\mu$ there are sets $C_\delta (\delta < \mu, \ \delta$ singular limit) such that:

(a) $C_\delta$ is a closed unbounded subset of $\delta$ of order type $< \delta$.

(b) If $\gamma$ is a limit ordinal and is in $C_\delta$, then

$$\sup(C_\delta \cap \gamma) = \gamma \text{ and } C_\gamma = C_\delta \cap \gamma.$$

**Claim 1.7** If $W$ has square, $\lambda \leq \mu$, let $S^\mu_{<\lambda} = \{\delta < \mu : \delta > \lambda, \mathrm{cf}\delta < \lambda\}$, then we can find $\langle C_\delta : \delta \in S^\mu_{<\lambda} \rangle$ such that:

(a) $C_\delta$ is a closed unbounded subset of $\delta$ of order type $< \lambda$.

(b) If $\gamma$ is a accumulation point of $C_\delta$ then $C_\delta \cap \gamma = C_\gamma$.

**Proof:** Let $\langle C_\delta : \delta < \mu$ a singular, limit ordinal$\rangle$ be as in Definition 1.6. W.l.o.g. $\delta > \lambda \Rightarrow C_\delta \cap \lambda = \emptyset$. For each $\delta$ for which $C_\delta$ is defined, let $f_\delta$ be the function with domain $C_\delta$, defined by $f_\delta(\alpha) = \mathrm{otp}(\alpha \cap C_\delta)$. Define $C^1_\delta$ by induction on $\delta < \mu$: if $C_\delta$ is not defined also $C^1_\delta$ is not defined, if $C_\delta$ is defined but $C_{\mathrm{otp}(C_\delta)}$ is not defined or $\mathrm{otp}(C_\delta) < \lambda$ let $C^1 = C_\delta$, and if $C_\delta$, $C_{\mathrm{otp}(C_\delta)}$ are defined but $\mathrm{otp}(C_\delta) \geq \lambda$, we let

$$C^1_\delta = \{\alpha \in C_\delta : f_\delta(\alpha) \in C^1_{\mathrm{otp}(C_\delta)}\}.$$

Now check that $\langle C^1_\delta : \delta \in S^\mu_{<\lambda}\rangle$ is as required. $\square_{1.7}$

**Definition 1.8** If the conclusion of 1.7 holds, (for every $\mu$) we say $W$ has $\lambda$-squares, and if this holds for every $\lambda \geq \aleph_2$, we say $W$ has squares.

**Claim 1.9** (1) If the pair $(W, V)$ satisfies the $\lambda$-covering lemma, ($\lambda$ a cardinal in $V$) then for every limit ordinal $\delta$ : if its cofinality in $W$ is $\geq \lambda$ then its cofinality in $V$ is $\geq \lambda$ (the inverse is trivial).

(2) If $W$ has $\lambda$-squares, $W \subseteq V$ and $(W, V)$ satisfies the $\lambda$-covering lemma, then $V$ has $\lambda$-squares.

**Definition 1.10** We say that the universe $W$ has scales if for every singular cardinal $\chi$, there is a set $G$ of $\chi^+$ functions, with domain

$$R_\chi = \{\theta : \theta < \chi \text{ regular}\},$$

$g(\theta) < \theta$ for $g \in G$, such that for every function $f$ satisfying

$$\mathrm{Dom}\, f \subseteq R_\chi,\ |\mathrm{Dom}\, f| < \chi \text{ and } (\forall\theta) f(\theta) < \theta,$$

there is $g \in G$, $f <^* g$ i.e.,

$$(\exists\sigma \in R_\chi)(\forall\theta)\big(\sigma < \theta \in \mathrm{Dom}\, f \to f(\theta) < g(\theta)\big).$$

If we restrict ourselves to one such $\chi$, we call this property "have $\chi^+$-scale".

**Remark 1.10A** It is easy to verify that if $W \models$GCH, then $W$ has scales.

**Claim 1.11** Let $(W, V)$ satisfy the covering lemma.

(1) If $W$ has $\lambda$-squares, $\lambda \geq \aleph^V_2$ regular in $V$, then $V$ has $\lambda$-squares.

(2) If $W$ has $\chi^+$-scale, $\chi$ a cardinal in $V$ (hence $\chi$ is singular in $W$, and $\chi^+$ in $W$'s sense is the successor of $\chi$ also in $V$) then $V$ has $\chi^+$-scale.

(3) If $W$ has squares then $V$ has squares.

(4) If $W$ has scales then $V$ has scales.

**Remark 1.11A** (1) The aim of 1.11 is that we will be able to get a strong covering lemma; for example for $(W, V)$ where $0^\# \notin V$, and not just for $(L, V)$.
(2) In 1.11 we can replace $\aleph_2$ by any other regular uncountable cardinal $\kappa$ of $V$ (if $(W, V)$ satisfies the $\lambda$-covering lemma for $\lambda \geq \kappa$ regular in $V$) and have other obvious variants.

**Proof:** Trivial.
For part 3) note that any universe $W$ has $\aleph_1$-squares: for every limit $\delta$ of cofinality $\aleph_\delta$ choose $C_\delta \subseteq \delta$ an unbounded subset of order type $\omega$.

**Definition 1.12** Let $D$ be a filter on $\lambda + 1$, $\mathrm{cf}\lambda > \aleph_0$ and always

$$[\alpha < \lambda \Rightarrow \lambda + 1 \backslash \alpha \in D].$$

(1) $D$ is called weakly normal = satisfies the $(\lambda^*, \lambda)$-demand 0 when: if $A_\zeta \in D$ for $\zeta < \lambda$ and $\zeta < \xi < \lambda \Rightarrow A_\xi \subseteq A_\zeta$ <u>then</u> $\{\zeta \leq \lambda : (\forall \xi < \zeta)[\zeta \in A_\xi]\}$ belongs to $D$.
(2) $D$ satisfies the $(\lambda^*, \lambda)$-demand 1 if: for every club $C$ of $\lambda$, $C \cup \{\lambda\} \in D$ and $\lambda^* > \lambda \Rightarrow C \in D$.
(3) $D$ satisfies the $(\lambda^*, \lambda)$-demand 2 when: if $C_\delta$ is a club of $\delta$ for every limit ordinal $\delta \leq \lambda$ of uncountable cofinality then

$$\cup\{C_\delta \cup \{\delta\} : \delta \leq \lambda,\ \aleph_0 < \mathrm{cf}\delta < \lambda^*\} \cup \{\alpha < \lambda : \mathrm{cf}\alpha > \aleph_0\} \in D.$$

(4) $D$ is said to satisfy the $(\lambda^*, \lambda)$-demand 3 when for every $\kappa = \mathrm{cf}\kappa < \lambda$ we have: $\{\delta : \delta < \lambda, \mathrm{cf}\delta \neq \kappa\} \in D$ <u>or</u> $\lambda^* > \lambda$ <u>or</u> if $C_\delta$ is a club of $\delta$ for each limit $\delta < \lambda$ then $\cup\{C_\delta \cup \{\delta\} : \delta < \lambda, \mathrm{cf}\delta \neq \kappa\} \in D$.

**Fact 1.13** Let $\kappa \leq \lambda \leq \lambda^*$ be regular and $\lambda > \aleph_0$.
(1) If $D = \{A \subseteq \lambda + 1 : \lambda \in A\}$ <u>then</u> $D$ is a filter, $\lambda$-complete satisfying the $(\lambda^*, \lambda)$ demand 0 and: $D$ satisfies the $(\lambda^*, \lambda)$-demand 1 iff $\lambda^* = \lambda$ and $D$ satisfies the $(\lambda^*, \lambda)$-demand 2 iff $\lambda^* > \lambda$.
(2) If $\kappa = \mathrm{cf}\kappa < \lambda$ and

$$D_{\lambda,\kappa} = \big\{A \subseteq \lambda + 1 : A \cup \{\delta < \lambda : \mathrm{cf}\delta \neq \kappa\} \text{ contains a club of } \lambda\big\}$$

<u>then</u>
($\alpha$) $D$ is normal and ($\lambda$-complete) and it satisfies the $(\lambda^*, \lambda)$-demands 0, 1.
($\beta$) $D$ satisfies the $(\lambda^*, \lambda)$-demand 2 if $\lambda^* > \lambda$ or $\kappa > \aleph_0$ or every stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ reflect in some $\delta < \lambda$.
($\gamma$) $D$ satsifies demand 3 if $\lambda^* > \lambda$ or every stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ reflect in some $\delta < \lambda$.

(3) If $\lambda^* > \lambda$ and $D$ satisfies $(\lambda^*, \lambda)$-demand 1 then $D$ satisfies $(\lambda^*, \lambda)$-demand 2.

(4) If $\lambda > \aleph_1$, and

$$D = \{A \subseteq \lambda + 1 : \text{for some club } C \text{ of } \lambda \text{ for every } \delta \in C \text{ of uncountable cofinality we have } \delta \in A\}$$

then $D$ satisfies the $(\lambda^*, \lambda)$-demands 0, 1, 2.

(5) Let $(W, V)$ be a pair. Assume $D$ is as in (2), for every $\alpha$, we have: $(W, V)$ satisfies the strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma iff it satisfies the $(\lambda^*, \lambda, D, \alpha)$-covering lemma.

## §2 Proof of the Strong Covering Lemma

This section is the crux of the chapter. Our aim is, essentially to prove that strong covering lemmas hold when the covering lemma holds. We can get more from the proofs. We prove trivial cases of the strong covering lemma (2.1) and two inductive lemmas, aim at enabling us to prove the strong covering by induction on cardinals of $W$. The first (2.2) saying that we can advance from $\mu$ to $\mu^+$, and the second (which is the main proof) saying that we can advance to a limit cardinal $\mu$ (really the proof split to cases by $\mathrm{cf}^V(\mu)$, so in some cases we get a little more).

**Lemma 2.1** *(1) Suppose $\chi$ is a regular cardinal in $V$, $(W, V)$ satisfies the strong $(\lambda, \mu)$-covering lemma for every $\mu < \chi$, $\mu > \lambda$. Then $(W, V)$ satisfies the strong $(\lambda, \chi)$-covering lemma.*

*(2) If $\kappa \leq \lambda < \lambda^*$ are regular cardinals in $V$, then $(W, V)$ satisfies the strong $(\lambda^*, \lambda, \kappa, \lambda^*)$-covering lemma.*

**Proof:**

(1) If $\lambda = \chi$, $M$ a model with universe $\chi$ and countably many functions, then in $V$, for some $\alpha < \chi$, $\alpha$ is closed under the functions of $M$, so it exemplifies the conclusion of the strong covering lemma.

If $\lambda > \chi$ the strong $(\lambda, \chi)$-covering lemma is trivial.

If $\lambda < \chi$ we can deduce the desired conclusion by 1.5(5) and the case $\lambda = \chi$ above.

(2) The proof is similar. $\square_{2.1}$

**Lemma 2.2** *Suppose:*

*(1) $(W, V)$ satisfies the $\lambda^*$-covering lemma and $W$ has $\lambda^*$-squares.*

*(2) $(W, V)$ satisfies the strong $(\lambda^*, \lambda, D, \mu)$-covering lemma*[28]

*(3) $D$ satifies the $(\lambda^*, \lambda)$-demand 2 (see Definition 1.12(3)).*

*(4) $\lambda$ uncountable (in $V$).*

*Then $(W, V)$ has the strong $(\lambda^*, \lambda, D, \mu^+)$-covering lemma ($\mu^+$ - in $W$).*

[28]hence in $V$ $\lambda \leq \lambda^*$ are regular

**Remark:** Remember $\lambda^* \geq \lambda > \aleph_0$ are regular cardinals in $V$, $D$ a filter on $\lambda + 1$ and without loss of generality $\mu \geq \lambda^*$, (by 2.1(2), 1.5(2)).

**Proof:** Before really proving 2.2, we shall give two facts, which are trivial but basic for our proofs: an observation, and a claim. We shall use assumption (2) only in the actual proof of 2.2.

**Fact 2.2A** In $W$ for each ordinal $\alpha$ there is a model $M^0_\alpha = (\alpha, F^0_\alpha, G^0_\alpha, S^0_\alpha, CF^0_\alpha, H^0_\alpha, 0)$, $F^0_\alpha$ is a two place function from $\alpha$ to $\alpha$, such that for every $\beta < \alpha$, $F^0_\alpha(\beta, -)$ is a one to one mapping from $\beta$ onto $|\beta|^W$ (its cardinality in $W$); $G^0_\alpha(\beta, -)$ is its inverse (on $|\beta|^W$), $S^0_\alpha$ is the successor function, $CF^0_\alpha$ is a one place function giving the cofinality for limit ordinals, and predecessors for successor ordinals; $H^0_\alpha$ is a two place function, such that for $\beta$ limit $\langle H^0_\alpha(\beta, i) : i < CF^0_\alpha(\beta)\rangle$ is an increasing continuous sequence converging to $\beta$; 0 is an individual constant denoting 0, i.e. a zero-place function; for $\beta$ successor, $H^0_\alpha(\beta, 0) = |\beta|$, $H^0_\alpha(\beta, 1) = (|\beta|^+)^W$ if this value is $< \alpha$ and 0 otherwise.

**Notation:** We say $a \subseteq \alpha$ is a submodel of $M^0_\alpha$ if it is closed under the functions of $M^0_\alpha$; and $\mathrm{cl}(a, M^0_\alpha)$ is the closure of $a \cap \alpha$ under the functions of $M^0_\alpha$; similarly for $M^1_\alpha$ which is defined below.

**Fact 2.2B** If $W$ has $\lambda^*$-squares (remember, $\lambda^*$ is a regular cardinal in $V$) then there is $M^1_\alpha = (M^0_\alpha, C^{1,\alpha})$, $C^{1,\alpha}$ a two place function such that: there is (in $W$) a sequence $\langle C^1_\beta : \beta < \alpha, \mathrm{cf}\beta < \lambda^*\rangle$ as in Claim 1.7 (with $\lambda^*, \alpha$ here standing for $\lambda, \mu$ there), such that:

$C^{1,\alpha}(\beta, \beta)$ is the order type of $C^1_\beta$ (if defined)

$C^{1,\alpha}(\beta, i)$ is the ith element of $C^1_\beta$ (if it exists).

$C^{1,\alpha}(\beta + 1, C^{1,\alpha}(\beta, i)) = i$.

**Notation:** We usually omit the subscript $\alpha$ in the above functions.

**Observation 2.2C** If $\mu$ is a cardinal of $W$, $\mu^+$ its successor in $W$, $a \subseteq \mu^+$ a submodel of $M^0_{\mu^+}$ and $b \subseteq a$ is unbounded in $a$ (i.e. , $(\forall\zeta \in \alpha)(\exists\xi \in b)$ $[\zeta \leq \xi]$), then:

(1) $a = \mathrm{cl}\left([a \cap \mu] \cup b, M^0_{\mu^+}\right)$ hence

(2) if $a \cap \mu \in W$, $b \in W$ then $a \in W$.

