BOREL SETS WITHOUT PERFECTLY MANY OVERLAPPING TRANSLATIONS IV

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ABSTRACT. We show that, consistently, there exists a Borel set $B \subseteq {}^\omega 2$ admitting a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle$ of distinct elements of ${}^\omega 2$ such that $(\eta_\alpha + B) \cap (\eta_\beta + B)$ is uncountable for all $\alpha, \beta < \lambda$ but with no perfect set P such that $|(\eta + B) \cap (\nu + B)| \geq 6$ for any distinct $\eta, \nu \in P$. This answers two questions from our previous works.

1. Introduction

In the series of articles [4, 5, 6] we investigated the existence of Borel sets with many, but not too many pairwise non-disjoint translations. For instance, in [5], for a countable ordinal $\varepsilon < \omega_1$ and an integer $2 \leq \iota < \omega$ we constructed a Σ^0_2 set $B \subseteq {}^\omega 2$ with the following property.

> *In some ccc forcing extension there is a sequence $\langle \rho_\alpha : \alpha < \aleph_\varepsilon \rangle$ of distinct elements of ${}^\omega 2$ such that*
>
> $$\big|(\rho_\alpha + B) \cap (\rho_\beta + B)\big| \geq 2\iota \text{ for all } \alpha, \beta < \lambda$$
>
> *but in no extension there is a perfect set of such ρ's.*

Similar resuts for the general case of perfect Abelian Polish groups were presented in [6]. However, in all those cases when discussing nonempty intersections we considered finite intersections only. It seemed that our arguments really needed a finite enumeration of "witnesses for nondisjointness". So in [4, Problem 5.1] and [6, Problem 7.6] we asked if there is a ccc forcing notion $\mathbb{P}$ adding a Σ^0_2 subset B of the Cantor space ${}^\omega 2$ such that

> *for some $H \subseteq {}^\omega 2$ of size λ, the intersections $(B+h) \cap (B+h')$ are infinite (uncountable, respectively) for all $h, h' \in H$, but for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$ with the intersection $(B+x) \cap (B+x')$ finite (countable, respectively).*

In the present paper we answer the above two questions positively. Our forcing construction slightly generalizes and simplifies that of [4, 5]. This allows us to show a stronger result:

> *If $\lambda < \lambda_{\omega_1}$ then some ccc forcing notion adds a Σ^0_2 set B which has λ translations with pairwise uncountable intersections, while for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$ with $|(B+x) \cap (B+x')| < 6$.*

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The article is organized as follows. First, in Section 2, we recall the splitting rank from Shelah [7]. This rank was fundamental for the question of no perfect squares and it is fundamental for problems of nondisjoint translations as well. Then, in the third section we introduce nice indexed bases $\bar{\mathcal{O}}$ and we define when translations of a Σ^0_2 set have $\bar{\mathcal{O}}$–large intersection. This allows us to put in the same framework sets with finite, infinite and uncountable intersections. We also analyze when a Σ^0_2 set may have a perfect set of translations with $\bar{\mathcal{O}}$–large intersections and we introduce a non-disjointness rank on finite approximations. Our main consistency theorem is presented in the fourth section. In the final part of the paper we summarize our results and pose a few relevant problems.

Notation: Our notation is standard and compatible with that of classical textbooks (like Jech [2] or Bartoszyński and Judah [1]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

1. For a set u we let $u^{\langle 2\rangle} = \{(x,y)\in u\times u : x\neq y\}$.
2. The Cantor space ${}^{\omega}2$ of all infinite sequences with values 0 and 1 is equipped with the natural product topology and the group operation of coordinate-wise addition + modulo 2.
3. Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta$ as well as ξ. Finite ordinals (non-negative integers) will be denoted by letters a,b,c,d,i,j,k,ℓ,m,n,M and ι.
4. The Greek letters κ,λ will stand for uncountable cardinals.
5. For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$), and $\underset{\sim}{G}_{\mathbb{P}}$ will stand for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$.

We fully utilize the algebraic properties of $({}^{\omega}2,+)$, in particular the fact that all elements of ${}^{\omega}2$ are self-inverse.

2. The splitting rank

In this section we recall some basic facts from [7, Section 1] concerning a rank (on models with countable vocabulary) which will be used in the construction of a forcing notion in the fourth section. This rank and relevant proofs were also presented in [4, Section 2].

Let λ be a cardinal and $\mathbb{M}$ be a model with the universe λ and a countable vocabulary τ.

Definition 2.1. (1) By induction on ordinals δ, for finite non-empty sets $w\subseteq\lambda$ we define when $\mathrm{rk}(w,\mathbb{M})\geq\delta$. Let $w=\{\alpha_0,\ldots,\alpha_n\}\subseteq\lambda$, $|w|=n+1$.

(a) $\mathrm{rk}(w)\geq 0$ if and only if for every quantifier free formula $\varphi\in\mathcal{L}(\tau)$ and each $k\leq n$, if $\mathbb{M}\models\varphi[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n]$ then the set

$$\big\{\alpha\in\lambda : \mathbb{M}\models\varphi[\alpha_0,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n]\big\}$$

is uncountable;

(b) if δ is limit, then $\mathrm{rk}(w,\mathbb{M})\geq\delta$ if and only if $\mathrm{rk}(w,\mathbb{M})\geq\gamma$ for all $\gamma<\delta$;

(c) $\mathrm{rk}(w,\mathbb{M}) \geq \delta+1$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n]$ then there is $\alpha^* \in \lambda \setminus w$ such that

$$\mathrm{rk}(w\cup\{\alpha^*\},\mathbb{M}) \geq \delta \quad \text{and} \quad \mathbb{M} \models \varphi[\alpha_0,\ldots,\alpha_{k-1},\alpha^*,\alpha_{k+1},\ldots,\alpha_n].$$

(2) The rank $\mathrm{rk}(w,\mathbb{M})$ of a finite non-empty set $w \subseteq \lambda$ is defined by:
- $\mathrm{rk}(w,\mathbb{M}) = -1$ if $\neg(\mathrm{rk}(w,\mathbb{M}) \geq 0)$, and
- $\mathrm{rk}(w,\mathbb{M}) = \infty$ if $\mathrm{rk}(w,\mathbb{M}) \geq \delta$ for all ordinals δ, and
- for an ordinal δ: $\mathrm{rk}(w,\mathbb{M}) = \delta$ if $\mathrm{rk}(w,\mathbb{M}) \geq \delta$ but $\neg(\mathrm{rk}(w,\mathbb{M}) \geq \delta+1)$.

Definition 2.2. For an ordinal ε and a cardinal λ let $\mathrm{NPr}_\varepsilon(\lambda)$ be the following statement:[1] "there is a model $\mathbb{M}^*$ with the universe λ and a countable vocabulary τ^* such that $\sup\{\mathrm{rk}(w,\mathbb{M}^*) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \varepsilon$."

$\mathrm{Pr}_\varepsilon(\lambda)$ is the negation of $\mathrm{NPr}_\varepsilon(\lambda)$.

Observation 2.3. *If λ is uncountable and $\mathrm{NPr}_\varepsilon(\lambda)$, then there is a model $\mathbb{M}^*$ with the universe λ and a countable vocabulary τ^* such that*
- $\mathrm{rk}(\{\alpha\},\mathbb{M}^*) \geq 0$ *for all* $\alpha \in \lambda$ *and*
- $\mathrm{rk}(w,\mathbb{M}^*) < \varepsilon$ *for every finite non-empty set* $w \subseteq \lambda$.

Proposition 2.4 (See [7, Claim 1.7] and/or [4, Proposition 2.6])**.**
(1) $\mathrm{NPr}_1(\omega_1)$.
(2) *If* $\mathrm{NPr}_\varepsilon(\lambda)$, *then* $\mathrm{NPr}_{\varepsilon+1}(\lambda^+)$.
(3) *If* $\mathrm{NPr}_\varepsilon(\mu)$ *for* $\mu < \lambda$ *and* $\mathrm{cf}(\lambda) = \omega$, *then* $\mathrm{NPr}_{\varepsilon+1}(\lambda)$.

Proposition 2.5 (See [7, Conclusion 1.8] and/or [4, Proposition 2.7])**.** *Assume $\beta < \alpha < \omega_1$, $\mathbb{M}$ is a model with a countable vocabulary τ and the universe μ, $m,n < \omega$, $n > 0$, $A \subseteq \mu$ and $|A| \geq \beth_{\omega\cdot\alpha}$. Then there is $w \subseteq A$ with $|w| = n$ and $\mathrm{rk}(w,\mathbb{M}) \geq \omega\cdot\beta + m$* [2].

