STRONG COVERING LEMMA AND CH IN $\mathbf{V}[r]$

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Abstract. For an inner model $\mathbf{W}$ of $\mathbf{V}$, the $(\mathbf{W},\mathbf{V})$-covering lemma states that for cardinals λ, κ with $\lambda > \kappa = \mathrm{cf}(\kappa)$ (usually $\kappa \geq \aleph_1$), the set

$$([\lambda]^{<\kappa})^{\mathbf{W}} := [\lambda]^{<\kappa} \cap \mathbf{W}$$

is cofinal in $[\lambda]^{<\kappa}$ (where $[\lambda]^{<\kappa} := \{A \subseteq \lambda : |A| < \kappa\}$, ordered by inclusion).

The *strong* $(\mathbf{W},\mathbf{V})$-covering lemma for (λ,κ) states that $([\lambda]^{<\kappa})^{\mathbf{W}}$ is a stationary subset of $[\lambda]^{<\kappa}$, which means that for every model $M \in \mathbf{V}$ with universe λ and vocabulary of cardinality $< \kappa$, there is $N \prec M$ with universe $\in ([\lambda]^{<\kappa})^{\mathbf{W}}$.

We give sufficient conditions for the strong $(\mathbf{W},\mathbf{V})$-covering lemma to hold, which are satisfied in the classical cases where the original lemma holds (i.e. covering, squares, and reals). In fact, we place stronger conditions on M. The proof does not use fine structure theory, but only some well-known combinatorial consequences thereof.

We use this to solve problems about the aspects of adding a real to a universe $\mathbf{V}$.

Earlier versions appeared as [She82, XIII,§1-4] in the author's book *Proper Forcing* (Springer-Verlag 940, 1982), and later versions as Chapter VII of *Cardinal Arithmetic* (Oxford University Press, Clerendon Press, Vol. 24).

§ 0. Introduction

We prove a strengthening of the covering lemma, not using the fine structure theory (only some well-known consequences; see Theorem 0.2). We prove it essentially in all cases in which the covering lemma holds.

This is essentially Chapter XIII, Sections 1-4 of *Proper Forcing* [She82] (the other sections, 5 and 6, are superseded by the other material in this book). My interest in the subject stems from Abraham's (see below), and the last sparks were discussions with Harrington and Woodin, and Harrington's willingness to hear the proof while it was being done. When revising [She82], I was told that it did not fit there[1] (not to say that the proof of $(\aleph_\omega)^{\aleph_0} < \aleph_{[2^{\aleph_0}]^+}$ in [She82, XIII,§5-6] was misplaced). As the proofs here inspire the proof of $\aleph_\omega^{\aleph_0} < \aleph_{[2^{\aleph_0}]^+}$ (i.e. reconstructing a submodel M by the characteristic function) and are combinatorial in character, we hope they will be more welcome here. Note that the main problem here is very close to

$$\min\{|S| : S \subseteq \mathcal{S}_{<\kappa}(\lambda) \text{ is stationary}\},$$

which plays an important role in the rest of the book, but is not identical. The characteristic function of a model, which has a major role here, is also used (for example) in [She94a, §1]: a difference being that here we use squares, whereas

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[1] Although see the remark below on a connection with properness.

in other places in the book we use weaker principles which hold in more general circumstances.

The changes from [She82, XIII,§1-4] are minor — local improvements in presentation (hopefully) and adding 0.5, 4.23.

The neatest case of the strong covering lemma is

Theorem 0.1. *Assume $\mathbf{0}^\#$ does not exist (in $\mathbf{V}$) and $A \subseteq \mathrm{Ord}^{\mathbf{V}}$. If $\aleph_2^{\mathbf{V}} = \aleph_2^{\mathbf{L}[A]}$ and M is a model in $\mathbf{V}$ with countably many finitary functions whose set of elements is an ordinal α, then for every $b \subseteq \alpha$ there is a set $a \subseteq \alpha$ which belongs to $\mathbf{L}[A]$, is closed under the functions of M, $b \subseteq a \subseteq \alpha$, and $|a| \leq |b|$ (in $\mathbf{V}$).*

[How can both of these be true, unless you have equality on the RHS?]

The theorem is really much more general — it speaks on a pair of universes $\mathbf{W} \subseteq \mathbf{V}$, and uses three hypotheses which are known to hold in the case above: the usual covering lemma, the existence of squares, and the existence of scales. (For successors of singular cardinals, see §1 for the definitions; it follows from GCH in the smaller universe.) Also, the conclusion is stronger: for regular $\kappa < \lambda < \lambda^*$, (with $\kappa > \aleph_0$ for simplicity) and an ordinal α, Player I has a winning strategy in the following game of length λ:

In the i^{th} move, Player I chooses $a_i \subseteq \alpha$ with $|a_i|^{\mathbf{V}} < \lambda^*$ and $\bigcup_{j<i} b_j \subseteq a_i$, and Player II chooses $b_i \subseteq \alpha$ with $|b_i|^{\mathbf{V}} < \lambda^*$ and $\bigcup_{j\leq i} a_j \subseteq b_i$.

In the end, Player I wins the play if for some closed unbounded $C \subseteq \lambda$ we have

$$\delta \in C \wedge \mathrm{cf}(\delta) = \kappa \Rightarrow \bigcup_{i<\delta} a_i \in \mathbf{W}.$$

We can conclude that (for example) if $\mathbf{0}^\# \notin \mathbf{V}$, then any forcing notion satisfies quite strong properness conditions. I.e. let $G \subseteq \mathbb{P}$ be generic over $\mathbf{V}$; we know that, for given cardinal χ and $x \in \mathcal{H}^{\mathbf{V}[G]}(\chi)$, there are (quite a few)

$$N \prec \left(\mathcal{H}^{\mathbf{V}[G]}(\chi), \in, <^*_\chi, \mathcal{H}^{\mathbf{V}}(\chi)\right)$$

such that $x \in N$ and $N \cap \mathcal{H}^{\mathbf{V}}(\chi) \in \mathbf{V}$, so there is $q \in G$ which forces this and forces a value to $N \cap \mathcal{H}^{\mathbf{V}}(\chi)$. Hence q is $(N \cap \mathcal{H}^{\mathbf{V}}(\chi), \mathbb{P})$-generic in $\mathbf{V}$. (Of course, this does not say that for any $N' \prec (\mathcal{H}(\chi)^{\mathbf{V}}, \in, <^*_\chi)$ we can find such a condition q). For example, there is such an N which has cardinality $\aleph_2^{\mathbf{V}[G]}$ in $\mathbf{V}[G]$. This was the rationale for putting this in [She82].

The problem arises as follows: Jensen and Solovay [JS70] asked how adding a real can affect a universe. Now adding $\mathbf{0}^\#$ to $\mathbf{L}$ causes the collapsing of many cardinals, and they knew that adding some real by forcing may collapse many cardinals. Later in Beller, Jensen and Weltch [ABW82] much more radical results were proved: if $\mathbf{V}$ satisfies GCH, then there is a generic extension of $\mathbf{V}$ (by a class forcing) which preserves cardinalities and has the form $\mathbf{L}[a]$ (first it was assumed $\mathbf{0}^\# \notin \mathbf{L}$). See more on this in Shelah-Stanley [SS95]. Still, $\mathbf{L}[a]$ always satisfies GCH. So it was natural to ask, as Jensen and Solovay [JS70] did:

Problem 0.2. If $\mathbf{W}$ satisfies GCH and $\mathbf{V} = \mathbf{W}[r]$, where r is a real and $\mathbf{V}$ and $\mathbf{W}$ have the same cardinals, does $\mathbf{V}$ satisfy CH?

There are also several other variants; for example,

Problem 0.3. 1) If $\mathbf{W}$ satisfies CH, $\mathbf{V} = \mathbf{W}[r]$ with r a real, and $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{W}}$, then does $\mathbf{V}$ satisfy CH?

2) Ask in addition that $\mathbf{V}$, $\mathbf{W}$ have the same cardinals $< 2^{\aleph_0}$ *and/or* $\mathbf{W}$ satisfies GCH.

Abraham [Abr79] was interested in this problem, he proved that the conclusion of 0.1 implies a positive answer to Question 0.2, and the author notes 0.1 holds if $\alpha < \aleph_\omega$. Harrington and Van Liere have similar results, working in parallel. Abraham [Abr79] also conjectured 0.1 when $\mathbf{V}$ and $\mathbf{L}$ have the same cardinals. He also gave another application:

If $\mathbf{L}[A]$, $\mathbf{L}[B]$ have no *non-constructible* reals, then neither does $\mathbf{L}[A,B]$ (provided that $\aleph_1^{\mathbf{L}[A,B]} = \aleph_1^{\mathbf{V}}$).

Just before the present work was completed, Shelah and Woodin [SW84] proved the consistency of a negative answer to problems 0.2 and 0.3. For example, adding a real to a universe $\mathbf{V}$ satisfying GCH may blow up the continuum while not collapsing cardinals, starting with a universe $\mathbf{W}$ with enough measurable cardinals (hence answering 0.2 in the negative). The other extreme variant is that from the consistency of ZFC we can get $\mathbf{V} = \mathbf{W}[r]$, with $\mathbf{W}$ satisfying CH, $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{W}}$, and $(2^{\aleph_0})^{\mathbf{V}}$ arbitrarily large (i.e. answering Problem 0.3(1)). Here, using the strong covering lemma we get several complementary results, so we know which large cardinals are necessary for which variant; for some variants we know exactly, and for some we know reasonable lower and upper bounds. This is done in Section 4, and one of the cases (see 4.13) involves proving somewhat more than the strong covering lemma.

The cases in which we do not have exact results are:

(A) For the first result, (for 0.2) a measurable cardinal is necessary, but Shelah and Woodin [SW84] use $(2^{\aleph_0})^{\mathbf{V}}$-many; we need a suitable inner model, so maybe Mitchell's work [Mit84] can help to close the case.

(B) The existence of $\mathbf{V} = \mathbf{W}[r]$, where $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{W}}$ and $\mathbf{W}$ satisfies GCH, but CH fails in $\mathbf{V}$. We need an inaccessible, and a 2-Mahlo cardinal suffices.

(C) For problem 0.2 when $\mathbf{W}$ satisfies GCH and $2^{\aleph_0} = \aleph_n$ in $\mathbf{V}$ (with $2 < n < \omega$), $\mathbf{0}^\#$ is necessary but '$\aleph_n$ measurables' will suffice.

The obvious approach to the strong covering lemma seemed to be to redo the covering lemma more carefully (and so it was thought); however, this is not our solution. Rather, we prove the statement described above by induction on α, using only some principles which follow and hold in many other situations.

After this work, two beautiful related covering theorems were proved. Carlson proved a stronger theorem from a stronger assumption: if $\mathbf{0}^\# \notin \mathbf{V}$, any increasing sequence of uncountable regular length of sets of ordinals from $\mathbf{L}$ belongs to $\mathbf{L}$. Magidor [Mag90] proved that any somewhat closed submodel of $(\mathbf{L}_\alpha, \in)$ is the union of $\leq \aleph_0$ sets from $\mathbf{L}$ if $\mathbf{0}^\# \notin \mathbf{V}$ (or at least the core model K has no Erdős cardinal).

* * *

Another question is due to Mathias [Mon96].

Question 0.4. Suppose $\mathbf{V}$ satisfies GCH and $A \subseteq \aleph_{\omega_1}$. If $\mathbf{V}[A]$ has the same cardinals as $\mathbf{V}$ and $2^{\aleph_0} > \aleph_{\omega_1}$ in $\mathbf{V}[A]$, can we have $\aleph_1^{\mathbf{V}[A]} = \aleph_1^{\mathbf{V}}$?

Note that if we replace $\aleph_{\omega_1}$ by a regular cardinal the answer is negative, and if we replace it by a singular cardinal of cofinality $\aleph_0$ (such as $\aleph_\omega$) the answer is positive. By the strong covering lemma, if $\mathbf{0}^\# \notin \mathbf{V}$, or even if $\mathbf{V}$ has no inner model with a measurable, the answer is no. In fact, even if $\mathbf{0}^\# \notin \mathbf{L}[A]$,

$$\mathbf{V} \vdash (\forall \alpha < \omega_1)\big[(\aleph_\alpha)^{\aleph_1} < \aleph_{\omega_1}\big],$$

$\aleph_\alpha^{\mathbf{V}} = \aleph_\alpha^{\mathbf{V}[A]}$ for $\alpha = \omega_1$ and for arbitrarily large $\alpha < \aleph_{\omega_1}$, then $\mathbf{V}[A] \vdash$ "$2^{\aleph_0} \leq \aleph_{\omega_1}$". It seemed very persuasive that using the inner models for hyper-measurable (see Mitchell [Mit84]) we can get stronger inner models for that question (and get the relevant exact equi-consistency result for the question of violating CH by adding a real mentioned above).

Recently, by [She94c], if we replace ω_1 by ω_4, the answer is no. Really, a negative answer of 0.4 will follow if we can prove, in ZFC:

$$(\forall \delta < \omega_1)[\delta \text{ is limit} \Rightarrow \mathrm{pp}(\aleph_\delta) < \aleph_{\omega_1}].$$

Both follow by the next theorem (see more in [She94c, §3]).

Theorem 0.5. 1) *Assume $\mathbf{V}$ is a model of set theory satisfying the GCH, λ a strong limit cardinal, $A \subseteq \lambda$ (not in $\mathbf{V}$), $\mathbf{V}[A]$ is a model of set theory with the same cardinals $\leq \lambda^+$, and*

(∗) *In $\mathbf{V}[A]$, there is a stationary $S \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$ of cardinality λ.*

Then $2^{\aleph_0} \leq \lambda$ *in* $\mathbf{V}[A]$.

2) *Assume $\mathbf{V}$ is a model of set theory, λ a strong limit cardinal, $\kappa < \lambda$, $A \subseteq \lambda$ (not in $\mathbf{V}$), $\mathbf{V}[A]$ is a model of set theory, $(\kappa^+)^{\mathbf{V}}$, λ, and $(\lambda^+)^{\mathbf{V}}$ are cardinals also in $\mathbf{V}[A]$, and*

(∗) *In $\mathbf{V}[A]$, there is a stationary subset of $\mathcal{S}_{\leq \kappa}(\lambda)$ of cardinality $\leq \lambda$.*

Then $\lambda^\kappa \leq \lambda$ *in* $\mathbf{V}[A]$.

Remark 0.6. The assumption (∗) holds, for example, when $\lambda = \aleph_{\omega_4}$ (by [She94c, 4.4+3.7]). The proof is similar to that of 4.11.

Proof. 1) Let

$$\mathfrak{A} := \left(\mathcal{H}(\lambda^+)^{\mathbf{V}[A]}, \mathcal{H}(\lambda^+)^{\mathbf{V}}, A, \in, <^*_\lambda\right)$$

(where $<^*_{\lambda^+} \in \mathbf{V}$ is a well-ordering of $\mathcal{H}(\lambda^+)^{\mathbf{V}}$).

We can represent $\mathfrak{A}$ (in $\mathbf{V}[A]$) as an increasing continuous chain $\mathfrak{A}_i$ (for $i < \lambda^+$) with $\|\mathfrak{A}_i\|^{\mathbf{V}[A]} < \lambda^+$ (because $\mathbf{V}[A] \vdash$ '$2^\lambda \leq \lambda^+$'). Similarly in $\mathbf{V}$, we may decompose $\mathcal{H}(\lambda^+) = \bigcup\limits_{i<\lambda^+} \mathbf{W}_i$ with $\mathbf{W}_i$ increasing continuous, $|\mathbf{W}_i| = \lambda < \lambda^+$, and

$$\langle \mathbf{W}_i : i < \lambda^+ \rangle \in \mathbf{V}.$$

In $\mathbf{V}[A]$, the set $\{i < \lambda^+ : \mathcal{H}(\lambda^+)^{\mathbf{V}} \cap \mathfrak{A}_i = \mathbf{W}_i\}$ is a club of λ^+, so for some club $E \in \mathbf{V}[A]$ of λ^+, for every $i \in E$, $\mathfrak{A}_i \prec \mathfrak{A}$ and $\mathcal{H}(\lambda^+)^{\mathbf{V}} \cap \mathfrak{A}_i = \mathbf{W}_i$. Let $\bar{f} = \langle f_i : i < \lambda^+ \rangle \in \mathbf{V}$ be such that each $f_i : \lambda \to \mathbf{W}_i$ is bijective.

Now for every $r \in ({}^\omega 2)^{\mathbf{V}[A]}$, we can find $i_r \in E$ such that $r \in \mathfrak{A}_{i_r}$ and a countable elementary submodel (N_r, f^r) of $(\mathfrak{A}_{i_r}, f_{i_r})$, with $N_r \cap \lambda \in S$, to which r belongs. Let $\mu_r < \lambda$ be such that $N_r \cap \mathcal{H}(\lambda)^{\mathbf{V}[A]} \subseteq \mathcal{H}(\mu_r)^{\mathbf{V}[A]}$, and let M_r be the elementary submodel of $\left(\mathcal{H}(\lambda^+)^{\mathbf{V}}, f_{i_r}, \in, <^*_{\lambda^+}\right)$ with universe the Skolem Hull of $\mathcal{H}(\mu_r)^{\mathbf{V}} \cup \{f_{i_r}\}$. (Note that $<^*_{\lambda^+} \in \mathbf{V}$ is a well-ordering of $\mathcal{H}(\lambda^+)^{\mathbf{V}}$.)

Clearly $M_r \in \mathbf{V}$ and $\|M_r\| \leq |\mathcal{H}(\mu_r)^{\mathbf{V}}| < \lambda$; as λ is strong limit in $\mathbf{V}$, the number of isomorphism types of possible M_r is $\leq \lambda$. Also, the number of possible $N_r \cap \lambda$ is $\leq |S| \leq \lambda$ (and the number of possible μ_r-s is $\leq \lambda$). So if the conclusion fails for some real r, the following set has cardinality λ^+ (in $\mathbf{V}[A]$):

$$R := \left\{s \in ({}^\omega 2)^{\mathbf{V}[A]} : M_s \cong M_r,\ \mu_s = \mu_r,\ N_s \cap \lambda = N_r \cap \lambda\right\}.$$

So it is enough to prove:[2]

(∗) If $s \in R$ then $s \in N_r$.

[2] Remember, N_r is countable!

As $s \in R$, there is an isomorphism g_s from M_r onto M_s; it is unique as M_r satisfies extensionality (being $\prec \big(\mathcal{H}(\lambda^+)^{\mathbf{V}}, f_{i_r}, \in, <^*_{\lambda^+}\big)$), and belongs to $\mathbf{V}$ as M_r and M_s belong to $\mathbf{V}$. Clearly g_s is necessarily the identity on $\mathcal{H}(\mu_r)^{\mathbf{V}}$ (as it is a transitive subset of $M_r \cap M_s$). Also note

- (α) $N_r \cap \lambda = N_s \cap \lambda \subseteq \mathcal{H}(\mu_r)^{\mathbf{V}}$ (an assumption).
- (β) $N_r \cap \mathcal{H}(\lambda^+)^{\mathbf{V}} = \{f_{i_r}(\alpha) : \alpha \in N_r \cap \lambda\}$
  (as $(N_r, f^r_i) \prec (\mathfrak{A}_{i_r}, f_{i_r})$ and by the choice of f_{i_r}).

Therefore, g_s clearly maps $\mathcal{H}(\lambda^+)^{\mathbf{V}} \cap N_r$ onto $\mathcal{H}(\lambda^+)^{\mathbf{V}} \cap N_s$.

Also, $g_s(A^{N_r}) = A^{N_s}$ as $A \subseteq \lambda$. Now N_r, being

$$\prec \mathfrak{A} = \big(\mathcal{H}(\lambda^+)^{\mathbf{V}[A]}, \mathcal{H}(\lambda^+)^{\mathbf{V}}, A, \in, <^*_{\lambda^+}\big),$$

"thinks" that $\mathcal{H}(\lambda^+)^{\mathbf{V}[A]}$ is $\mathcal{H}(\lambda^+)^{\mathbf{V}}[A]$. But constructing $\mathcal{H}(\lambda^+)^{\mathbf{V}[A]}$ as '$\mathcal{H}(\lambda^+)^{\mathbf{V}}$ extended by A' is a unique process, so g_s can be extended to an isomorphism from N_r onto N_s. But necessarily $s = g_s^{-1}(s)$, so $s = g_s^{-1}(s) \in N_r$ as required.