**Proof:**

(1) As $a \cap \mu \subseteq a$, $b \subseteq a$ and $a$ is a submodel of $M^0_{\mu^+}$, trivially

$$\mathrm{cl}\big([a \cap \mu] \cup b, M^0_{\mu^+}\big) \subseteq a.$$

For the other inclusion assume $\zeta \in a$, hence there is $\xi \in b$, $\zeta \leq \xi$; if $\zeta = \xi$ there is nothing to prove, so let $\zeta < \xi$. Hence $F^0(\xi, \zeta) < \mu$ (as $|\xi| \leq \xi < \mu^+$)

and $F^0(\xi,\zeta) \in a$ (as $a$ is a submodel of $M^0_{\mu^+}$) hence $F^0(\xi,\zeta) \in a \cap \mu$. But $G^0(\xi, F^0(\xi,\zeta)) = \zeta$, and $\xi \in b$, so $\zeta \in \mathrm{cl}\big([a\cap\mu] \cup b, M^0_{M^+}\big)$, as required.
(2) Easy.

**Claim 2.2D** Suppose $D$ is a filter on $\lambda+1$ which satisfies the $(\lambda^*,\lambda)$-demand 2 (see Definition 1.12(3)) and $\lambda, \lambda^*$ are regular cardinals (in $V$), $\lambda \le \lambda^* \le \alpha$, $\alpha$ an ordinal and $W$ has $\lambda^*$-squares (so $C^1_\alpha$, $M^1_\alpha$ are well defined). Suppose further that (in $V$) $\langle a_\zeta : \zeta \le \lambda\rangle$ is an increasing continuous sequence of subsets of $\alpha$, $\zeta < \lambda^* \Rightarrow |a_\zeta| < \lambda^*$, each $a_\zeta$ a submodel of $M^1_\alpha$, $\sup(a_\zeta \cap \lambda^*) \subseteq a_{\zeta+1}$.Lastly suppose the closure (in the order topology on the ordinals) of $a_\zeta$ is included in $a_{\zeta+1}$ or at least (for a fixed $\delta \le \alpha$) $\bigwedge_{\zeta<\lambda} \sup(\delta \cap a_\zeta) \in a_\zeta + 1$. If $\delta \in a_\lambda$, $\mathrm{cf}(\delta) \ge \lambda^*$ <u>then</u> $S = \{\zeta \le \lambda : C^1_{\sup(\delta\cap a_\zeta)} \subseteq a_\zeta\}$ belongs to $D$.

**Proof:** Let $\delta(\zeta) = \sup(a_\zeta \cap \delta)$ for $\zeta \le \lambda$.

Assume $\lambda^* > \mathrm{cf}\zeta > \aleph_0$ and $\zeta \le \lambda$. Now clearly $\langle \delta(i) : i < \zeta\rangle$ is a (strictly) increasing continuous sequence converging to $\delta(\zeta)$, so as $C^1_{\delta(\zeta)}$ is a closed unbounded subset of $\delta(\zeta)$, $C^1_{\delta(\zeta)} \cap \{\delta(i) : i < \zeta\}$ is a closed unbounded subset of $\delta(\zeta)$. But $\delta(i) \in a_{i+1} \subseteq a_\zeta$ (for $i < \zeta$). Hence for a closed unbounded set of $i < \zeta$, $\delta(i) \in C^1_{\delta(\zeta)} \cap a_\zeta$. But $a_\zeta$ is a submodel of $M^1_\alpha$, and $a_\zeta \cap \lambda^*$ is an initial segment of $\lambda^*$ (see assumptions on $a_i$ : $\sup(\lambda^* \cap a_\xi) \subseteq a_{\xi+1}$). So by the definition of $M^1_\alpha$, for a closed unbounded set $E$ of limit $i < \zeta$, $C^1(\delta(i), \delta(i))$ belongs to $a_{i+1} \subseteq a_\zeta$, hence (see $M^1$'s definition)

$$\{\gamma : \gamma < \text{ the order type of } C^1_{\delta(i)}\} \subseteq a_{i+1} \subseteq a_\zeta,$$

hence (using $C^1(\delta(i),\gamma)$), $C^1_{\delta(i)} \subseteq a_{i+1} \subseteq a_\zeta$, and of course, $\delta(i) \in C^1_{\delta(\zeta)}$ and is even an accumulation point of $C^1_{\delta(\zeta)}$. By the definition of squares $C^1_{\delta(\zeta)} \subseteq a_\zeta$, and for $i$ an accumulation point of $E$, $C^1_{\delta(i)} \subseteq a_i$. So $\zeta \le \lambda$, $\aleph_0 < \mathrm{cf}\zeta < \lambda^*$ implies: $\zeta \in S$ and a club of $i < \zeta$ belongs to $S$. This clearly suffices.

So we have proved 2.2D. $\square_{2.2D}$

**Proof of 2.2:** By the hypothesis, player I has a winning strategy in the $(\lambda^*, \lambda, D, \mu)$-covering game, which we denote by $K_i(i<\lambda)$; i.e., if $b_i \subseteq \mu$ for $i < \lambda$, $|b_i|^V < \lambda^*$, <u>then</u> $a_i = K_i(b_0, b_1, ..., b_j \cdots)_{j<i}$ is a subset of $\mu$ of cardinality $< \lambda^*$, $b_j \subseteq a_i$ for $j < i$, and if in addition for $i < \lambda$ we have $\bigcup_{j\le i} a_j \subseteq b_i$ <u>then:</u>

$$\Big\{\delta \le \lambda : \bigcup_{j<\delta} a_j \in W\Big\} \in D.$$

Let us describe the winning strategy of player I in the $(\lambda^*, \lambda, D, \mu^+)$-covering game.

In the $\zeta$-th move, $a_j \subseteq b_j \subseteq a_i (j < i < \zeta)$ are given, player I let:
(i) $a^0_\zeta = \bigcup_{j<\zeta} b_j$
(ii) $a^1_\zeta = K_\zeta(b_0 \cap \mu, b_1 \cap \mu, ..., b_i \cap \mu, \cdots)_{i<\zeta}$
(iii) $a^2_\zeta = a^0_\zeta \cup a^1_\zeta \cup \{\sup(a^0_\zeta)\} \cup \{\gamma : \gamma < \sup(a^0_\zeta \cap \lambda^*)\}$
and he chooses $a_\zeta = \mathrm{cl}\big(a^2_\zeta, M^1_{\mu^+}\big)$. Note that $\sup(a^0_\zeta) < (\mu^+)^W$ as $|a^0_\zeta|^V < \lambda^*$ because $(W, V)$ satisfies the $\lambda^*$-covering lemma.

Let us show that this strategy is a winning one, so let $\langle a_i, b_i : i < \lambda\rangle$ be a play in which player I uses the strategy described above. Clearly by the choice of the $K_\zeta$'s, there is $C \in D$ such that if $\zeta \in C$ then

$$a^0_\zeta \cap \mu = \bigcup_{i<\zeta} b_i \cap \mu \in W.$$

Let $\delta(i) = \sup(a^0_i)$.

For any limit $\zeta \in C$, clearly $a^0_\zeta$ is a submodel of $M^1_{\mu^+}$, hence by Observation 2.2C, part 2 in order to prove $a^0_\zeta \in W$ it is enough to find an unbounded subset $b \subseteq a^0_\zeta$ as there i.e., $b \in W$; our $b$ here will be $C^1_{\delta(\zeta)}$ from Fact 2.2*B*. Hence it suffices to prove that for some $C' \in D$, $(C' \subseteq C)$ and for every $\zeta \in C'$ we have: $C^1_{\delta(\zeta)}$ is a subset of $a^0_\zeta$. By 2.2D we finish. $\square_{2.2}$

**Remark 2.2F** Note that if there are $\lambda^*$-squares then for each $\mu$ there is $\langle C^1_\delta : \lambda^* < \delta < \mu, \mathrm{cf}\delta < \lambda^*\rangle$ as required, with: otp $C^1_\delta$ not divisible by $\omega^2$ implies $C^1_\delta$ include some end segment of $\delta$.

**Lemma 2.3** *Suppose*

*(A) (1)* $(W, V)$ *satisfies the* $\lambda^*$*-covering lemma, and* $W$ *have* $\lambda^*$*-squares and have scales (at least for* $\theta \geq \lambda^* > \mathrm{cf}\theta$, $\theta$ *a* $W$*-cardinal).*
*(2)* $\mu > \lambda^*$ *is a limit cardinal (in* $W$*).*
*(3)* $(W, V)$ *satisfies the strong* $(\lambda^*, \lambda, D, \alpha)$*-covering lemma for every* $\alpha < \mu$ *(where* $D$ *is a filter on* $\{\zeta : \zeta \leq \lambda\}$ *and* $\aleph_0 < \lambda \leq \lambda^*$ *are regular cardinals in* $V$*).*

*(B) At least one of the following holds:*
*(4)* $\mathrm{cf}\mu < \lambda^*$ *and* $D$ *satisfies the* $(\lambda^*, \lambda)$*-demands 0, 1, 2, 3 and is* $\lambda$*-complete.*
*(5)* $\mathrm{cf}\mu \geq \lambda^*$*, and* $D$ *satisfies demands 0, 1, 2 and* $\aleph_0 < \lambda < \lambda^*$*,*

$$\{\xi : \zeta < \lambda, \mathrm{cf}\zeta > \aleph_0\} \in D, \ D \textit{ is normal}$$

*[i.e., if* $S_\zeta \in D$ *for* $\zeta < \lambda$ *then* $\{\zeta : (\forall \xi < \zeta)\zeta \in S_\xi\} \in D$*] and* $(W, V)$ *satisfies the* $\lambda$*-covering lemma and* $W$ *has* $\lambda$*-squares.*

*Then* $(W, V)$ *satisfies the strong* $(\lambda^*, \lambda, D, \mu^+)$*-covering lemma (*$\mu^+$ *in* $W$*'s sense).*

**Remark:**

(1) Suppose $\lambda = \lambda^*$. If $\lambda$ has the same successor in $V$ and $W$, the situation is much simpler as for example we can use $\lambda^*$-squares with every $C_\delta$ of order type $\leq \lambda$ (see 4.17).

(2) This lemma is the heart of the matter.

(3) The proof is broken to smaller parts. Part (A) of the assumption (i.e. (1), (2), (3)) is used freely but we shall say when we use an assumption from (B).

We work for a while in $W$, present some definitions and facts, and only later return to the lemma.

**Notation 2.3A** We let $R$ denote the class of regular cardinals of $W$, $R(\mu_1, \mu) = \{\chi \in R : \mu_1 < \chi < \mu\}$.
Let $T$ be (the class of) functions $f$, with domain a subset of $R$, $f(\chi) < \chi$.
We have two natural relations on $T$ :

(1) $f < g$ if Dom $f \subseteq$ Dom $g$ and $f(\chi) < g(\chi)$ for $\chi \in$ Dom $f$ (similarly $f \leq g$). This is a partial order.

(2) $f <^* g$ if Dom $f \subseteq$ Dom $g$, Dom $f$ has no last element, and for some $\chi_0 \in$ Dom $f$, and for every $\chi \geq \chi_0$,

$$[\chi \in \text{Dom } f \Rightarrow f(\chi) < g(\chi)]$$

($<^*$ is a partial order on each $T_I$ (see below)).

(3) If $I \subseteq R$ is a set with no last element,

$$T_I = T(I) = \{f \in T : \text{Dom } f \subseteq I \text{ and } \sup \text{Dom} f = \sup\ I\}$$

and $T(\mu_1, \mu) = T(R(\mu_1, \mu))$.

**Fact 2.3B** In the universe $W$ :

(1) If $f_j \in T_I$ for $j < \delta$, $\delta < \min I$, then there is $f_\delta \in T_I$, such that $f_i < f_\delta$ for every $i < \delta$.

(2) Assume $f_j \in T_I$ for $j < \delta$, $I$ has no last element and one of the following holds: $\delta < (\sup\ I)^+$ and $\sup(I)$ is singular <u>or</u> $\delta < \sup(I)$ <u>or</u> $\delta = \sup(I)$, $\sup(I)$ is regular but $I$ is a non-stationary subset of $\sup(I)$. <u>Then</u> there is $f_\delta \in T_I$ such that $f_i <^* f_\delta$ for every $i < \delta$.

**Fact 2.3C** In the universe $W$ suppose $\theta = \sup(I)$, $I \subseteq R$, $\theta$ a singular cardinal of cofinality $< \lambda^*$ (remember that by clause (1) of 2.3, $W$ have a $\theta^+$-scale if $\theta \geq \lambda^* > \text{cf}\theta$). <u>Then:</u>

(1) there are functions $f_i \in T_I (i < \theta^+)$ such that for every $i < j < \theta^+$, $f_i <^* f_j$ and for every $f \in T_I$ for some $i$, $f <^* f_i$, provided that $|\text{Dom } f| < \theta$.

(2) If in addition $\langle C_\delta : \delta \in S^{\theta^+}_{<\lambda^*}\rangle$ is a $\lambda^*$-square, $\lambda^* < \theta$, we can in part (1) demand that:
(a) if $i < j < \theta^+$, $i \in C_j$ and $\chi \in I$, $\chi \geq \lambda^*$ and $\chi$ is regular <u>then</u> $f_i(\chi) < f_j(\chi)$.
(b) if $j < \theta^+$, $j$ is a limit ordinal and $j \in S^{\theta^+}_{<\lambda^*}$, <u>then</u> for $\chi \in I$, $\chi \geq \lambda^*$ we have $f_j(\chi) = \sup\{f_i(\chi) + 1 : i \in C_j\}$.

**Definition 2.3D** (1) For every $\alpha$ we can define a model $M^2_\alpha \in W$, an expansion of $M^1_\alpha$ by the functions $F^2 = F^2_\alpha$, where:
for each singular cardinal $\theta$ of $W$, such that $\mathrm{cf}(\theta) < \lambda^* < \theta$, $\theta^+ \leq \alpha$, let $f^{2,\theta}_i (i < \theta^+)$ be as in Fact 2.3C (for $I = R \cap \theta$, and the $\lambda^*$-squares $\langle C^1_\delta : \delta \in S^\alpha_{<\lambda^*}\rangle$ we have used in the definition of $M^1_\alpha$), and

$$F^2(\theta, i, \chi) = f^{2,\theta}_i(\chi).$$

Of course, $\{(i, \theta, f^{2,\theta}_i) : i < \theta^+, \theta^+ < \alpha\} \in W)$.
(2) If $(W, V)$ has $\lambda$-squares (see clause (5) of 2.3) and satisfies the $\lambda$-covering lemma then $M^3_\alpha$ is the expansion of $M^2_\alpha$ by $C^{2,\alpha}$, where $C^{2,\alpha}$ is like $C^{1,\alpha}$ (see 2.2B), but using a $\lambda$-square $\langle C^2_i : i < \alpha$ and $W \models \mathrm{cf}(i) < \lambda\rangle$.
(3) Without loss of generality $C^1_\delta$ are as in 2.2F.

**Fact 2.3E** Let $\mu$ be singular cardinal of $W, \mu \geq \lambda^* > \mathrm{cf}\mu$, we let $M^2 = M^2_{\mu^+}$, etc. Suppose $a \subseteq M^2$ is an elementary submodel in $V$, $\mu \in a$, $A \subseteq a$ is unbounded, $\chi_0 < \mu$, and for every $\chi \in R \cap \alpha \cap (\chi_0, \mu)$, $\sup(a \cap \chi) \leq \sup_{i \in A} f^{2,\mu}_i(\chi)$. <u>Then</u>:
(1) $a = \mathrm{cl}\big([\alpha \cap \chi_0] \cup A, M^2\big)$, hence
(2) if $a \cap \chi_0$, $A \in W$ then $a \in W$.