Definition 2.6. Let λ_{ω_1} be the smallest cardinal λ such that $\mathrm{Pr}_{\omega_1}(\lambda)$.

Corollary 2.7. (1) *If* $\alpha < \omega_1$, *then* $\mathrm{NPr}_{\omega_1}(\aleph_\alpha)$.
(2) $\mathrm{Pr}_{\omega_1}(\beth_{\omega_1})$ *holds true.*
(3) $\aleph_{\omega_1} \leq \lambda_{\omega_1} \leq \beth_{\omega_1}$.

Corollary 2.8 (See [4, Proposition 2.10 and Corollary 2.11])**.** *Let $\mu = \beth_{\omega_1} \leq \kappa$. If $\mathbb{P}$ is a ccc forcing notion, then $\Vdash_{\mathbb{P}} \mathrm{Pr}_{\omega_1}(\mu)$. In particular, if $\mathbb{C}_\kappa$ is the forcing notion adding κ Cohen reals, then $\Vdash_{\mathbb{C}_\kappa} \lambda_{\omega_1} \leq \mu \leq \mathfrak{c}$.*

3. Spectrum of translation non-disjointness

We want to analyze sets with many non-disjoint translations in more detail, restricting ourselves to Σ^0_2 subsets of ${}^\omega 2$. In this section we will keep the following assumptions.

[1]The notation $\mathrm{Pr}(\lambda)/\mathrm{NPr}(\lambda)$ was introduced in [7]. It originated in asserting that the cardinal λ has or does not have the **PR**operty under consideration. It is somewhat unfortunate that as a result of not choosing better names in the past, today many properties are called PR.

[2]" . " stands for the ordinal multiplication

Assumptions 3.1. Let $\bar{T} = \langle T_n : n < \omega \rangle$, where each $T_n \subseteq {}^{\omega>}2$ is a tree with no maximal nodes (for $n < \omega$). Let $B = \bigcup\limits_{n<\omega} \lim(T_n)$.

Definition 3.2. (1) Let $\mathcal{L}$ consist of all non-empty sets $u \subseteq {}^{\omega>}2$ such that $u \subseteq {}^{\ell}2$ for some $\ell = \ell(u) < \omega$.

(2) *A simple base* is a (strict) partial order $\mathcal{O} = (\mathcal{O}, \prec)$ such that $\mathcal{O} \subseteq \mathcal{L}$ and for $u, u' \in \mathcal{O}$:
 (a) if $u \prec u'$ then $\ell(u) < \ell(u')$ and $u = \{\eta \restriction \ell(u) : \eta \in u'\}$,
 (b) there is a $v \in \mathcal{O}$ such that $u \prec v$,
 (c) if $\rho \in {}^{\ell(u)}2$ then $u + \rho \in \mathcal{O}$, and if $\rho \in {}^{\ell(u')}2$ and $u \prec u'$ then $u + \rho \restriction \ell(u) \prec u' + \rho$.

(3) Let $(\mathcal{O}, \prec)$ be a simple base. An *$\mathcal{O}$–tower* is a $\prec$–increasing sequence $\bar{u} = \langle u_n : n < \omega \rangle \subseteq \mathcal{O}$ (so $u_n \prec u_{n+1}$ for all $n < \omega$). *The cover of an $\mathcal{O}$–tower* $\bar{u}$ is the set $\mathcal{C}(\bar{u}) \stackrel{\text{def}}{=} \big\{\eta \in {}^{\omega}2 : \big(\forall n < \omega\big)\big(\eta \restriction \ell(u_n) \in u_n\big)\big\}$.

(4) *An indexed base* is a sequence $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ where $0 < i^* \leq \omega$ and each $\mathcal{O}_i$ is a simple base.

Definition 3.3. Let $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ be an indexed base.

(1) We say that two translations $B + x$ and $B + y$ of the set[3] B (for $x, y \in {}^{\omega}2$) have *$\bar{\mathcal{O}}$–large intersection* if for some $\langle \bar{u}_i : i < i^* \rangle$ for every $i < i^*$ we have:
- $\bar{u}_i$ is an $\mathcal{O}_i$–tower,
- for some $n_1, n_2 < \omega$,
$$\mathcal{C}(\bar{u}_i) \subseteq \big(\lim(T_{n_1}) + x\big) \cap \big(\lim(T_{n_2}) + y\big),$$
- $\mathcal{C}(\bar{u}_i) \cap \mathcal{C}(\bar{u}_j) = \emptyset$ whenever $j < i^*$, $j \neq i$.

In the above situation we may also say that $(B + x) \cap (B + y)$ *is $\bar{\mathcal{O}}$–large.*

(2) We say that B is *perfectly orthogonal to $\bar{\mathcal{O}}$–small* (or a $\bar{\mathcal{O}}$–**pots**–set) if there is a perfect set $P \subseteq {}^{\omega}2$ such that the translations $B + x$, $B + y$ have a $\bar{\mathcal{O}}$–large intersection for all $x, y \in P$.
The set B is *a $\bar{\mathcal{O}}$–**npots**–set* if it is not $\bar{\mathcal{O}}$–**pots**.

(3) We say that B has λ *many pairwise $\bar{\mathcal{O}}$–nondisjoint translations* if for some set $X \subseteq {}^{\omega}2$ of cardinality λ, for all $x, y \in X$ the translations $B + x$, $B + y$ have a $\bar{\mathcal{O}}$–large intersection.

(4) We define the *spectrum of translation $\bar{\mathcal{O}}$–nondisjointness of B* as
$$\mathrm{stnd}_{\bar{\mathcal{O}}}(B) = \big\{(x, y) \in {}^{\omega}2 \times {}^{\omega}2 : \begin{array}{l}\text{the translations } B + x, B + y\\ \text{have a } \bar{\mathcal{O}}\text{–large intersection}\end{array}\big\}.$$

Example 3.4. (1) Let $6 \leq \iota \leq \omega$. Put $\mathcal{O}^0 = \{u \in \mathcal{L} : |u| = 1\}$ and let a relation $\prec^0$ be defined by:
 $u \prec^0 v$ if and only if $\ell(u) < \ell(v) \ \wedge \ u = \{\eta \restriction \ell(u) : \eta \in v\}$.
Then $(\mathcal{O}^0, \prec^0)$ is a simple base and $\bar{\mathcal{O}}^\iota = \langle \mathcal{O}^0 : i < \iota \rangle$ is an indexed base. Two translations $B + x$ and $B + y$ of the set B (for $x, y \in {}^{\omega}2$) have $\bar{\mathcal{O}}^\iota$–large intersection if and only if $(B + x) \cap (B + y)$ has at least ι members.

[3] Remember Assumptions 3.1

(2) Let $\mathcal{O}^{\rm per}=\{u\in\mathcal{L}:|u|\geq 3\}$ and let a relation $\prec^{\rm per}$ be defined by
$u\prec^{\rm per} v$ if and only if
$u=\{\eta\restriction\ell(u):\eta\in v\}\ \wedge\ (\forall\nu\in u)(|\{\eta\in v:\nu\vartriangleleft\eta\}|\geq 2)$.
Then $(\mathcal{O}^{\rm per},\prec^{\rm per})$ is a simple base and $\bar{\mathcal{O}}^{\rm per}=\langle\mathcal{O}^{\rm per}\rangle$ is an indexed base. Two translations $B+x$ and $B+y$ of the set B (for $x,y\in{}^\omega 2$) have $\bar{\mathcal{O}}^{\rm per}$–large intersection if and only if $(B+x)\cap(B+y)$ is uncountable.