2) Similarly. (Note that without loss of generality $\mathbf{V} = \mathbf{L}[B]$ for some $B \subseteq \lambda^+$, hence $\mathbf{V}$ and $\mathbf{V}[A]$ satisfy '$2^\lambda = \lambda^+$'.) $\square_{0.5}$

See more in [She93].

§ 1. The Strong Covering Lemma: Definition and implications

This section defines our central notions and gives the easy relevant facts.

Context 1.1. Let $\mathbf{V}$ be a universe (of set theory) and $\mathbf{W} \subseteq \mathbf{V}$ a transitive class of $\mathbf{V}$ which is a model of ZFC (with the same ordinals). Writing (for example) $\mathbf{W}_0 \subseteq \mathbf{W}_1$, we implicitly assume the corresponding hypothesis.

Definition 1.2. The pair $(\mathbf{W}, \mathbf{V})$ satisfies the λ-*covering lemma* (where λ is a cardinal of $\mathbf{V}$) if for every set $a \in \mathbf{V}$ with $a \subseteq \lambda$ (or $a \subseteq \mathbf{W}$) of cardinality $< \lambda$ (in $\mathbf{V}$), there is a set $b \in \mathbf{W}$ such that $a \subseteq b$ and $\mathbf{V} \vdash$ "$|b| < \lambda$".

If we omit λ, this means "for every $\lambda \geq \aleph_2^{\mathbf{V}}$". Without loss of generality a and b are sets of ordinals.

Definition 1.3. The pair $(\mathbf{W}, \mathbf{V})$ satisfies the strong (λ, α)-*covering lemma* (λ a regular cardinal in $\mathbf{V}$ and α an ordinal) if for every model M in $\mathbf{V}$ with universe α (always with countably many finitary functions and relations) and $a \in \left([\alpha]^{<\lambda}\right)^{\mathbf{V}}$, there is $b \in \mathbf{W}$ such that $a \subseteq b \subseteq \alpha$, b is an elementary submodel of M (i.e. the set of elements of such a submodel) and

$$\mathbf{V} \vdash \text{“}|b| < \lambda\text{”}.$$

Instead of saying '(λ, α) for every α,' we write ∞ instead of α, or write "the strong λ-covering".

Of course, we can replace α by any set in $\mathbf{W}$ of the same cardinality, so without loss of generality α is a cardinal of $\mathbf{W}$; we may assume M has Skolem functions, so it is enough that b is a submodel.

Definition 1.4. 1) The pair $(\mathbf{W}, \mathbf{V})$ satisfies the *strong* $(\lambda^*, \lambda, \kappa, \alpha)$-*covering lemma* (where $\kappa \leq \lambda \leq \lambda^*$ are regular cardinals in $\mathbf{V}$, and α an ordinal) if Player I wins the following game in $\mathbf{V}$ (i.e. has a winning strategy).

The $(\lambda^*, \lambda, \kappa, \alpha)$-covering game:

A play lasts λ moves. In the i^{th} move, Player I chooses $a_i \in \mathbf{V}$, a subset of α of cardinality $< \lambda^*$ (in $\mathbf{V}$) which includes $\bigcup\limits_{j<i} b_j$, and Player II responds with b_i, a subset of α of cardinality $< \lambda^*$ which contains $\bigcup\limits_{j\leq i} a_j$.

Player I wins if there is a closed unbounded subset $C \subseteq \lambda$ such that for every $i \in C \cup \{\lambda\}$,

$$\mathrm{cf}(i) = \kappa \Rightarrow \bigcup_{j<i} a_j \in \mathbf{W}$$

(if $\kappa = \lambda$, then only $i = \lambda$ will count). We omit α if we mean "for every α."

2) Let D be a filter on $\lambda + 1$ (i.e. on $\{i : i \leq \lambda\}$) and λ^*, λ, α be as before. The pair $(\mathbf{W}, \mathbf{V})$ satisfies the *strong* $(\lambda^*, \lambda, D, \alpha)$-*covering lemma* if Player I wins in the following game (i.e. has a winning strategy in $\mathbf{V}$).

The $(\lambda^*, \lambda, D, \alpha)$-covering game:

The play last λ moves. In the i^{th} move Player I chooses $a_i \in \mathbf{V}$, a subset of α of cardinality λ^* (in $\mathbf{V}$) which includes $\bigcup\limits_{j<i} b_j$, and then Player II chooses b_i, a subset of α of cardinality $< \lambda^*$ which includes $\bigcup\limits_{j\leq i} a_j$. Player I wins the game if $\left\{i \leq \lambda : \bigcup\limits_{j<i} a_j \in \mathbf{W}\right\} \in D$.

Remark 1.5. The two popular cases are

$$D = \{A \subseteq \lambda + 1 : \lambda \in A\}$$

(then we get the $(\lambda^*, \lambda, \lambda, \alpha)$-covering game) and

$$D = \big\{A \subseteq \lambda + 1 : \text{there is a club } C \subseteq \lambda \text{ such that } \{\delta \in C : \operatorname{cf}(\delta) = \kappa\} \subseteq A\big\}$$

(then we get the $(\lambda^*, \lambda, \kappa, \alpha)$-covering game).

Claim 1.6. 1) *The strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma implies the strong (λ^*, α)-covering lemma when $[\lambda^* > \lambda \ \text{ or } \ \lambda > \kappa]$, and it implies the strong $\big((\lambda^*)^+, \alpha\big)$-covering lemma when $\lambda^* = \lambda = \kappa$ (λ^+, in $\mathbf{V}$'s sense).*

2) *The strong $(\lambda^*, \lambda, \kappa, \alpha_0)$-covering lemma implies the strong $(\lambda^*, \lambda, \kappa, \alpha_1)$-covering lemma when $\alpha_0 \geq \alpha_1$.*

3) *If $\mathbf{W}_1 \subseteq \mathbf{W} \subseteq \mathbf{V} \subseteq \mathbf{V}_1$ are universes of set theory with the same ordinals, then:*

- (*a*) *The strong (λ, α)-covering lemma for $(\mathbf{W}_1, \mathbf{V}_1)$ implies the strong (λ, α)-covering lemma for $(\mathbf{W}, \mathbf{V})$.*
- (*b*) *The strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma for $(\mathbf{W}_1, \mathbf{V})$ implies the strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma for $(\mathbf{W}, \mathbf{V})$ (see 1.7).*

4) *In the $(\lambda^*, \lambda, \kappa, \alpha)$-covering game, it does not hurt any player to choose bigger sets (as long as they are still subsets of α of cardinality $< \lambda^*$). I.e. if a player has a winning strategy, then choosing even larger sets will still ensure victory.*

5) *If $\lambda_1 \leq \lambda_2 \leq \lambda_3$, and $(\mathbf{W}, \mathbf{V})$ satisfies the* [strong] *(λ_1, λ)-covering lemma for every $\lambda < \lambda_2$ and also the* [strong] *(λ_2, λ_3)-covering lemma, then $(\mathbf{W}, \mathbf{V})$ satisfies the* [strong] *(λ_1, λ_3)-covering lemma.*

6) *Let $\mathbf{W}_1 \subseteq \mathbf{W}_2 \subseteq \mathbf{W}_3$ and $\lambda_1 \leq \lambda_2$. If $(\mathbf{W}_1, \mathbf{W}_2)$ and $(\mathbf{W}_2, \mathbf{W}_3)$ satisfy the (λ_1, λ_2)-covering lemma, then $(\mathbf{W}_1, \mathbf{W}_3)$ does as well.*

7) *We can replace κ by a filter D on $\lambda + 1$ in parts* (1)-(4).

Proof. Left to the reader, being trivial. $\square_{1.6}$

Remark 1.7. In 1.6(3)(b), why do we speak about $(\mathbf{W}_1, \mathbf{V})$ and not $(\mathbf{W}_1, \mathbf{V}_1)$? The winning strategy may be missing from $\mathbf{V}_1$ (also the club C).

Definition 1.8. We say $\mathbf{W}$ has a *square* if for any cardinal μ there are sets C_δ (with $\delta < \mu$, δ singular limit) such that:

- (a) C_δ is a closed unbounded subset of δ of order type $< \delta$.
- (b) If γ is a limit ordinal and is in C_δ, then

$$\sup(C_\delta \cap \gamma) = \gamma \text{ and } C_\gamma = C_\delta \cap \gamma.$$

Claim 1.9. *If $\mathbf{W}$ has a square, $\lambda \leq \mu$, and*

$$S^\mu_{<\lambda} := \big\{\delta \in (\lambda, \mu) : \operatorname{cf}(\delta) < \lambda\big\},$$

then we can find $\langle C_\delta : \delta \in S^\mu_{<\lambda}\rangle$ such that:

- (*a*) *C_δ is a closed unbounded subset of δ of order type $< \lambda$.*
- (*b*) *If γ is a accumulation point of C_δ then $C_\delta \cap \gamma = C_\gamma$.*

Proof. Let $\langle C_\delta : \delta < \mu$ a singular limit ordinal$\rangle$ be as in Definition 1.8. Without loss of generality, $\delta > \lambda \Rightarrow C_\delta \cap \lambda = \varnothing$. For each δ for which C_δ is defined, let f_δ be the function with domain C_δ defined by $f_\delta(\alpha) := \mathrm{otp}(\alpha \cap C_\delta)$. Define C^1_δ by induction on $\delta < \mu$: if C_δ is not defined then C^1_δ is also not defined. If C_δ is defined but $C_{\mathrm{otp}(C_\delta)}$ is not defined or $\mathrm{otp}(C_\delta) < \lambda$, let $C^1_\delta := C_\delta$. And if C_δ and $C_{\mathrm{otp}(C_\delta)}$ are defined but $\mathrm{otp}(C_\delta) \geq \lambda$, we let

$$C^1_\delta := \big\{\alpha \in C_\delta : f_\delta(\alpha) \in C^1_{\mathrm{otp}(C_\delta)}\big\}.$$

Now check that $\langle C^1_\delta : \delta \in S^\mu_{<\lambda}\rangle$ is as required. $\square_{1.9}$

Definition 1.10. If the conclusion of 1.9 holds (for every μ), we say $\mathbf{W}$ *has λ-squares*, and if this holds for every $\lambda \geq \aleph_2$, we say $\mathbf{W}$ *has squares.*

Claim 1.11. 1) *If the pair* $(\mathbf{W}, \mathbf{V})$ *satisfies the λ-covering lemma, (λ a cardinal in $\mathbf{V}$) then for every limit ordinal δ, if its cofinality in $\mathbf{W}$ is $\geq \lambda$ then its cofinality in $\mathbf{V}$ is as well (the inverse is trivial).*

2) *If $\mathbf{W}$ has λ-squares, $\mathbf{W} \subseteq \mathbf{V}$, and $(\mathbf{W}, \mathbf{V})$ satisfies the λ-covering lemma, then $\mathbf{V}$ has λ-squares.*

Definition 1.12. We say that the universe $\mathbf{W}$ has *scales* if for every singular cardinal χ there is a set G of cardinality χ^+, consisting of functions with the following properties.
For all $g \in G$:

- $\mathrm{dom}(g) = R_\chi := \chi \cap \mathrm{Reg}$
- $g(\theta) < \theta$
- For every function f satisfying $\mathrm{dom}(f) \in [R_\chi]^{<\chi}$ and $(\forall\theta)[f(\theta) < \theta]$, there is $g \in G$ such that $f <^* g$.

(By '$<^*$,' we mean $(\exists\sigma \in R_\chi)(\forall\theta > \sigma)\big[\theta \in \mathrm{dom}(f) \Rightarrow f(\theta) < g(\theta)\big]$.)
If we restrict ourselves to one such χ, we call this property "has χ^+-scales."

Remark 1.13. It is easy to verify that if $\mathbf{W} \vdash \mathsf{GCH}$, then $\mathbf{W}$ has scales.

Claim 1.14. *Let $(\mathbf{W}, \mathbf{V})$ satisfy the covering lemma.*

1) *If $\mathbf{W}$ has λ-squares, where $\lambda \geq \aleph_2^{\mathbf{V}}$ is regular in $\mathbf{V}$, then $\mathbf{V}$ has λ-squares.*

2) *If $\mathbf{W}$ has χ^+-scales, where χ a cardinal in $\mathbf{V}$ (hence χ is singular in $\mathbf{W}$, and χ^+ — in $\mathbf{W}$'s sense — is the successor of χ also in $\mathbf{V}$) then $\mathbf{V}$ has χ^+-scales.*

3) *If $\mathbf{W}$ has squares then $\mathbf{V}$ has squares.*

4) *If $\mathbf{W}$ has scales then $\mathbf{V}$ has scales.*

Remark 1.15. 1) The aim of 1.14 is that we will be able to get a strong covering lemma; for example for $(\mathbf{W}, \mathbf{V})$ where $\mathbf{0}^\# \notin \mathbf{V}$, and not just for $(\mathbf{L}, \mathbf{V})$.

2) We can replace $\aleph_2$ by any other regular uncountable cardinal κ of $\mathbf{V}$ (if $(\mathbf{W}, \mathbf{V})$ satisfies the λ-covering lemma for $\lambda \geq \kappa$ regular in $\mathbf{V}$). We have other obvious variants.

Proof. Trivial.
For part (3), note that any universe $\mathbf{W}$ has $\aleph_1$-squares: for every limit δ of cofinality $\aleph_\delta$, choose $C_\delta \subseteq \delta$ an unbounded subset of order type ω. $\square_{1.14}$

Definition 1.16. Let D be a filter on $\lambda + 1$, where $\mathrm{cf}(\lambda) > \aleph_0$ and (always)

$$\alpha < \lambda \Rightarrow (\lambda + 1) \setminus \alpha \in D.$$

1) D is called *weakly normal* (equivalently, satisfies (λ^*, λ)-demand Zero) when:
if $\zeta < \xi < \lambda \Rightarrow A_\zeta \in D \wedge A_\xi \subseteq A_\zeta$
then $\{\zeta \leq \lambda : (\forall \xi < \zeta)[\zeta \in A_\xi]\}$ belongs to D.
[Do you want these guys to be $\subseteq$-increasing or $\subseteq$-*decreasing*?]

2) D satisfies the (λ^*, λ)-demand One if for every club C of λ, we have $C \cup \{\lambda\} \in D$ and $\lambda^* > \lambda \Rightarrow C \in D$.

3) D satisfies the (λ^*, λ)-demand Two when: if C_δ is a club of δ for every limit ordinal $\delta \leq \lambda$ of uncountable cofinality, then

$$\bigcup \{C_\delta \cup \{\delta\} : \delta \leq \lambda,\ \mathrm{cf}(\delta) \in (\aleph_0, \lambda^*)\} \cup \{\alpha < \lambda : \mathrm{cf}(\alpha) > \aleph_0\} \in D.$$

4) D is said to satisfy the (λ^*, λ)-demand Three when for every $\kappa = \mathrm{cf}(\kappa) < \lambda$, we have '(A) or (B) or (C),' where:

(A) $\{\delta < \lambda : \mathrm{cf}(\delta) \neq \kappa\} \in D$
(B) $\lambda^* > \lambda$
(C) If C_δ is a club of δ for each limit $\delta < \lambda$, then

$$\bigcup \{C_\delta \cup \{\delta\} : \delta < \lambda,\ \mathrm{cf}(\delta) \neq \kappa\} \in D.$$

[Demand Zero doesn't depend on λ^*, and Demand Three can be trivially satisfied by $\lambda^* > \lambda$ (which appears to be the default). Otherwise, it does not depend on λ^* either.]

Fact 1.17. Let $\kappa \leq \lambda \leq \lambda^*$ be regular and $\lambda > \aleph_0$.

1) If $D := \{A \subseteq \lambda + 1 : \lambda \in A\}$ then D is a λ-complete filter satisfying the (λ^*, λ)-demand Zero. Furthermore, D satisfies the (λ^*, λ)-demand One iff $\lambda^* = \lambda$, and D satisfies the (λ^*, λ)-demand Two iff $\lambda^* > \lambda$.
[If something satisfies (λ^*, λ)-demand One when $\lambda^* = \lambda$, it must necessarily hold for $\lambda^* < \lambda$.]

2) If $\kappa = \mathrm{cf}(\kappa) < \lambda$ and

$$D_{\lambda,\kappa} := \{A \subseteq \lambda + 1 : A \cup \{\delta < \lambda : \mathrm{cf}(\delta) \neq \kappa\} \text{ contains a club of } \lambda\},$$

then:

(α) D is normal (and λ-complete) and it satisfies the (λ^*, λ)-demands Zero and One.
(β) D satisfies the (λ^*, λ)-demand Two if $\lambda^* > \lambda$ or $\kappa > \aleph_0$ or every stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ reflects in some $\delta < \lambda$.
(γ) D satisfies demand Three if $\lambda^* > \lambda$ or every stationary

$$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$$

reflects in some $\delta < \lambda$.

3) If $\lambda^* > \lambda$ and D satisfies (λ^*, λ)-demand One, then D satisfies (λ^*, λ)-demand Two.

4) If $\lambda > \aleph_1$, and

$$D := \{A \subseteq \lambda + 1 : \text{for some club } C \text{ of } \lambda, \text{ for every } \delta \in C \text{ of uncountable cofinality, we have } \delta \in A\},$$

then D satisfies the (λ^*, λ)-demands Zero, One, and Two.

5) Let $(\mathbf{W}, \mathbf{V})$ be a pair and assume D is as in part (2). For every α, we have that $(\mathbf{W}, \mathbf{V})$ satisfies the strong $(\lambda^*, \lambda, \kappa, \alpha)$-covering lemma iff it satisfies the $(\lambda^*, \lambda, D, \alpha)$-covering lemma.

§ 2. Proof of the Strong Covering Lemma

This section is the crux of the chapter. Our aim is essentially to prove that [some / all] of the strong covering lemmas hold when the covering lemma holds. We can get more from the proofs. We prove trivial cases of the strong covering lemma (2.1) and two inductive lemmas, with the aim of enabling us to prove the strong covering by induction on the cardinals of $\mathbf{W}$. The first (2.2) says that we can advance from μ to μ^+, and the second (which is the main proof) says that we can advance to a limit cardinal μ (really, the proof splits into cases by $\mathrm{cf}^{\mathbf{V}}(\mu)$, so in some cases we get a little more).

Lemma 2.1. 1) *Suppose χ is a regular cardinal in $\mathbf{V}$ and $(\mathbf{W},\mathbf{V})$ satisfies the strong (λ,μ)-covering lemma for every $\mu \in (\lambda,\chi)$. Then $(\mathbf{W},\mathbf{V})$ satisfies the strong (λ,χ)-covering lemma.*

2) *If $\kappa \le \lambda < \lambda^*$ are regular cardinals in $\mathbf{V}$, then $(\mathbf{W},\mathbf{V})$ satisfies the strong $(\lambda^*,\lambda,\kappa,\lambda^*)$-covering lemma.*

Proof. 1) If $\lambda = \chi$ and M a model with universe χ and countably many functions, then in $\mathbf{V}$ we have some $\alpha < \chi$ which is closed under the functions of M, so it exemplifies the conclusion of the strong covering lemma.

If $\lambda > \chi$, the strong (λ,χ)-covering lemma is trivial.

If $\lambda < \chi$, we can deduce the desired conclusion by 1.6(5) and the case $\lambda = \chi$ above.

2) The proof is similar. $\square_{2.1}$

Theorem 2.2. *Suppose:*

1) *$(\mathbf{W},\mathbf{V})$ satisfies the λ^*-covering lemma and $\mathbf{W}$ has λ^*-squares.*

2) *$(\mathbf{W},\mathbf{V})$ satisfies the strong $(\lambda^*,\lambda,D,\mu)$-covering lemma.*[3]

3) *D satisfies the (λ^*,λ)-demand Two (see Definition 1.16(3)).*

4) *λ is uncountable (in $\mathbf{V}$).*

Then $(\mathbf{W},\mathbf{V})$ has the strong $(\lambda^*,\lambda,D,\mu^+)$-covering lemma (with μ^+ in $\mathbf{W}$).

Remark 2.3. Remember $\lambda^* \ge \lambda > \aleph_0$ are regular cardinals in $\mathbf{V}$, D is a filter on $\lambda+1$, and without loss of generality $\mu \ge \lambda^*$ (by 2.1(2), 1.6(2)).