**Proof:**
(1) Let $b =: \mathrm{cl}\big([a \cap \chi_0] \cup A, M^2\big)$, so clearly $b \subseteq a$; suppose $b \neq a$ and eventually we shall get a contradiction. Let $\zeta$ be the first element in $a \backslash b$ and $\xi$ the first element in $b \backslash \zeta$ (it exists as by assumption $A$ is unbounded in $a$); so there is no member of $b$ in the interval $[\zeta, \xi)$ and $\zeta < \xi$ (so $b \cap \xi \subseteq \zeta$).

<u>Case I:</u> Let $\xi$ be a successor ordinal.
Then as $\xi \in b$ also $\xi - 1 \in b$ (as $CF^0$ is one of the functions even of $M^0_{\mu^+}$, $CF^0(\xi) = \xi - 1$, see Fact 2.2A), but $\zeta \leq \xi - 1 < \xi$, contradiction.
<u>Case II:</u> Let $\xi$ be a limit ordinal, singular in $W$ (i.e. in $W$ either, $|\xi| < \xi$, or $\xi$ is a singular cardinal).
Then as $CF^0(\xi) < \xi$, and $\xi \in b \Rightarrow CF^0(\xi) \in b$, clearly $CF^0(\xi) < \zeta$. Now

$$M^2 \models \text{“}(\exists x)(x < CF^0(\xi) \ \& \ \zeta < H^0(\xi, x) < \xi)\text{”}$$

(by $H^0$'s choice, see 2.2A) hence as $\zeta, \xi \in a$ :

$$M^2 \restriction a \models \text{“}(\exists x)(x < CF^0(\xi) \ \& \ \zeta < H^0(\xi, x) < \xi)\text{”}.$$

So let $\alpha \in a$ be such that

$$\alpha < CF^0(\xi) \ \& \ \zeta < H^0(\xi, \alpha) < \xi.$$

As $CF^0(\xi) < \zeta$ (see above) $\alpha < \zeta$; but by the choice of $\zeta$, $\alpha \in a$ implies $\alpha \in b$. As $\alpha \in b$, $H^0(\xi, \alpha) \in b$, but $\zeta < H^0(\xi, \alpha) < \xi$, contradiction to the choice of $\xi$.

Case III: Let $\xi$ be a regular cardinal in $W$.

Then $\xi > \chi_0$ as $\zeta \geq \chi_0$, as $a \cap \chi_0 \subseteq b$. So

$$\sup(a \cap \xi) \leq \sup_{i \in A} \left( f_i^{2,\mu}(\xi) \right) = \sup_{i \in A} \left( F^2(\mu, i, \xi) \right) \leq \sup(b \cap \xi).$$

The last inequality holds as $\mu \in b$, $\xi \in b$, $A \subseteq b$ (why? $\mu \in b$ as there is $\gamma \in A$ & $\mu < \gamma$ (as $\sup(A) = \sup(a)$, and $\mu \in a$ by a hypothesis of the Fact) hence $\mu = |\gamma| = H^0_{\mu^+}(\gamma, 0) \in b$; $\xi \in b$ by the choice of $\xi$, $A \subseteq b$ by the definition of $b$).

As trivially $b \subseteq a$ we can conclude $\sup(a \cap \xi) = \sup(b \cap \xi)$; however, we know that $\zeta \in a \cap \xi$ hence $\zeta + 1 \in a \cap \xi$ hence $\zeta < \sup(a \cap \xi)$ whereas $b \cap \xi \subseteq \zeta$ hence $\sup(b \cap \xi) \leq \zeta$. Contradiction.

(2) Follows from part (1) of 2.3E. $\square_{2.3E}$

**Proof of Lemma 2.3:** By the hypothesis of the Lemma, for every $\alpha < \mu$, player I has a winning strategy in the $(\lambda^*, \lambda, D, \alpha)$-covering game, which we denote by $\bar{K}^\alpha = \langle K_i^\alpha : i < \lambda \rangle$; i.e., if $b_i \subseteq \alpha$ for $i < \lambda$, $|b_i|^V < \lambda^*$ then $a_i = K_i^\alpha(b_0, b_1, ..., b_j, \cdots)_{j<i}$ is a subset of $\alpha$, of $V$-cardinality $< \lambda^*$, $b_j \subseteq a_i$ for $j < i$; if in addition $a_i \subseteq b_i$ for $i < \lambda$ then:

$$\left\{ \delta \leq \lambda : \bigcup_{j<\delta} a_j \in W \right\} \in D.$$

**Defining the Strategy 2.3F** Let us describe a winning strategy of player I in the $(\lambda^*, \lambda, D, \mu^+)$-covering game.

In the $\zeta$-th move, $a_j \subseteq b_j \subseteq a_i \subseteq b_i$ (for $j < i < \zeta$) are given, player I let:

(i) $a_\zeta^0 = \bigcup_{j<\zeta} b_j$

(ii) $a_\zeta^1 = \cup \Big\{ K_\xi^\alpha(b_j \cap \alpha, b_{j+1} \cap \alpha, ..., b_i \cap \alpha, \cdots)_{j \leq i < \zeta} :$ for some $j < \zeta$, we have $\zeta = j + \xi$, $\alpha \in a_j \backslash \bigcup_{\gamma<j} a_\gamma$ and $\alpha < \mu \Big\}$

(iii) $a_\zeta^2 = a_\zeta^0 \cup a_\xi^1 \cup \{\sup(a_\zeta^0)\} \cup \{\gamma : \gamma < \sup(a_\zeta^0 \cap \lambda^*)\}$.

As $(W, V)$ satisfies the $\lambda^*$-covering lemma, and the set $a^0_\zeta$ has cardinality $< \lambda^*$ (in $V$), there is $a^3_\zeta$ such that:

(iv) $a^3_\zeta \in W$, $a^0_\zeta \subseteq a^3_\zeta$, $|a^3_\zeta|^V < \lambda^*$; moreover $a^3_\zeta$ is an elementary submodel of $M^2_{\mu^+}$ (remember $M^2_{\mu^+} \in W$), (and of $M^3_{\mu^+}$ if well defined) and include the topological closure of $a^0_\zeta$ (in the order topology on the ordinals).

Let $\mathrm{Ch}_\zeta$ be a function, with domain $a^0_\zeta \cap (R \backslash \lambda^*)$, $\mathrm{Ch}_\zeta(\chi) = \sup(a^0_\zeta \cap \chi) < \chi$ (remember that by the $\lambda^*$-covering lemma $\chi \in R\backslash\lambda^*$ implies $\mathrm{cf}^V(\chi) \geq \lambda^*$ as: $\lambda^*$ is a regular cardinal in $V$, $\chi$ regular cardinal in $W$, hence if $a \in V$, $a \subseteq \chi$ $|a|^V < \lambda^*$, then there is $b \in W$, $a \subseteq b \subseteq \chi$, $|b|^V < \lambda^*$, hence $\mathrm{otp}(b) < \lambda^*$ so $|b|^W < \lambda^*$ but $W \models$ "$\chi = \mathrm{cf}\chi \geq \lambda^*$, so $a \cap \chi$ is a bounded subset of $\chi$). By the $\lambda^*$-covering lemma there is a function $f_\zeta \in (T_{R\cap\mu})^W$, $\mathrm{Dom}(f_\zeta)$ a subset of $R \cap \mu$ of cardinality $< \lambda^*$, $\mathrm{Ch}_\zeta < f_\zeta$, i.e., $\mathrm{Ch}_\zeta(\chi) < f_\zeta(\chi)$ when $\chi \in a^0_\zeta \cap R$, $\chi \geq \lambda^*$. Let, for each cardinal $\theta$ of $W$, $\mathrm{Ch}^\theta_\xi = \mathrm{Ch}_\zeta{\restriction}\theta$. For $\theta \in (\lambda^*, \mu]$ singular in $W$, by the choice of $\langle f^{2,\theta}_i : i < \theta^+\rangle$, for some $i_\theta(\zeta) < \theta^+$ we have $f_\zeta{\restriction}[\lambda^*, \theta) <^* f^{2,\theta}_{i_\theta(\zeta)}$ or $(\mathrm{Dom}\ f_\zeta) \cap \theta$ is a bounded subset of $\theta$.

Lastly player I chooses

$$a_\zeta = \mathrm{cl}\Big(a^2_\zeta \cap a^3_\zeta \cup \Big\{i_\theta(\zeta) : \theta \leq \mu, \theta \in a^0_\zeta \text{ and } \theta \text{ is singular in } W\Big\},\ M^2_{\mu^+}\Big).$$

* * *

The "only" thing left is to show that this strategy is a winning one; i.e.:

**Framework and Notation 2.3G** Let $\langle a_i, b_i : i < \lambda\rangle$ be a play in which player I uses the strategy described above. Let $a^0_\lambda =: \bigcup_{i<\lambda} a^0_i$. Note: for limit $\zeta < \lambda$, $a^0_\zeta = \bigcup_{\xi<\zeta} a_\xi$, and $\langle a_\xi : \zeta < \lambda\rangle$ is increasing and $\langle a^0_\zeta : \zeta \leq \lambda\rangle$ is increasing continuous, $a^0_\zeta \subseteq a_\zeta \subseteq a^0_{\zeta+1}$.

Let us introduce some more notations. For $\theta \in (\lambda^*, \mu]$, a singular cardinal of $W$ which belong to $a^0_\lambda$ and for an ordinal $\zeta \leq \lambda$ let

$$\delta_\theta(\zeta) = \delta(\zeta, \theta) = \sup(a^0_\zeta \cap \theta^+),$$

so $\delta_\theta(\zeta) = \mathrm{Ch}_\zeta(\theta^+)$ if $\theta \in \mathrm{Dom}(\mathrm{Ch}_\zeta)$. If $x \in a^0_\lambda$, let

$$j(x) = \min\{j < \lambda : x \in a^0_j\} < \lambda.$$

If $\theta \in a^0_\lambda$ and $\mathrm{cf}^W\theta < \lambda^*$ (equivalently, $\mathrm{cf}^V\theta < \lambda^*$), clearly $\theta \in a^0_{j(\theta)}$ and $a^0_{j(\theta)} \cap \theta$ is an unbounded subset of $\theta$. Let $\Theta$ [let $\Theta^*$] be the set of all $W$-cardinals $\theta \in \bigcup_{\zeta<\lambda} a_\zeta \backslash \lambda^*$ for which $\mathrm{cf}\theta < \lambda$ [for which $\mathrm{cf}\theta < \lambda^*$]. Note that $\langle \delta_\theta(\zeta) : j(\theta) \leq \zeta \leq \lambda\rangle$ is strictly increasing continuous (see (iv) above) hence for limit $\zeta$, $\mathrm{cf}^V(\delta_\theta(\zeta)) = \mathrm{cf}^V(\zeta)$. Note also that $\theta \in a^0_\zeta \Rightarrow$

$\delta_\theta(\zeta) \in a^0_{\zeta+1}$ (remember that $a^3_\zeta$ contains the topological closure of $a^0_\zeta$ and obviously $\delta_\theta(\zeta)$ is in the closure of $a^0_\zeta$ by its definition).

**Subfact 2.3H** For each $\alpha \in a^0_\lambda \cap \mu$ for the $D$-majority of $i \leq \lambda$, $a^0_i \cap \alpha \in W$.

**Proof:** This is by (ii) (of the definition of the strategy of player I; i.e. in 2.3F) as $a^1_\xi \subseteq a_{\xi+1}$, and as $K^\alpha_\xi(\xi < \lambda)$ is a winning strategy of player I in the strong $(\lambda^*, \lambda, D, \alpha)$-covering game. $\square_{2.3H}$

**Subfact 2.3I** (1) Suppose $\theta \in \Theta^*$, $j(\theta) \leq \zeta \leq \lambda$ (on $j(\theta)$ – *see* above 2.3G), $\aleph_0 < \mathrm{cf}^V \zeta < \lambda^*$. Then for some closed unbounded subset $C$ of $\zeta$, for every $\xi \in C \cup \{\zeta\}$, the set $C^1_{\delta(\xi,\theta)}$ (is defined and) is an unbounded subset of $a^0_\xi \cap \theta^+$.

(2) If $D$ satisfies $(\lambda^*, \lambda)$-demand 2 then for $\theta \in \Theta^*$, we have:
$\{\zeta \leq \lambda : C^1_{\delta(\xi,\theta)}$ is an unbounded subset of $a^0_\xi \cap \theta^+\} \in D$.

**Proof:**
(1) We can prove this as in the proof of 2.2D.
(2) For each $\theta \in \Theta^*$, $j(\theta) < \lambda$, and by (1) and "$D$ satisfies the $(\lambda^*, \lambda)$-demand 2" the conclusion follows. $\square_{2.3I}$

**Fact 2.3J** If $\theta \in \Theta^*$, $\theta \in a^0_{j(\theta)}$, $j(\theta) < \zeta < \xi < \lambda$ then:

(1) $\mathrm{Ch}^\theta_\zeta < \mathrm{Ch}^\theta_\xi$,
(2) $f^{2,\theta}_{\delta_\theta(\zeta)} <^* f^{2,\theta}_{\delta_\theta(\xi)}$
(3) $f^{2,\theta}_{\delta_\theta(\zeta)} \restriction a^0_\xi < \mathrm{Ch}^\theta_\xi$
(4) $\mathrm{Ch}^\theta_\zeta <^* f^{2,\theta}_{\delta_\theta(\xi)}$
(5) if $i \leq \lambda$, $\mathrm{cf} i < \lambda^*$, $i$ a limit ordinal, $C^1_{\delta(i,\theta)} \subseteq a^0_i$ then $f^{2,\theta}_{\delta(i,\theta)} \restriction a^0_i \leq \mathrm{Ch}^\theta_i$.