Proposition 3.5. *Let $\bar{\mathcal{O}}$ be an indexed base and let $\bar{T},B$ be as in Assumptions 3.1.*

1. *The set B is a $\bar{\mathcal{O}}$–**pots**–set if and only if there is a perfect set $P\subseteq{}^\omega 2$ such that $P\times P\subseteq \mathrm{stnd}_{\bar{\mathcal{O}}}(B)$.*
2. *The set $\mathrm{stnd}_{\bar{\mathcal{O}}}(B)$ is Σ^1_1.*
3. *Let $\mathfrak{c}<\lambda\leq\mu$ and let $\mathbb{C}_\mu$ be the forcing notion adding μ Cohen reals. Then, remembering Definition 3.3(2),*

$$\Vdash_{\mathbb{C}_\mu}\text{“ if } B \text{ has } \lambda \text{ many pairwise } \bar{\mathcal{O}}\text{–nondisjoint translations, then } B \text{ is a } \bar{\mathcal{O}}\text{–}\mathbf{pots}\text{–set ”.}$$

4. *Assume $\mathrm{Pr}_{\omega_1}(\lambda)$. If B has λ many pairwise $\bar{\mathcal{O}}$–nondisjoint translations, then it is a $\bar{\mathcal{O}}$–**pots**–set.*

Proof. (1,2) Straightforward; in evaluation of the complexity of $\mathrm{stnd}_{\bar{\mathcal{O}}}(B)$ note that for $\mathcal{O}_i$–towers $\bar{u}_i=\langle u^i_n:n<\omega\rangle$, $x\in{}^\omega 2$ and $k<\omega$:
$\mathcal{C}(\bar{u}_i)\subseteq\lim(T_k)+x$ if and only if $(\forall n<\omega)(u^i_n\subseteq T_k+x)$, and
$\mathcal{C}(\bar{u}_{i_1})\cap\mathcal{C}(\bar{u}_{i_2})=\emptyset$ if and only if $(\exists\ell<\omega)(\forall n_1,n_2>\ell)(u^{i_1}_{n_1}\restriction\ell\cap u^{i_2}_{n_2}\restriction\ell=\emptyset)$.

(3) This is a consequence of (1,2) above and Shelah [7, Fact 1.16].

(4) By [7, Claim 1.12(1)]. □

To carry out our arguments we need to assume that our indexed base $\bar{\mathcal{O}}$ satisfies some additional properties.

Definition 3.6. An indexed base $\bar{\mathcal{O}}=\langle\mathcal{O}_i:i<i^*\rangle$ is *nice* if it satisfies the following demands (i)–(v).

(i) Either $i^*\geq 6$ or for some $i<i^*$ we have
$$\big(\forall u\in\mathcal{O}_i\big)\big(\exists v\in\mathcal{O}_i\big)\big(u\prec v\ \wedge\ |v|\geq 6\big).$$
(ii) If $i<i^*$, $u\prec_i v\prec_i v'\prec_i v''$, and $\ell(v)\leq\ell\leq\ell(v')$, then $\{\eta\restriction\ell:\eta\in v'\}\in\mathcal{O}_i$ and $u\prec_i\{\eta\restriction\ell:\eta\in v'\}\prec_i v''$.
(iii) If $i<i^*$, $u\prec_i v$, $\ell(v)<\ell$ and $v'\subseteq{}^\ell 2$ is such that for each $\nu\in v$ the set $\{\eta\in v':\nu\vartriangleleft\eta\}$ has exactly one element, then $v'\in\mathcal{O}_i$ and $u\prec_i v'$.
(iv) Suppose $u\prec_i v$ and $u'\subseteq u$ is such that $u'\in\mathcal{O}_i$. Let $v'=\{\eta\in v:\eta\restriction\ell(u)\in u'\}$. Then $v'\in\mathcal{O}_i$ and $u'\prec_i v'$.
(v) If $i^*=\omega$, then for each $i<i^*$ there are infinitely many $j<i^*$ such that $\mathcal{O}_i=\mathcal{O}_j$.

Observation 3.7. *The indexed bases $\bar{\mathcal{O}}^\iota$ (for $6 \leq \iota \leq \omega$) and $\bar{\mathcal{O}}^{\rm per}$ introduced in Example 3.4 are nice.*

Proposition 3.8. *Suppose an indexed base $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ is nice. Then:*

(⊛) *If $2 \leq K < \omega$ and $\bar{u}^k$ (for $k < K$) is an $\mathcal{O}_{i(k)}$–tower for some $i(k) < i^*$, then there are $\mathcal{O}_{i(k)}$–towers $\bar{v}^k = \langle v^k_n : n < \omega \rangle$ (for $k < K$) such that*
- *$\mathcal{C}(\bar{v}^k) = \mathcal{C}(\bar{u}^k)$, $v^k_0 = u^k_0$ and*
- *$\bigcap\limits_{k \in K} \{\ell(v^k_n) : n < \omega\}$ is infinite.*

Proof. Induction on K. For $K = 2$ we proceed as follows. Let $\bar{u}^0$ be an $\mathcal{O}_{i(0)}$–tower and $\bar{u}^1$ be an $\mathcal{O}_{i(1)}$–tower. Choose inductively a sequence $\langle n_k : k < \omega \rangle$ so that

- $5 < n_0 < n_1 < n_2 < n_3 < \ldots$,
- $\ell(u^1_5) < \ell(u^0_{n_0})$,
- if $\ell(u^1_j) \leq \ell(u^0_{n_k}) < \ell(u^1_{j+1})$, then $\ell(u^1_{j+5}) \leq \ell(u^0_{n_{k+1}})$.

For $k < \omega$ let $j(k)$ be such that $\ell(u^1_{j(k)}) \leq \ell(u^0_{n_k}) < \ell(u^1_{j(k)+1})$. Put $v_k = \{\eta\restriction\ell(u^0_{n_k}) : \eta \in u^1_{j(k)+1}\}$. By 3.6(ii), $v_k \in \mathcal{O}_{i(1)}$ and $u^1_{j(k)-1} \prec_{i(1)} v_k \prec_{i(1)} u^1_{j(k)+2}$. The rest should be clear. □

For the rest of this section we will be assuming the following.

Assumptions 3.9.
(1) $\bar{T} = \langle T_n : n < \omega \rangle$, B are as in Assumptions 3.1,
(2) $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ is a nice indexed base with $\mathcal{O}_i = (\mathcal{O}_i, \prec_i)$,
(3) there are distinct $x, y \in {}^\omega 2$ such that $(B + x) \cap (B + y)$ is $\bar{\mathcal{O}}$–large.

Definition 3.10. Let $\mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ consist of all tuples

$$\mathbf{m} = (\ell^{\mathbf{m}}, \iota^{\mathbf{m}}, u^{\mathbf{m}}, \bar{h}^{\mathbf{m}}, \bar{g}^{\mathbf{m}}) = (\ell, \iota, u, \bar{h}, \bar{g})$$

such that:

(a) $0 < \ell < \omega$, $u \subseteq {}^\ell 2$ and $2 \leq |u|$, and $\iota = i^*$ if $i^* < \omega$, and $3 \leq \iota < \omega$ otherwise;
(b) $\bar{g} = \langle g_i : i < \iota \rangle$, where[4] $g_i : u^{\langle 2 \rangle} \longrightarrow \mathcal{O}_i$ is such that $g_i(\eta, \nu) = g_i(\nu, \eta)$ and $\ell\big(g_i(\eta, \nu)\big) = \ell$ for each $(\eta, \nu) \in u^{\langle 2 \rangle}$;
(c) if $(\eta, \nu) \in u^{\langle 2 \rangle}$ and $i < i' < \iota$, then $g_i(\eta, \nu) \cap g_{i'}(\eta, \nu) = \emptyset$,
(d) $\bar{h} = \langle h_i : i < \iota \rangle$, where $h_i : u^{\langle 2 \rangle} \longrightarrow \omega$;
(e) for each $(\eta, \nu) \in u^{\langle 2 \rangle}$, if $\sigma \in g_i(\eta, \nu)$ then $\eta + \sigma \in T_{h_i(\eta,\nu)}$.

Definition 3.11. Assume $\mathbf{m} = (\ell, \iota, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ and $\rho \in {}^\ell 2$. We define $\mathbf{m} + \rho = (\ell', \iota', u', \bar{h}', \bar{g}')$ by

- $\ell' = \ell$, $\iota' = \iota$, $u' = \{\eta + \rho : \eta \in u\}$,
- $\bar{g}' = \langle g'_i : i < \iota' \rangle$, where $g'_i : (u')^{\langle 2 \rangle} \longrightarrow \mathcal{O}_i : (\eta + \rho, \nu + \rho) \mapsto g_i(\eta, \nu) + \rho$,
- $\bar{h}' = \langle h'_i : i < \iota' \rangle$, where $h'_i : (u')^{\langle 2 \rangle} \longrightarrow \omega$ are such that $h'_i(\eta + \rho, \nu + \rho) = h_i(\eta, \nu)$ for $(\eta, \nu) \in u^{\langle 2 \rangle}$.

Also if $\rho \in {}^\omega 2$, then we set $\mathbf{m} + \rho = \mathbf{m} + (\rho\restriction\ell)$.

[4] remember $u^{\langle 2 \rangle} = \{(\eta, \nu) \in u \times u : \eta \neq \nu\}$

Observation 3.12. (1) *If* $\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *and* $\rho\in{}^{\ell^{\mathbf{m}}}2$, *then* $\mathbf{m}+\rho\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$.