Proof. Before starting the proof in earnest we shall give two facts (which are trivial, but basic for our proofs), an observation, and a claim. We shall use assumption (2) only in the actual proof of 2.2.

Fact 2.4. In $\mathbf{W}$, for each ordinal α there is a model
$$M^0_\alpha := \big(\alpha, F^0_\alpha, G^0_\alpha, S^0_\alpha, \mathrm{CF}^0_\alpha, H^0_\alpha, 0\big)$$
such that

(a) $F^0_\alpha : \alpha\times\alpha \to \alpha$ is such that for every $\beta<\alpha$, $F^0_\alpha(\beta,-)$ is a one-to-one mapping from β onto $|\beta|^{\mathbf{W}}$ (its cardinality in $\mathbf{W}$).

(b) $G^0_\alpha(\beta,-)$ is the inverse of $F^0_\alpha(\beta,-)$ (on $|\beta|^{\mathbf{W}}$).

(c) S^0_α is the successor function.

(d) CF^0_α is a one-place function returning the cofinality for limit ordinals and the predecessor for successor ordinals.

[3] Hence, in $\mathbf{V}$, $\lambda \le \lambda^*$ are regular.

(e) H^0_α is a two-place function such that for β limit,
$$\langle H^0_\alpha(\beta,i) : i < \mathrm{CF}^0_\alpha(\beta)\rangle$$
is an increasing continuous sequence converging to β.
For β successor, $H^0_\alpha(\beta,0) := |\beta|$ and
$$H^0_\alpha(\beta,1) := \begin{cases} (|\beta|^+)^{\mathbf{W}} & \text{if } (|\beta|^+)^{\mathbf{W}} < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

(f) 0 is an individual constant denoting 0 (i.e. a zero-place function).

Notation 2.5. 1) We say $a \subseteq \alpha$ is a *submodel* of M^0_α if it is closed under the functions of M^0_α.
2) $c\ell(a, M^0_\alpha)$ is the closure of $a \cap \alpha$ under the functions of M^0_α. (Similarly for M^1_α, which is defined below.)

Fact 2.6. If $\mathbf{W}$ has λ^*-squares[4] then there exists $M^1_\alpha := (M^0_\alpha, C^{1,\alpha})$, where $C^{1,\alpha}$ is a two-place function satisfying the following.
There is (in $\mathbf{W}$) a sequence of clubs $\langle C^1_\beta : \beta < \alpha,\ \mathrm{cf}(\beta) < \lambda^*\rangle$ as in Claim 1.9 (with λ^* and α here standing in for λ, μ there) such that:

- $C^{1,\alpha}(\beta,\beta)$ is the order type of C^1_β (if defined).
- $C^{1,\alpha}(\beta,i)$ is the i^{th} element of C^1_β (if it exists).
- $C^{1,\alpha}(\beta+1, C^{1,\alpha}(\beta,i)) := i$.

Notation 2.7. We usually omit the subscript α in the above functions.

Observation 2.8. *Suppose μ is a cardinal of $\mathbf{W}$, μ^+ its successor in $\mathbf{W}$, $a \subseteq \mu^+$ a submodel of $M^0_{\mu^+}$, and $b \subseteq a$ is unbounded in a (i.e. $(\forall \zeta \in \alpha)(\exists \xi \in b)[\zeta \le \xi]$).*
Then:
1) $a = c\ell\big((a \cap \mu) \cup b, M^0_{\mu^+}\big)$
2) *Hence, if $a \cap \mu \in \mathbf{W}$ and $b \in \mathbf{W}$ then $a \in \mathbf{W}$.*

Proof. 1) As $a \cap \mu \subseteq a$, $b \subseteq a$, and a is a submodel of $M^0_{\mu^+}$, trivially
$$c\ell\big((a \cap \mu) \cup b, M^0_{\mu^+}\big) \subseteq a.$$
For the other inclusion, assume $\zeta \in a$; hence there is $\xi \in b$ such that $\zeta \le \xi$. If $\zeta = \xi$ there is nothing to prove, so let $\zeta < \xi$. Hence $F^0(\xi,\zeta) < \mu$ (as $|\xi| \le \xi < \mu^+$) and $F^0(\xi,\zeta) \in a$ (as a is a submodel of $M^0_{\mu^+}$) hence $F^0(\xi,\zeta) \in a \cap \mu$. But $G^0(\xi, F^0(\xi,\zeta)) = \zeta$ and $\xi \in b$, so $\zeta \in c\ell\big([a \cap \mu] \cup b, M^0_{M^+}\big)$, as required.
2) Easy. $\square_{2.8}$

Claim 2.9. *Suppose D is a filter on $\lambda + 1$ which satisfies the (λ^*, λ)-demand Two (see Definition 1.16(3)), λ, λ^* are regular cardinals (in $\mathbf{V}$) and α an ordinal such that $\lambda \le \lambda^* \le \alpha$, and $\mathbf{W}$ has λ^*-squares (so C^1_α and M^1_α are well defined).*
Suppose further that (in $\mathbf{V}$) $\langle a_\zeta : \zeta \le \lambda\rangle$ is an increasing continuous sequence of subsets of α, $\zeta < \lambda^* \Rightarrow |a_\zeta| < \lambda^*$, each a_ζ is a submodel of M^1_α, and
$$\sup(a_\zeta \cap \lambda^*) \subseteq a_{\zeta+1}.$$

[4] Remember, λ^* is a regular cardinal in $\mathbf{V}$.

Lastly, suppose that the closure of a_ζ (in the order topology on the ordinals) is included in $a_{\zeta+1}$, or at least (for a fixed $\delta \leq \alpha$)

$$\bigwedge_{\zeta<\lambda} \big[\sup(\delta \cap a_\zeta) \in a_\zeta + 1\big].$$

If $\delta \in a_\lambda$ and $\mathrm{cf}(\delta) \geq \lambda^*$, then $S := \big\{\zeta \leq \lambda : C^1_{\sup(\delta\cap a_\zeta)} \subseteq a_\zeta\big\}$ belongs to D.

Proof. Let $\delta(\zeta) := \sup(a_\zeta \cap \delta)$ for $\zeta \leq \lambda$. Assume $\lambda^* > \mathrm{cf}(\zeta) > \aleph_0$ and $\zeta \leq \lambda$.

Now clearly $\langle \delta(i) : i < \zeta\rangle$ is a (strictly) increasing continuous sequence converging to $\delta(\zeta)$, so (as $C^1_{\delta(\zeta)}$ is a closed unbounded subset of $\delta(\zeta)$) $C^1_{\delta(\zeta)} \cap \{\delta(i) : i < \zeta\}$ is a closed unbounded subset of $\delta(\zeta)$. But $\delta(i) \in a_{i+1} \subseteq a_\zeta$ (for $i < \zeta$). Hence for a closed unbounded set of $i < \zeta$, $\delta(i) \in C^1_{\delta(\zeta)} \cap a_\zeta$. But a_ζ is a submodel of M^1_α, and $a_\zeta \cap \lambda^*$ is an initial segment[5] of λ^*. So by the definition of M^1_α, for a closed unbounded set E of limit ordinals $i < \zeta$, $C^1\big(\delta(i), \delta(i)\big)$ belongs to $a_{i+1} \subseteq a_\zeta$, hence (see the definiton of M^1)

$$\big\{\gamma : \gamma < \mathrm{otp}(C^1_{\delta(i)})\big\} \subseteq a_{i+1} \subseteq a_\zeta.$$

Hence (using $C^1(\delta(i), \gamma)$), $C^1_{\delta(i)} \subseteq a_{i+1} \subseteq a_\zeta$. Of course $\delta(i) \in C^1_{\delta(\zeta)}$, and is even an accumulation point of $C^1_{\delta(\zeta)}$. By the definition of squares, $C^1_{\delta(\zeta)} \subseteq a_\zeta$, and for i an accumulation point of E we have $C^1_{\delta(i)} \subseteq a_i$. So $\zeta \leq \lambda$ and $\aleph_0 < \mathrm{cf}(\zeta) < \lambda^*$ imply that $\zeta \in S$ and a club of $i < \zeta$ belongs to S. This clearly suffices.

So we have proved 2.9. $\square_{2.9}$

Proof of 2.2:

By the hypothesis, Player I has a winning strategy in the $(\lambda^*, \lambda, D, \mu)$-covering game, which we denote by $\langle K_i : i < \lambda\rangle$. I.e. if $b_i \subseteq \mu$ for $i < \lambda$, $|b_i|^{\mathbf{V}} < \lambda^*$, <u>then</u>

$$a_i := K_i(b_0, b_1, \ldots, b_j, \ldots)_{j<i}$$

is a subset of μ of cardinality $< \lambda^*$, $b_j \subseteq a_i$ for $j < i$, and if in addition for $i < \lambda$ we have $\bigcup_{j\leq i} a_j \subseteq b_i$ <u>then</u>

$$\big\{\delta \leq \lambda : \bigcup_{j<\delta} a_j \in \mathbf{W}\big\} \in D.$$

Let us describe the winning strategy of Player I in the $(\lambda^*, \lambda, D, \mu^+)$-covering game. In the ζ-th move, where $a_j \subseteq b_j \subseteq a_i$ (for $j < i < \zeta$) are given, Player I will choose

$$a_\zeta := c\ell\big(a^2_\zeta, M^1_{\mu^+}\big),$$

where

(i) $a^0_\zeta := \bigcup_{j<\zeta} b_j$

(ii) $a^1_\zeta := K_\zeta(b_0 \cap \mu, b_1 \cap \mu, \ldots, b_i \cap \mu, \ldots)_{i<\zeta}$

(iii) $a^2_\zeta := a^0_\zeta \cup a^1_\zeta \cup \big\{\sup(a^0_\zeta)\big\} \cup \big\{\gamma : \gamma < \sup(a^0_\zeta \cap \lambda^*)\big\}$.

Note that $\sup(a^0_\zeta) < (\mu^+)^{\mathbf{W}}$, as $|a^0_\zeta|^{\mathbf{V}} < \lambda^*$, because $(\mathbf{W}, \mathbf{V})$ satisfies the λ^*-covering lemma.

We need to show that this strategy is a winning one, so let $\langle a_i, b_i : i < \lambda\rangle$ be a play in which Player I uses the strategy described above. Clearly by the choice of the K_ζ-s, there is $C \in D$ such that if $\zeta \in C$ then

$$a^0_\zeta \cap \mu = \bigcup_{i<\zeta} b_i \cap \mu \in \mathbf{W}.$$

[5] See the assumptions on a_i: $\sup(\lambda^* \cap a_\xi) \subseteq a_{\xi+1}$.

Let $\delta(i) := \sup(a^0_i)$.

For any limit $\zeta \in C$, clearly a^0_ζ is a submodel of $M^1_{\mu^+}$; hence by 2.8(2), in order to prove $a^0_\zeta \in \mathbf{W}$ it is enough to find an unbounded subset $b \subseteq a^0_\zeta$ as there (i.e. $b \in \mathbf{W}$). Our b here will be $C^1_{\delta(\zeta)}$ from Fact 2.6. Hence it suffices to prove that for some $C' \in D$, ($C' \subseteq C$ and) for every $\zeta \in C'$, we have $C^1_{\delta(\zeta)} \subseteq a^0_\zeta$. By 2.9 we finish. $\square_{2.2}$

Remark 2.10. Note that if there are λ^*-squares, then for each μ there is

$$\left\langle C^1_\delta : \lambda^* < \delta < \mu,\ \mathrm{cf}(\delta) < \lambda^* \right\rangle$$

as required, with "otp(C^1_δ) not divisible by ω^2 implies C^1_δ includes some end segment of δ."

Lemma 2.11. *Suppose:*

1) $(\mathbf{W}, \mathbf{V})$ *satisfies the λ^*-covering lemma, and $\mathbf{W}$ has λ^*-squares and scales (at least for $\theta \geq \lambda^* > \mathrm{cf}(\theta)$, θ a $\mathbf{W}$-cardinal).*

2) $\mu > \lambda^*$ *is a limit cardinal (in $\mathbf{W}$).*

3) $(\mathbf{W}, \mathbf{V})$ *satisfies the strong $(\lambda^*, \lambda, D, \alpha)$-covering lemma for every $\alpha < \mu$ (where D is a filter on $\{\zeta : \zeta \leq \lambda\}$ and $\aleph_0 < \lambda \leq \lambda^*$ are regular cardinals in $\mathbf{V}$).*

4) *At least one of the following holds:*

- (*A*) $\mathrm{cf}(\mu) < \lambda^*$, *and D is λ-complete and satisfies the (λ^*, λ)-demands Zero through Three.*
- (*B*) $\mathrm{cf}(\mu) \geq \lambda^*$, *and D satisfies demands Zero through Two, $\aleph_0 < \lambda < \lambda^*$,*

$$\left\{\xi : \zeta < \lambda,\ \mathrm{cf}(\zeta) > \aleph_0\right\} \in D,$$

D is normal,[6] *$(\mathbf{W}, \mathbf{V})$ satisfies the λ-covering lemma, and $\mathbf{W}$ has λ-squares.*

<u>Then</u> *$(\mathbf{W}, \mathbf{V})$ satisfies the strong $(\lambda^*, \lambda, D, \mu^+)$-covering lemma ($\mu^+$ the successor of μ, in $\mathbf{W}$'s sense).*

Remark 2.12. 1) Suppose $\lambda = \lambda^*$. If λ has the same successor in $\mathbf{V}$ and $\mathbf{W}$, the situation is much simpler (as, for example, we can use λ^*-squares with every C_δ of order type $\leq \lambda$ — see 4.22).

2) This lemma is the heart of the matter.

3) The proof is broken into smaller parts. Assumptions (1)-(3) are used freely, but we shall say when we use part (A) or (B) of assumption (4).

We will work for a while in $\mathbf{W}$, present some definitions and facts, and only later return to the lemma.

Notation 2.13. 1) We let Reg denote the class of regular cardinals of $\mathbf{W}$, and

$$R(\mu_1, \mu_2) := \mathrm{Reg} \cap (\mu_1, \mu_2).$$

2) Let T be (the class of) functions f with domain a subset of Reg, $f(\chi) < \chi$. We have two natural relations on T:

- (A) $f < g$ if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(\chi) < g(\chi)$ for $\chi \in \mathrm{dom}(f)$. (Similarly for $f \leq g$.) This is a partial order.

[6] I.e. if $S_\zeta \in D$ for $\zeta < \lambda$ then $\left\{\zeta : (\forall \xi < \zeta)[\zeta \in S_\xi]\right\} \in D$.

(B) $f <^* g$ if $\operatorname{dom}(f) \subseteq \operatorname{dom}(g)$, $\operatorname{dom}(f)$ has no last element, and for some $\chi_0 \in \operatorname{dom}(f)$, for every $\chi \geq \chi_0$,

$$\chi \in \operatorname{dom}(f) \Rightarrow f(\chi) < g(\chi).$$

$<^*$ is a partial order on each T_I (see below).

3) If $I \subseteq R$ is a set with no last element, then

$$T_I = T(I) := \{f \in T : \operatorname{dom}(f) \subseteq I \text{ and } \sup \operatorname{dom}(f) = \sup I\}$$

and $T(\mu_1, \mu_2) := T(R(\mu_1, \mu_2))$.

Fact 2.14. In the universe $\mathbf{W}$:

1) If $\delta < \min I$ and $f_j \in T_I$ for all $j < \delta$, then there is $f_\delta \in T_I$ such that $f_i < f_\delta$ for every $i < \delta$.

2) Assume $f_j \in T_I$ for $j < \delta$, I has no last element, and one of the following holds:

- $\delta < (\sup I)^+$ and $\sup I$ is singular.
- $\delta < \sup I$
- $\delta = \sup I$ and $\sup I$ is regular, but I is a non-stationary subset of $\sup(I)$.

Then there is $f_\delta \in T_I$ such that $f_i <^* f_\delta$ for every $i < \delta$.

Fact 2.15. In the universe $\mathbf{W}$, suppose $\theta = \sup(I)$ for some $I \subseteq \mathrm{Reg}$, and θ is a singular cardinal of cofinality $< \lambda^*$.

(Recall that by clause (1) of 2.11, $\mathbf{W}$ has a θ^+-scale if $\theta \geq \lambda^* > \operatorname{cf}(\theta)$). Then:

1) There are functions $f_i \in T_I$ (for $i < \theta^+$) such that $f_i <^* f_j$ for every $i < j < \theta^+$, and

$$(\forall g \in T_I)(\exists i < \theta^+)[g <^* f_i]$$

(provided that $\left|\operatorname{dom}(g)\right| < \theta$).

2) If in addition $\langle C_\delta : \delta \in S^{\theta^+}_{<\lambda^*} \rangle$ is a λ^*-square, with $\lambda^* < \theta$, then in part (1) we can demand that:

(a) If $i < j < \theta^+$, $i \in C_j$, $\chi \in I$, $\chi \geq \lambda^*$, and χ is regular, then $f_i(\chi) < f_j(\chi)$.

(b) If $j < \theta^+$, j is a limit ordinal, and $j \in S^{\theta^+}_{<\lambda^*}$, then for $\chi \in I$ with $\chi \geq \lambda^*$ we have

$$f_j(\chi) = \sup\{f_i(\chi) + 1 : i \in C_j\}.$$

Definition 2.16. 1) For every α we can define a model $M^2_\alpha \in \mathbf{W}$, an expansion of M^1_α by the functions $F^2 = F^2_\alpha$, where:

For each singular cardinal θ of $\mathbf{W}$ such that $\operatorname{cf}(\theta) < \lambda^* < \theta < \alpha$, let $\langle f^{2,\theta}_i : i < \theta^+ \rangle$ be as in Fact 2.15 (for $I := \mathrm{Reg} \cap \theta$, and the λ^*-squares $\langle C^1_\delta : \delta \in S^\alpha_{<\lambda^*} \rangle$ we have used in the definition of M^1_α), and

$$F^2(\theta, i, \chi) := f^{2,\theta}_i(\chi).$$

(Of course, $\big\{\big(i, \theta, f^{2,\theta}_i\big) : i < \theta^+ < \alpha\big\} \in \mathbf{W}$).

2) If $(\mathbf{W}, \mathbf{V})$ has λ-squares (see clause (5) of 2.3) and satisfies the λ-covering lemma, then we define M^3_α as the expansion of M^2_α by $C^{2,\alpha}$, where $C^{2,\alpha}$ is like $C^{1,\alpha}$ (see 2.6), but using a λ-square

$$\big\langle C^2_i : i < \alpha,\ \mathbf{W} \vdash \text{‘}\operatorname{cf}(i) < \lambda\text{’}\big\rangle.$$

3) Without loss of generality C^1_δ are as in 2.10.

Fact 2.17. Let μ be a singular cardinal of $\mathbf{W}$ with $\mu \geq \lambda^* > \mathrm{cf}(\mu)$; we let $M^2 := M^2_{\mu^+}$, etc. Suppose $a \subseteq M^2$ is an elementary submodel in $\mathbf{V}$ with $\mu \in a$, $A \subseteq a$ is unbounded, $\chi_0 < \mu$, and for every $\chi \in \mathrm{Reg} \cap \alpha \cap (\chi_0, \mu)$,

$$\sup(a \cap \chi) \leq \sup_{i \in A} f_i^{2,\mu}(\chi).$$

Then:

1) $a = c\ell\big((\alpha \cap \chi_0) \cup A, M^2\big)$

2) If $a \cap \chi_0$ and $A \in \mathbf{W}$, then $a \in \mathbf{W}$.

Proof. 1) Let $b := c\ell\big((a \cap \chi_0) \cup A, M^2\big)$, so clearly $b \subseteq a$; suppose $b \neq a$ and eventually we shall get a contradiction. Let ζ be the first element in $a \setminus b$ and ξ the first element in $b \setminus \zeta$ (it exists, as by assumption A is unbounded in a). There is no member of b in the interval $[\zeta, \xi)$ and $\zeta < \xi$, so $b \cap \xi \subseteq \zeta$.

Case I: Let ξ be a successor ordinal.

Then as $\xi \in b$ also $\xi - 1 \in b$ (as CF^0 is one of the functions even of $M^0_{\mu^+}$, $\mathrm{CF}^0(\xi) = \xi - 1$; see Fact 2.4). But $\zeta \leq \xi - 1 < \xi$, a contradiction.