**Proof:** This can be proved quite easily. The first part holds as $a^0_{\zeta+1} \subseteq a^0_\xi$ as $\zeta < \xi$ and the definition of $a_{\zeta+1}$ above (as the closure of $a^0_\zeta$ in the order topology of ordinals is a subset of $a^3_\zeta \subseteq a^0_{\zeta+1}$ hence of $a^0_\xi$). The second part by the choice of the $f^{2,\theta}_i$ (see 2.3($C$)(1)) as $[j(\theta) \leq \zeta < \xi \Rightarrow \delta_\theta(\zeta) < \delta_\alpha(\xi)]$. The third part is true as $\delta_\theta(\zeta) \in a^0_{\zeta+1} \subseteq a^0_\xi$ (as $\delta_\theta(\zeta)$ is in the topological closure of $a^0_\zeta$ which is a subset of $a^3_\zeta \subseteq a^0_{\zeta+1}$) hence

$$[\sigma \in R \cap a_\xi \cap \theta \Rightarrow f^{2,\theta}_{\delta_\theta(\zeta)}(\sigma) \in a_\xi \cap \sigma].$$

The fourth part holds as $\mathrm{Ch}^\zeta_\theta <^* f^{2,\theta}_{i_\theta(\zeta)}$ (by the choice of $i_\theta(\zeta)$ in 2.3F) and $i_\theta(\zeta) \in a_{\zeta+1} \subseteq a^0_\xi$ (see choice of $a_{\zeta+1}$) hence $i_\theta(\zeta) < \delta_\theta(\xi)$ and so

$f^{2,\theta}_{i_\theta(\zeta)} <^* f^{2,\theta}_{\delta_\theta(\xi)}$ (see 2.3C(1)). As $<^*$ is transitive, we finish proving (4). As for the fifth, we know that, for every $\chi \in R \cap \theta \cap a^0_i \setminus \lambda^*$,

$$f^{2,\theta}_{\delta(i,\theta)}(\chi) = \sup\{f^{2,\theta}_j(\chi) + 1 : j \in C^1_i\} = \sup\{F^2(\theta, j, \chi) + 1 : j \in C^1_i\}$$

$$\leq \sup(a^0_i \cap \theta^+) = \mathrm{Ch}^\theta_i(\chi).$$

$\square_{2.3J}$

**Fact 2.3K** Suppose $\theta \in \Theta^*$.
Notation: For $\zeta \leq \lambda$, $j(\theta) < \zeta$ let $\chi_\theta(\zeta) = \chi(\zeta,\theta) \in \theta \cap R \cap a^0_\zeta$ be the minimal cardinal $\geq \lambda^*$ of $W$ satisfying $(*)_{\zeta,\theta}$ below (if there is one):
$(*)_{\zeta,\theta}$ $(\forall \chi)\left[\chi_\theta(\zeta) \leq \chi < \theta \ \&\ \chi \in R \cap a^0_\zeta \Rightarrow \mathrm{Ch}^\theta_\zeta(\chi) = f^{2,\theta}_{\delta(\zeta,\theta)}(\chi)\right]$.
Now we claim:

(1) If $\zeta \leq \lambda$ is a limit ordinal, $\mathrm{cf}^V(\zeta) \neq \mathrm{cf}^V(\theta)$ and $\mathrm{cf}^V(\zeta) < \lambda^*$ <u>then</u> $\chi_\theta(\zeta)$ exists.

(2) If $\zeta \leq \lambda$, $\aleph_0 < \mathrm{cf}^V(\zeta) < \lambda^*$, $\theta \in a^0_\zeta$ and $\mathrm{cf}\theta \neq \mathrm{cf}\zeta$ <u>then</u> for a closed unbounded set of $\xi < \zeta$, $(*)_{\xi,\theta}$ above is satisfied for $\chi = \chi_\theta(\zeta)$ (so $\chi_\theta(\xi) \leq \chi_\theta(\zeta)$).

(3) If $\theta \in \bigcup_{\zeta<\lambda} a^0_\zeta$ and $D$ satisfies the $(\lambda^*,\lambda)$-demands 1,3 <u>then</u> the set $\{\zeta < \lambda : \chi_\theta(\zeta)$ well defined$\}$ belongs to $D$.

**Proof:**
(1) As $\mathrm{cf}^V(\zeta) < \lambda^*$, $\zeta$ a limit ordinal, clearly $\mathrm{cf}(\delta_\theta(\zeta)) = \mathrm{cf}(\zeta) < \lambda^*$, hence $C^1_{\delta(\zeta,\theta)}$ is defined. Let $\xi(\epsilon) < \zeta$ (for $\epsilon < \mathrm{cf}^V(\zeta)$) be increasing continuous, $\bigcup_{\epsilon<cf\zeta} \xi(\epsilon) = \zeta$ and be such that for each $\epsilon$ for some $\alpha(\epsilon)$, a limit ordinal from $C^1_{\delta(\zeta,\theta)}$, we have $\delta_\theta(\xi(\epsilon)) < \alpha(\epsilon) < \delta_\theta(\xi(\epsilon+1))$, and let $\xi(\mathrm{cf}\zeta) = \delta_\theta(\zeta)$ (remember 2.2F, 2.3D(3)). For each $\epsilon < \mathrm{cf}\zeta$, by 2.3J,

$$\mathrm{Ch}^\theta_{\xi(\epsilon)} \leq^* f^{2,\theta}_{\delta(\xi(\epsilon+1),\theta)} \restriction a^0_{\xi(\epsilon+2)} \leq^* f^{2,\theta}_{\alpha(\varepsilon+1)} \restriction a^0_{\xi(\epsilon+2)}$$

$$\leq^* f^{2,\theta}_{\delta_\theta(\xi(\epsilon+3))} \restriction a^0_{\xi(\epsilon+3)} \leq \mathrm{Ch}^\theta_{\xi(\epsilon+3)}$$

hence for some $\chi_\epsilon < \theta$ :

$$\begin{aligned}(*)\ \mathrm{Ch}^\theta_{\xi(\epsilon)} \restriction [\chi_\epsilon, \theta) &\leq f^{2,\theta}_{\delta(\xi(\epsilon+1),\theta)} \restriction ([\chi_\epsilon,\theta) \cap a^0_{\xi(\epsilon)}) \\ &\leq f^{2,\theta}_{\alpha(\epsilon+1)} \restriction ([\chi_\epsilon,\theta) \cap a^0_{\xi(\epsilon)}) \\ &\leq Ch^\theta_{\xi(\epsilon+3)} \restriction ([\chi_\epsilon,\theta) \cap a^0_{\xi(\epsilon)}).\end{aligned}$$

As $\mathrm{cf}\theta \neq \mathrm{cf}\zeta$, there is $\chi^* = \chi(\zeta, \theta) < \theta$ such that

$$S = \{\epsilon < \mathrm{cf}\zeta : \chi_\epsilon \leq \chi^*\}$$

is an unbounded subset of $\mathrm{cf}(\zeta)$. Without loss of generality

$$\epsilon \in S \Rightarrow \epsilon + 1,\ \epsilon + 2,\ \epsilon + 3 \notin S.$$

Now notice:

(a) For each $\chi \in [\chi^*, \theta) \cap a^0_\zeta$ the sequence

$$\langle \mathrm{Ch}^\theta_{\xi(\epsilon)}(\chi) : \epsilon \leq \mathrm{cf}(\zeta) \text{ and } \xi(\epsilon) \geq j(\chi)\rangle$$

is strictly increasing and continuous (as $\langle \xi(\epsilon) : \epsilon \leq \mathrm{cf}(\zeta)\rangle$ and $\langle a^0_{\xi(\epsilon)} : \epsilon \leq \mathrm{cf}(\zeta)\rangle$ are increasing and continuous and see 2.3J).

(b) For each $\chi \in [\chi^*, \theta) \cap a^0_\zeta$,

$$\langle f^{2,\theta}_\beta(\chi) : \beta \in \{\xi(\epsilon+1), \alpha(\epsilon+1) : \epsilon \in S \text{ and } \xi(\epsilon) \geq j(\chi)\}\rangle$$

is increasing (by (a) and the inequalities above).

(c) For each $\epsilon_1 < \epsilon_2$ from $S$ and $\chi \in [\chi^*, \theta) \cap a^0_{\xi(\epsilon)}$ we have

$$\mathrm{Ch}^\theta_{\xi(\epsilon_1)}(\chi) < f^{2,\theta}_{\xi(\epsilon_1+2)}(\chi) < \mathrm{Ch}^\theta_{\xi(\epsilon_2)}(\chi)$$

(by $(*)$ above).

(d) For each $\chi \in [\chi^*, \theta) \cap a^0_\zeta$ we have

$$f^{2,\theta}_{\delta(\zeta,\theta)}(\chi) = \sup\{f^{2,\theta}_{\alpha(\epsilon+1)}(\chi) + 1 : \epsilon \in S\} = \sup\{f^{2,\theta}_{\alpha(\epsilon+1)}(\chi) : \epsilon \in S\}$$

(by 2.3C(2)).

As $S \subseteq \mathrm{cf}(\zeta)$ is unbounded, a), b), c), d) together give the desired result.

(2) By subfact 2.3I $C^1_{\delta(\zeta,\theta)} \subseteq a^0_\zeta$ and for some closed unbounded $C \subseteq \zeta$,

$$(\forall \xi \in C \cup \{\zeta\})[C^1_{\delta(\xi,\theta)} = \delta_\theta(\xi) \cap C^1_{\delta(\zeta,\theta)} \subseteq a^0_\xi],$$

and let $\xi(\epsilon)$ in the proof of (1) be such that $\delta_\theta(\xi(\epsilon)) \in C$, and let $\chi^*$, $S$ be as there. Now if $\epsilon^* < \mathrm{cf}(\zeta)$ is a limit ordinal and $S \cap \epsilon^*$ an unbounded subset of $\epsilon^*$, the proof there gives the results for $\xi = \xi(\epsilon^*)$, but the set of such $\xi(\epsilon^*)$ is a closed unbounded subset of $\zeta$ (as $\mathrm{cf}\zeta > \aleph_0$). So $\chi_\theta(\xi)$ is well defined and $\leq \chi_\theta(\zeta)$ for a closed unbounded set of $\xi < \zeta$.

(3) Should be clear (see 1.12). If $\lambda^* > \lambda$, $\mathrm{cf}(\theta) \neq \lambda$ we can apply part 2 to $\zeta = \lambda$, and get a club $C$ of $\lambda$ such that $\xi \in C \Rightarrow (*)_{\xi,\theta}$, by part

1 we know $(*)_{\lambda,\theta}$, so it is enough to have $C \cup \{\lambda\} \in E$ which holds by "$D$ satisfies the $(\lambda^*, \lambda)$-demand 1". If $\lambda^* = \lambda$, let $\kappa$ for 1.12(4) be $\mathrm{cf}(\theta)$: for every $\zeta \in \{\delta < \lambda : \mathrm{cf}^V(\delta) \neq \mathrm{cf}(\theta)\}$ as above for some club $C_\delta$ of $\delta$ $[\zeta \in C_\delta \cup \{\delta\} \Rightarrow (*)_{\zeta,\theta}]$, and apply "$D$ satisfies $(\lambda^*, \lambda)$-demand 3." We are left with the case $\lambda^* > \lambda = \mathrm{cf}\theta$ which is like the second case but easier (use for example $\kappa = \aleph_0$; note that in this case $\lambda \in D$ as in 1.12(4) we have three possibilities; the second is excluded, the first and third imply $\lambda \in D$). $\square_{2.3K}$

**Fact 2.3L** For the $D$-majority of $\zeta \leq \lambda$, $a^0_\zeta \in W$, provided that $\mathrm{cf}^V(\mu) < \lambda^*$ (assuming (4) of (B) of 2.3).

**Proof of 2.3L:** The proof is split into cases (they cover more than demanded in (4) of 2.3; (5) of 2.3 is irrelevant).
Case A: $\lambda < \lambda^*$, $\mathrm{cf}\mu \neq \lambda$ and $D$ satisfies the $(\lambda^*, \lambda)$-demands 1, 2.

First, by Fact 2.3K(2) (applied with $\mu, \lambda$ here standing for $\theta, \zeta$ there) as $\lambda < \lambda^*$, there is a closed unbounded $C \subseteq \lambda$, and $\chi^* < \mu$ such that for every $\zeta \in C \cup \{\lambda\}$, $(*)_{\zeta,\mu}$ holds for $\chi(\zeta, \mu) \leq \chi^*$. Note that by the hypothesis of this case (A), demand 1 (see 1.12(2)) holds hence $C \cup \{\lambda\} \in D$ and without loss of generality every member of $C$ is a limit ordinal.

Secondly by Subfact 2.3H the set $S = \{\zeta \leq \lambda : a^0_\zeta \cap \chi^* \in W\}$ belongs to $D$.

Lastly, by Subfact 2.3I(2), for some set $C^1 \in D$, for every $\zeta \in C^1$, $C^1_{\delta(\zeta,\mu)}$ is an unbounded subset of $a^0_\zeta$.

As $D$ is a filter, $S^* = (C \cup \{\lambda\}) \cap S \cap C^1$ belong to $D$, and we shall prove that for every $\zeta \in S^*$, $a^0_\zeta \in W$, thus proving 2.3L,Case A. Let $A = \mathrm{cl}\big((a^0_\zeta \cap \chi^*) \cup C^1_{\delta(\zeta,\mu)}, M^2_{\mu^+}\big)$. As $\zeta \in C \cup \{\lambda\}$, $\zeta$ is a limit ordinal. As $\zeta \in S$, $a^0_\zeta \cap \chi^* \in W$, and obviously $C^1_{\delta(\zeta,\mu)} \in W$, hence $A \in W$, so it is enough to prove $A = a^0_\zeta$. As $a^0_\zeta \cap \chi^* \subseteq a^0_\zeta$, and $C^1_{\delta(\zeta,\mu)} \subseteq a^0_\zeta$ (because $\delta \in C^1$) and as $a^0_\zeta$ is a submodel of $M^2_{\mu^+}$, clearly $A \subseteq a^0_\zeta$. We shall prove the other inclusion by Fact 2.3E, so we have just to check that for every $\chi \in R \cap A \cap a^0_\zeta \cap (\chi^*, \mu)$,

$$\sup(a^0_\zeta \cap \chi) \leq \sup\{f^{2,\mu}_i(\chi) : i \in A\}.$$

For this remember that $\zeta \in C \cup \{\lambda\}$, $\chi(\zeta, \mu) \leq \chi^*$, so by $(*)_{\zeta,\mu}$ (from Fact 2.3K) $\mathrm{Ch}^\mu_\zeta(\chi) = f^{2,\mu}_{\delta(\zeta,\mu)}(\chi)$. So

| | |
|---|---|
| $\sup(a^0_\zeta \cap \chi) =$ | (by $\mathrm{Ch}^\mu_\zeta(\chi)$'s definition) |
| $\mathrm{Ch}^\mu_\zeta(\chi) =$ | (by $(*)_{\zeta,\mu}$) |
| $f^{2,\mu}_{\delta(\zeta,\mu)}(\chi) =$ | (by the definition of $f^{2,\mu}_i$, see 2.3D and 2.3D(2)(1)) |
| $\sup\{f^{2,\mu}_i(\chi) : i \in C^1_{\delta(\zeta,\mu)}\} \leq \sup\{f^{2,\mu}_i(\chi) : i \in A\}$ | |

So we have proved the inequality required for applying Fact 2.3E, hence $A = a^0_\zeta$, hence $a^0_\zeta \in W$. As this holds for every $\zeta \in S^*$ and $S^* \in D$, we finish the proof of Fact 2.3K, Case A.

Case B: $D$ is $(\mathrm{cf}\mu)^+$-complete (or at least closed under intersection of decreasing sequences of length $\mathrm{cf}(\mu)$) and satisfies the $(\lambda^*, \lambda)$-demands 1,2,3 (3: for $\kappa = \mathrm{cf}\mu$).

As "$D$ satisfies $(\lambda^*, \lambda)$-demands 1,3 (for $\kappa = \mathrm{cf}\mu$)", by 2.3K(3) for some $C \in D$, for every $\zeta \in C$, $\chi_\mu(\zeta)$ is well defined (this is a weaker conclusion than in the first paragraph of the proof of Case A, so we strengthen the conclusion of the second paragraph). Define for $\chi < \mu$,

$$S_\chi =: \{\zeta \le \lambda : a^0_\zeta \cap \chi \in W\}.$$

Now define $S$, $S = \bigcap_{\chi<\mu} S_\chi$.

Note that $\chi_1 \le \chi_2 < \mu \Rightarrow S_{\chi_2} \subseteq S_{\chi_1}$. Hence as $D$ is $(\mathrm{cf}\mu)^+$-complete, $S \in D$.