(2) *For each* $\rho\in{}^{\omega}2$ *the mapping* $\mathbf{M}_{\bar T,\bar{\mathcal{O}}}\longrightarrow\mathbf{M}_{\bar T,\bar{\mathcal{O}}}:\mathbf{m}\mapsto\mathbf{m}+\rho$ *is a bijection.*

Definition 3.13. Assume $\mathbf{m},\mathbf{n}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$. We say that $\mathbf{n}$ *strictly extends* $\mathbf{m}$ ($\mathbf{m}\sqsubset\mathbf{n}$ in short) if and only if:

- $\ell^{\mathbf{m}}<\ell^{\mathbf{n}}$, $\iota^{\mathbf{m}}\le\iota^{\mathbf{n}}$, $u^{\mathbf{m}}=\{\eta\restriction\ell^{\mathbf{m}}:\eta\in u^{\mathbf{n}}\}$, and
- for every $(\eta,\nu)\in(u^{\mathbf{n}})^{\langle 2\rangle}$ such that $\eta\restriction\ell^{\mathbf{m}}\neq\nu\restriction\ell^{\mathbf{m}}$ and each $i<\iota^{\mathbf{m}}$ we have
  - $g^{\mathbf{m}}_i(\eta\restriction\ell^{\mathbf{m}},\nu\restriction\ell^{\mathbf{m}})\prec g^{\mathbf{n}}_i(\eta,\nu)$, and
  - $h^{\mathbf{m}}_i(\eta\restriction\ell^{\mathbf{m}},\nu\restriction\ell^{\mathbf{m}})=h^{\mathbf{n}}_i(\eta,\nu)$.

Definition 3.14. (1) By induction on ordinals α we define $D^{\bar T}(\alpha)\subseteq\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$. We declare that:
  - $D^{\bar T}(0)=\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$,
  - if α is a limit ordinal, then $D^{\bar T}(\alpha)=\bigcap\limits_{\beta<\alpha}D^{\bar T}(\beta)$,
  - if $\alpha=\beta+1$, then $D^{\bar T}(\alpha)$ consists of all $\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ such that for each for each $\nu\in u^{\mathbf{m}}$ there is an $\mathbf{n}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ satisfying
    - $\mathbf{m}\sqsubset\mathbf{n}$ and $\mathbf{n}\in D^{\bar T}(\beta)$, and if $i^*=\omega$ then $\iota^{\mathbf{m}}<\iota^{\mathbf{n}}$, and
    - the set $\{\eta\in u^{\mathbf{n}}:\nu\vartriangleleft\eta\}$ has at least two elements.

(2) We define a function[5] $\mathrm{ndrk}^{\bar T}_{\bar{\mathcal{O}}}=\mathrm{ndrk}:\mathbf{M}_{\bar T,\bar{\mathcal{O}}}\longrightarrow\mathrm{ON}\cup\{\infty\}$ as follows.
If $\mathbf{m}\in D^{\bar T}(\alpha)$ for all ordinals α, then we say that $\mathrm{ndrk}(\mathbf{m})=\infty$.
Otherwise, $\mathrm{ndrk}(\mathbf{m})$ is the first ordinal α for which $\mathbf{m}\notin D^{\bar T}(\alpha+1)$.

(3) We also define

$$\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar T)=\mathrm{NDRK}(\bar T)=\sup\{\mathrm{ndrk}(\mathbf{m})+1:\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}\}.$$

Lemma 3.15. (1) *The relation* $\sqsubset$ *is a strict partial order on* $\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$.

(2) *If* $\mathbf{m},\mathbf{n}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *and* $\mathbf{m}\sqsubset\mathbf{n}$ *and* $\mathbf{n}\in D^{\bar T}(\alpha)$, *then* $\mathbf{m}\in D^{\bar T}(\alpha)$.

(3) *If* $\alpha<\beta$ *then* $D^{\bar T}(\beta)\subseteq D^{\bar T}(\alpha)$. *Hence for* $\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$, $\mathbf{m}\in D^{\bar T}(\alpha)$ *if and only if* $\alpha\le\mathrm{ndrk}(\mathbf{m})$.

(4) *If* $\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *and* $\rho\in{}^{\omega}2$ *then* $\mathrm{ndrk}(\mathbf{m})=\mathrm{ndrk}(\mathbf{m}+\rho)$.

(5) *If* $\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *and* $\mathrm{ndrk}(\mathbf{m})\ge\omega_1$, *then there is an* $\mathbf{n}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *such that* $\mathbf{m}\sqsubset\mathbf{n}$, $|\{\eta\in u^{\mathbf{n}}:\nu\vartriangleleft\eta\}|\ge 2$ *for each* $\nu\in u^{\mathbf{m}}$, *if* $i^*=\omega$ *then* $\iota^{\mathbf{m}}<\iota^{\mathbf{n}}$, *and* $\mathrm{ndrk}(\mathbf{n})\ge\omega_1$.

(6) *If* $\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *and* $\infty>\mathrm{ndrk}(\mathbf{m})=\beta>\alpha$, *then there is* $\mathbf{n}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *such that* $\mathbf{m}\sqsubset\mathbf{n}$ *and* $\mathrm{ndrk}(\mathbf{n})=\alpha$.

(7) *If* $\mathrm{NDRK}(\bar T)\ge\omega_1$, *then* $\mathrm{NDRK}(\bar T)=\infty$.

(8) *Assume* $\mathbf{m}\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *and* $u'\subseteq u^{\mathbf{m}}$, $|u'|\ge 2$. *Put* $\ell'=\ell^{\mathbf{m}}$, $\iota'=\iota^{\mathbf{m}}$, *and for* $i<\iota'$ *let* $h'_i=h^{\mathbf{m}}_i\restriction(u')^{\langle 2\rangle}$ *and* $g'_i=g^{\mathbf{m}}_i\restriction(u')^{\langle 2\rangle}$. *Let* $\mathbf{m}\restriction u'=(\ell',u',i',\bar h',\bar g')$. *Then* $\mathbf{m}\restriction u'\in\mathbf{M}_{\bar T,\bar{\mathcal{O}}}$ *and* $\mathrm{ndrk}(\mathbf{m})\le\mathrm{ndrk}(\mathbf{m}\restriction u')$.

Proof. Exactly the same as for [4, Lemma 3.10]. □

Proposition 3.16. *For a nice indexed base* $\bar{\mathcal{O}}$ *the following conditions (a) – (d) are equivalent.*

[5] ndrk stands for **n**on**d**isjointness **r**an**k**

(a) $\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar{T}) \geq \omega_1$.
(b) $\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar{T}) = \infty$.
(c) *B is perfectly orthogonal to $\bar{\mathcal{O}}$–small (see 3.3(2)).*
(d) *In some ccc forcing extension, the set B has λ_{ω_1} many pairwise $\bar{\mathcal{O}}$–nondisjoint translations (see 3.3(3)).*

Proof. The proof follows closely the lines of [4, Proposition 3.11].
(c) $\Rightarrow$ (d) Assume (c) and let $P \subseteq {}^\omega 2$ be a perfect set such that the translations $B+x$, $B+y$ have $\bar{\mathcal{O}}$–large intersection for all $x, y \in P$. Let $\kappa = \beth_{\omega_1}$. By Corollary 2.8, $\Vdash_{\mathbb{C}_\kappa} \lambda_{\omega_1} \leq \mathfrak{c}$. By Proposition 3.5(2), the formula "$P \times P \subseteq \mathrm{stnd}_{\bar{\mathcal{O}}}(B)$" is Π^1_2, so it holds in the forcing extension by $\mathbb{C}_\kappa$. Now we easily conclude (d).

(d) $\Rightarrow$ (a) Assume (d) and let $\mathbb{P}$ be the ccc forcing notion witnessing this assumption, $G \subseteq \mathbb{P}$ be generic over $\mathbf{V}$. Let us work in $\mathbf{V}[G]$.

Let $\langle \eta_\alpha : \alpha < \lambda_{\omega_1}\rangle$ be a sequence of distinct elements of ${}^\omega 2$ such that

$$\big(\forall \alpha < \beta < \lambda_{\omega_1}\big)\big((B+\eta_\alpha) \cap (B+\eta_\beta) \text{ is } \bar{\mathcal{O}}\text{–large }\big).$$

Remember Definition 3.2(3): an $\mathcal{O}_i$–tower is an $\prec_i$–increasing sequence $\bar{u} = \langle u_n : n < \omega\rangle$ and its cover $\mathcal{C}(\bar{u})$ is the set $\{\eta \in {}^\omega 2 : (\forall n < \omega)(\eta{\restriction}\ell(u_n) \in u_n)\}$.