Case II: Let ξ be a limit ordinal, singular in $\mathbf{W}$ (i.e. in $\mathbf{W}$, either $|\xi| < \xi$ or ξ is a singular cardinal).

Then, as $\mathrm{CF}^0(\xi) < \xi$ and $\xi \in b \Rightarrow \mathrm{CF}^0(\xi) \in b$, clearly $\mathrm{CF}^0(\xi) < \zeta$. Now

$$M^2 \vdash \big(\exists x < \mathrm{CF}^0(\xi)\big)\big[\zeta < H^0(\xi, x) < \xi\big]$$

(by the choice of H^0; see 2.4). Hence, as $\zeta, \xi \in a$,

$$M^2 \restriction a \vdash \big(\exists x < \mathrm{CF}^0(\xi)\big)\big[\zeta < H^0(\xi, x) < \xi\big].$$

So let $\alpha \in a$ be such that

$$\alpha < \mathrm{CF}^0(\xi) \wedge \zeta < H^0(\xi, \alpha) < \xi.$$

As $\mathrm{CF}^0(\xi) < \zeta$ (see above) we have $\alpha < \zeta$; but by the choice of ζ, $\alpha \in a$ implies $\alpha \in b$. As $\alpha \in b$, $H^0(\xi, \alpha) \in b$, but $\zeta < H^0(\xi, \alpha) < \xi$, contradicting the choice of ξ.

Case III: Let ξ be a regular cardinal in $\mathbf{W}$.

Then $\xi > \chi_0$, as $\zeta \geq \chi_0$ (as $a \cap \chi_0 \subseteq b$). So

$$\sup(a \cap \xi) \leq \sup_{i \in A} \big(f_i^{2,\mu}(\xi)\big) = \sup_{i \in A} \big(F^2(\mu, i, \xi)\big) \leq \sup(b \cap \xi).$$

The last inequality holds as $\mu \in b$, $\xi \in b$, and $A \subseteq b$.
[Why? $\mu \in b$ as there is $\gamma \in A$ with $\mu < \gamma$ (as $\sup(A) = \sup(a)$, and $\mu \in a$ by one of our hypotheses) hence $\mu = |\gamma| = H^0_{\mu^+}(\gamma, 0) \in b$. Lastly, $\xi \in b$ by the choice of ξ and $A \subseteq b$ by the definition of b.]

As (trivially) $b \subseteq a$, we can conclude $\sup(a \cap \xi) = \sup(b \cap \xi)$; however, we know that $\zeta \in a \cap \xi$ hence $\zeta + 1 \in a \cap \xi$ hence $\sup(a \cap \xi) > \zeta$, whereas $b \cap \xi \subseteq \zeta$ hence $\sup(b \cap \xi) \leq \zeta$. This gives us our contradiction.

2) Follows from part (1) of 2.17. $\square_{2.17}$

Proof. Proof of Lemma 2.11.

By the hypothesis of the Lemma, for every $\alpha < \mu$, Player I has a winning strategy in the $(\lambda^*, \lambda, D, \alpha)$-covering game, which we denote by $\overline{K}^\alpha = \langle K_i^\alpha : i < \lambda \rangle$.

I.e. if $b_i \subseteq \alpha$ for $i < \lambda$ with $|b_i|^{\mathbf{V}} < \lambda^*$, then $a_i = K_i^\alpha(b_0, b_1, \ldots, b_j, \ldots)_{j<i}$ is a subset of α of $\mathbf{V}$-cardinality $< \lambda^*$, with $b_j \subseteq a_i$ for $j < i$. If in addition $a_i \subseteq b_i$ for $i < \lambda$, then

$$\big\{\delta \leq \lambda : \bigcup_{j<\delta} a_j \in \mathbf{W}\big\} \in D.$$

Remark 2.18. Defining the Strategy.

Let us describe a winning strategy of Player I in the $(\lambda^*, \lambda, D, \mu^+)$-covering game. In the ζ-th move, where $a_j \subseteq b_j \subseteq a_i \subseteq b_i$ (for $j < i < \zeta$) are given, Player I will construct a_ζ as follows.

First, some notation:

(i) $a_\zeta^0 := \bigcup_{j<\zeta} b_j$

(ii) $a_\zeta^1 :=$

$$\bigcup \big\{K_\xi^\alpha(b_j \cap \alpha, b_{j+1} \cap \alpha, \ldots, b_i \cap \alpha, \ldots)_{j \leq i < \zeta} : \alpha < \mu,\ j \text{ satisfies } \zeta = j + \xi, \text{ and } \alpha \in a_j \setminus \bigcup_{\gamma<j} a_\gamma\big\}.$$

(iii) $a_\zeta^2 := a_\zeta^0 \cup a_\xi^1 \cup \big\{\sup(a_\zeta^0)\big\} \cup \big\{\gamma : \gamma < \sup(a_\zeta^0 \cap \lambda^*)\big\}$.

As $(\mathbf{W}, \mathbf{V})$ satisfies the λ^*-covering lemma, and the set a_ζ^0 has cardinality $< \lambda^*$ (in $\mathbf{V}$), there is a_ζ^3 such that:

(iv) $a_\zeta^3 \in \mathbf{W}$, $a_\zeta^0 \subseteq a_\zeta^3$, and $|a_\zeta^3|^{\mathbf{V}} < \lambda^*$; moreover, a_ζ^3 is an elementary submodel of[7] $M_{\mu^+}^2$ (and of $M_{\mu^+}^3$, if well-defined), and includes the topological closure of a_ζ^0 (in the order topology on the ordinals).

Let Ch_ζ be a function with domain $a_\zeta^0 \cap (\mathrm{Reg} \setminus \lambda^*)$ defined by

$$\mathrm{Ch}_\zeta(\chi) := \sup(a_\zeta^0 \cap \chi) < \chi.$$

Remember that by the λ^*-covering lemma, $\chi \in \mathrm{Reg} \setminus \lambda^*$ implies $\mathrm{cf}^{\mathbf{V}}(\chi) \geq \lambda^*$.
[Why? λ^* is a regular cardinal in $\mathbf{V}$ and χ a regular cardinal in $\mathbf{W}$. Hence if $a \in \mathbf{V}$ with $a \subseteq \chi$ and $|a|^{\mathbf{V}} < \lambda^*$, then there is $b \in \mathbf{W}$ with $a \subseteq b \subseteq \chi$ and $|b|^{\mathbf{V}} < \lambda^*$, hence $\mathrm{otp}(b) < \lambda^*$, so $|b|^{\mathbf{W}} < \lambda^*$, but $\mathbf{W} \models$ "$\chi = \mathrm{cf}(\chi) \geq \lambda^*$, so $a \cap \chi$ is a bounded subset of χ.]

By the λ^*-covering lemma there is a function $f_\zeta \in (T_{\mathrm{Reg}\cap\mu})^{\mathbf{W}}$ with $\mathrm{dom}(f_\zeta)$ a subset of $\mathrm{Reg} \cap \mu$ of cardinality $< \lambda^*$ and[8] $\mathrm{Ch}_\zeta <^* f_\zeta$. For each cardinal θ of $\mathbf{W}$, let $\mathrm{Ch}_\xi^\theta := \mathrm{Ch}_\zeta \restriction \theta$. For $\theta \in (\lambda^*, \mu]$ singular in $\mathbf{W}$, by the choice of $\langle f_i^{2,\theta} : i < \theta^+\rangle$, for some $i_\theta(\zeta) < \theta^+$ we have either $f_\zeta \restriction [\lambda^*, \theta) <^* f_{i_\theta(\zeta)}^{2,\theta}$ or $\mathrm{dom}(f_\zeta) \cap \theta$ is a bounded subset of θ.

Finally, Player I chooses

$$a_\zeta := c\ell\big(a_\zeta^2 \cap a_\zeta^3 \cup \big\{i_\theta(\zeta) : \theta \leq \mu,\ \theta \in a_\zeta^0, \text{ and } \theta \text{ is singular in } \mathbf{W}\big\}, M_{\mu^+}^2\big).$$

* * *

The "only" thing left is to show that this strategy is a winning one — i.e.:

---

[7] Remember $M_{\mu^+}^2 \in \mathbf{W}$.

[8] I.e. $\mathrm{Ch}_\zeta(\chi) < f_\zeta(\chi)$ when $\chi \in a_\zeta^0 \cap \mathrm{Reg}$, $\chi \geq \lambda^*$.

Notation 2.19. FRAMEWORK AND NOTATION.

Let $\langle a_i, b_i : i < \lambda \rangle$ be a play in which Player I uses the strategy described above. Let $a^0_\lambda := \bigcup_{i<\lambda} a^0_i$. Note: $a^0_\zeta = \bigcup_{\xi<\zeta} a_\xi$ for limit $\zeta < \lambda$, $\langle a_\xi : \zeta < \lambda \rangle$ is increasing and $\langle a^0_\zeta : \zeta \le \lambda \rangle$ is increasing continuous, and $a^0_\zeta \subseteq a_\zeta \subseteq a^0_{\zeta+1}$.

Let us introduce some more notation. For $\theta \in (\lambda^*, \mu]$ a singular cardinal of $\mathbf{W}$ which belongs to a^0_λ and for an ordinal $\zeta \le \lambda$, let

$$\delta_\theta(\zeta) = \delta(\zeta, \theta) := \sup(a^0_\zeta \cap \theta^+)$$

(so $\delta_\theta(\zeta) = \mathrm{Ch}_\zeta(\theta^+)$ for all $\theta \in \mathrm{dom}(\mathrm{Ch}_\zeta)$). If $x \in a^0_\lambda$, let

$$j(x) := \min\{j < \lambda : x \in a^0_j\} < \lambda.$$

If $\theta \in a^0_\lambda$ and $\mathrm{cf}^{\mathbf{W}}(\theta) < \lambda^*$ (equivalently, $\mathrm{cf}^{\mathbf{V}}(\theta) < \lambda^*$), clearly $\theta \in a^0_{j(\theta)}$ and $a^0_{j(\theta)} \cap \theta$ is an unbounded subset of θ. Let Θ [let Θ^*] be the set of all $\mathbf{W}$-cardinals $\theta \in \bigcup_{\zeta<\lambda} a_\zeta \setminus \lambda^*$ for which $\mathrm{cf}(\theta) < \lambda$ [for which $\mathrm{cf}(\theta) < \lambda^*$]. Note that

$$\langle \delta_\theta(\zeta) : j(\theta) \le \zeta \le \lambda \rangle$$

is strictly increasing continuous (see clause (iv) above). Hence for ζ limit we have $\mathrm{cf}^{\mathbf{V}}(\delta_\theta(\zeta)) = \mathrm{cf}^{\mathbf{V}}(\zeta)$. Note also that

$$\theta \in a^0_\zeta \Rightarrow \delta_\theta(\zeta) \in a^0_{\zeta+1}.$$

(Remember that a^3_ζ contains the topological closure of a^0_ζ, and obviously $\delta_\theta(\zeta)$ is in the closure of a^0_ζ by its definition.)

Subfact 2.20. For each $\alpha \in a^0_\lambda \cap \mu$, for the D-majority of $i \le \lambda$, $a^0_i \cap \alpha \in \mathbf{W}$.

Proof. This is by 2.18(ii) (i.e. in the definition of the strategy of Player I) as $a^1_\xi \subseteq a_{\xi+1}$, and as $\langle K^\alpha_\xi : \xi < \lambda \rangle$ is a winning strategy of Player I in the strong $(\lambda^*, \lambda, D, \alpha)$-covering game. $\square_{2.20}$

Subfact 2.21. 1) Suppose $\theta \in \Theta^*$, $j(\theta) \le \zeta \le \lambda$ (for the definition of $j(\theta)$, see 2.19 above), and $\aleph_0 < \mathrm{cf}^{\mathbf{V}}(\zeta) < \lambda^*$. <u>Then</u> for some closed unbounded subset C of ζ, for every $\xi \in C \cup \{\zeta\}$, the set $C^1_{\delta(\xi,\theta)}$ (is defined and) is an unbounded subset of $a^0_\xi \cap \theta^+$.

2) If D satisfies (λ^*, λ)-demand Two, <u>then</u> for $\theta \in \Theta^*$ we have

$$\{\zeta \le \lambda : C^1_{\delta(\xi,\theta)} \text{ is an unbounded subset of } a^0_\xi \cap \theta^+\} \in D.$$

Proof. 1) We can prove this as in the proof of 2.9.

2) $j(\theta) < \lambda$ for each $\theta \in \Theta^*$, and by part (1) and "D satisfies the (λ^*, λ)-demand Two," the conclusion follows. $\square_{2.21}$

Fact 2.22. If $\theta \in \Theta^*$, $\theta \in a^0_{j(\theta)}$, and $j(\theta) < \zeta < \xi < \lambda$, <u>then</u>:

1) $\mathrm{Ch}^\theta_\zeta < \mathrm{Ch}^\theta_\xi$

2) $f^{2,\theta}_{\delta_\theta(\zeta)} <^* f^{2,\theta}_{\delta_\theta(\xi)}$

3) $f^{2,\theta}_{\delta_\theta(\zeta)} \restriction a^0_\xi < \mathrm{Ch}^\theta_\xi$

4) $\mathrm{Ch}^\theta_\zeta <^* f^{2,\theta}_{\delta_\theta(\xi)}$

5) If $i \le \lambda$ is a limit ordinal, $\mathrm{cf}(i) < \lambda^*$, and $C^1_{\delta(i,\theta)} \subseteq a^0_i$, then $f^{2,\theta}_{\delta(i,\theta)} \restriction a^0_i \le \mathrm{Ch}^\theta_i$.

Proof. This can be proved quite easily. The first part holds as $a^0_{\zeta+1} \subseteq a^0_\xi$, as $\zeta < \xi$ and the definition of $a_{\zeta+1}$ above (as the closure of a^0_ζ in the order topology of ordinals is a subset of $a^3_\zeta \subseteq a^0_{\zeta+1}$, and hence of a^0_ξ). The second part holds by the choice of the $f^{2,\theta}_i$ (see 2.11(C)(1))

[Does not exist. Do you mean 2.15(1)?]

as

$$j(\theta) \le \zeta < \xi \Rightarrow \delta_\theta(\zeta) < \delta_\alpha(\xi).$$

The third part is true as $\delta_\theta(\zeta) \in a^0_{\zeta+1} \subseteq a^0_\xi$ (as $\delta_\theta(\zeta)$ is in the topological closure of a^0_ζ which is a subset of $a^3_\zeta \subseteq a^0_{\zeta+1}$) hence

$$\sigma \in \mathrm{Reg} \cap a_\xi \cap \theta \Rightarrow f^{2,\theta}_{\delta_\theta(\zeta)}(\sigma) \in a_\xi \cap \sigma.$$

The fourth part holds as $\mathrm{Ch}^\zeta_\theta <^* f^{2,\theta}_{i_\theta(\zeta)}$ (by the choice of $i_\theta(\zeta)$ in 2.18) and $i_\theta(\zeta) \in a_{\zeta+1} \subseteq a^0_\xi$ (see our choice of $a_{\zeta+1}$), hence $i_\theta(\zeta) < \delta_\theta(\xi)$ and so $f^{2,\theta}_{i_\theta(\zeta)} <^* f^{2,\theta}_{\delta_\theta(\xi)}$ (see 2.11(C)(1)). As $<^*$ is transitive, we finish proving (4).

As for the fifth, we know that for every $\chi \in \mathrm{Reg} \cap \theta \cap a^0_i \setminus \lambda^*$,

$$f^{2,\theta}_{\delta(i,\theta)}(\chi) = \sup\{f^{2,\theta}_j(\chi) + 1 : j \in C^1_i\} = \sup\{F^2(\theta, j, \chi) + 1 : j \in C^1_i\}$$
$$\le \sup(a^0_i \cap \theta^+) = \mathrm{Ch}^\theta_i(\chi).$$

$\square_{2.22}$

Fact 2.23. Suppose $\theta \in \Theta^*$.

> NOTATION: For $\zeta \in (j(\theta), \lambda]$, let $\chi_\theta(\zeta) = \chi(\zeta, \theta) \in \theta \cap \mathrm{Reg} \cap a^0_\zeta$ be the minimal cardinal $\ge \lambda^*$ of $\mathbf{W}$ satisfying $(*)_{\zeta,\theta}$ below (if there is one).

$(*)_{\zeta,\theta}$ $(\forall \chi \in [\chi_\theta(\zeta), \theta))[\chi \in \mathrm{Reg} \cap a^0_\zeta \Rightarrow \mathrm{Ch}^\theta_\zeta(\chi) = f^{2,\theta}_{\delta(\zeta,\theta)}(\chi)]$.

Now we claim:

1) If $\zeta \le \lambda$ is a limit ordinal, $\mathrm{cf}^{\mathbf{V}}(\zeta) \ne \mathrm{cf}^{\mathbf{V}}(\theta)$, and $\mathrm{cf}^{\mathbf{V}}(\zeta) < \lambda^*$ then $\chi_\theta(\zeta)$ exists.

2) If $\zeta \le \lambda$, $\aleph_0 < \mathrm{cf}^{\mathbf{V}}(\zeta) < \lambda^*$, $\theta \in a^0_\zeta$ and $\mathrm{cf}(\theta) \ne \mathrm{cf}(\zeta)$ then for a closed unbounded set of $\xi < \zeta$, $(*)_{\xi,\theta}$ above is satisfied by $\chi_\theta(\zeta)$ (so $\chi_\theta(\xi) \le \chi_\theta(\zeta)$).

3) If $\theta \in \bigcup_{\zeta<\lambda} a^0_\zeta$ and D satisfies the (λ^*, λ)-demands One and Three, then the set $\{\zeta < \lambda : \chi_\theta(\zeta) \text{ well-defined}\}$ belongs to D.

Proof. 1) As $\mathrm{cf}^{\mathbf{V}}(\zeta) < \lambda^*$ and ζ a limit ordinal, clearly $\mathrm{cf}(\delta_\theta(\zeta)) = \mathrm{cf}(\zeta) < \lambda^*$, hence $C^1_{\delta(\zeta,\theta)}$ is defined. Let $\langle \xi(\varepsilon) < \zeta : \varepsilon < \mathrm{cf}^{\mathbf{V}}(\zeta)\rangle$ be increasing continuous with $\bigcup_{\varepsilon<\mathrm{cf}(\zeta)} \xi(\varepsilon) = \zeta$ such that for each ε, for some limit ordinal $\alpha(\varepsilon) \in C^1_{\delta(\zeta,\theta)}$, we have

$$\delta_\theta(\xi(\varepsilon)) < \alpha(\varepsilon) < \delta_\theta(\xi(\varepsilon+1)).$$

Let $\xi(\mathrm{cf}(\zeta)) := \delta_\theta(\zeta)$ (remember 2.10, 2.16(3)). For each $\varepsilon < \mathrm{cf}(\zeta)$, by 2.22,

$$\mathrm{Ch}^\theta_{\xi(\varepsilon)} \le^* f^{2,\theta}_{\delta(\xi(\varepsilon+1),\theta)} \restriction a^0_{\xi(\varepsilon+2)} \le^* f^{2,\theta}_{\alpha(\varepsilon+1)} \restriction a^0_{\xi(\varepsilon+2)}$$
$$\le^* f^{2,\theta}_{\delta_\theta(\xi(\varepsilon+3))} \restriction a^0_{\xi(\varepsilon+3)} \le \mathrm{Ch}^\theta_{\xi(\varepsilon+3)}.$$

Hence for some $\chi_\varepsilon < \theta$:

$$(*)\ \ \mathrm{Ch}^\theta_{\xi(\varepsilon)} \restriction [\chi_\varepsilon, \theta) \le f^{2,\theta}_{\delta(\xi(\varepsilon+1),\theta)} \restriction ([\chi_\varepsilon, \theta) \cap a^0_{\xi(\varepsilon)})$$
$$\le f^{2,\theta}_{\alpha(\varepsilon+1)} \restriction ([\chi_\varepsilon, \theta) \cap a^0_{\xi(\varepsilon)}) \le \mathrm{Ch}^\theta_{\xi(\varepsilon+3)} \restriction ([\chi_\varepsilon, \theta) \cap a^0_{\xi(\varepsilon)}).$$

As $\mathrm{cf}(\theta) \neq \mathrm{cf}(\zeta)$, there is $\chi^* := \chi(\zeta, \theta) < \theta$ such that

$$S := \{\varepsilon < \mathrm{cf}(\zeta) : \chi_\varepsilon \leq \chi^*\}$$

is an unbounded subset of $\mathrm{cf}(\zeta)$. Without loss of generality,

$$\varepsilon \in S \Rightarrow \varepsilon + 1,\ \varepsilon + 2,\ \varepsilon + 3 \notin S.$$

Now notice:

(a) For each $\chi \in [\chi^*, \theta) \cap a^0_\zeta$, the sequence

$$\big\langle \mathrm{Ch}^\theta_{\xi(\varepsilon)}(\chi) : \varepsilon \leq \mathrm{cf}(\zeta) \text{ and } \xi(\varepsilon) \geq j(\chi)\big\rangle$$

is strictly increasing and continuous (as $\big\langle \xi(\varepsilon) : \varepsilon \leq \mathrm{cf}(\zeta)\big\rangle$ and $\big\langle a^0_{\xi(\varepsilon)} : \varepsilon \leq \mathrm{cf}(\zeta)\big\rangle$ are increasing continuous — see 2.22).