Lastly, by 2.3I, as $D$ satisfies the $(\lambda^*, \lambda)$-demand 2 for some $C^1 \in D$, we have $[\zeta \in C^1 \Rightarrow C^1_{\delta(\zeta,\mu)}$ is an unbounded subset of $a^0_\zeta]$. We can continue as in Case A.

Case C: $\mathrm{cf}(\mu) = \lambda$, $\lambda \in D$ and $D$ satisfies the $(\lambda^*, \lambda)$-demands 0, 1, 2.

Note — necessarily $\lambda = \mathrm{cf}\mu < \lambda^*$. We define $C$ as in Case $B$, and also $S_\chi(\chi < \mu)$. Let $\langle \theta_\zeta : \zeta < \lambda\rangle$ be an increasing continuous sequence of cardinals $< \mu$, $\mu = \bigcup_{\zeta<\lambda} \theta_\zeta$; and without loss of generality $\bigwedge_\zeta \theta_\zeta \in a^0_{\zeta+1}$. Let

$$S = \left\{\zeta \le \lambda : (\forall \xi < \zeta)[\zeta \in S_{\theta_\xi}]\right\};$$

as $[\chi_1 < \chi_2 < \mu \Rightarrow S_{\chi_1} \subseteq S_{\chi_2}]$ and each $S_\chi$ is in $D$, and $D$ weakly normal (i.e. satisfies demand 0 from 2.12(1)) we get $S \in D$. For each $\zeta < \lambda$,

$$\mathrm{Ch}^\mu_\zeta \restriction [\theta_\zeta, \mu) = f^{2,\mu}_{\delta(\zeta,\mu)} \restriction \big([\theta_\zeta, \mu) \cap a^0_\zeta\big)$$

(note: $\lambda < \lambda^*$ hence $\sup(a^0_\zeta \cap \mu) = \mu$).
The rest is as in Case A. $\square_{2.3L}$

So without loss of generality we could have assumed (5) of 2.3 hence:

**Hypothesis 2.3M** $\mathrm{cf}\mu \ge \lambda^*$ and $\{\zeta \le \lambda : \mathrm{cf}\zeta > \aleph_0\} \in D$, $\lambda < \lambda^*$.

**Fact 2.3N** We can find $\theta(\zeta) \in \Theta$ for $\zeta < \lambda$ such that:
(a) $\langle \theta(\zeta) : \zeta < \lambda\rangle$ is strictly increasing continuous.
(b) $\bigcup_{\zeta<\lambda} \theta(\zeta) = \sup(a^0_\lambda \cap \mu)$
(c) $\mathrm{cf}^V[\theta(\zeta)] < \lambda$
(d) $\theta(\zeta) = \sup(a^0_\zeta \cap \mu)$.

We leave the proof of this fact to the reader. (Note: the non-limit $\zeta$ are not important).

**Fact 2.3P** If $(W,V)$ satisfies $\lambda$-covering, $W$ has $\lambda$-squares, $\lambda < \lambda^* \leq \mathrm{cf}\mu$, $D$ a normal filter on $\lambda+1$, so $\lambda \in D$, satisfying the $(\lambda^*,\lambda)$-demands 0, 1, 2 <u>then</u> for the $D$-majority of $\zeta$, $a^0_\zeta \cap \mu^+ \in W$.

**Proof:** By Fact 2.3K(2) (with $\zeta$ there standing for $\mu$ here) for each $\theta \in \Theta$ (i.e. $\mathrm{cf}^V\theta < \lambda$ hence $\mathrm{cf}^W\theta < \lambda$ by $\lambda$-covering hence $\theta \in \Theta^*$) there is a closed unbounded subset $E^0_\theta$ of $\lambda$ such that for every $\xi \in E^0_\theta \cup \{\lambda\}$, $\chi_\theta(\xi) \leq \chi_\theta(\lambda) < \theta$ and by 2.3I, (using "$D$ satisfies $(\lambda^*,\lambda)$-demand 2") we get: for some $Y \in D$, for every $\xi \in Y$ also $C^1_{\delta(\xi,\theta)} = C^1_{\delta(\lambda,\theta)} \cap a^0_\xi$ is an unbounded subset of $a^0_\xi \cap \theta^+$. Let $E^1_\zeta = \cap\{E^0_\theta : \theta \in C^2_{\theta(\zeta)} \cap \Theta\}$, which is also a closed unbounded subset of $\lambda$, (remember $C^2_{\theta(\zeta)}$ has power $< \lambda$) and at last

$$E^2 = \big\{\zeta < \lambda\text{: (i) for every } \xi < \zeta,\ \zeta \in E^1_\xi \text{ and} \\ \text{(ii) } \theta(\zeta) = \sup(a^0_\zeta \cap \mu) \text{ (use } 2.3N(d))\big\},$$

which again is a closed unbounded subset of $\lambda$, hence $E^2 \in D$ (as $D$ satisfies $(\lambda^*,\lambda)$-demand 1, $\lambda \in D$). Similarly as $D$ satisfies the $(\lambda^*,\lambda)$-demands 0, 2 and by Subfact 2.3H and a variant of Subfact 2.3I (for $C^2_i$ instead $C^1_i$) we have:

$$E^3 = \big\{\zeta \in E^2 : \text{ for every } \theta \in C^2_{\theta(\zeta)} \cap \Theta,\ a^0_\zeta \cap \theta^+ \in W \text{ and } C^2_{\theta(\zeta)} \subseteq a^0_\zeta\big\}$$

belongs to $D$ (weak normality suffices as an initial segment of $b \subseteq \alpha$, $b \in W$ is in $W$). We shall prove now that for each $\zeta \in E^3$ : if $\mathrm{cf}\zeta > \aleph_0$, $\zeta \in Y$ then $a^0_\zeta \in W$. By the proof of Lemma 2.2 (i.e. 2.2C(2)) it is enough to prove that $a^0_\zeta \cap \mu \in W$.

Clearly there is $\epsilon < \lambda$ such that $\chi_{\theta(\epsilon)}(\lambda) < \theta(\zeta) \leq \theta(\epsilon)$ (for example $\epsilon = \zeta$, but even if we want to use $\chi_{\theta(\epsilon)}(\lambda)$ for some stationary set of $\epsilon$'s, we can use Fodor's Lemma decreasing a little $E^3$). As $\mathrm{cf}^V\zeta > \aleph_0$, and $\chi_\sigma(\lambda)$ (with $\sigma$ varying) is a regressive function on $C^2_{\theta(\zeta)} \cap \Theta$, for some $\chi^*$, $\chi_{\theta(\epsilon)}(\lambda) \leq \chi^* < \theta(\zeta)$ and $\chi^* \in C^2_{\theta(\zeta)}$ and $S =: \{\theta \in C^2_{\theta(\zeta)} \cap \Theta : \chi_\theta(\lambda) \leq \chi^* < \theta\}$ is a stationary subset of $\theta(\zeta)$.

Let $A = (a^0_\zeta \cap \chi^*) \cup \bigcup\{C^1_{\delta(\zeta,\theta)} : \theta \in S \cap \Theta\} \cup C^1_{\delta(\zeta,\theta(\zeta))}$, then clearly $A \subseteq a^0_\zeta$ (note: $C^1_{\delta(\zeta,\theta)} \subseteq a^0_\zeta$ by Subfact 2.3I as $\mathrm{cf}^V\zeta > \aleph_0$). For each $\theta \in S$, $\theta^+ \cap \mathrm{cl}\big((a^0_\zeta \cap \chi^*) \cup C^1_{\delta(\zeta,\theta)}, M^2_{\mu^+}\big)$ is equal to $\theta^+ \cap a^0_\zeta$ as $\zeta \in E^3$, $\theta \in C^2_{\theta(\zeta)}$ (as in the proof of 2.3L). As $S$ is unbounded (in $C^2_{\theta(\zeta)}$), it follows that $a^0_\zeta \cap \theta(\zeta) \subseteq \mathrm{cl}(A, M^2_{\mu^+}) \subseteq a^0_\zeta$. As $\mathrm{cf}^V\mu \geq \lambda^*$, by 2.3N(d) we have $\theta(\zeta) = \sup(a^0_\zeta \cap \mu)$, so $a^0_\zeta \cap \mu = \mathrm{cl}(A, M^3_{\mu^+}) \cap \mu$. So it suffices to prove that $A \in W$, and for this it suffices to prove that $S$ and $\langle \delta_\theta(\zeta) : \theta \in S \cap \Theta\rangle$ belongs to

$W$. Note that $C^2_{\theta(\zeta)} \cap \Theta = \{\theta : \theta \in C^2_\theta(\theta),\ W \models$ "$\theta$ a cardinal of cofinality $< \lambda\}$ hence $C^2_{\theta(\zeta)} \cap \Theta$ belongs to $W$.

Why $S \in W$?

Remember $\theta(\epsilon), \chi_{\theta(\epsilon)}(\lambda)$ used above and compare for $\theta \in C^2_{\theta(\zeta)} \cap \Theta \backslash \chi^*$, the functions $f^{2,\theta(\epsilon)}_{\delta(\lambda,\theta(\epsilon))}$ and $f^{2,\theta}_{\delta(\lambda,\theta)}$. We know that $f^{2,\theta(\epsilon)}_{\delta(\lambda,\theta(\epsilon))} \restriction \Big([\chi^*, \theta(\epsilon)) \cap a^0_\lambda\Big)$ is equal to $\mathrm{Ch}^{\theta(\epsilon)}_\lambda \restriction [\chi^*, \theta(\epsilon))$. So for $\xi \in [\zeta, \lambda)$ we have

$$f^*_\xi =: f^{2,\theta(\epsilon)}_{\delta(\lambda,\theta(\epsilon))} \restriction \big([\chi^*, \theta(\zeta)) \cap a^3_\xi\big) \in W$$

(see clause (iv) in the definition of the first player's strategy, $a^0_\xi \subseteq a^3_\xi \subseteq a^0_{\xi+1}$, $a^3_\xi \in W$), and $f^*_\xi$ is equal to

$$\mathrm{Ch}^{\theta(\zeta)}_\lambda \restriction \big([\chi^*, \theta(\zeta)) \cap a^3_\xi\big)\,.$$

Now if $\theta \in S$ and $\xi \in [\zeta, \lambda)$ then $f^{2,\theta}_{\delta(\lambda,\theta)} \restriction \Big([\chi^*, \theta) \cap a^3_\xi\Big)$ is equal to $\mathrm{Ch}^\theta_\lambda \restriction \Big([\chi^*, \theta) \cap a^3_\xi\Big) = f^*_\xi \restriction [\chi^*, \theta)$. But if $\theta \in C^2_{\theta(\zeta)} \cap \Theta \backslash \chi^* \backslash S$, then by the definition of $\chi_\theta(\lambda)$, as $\chi_\theta(\lambda) > \chi^*$, for every $\xi < \lambda$ large enough

$$f^*_\xi \restriction [\chi^*, \theta) \neq f^{2,\theta}_{\delta(\lambda,\theta)} \restriction \Big([\chi^*, \theta) \cap a^3_\xi\Big).$$

As $|C^2_{\theta(\zeta)}| < \lambda = \mathrm{cf}^V \lambda$, one $\xi(*) \in [\zeta, \lambda)$ is large enough for all. Also, by the choice of $\epsilon$, for $\theta \in C^2_{\theta(\zeta)} \cap \Theta$ we have

$$\delta_\theta(\lambda) = \mathrm{Ch}^\mu_\lambda((\theta^+)^W) = f^*_{\xi(*)}((\theta^+)^W)$$

so as $f^*_{\xi(*)} \in W$ the function

$$\langle \delta_\theta(\lambda) : \theta \in [\chi, \theta(\zeta)) \cap \Theta \text{ and } \mathrm{cf}^W \theta < \lambda \text{ and } \theta \in C^2_{\theta(\zeta)} \rangle$$

belongs to $W$. So we have a definition of $S$ in $W$, hence $S \in W$.

Why $\langle \delta_\theta(\zeta) : \theta \in S \cap \Theta \rangle$ belongs to $W$?

For each $\theta \in S \cap \Theta$, $\delta_\theta(\zeta) \in C^1_{\delta(\zeta,\lambda)}$ (as $\zeta \in E^0_\zeta$). We know that $\langle f^{2,\theta}_i(\chi^*) : i \in \mathrm{acc}\, C^1_{\delta(\lambda,\theta)} \rangle$ is strictly increasing, and is continuous. Now $\langle \delta_\theta(\lambda) : \theta \in S \rangle$ (as a function) belong to $W$ (as $f^*_{\xi(*)} \in W$), hence

$$\delta_\theta(\zeta) = \min\{\gamma : \gamma \in C^1_{\delta(\lambda,\theta)} \text{ and } f^{2,\theta}_\gamma(\chi^*) \geq \sup(a^0_\zeta \cap \chi^*)\}.$$

This definition can be carried in $W$ hence $\langle \delta_\theta(\zeta) : \theta \in S \cap \Theta \rangle \in W$. So we finish the proof of 2.3P. $\square_{2.3P}$

**End of the Proof of 2.3:** It is easy to check that 2.3L, 2.3P proved 2.3 (see 2.3(4), 2.3(5)). $\square_{2.3}$

**Remark 2.4** If we want to get the result for $\kappa = \aleph_0 < \lambda < \lambda^*$ (for example, for $\lambda = \aleph_1$, $\lambda^* = \aleph_2$ when $0^\# \notin V$) we can drop from the hypothesis on $\lambda$ (i.e., $\lambda$-covering and $\lambda$-squares) and add that the $\lambda^+$-squared scales (defined below) exists for $W$.
It was not clear whether they exist when [Sh- b, XIII]'s writting was essentially finished (early 1981). Later Abraham who was converging toward it and the author looked at it and tried to develop it and with Stanley seemingly proves its consistency. Subsequently Donder, Jensen and Stanley [DJS] proved it.

**Definition 2.5** $W$ has $\lambda^*$-squared scales, if there are for each singular $\theta$, a scale $\langle f_i^\theta : i < \theta^+\rangle$, and a $\lambda^*$-square $\langle C_\delta^{2,\theta} : \delta < \theta^+, \mathrm{cf}\delta < \lambda^*\rangle$ , and a $\lambda^*$-square $C_\theta^3$ ($\theta$ a cardinal in $W$, $\theta > \lambda^* > \mathrm{cf}\theta$) such that:

$(*)$ if $\theta(1) \in C_\theta^3$, $\zeta \in C_\delta^{2,\theta}$ then $f_\zeta^\theta(\theta(1)^+) \in C_\xi^{2,\theta}$ when $\xi = f_\delta^\theta(\theta(1)^+)$

**Remark 2.5A** We can restrict ourselves to $\theta < \alpha^*$ for any fixed $\alpha^*$.

**Theorem 2.6** *Suppose $(V, W)$ is a pair of universes of set theory, $\aleph_0 < \kappa < \lambda < \lambda^*$ are regular cardinals in $V$, $W$ have square (or just $\lambda^*$-squares and $\lambda$-squares) and scales.*
*Then $(V, W)$ satisfies the strong $(\lambda^*, \lambda, \kappa, \infty)$-covering lemma, if it satisfies the $\lambda^*$-covering lemma and the $\lambda$-covering lemma.*

**Proof:** Let $D$ be as in 1.13(2), by which it satisfies demand 0,1,2,3. As the strong $(\lambda^*, \lambda, \kappa, \infty)$-covering lemma is equivalent to the strong $(\lambda^*, \lambda, D, \infty)$-covering lemma, it suffices to prove the later. We prove by induction on $\mu$ ( a cardinal in $W$) that $(V, W)$ has the strong $(\lambda^*, \lambda, \kappa, \mu)$-covering lemma. For $\mu \le \lambda^*$ see 2.1 for successor $\mu$ (in $W$) use 2.2 and for limit $\mu$ use 2.3. $\square_{2.6}$

**Conclusion 2.7** If in $V$, $0^\#$ does not exist, then $(L, V)$ satisfies the strong $(\aleph_3^V, \aleph_2^V, \aleph_1^V, \infty)$-covering lemma.