Let $\tau = \{R_{\mathbf{m}} : \mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}\}$ be a vocabulary where each $R_{\mathbf{m}}$ is a $|u^{\mathbf{m}}|$–ary relational symbol. Let $\mathbb{M} = \big(\lambda_{\omega_1}, \big\{R^{\mathbb{M}}_{\mathbf{m}}\big\}_{\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}}\big)$ be the model in the vocabulary τ, where for $\mathbf{m} = (\ell, \iota, u, h, g) \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ the relation $R^{\mathbb{M}}_{\mathbf{m}}$ is defined by

$$R^{\mathbb{M}}_{\mathbf{m}} = \Big\{(\alpha_0,\ldots,\alpha_{|u|-1}) \in (\lambda_{\omega_1})^{|u|} : \{\eta_{\alpha_0}{\restriction}\ell,\ldots,\eta_{|u|-1}{\restriction}\ell\} = u \text{ and}$$
$$\text{for each distinct } j_1, j_2 < |u| \text{ and every } i < \iota$$
$$\text{there is an } \mathcal{O}_i\text{–tower } \bar{u}^i(j_1,j_2) = \langle u^i_n(j_1,j_2) \colon n < \omega\rangle \text{ such that}$$
$$g_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) = u^i_0(j_1,j_2) \text{ and } \mathcal{C}\big(\bar{u}^i(j_1,j_2)\big) \text{ is included in}$$
$$[\lim(T_{h_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell)}) + \eta_{\alpha_{j_1}}] \cap [\lim(T_{h_i(\eta_{\alpha_{j_2}}, \eta_{\alpha_{j_1}})}) + \eta_{\alpha_{j_2}}]\Big\}.$$

Claim 3.16.1. (1) *If $\alpha_0, \alpha_1, \ldots, \alpha_{j-1} < \lambda_{\omega_1}$ are distinct, $j \geq 2$, then for infinitely many $k < \omega$ there is $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ such that*

$$\ell^{\mathbf{m}} = k, \quad u^{\mathbf{m}} = \{\eta_{\alpha_0}{\restriction}k, \ldots, \eta_{\alpha_{j-1}}{\restriction}k\} \quad \text{and} \quad \mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{j-1}].$$

(2) *Assume that $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$, $j < |u^{\mathbf{m}}|$, $\alpha_0, \alpha_1, \ldots, \alpha_{|u^{\mathbf{m}}|-1} < \lambda_{\omega_1}$ and $\alpha^* < \lambda_{\omega_1}$ are all pairwise distinct and such that*
- $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_j,\ldots,\alpha_{|u^{\mathbf{m}}|-1}]$ *and*
- $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{j-1},\alpha^*,\alpha_{j+1},\ldots\alpha_{|u^{\mathbf{m}}|-1}]$.

Then for infinitely many $k < \omega$ there is an $\mathbf{n} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ such that $\mathbf{m} \sqsubset \mathbf{n}$ and $\ell^{\mathbf{n}} = k$, $u^{\mathbf{n}} = \{\eta_{\alpha_0}{\restriction}k, \ldots, \eta_{\alpha_{|u^{\mathbf{m}}|-1}}{\restriction}k, \eta_{\alpha^*}{\restriction}k\}$ and $\mathbb{M} \models R_{\mathbf{n}}[\alpha_0,\ldots,\alpha_{|u^{\mathbf{m}}|-1},\alpha^*]$, and if $i^* = \omega$ then also $\iota^{\mathbf{m}} < \iota^{\mathbf{n}}$.

(3) *If $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0,\ldots,\alpha_{|u^{\mathbf{m}}|-1}]$, then*

$$\mathrm{rk}(\{\alpha_0,\ldots,\alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M}) \leq \mathrm{ndrk}^{\bar{T}}_{\bar{\mathcal{O}}}(\mathbf{m}).$$

Proof of the Claim. (1) It is a simpler version of the proof below.

(2) By the definition of $R^{\mathbb{M}}_{\mathbf{m}}$, since $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots \alpha_{|u^{\mathbf{m}}|-1}]$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_j, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$, we may choose a sequence

$$\langle \bar{u}^i(j_1, j_2) : (j_1, j_2) \in \big(|u^{\mathbf{m}}| + 1\big)^{\langle 2 \rangle} \ \wedge \ i < \iota^{\mathbf{m}} \rangle$$

satisfying the following demands. Letting $\alpha_{|u^{\mathbf{m}}|} = \alpha^*$, for $(j_1, j_2) \in \big(|u^{\mathbf{m}}| + 1\big)^{\langle 2 \rangle}$ and $i < \iota^{\mathbf{m}}$:

- $\bar{u}^i(j_1, j_2) = \bar{u}^i(j_2, j_1)$ is a $\mathcal{O}_i$–tower,
- if $\{j_1, j_2\} \neq \{j, |u^{\mathbf{m}}|\}$, then $u^i_0(j_1, j_2) = g^{\mathbf{m}}_i(\eta_{\alpha_{j_1}} \restriction \ell^{\mathbf{m}}, \eta_{\alpha_{j_2}} \restriction \ell^{\mathbf{m}})$,
- if $i_1 < i_2 < \iota^{\mathbf{m}}$, then $\mathcal{C}\big(\bar{u}^{i_1}(j_1, j_2)\big) \cap \mathcal{C}\big(\bar{u}^{i_2}(j_1, j_2)\big) = \emptyset$,
- if $\{j_1, j_2\} \neq \{j, |u^{\mathbf{m}}|\}$, then $\mathcal{C}\big(\bar{u}^i(j_1, j_2)\big)$ is included in

$$[\lim(T_{h^{\mathbf{m}}_i(\eta_{\alpha_{j_1}} \restriction \ell^{\mathbf{m}}, \eta_{\alpha_{j_2}} \restriction \ell^{\mathbf{m}})}) + \eta_{\alpha_{j_1}}] \cap [\lim(T_{h^{\mathbf{m}}_i(\eta_{\alpha_{j_2}} \restriction \ell^{\mathbf{m}}, \eta_{\alpha_{j_1}} \restriction \ell^{\mathbf{m}})}) + \eta_{\alpha_{j_2}}],$$

- for some N'_i, N''_i we have

$$\mathcal{C}\big(\bar{u}^i(j, |u^{\mathbf{m}}|)\big) \subseteq [\lim(T_{N'_i}) + \eta_{\alpha_j}] \cap [\lim(T_{N''_i}) + \eta_{\alpha^*}].$$

Since $\bar{\mathcal{O}}$ is nice (and $\iota^{\mathbf{m}}$ and $u^{\mathbf{m}}$ are finite), we may use 3.8(⊛) and modify $\bar{u}^i(j_1, j_2)$ (without changing $u^i_0(j_1, j_2)$) and demand that the set

$$A = \bigcap_{i < \iota^{\mathbf{m}}} \ \bigcap_{j_1 < j_2 \leq |u^{\mathbf{m}}|} \big\{\ell\big(u^i_n(j_1, j_2)\big) : n \in \omega\big\}$$

is infinite. Let $\ell_0 \in A \setminus (\ell^{\mathbf{m}} + 1)$ be bigger than the second element of $A \setminus (\ell^{\mathbf{m}} + 1)$ and such that $\eta_{\alpha_{|u^{\mathbf{m}}|}} \restriction \ell_0 \neq \eta_{\alpha_j} \restriction \ell_0$, and $x \restriction \ell_0 \neq y \restriction \ell_0$ whenever $x \in \mathcal{C}(\bar{u}^{i_1}(j_1, j_2))$, $y \in \mathcal{C}(\bar{u}^{i_2}(j_1, j_2))$, $(j_1, j_2) \in \big(|u^{\mathbf{m}}| + 1\big)^{\langle 2 \rangle}$ and $i_1 < i_2 < \iota^{\mathbf{m}}$.