(b) For each $\chi \in [\chi^*, \theta) \cap a^0_\zeta$,

$$\big\langle f^{2,\theta}_\beta(\chi) : \beta \in \big\{\xi(\varepsilon + 1), \alpha(\varepsilon + 1) : \varepsilon \in S,\ \xi(\varepsilon) \geq j(\chi)\big\}\big\rangle$$

is increasing (by (a) and the inequalities above).

(c) For each $\varepsilon_1 < \varepsilon_2$ from S and $\chi \in [\chi^*, \theta) \cap a^0_{\xi(\varepsilon)}$, we have

$$\mathrm{Ch}^\theta_{\xi(\varepsilon_1)}(\chi) < f^{2,\theta}_{\xi(\varepsilon_1+2)}(\chi) < \mathrm{Ch}^\theta_{\xi(\varepsilon_2)}(\chi)$$

(by $(*)$ above).

(d) For each $\chi \in [\chi^*, \theta) \cap a^0_\zeta$, we have

$$f^{2,\theta}_{\delta(\zeta,\theta)}(\chi) = \sup\big\{f^{2,\theta}_{\alpha(\varepsilon+1)}(\chi) + 1 : \varepsilon \in S\big\} = \sup\big\{f^{2,\theta}_{\alpha(\varepsilon+1)}(\chi) : \varepsilon \in S\big\}$$

(by 2.15(2)).

As $S \subseteq \mathrm{cf}(\zeta)$ is unbounded, (a)-(d) together give the desired result.

2) By subfact 2.21, $C^1_{\delta(\zeta,\theta)} \subseteq a^0_\zeta$, and for some closed unbounded $C \subseteq \zeta$,

$$(\forall \xi \in C \cup \{\zeta\})\,\big[C^1_{\delta(\xi,\theta)} = \delta_\theta(\xi)\ \cap\ C^1_{\delta(\zeta,\theta)} \subseteq a^0_\xi\big].$$

[Should that be a $\wedge$?]

Let $\xi(\varepsilon)$ in the proof of (1) be such that $\delta_\theta(\xi(\varepsilon)) \in C$, and let χ^*, S be as there. Now if $\varepsilon^* < \mathrm{cf}(\zeta)$ is a limit ordinal and $S \cap \varepsilon^*$ an unbounded subset of ε^*, the proof there gives the results for $\xi = \xi(\varepsilon^*)$, but the set of such $\xi(\varepsilon^*)$ is a closed unbounded subset of ζ (as $\mathrm{cf}(\zeta) > \aleph_0$). So $\chi_\theta(\xi)$ is well defined and $\leq \chi_\theta(\zeta)$ for a closed unbounded set of $\xi < \zeta$.

3) Should be clear (see 1.16). If $\lambda^* > \lambda$ and $\mathrm{cf}(\theta) \neq \lambda$ we can apply part (2) to $\zeta = \lambda$, and get a club C of λ such that $\xi \in C \Rightarrow (*)_{\xi,\theta}$. by part (1) we know $(*)_{\lambda,\theta}$, so it is enough to have $C \cup \{\lambda\} \in E$ (which holds by "D satisfies the (λ^*, λ)-demand One").

If $\lambda^* = \lambda$, let the κ from 1.16(4) be $\mathrm{cf}(\theta)$:

[Demand Three from 1.16(4) is a statement about *all* $\kappa \in \mathrm{Reg} \cap \lambda$.]

for every $\zeta \in \{\delta < \lambda : \mathrm{cf}^{\mathbf{V}}(\delta) \neq \mathrm{cf}(\theta)\}$ as above, for some club C_δ of δ, we have

$$\zeta \in C_\delta \cup \{\delta\} \Rightarrow (*)_{\zeta,\theta}.$$

Now apply "D satisfies (λ^*, λ)-demand Three."

We are left with the case $\lambda^* > \lambda = \mathrm{cf}(\theta)$, which is like the second case but easier. (Use, for example, $\kappa = \aleph_0$. Note that in this case $\lambda \in D$, as in 1.16(4) we have three possibilities: the second is excluded, and the first and third imply $\lambda \in D$.) $\square_{2.23}$

Fact 2.24. For the D-majority of $\zeta \leq \lambda$ we have $a^0_\zeta \in \mathbf{W}$, provided that $\mathrm{cf}^{\mathbf{V}}(\mu) < \lambda^*$ (assuming 2.11(4)(A)).

Proof. PROOF OF 2.24.

The proof is split into cases (they cover more than demanded in 2.11(4)(A); clause (4)(B) is irrelevant).

Case A: $\lambda < \lambda^*$, $\mathrm{cf}(\mu) \neq \lambda$, and D satisfies the (λ^*, λ)-demands One and Two.

First, by Fact 2.23(2) (applied with μ, λ here standing for θ, ζ there) as $\lambda < \lambda^*$, there is a closed unbounded $C \subseteq \lambda$ and a $\chi^* < \mu$ such that for every $\zeta \in C \cup \{\lambda\}$, $(*)_{\zeta,\mu}$ holds for $\chi(\zeta, \mu) \leq \chi^*$. Note that demand One (see 1.16(2)) holds by hypothesis, hence $C \cup \{\lambda\} \in D$, and without loss of generality every member of C is a limit ordinal.

Second, by Subfact 2.20 the set $S := \{\zeta \leq \lambda : a^0_\zeta \cap \chi^* \in \mathbf{W}\}$ belongs to D.

Lastly, by Subfact 2.21(2), for some set $C^1 \in D$, for every $\zeta \in C^1$, $C^1_{\delta(\zeta,\mu)}$ is an unbounded subset of a^0_ζ.

As D is a filter, $S^* := \big(C \cup \{\lambda\}\big) \cap S \cap C^1$ belongs to D, and we shall prove that $a^0_\zeta \in \mathbf{W}$ for every $\zeta \in S^*$. Let

$$A := c\ell\big((a^0_\zeta \cap \chi^*) \cup C^1_{\delta(\zeta,\mu)}, M^2_{\mu^+}\big).$$

As $\zeta \in C \cup \{\lambda\}$, ζ is a limit ordinal. As $\zeta \in S$, $a^0_\zeta \cap \chi^* \in \mathbf{W}$, and obviously $C^1_{\delta(\zeta,\mu)} \in \mathbf{W}$, hence $A \in \mathbf{W}$, so it is enough to prove $A = a^0_\zeta$. As $a^0_\zeta \cap \chi^* \subseteq a^0_\zeta$, and $C^1_{\delta(\zeta,\mu)} \subseteq a^0_\zeta$ (because $\delta \in C^1$) and as a^0_ζ is a submodel of $M^2_{\mu^+}$, clearly $A \subseteq a^0_\zeta$. We shall prove the other inclusion by Fact 2.17, so we have just to check that for every $\chi \in \mathrm{Reg} \cap A \cap a^0_\zeta \cap (\chi^*, \mu)$,

$$\sup(a^0_\zeta \cap \chi) \leq \sup\{f^{2,\mu}_i(\chi) : i \in A\}.$$

For this remember that $\zeta \in C \cup \{\lambda\}$, $\chi(\zeta, \mu) \leq \chi^*$, so by $(*)_{\zeta,\mu}$ (from Fact 2.23) $\mathrm{Ch}^\mu_\zeta(\chi) = f^{2,\mu}_{\delta(\zeta,\mu)}(\chi)$. So

$$\sup(a^0_\zeta \cap \chi) = \mathrm{Ch}^\mu_\zeta(\chi) = f^{2,\mu}_{\delta(\zeta,\mu)}(\chi) = \sup\big\{f^{2,\mu}_i(\chi) : i \in C^1_{\delta(\zeta,\mu)}\big\} \leq \sup\{f^{2,\mu}_i(\chi) : i \in A\}.$$

(The first equality is by the definition of $\mathrm{Ch}^\mu_\zeta(\chi)$, the second by $(*)_{\zeta,\mu}$, and the third by the definition of $f^{2,\mu}_i$; see 2.16 and 2.16(2)(1).)

[Does not exist; did you mean '2.16(1) and (2)?']

So we have proved the inequality required for applying Fact 2.17, hence $A = a^0_\zeta$, hence $a^0_\zeta \in \mathbf{W}$. As this holds for every $\zeta \in S^*$ and $S^* \in D$, we have proved the Fact for Case A.

Case B: D is $\mathrm{cf}(\mu)^+$-complete (or at least closed under intersection of decreasing sequences of length $\mathrm{cf}(\mu)$) and satisfies the (λ^*, λ)-demands One-Three (at least for $\kappa = \mathrm{cf}(\mu)$).

As "D satisfies (λ^*, λ)-demands One and Three (for $\kappa = \mathrm{cf}(\mu)$)," by 2.23(3) for some $C \in D$, for every $\zeta \in C$, $\chi_\mu(\zeta)$ is well defined (this is a weaker conclusion than in the first paragraph of the proof of Case A, so we strengthen the conclusion of the second paragraph). For $\chi < \mu$, define

$$S_\chi := \big\{\zeta \leq \lambda : a^0_\zeta \cap \chi \in \mathbf{W}\big\}.$$

Now define $S := \bigcap_{\chi < \mu} S_\chi$.

Note that $\chi_1 \leq \chi_2 < \mu \Rightarrow S_{\chi_2} \subseteq S_{\chi_1}$. Hence as D is $\mathrm{cf}(\mu)^+$-complete, $S \in D$.

Lastly, by 2.21, as D satisfies the (λ^*, λ)-demand Two for some $C^1 \in D$, we have

$$\zeta \in C^1 \Rightarrow C^1_{\delta(\zeta,\mu)} \text{ is an unbounded subset of } a^0_\zeta.$$

We can continue as in Case A.

Case C: $\mathrm{cf}(\mu) = \lambda$, $\lambda \in D$ and D satisfies the (λ^*, λ)-demands Zero-Two.

Note: necessarily $\lambda = \mathrm{cf}(\mu) < \lambda^*$. We define C as in Case B, as well as

$$\langle S_\chi : \chi < \mu \rangle.$$

Let $\langle \theta_\zeta : \zeta < \lambda \rangle$ be an increasing continuous sequence of cardinals $< \mu$ such that $\mu = \bigcup_{\zeta < \lambda} \theta_\zeta$, and without loss of generality $\bigwedge_\zeta [\theta_\zeta \in a^0_{\zeta+1}]$. Let

$$S = \{\zeta \leq \lambda : (\forall \xi < \zeta)[\zeta \in S_{\theta_\xi}]\};$$

as $\chi_1 < \chi_2 < \mu \Rightarrow S_{\chi_1} \subseteq S_{\chi_2}$, each S_χ is in D, and D weakly normal (i.e. satisfies demand Zero from 1.16(1)) we get $S \in D$. For each $\zeta < \lambda$,

$$\mathrm{Ch}^\mu_\zeta \restriction [\theta_\zeta, \mu) = f^{2,\mu}_{\delta(\zeta,\mu)} \restriction \big([\theta_\zeta, \mu) \cap a^0_\zeta\big).$$

(Note: $\lambda < \lambda^*$, hence $\sup(a^0_\zeta \cap \mu) = \mu$.)

The rest is as in Case A. $\square_{2.24}$

So without loss of generality we could have assumed 2.11(5), hence:

Hypothesis 2.25. $\mathrm{cf}(\mu) \geq \lambda^*$, $\{\zeta \leq \lambda : \mathrm{cf}(\zeta) > \aleph_0\} \in D$, and $\lambda < \lambda^*$.

Fact 2.26. We can find $\theta(\zeta) \in \Theta$ for $\zeta < \lambda$ such that:

(a) $\langle \theta(\zeta) : \zeta < \lambda \rangle$ is strictly increasing continuous.

(b) $\bigcup_{\zeta<\lambda} \theta(\zeta) = \sup(a^0_\lambda \cap \mu)$

(c) $\mathrm{cf}^{\mathbf{V}}\big(\theta(\zeta)\big) < \lambda$

(d) $\theta(\zeta) = \sup(a^0_\zeta \cap \mu)$.

[This last clause looks like a *definition*, not just another property they need to satisfy. Shouldn't you open with "Let $\theta(\zeta) := \sup(a^0_\zeta \cap \mu)$. Then . . . ," and list the three other properties?]

We leave the proof of this fact to the reader. (Note: the non-limit ζ are not important).

Fact 2.27. If $(\mathbf{W}, \mathbf{V})$ satisfies the λ-covering lemma, $\mathbf{W}$ has λ-squares,

$$\lambda < \lambda^* \leq \mathrm{cf}(\mu),$$

D is a normal filter on $\lambda + 1$ (so $\lambda \in D$) satisfying the (λ^*, λ)-demands Zero-Two, then for the D-majority of ζ, $a^0_\zeta \cap \mu^+ \in \mathbf{W}$.

Proof. By Fact 2.23(2) (with ζ there standing for μ here) for each[9] $\theta \in \Theta$ there is a closed unbounded subset E^0_θ of λ such that

$$\big(\forall \xi \in E^0_\theta \cup \{\lambda\}\big)\big[\chi_\theta(\xi) \leq \chi_\theta(\lambda) < \theta\big].$$

By 2.21 (using "D satisfies (λ^*, λ)-demand Two") we get that for some $Y \in D$, for every $\xi \in Y$, $C^1_{\delta(\xi,\theta)} = C^1_{\delta(\lambda,\theta)} \cap a^0_\xi$ is also an unbounded subset of $a^0_\xi \cap \theta^+$. Let $E^1_\zeta := \bigcap\{E^0_\theta : \theta \in C^2_{\theta(\zeta)} \cap \Theta\}$, which is also a closed unbounded subset of λ (remember, $C^2_{\theta(\zeta)}$ has cardinality $< \lambda$). Lastly, define E^2 as the set of $\zeta < \lambda$ such that:

(i) $\zeta \in E^1_\xi$ for every $\xi < \zeta$.

(ii) $\theta(\zeta) = \sup(a^0_\zeta \cap \mu)$ (use 2.26(d))

[9] I.e. $\mathrm{cf}^{\mathbf{V}}(\theta) < \lambda$ hence $\mathrm{cf}^{\mathbf{W}}(\theta) < \lambda$ by λ-covering, hence $\theta \in \Theta^*$.

[Again, clause (d) states that this holds for *all* $\zeta < \lambda$.]

This is again is a closed unbounded subset of λ, hence $E^2 \in D$ (as D satisfies (λ^*, λ)-demand One and $\lambda \in D$). Similarly, as D satisfies the (λ^*, λ)-demands Zero and Two, and by Subfact 2.20 and a variant of Subfact 2.21 (for C^2_i instead C^1_i), the set

$$E^3 := \left\{\zeta \in E^2 : C^2_{\theta(\zeta)} \subseteq a^0_\zeta \text{ and } \left(\forall \theta \in C^2_{\theta(\zeta)} \cap \Theta\right)\left[a^0_\zeta \cap \theta^+ \in \mathbf{W}\right]\right\}$$

belongs to D.

[Weak normality suffices, as an initial segment of $b \subseteq \alpha$, $b \in \mathbf{W}$ is in $\mathbf{W}$.]

We shall prove now that for each $\zeta \in E^3$, if $\mathrm{cf}(\zeta) > \aleph_0$ and $\zeta \in Y$ then $a^0_\zeta \in \mathbf{W}$. By the proof of Lemma 2.2 (i.e. 2.8(2)) it is enough to prove that $a^0_\zeta \cap \mu \in \mathbf{W}$.

Clearly there is $\varepsilon < \lambda$ such that

$$\chi_{\theta(\varepsilon)}(\lambda) < \theta(\zeta) \leq \theta(\varepsilon).$$

(For example, use $\varepsilon = \zeta$. But even if we want to use $\chi_{\theta(\varepsilon)}(\lambda)$ for some stationary set of ε-s, we can use Fodor's Lemma, slightly decreasing E^3). As $\mathrm{cf}^{\mathbf{V}}(\zeta) > \aleph_0$ and $\chi_\sigma(\lambda)$ (with σ varying) is a regressive function on $C^2_{\theta(\zeta)} \cap \Theta$, there is some $\chi^* \in \left[\chi_{\theta(\varepsilon)}(\lambda), \theta(\zeta)\right) \cap C^2_{\theta(\zeta)}$ such that

$$S := \left\{\theta \in C^2_{\theta(\zeta)} \cap \Theta : \theta > \chi^*,\ \chi_\theta(\lambda) \leq \chi^*\right\}$$

is a stationary subset of $\theta(\zeta)$.

Let

$$A := (a^0_\zeta \cap \chi^*) \cup C^1_{\delta(\zeta, \theta(\zeta))} \cup \bigcup_{\theta \in S \cap \Theta} C^1_{\delta(\zeta,\theta)}.$$

Clearly[10] $A \subseteq a^0_\zeta$. For each $\theta \in S$,

$$\theta^+ \cap c\ell\big((a^0_\zeta \cap \chi^*) \cup C^1_{\delta(\zeta,\theta)}, M^2_{\mu^+}\big) = \theta^+ \cap a^0_\zeta$$

as $\zeta \in E^3$ and $\theta \in C^2_{\theta(\zeta)}$ (as in the proof of 2.24). As S is unbounded (in $C^2_{\theta(\zeta)}$), it follows that $a^0_\zeta \cap \theta(\zeta) \subseteq c\ell(A, M^2_{\mu^+}) \subseteq a^0_\zeta$. As $\mathrm{cf}^{\mathbf{V}}(\mu) \geq \lambda^*$, by 2.26(d) we have $\theta(\zeta) = \sup(a^0_\zeta \cap \mu)$, so $a^0_\zeta \cap \mu = c\ell(A, M^3_{\mu^+}) \cap \mu$. So it suffices to prove that $A \in \mathbf{W}$, and for this it suffices to prove that S and $\langle \delta_\theta(\zeta) : \theta \in S \cap \Theta\rangle$ belong to $\mathbf{W}$. Note that

$$C^2_{\theta(\zeta)} \cap \Theta = \{\theta \in C^2_\theta(\theta) : \mathbf{W} \models \text{“}\theta \text{ is a cardinal of cofinality} < \lambda\text{”}\}$$

hence $C^2_{\theta(\zeta)} \cap \Theta$ belongs to $\mathbf{W}$.

Why is $S \in \mathbf{W}$?

Remember $\theta(\varepsilon)$ and $\chi_{\theta(\varepsilon)}(\lambda)$ used above and compare the functions $f^{2,\theta(\varepsilon)}_{\delta(\lambda,\theta(\varepsilon))}$ and $f^{2,\theta}_{\delta(\lambda,\theta)}$ for $\theta \in C^2_{\theta(\zeta)} \cap \Theta \setminus \chi^*$. We know that

$$f^{2,\theta(\varepsilon)}_{\delta(\lambda,\theta(\varepsilon))} \restriction \big(\left[\chi^*, \theta(\varepsilon)\right) \cap a^0_\lambda\big) = \mathrm{Ch}^{\theta(\varepsilon)}_\lambda \restriction \left[\chi^*, \theta(\varepsilon)\right).$$

So for $\xi \in [\zeta, \lambda)$ we have

$$f^*_\xi := f^{2,\theta(\varepsilon)}_{\delta(\lambda,\theta(\varepsilon))} \restriction \big(\left[\chi^*, \theta(\zeta)\right) \cap a^3_\xi\big) \in \mathbf{W}$$

(see clause (iv) in the definition of the first player's strategy, and note $a^0_\xi \subseteq a^3_\xi \subseteq a^0_{\xi+1}$ and $a^3_\xi \in \mathbf{W}$).