**Theorem 2.8** *Suppose $(W, V)$ satisfies the $\lambda^*$-covering lemma, $W$ has square and has scales. If there is no cardinal $\mu$ of $W$, $\lambda < \mu < \lambda^*$ and $\kappa < \lambda < \lambda^*$ are regular cardinals of $V$ then $(W, V)$ has the $(\lambda^*, \lambda, \kappa, \infty)$-strong covering property.*

**Proof:** Note that if $V \models$"$\mathrm{cf}\alpha = \lambda$" then $W \models$"$\lambda \le \mathrm{cf}\alpha < (\lambda^+)^V$", hence in our case $W \models$"$\mathrm{cf}\alpha = \lambda$". So we can strengthen a little the Claim 1.7 demanding: if $\mathrm{cf}\delta = \lambda$ then $C_\delta^1$ has order type $\lambda$. Now repeat the proofs of 2.2, 2.3 (or see 4.17). $\square_{2.8}$

**Conclusion 2.9** If $0^\# \notin L$, and there is no cardinal $\mu$ of $L$ for which $\aleph_1^V < \mu < \aleph_2^V$ then $(L, V)$ satisfies the strong $\aleph_2$-covering lemma and the strong $\aleph_1$-covering lemma.

**Proof:** The strong $\aleph_2$ covering is by 2.8, the strong $\aleph_1$-covering follows immediately. $\square_{2.9}$

## §3 A counterexample

The following lemma says that even if $V$ and $L$ have the same cardinals, except $\aleph_2^L$ and $\mathrm{cf}^V(\aleph_2^L) = \aleph_1$, the strong $\aleph_1$-covering lemma may fail. It uses forcing but its role is just to show some theorems cannot be proved.

**Lemma 3.1** *Assume $V$ satisfies* CH*, then there is a forcing notion $R$, of power $\aleph_2$, which does not collapse $\aleph_1$ (and even satisfies the condition from [Sh- b, XI]; better see [Sh-f,XI,XV]) and does not collapse any $\aleph_\alpha > \aleph_2$, such that $(V, V^R)$ does not satisfy the strong $(\aleph_1, \aleph_2^V)$-covering lemma (note: if $V$ satisfies* GCH*, $W$ satisfies* GCH*, too).*

**Proof:** Let $P$ be for example, the forcing of adding a Cohen real. In $V^P$ we define a forcing notion $Q$ :

$$Q = \big\{ f : f \text{ is a function, with domain an ordinal } \alpha < \aleph_1, \text{ and range included in } \aleph_2, \text{ and for every limit } \delta \le \alpha, \ \mathrm{Rang}(f \restriction \delta) \notin V \big\}$$

$Q$ is ordered by inclusion.

First note that $Q \ne \emptyset$ (as the empty function belongs to $Q$) and we shall prove that for every $p \in Q$ and $\beta < \aleph_1$ there is $q \in Q$, $p \le q$, $\beta \subseteq \mathrm{Dom}\ q$. Let $\mathrm{Dom}\ p = \alpha$, and choose $i > \sup \mathrm{Rang}(p)$ (and $i < \aleph_2$) and choose $A \subseteq [i, i + \beta + \omega]$ such that: for every limit $\delta$ if $i < \delta < i + \beta + \omega$ then $A \cap [\delta, \delta + \omega] \notin V$ and so $A$ has order type $\beta + \omega$ (easy as $P$ add reals). Now define $q : \mathrm{Dom}\ q = \alpha + \beta$, $q(j) = p(j)$ for $j < \alpha$ and $q(\alpha + j)$ is the $j$-th element of A. Also trivially for every $i < \alpha_2$ $\{p \in Q : i \in \mathrm{Rang}(p)\}$ is a dense subset of $Q$.
We work for a while in $V^P$.

As $V^P$ satisfies $2^{\aleph_0} = \aleph_1$, clearly $Q$ has power $\aleph_2$ and it is easy to check $P * \underset{\sim}{Q}$ has power $\aleph_2$, and it will be our $R$. It suffices to prove that $Q$ does not add reals hence does not collapse $\aleph_1$, as the generic function from $\aleph_1$ to $\aleph_2$ will be the evidence of the failure of the strong $(\aleph_1, \aleph_2^V)$-covering lemma.

So let $\underset{\sim}{h}$ be a $Q$-name (in $V^P$), and $p \in Q$ force that it is a function from $\omega$ into $\aleph_1$. We define by induction $n < \omega$ for every $\eta \in {}^n(\omega_2)$ (i.e., a sequence of ordinals $< \omega_2$ of length $n$) a condition $p_\eta \in Q$ such that:
(1) $p_{<>} = p$, $p_\eta \restriction \ell \le p_\eta$ for $\ell \le \ell g(\eta)$

(2) $p_\eta \Vdash_Q$ "$\underset{\sim}{h}(m) = \gamma_\eta$" when $\ell g(\eta) = m+1$, for some $\gamma_\eta < \omega_1$
(3) Rang $\eta \subseteq$ Rang $p_\eta$, moreover if $\ell g(\eta) = m+1$ then

$$p_\eta(\sup \ \text{Rang} \ p_{\eta\restriction m}) = \eta(m).$$

There are no problems in the definition.

By Rubin and Shelah [RuSh117] (the theorem on $\Delta$-systems) there is $T \subseteq {}^{\omega>}(\omega_2)$ such that:

($\alpha$) $<>\in T$ and for every $\eta \in T$ the number of $\eta\hat{\ } < \alpha > \in T$ is $\aleph_2$ and $T$ closed under initial segments

($\beta$) Dom $p_\eta = \delta_{\ell g(\eta)} < \omega_1$, (i.e. Dom $p_\eta$ depends on the length of $\eta$ only)

($\gamma$) there are countable sets $a_\eta \subseteq \omega_2$, for $\eta \in T$ such that: (for $\eta, \rho \in T$) we have Rang $p_\eta \subseteq a_\eta$ and if $\eta\restriction\ell = \rho\restriction\ell$, $\eta(\ell) \neq \rho(\ell)$ then $a_\eta \cap a_\rho = a_{\eta\restriction\ell}$.

Let $\alpha_\eta = \sup(a_\eta)$, so $p_{\eta\hat{\ }\langle\beta\rangle}(\alpha_\eta) = \beta$ (see (3) above) and without loss of generality

($*$) if $\alpha < \beta$, $\eta\hat{\ } < \alpha > \in T$, $\eta\hat{\ } < \beta > \in T$ then $\min(a_{\eta\hat{\ }<\beta>} \backslash a_\eta) > \alpha$.

Let $C = \big\{\zeta < \aleph_2 :$ for every $\eta \in ({}^{\omega>}\zeta) \cap T$ we have $a_\eta \subseteq \zeta$ and $\{\alpha < \zeta : \eta\hat{\ } < \alpha > \in T\}$ is unbounded in $\zeta\big\}$.

Clearly it is a closed unbounded subset of $\aleph_2$, hence it contains a closed unbounded subset $C$ which belongs to $V$ (we are working in $V^P$). As the cardinality of $P$ is $\leq \aleph_1 < \aleph_2$, there is $\langle \zeta_n : n < \omega \rangle \in V$ increasing and included in $C$.

Now let $a = \{n(\ell) : \ell < \omega\} \notin V$ (but $\in V^P$), $n(\ell) < n(\ell+1)$. We define by induction on $\ell$, an ordinal $\alpha_\ell$ such that:
(1) $\zeta_{n(\ell)} < \alpha_\ell < \zeta_{n(\ell)+1}$
(2) $\langle \alpha_0, ..., \alpha_\ell \rangle \in T$, $a_{\langle\alpha_0,\ldots,\alpha_{\ell+1}\rangle} \cap \zeta_{n(\ell+1)} \subseteq \zeta_{n(\ell)+1}$.
This is easy by ($*$) above. Let $p^* = \bigcup_\ell p_{\langle\alpha_0,\ldots,\alpha_\ell\rangle}$. Now $p^*$ is a function from $\delta = \bigcup_\ell \delta_\ell$ to $\omega_2$; if $b =$ Rang $p^* \in V$, then $a = \{\ell : (\zeta_\ell, \zeta_{\ell+1}) \cap b \neq \emptyset\} \in V$ as $\langle \zeta_\ell : \ell < \omega \rangle \in V$, contradiction. Hence $b \notin V$, and it is easy to check $p^* \in Q$, and clearly $p^*$ forces a value to $\underset{\sim}{h}$, so we finish to prove that $Q$ does not add reals, hence does not collapse $\aleph_1$, hence we finish the proof of 3.1. $\square_{3.1}$

**Remark 3.2A** The choice of "$P$ ia cohen forcing" is as it is the simplest. For example, assume $\kappa = \kappa^{\aleph_0}$, $P$ is a forcing notion of cardinality at most $\kappa$ adding a new real and $Q$ is the forcing defined in the proof of 3.1 with $\kappa^+$ replacing $\aleph_2$. Then the forcing by $P * \underset{\sim}{Q}$ collapse $\kappa$ to $\aleph_1$, collapse no other cardinality (nor change cofinality) (in particular do not collapse $\aleph_1$), has cardinality $\kappa^+$, all the reals of $V^{P*\underset{\sim}{Q}}$ are from $V^P$. So if $P$ is ${}^\omega\omega$-bounding (for example Sacks forcing) then so is $P * Q$. On the other hand, the strong $(\aleph_1, \kappa^+)$-covering lemma fails i.e. the family of old countable subsets of $\kappa^+$ is not stationary; this answer a question of Kamburelis. Really assuming CH, any proper forcing adding a new real of cardinality $\aleph_1$ is OK. The proof give more than stated in 3.1, in particular answering a question of Kamburelis.

## §4 When adding a real cannot destroy CH

Here we draw conclusions concerning consistency strength, but the section is not used later, so knowledge of inner model is required.

On core models see Dodd and Jensen [DJ1]; they prove

**Theorem 4.1** *For every model (of set theory) $V$ there is a core model $K(V) \subseteq V$, such that:*

*(1) $K(V)$ is a transitive class containing all ordinals, and $W \subseteq V$ implies $K(W) \subseteq K(V)$.*

*(2) $K(V)$ satisfies* GCH *(hence has scales), has squares; let $K_\lambda(V)$ be the family of sets in $K(V)$ of hereditary power $< \lambda$.*

*(3) If in V there is no inner model with a measurable cardinal, then $(K(V), V)$ satisfies the covering lemma (see Definition 1.2).*

*(4) $K(V)$ has a definable well ordering (hence definable Skolem functions).*

The following is known:

**Theorem 4.2** *(1) Suppose $W \subseteq V$ have the same cardinals, then they have the same core model.*

*(2) Moreover, if $W \subseteq V$ have the same cardinals $\leq \lambda$, where $\lambda$ is a limit cardinal (in both) then $K_\lambda(V) = K_\lambda(W)$ (see 4.1(2)).*

**Proof:**
(1) Suppose $K(W) \neq K(V)$: clearly $K(W) \subseteq K(V)$, so let $A \subseteq \alpha$, $A \in K(V)$, $A \notin K(W)$. So there is a mice of $K(V)$ to which A belongs, hence there is such a mice of $K(V)$-power $|\alpha|$; but we can extend it, hence for every limit cardinal $\lambda > \alpha$ of $V$ there is a mice with critical point $\lambda$, to which A belongs, and the filter is generated by end segments of

$$\{\chi : \chi < \lambda,\ \chi \text{ a cardinal in } V\}.$$

But then this mice is in $W$ hence in $K(W)$.
(2) The same proof. $\square_{4.2}$

**Conclusion 4.3** Suppose in $V$ there is no inner model with a measurable cardinal. Then:

(1) $(K(V), V)$ satisfies the strong $\lambda$-covering for every $\lambda > \aleph_2$.

(2) If $W \subseteq V$ have the same cardinals then $(W, V)$ satisfies the strong $\lambda$-covering lemma for every cardinal $\lambda \geq \aleph_1$ of $V$.

(3) If $W \subseteq V$ have the same cardinals $\leq \mu$ or even $K_\mu(W) = K_\mu(V)$, where $\mu$ is a limit cardinal (of $V$) then $(W, V)$ satisfies the strong $(\lambda, \mu)$-covering lemma for any cardinal $\lambda$ of $V (\lambda > \aleph_0)$.

**Proof:**
(1) By hypothesis $(K(V), V)$ satisfies the $\lambda$-covering lemma for every $\lambda \geq \aleph_2^V$ (cardinal in $V$), by fine structure theory $K(V)$ has squares and scales. So our main theorem 2.6 give the desired conclusion.
(2) $K(V) = K(W)$ by the previous theorem, hence $K(V) \subseteq W$. We can finish by part (1) using 2.8.
(3) Similar proof. $\square_{4.3}$

Remember:

**Theorem 4.4. (Magidor)** *If $W \subseteq V$, $K(W) \neq K(V)$ then for some cardinal $\lambda$ of $K(V)$ and $A \subseteq \lambda$, $A \in K(W)$, $K_\lambda(W) = K_\lambda(V)$ but there is a class $C$ (in $V$) of ordinals in $K(W)$, such that in $K(W)$, $C$ is indiscernible over $A$, and $K(W)$ is the Skolem Hull (see 4.1(4)) of $A \cup C$.*

**Theorem 4.5**
*(1) If $V = W[r]$, $r$ a real and $(W, V)$ satisfies the strong $\lambda$-covering lemma ($\lambda$ a cardinal of $V$) <u>then:</u>*

*(i) $\sum_{\mu<\lambda}(2^\mu)^V = |\sum_{\mu<\lambda}(2^\mu)^W|^V$ and*

*(ii) $\left(\chi^{<\lambda}\right)^V = |\left(\chi^{<\lambda}\right)^W|^V$ for every $\chi$.*

*(iii) Assume $\lambda$ is regular in $V$, $A \in W$, $A \subseteq \lambda$ and $H(\lambda)^W \subseteq L_\lambda[A]$. Then any bounded subset $B$ of $\lambda$ from $V$ belongs to $L_\alpha[A \cap \alpha, r]$ for some $\alpha < \lambda$.*

*(2) For having (ii) it suffices to have the strong $(\lambda, \alpha)$-covering lemma for $\alpha < \left(\left[(\chi^{<\lambda})^W\right]^+\right)^V$; (note that (i) is a particular case (ii)).*

**Proof:** Easy.

**Conclusion 4.6** If $V$ has no inner model with a measurable cardinal, $V = W[r]$, $r$ a real, $W$, $V$ have the same cardinals $\leq \lambda$ were $\left(2^{\aleph_0}\right)^V \geq \lambda \geq \aleph_\omega^V$, $\lambda$ a limit cardinal and $W$ satisfies CH (but $V$ does not) (or at least $W \models 2^{\aleph_0} < \lambda$), <u>then</u>

$$K_\lambda(W) = K_\lambda(V), \quad K(W) \neq K(V)$$

(this is stronger than $0^\# \in V$, see 4.4).