Let $\iota = \iota^{\mathbf{m}} = i^*$ if $i^* < \omega$ and let $\iota = \iota^{\mathbf{m}} + 1$ otherwise. In the latter case we also have to choose $\mathcal{O}_{\iota^{\mathbf{m}}}$–towers $\bar{u}^{\iota^{\mathbf{m}}}(j_1, j_2)$, but to ensure the demand 3.10(c) we will have to modify the already chosen towers $\bar{u}^i(j_1, j_2)$ (for $i < \iota^{\mathbf{m}}$). Fix $(j_1, j_2) \in \big(|u^{\mathbf{m}}| + 1\big)^{\langle 2 \rangle}$ for a moment. Let

$$K = \sum \big\{\big|u^i_n(j_1, j_2)\big| : \ell(u^i_n(j_1, j_2)) = \ell_0 \ \wedge \ i < \iota^{\mathbf{m}} \ \wedge \ n < \omega\big\}.$$

By 3.6(v) and the assumptions on $\langle \eta_\alpha : \alpha < \lambda_{\omega_1} \rangle$, there are infinitely many $\mathcal{O}_{\iota^{\mathbf{m}}}$–towers $\bar{v}^k$ such that their covers are pairwise disjoint and included in $\big(\lim(T_{k_1}) + \eta_{\alpha_{j_1}}\big) \cap \big(\lim(T_{k_2}) + \eta_{\alpha_{j_2}}\big)$ for some k_1, k_2.

Choose $\ell(j_1, j_2) \in A \setminus (\ell_0 + 1)$ so large, that there are more than $K + 1$ many k's for which the sets $\{\eta \restriction \ell(j_1, j_2) : \eta \in v^k_n\}$ are pairwise disjoint (for large n) and $\ell(v^k_5) < \ell(j_1, j_2)$ for all those k's. For $i < i^{\mathbf{m}}$ let $n(i), m(i)$ be such that $\ell(u^i_{n(i)}(j_1, j_2)) = \ell_0$ and $\ell(u^i_{m(i)}(j_1, j_2)) = \ell(j_1, j_2)$, and let $v^i \subseteq u^i_{m(i)}(j_1, j_2)$ be such that for each $\nu \in u^i_{n(i)}(j_1, j_2)$ the set $\{\eta \in v^i : \nu \vartriangleleft \eta\}$ has exactly one element. By 3.6(iii) we have

$$v^i \in \mathcal{O}_i \quad \text{ and } \quad u^i_{n(i)-1}(j_1, j_2) \prec_i v^i.$$

Hence, using repeatedly 3.6(iv), we may modify the towers $\bar{u}^i(j_1, j_2)$ (for $i < i^{\mathbf{m}}$) and demand that

- for each $i < \iota^{\mathbf{m}}$, for some $n^*(i)$,

$$\ell(u^i_{n^*(i)}(j_1, j_2)) = \ell(j_1, j_2) \ \text{ and } \ \big|u^i_{n^*(i)}(j_1, j_2)\big| = \big|\{\eta \restriction \ell_0 : \eta \in u^i_{n^*(i)}(j_1, j_2)\}\big|.$$

Looking back at the towers $\bar{v}^k$, we may choose one, $\bar{v}^{k^*} = \bar{v}(j_1, j_2)$, which has the property that for all large n

$$\{\eta{\restriction}\ell(j_1,j_2) : \eta \in v_n(j_1,j_2)\} \cap \bigcup \{u^i_{n^*(i)}(j_1,j_2) : i < \iota^{\mathbf{m}}\} = \emptyset.$$

Now unfix (j_1, j_2) and set $\ell = \max\{\ell(j_1,j_2) : (j_1,j_2) \in (|u^{\mathbf{m}}|+1)^{\langle 2\rangle}\}$.

Suppose $j_1 < j_2 \le |u^{\mathbf{m}}|$ and let n be such that $\ell(v_{n-1}(j_1,j_2)) < \ell \le \ell(v_n(j_1,j_2))$. By 3.6(ii), we may let

- $u^{\iota^{\mathbf{m}}}_0(j_1,j_2) = u^{\iota^{\mathbf{m}}}_0(j_2,j_1) = \{\eta{\restriction}\ell : \eta \in v_n(j_1,j_2)\}$,
- $u^{\iota^{\mathbf{m}}}_m(j_1,j_2) = u^{\iota^{\mathbf{m}}}_m(j_2,j_1) = v_{n+m}(j_1,j_2)$ for $m > 0$,

getting a $\mathcal{O}_{\iota^{\mathbf{m}}}$–tower $\bar{u}^{\iota^{\mathbf{m}}}(j_1,j_2)$. We also fix $k(j_1,j_2), k(j_2,j_1)$ such that

$$\mathcal{C}(\bar{u}^{\iota^{\mathbf{m}}}(j_1,j_2)) \subseteq \big(\lim(T_{k(j_1,j_2)} + \eta_{\alpha_{j_1}}\big) \cap \big(\lim(T_{k(j_2,j_1)} + \eta_{\alpha_{j_2}}\big).$$

If $i^* = \iota^{\mathbf{m}} < \omega$, then the procedure leading to the choice of $\bar{u}^{\iota^{\mathbf{m}}}(j_1,j_2)$ is not present and we just let $\ell = \min(A \setminus (\ell_0+1))$.

Let $u = \big\{\eta_{\alpha_0}{\restriction}\ell, \ldots, \eta_{\alpha_{|u^{\mathbf{m}}|-1}}{\restriction}\ell, \eta_{\alpha^*}{\restriction}\ell\big\}$.

For each $i < \iota$ and $(j_1,j_2) \in \big(|u^{\mathbf{m}}|+1\big)^{\langle 2\rangle}$ put $g_i(\eta_{\alpha_{j_1}}{\restriction}\ell, \eta_{\alpha_{j_2}}{\restriction}\ell) = u^i_n(j_1,j_2)$, where n is such that $\ell\big(u^i_n(j_1,j_2)\big) = \ell$. This defines $g_i : u^{\langle 2\rangle} \longrightarrow \mathcal{O}_i$ for $i < \iota$. For $(\nu_1,\nu_2) \in u^{\langle 2\rangle}$ we also set

$$h_i(\nu_1,\nu_2) = \begin{cases} h^{\mathbf{m}}_i(\nu_1{\restriction}\ell^{\mathbf{m}}, \nu_2{\restriction}\ell^{\mathbf{m}}) & \text{if } \nu_1{\restriction}\ell^{\mathbf{m}} \neq \nu_2{\restriction}\ell^{\mathbf{m}},\ i < \iota^{\mathbf{m}}, \\ N'_i & \text{if } \nu_1 \vartriangleleft \eta_{\alpha_j},\ \nu_2 \vartriangleleft \eta_{\alpha^*},\ i < \iota^{\mathbf{m}}, \\ N''_i & \text{if } \nu_1 \vartriangleleft \eta_{\alpha^*},\ \nu_2 \vartriangleleft \eta_{\alpha_j},\ i < \iota^{\mathbf{m}}, \\ k(j_1,j_2) & \text{if } \nu_1 \vartriangleleft \eta_{\alpha_{j_1}},\ \nu_2 \vartriangleleft \eta_{\alpha_{j_2}},\ i = \iota^{\mathbf{m}} < \iota. \end{cases}$$

It should be clear that $\mathbf{n} = (\ell, \iota, u, g, h) \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ is as required.

(3) By induction on β we show that

for every $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ and *all* $\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1} < \lambda_{\omega_1}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$:
$\beta \le \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M})$ implies $\beta \le \mathrm{ndrk}(\mathbf{m})$.

Steps $\beta = 0$ and β is limit: Straightforward.

Step $\beta = \gamma + 1$: Suppose $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ and $\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1} < \lambda_{\omega_1}$ are such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$ and $\gamma+1 \le \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M})$. Let $\nu \in u^{\mathbf{m}}$, so $\nu = \eta_{\alpha_j}{\restriction}\ell^{\mathbf{m}}$ for some $j < |u^{\mathbf{m}}|$. Since $\gamma+1 \le \mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M})$ we may find $\alpha^* \in \lambda_{\omega_1} \setminus \{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}$ such that

$$\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$$

and $\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}, \alpha^*\}, \mathbb{M}) \ge \gamma$. By clause (2) we may find $\mathbf{n} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ such that $\mathbf{m} \sqsubset \mathbf{n}$ and $u^{\mathbf{n}} = \{\eta_{\alpha_0}{\restriction}\ell^{\mathbf{n}}, \ldots, \eta_{\alpha_{|u^{\mathbf{m}}|-1}}{\restriction}\ell^{\mathbf{n}}, \eta_{\alpha^*}{\restriction}\ell^{\mathbf{n}}\}$, $\eta_{\alpha_j}{\restriction}\ell^{\mathbf{n}} \neq \eta_{\alpha^*}{\restriction}\ell^{\mathbf{n}}$, and if $i^* = \omega$ then $\iota^{\mathbf{m}} < \iota^{\mathbf{n}}$, and $\mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}, \alpha^*]$. Then also $|\{\eta \in u^{\mathbf{n}} : \nu \vartriangleleft \eta\}| = 2$. By the inductive hypothesis we have also $\gamma \le \mathrm{ndrk}(\mathbf{n})$. Now we may easily conclude that $\gamma+1 \le \mathrm{ndrk}(\mathbf{m})$. □

By the definition of λ_{ω_1},

(⊙) $\sup\{\mathrm{rk}(w, \mathbb{M}) : \emptyset \neq w \in [\lambda_{\omega_1}]^{<\omega}\} \ge \omega_1$

Now, suppose that $\beta < \omega_1$. By ($\odot$), there are distinct $\alpha_0, \ldots, \alpha_{j-1} < \lambda_{\omega_1}$, $j \geq 2$, such that $\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{j-1}\}, \mathbb{M}) \geq \beta$. By Claim 3.16.1(1) we may find $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}]$. Then by Claim 3.16.1(3) we also have $\mathrm{ndrk}^{\bar{T}}_{\bar{\mathcal{O}}}(\mathbf{m}) \geq \beta$. Consequently, $\mathrm{NDRK}(\bar{T}) \geq \omega_1$.