From the line above, we know f^*_ξ is equal to

$$\mathrm{Ch}^{\theta(\zeta)}_\lambda \restriction \big(\left[\chi^*, \theta(\zeta)\right) \cap a^3_\xi\big).$$

---

[10] Note: $C^1_{\delta(\zeta,\theta)} \subseteq a^0_\zeta$ by Subfact 2.21 as $\mathrm{cf}^{\mathbf{V}}(\zeta) > \aleph_0$.

Now if $\theta \in S$ and $\xi \in [\zeta, \lambda)$ then

$$f^{2,\theta}_{\delta(\lambda,\theta)} \restriction \big([\chi^*, \theta) \cap a^3_\xi\big) = \mathrm{Ch}^\theta_\lambda \restriction \big([\chi^*, \theta) \cap a^3_\xi\big) = f^*_\xi \restriction [\chi^*, \theta).$$

But if $\theta \in C^2_{\theta(\zeta)} \cap \Theta \setminus \chi^* \setminus S$, then by the definition of $\chi_\theta(\lambda)$ (as $\chi_\theta(\lambda) > \chi^*$) for every $\xi < \lambda$ large enough,

$$f^*_\xi \restriction [\chi^*, \theta) \neq f^{2,\theta}_{\delta(\lambda,\theta)} \restriction \big([\chi^*, \theta) \cap a^3_\xi\big).$$

As $\big|C^2_{\theta(\zeta)}\big| < \lambda = \mathrm{cf}^{\mathbf{V}}(\lambda)$, one $\xi_* \in [\zeta, \lambda)$ is large enough for all. Also, by the choice of ε, for all $\theta \in C^2_{\theta(\zeta)} \cap \Theta$ we have

$$\delta_\theta(\lambda) = \mathrm{Ch}^\mu_\lambda\big((\theta^+)^{\mathbf{W}}\big) = f^*_{\xi_*}\big((\theta^+)^{\mathbf{W}}\big).$$

So as $f^*_{\xi_*} \in \mathbf{W}$, the function

$$\big\langle \delta_\theta(\lambda) : \theta \in \big[\chi, \theta(\zeta)\big) \cap \Theta \cap C^2_{\theta(\zeta)} \text{ and } \mathrm{cf}^{\mathbf{W}}(\theta) < \lambda \big\rangle$$

belongs to $\mathbf{W}$. So we have a definition of S in $\mathbf{W}$, hence $S \in \mathbf{W}$.

<u>Why does $\langle \delta_\theta(\zeta) : \theta \in S \cap \Theta \rangle$ belong to $\mathbf{W}$?</u>

For each $\theta \in S \cap \Theta$, $\delta_\theta(\zeta) \in C^1_{\delta(\zeta,\lambda)}$ (as $\zeta \in E^0_\zeta$). We know that

$$\big\langle f^{2,\theta}_i(\chi^*) : i \in \mathrm{acc}(C^1_{\delta(\lambda,\theta)}) \big\rangle$$

is strictly increasing and continuous. Now $\langle \delta_\theta(\lambda) : \theta \in S \rangle$ (as a function) belongs to $\mathbf{W}$ (as $f^*_{\xi_*} \in \mathbf{W}$), hence

$$\delta_\theta(\zeta) = \min\big\{\gamma \in C^1_{\delta(\lambda,\theta)} : f^{2,\theta}_\gamma(\chi^*) \geq \sup(a^0_\zeta \cap \chi^*)\big\}.$$

This definition can be carried in $\mathbf{W}$, hence $\langle \delta_\theta(\zeta) : \theta \in S \cap \Theta \rangle \in \mathbf{W}$. So we finish the proof of 2.27. $\square_{2.27}$

End of the Proof of 2.11:

It is easy to check that 2.24 and 2.27 suffice to prove 2.11 (see 2.11(4)(A),(B)). $\square_{2.11}$

Remark 2.28. If we want to get the result for $\kappa = \aleph_0 < \lambda < \lambda^*$ (for example, for $\lambda = \aleph_1$ and $\lambda^* = \aleph_2$, when $\mathbf{0}^\# \notin \mathbf{V}$) we can drop from the hypothesis on λ (i.e. λ-covering and λ-squares) and add that the λ^+-squared scales (defined below) exists for $\mathbf{W}$.

It was not clear whether they existed when [She82, XIII] was essentially finished (early 1981). Later, Abraham – who was converging toward it – and the author looked at it and tried to develop it; with Stanley, they seemingly proved its consistency. Subsequently, Donder, Jensen and Stanley [DJS] proved it.

[Not in the bibfile. Presumably this is 'Donder, Hans-Dieter, Ronald B. Jensen, Lee J. Stanley, A. Nerode, and R. A. Shore. "Condensation-coherent global square systems." *Recursion theory* 42 (1985): 237-258.?' If so, I have the BibTeX as a comment below this line.]

Definition 2.29. We say that $\mathbf{W}$ has λ^*-*squared scales* if for each singular θ there are a scale $\langle f^\theta_i : i < \theta^+ \rangle$, a λ^*-square $\langle C^{2,\theta}_\delta : \delta < \theta^+, \mathrm{cf}(\delta) < \lambda^* \rangle$, and (for θ a cardinal in $\mathbf{W}$, $\theta > \lambda^* > \mathrm{cf}(\theta)$) a λ^*-square C^3_θ such that:

(∗) If $\theta_1 \in C^3_\theta$, $\zeta \in C^{2,\theta}_\delta$, and $\xi := f^\theta_\delta(\theta^+_1)$ <u>then</u> $f^\theta_\zeta(\theta^+_1) \in C^{2,\theta}_\xi$.

Remark 2.30. We can restrict ourselves to $\theta < \alpha^*$ for any fixed α^*.

Theorem 2.31. *Suppose $(\mathbf{V}, \mathbf{W})$ is a pair of universes of set theory, $\aleph_0 < \kappa < \lambda < \lambda^*$ are regular cardinals in $\mathbf{V}$, and $\mathbf{W}$ has squares (or just λ^*-squares and λ-squares) and scales.*

Then $(\mathbf{V}, \mathbf{W})$ satisfies the strong $(\lambda^*, \lambda, \kappa, \infty)$-covering lemma, if it satisfies the λ^*-covering lemma and the λ-covering lemma.

Proof. Let D be as in 1.17(2), by which it satisfies demands Zero through Three. As the strong $(\lambda^*, \lambda, \kappa, \infty)$-covering lemma is equivalent to the strong $(\lambda^*, \lambda, D, \infty)$-covering lemma, it suffices to prove the later. We prove by induction on μ (a cardinal in $\mathbf{W}$) that $(\mathbf{V}, \mathbf{W})$ has the strong $(\lambda^*, \lambda, \kappa, \mu)$-covering lemma. For $\mu \leq \lambda^*$, see 2.1; for successor μ (in $\mathbf{W}$) use 2.2, and for limit μ use 2.11. $\square_{2.31}$

Conclusion 2.32. *If in $\mathbf{V}$, $\mathbf{0}^\#$ does not exist, then $(\mathbf{L}, \mathbf{V})$ satisfies the strong $(\aleph_3^{\mathbf{V}}, \aleph_2^{\mathbf{V}}, \aleph_1^{\mathbf{V}}, \infty)$-covering lemma.*

Theorem 2.33. *Suppose $(\mathbf{W}, \mathbf{V})$ satisfies the λ^*-covering lemma and $\mathbf{W}$ has squares and scales. If there is no cardinal μ of $\mathbf{W}$ such that $\lambda < \mu < \lambda^*$, and $\kappa < \lambda < \lambda^*$ are regular cardinals of $\mathbf{V}$, then $(\mathbf{W}, \mathbf{V})$ has the $(\lambda^*, \lambda, \kappa, \infty)$-strong covering property.*

Proof. Note that if $\mathbf{V} \vdash$ "$\mathrm{cf}(\alpha) = \lambda$" then $\mathbf{W} \vdash$ "$\lambda \leq \mathrm{cf}(\alpha) < (\lambda^+)^{\mathbf{V}}$", hence in our case $\mathbf{W} \vdash$ "$\mathrm{cf}(\alpha) = \lambda$". So we can slightly strengthen Claim 1.9, by demanding:

'If $\mathrm{cf}(\delta) = \lambda$ then C^1_δ has order type λ.'

Now repeat the proofs of 2.2 and 2.11 (or see 4.22). $\square_{2.33}$

Conclusion 2.34. *If $\mathbf{0}^\# \notin \mathbf{L}$ and there is no cardinal μ of $\mathbf{L}$ for which $\aleph_1^{\mathbf{V}} < \mu < \aleph_2^{\mathbf{V}}$, then $(\mathbf{L}, \mathbf{V})$ satisfies the strong $\aleph_2$-covering lemma and the strong $\aleph_1$-covering lemma.*

Proof. The strong $\aleph_2$-covering is by 2.33, and the strong $\aleph_1$-covering follows immediately. $\square_{2.34}$

§ 3. A counterexample

The following lemma says that even if $\mathbf{V}$ and $\mathbf{L}$ have the same cardinals, except $\aleph_2^{\mathbf{L}}$ and $\mathrm{cf}^{\mathbf{V}}(\aleph_2^{\mathbf{L}}) = \aleph_1$, the strong $\aleph_1$-covering lemma may fail. It uses forcings, but its role is just to show that some theorems cannot be proved.

Lemma 3.1. *Assume $\mathbf{V}$ satisfies* CH*. Then there is a forcing notion $\mathbb{R}$ of cardinality $\aleph_2$ which does not collapse $\aleph_1$ (and even satisfies the condition from* [She82, XI][11]*) and does not collapse any $\aleph_\alpha > \aleph_2$, such that $(\mathbf{V}, \mathbf{V}^{\mathbb{R}})$ does not satisfy the strong $(\aleph_1, \aleph_2^{\mathbf{V}})$-covering lemma. (Note: if $\mathbf{V}$ satisfies* GCH *then so does $\mathbf{W}$.)*

Proof. Let $\mathbb{P}$ be (for example) the forcing of adding a Cohen real. In $\mathbf{V}^{\mathbb{P}}$ we define a forcing notion $\mathbb{Q}$ as the following set of functions from some ordinal $\alpha < \aleph_1$ with range contained in $\aleph_2$:

$$\mathbb{Q} := \bigcup_{\alpha<\aleph_1} \big\{ f \in {}^{\alpha}(\aleph_2) : (\forall \delta \leq \alpha)\big[\delta \text{ limit} \Rightarrow \mathrm{rang}(f \restriction \delta) \notin \mathbf{V}\big] \big\}$$

$\mathbb{Q}$ is ordered by inclusion.

First note that $\mathbb{Q} \neq \varnothing$ (as the empty function belongs to $\mathbb{Q}$), and we shall prove that for every $p \in \mathbb{Q}$ and $\beta < \aleph_1$ there is $q \in \mathbb{Q}$ such that $p \leq q$ and $\beta \subseteq \mathrm{dom}(q)$.

Let $\mathrm{dom}(p) := \alpha$, and choose $i > \sup\mathrm{rang}(p)$ (with $i < \aleph_2$) and $A \subseteq [i, i+\beta+\omega]$ such that for every limit δ,

$$\delta \in (i, i+\beta+\omega) \Rightarrow A \cap [\delta, \delta+\omega] \notin \mathbf{V}$$

and so A has order type $\beta+\omega$ (easy [to see?], as $\mathbb{P}$ adds reals). Now define q as follows: $\mathrm{dom}(q) := \alpha+\beta$, $q(j) := p(j)$ for $j < \alpha$, and $q(\alpha+j)$ is the j-th element of A. Also (trivially), for every $i < \alpha_2$ we know that $\big\{p \in \mathbb{Q} : i \in \mathrm{rang}(p)\big\}$ is a dense subset of $\mathbb{Q}$.

We will work for a while in $\mathbf{V}^{\mathbb{P}}$. As $\mathbf{V}^{\mathbb{P}}$ satisfies '$2^{\aleph_0} = \aleph_1$,' clearly $\mathbb{Q}$ has cardinality $\aleph_2$ and it is easy to check $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ has cardinality $\aleph_2$, and it will be our $\mathbb{R}$. It suffices to prove that $\mathbb{Q}$ does not add reals (and hence does not collapse $\aleph_1$), as the generic function from $\aleph_1$ to $\aleph_2$ will be the evidence of the failure of the strong $(\aleph_1, \aleph_2^{\mathbf{V}})$-covering lemma.

So let $\underset{\sim}{h}$ be a $\mathbb{Q}$-name (in $\mathbf{V}^{\mathbb{P}}$), and $p \in \mathbb{Q}$ forces that it is a function from ω into $\aleph_1$. We define a condition $p_\eta \in \mathbb{Q}$ for every $\eta \in {}^{n}(\omega_2)$ (i.e. a sequence of ordinals $< \omega_2$ of length n) by induction on $n < \omega$ such that:

- (A) $p_{\langle\,\rangle} := p$
- (B) $p_\eta \restriction \ell \leq p_\eta$ for $\ell \leq \ell g(\eta)$.
- (C) $p_\eta \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{h}(m) = \gamma_\eta$" when $\ell g(\eta) = m+1$, for some $\gamma_\eta < \omega_1$.
- (D) $\mathrm{rang}(\eta) \subseteq \mathrm{rang}(p_\eta)$; moreover, if $\ell g(\eta) = m+1$ then
$$p_\eta(\sup\mathrm{rang}(p_{\eta\restriction m})) = \eta(m).$$

There are no problems in the definition.

By Rubin and Shelah [RS87] (the theorem on Δ-systems) there is $T \subseteq {}^{\omega>}(\omega_2)$ such that:

- (α) $\langle\,\rangle \in T$, $\big|\mathrm{suc}_T(\eta)\big| = \aleph_2$ for every $\eta \in T$, and T is closed under initial segments.
- (β) $\mathrm{dom}(p_\eta) = \delta_{\ell g(\eta)} < \omega_1$ (I.e. $\mathrm{dom}(p_\eta)$ depends only on the length of η.)
- (γ) There are countable sets $A_\eta \subseteq \omega_2$ for $\eta \in T$ such that $\mathrm{rang}(p_\eta) \subseteq A_\eta$ and
$$\big(\forall \eta, \rho \in T\big)\big[\eta(\ell) \neq \rho(\ell) \wedge \eta \restriction \ell = \rho \restriction \ell \Rightarrow A_\eta \cap A_\rho = A_{\eta\restriction\ell}\big].$$

[11] Even better, see [She98, XI,XV].

Let $\alpha_\eta := \sup(A_\eta)$, so $p_{\eta\hat{}\langle\beta\rangle}(\alpha_\eta) = \beta$ (see clause (D) above), and without loss of generality

(∗) If $\alpha < \beta$ and $\eta\hat{}\langle\alpha\rangle, \eta\hat{}\langle\beta\rangle \in T$, then $\min(A_{\eta\hat{}\langle\beta\rangle} \setminus A_\eta) > \alpha$.

Let

$$C := \big\{\zeta < \aleph_2 : \text{for all } \eta \in ({}^{\omega>}\zeta) \cap T,\ a_\eta \subseteq \zeta \text{ and } \{\alpha < \zeta : \eta\hat{}\langle\alpha\rangle \in T\} \text{ is unbounded in } \zeta\big\}.$$

Clearly it is a closed unbounded subset of $\aleph_2$, hence it contains a closed unbounded subset C which belongs to $\mathbf{V}$ (remember, we are working in $\mathbf{V}^{\mathbb{P}}$).

[Should I call the first guy C' or something?]

As the cardinality of $\mathbb{P}$ is $\leq \aleph_1 < \aleph_2$, there is an increasing sequence $\langle\zeta_n : n < \omega\rangle \in \mathbf{V}$ included in C.

Now let $A := \{n(\ell) : \ell < \omega\} \in \mathbf{V}^{\mathbb{P}} \setminus \mathbf{V}$, with $n(\ell) < n(\ell+1)$. We define an ordinal α_ℓ by induction on ℓ such that:

- $\alpha_\ell \in \big(\zeta_{n(\ell)}, \zeta_{n(\ell)+1}\big)$
- $\langle\alpha_0, \ldots, \alpha_\ell\rangle \in T$ and $A_{\langle\alpha_0,\ldots,\alpha_{\ell+1}\rangle} \cap \zeta_{n(\ell+1)} \subseteq \zeta_{n(\ell)+1}$.

This is easy by (∗) above. Let $p^* := \bigcup_\ell p_{\langle\alpha_0,\ldots,\alpha_\ell\rangle}$. Now p^* is a function from $\delta := \bigcup_\ell \delta_\ell$ to ω_2; if $\operatorname{rang}(p^*) \in \mathbf{V}$, then

$$A = \big\{\ell : (\zeta_\ell, \zeta_{\ell+1}) \cap \operatorname{rang}(p^*) \neq \varnothing\big\} \in \mathbf{V}$$

as $\langle\zeta_\ell : \ell < \omega\rangle \in \mathbf{V}$, a contradiction. Hence $b \notin \mathbf{V}$, and it is easy to check $p^* \in \mathbb{Q}$, and clearly p^* forces a value to $\underset{\sim}{h}$, so we have proved that $\mathbb{Q}$ does not add reals, hence does not collapse $\aleph_1$, hence we finish the proof of 3.1. $\square_{3.1}$

Remark 3.2. The choice of "$\mathbb{P}$ is a Cohen forcing" is because it is the simplest option. For example, assume $\kappa = \kappa^{\aleph_0}$, $\mathbb{P}$ is a forcing notion of cardinality $\leq \kappa$ adding a new real, and $\mathbb{Q}$ is the forcing defined in the proof of 3.1 (with κ^+ replacing $\aleph_2$). Then the forcing by $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ collapses κ to $\aleph_1$ but collapses no other cardinality nor changes any cofinality (in particular, it does not collapse $\aleph_1$), has cardinality κ^+, and all the reals of $\mathbf{V}^{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ are from $\mathbf{V}^{\mathbb{P}}$. So if $\mathbb{P}$ is ${}^\omega\omega$-bounding (for example, Sacks forcing) then so is $\mathbb{P} * \mathbb{Q}$. On the other hand, the strong $(\aleph_1, \kappa^+)$-covering lemma fails (i.e. the family of old countable subsets of κ^+ is not stationary); this answers a question of Kamburelis. Really, assuming CH, any proper forcing adding a new real of cardinality $\aleph_1$ is okay. The proof gives us more than stated in 3.1; in particular, it answers that question of Kamburelis.

§ 4. When adding a real cannot destroy CH

Here we draw conclusions concerning consistency strength, but the section is not used later, and knowledge of inner models is required.

On core models, see Dodd and Jensen [DJ81]; they prove

Theorem 4.1. *For every model (of set theory)* $\mathbf{V}$ *there is a core model* $K(\mathbf{V}) \subseteq \mathbf{V}$*, such that:*

1) $K(\mathbf{V})$ *is a transitive class containing all ordinals, and* $\mathbf{W} \subseteq \mathbf{V}$ *implies* $K(\mathbf{W}) \subseteq K(\mathbf{V})$*.*

2) $K(\mathbf{V})$ *has squares and satisfies* GCH *(and hence has scales).*
Let $K_\lambda(\mathbf{V})$ *be the family of sets in* $K(\mathbf{V})$ *of hereditary power* $< \lambda$*.*

3) *If in* $\mathbf{V}$ *there is no inner model with a measurable cardinal, then* $(K(\mathbf{V}), \mathbf{V})$ *satisfies the covering lemma (see Definition* 1.2*).*

4) $K(\mathbf{V})$ *has a definable well-ordering (and hence definable Skolem functions).*

The following is known:

Theorem 4.2. 1) *Suppose* $\mathbf{W} \subseteq \mathbf{V}$ *have the same cardinals, then they have the same core model.*

2) *Moreover, if* $\mathbf{W} \subseteq \mathbf{V}$ *have the same cardinals* $\leq \lambda$*, where* λ *is a limit cardinal (in both models), then* $K_\lambda(\mathbf{V}) = K_\lambda(\mathbf{W})$ *(see* 4.1(2)*).*

Proof. 1) Suppose $K(\mathbf{W}) \neq K(\mathbf{V})$: clearly $K(\mathbf{W}) \subseteq K(\mathbf{V})$, so let $A \subseteq \alpha$, $A \in K(\mathbf{V})$, $A \notin K(\mathbf{W})$. So there is a mice of $K(\mathbf{V})$ to which A belongs, hence there is such a mice of $K(\mathbf{V})$-power $|\alpha|$. But we can extend it, hence for every limit cardinal $\lambda > \alpha$ of $\mathbf{V}$ there is a mice with critical point λ to which A belongs, and the filter is generated by end segments of

$$\{\chi < \lambda : \chi \text{ a cardinal in } \mathbf{V}\}.$$

But then this mice is in $\mathbf{W}$ hence in $K(\mathbf{W})$.