**Proof:** We know that $K_\lambda(W) = K_\lambda(V)$ by 4.2(2). On the other hand if $K(W) = K(V)$ then by 4.3(3) the pair $(W, V)$ satisfies the strong $\lambda$-covering lemma. So by Theorem 4.5 (above) $\left(2^{\aleph_0}\right)^V = \left(2^{\aleph_0}\right)^W$, contradiction to "$W$ satisfies CH but $V$ does not". $\square_{4.6}$

**Conclusion 4.7** If $V = W[r]$, $r$ a real, $W$ satisfies CH and in $V$, $2^{\aleph_0} > \aleph_2$ then $0^\# \in V$.

**Proof:** Suppose $0^\# \notin V$. We know $\lambda^{\aleph_0} \leq \lambda^+ + 2^{\aleph_0} = \lambda^+$ for every $\lambda$. By 2.5 $(W, V)$ satisfies the strong $\aleph_3^V$-covering lemma, hence by 4.5 in $V$, $2^{\aleph_0} \leq \aleph_3$ in $V$. To get the exact result we should use a finer theorem, 4.15 below. $\square_{4.7}$

**Conclusion 4.8** If $V = W[r]$, $r$ a real, $V$ and $W$ have the same $\aleph_1$ and $\aleph_2$, $W$ satisfies CH but $V$ does not then $0^\# \in V$.

**Proof:** Use 2.9 and 4.5.

**Lemma 4.9** *Suppose $W \subseteq V$, $\lambda$ a cardinal of $W$ and*
*(a) (i) $\lambda \in W$ is a regular cardinal in $W$, or*
*(ii) the square principle for $\lambda$ holds in $W$, i.e., there are $C_\delta \subseteq \delta$ for $\delta$ limit $< (\lambda^+)^W$, $C_\delta$ closed unbounded and:*

$$\gamma = \sup(\gamma \cap C_\delta) \Rightarrow C_\gamma = (\gamma \cap C_\delta).$$

*(b) $V \models$"$\mathrm{cf}^V \lambda \neq \mathrm{cf}^V(|\lambda|^V)$".*
*Then in $V$ the $W$-successor of $\lambda$ is not a cardinal.*

**Remark 4.9A** In (a), also "$\mathrm{pp}(\lambda) > \lambda^+$ & $\mathrm{cf}\lambda < \lambda$" suffices (see [Sh355, 1.5A]).

**Proof:** By hypothesis (a) in $W$ we can easily find $\langle A_i : i < \lambda^+\rangle$ such that: $A_i \subseteq \lambda$, $A_i$ unbounded in $\lambda$, and for every $i < \lambda^+$ there is a function $f_i : i \to \lambda$, such that the sets $A_j \backslash f_i(j)$ (for $j < i$) are pairwise disjoint. (If $\lambda$ regular: trivially (choose for $i < \lambda$ $A_i \subseteq \lambda$ pairwise disjoint of cardinality $\lambda$, and then choose by induction on $i \in [\lambda, \lambda^+)$, $A_i \subseteq \lambda$ pairwise disjoint of cardinality $\lambda$ such that $j < i \Rightarrow |A_j \cap A_i| < \lambda$), if not: by Litman [Li], using Jensen's theorem on gap one transfer theorem (see Ben David [BD]), or directly let $\langle \lambda_i : i < \mathrm{cf}\lambda\rangle$ be an increasing sequence of regular cardinals with limit $\lambda$; choose by induction on $\alpha < \lambda^+$, $f_\alpha \in \prod_i \lambda_i$ such that

$$\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J^{\mathrm{bd}}_{\mathrm{cf}\lambda},\ \alpha \in C_\beta \ \&\ |C_\beta| < \lambda_i \Rightarrow f_\alpha(i) < f_\beta(i);$$

let $A_\alpha = \mathrm{Rang}\ f_\alpha$).

Suppose $\lambda^+$ (in $W$'s sense) is a cardinal of $V$. Let us work in $V$. Let $\chi = \mathrm{cf}^V(|\lambda|)$, $\mathrm{cf}^V(\lambda) = \mu$. So $\lambda = \bigcup_{\alpha<\chi} B_\alpha$, $B_\alpha$ increasing continuous with $\alpha$, $|B_\alpha| < |\lambda|$, all in $V$. Now each $A_i$, as an ordered subset of $\lambda$, has cofinality $\mu$ (as $A_i$ is unbounded in $\lambda$) and by assumption (b), $\mu \neq \chi$. Hence for each $i$ for some $\alpha(i) < \chi$, $A_i \cap B_{\alpha(i)}$ is an unbounded subset of $A_i$ (if $\chi < \mu$, trivially, and if $\chi > \mu$ remember $B_\alpha$ is increasing). We are assuming that (in $V$) the number of $A_i$'s is $\lambda^+$, $|\lambda^+| > |\lambda|$, hence for some $\alpha$, $C = \{i : \alpha(i) = \alpha\}$ has power $> |\lambda|$. Let $i$ be the $\lambda$-th member of $C$; so $\{[A_j \backslash f_i(j)] \cap B_\alpha : j < i, j \in C\}$ is a family of $|\lambda|$ pairwise disjoint subsets of $B_\alpha$, each non-empty, contradiction to $|B_\alpha| < |\lambda|$. $\square_{4.4}$

**Theorem 4.10** *Suppose* $W \subseteq V = W[r]$, $r$ *a real and*
*(a) in* $V$ *the continuum hypothesis fails.*
*(b) In* $W$, GCH *holds.*
*(c)* $W$ *has squares.*
*(d)* $(W, V)$ *satisfies the strong* $\aleph_2$*-covering lemma.*
*Then in* $W$ *there is an inaccessible cardinal, in fact* $\aleph_2^V$ *is inaccessible in* $W$*.*

**Remark:** Note, clause $(c)$ really is not necessary (if the conclusion fails then $0^\# \notin V$).

**Proof:** Let $\kappa = \left(2^{\aleph_0}\right)^V$, $\chi = \aleph_1^V$. By 4.7 without loss of generality $\kappa \leq (\aleph_2)^V$ hence by clause $(a)$ we know $\kappa = \aleph_2^V$, hence $\kappa$ is a regular cardinal in $V$ hence in $W$. If the conclusion of the theorem fails, $\kappa$ is a successor cardinal in $W$, so let it be $\kappa = \lambda^+$. So by the previous lemma $\mathrm{cf}^V \lambda = \mathrm{cf}^V(|\lambda|^V)$. However, $|\lambda|^V$ is necessarily $\aleph_1^V = \chi$ (as $\aleph_1^V \leq \lambda < \aleph_2^V$) hence $\mathrm{cf}^V \lambda = \aleph_1^V$.

Let $\bar{C} = \langle C_i : i < \kappa \rangle \in W$ be a list of all bounded subsets of $\kappa$ in $W$. By 4.5(1)(iii), every real $s$ of $V$ is in $L_\alpha[\bar{C}, r]$ for some $\alpha < \kappa$ (so really we can replace $W$ by $L[\bar{C}]$). Let in $V$, $\lambda = \bigcup_{i<\chi} A_i$, $|A_i|^V < \chi$ (remember $\chi = \aleph_1^V$), $A_i (i < \chi)$ increasing continuous. Let $s$ be a real of $V$, then $s \in L_{\alpha(s)}[\bar{C}, r]$ for some $\alpha(s) < \kappa$, without loss of generality $\alpha(s) \geq \lambda$. Let $f_{\alpha(s)} \in W$ be a one-to-one function from $L_{\alpha(s)}[\bar{C}]$ onto $\lambda$. Still working in $V$, $L_{\alpha(s)}[\bar{C}, r] = \bigcup_{\gamma<\chi} N_\gamma^s$, $N_\gamma^s (\gamma < \chi)$ an increasing continuous sequence of countable elementary submodels of $L_{\alpha(s)}[\bar{C}, r]$, closed under $f_{\alpha(s)}$, $f_{\alpha(s)}^{-1}$. So $\langle A_i : i < \chi \rangle$ and $\langle N_\gamma^s \cap \lambda : \gamma < \chi \rangle$ are sequences (in $V$) of countable sets increasing, continuous with the same union: $\lambda$ and of length $\chi = \aleph_1^V$. Clearly for some $\gamma(s)$, $N_{\gamma(s)}^s \cap \lambda = A_{\gamma(s)}$, and let $\delta_\gamma = \sup\ A_\gamma < \lambda$.

Now in $V$ the continuum hypothesis fails, hence there is a list of $\kappa$ distinct reals, $\{s_\zeta : \zeta < \kappa\}$, and we can replace it by any subfamily of power $\kappa$. So without loss of generality $\gamma(s_\zeta) = \gamma(*)$ for every $\zeta < \kappa$ and for each $\zeta < \kappa$, let $A^\zeta$ be the closure of $\delta_{\gamma(*)}$ by $f_{\alpha(s_\zeta)}$, $f_{\alpha(s_\zeta)}^{-1}$ so $A^\zeta \in W$. Now in $W$ the number of possible isomorphism types of

$$M_\zeta =: (A^\zeta, f_{\alpha(s_\zeta)}, f_{\alpha(s_\zeta)}^{-1}, <, \text{“}i \in C_j\text{”}, \delta_{\gamma(*)})$$

is $\leq 2^{|\delta_{\gamma(*)}|} \leq \lambda$ (as $W$ satisfies GCH). So without loss of generality this isomorphism type is the same for all ordinals $\zeta < \kappa$.

Now we show that all $N_{\gamma(*)}^{s_\zeta}$ (for $\zeta < \kappa$) are isomorphic (in $V$) : let $\zeta$, $\xi < \kappa$, now any isomorphism from $M_\zeta$ onto $M_\xi$ is the identity on $A_{\gamma(*)}$ (as

$A_{\gamma(*)} \subseteq \delta_{\gamma(*)} \subseteq M_\xi$), hence take $N^{s_\zeta}_{\gamma(*)} \cap \lambda = A_{\gamma(*)}$ onto $N^{s_\xi}_{\gamma(*)} \cap \lambda = A_{\gamma(*)}$; but $|N^{s_\ell}_{\gamma(*)}|$ is the closure of $N^{s_\ell}_{\gamma(*)} \cap \lambda$ by $f_{\alpha(s_\zeta)}$, $f^{-1}_{\alpha(s_\zeta)}$; so looking at the definition of $M_\zeta$ we see that the isomorphism takes $N^{s_\zeta}_{\gamma(*)} \cap \kappa$ onto $N^{s_\xi}_{\gamma(*)} \cap \kappa$ and preserve the relation "$i \in C_j$" and map $r$ to $r$. But $N^{s_\ell}_{\gamma(*)}$ "think" it is $L_{\alpha(s)}[\bar{C}, r]$, so the isomorphism can be extended to an isomorphism from $N^{s_\zeta}_{\gamma(*)}$ onto $N^{s_\xi}_{\gamma(*)}$, as promised. But $N^{s_\zeta}_{\gamma(*)}$ is countable, and we have too many reals, contradiction. $\square_{4.10}$

**Conclusion 4.11** If there are universes $W \subseteq V = W[r]$, $r$ a real, $W$ satisfies GCH, and CH fails in $V$ <u>then</u> in $L$ there is an inaccessible cardinal, in fact $\aleph^V_2$ is inaccessible in $L$.

**Proof:** Suppose in $L$ there is no inaccessible cardinal or just $\aleph^V_1$ is not unaccessible in $L$. Then $0^\# \notin V$ hence $0^\# \notin W$ and as $W$ satisfies GCH, $W$ has squares and scales. If $(W, V)$ satisfies the strong $\aleph^V_2$-covering lemma, then all the hypothesis of 4.10 are satisfied, hence its conclusion, which is the conclusion of 4.11. Still by §2 we do not know that the strong $\aleph_2$-covering lemma holds. However, (letting $\chi = \aleph^V_1$, $\kappa = \aleph^V_2, \kappa = (\lambda^+)^\kappa$) by 4.15 below, we know that for every real $s \in V$, for some increasing continuous sequence $\langle N^s_i : i < \chi \rangle$ of countable models (in $V$, $N_i \subseteq V$) we have $s \in N^s_i$, $N^s_i = (N^s_i \cap W)[r]$, $\bigcup_{i<\chi} N_i \cap \kappa$ is an ordinal$> \lambda$, and each $N_i$ is 2-trivially defined from $N_i \cap \kappa$ (see 4.12 for meaning). The rest is as in the proof of 4.10. $\square_{4.11}$

**Remark 4.11A** (1) So why 4.10, 4.11 comes before 4.15? We think the proof of 4.10 makes the understanding of 4.12 — 4.15 easier (using the notation of the proof of 4.10).

(2) But 4.11 is later reproved (in 4.17).

**Definition 4.12** Let $W \subseteq V$, $A \in V$, $A \subseteq \lambda^*$, $\alpha$ an ordinal, $B \subseteq \alpha$, $B \in V$. We define when "$B$ is $\ell$-trivially defined over $(W, A, \alpha)$" or $B \in W^\ell_{tr}[A, \alpha)$ for $\ell = 0, 1, 2$ (where cl, $M^2_\alpha$, $C^\ell_\alpha$ are as in §2, specifically see 2.3D, 2.2B, 2.2A).

$\ell = 0$ : for some $\delta$, $B = \mathrm{cl}\big(A \cup \{\delta\}, M^2_\alpha\big) \cap C^1_\delta$

$\ell = 1$ : for some $B_1 \in W^0_{tr}[A, \alpha)$, and a function $f \in W$,

$$B = \mathrm{cl}\Big(A \cup \{f(i) : i \in B_1\} \cup (\textstyle\bigcup_{i \in B_1} C^1_{f(i)}), M^2_\alpha\Big)$$

$\ell = 2$ : for some $n < \omega$, $B_1, ..., B_n \in W^1_{tr}[A, \alpha)$, and $\beta \leq \alpha$,

$$B = \mathrm{cl}\Big((A \cup \textstyle\bigcup^n_{m=1} B_m), M^2_\alpha\Big) \cap \beta.$$

**Definition 4.13** In $V$ let $D$ be a filter on $\mathcal{S}_{<\lambda^*}(\lambda^*)$, $\lambda^*$ be regular cardinal. We define the strong $(\lambda^*, D, \alpha)$-covering game; it last $\lambda^*$ moves; in the $i$-th move, player I chooses $a_i \in V$, a subset of $\alpha$ of power $< \lambda^*$ (in $V$) and

a function $f_i$ from an ordinal $< \lambda^*$ onto $a_i$, such that $a_i \supseteq \bigcup_{j<i} b_j$ and $f_i \supseteq \bigcup_{j<i} g_j$ and then player II chooses $b_i$, a subset of $\alpha$ of power $< \lambda^*$ (in $V$) and a function $g_i$ from an ordinal $< \lambda^*$ onto $b_i$, such that $b_i \supseteq \bigcup_{j\leq i} a_j$, $g_i \supseteq \bigcup_{j\leq i} f_j$.
In the end player I wins if the following set belongs to $D$ :

$$\Big\{A \in \mathcal{S}_{<\lambda^*}(\lambda^*) : \{f_i(\alpha) : i < \lambda^*, \alpha \in A\} \in W^2_{tr}[A, \alpha)\Big\}.$$

$(W, V)$ has the $(\lambda^*, D, \alpha)$-strong covering property if player I has a winning strategy in the $(\lambda^*, D, \alpha)$-strong covering game. We omit $\alpha$ if it is true for every $\alpha$.