All the considerations above where carried out in $\mathbf{V}[G]$. However, the rank function $\mathrm{ndrk}^{\bar{T}}_{\bar{\mathcal{O}}}$ is absolute, so we may also claim that in $\mathbf{V}$ we have $\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar{T}) \geq \omega_1$. □

4. The main result

In this section we construct a forcing notion adding a sequence $\bar{T}$ of subtrees of ${}^{\omega>}2$ such that $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T}) < \omega_1$ and yet with many $\bar{\mathcal{O}}$–nondisjoint translations (for a nice $\bar{\mathcal{O}}$). The sequence $\bar{T}$ will be added by finite approximations, so we will need a finite version of Definition 3.10.

Definition 4.1. Assume that

(a) $0 < n, M < \omega$, $\bar{t} = \langle t_m : m < M \rangle$, and each t_m is a subtree of ${}^{n\geq}2$ in which all terminal branches are of length n,

(b) $T_j \subseteq {}^{\omega>}2$ (for $j < \omega$) are trees with no maximal nodes, $\bar{T} = \langle T_j : j < \omega \rangle$ and $t_m = T_m \cap {}^{n\geq}2$ for $m < M$,

(c) $\mathbf{M}_{\bar{T},\bar{\mathcal{O}}^6}$ is defined as in Definition 3.10 for $\bar{\mathcal{O}}^6$ introduced in Example 3.4(1).

We let $\mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ consist of all tuples $\mathbf{m} = (\ell^{\mathbf{m}}, 6, u^{\mathbf{m}}, \bar{h}^{\mathbf{m}}, \bar{g}^{\mathbf{m}}) \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}^6}$ such that $\ell^{\mathbf{m}} \leq n$ and $\mathrm{rng}(h^{\mathbf{m}}_i) \subseteq M$ for each $i < 6$.

The extension relation $\sqsubset$ on $\mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ is inherited from $\mathbf{M}_{\bar{T},\bar{\mathcal{O}}^6}$ (see Definition 3.13).

Observation 4.2. (1) *The Definition of $\mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ does not depend on the choice of $\bar{T}$, as long as the clause 4.1(b) is satisfied.*

(2) *If $\mathbf{m} \in \mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ and $\rho \in {}^{\ell^{\mathbf{m}}}2$, then $\mathbf{m}+\rho \in \mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ (remember Definition 3.11).*

Lemma 4.3 (See [3, Lemma 2.3].). *Let $0 < \ell < \omega$ and let $\mathcal{B} \subseteq {}^{\ell}2$ be a linearly independent set of vectors (in $({}^{\ell}2, +)$ over $\mathbb{Z}_2$). If $\mathcal{A} \subseteq {}^{\ell}2$, $|\mathcal{A}| \geq 5$ and $\mathcal{A} + \mathcal{A} \subseteq \mathcal{B} + \mathcal{B}$, then for a unique $x \in {}^{\ell}2$ we have $\mathcal{A} + x \subseteq \mathcal{B}$.*

Theorem 4.4. *Assume that an uncountable cardinal λ satisfies $\mathrm{NPr}_{\omega_1}(\lambda)$ and suppose that $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ is a nice indexed base. Then there is a ccc forcing notion $\mathbb{P}$ of size λ such that*

$$\Vdash_{\mathbb{P}} \text{“for some } \Sigma^0_2\ \bar{\mathcal{O}}^6\text{–}\mathbf{npots}\text{–set } B = \bigcup_{n<\omega} \lim(T_n) \subseteq {}^{\omega}2 \text{ there is a sequence } \langle \eta_\alpha : \alpha < \lambda \rangle \text{ of distinct elements of } {}^{\omega}2 \text{ such that all intersections } (\eta_\alpha + B) \cap (\eta_\beta + B) \text{ are } \bar{\mathcal{O}}\text{–large for } \alpha, \beta < \lambda\text{”.}$$

Proof. Fix a countable vocabulary $\tau = \{R_{n,\zeta} : n, \zeta < \omega\}$, where $R_{n,\zeta}$ is an n–ary relational symbol (for $n, \zeta < \omega$). By the assumption on λ, we may fix a model $\mathbb{M} = (\lambda, \{R^{\mathbb{M}}_{n,\zeta}\}_{n,\zeta<\omega})$ in the vocabulary τ with the universe λ and an ordinal $\alpha^* < \omega_1$ such that:

($\circledast$)$_{\rm a}$ for every n and a quantifier free formula $\varphi(x_0,\ldots,x_{n-1})\in\mathcal{L}(\tau)$ there is $\zeta<\omega$ such that for all $a_0,\ldots,a_{n-1}\in\lambda$,

$$\mathbb{M}\models\varphi[a_0,\ldots,a_{n-1}]\Leftrightarrow R_{n,\zeta}[a_0,\ldots,a_{n-1}],$$

($\circledast$)$_{\rm b}$ $\sup\{{\rm rk}(v,\mathbb{M}):\emptyset\neq v\in[\lambda]^{<\omega}\}<\alpha^*$,

($\circledast$)$_{\rm c}$ the rank of every singleton is at least 0.

For a nonempty finite set $v\subseteq\lambda$ let ${\rm rk}(v)={\rm rk}(v,\mathbb{M})$, and let $\zeta(v)<\omega$ and $k(v)<|v|$ be such that $R_{|v|,\zeta(v)},k(v)$ witnesses the rank of v. Thus letting $\{a_0,\ldots,a_k,\ldots a_{n-1}\}$ be the increasing enumeration of v and $k=k(v)$ and $\zeta=\zeta(v)$, we have

($\circledast$)$_{\rm d}$ if ${\rm rk}(v)\geq 0$, then $\mathbb{M}\models R_{n,\zeta}[a_0,\ldots,a_k,\ldots,a_{n-1}]$ but there is no $a\in\lambda\setminus v$ such that

$${\rm rk}(v\cup\{a\})\geq{\rm rk}(v)\ \text{ and }\ \mathbb{M}\models R_{n,\zeta}[a_0,\ldots,a_{k-1},a,a_{k+1},\ldots,a_{n-1}],$$

($\circledast$)$_{\rm e}$ if ${\rm rk}(v)=-1$, then $\mathbb{M}\models R_{n,\zeta}[a_0,\ldots,a_k,\ldots,a_{n-1}]$ but the set

$$\big\{a\in\lambda:\mathbb{M}\models R_{n,\zeta}[a_0,\ldots,a_{k-1},a,a_{k+1},\ldots,a_{n-1}]\big\}$$

is countable.

Without loss of generality we may also require that (for $\zeta=\zeta(v)$, $n=|v|$)

($\circledast$)$_{\rm f}$ for every $b_0,\ldots,b_{n-1}<\lambda$

$$\text{if }\ \mathbb{M}\models R_{n,\zeta}[b_0,\ldots,b_{n-1}]\text{ then }\ b_0<\ldots<b_{n-1}.$$

Now we will define a forcing notion $\mathbb{P}$. *A condition* p in $\mathbb{P}$ is a tuple

$$\big(w^p,n^p,\iota^p,M^p,\bar\eta^p,\bar t^p,\bar r^p,\bar h^p,\bar g^p,\mathcal{M}^p\big)=\big(w,n,\iota,M,\bar\eta,\bar t,\bar r,\bar h,\bar g,\mathcal{M}\big)$$

such that the following demands $(*)_1$–$(*)_{11}$ are satisfied.

$(*)_1$ $w\in[\lambda]^{<\omega}$, $|w|\geq 5$, $5\leq n$, $M<\omega$, $\iota<\omega$ and if $i^*<\omega$ then $\iota=i^*$.