2) The same proof. $\square_{4.2}$

Conclusion 4.3. *Suppose in* $\mathbf{V}$ *there is no inner model with a measurable cardinal. Then:*

1) $(K(\mathbf{V}), \mathbf{V})$ *satisfies the strong* λ*-covering for every* $\lambda > \aleph_2$*.*

2) *If* $\mathbf{W} \subseteq \mathbf{V}$ *have the same cardinals, then* $(\mathbf{W}, \mathbf{V})$ *satisfies the strong* λ*-covering lemma for every cardinal* $\lambda \geq \aleph_1$ *of* $\mathbf{V}$*.*

3) *If* $\mathbf{W} \subseteq \mathbf{V}$ *have the same cardinals* $\leq \mu$ *(or even* $K_\mu(\mathbf{W}) = K_\mu(\mathbf{V})$*, where* μ *is a limit cardinal of* $\mathbf{V}$*) then* $(\mathbf{W}, \mathbf{V})$ *satisfies the strong* (λ, μ)*-covering lemma for any cardinal* $\lambda > \aleph_0$ *of* $\mathbf{V}$*.*

Proof. 1) By hypothesis $(K(\mathbf{V}), \mathbf{V})$ satisfies the λ-covering lemma for every $\lambda \geq \aleph_2^{\mathbf{V}}$; by fine structure theory, $K(\mathbf{V})$ has squares and scales. So our main theorem 2.31 gives the desired conclusion.

2) $K(\mathbf{V}) = K(\mathbf{W})$ by the previous theorem, hence $K(\mathbf{V}) \subseteq \mathbf{W}$. We can finish by part (1) using 2.33.

3) Similar proof. $\square_{4.3}$

Remember:

Theorem 4.4 (**Magidor**). *If $\mathbf{W} \subseteq \mathbf{V}$ and $K(\mathbf{W}) \neq K(\mathbf{V})$, then for some cardinal λ of $K(\mathbf{V})$ and $A \subseteq \lambda$, we have $A \in K(\mathbf{W})$ and $K_\lambda(\mathbf{W}) = K_\lambda(\mathbf{V})$, but there is a class C (in $\mathbf{V}$) of ordinals from $K(\mathbf{W})$ such that in $K(\mathbf{W})$, C is indiscernible over A and $K(\mathbf{W})$ is the Skolem Hull (see 4.1(4)) of $A \cup C$.*

Theorem 4.5. 1) *If $\mathbf{V} := \mathbf{W}[r]$, r a real, and $(\mathbf{W}, \mathbf{V})$ satisfies the strong λ-covering lemma (λ a cardinal of $\mathbf{V}$) then:*

(*i*) $\sum_{\mu<\lambda} (2^\mu)^{\mathbf{V}} = \left| \sum_{\mu<\lambda} (2^\mu)^{\mathbf{W}} \right|^{\mathbf{V}}$

(*ii*) $\left(\chi^{<\lambda}\right)^{\mathbf{V}} = \left|(\chi^{<\lambda})^{\mathbf{W}}\right|^{\mathbf{V}}$ *for every χ.*

(*iii*) *Assume λ is regular in $\mathbf{V}$, $A \in \mathbf{W}$, $A \subseteq \lambda$, and $\mathcal{H}(\lambda)^{\mathbf{W}} \subseteq \mathbf{L}_\lambda[A]$. Then any bounded subset B of λ from $\mathbf{V}$ belongs to $\mathbf{L}_\alpha[A \cap \alpha, r]$ for some $\alpha < \lambda$.*

2) *For getting (ii), it suffices to have the strong (λ, α)-covering lemma for $\alpha < \left(\left[(\chi^{<\lambda})^{\mathbf{W}}\right]^+\right)^{\mathbf{V}}$. (Note that (i) is a special case of (ii).)*

Proof. Easy. $\square_{4.5}$

Conclusion 4.6. *If $\mathbf{V}$ has no inner model with a measurable cardinal, $\mathbf{V} := \mathbf{W}[r]$ with r a real, $\mathbf{W}$ and $\mathbf{V}$ have the same cardinals $\leq \lambda$ (where λ is a limit cardinal in the interval $\left[\aleph_\omega, 2^{\aleph_0}\right]^{\mathbf{V}}$), and $\mathbf{W}$ satisfies* CH[12] *(but $\mathbf{V}$ does not), then*

$$K_\lambda(\mathbf{W}) = K_\lambda(\mathbf{V}), \quad K(\mathbf{W}) \neq K(\mathbf{V}).$$

(This is stronger than '$\mathbf{0}^\# \in \mathbf{V}$'; see 4.4.)

Proof. We know that $K_\lambda(\mathbf{W}) = K_\lambda(\mathbf{V})$ by 4.2(2). On the other hand, if $K(\mathbf{W}) = K(\mathbf{V})$ then by 4.3(3) the pair $(\mathbf{W}, \mathbf{V})$ satisfies the strong λ-covering lemma. So by Theorem 4.5 above, $\left(2^{\aleph_0}\right)^{\mathbf{V}} = \left(2^{\aleph_0}\right)^{\mathbf{W}}$, in contradiction to "$\mathbf{W}$ satisfies CH but $\mathbf{V}$ does not." $\square_{4.6}$

Conclusion 4.7. *If $\mathbf{V} := \mathbf{W}[r]$ with r a real, $\mathbf{W}$ satisfies CH, and '$2^{\aleph_0} > \aleph_2$' holds in $\mathbf{V}$, then $\mathbf{0}^\# \in \mathbf{V}$.*

Proof. Suppose $\mathbf{0}^\# \notin \mathbf{V}$. We know $\lambda^{\aleph_0} \leq \lambda^+ + 2^{\aleph_0} = \lambda^+$ for every λ. By 2.29, $(\mathbf{W}, \mathbf{V})$ satisfies the strong $\aleph_3^{\mathbf{V}}$-covering lemma, hence by 4.5, $2^{\aleph_0} \leq \aleph_3$ in $\mathbf{V}$. To get the exact result, we should use a finer theorem: 4.20 below. $\square_{4.7}$

Conclusion 4.8. *If $\mathbf{V} := \mathbf{W}[r]$ with r a real, $\mathbf{V}$ and $\mathbf{W}$ have the same $\aleph_1$ and $\aleph_2$, and $\mathbf{W}$ satisfies CH but $\mathbf{V}$ does not, then $\mathbf{0}^\# \in \mathbf{V}$.*

Proof. Use 2.34 and 4.5. $\square_{4.8}$

Lemma 4.9. *Suppose $\mathbf{W} \subseteq \mathbf{V}$, λ a cardinal of $\mathbf{W}$ and*

(*a*) *'(i) or (ii),' where:*

  (*i*) $\lambda \in \mathrm{Reg}^{\mathbf{W}}$

  (*ii*) *The square principle for λ holds in $\mathbf{W}$.*
  (I.e. there are $C_\delta \subseteq \delta$ for $\delta < (\lambda^+)^{\mathbf{W}}$ limit, with C_δ closed and unbounded and

$$\gamma = \sup(\gamma \cap C_\delta) \Rightarrow C_\gamma = \gamma \cap C_\delta.)$$

(*b*) $\mathbf{V} \vdash$ "$\mathrm{cf}^{\mathbf{V}}(\lambda) \neq \mathrm{cf}^{\mathbf{V}}(|\lambda|^{\mathbf{V}})$".

[12] Or at least $\mathbf{W} \vdash$ '$2^{\aleph_0} < \lambda$'.

Then in $\mathbf{V}$, the $\mathbf{W}$-successor of λ is not a cardinal.

Remark 4.10. In clause (a), "$\mathrm{pp}(\lambda) > \lambda^+ \wedge \mathrm{cf}(\lambda) < \lambda$" will also suffice (see [She94b, 1.5A]).

Proof. By hypothesis (a), we can easily find $\langle A_i : i < \lambda^+ \rangle$ in $\mathbf{W}$ such that $A_i \subseteq \lambda$, A_i is unbounded in λ, and for every $i < \lambda^+$ there is a function $f_i : i \to \lambda$ such that the sets $A_j \setminus f_i(j)$ (for $j < i$) are pairwise disjoint.

If λ is regular, this is trivial. Choose $A_i \in [\lambda]^\lambda$ (for $i < \lambda$) pairwise disjoint, and then continue choosing $A_i \in [\lambda]^\lambda$ by induction on $i \in [\lambda, \lambda^+)$

[How many i-s *are there* in the interval $[\lambda, \lambda^+)$?]

pairwise disjoint such that

$$j < i \Rightarrow |A_j \cap A_i| < \lambda.$$

If not: by Litman [Lit81], using Jensen's theorem on gap-one transfer theorem (or see Ben David [BD78]). Specifically, let $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of regular cardinals with limit λ; choose $f_\alpha \in \prod_i \lambda_i$ by induction on $\alpha < \lambda^+$ such that

$$\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J^{\mathrm{bd}}_{\mathrm{cf}(\lambda)} \text{ and}$$
$$\alpha \in C_\beta \wedge |C_\beta| < \lambda_i \Rightarrow f_\alpha(i) < f_\beta(i).$$

let $A_\alpha := \mathrm{rang}(f_\alpha)$.

Suppose λ^+ (in $\mathbf{W}$'s sense) is a cardinal of $\mathbf{V}$. Let us work in $\mathbf{V}$. Let $\chi := \mathrm{cf}^{\mathbf{V}}(|\lambda|)$ and $\mu := \mathrm{cf}^{\mathbf{V}}(\lambda)$. So $\lambda = \bigcup_{\alpha<\chi} B_\alpha$, with B_α increasing continuous with α and $|B_\alpha| < |\lambda|$ (all in $\mathbf{V}$). Now each A_i, as an ordered subset of λ, has cofinality μ (as A_i is unbounded in λ) and by assumption (b), $\mu \neq \chi$. Hence for each i, for some $\alpha(i) < \chi$, $A_i \cap B_{\alpha(i)}$ is an unbounded subset of A_i (if $\chi < \mu$ this is trivial, and if $\chi > \mu$ remember that B_α is increasing). We are assuming that (in $\mathbf{V}$) the number of A_i-s is λ^+ and $|\lambda^+| > |\lambda|$; hence for some α, $C := \{i : \alpha(i) = \alpha\}$ has cardinality $> |\lambda|$. Let i_* be the λ-th member of C, so

$$\{(A_j \setminus f_{i_*}(j)) \cap B_\alpha : j < i_*,\, j \in C\}$$

is a family of $|\lambda|$-many pairwise disjoint non-empty subsets of B_α, contradicting '$|B_\alpha| < |\lambda|$.' $\square_{4.4}$

Theorem 4.11. *Suppose $\mathbf{W} \subseteq \mathbf{V} := \mathbf{W}[r]$, r a real, and*

- *(a) The continuum hypothesis fails in $\mathbf{V}$.*
- *(b) GCH holds in $\mathbf{W}$.*
- *(c) $\mathbf{W}$ has squares.*
- *(d) $(\mathbf{W}, \mathbf{V})$ satisfies the strong $\aleph_2$-covering lemma.*

Then in $\mathbf{W}$ there is an inaccessible cardinal; in fact, $\aleph_2^{\mathbf{V}}$ is inaccessible in $\mathbf{W}$.

Remark 4.12. Note that clause (c) is not really necessary (if the conclusion fails then $\mathbf{0}^\# \notin \mathbf{V}$).

Proof. Let $\kappa := (2^{\aleph_0})^{\mathbf{V}}$ and $\chi := \aleph_1^{\mathbf{V}}$. By 4.7, without loss of generality $\kappa \leq (\aleph_2)^{\mathbf{V}}$ hence by clause (a) we know $\kappa = \aleph_2^{\mathbf{V}}$, hence κ is a regular cardinal in $\mathbf{V}$ and hence in $\mathbf{W}$. If the conclusion of the theorem fails, κ is a successor cardinal in $\mathbf{W}$, so let $\kappa = \lambda^+$. By the previous lemma, $\mathrm{cf}^{\mathbf{V}}(\lambda) = \mathrm{cf}^{\mathbf{V}}(|\lambda|^{\mathbf{V}})$. However, $|\lambda|^{\mathbf{V}}$ is necessarily $\aleph_1^{\mathbf{V}} = \chi$ (as $\aleph_1^{\mathbf{V}} \leq \lambda < \aleph_2^{\mathbf{V}}$) hence $\mathrm{cf}^{\mathbf{V}}(\lambda) = \aleph_1^{\mathbf{V}}$.

Let $\overline{C} = \langle C_i : i < \kappa \rangle \in \mathbf{W}$ be a list of all bounded subsets of κ in $\mathbf{W}$. By 4.5(1)(iii), every real s of $\mathbf{V}$ is in $\mathbf{L}_\alpha[\overline{C}, r]$ for some $\alpha < \kappa$ (so really, we can replace $\mathbf{W}$ by $\mathbf{L}[\overline{C}]$). In $\mathbf{V}$, let $\lambda = \bigcup\limits_{i<\chi} A_i$ with $|A_i|^{\mathbf{V}} < \chi$ (remember, $\chi = \aleph_1^{\mathbf{V}}$) and $\langle A_i : i < \chi \rangle$ increasing continuous.

Let s be a real of $\mathbf{V}$; then $s \in \mathbf{L}_{\alpha(s)}[\overline{C}, r]$ for some $\alpha(s) < \kappa$, and without loss of generality $\alpha(s) \geq \lambda$. Let $f_{\alpha(s)} \in \mathbf{W}$ be a one-to-one function from $\mathbf{L}_{\alpha(s)}[\overline{C}]$ onto λ. Still working in $\mathbf{V}$,

$$\mathbf{L}_{\alpha(s)}[\overline{C}, r] = \bigcup_{\gamma<\chi} N^s_\gamma$$

with $\langle N^s_\gamma : \gamma < \chi \rangle$ an increasing continuous sequence of countable elementary submodels of $\mathbf{L}_{\alpha(s)}[\overline{C}, r]$, closed under $f_{\alpha(s)}$ and $f^{-1}_{\alpha(s)}$.

So $\langle A_i : i < \chi \rangle$ and $\langle N^s_\gamma \cap \lambda : \gamma < \chi \rangle$ are sequences (in $\mathbf{V}$) of countable sets, increasing and continuous, of length $\chi = \aleph_1^{\mathbf{V}}$, with the same union: λ. Clearly $N^s_{\gamma(s)} \cap \lambda = A_{\gamma(s)}$ for some $\gamma(s)$, and let $\delta_\gamma := \sup A_\gamma < \lambda$.

Now in $\mathbf{V}$ the continuum hypothesis fails, hence there is a list of κ distinct reals $\{s_\zeta : \zeta < \kappa\}$, and we can replace it by any subfamily of cardinality κ. So without loss of generality $\gamma(s_\zeta) = \gamma_*$ for every $\zeta < \kappa$, and for each $\zeta < \kappa$ we let A^ζ be the closure of δ_{γ_*} by $f_{\alpha(s_\zeta)}$ and $f^{-1}_{\alpha(s_\zeta)}$ (so $A^\zeta \in \mathbf{W}$). Now in $\mathbf{W}$, the number of possible isomorphism types of

$$M_\zeta := \left(A^\zeta, f_{\alpha(s_\zeta)}, f^{-1}_{\alpha(s_\zeta)}, <, \text{“}i \in C_j\text{”}, \delta_{\gamma_*}\right)$$

is $\leq 2^{|\delta_{\gamma_*}|} \leq \lambda$ (as $\mathbf{W}$ satisfies GCH). So without loss of generality this isomorphism type is the same for all ordinals $\zeta < \kappa$.

Now we show that all $N^{s_\zeta}_{\gamma_*}$ (for $\zeta < \kappa$) are isomorphic (in $\mathbf{V}$). Let ζ, $\xi < \kappa$; now any isomorphism from M_ζ onto M_ξ is the identity on A_{γ_*} (as $A_{\gamma_*} \subseteq \delta_{\gamma_*} \subseteq M_\xi$), and hence maps $N^{s_\zeta}_{\gamma_*} \cap \lambda = A_{\gamma_*}$ onto $N^{s_\xi}_{\gamma_*} \cap \lambda = A_{\gamma_*}$. But $\left|N^{s_\ell}_{\gamma_*}\right|$ is the closure of $N^{s_\ell}_{\gamma_*} \cap \lambda$ by $f_{\alpha(s_\zeta)}$ and $f^{-1}_{\alpha(s_\zeta)}$, so looking at the definition of M_ζ we see that the isomorphism takes $N^{s_\zeta}_{\gamma_*} \cap \kappa$ onto $N^{s_\xi}_{\gamma_*} \cap \kappa$, preserves the relation “$i \in C_j$”, and maps r to r. But $N^{s_\ell}_{\gamma_*}$ “thinks” it is $\mathbf{L}_{\alpha(s)}[\overline{C}, r]$, so the isomorphism can be extended to an isomorphism from $N^{s_\zeta}_{\gamma_*}$ onto $N^{s_\xi}_{\gamma_*}$, as promised. But $N^{s_\zeta}_{\gamma_*}$ is countable and we have too many reals; a contradiction. $\square_{4.11}$

Conclusion 4.13. *If there are universes* $\mathbf{W} \subseteq \mathbf{V} := \mathbf{W}[r]$, r *a real,* $\mathbf{W}$ *satisfies* GCH*, and* CH *fails in* $\mathbf{V}$*,* <u>*then*</u> *in* $\mathbf{L}$ *there is an inaccessible cardinal. In fact,* $\aleph_2^{\mathbf{V}}$ *is inaccessible in* $\mathbf{L}$*.*

Proof. Suppose in $\mathbf{L}$ there is no inaccessible cardinal or just $\aleph_1^{\mathbf{V}}$ is not an inaccessible in $\mathbf{L}$. Then $\mathbf{0}^\# \notin \mathbf{V}$, hence $\mathbf{0}^\# \notin \mathbf{W}$, and as $\mathbf{W}$ satisfies GCH, $\mathbf{W}$ has squares and scales. If $(\mathbf{W}, \mathbf{V})$ satisfies the strong $\aleph_2^{\mathbf{V}}$-covering lemma, then all the hypotheses of 4.11 are satisfied (and hence its conclusion, which is the conclusion of 4.13). Still, by §2 we do not know that the strong $\aleph_2$-covering lemma holds. However, (letting $\chi := \aleph_1^{\mathbf{V}}$, $\kappa := \aleph_2^{\mathbf{V}}$, and $\kappa = (\lambda^+)^\kappa$) by 4.20 below, we know that for every real $s \in \mathbf{V}$, for some increasing continuous sequence $\langle N^s_i : i < \chi \rangle$ of countable models (in $\mathbf{V}$, $N_i \subseteq \mathbf{V}$) we have $s \in N^s_i$, $N^s_i = (N^s_i \cap \mathbf{W})[r]$, $\bigcup\limits_{i<\chi} N_i \cap \kappa$ is an ordinal $> \lambda$, and each N_i is 2-*trivially defined* from $N_i \cap \kappa$ (see 4.15 for the definition). The rest is as in the proof of 4.11. $\square_{4.13}$

Remark 4.14. 1) So why do 4.11 and 4.13 come before 4.20? We think the proof of 4.11 makes 4.15–4.20 easier to understand (using the notation of the proof of 4.11).

2) However, 4.13 is later reproved (in 4.22).

Definition 4.15. Let $\mathbf{W} \subseteq \mathbf{V}$, $A \in \mathbf{V}$, $A \subseteq \lambda^*$, α an ordinal, and $B \subseteq \alpha$ such that $B \in \mathbf{V}$.

We say that B is *ℓ-trivially defined over* $(\mathbf{W}, A, \alpha)$ (or '$B \in \mathbf{W}^\ell_{\rm tr}[A, \alpha]$') for $\ell = 0, 1, 2$ when:[13]

$\underline{\ell = 0}$: For some δ, $B = c\ell\big(A \cup \{\delta\}, M^2_\alpha\big) \cap C^1_\delta$.