**Remark 4.13A** Without loss of generality Dom $f_i = a_i \cap \lambda^*$, Dom $g_i = b_i \cap \lambda^*$. This definition does not contradict the earlier one as the filter here is not on some cardinal (but on $\mathcal{S}_{<\lambda^*}(\lambda^*)$).

**Definition 4.14** Suppose in $V$, $\lambda^* = \lambda^+$, $\lambda$ (and $\lambda^*$) are regular. We shall define a filter $D[\lambda^*, \lambda]$.
Let for each $\alpha < \lambda^*$, $\alpha = \bigcup_{i<\lambda} A^\alpha_i$, $A^\alpha_i$ increasing continuous, $|A^\alpha_i| < \lambda$.

$$D[\lambda^*, \lambda] = \Big\{S \subseteq \mathcal{S}_{<\lambda^*}(\lambda^*) : \begin{array}{l}\text{for some closed unbounded } C \subseteq \lambda^*, \text{ for}\\ \text{every } \alpha \in C, \text{ if } \mathrm{cf}\alpha = \lambda \ \underline{\text{then}}\\ \{i < \lambda : A^\alpha_i \in S\} \in D_\lambda\Big\}.\end{array}$$

**Remark 4.14A** (1) This definition appears essentially in [Sh52, §3].
(2) The filter does not depend on the choice of the $A^\alpha_i$'s.

**Theorem 4.15** *Suppose in $V$, $\lambda^* = \lambda^+$ and $\lambda$, $\lambda^*$ are regular cardinals $\lambda > \aleph_0$ and let $D = D[\lambda^*, \lambda]$. If $W$ has $\lambda^*$-squares, has scales, and $(W, V)$ satisfies the $\lambda^*$-covering lemma* <u>*then*</u> *$(W, V)$ has the $(\lambda^*, D)$-strong covering property.*

**Proof:** We just repeat the proof of 2.2, 2.3.
Note that we use $\beta \in W^0_{tr}[A, \alpha)$ for the parallel of "$C^1_{\delta(\zeta,\theta)} \subseteq a^0_\zeta$" and $B \in W^1_{tr}[A, \alpha)$ for the parallel of "$\bigcup_{\theta\in S} C^1_{\delta(\zeta,\theta)}$". $\square_{4.15}$

**Corollary 4.16** *If the hypothesis of 4.15 holds and $\aleph^W_2 = \aleph^V_2$ then $(W, V)$ satisfies the strong $\aleph_2$-covering lemma and the strong $\aleph_1$-covering lemma.*

* * *

We have remarked that if $\lambda^*$ is the successor of $\lambda$ in $W$, things are much simpler. Let us present this

**Lemma 4.17** *Assume $W \subseteq V$, $\lambda$ a regular uncountable cardinal in $V$ and $(\lambda^+)^V = (\lambda^+)^W$ and $(W,V)$ satisfies the $\lambda^+$-covering lemma.*
*(1) If $W$ has $\lambda$-squares, $D = D_\lambda + \{\delta < \lambda : \mathrm{cf}^V \delta > \aleph_0\}$ then $(W,V)$ satisfies the strong $(\lambda,\lambda,D)$-covering lemma.*
*(2) If $W$ has $(\lambda^+)^W$-squares, $D = D_\lambda$, then $(W,V)$ satisfies the strong $(\lambda,\lambda,D)$-covering lemma.*
*(3) If $W$ has $\lambda$-squares,*
*$D = \{\lambda \backslash A : \{\delta < \lambda : \delta \in A$ or $A \cap \delta$ is stationary in $\delta\}$ is not stationary$\}$.*
*Then $(W,V)$ satisfies the strong $(\lambda,\lambda,D)$-covering lemma.*

**Proof:** For any ordinal we can find $\mu$ bigger than it, $\mu$ a regular cardinal in $W$, $V \models$"$\mu^\lambda = \mu$", and let $\alpha(*) = (\mu^+)^W$. Clearly it suffices to deal with subsets of $\alpha(*)$ (in (1) — prove that player I wins the $(\lambda,\lambda,D,\alpha(*))$-covering game).

We define a model $M_{\alpha(*)}$. Let, in $W$, $\{f_\beta : \beta < \alpha(*)\} \in W$ list all functions $f \in W$ such that: $\mathrm{Dom}\, f \subseteq \{\kappa : \lambda \le \kappa \le \mu,\ \ \kappa$ regular in $W\}$ and $W \models |\mathrm{Dom}\, f| < \lambda$ and $f(\kappa) < \kappa$ for $\kappa \in \mathrm{Dom}\, f$ (there is such a list as $V \models$"$\mu^{<\lambda} = \mu$". For (1) let $M_{\alpha(*)}$ be $M^0_{\alpha(*)}$ (from 2.2A) expanded by $F$, a partial two place function, $F(\beta,\gamma) = f_\beta(\gamma)$. For (2) , we replace

$$\langle C^1_\alpha : \lambda \le \alpha < \alpha(*) \text{ and } \mathrm{cf}^V(\alpha) < \lambda\rangle$$

($\lambda$-square) by a $\lambda^+$-square

$$\langle C^1_\alpha : \lambda \le \alpha < \alpha(*),\ \mathrm{cf}^V(\alpha) < \lambda^+\rangle,$$

otp $C^1_\alpha \le \lambda$ (equality holds when $\mathrm{cf}^V \alpha = \lambda$).

Note that $(W,V)$ satisfies the $\lambda$-covering lemma [if $a \subseteq \mathrm{Ord}$, $V \models$"$|a| < \lambda$" by assumption there is $b \in W$, $V \models$"$|b| < \lambda^+$" and $a \subseteq b$. So $|b|^W < (\lambda^+)^V = (\lambda^+)^W$ hence by an assumption $W \models |b| \le \lambda$. So in $W$ we have an increasing sequence $\langle b_i : i < \lambda\rangle$, $b = \bigcup_{i<\lambda} b_i$, $W \models |b_i| < \lambda$. Now for some $i$, $a \subseteq b_i$ (as in $V |a| < \lambda$ & $\mathrm{cf}\lambda = \lambda$) so we finish]. Now let for any set $a \subseteq a(*)$, $\mathrm{Ch}_a$ be the function with domain

$$\{\kappa < \alpha : \kappa \text{ regular uncountable in } W,\ \kappa \in a\},$$

$\mathrm{Ch}_a(\kappa) = \sup(a \cap \kappa)$.

We now define a strategy for player I in the $(\lambda,\lambda,D,\alpha)$-covering game: he chose $a_i \subseteq \alpha$ such that: $a_i \in W$, $a_i \subseteq \alpha$, $\mu \in a_i$, $|a_i| < \lambda$ and $a_i$ include the closure in order topology of the Skolem Hull of $\bigcup_{j<i} b_j$ in $M_\alpha$, and for each $j < i$ for some $\beta_j < \alpha(*)$, $\mathrm{Ch}_{a_j} = f_{\beta_j}$ and $\beta_j \in a_{j+1}$. Clearly this is possible.

Let us show that this is a winning strategy. So let $\langle a_i, b_i : i < \lambda\rangle$ be a play of the $(\lambda,\lambda,D,\alpha)$-covering game in which player I uses his strategy.

By the assumption [i.e. $(W,V)$ has $\lambda^+$-covering, applied to the set $\bigcup_{i<\lambda} a_i$] there is a set $d \subseteq \alpha$, $d \in W$, $|d| < \lambda^+$ and $\bigcup_{i<\lambda} a_i \subseteq d$. As before (because $(\lambda^+)^V = (\lambda^+)^W$] we have $W \models$"$|d| \le \lambda$" so there is an increasing continuous sequence $\langle d_i : i < \lambda\rangle \in W$ of subsets of $d$ such that: $d = \bigcup_{i<\lambda} d_i$ and $[i < \lambda \Rightarrow W \models$"$|d_i| < \lambda$"$]$.

Clearly $C_0 = \{\delta < \lambda : \delta$ a limit ordinal and $d_\delta \cap \bigcup_{j<\lambda} a_j = \bigcup_{j<\delta} a_j\}$ is a club of $\lambda$. Also

$C_1 = \{\delta < \lambda :$ <u>if</u> $\beta \in d_\delta$ and for some $j < \lambda$, $f_\beta < f_{\alpha_j}$
(i.e. Dom $f_\beta \subseteq$ Dom $f_{\alpha_j}$ and $(\forall \kappa \in \text{Dom } f_\beta)[f_\beta(\kappa) < f_j(\kappa)]$) <u>then</u>
there is such $j < \delta\}$

is a club of $\lambda$. Hence it suffices to prove that for every $\delta \in C_0 \cap C_1$ we have $\bigcup_{j<\delta} a_j \in W$. Let $\delta \in C_0 \cap C_1$, define $Y_\delta =: \{\beta \in d_\delta : f_\beta < f_{\alpha_\delta}\}$.

Now for each $\zeta < \delta$, we know that

$$\beta_\zeta \in a_{\zeta+1} \subseteq \bigcup_{j<\delta} a \text{ and } f_{\beta_\zeta} = \text{Ch}_{a_\zeta} < \text{Ch}_{a_\delta} = f_{\alpha_\delta}$$

hence $[\zeta < \delta \Rightarrow \beta_\zeta \in Y_\delta]$.

On the other hand (as $\delta \in C_1$)

$$\beta \in Y_\delta \Rightarrow f_\beta < f_{\beta_\delta} \Rightarrow \bigvee_{j<\delta} f_\beta < f_{\beta_j}.$$

Hence for every $\kappa \in \bigcup_{j<\delta} a_j \backslash \lambda$ regular in $W$

$$\text{Ch}_{\bigcup_{j<\delta} a_j}(\kappa) = \sup_{j<\delta} \text{Ch}_{a_j}(\kappa) = \sup_{\beta \in Y_\delta} f_\beta(\kappa).$$

So $g^*_\delta$, the function with domain

$$\{\kappa : \kappa \in d_\delta \backslash \lambda, \kappa \le \mu, \quad \kappa \text{ regular in } W\},$$

$g^*_\delta(\kappa) = \sup_{\beta \in Y_\delta} f_\beta(\kappa)$ belongs to $W$ (as $Y_\delta$ and $d_\delta$ belongs) and

$$\text{Ch}_{\bigcup_{j<\delta} a_j} \subseteq g^*_\delta.$$

**Proof of 4.17(1):** Remember by assumption $W$ has $\lambda$-square, say

$$\langle C^1_\delta : \delta < \alpha(*), \text{cf}\delta < \lambda\rangle,$$

and they "appear" in $M_{\alpha(*)}$. By the strategy for every $j < \lambda$ of uncountable cofinality and $\theta \in a_j \backslash \lambda$ (regular in $W$, $\in [\lambda, \mu]$) $C^1_{\text{Ch}_{a_j}(\theta)} \subseteq a_{j+2}$. Hence as in 2.2$(D)$ for limit $\delta \in C_0 \cap C_1$ of uncountable cofinality $C^1_{\text{Ch}_{\cup_{j<\delta} a_j}(\theta)} \subseteq$

$\bigcup_{j<\delta} a_j$, so by 2.2 we finish similarly to 2.3E. I.e. define by induction on $n$ :

$a_0$ is the Skolem Hull of $\emptyset$ in $M_{\alpha(*)}$
$a_{n+1}$ is the Skolem Hull in $M_{\alpha(*)}$ of
$a_n \cup \{C^1_{g^*(\theta)} : \theta \in a_n$ a regular cardinal $\geq \lambda$ of $W$, in the domain of $g^*\}$.

Clearly $\langle a_n : n < \omega\rangle \in W$ hence $a_\omega =: \bigcup_{n<\omega} a_n \in W$, and each $a_n$ is a subset of a so $a_\omega \subseteq a$. Lastly $a_\omega = a$ similarly to 2.3E.

**Proof of 4.17(2),(3):**
Similar. $\square_{4.17}$

* * *

**Lemma 4.18** *Suppose* $W \subseteq V = W[r]$, $\aleph_1^V = \aleph_1^W$, *$r$ a real, $W$ satisfies* CH *while $V$ fails* CH. <u>*Then*</u> $\aleph_2^V$ *is inaccessible in $L$.*

**Proof:** Assume the conclusion fails, so $\kappa =: \aleph_2^V = (\lambda^+)^W$, $\lambda$ a cardinal in $L$. Let $\chi = \aleph_1^V = \aleph_1^W$. By 4.9 $\mathrm{cf}^V \lambda = \aleph_1$. Also as $\aleph_2^V$ is not inaccessible in $L$, necessarily $0^\# \notin V$ hence by 4.7 $V \models 2^{\aleph_0} \leq \aleph_2$ hence $V \models 2^{\aleph_0} = \aleph_2$. Choose $A \in V$, $A \subseteq \lambda$ such that $\aleph_1^{L[A]} = \aleph_1^V (= \aleph_1^W)$ and $L[A] \models$"$|\lambda| = \aleph_1$", (so we cannot exclude the possibility "$A \notin W$"). Now by Lemma 4.19 below, $L[A] \models$"$2^{\aleph_0} = \aleph_1$", (note $L, L[A]$, $\lambda$, A here stand for $W$, $V$, $\lambda$, $r$ there).

By 2.8 (with $L[A]$, $V$, $\aleph_1^V$, $\aleph_2^V$, $\aleph_0$ here standing for $W$, $V$, $\lambda^*$, $\lambda$, $\kappa$ there) the pair $(L[A], V)$ satisfies the strong $\aleph_1^V$-covering lemma. As $L[A] \models$CH by 4.5 also $V$ satisfies CH, contradiction. $\square_{4.18}$

**Claim 4.19** Suppose $W \subseteq V = W[r]$, $r$ a subset of $\lambda$, $\lambda$ a cardinal of $W$, $(\lambda^+)^W = \aleph_2^V$ and $W$ satisfies GCH.
<u>Then</u> $V$ satisfies CH.

**Proof:** Let $\kappa =: \aleph_2^V$, $\chi = \aleph_1^V$, so $W \models$"$\kappa = \lambda^+$". Now $V \models$"$\mathrm{cf}\lambda = \aleph_1$" by 4.9; and assume $V \models$"$2^{\aleph_0} > \aleph_1$" and we shall get a contradiction.

Now repeat the proof of 4.10 (after the first paragraph). The additional point is in proving $N_\gamma^{s_\zeta}$, $N_\gamma^{s_\xi}$ are isomorphic. We have to check that the mapping preserves "$i \in r$", but $r \subseteq \lambda$ and $N_\gamma^{s_\zeta} \cap \lambda = N_\gamma^{s_\xi} \cap \lambda \subseteq A_{\delta_\gamma}$, and the mapping is the identity on $A_{\delta_\gamma}$. $\square_{4.19}$