$\underline{\ell = 1}$: For some $B_1 \in \mathbf{W}^0_{\rm tr}[A, \alpha]$ and a function $f \in \mathbf{W}$,

$$B = c\ell\big(A \cup \{f(i) : i \in B_1\} \cup \bigcup_{i \in B_1} C^1_{f(i)}, M^2_\alpha\big).$$

$\underline{\ell = 2}$: For some $n < \omega$, $B_1, \ldots, B_n \in \mathbf{W}^1_{\rm tr}[A, \alpha]$, and $\beta \leq \alpha$,

$$B = c\ell\big(A \cup \bigcup_{m=1}^{n} B_m, M^2_\alpha\big) \cap \beta.$$

Definition 4.16. In $\mathbf{V}$, let D be a filter on $\mathcal{S}_{<\lambda^*}(\lambda^*)$, where λ^* is a regular cardinal. We define the *strong* (λ^*, D, α)-*covering game* as follows.

(A) A play lasts λ^* moves.

(B) In the $i^{\rm th}$ move:

  (a) Player I chooses

    $\bullet_1$ $a_i \in \mathbf{V}$, a subset of α of cardinality $< \lambda^*$ (in $\mathbf{V}$).

    $\bullet_2$ A function f_i from an ordinal $< \lambda^*$ onto a_i, such that $a_i \supseteq \bigcup_{j<i} b_j$ and $f_i \supseteq \bigcup_{j<i} g_j$.

  (b) Then Player II responds with

    $\bullet_1$ b_i, a subset of α of cardinality $< \lambda^*$ (in $\mathbf{V}$).

    $\bullet_2$ A function g_i from an ordinal $< \lambda^*$ onto b_i, such that $b_i \supseteq \bigcup_{j \leq i} a_j$ and $g_i \supseteq \bigcup_{j \leq i} f_j$.

(C) In the end, Player I wins if the following set belongs to D:

$$\Big\{A \in \mathcal{S}_{<\lambda^*}(\lambda^*) : \big\{f_i(\alpha) : i < \lambda^*, \alpha \in A\big\} \in W^2_{\rm tr}[A, \alpha]\Big\}.$$

We say $(\mathbf{W}, \mathbf{V})$ has the (λ^*, D, α)-*strong covering property* if Player I has a winning strategy in the (λ^*, D, α)-strong covering game. We omit α if it is true for every α.

Remark 4.17. Without loss of generality, $\mathrm{dom}(f_i) = a_i \cap \lambda^*$ and $\mathrm{dom}(g_i) = b_i \cap \lambda^*$. This definition does not contradict the earlier one, as the filter here is not on some cardinal (but on $\mathcal{S}_{<\lambda^*}(\lambda^*)$).

Definition 4.18. Suppose λ and $\lambda^* := \lambda^+$ are regular cardinals in $\mathbf{V}$. We shall define a filter $D[\lambda^*, \lambda]$.

For each $\alpha < \lambda^*$, construct a decomposition $\alpha = \bigcup_{i<\lambda} A^\alpha_i$, with $|A^\alpha_i| < \lambda$ and $\langle A^\alpha_i : i < \lambda\rangle$ increasing continuous.

$$D[\lambda^*, \lambda] := \big\{S \subseteq \mathcal{S}_{<\lambda^*}(\lambda^*) : \text{for some club } C \subseteq \lambda^*, (\forall \alpha \in C)\big[\mathrm{cf}(\alpha) = \lambda \Rightarrow \{i < \lambda : A^\alpha_i \in S\} \in D_\lambda\big]\big\}.$$

[I don't see a definition for D_λ.]

[13] $c\ell$, M^2_α, and C^ℓ_α are as in §2; specifically, see 2.16, 2.6, 2.4.

Remark 4.19. 1) This definition appears in essentially the same form in [She75, §3].
2) The filter does not depend on the choice of the A_i^α-s.

Theorem 4.20. *Suppose that in* $\mathbf{V}$, $\lambda^* := \lambda^+$ *and* λ, λ^* *are regular cardinals with* $\lambda > \aleph_0$. *Let* $D := D[\lambda^*, \lambda]$. *If* $\mathbf{W}$ *has* λ^**-squares, has scales, and* $(\mathbf{W}, \mathbf{V})$ *satisfies the* λ^**-covering lemma* *then* $(\mathbf{W}, \mathbf{V})$ *has the* (λ^*, D)*-strong covering property.*

Proof. We just repeat the proof of 2.2, 2.11.
Note that we use $\beta \in \mathbf{W}^0_{\rm tr}[A, \alpha]$ for the parallel of "$C^1_{\delta(\zeta,\theta)} \subseteq a^0_\zeta$" and $B \in \mathbf{W}^1_{\rm tr}[A, \alpha]$ for the parallel of "$\bigcup_{\theta \in S} C^1_{\delta(\zeta,\theta)}$". $\square_{4.20}$

Corollary 4.21. *If the hypotheses of* 4.20 *hold and* $\aleph_2^{\mathbf{W}} = \aleph_2^{\mathbf{V}}$, *then* $(\mathbf{W}, \mathbf{V})$ *satisfies the strong* $\aleph_2$*-covering lemma and the strong* $\aleph_1$*-covering lemma.*

* * *

We have remarked that if λ^* is the successor of λ in $\mathbf{W}$, things are much simpler. Let us present this.

Lemma 4.22. *Assume* $\mathbf{W} \subseteq \mathbf{V}$, λ *a regular uncountable cardinal in* $\mathbf{V}$, $(\lambda^+)^{\mathbf{V}} = (\lambda^+)^{\mathbf{W}}$, *and* $(\mathbf{W}, \mathbf{V})$ *satisfies the* λ^+*-covering lemma.*

1) *If* $\mathbf{W}$ *has* λ*-squares and* $D := D_\lambda + \{\delta < \lambda : \mathrm{cf}^{\mathbf{V}}(\delta) > \aleph_0\}$, *then* $(\mathbf{W}, \mathbf{V})$ *satisfies the strong* (λ, λ, D)*-covering lemma.*

2) *If* $\mathbf{W}$ *has* $(\lambda^+)^{\mathbf{W}}$*-squares* *then* $(\mathbf{W}, \mathbf{V})$ *satisfies the strong* $(\lambda, \lambda, D_\lambda)$*-covering lemma.*

3) *Suppose* $\mathbf{W}$ *has* λ*-squares and*

$$D := \big\{\lambda \setminus A : \{\delta < \lambda : \delta \in A \text{ or } A \cap \delta \text{ is stationary in } \delta\} \text{ is not stationary}\big\}.$$

Then $(\mathbf{W}, \mathbf{V})$ *satisfies the strong* (λ, λ, D)*-covering lemma.*

Proof. For any ordinal, we can find a regular cardinal μ in $\mathbf{W}$ bigger than it satisfying $\mathbf{V} \vdash$ "$\mu^\lambda = \mu$", and let $\alpha_* := (\mu^+)^{\mathbf{W}}$. Clearly it suffices to deal with subsets of α_* (in part (1), prove that Player I wins the $(\lambda, \lambda, D, \alpha_*)$-covering game).
We define a model M_{α_*}. Let (in $\mathbf{W}$) $\{f_\beta : \beta < \alpha_*\} \in \mathbf{W}$ be an enumeration of the set

$$\big\{f \in \mathbf{W} : \mathrm{dom}(f) \subseteq [\lambda, \mu] \cap \mathrm{Reg}^{\mathbf{W}},\ \mathbf{W} \vdash \text{`}\big|\mathrm{dom}(f)\big| < \lambda\text{', and } (\forall\kappa)[f(\kappa) < \kappa]\big\}.$$

(There is such a list as $\mathbf{V} \vdash$ "$\mu^{<\lambda} = \mu$".)
For proving part (1), let M_{α_*} be $M^0_{\alpha_*}$ (from 2.4) expanded by a partial two-place function $F(\beta, \gamma) := f_\beta(\gamma)$. For (2), we replace the λ-square

$$\big\langle C^1_\alpha : \lambda \le \alpha < \alpha_*,\ \mathrm{cf}^{\mathbf{V}}(\alpha) < \lambda\big\rangle$$

by a λ^+-square

$$\big\langle C^1_\alpha : \lambda \le \alpha < \alpha_*,\ \mathrm{cf}^{\mathbf{V}}(\alpha) < \lambda^+\big\rangle,$$

where $\mathrm{otp}(C^1_\alpha) \le \lambda$ (equality holds when $\mathrm{cf}^{\mathbf{V}}(\alpha) = \lambda$).
Note that $(\mathbf{W}, \mathbf{V})$ satisfies the λ-covering lemma.
[If $a \subseteq \mathrm{Ord}$ and $\mathbf{V} \vdash$ '$|a| < \lambda$', then by assumption there is $b \in \mathbf{W}$ such that $\mathbf{V} \vdash$ '$|b| < \lambda^+$' and $a \subseteq b$. So $|b|^{\mathbf{W}} < (\lambda^+)^{\mathbf{V}} = (\lambda^+)^{\mathbf{W}}$, hence (by an assumption) $\mathbf{W} \vdash$ '$|b| \le \lambda$'. So in $\mathbf{W}$ we have an increasing sequence $\langle b_i : i < \lambda\rangle$ such that

$b = \bigcup_{i<\lambda} b_i$ and $\mathbf{W} \vdash$ '$|b_i| < \lambda$'. Now for some i, $a \subseteq b_i$ (as $|a| < \lambda$ and $\mathrm{cf}(\lambda) = \lambda$ in $\mathbf{V}$), so we finish.]

Now, for any set $a \subseteq a(*)$, let Ch_a be the function which maps $\kappa \mapsto \sup(a \cap \kappa)$, with domain

[Undefined. Did you mean α_*?]

$$\{\kappa < \alpha : \kappa \text{ regular uncountable in } \mathbf{W},\ \kappa \in a\},$$

We now define a strategy for Player I in the $(\lambda, \lambda, D, \alpha)$-covering game. He will choose $a_i \in \mathbf{W}$ such that $a_i \subseteq \alpha$, $\mu \in a_i$, $|a_i| < \lambda$, a_i includes the closure (in the order topology) of the Skolem Hull of $\bigcup_{j<i} b_j$ in M_α, and

$$(\forall j < i)(\exists \beta_j < \alpha_*)\big[\mathrm{Ch}_{a_j} = f_{\beta_j} \wedge \beta_j \in a_{j+1}\big].$$

Clearly this is possible.

Let us show that this is a winning strategy. Let $\langle a_i, b_i : i < \lambda\rangle$ be a play of the $(\lambda, \lambda, D, \alpha)$-covering game in which Player I uses his strategy (and for ease of notation, let $A_\delta := \bigcup_{i<\delta} a_i$).

By the assumption [i.e. '$(\mathbf{W}, \mathbf{V})$ has λ^+-covering,' applied to the set $\bigcup_{i<\lambda} a_i$] there is a set $d \in \mathbf{W}$ such that $d \subseteq \alpha$, $|d| < \lambda^+$, and $\bigcup_{i<\lambda} a_i \subseteq d$. As before (because $(\lambda^+)^{\mathbf{V}} = (\lambda^+)^{\mathbf{W}}$) we have $\mathbf{W} \vdash$ '$|d| \leq \lambda$', so there is an increasing continuous sequence $\langle d_i : i < \lambda\rangle \in \mathbf{W}$ of subsets of d such that $d = \bigcup_{i<\lambda} d_i$ and

$$i < \lambda \Rightarrow \mathbf{W} \vdash \text{“}|d_i| < \lambda\text{”}.$$

Clearly

$$C_0 := \Big\{\delta < \lambda : \delta \text{ a limit ordinal and } d_\delta \cap \bigcup_{j<\lambda} a_j = \bigcup_{j<\delta} a_j\Big\}$$

and

$$C_1 := \big\{\delta < \lambda : \underline{\text{if}}\ \beta \in d_\delta \text{ and } (\exists j < \lambda)[f_\beta < f_{\alpha_j}]\ \underline{\text{then}} \text{ there is such a } j < \delta\big\}$$

are both clubs of λ.[14] Hence it suffices to prove that for every $\delta \in C_0 \cap C_1$ we have $A_\delta := \bigcup_{j<\delta} a_j \in \mathbf{W}$. Let $\delta \in C_0 \cap C_1$, and define $Y_\delta := \big\{\beta \in d_\delta : f_\beta < f_{\alpha_\delta}\big\}$.

[I don't see α_δ defined anywhere. (Note that it's not a_δ, as f_{a_δ} is nonsense.)]

Now for each $\zeta < \delta$, we know that

$$\beta_\zeta \in a_{\zeta+1} \subseteq A_\delta = \bigcup_{j<\delta} a_j \text{ and } f_{\beta_\zeta} = \mathrm{Ch}_{a_\zeta} < \mathrm{Ch}_{a_\delta} = f_{\alpha_\delta},$$

hence $\zeta < \delta \Rightarrow \beta_\zeta \in Y_\delta$. On the other hand (as $\delta \in C_1$),

$$\beta \in Y_\delta \Rightarrow f_\beta < f_{\beta_\delta} \Rightarrow \bigvee_{j<\delta} [f_\beta < f_{\beta_j}].$$

Hence for every $\kappa \in A_\delta \setminus \lambda$ regular in $\mathbf{W}$,

$$\mathrm{Ch}_{A_\delta}(\kappa) = \sup_{j<\delta} \mathrm{Ch}_{a_j}(\kappa) = \sup_{\beta \in Y_\delta} f_\beta(\kappa).$$

Let g^*_δ be the function $\kappa \mapsto \sup_{\beta \in Y_\delta} f_\beta(\kappa)$ with domain

$$\{\kappa \in d_\delta \setminus \lambda : \kappa \leq \mu,\ \kappa \text{ regular in } \mathbf{W}\}.$$

[Can I say '$[0, \mu] \cap \mathrm{Reg}^{\mathbf{W}} \cap d_\delta \setminus \lambda$?']

[14] As always, by '$<$' we mean $\mathrm{dom}(f_\beta) \subseteq \mathrm{dom}(f_{\alpha_j}) \wedge \big(\forall \kappa \in \mathrm{dom}(f_\beta)\big)\big[f_\beta(\kappa) < f_j(\kappa)\big]$.

We know g^*_δ belongs to $\mathbf{W}$ (because Y_δ and d_δ do) and

$$\mathrm{Ch}_{A_\delta} \subseteq g^*_\delta.$$

With these preliminaries out of the way, we may begin the proof proper.

Proof. Proof of 4.22(1).

Remember that by assumption $\mathbf{W}$ has a λ-square — say,

$$\langle C^1_\delta : \delta < \alpha_*,\ \mathrm{cf}(\delta) < \lambda\rangle,$$

and they "appear" in M_{α_*}. By the strategy, for every $j < \lambda$ of uncountable cofinality and $\theta \in a_j \setminus \lambda$ (regular in $\mathbf{W}$, $\in [\lambda, \mu]$), $C^1_{\mathrm{Ch}_{a_j}(\theta)} \subseteq a_{j+2}$. Hence as in 2.2(D), for limit $\delta \in C_0 \cap C_1$ of uncountable cofinality, we have $C^1_{\mathrm{Ch}_{A_\delta}(\theta)} \subseteq A_\delta := \bigcup\limits_{j<\delta} a_j$, so by 2.2 we finish similarly to 2.17.

[This next part appears to be defining a new sequence $\langle a_n : n < \omega\rangle \in \mathbf{W}$, which is *not* the same as $\langle a_i : i < \lambda\rangle \in \mathbf{W}$ which we've been talking about for the entire last page. Furthermore, the 'a' that is referenced in the last two sentences does not appear in this proof; it appears to be an elementary submodel of $M^2_{\mu^+}$ from the hypotheses of 2.17. g^* is not defined anywhere, but could be an allusion to g^*_δ above. I don't know what a natural definition for g^*_δ with δ undefined would be; maybe $\sup\limits_{\beta<\alpha_*} f_\beta$?]

I.e. define by induction on n:

- a_0 is the Skolem Hull of $\varnothing$ in M_{α_*}.
- a_{n+1} is the Skolem Hull in M_{α_*} of

$$a_n \cup \left\{C^1_{g^*(\theta)} : \theta \in a_n \cap \mathrm{Reg}^{\mathbf{W}} \cap \mathrm{dom}(g^*),\ \theta \geq \lambda\right\}.$$

Clearly $\langle a_n : n < \omega\rangle \in \mathbf{W}$, hence $a_\omega := \bigcup\limits_{n<\omega} a_n \in \mathbf{W}$, and each a_n is a subset of a so $a_\omega \subseteq a$. Lastly, $a_\omega = a$; similarly to 2.17. $\square_{4.22(1)}$

Proof of 4.22(2),(3):

Similar. $\square_{4.22}$

* * *

Lemma 4.23. *Suppose $\mathbf{W} \subseteq \mathbf{V} := \mathbf{W}[r]$ with r a real, $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{W}}$, and $\mathbf{W}$ satisfies* CH *while $\mathbf{V}$ fails* CH. *<u>Then</u> $\aleph_2^{\mathbf{V}}$ is inaccessible in $\mathbf{L}$.*

Proof. Assume the conclusion fails, so $\kappa := \aleph_2^{\mathbf{V}} = (\lambda^+)^{\mathbf{W}}$ with λ a cardinal in $\mathbf{L}$. Let $\chi := \aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{W}}$. By 4.9, $\mathrm{cf}^{\mathbf{V}}(\lambda) = \aleph_1$. Also, as $\aleph_2^{\mathbf{V}}$ is not inaccessible in $\mathbf{L}$, necessarily $\mathbf{0}^\# \notin \mathbf{V}$. Hence by 4.7, $\mathbf{V} \vdash$ '$2^{\aleph_0} \leq \aleph_2$' and so $\mathbf{V} \vdash$ '$2^{\aleph_0} = \aleph_2$'.

Choose $A \in \mathbf{V}$ a subset of λ such that $\aleph_1^{\mathbf{L}[A]} = \aleph_1^{\mathbf{V}}$ $(= \aleph_1^{\mathbf{W}})$ and $\mathbf{L}[A] \vdash$ "$|\lambda| = \aleph_1$" (so we cannot exclude the possibility "$A \notin \mathbf{W}$"). Now by Lemma 4.24 below, $\mathbf{L}[A] \vdash$ "$2^{\aleph_0} = \aleph_1$", (note that $\mathbf{L}, L[A], \lambda, A$ here are standing in for $\mathbf{W}, \mathbf{V}, \lambda, r$ there).

By 2.33 (with $\mathbf{L}[A]$, $\mathbf{V}$, $\aleph_1^{\mathbf{V}}$, $\aleph_2^{\mathbf{V}}$, $\aleph_0$ here standing for $\mathbf{W}$, $\mathbf{V}$, λ^*, λ, κ there) the pair $(\mathbf{L}[A], \mathbf{V})$ satisfies the strong $\aleph_1^{\mathbf{V}}$-covering lemma. As $\mathbf{L}[A] \vdash$ CH by 4.5, $\mathbf{V}$ also satisfies CH — a contradiction. $\square_{4.23}$

Claim 4.24. *Suppose* $\mathbf{W} \subseteq \mathbf{V} = \mathbf{W}[r]$ *with* r *a subset of* λ, λ *a cardinal in* $\mathbf{W}$, $(\lambda^+)^{\mathbf{W}} = \aleph_2^{\mathbf{V}}$, *and* $\mathbf{W}$ *satisfies* GCH.

Then $\mathbf{V}$ *satisfies* CH.

Proof. Let $\kappa := \aleph_2^{\mathbf{V}}$ and $\chi := \aleph_1^{\mathbf{V}}$ (so $\mathbf{W} \vdash$ "$\kappa = \lambda^+$"). Now $\mathbf{V} \vdash$ "$\mathrm{cf}(\lambda) = \aleph_1$" by 4.9; assume $\mathbf{V} \vdash$ "$2^{\aleph_0} > \aleph_1$", and we shall get a contradiction.

Now repeat the proof of 4.11 (after the first paragraph). The additional point is in proving that $N_\gamma^{s_\zeta}$ and $N_\gamma^{s_\xi}$ are isomorphic. We have to check that the mapping preserves "$i \in r$", but $r \subseteq \lambda$ and $N_\gamma^{s_\zeta} \cap \lambda = N_\gamma^{s_\xi} \cap \lambda \subseteq A_{\delta_\gamma}$, and the mapping is the identity on A_{δ_γ}. $\square_{4.24}$

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