$(*)_2$ $\bar\eta=\langle\eta_\alpha:\alpha\in w\rangle\subseteq{}^n2$.

$(*)_3$ $\bar t=\langle t_m:m<M\rangle$, where $\emptyset\neq t_m\subseteq{}^{n\geq}2$ for $m<M$ is a tree in which all terminal branches are of length n and $t_m\cap t_{m'}\cap{}^n2=\emptyset$ for $m<m'<M$.

$(*)_4$ $\bar r=\langle r_m:m<M\rangle$, where $0<r_m\leq n$ for $m<M$.

$(*)_5$ $\bar h=\langle h_i:i<\iota\rangle$, where $h_i:w^{\langle 2\rangle}\longrightarrow M$ are such that $h_i(\alpha,\beta)=h_i(\beta,\alpha)$.

$(*)_6$ $\bar g=\langle g_i:i<\iota\rangle$, where $g_i:w^{\langle 2\rangle}\longrightarrow\mathcal{O}_i$ are such that $\ell\big(g_i(\alpha,\beta)\big)=n$, $g_i(\alpha,\beta)=g_i(\beta,\alpha)$ and, for each $(\alpha,\beta)\in w^{\langle 2\rangle}$, $\big|\bigcup\limits_{i<\iota}g_i(\alpha,\beta)\big|\geq 6$.

$(*)_7$ For each $m<M$,

$$t_m\cap{}^n2=\bigcup\big\{\eta_\alpha+g_i(\alpha,\beta):(\alpha,\beta)\in w^{\langle 2\rangle}\text{ and }i<\iota\text{ and }h_i(\alpha,\beta)=m\big\}.$$

$(*)_8$ The family

$$\{\eta_\alpha:\alpha\in w\}\cup\bigcup\big\{g_i(\alpha,\beta):(\alpha,\beta)\in w^{\langle 2\rangle}\ \wedge\ i<\iota\big\}$$

is a linearly independent set of vectors in ${}^n2$ (over the field $\mathbb{Z}_2$); in particular there are no repetitions in the representation above and all elements are non-zero vectors.

$(*)_9$ $\mathcal{M}$ consists of all triples $\mathfrak{d}=(\ell^{\mathfrak{d}},v^{\mathfrak{d}},\mathbf{m}^{\mathfrak{d}})=(\ell,v,\mathbf{m})$ such that

$(*)_9^{\rm a}$ $0<\ell\le n$, $v\subseteq w$, $5\le |v|$, and $\eta_\alpha\restriction\ell\neq\eta_\beta\restriction\ell$ for distinct $\alpha,\beta\in v$,
$(*)_9^{\rm b}$ $\mathbf{m}\in\mathbf{M}^n_{\bar t,\bar{\mathcal{O}}^6}$, $\ell^{\mathbf{m}}=\ell$, $u^{\mathbf{m}}=\{\eta_\alpha\restriction\ell:\alpha\in v\}$,
$(*)_9^{\rm c}$ for each $(\alpha,\beta)\in(v)^{\langle 2\rangle}$ and $i<6$ we have $r_{h_i^{\mathbf{m}}(\eta_\alpha\restriction\ell,\eta_\beta\restriction\ell)}\le\ell^{\mathfrak{d}}$,
$(*)_9^{\rm d}$ $\big(\forall(\alpha,\beta)\in v^{\langle 2\rangle}\big)\big(\forall i<6\big)\big(\exists j<\iota\big)\big(h_i^{\mathbf{m}}(\eta_\alpha\restriction\ell,\eta_\beta\restriction\ell)=h_j(\alpha,\beta)\big)$.

$(*)_{10}$ If $\mathfrak{d}_0,\mathfrak{d}_1\in\mathcal{M}$, $\ell^{\mathfrak{d}_0}=\ell^{\mathfrak{d}_1}=\ell$, $\rho\in{}^\ell 2$, and $\mathbf{m}^{\mathfrak{d}_1}=\mathbf{m}^{\mathfrak{d}_0}+\rho$, then $\mathrm{rk}(v^{\mathfrak{d}_0})=\mathrm{rk}(v^{\mathfrak{d}_1})$, $\zeta(v^{\mathfrak{d}_0})=\zeta(v^{\mathfrak{d}_1})$, $k(v^{\mathfrak{d}_0})=k(v^{\mathfrak{d}_1})$ and if $\alpha\in v^{\mathfrak{d}_0}$, $\beta\in v^{\mathfrak{d}_1}$ are such that $|\alpha\cap v^{\mathfrak{d}_0}|=k(v^{\mathfrak{d}_0})=k(v^{\mathfrak{d}_1})=|\beta\cap v^{\mathfrak{d}_1}|$, then $(\eta_\alpha\restriction\ell)+\rho=\eta_\beta\restriction\ell$.

$(*)_{11}$ Suppose that
- $\mathfrak{d}_0,\mathfrak{d}_1\in\mathcal{M}$, $\mathbf{m}^{\mathfrak{d}_0}\sqsubset\mathbf{m}^{\mathfrak{d}_1}$ and $v^{\mathfrak{d}_0}\subseteq v^{\mathfrak{d}_1}$, and
- $\alpha_0\in v^{\mathfrak{d}_0}$, $|\alpha_0\cap v^{\mathfrak{d}_0}|=k(v^{\mathfrak{d}_0})$, $\mathrm{rk}(v^{\mathfrak{d}_0})=-1$.

Then $|\{\nu\in u^{\mathbf{m}^{\mathfrak{d}_1}}:(\eta_{\alpha_0}\restriction\ell^{\mathfrak{d}_0})\trianglelefteq\nu\}|=1$.

To define *the order* $\le$ *of* $\mathbb{P}$ we declare for $p,q\in\mathbb{P}$ that $p\le q$ if and only if

- $w^p\subseteq w^q$, $n^p\le n^q$, $M^p\le M^q$, $\iota^p\le\iota^q$ and
- $t^p_m=t^q_m\cap{}^{n^p\ge}2$ and $r^p_m=r^q_m$ for all $m<M^p$, and
- $\eta^p_\alpha\trianglelefteq\eta^q_\alpha$ for all $\alpha\in w^p$, and
- $h^q_i\restriction(w^p)^{\langle 2\rangle}=h^p_i$ and $g^p_i(\alpha,\beta)\preceq_i g^q_i(\alpha,\beta)$ for $i<\iota^p$ and $(\alpha,\beta)\in(w^p)^{\langle 2\rangle}$.

Claim 4.4.1. (1) *$(\mathbb{P},\le)$ is a partial order of size λ.*

(2) *For each $\beta<\lambda$ and $n_0,M_0<\omega$ the set*

$$D^{n_0,M_0}_\beta=\big\{p\in\mathbb{P}:n^p>n_0\ \wedge\ M^p>M_0\ \wedge\ \beta\in w^p\big\}$$

is open dense in $\mathbb{P}$.

(3) *If $i^*=\omega$, then for each $\iota<\omega$ the set $D_\iota=\{p\in\mathbb{P}:\iota^p\ge\iota\}$ is open dense in $\mathbb{P}$.*

Proof of the Claim. (1) First let us argue that $\mathbb{P}\neq\emptyset$. Let $\iota=i^*$ if it is finite, and $\iota=6$ if $i^*=\omega$. Let $w=\{\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ be any 5 element subset of λ. Using 3.2(2b)+3.6(ii) we may find $v(i,b)$ for $i<\iota$ and $b<2$ such that for some $\ell<\omega$ for all $i<\iota$ and $b<2$ we have

$$v(i,b)\in\mathcal{O}_i,\quad v(i,0)\prec_i v(i,1),\quad\text{ and }\quad \ell\big(v(i,1)\big)=\ell.$$

By 3.6(i), we may also require that if $i^*<6$ then for some $i<\iota$ we have $|v(i,1)|\ge 6$. Fix an enumeration

$$\big\{(\sigma_a,i_a,j_a,k_a):a<A\big\}=\big\{(\sigma,i,j,k):j<k<5\ \wedge\ i<\iota\ \wedge\ \sigma\in v(i,1)\big\}.$$

Choose $n>\ell+5$ and a sequence $\langle\rho_a:a<A+5\rangle\subseteq{}^n2$ so that

- $\langle\rho_a\restriction[\ell,n):a<A+5\rangle$ is linearly independent in ${}^{[\ell,n)}2$ over $\mathbb{Z}_2$, and
- $\sigma_a\triangleleft\rho_a$ for each $a<A$.

Put

- $\eta_{\alpha_b}=\rho_{A+b}$ (for $b<5$) and $\bar\eta=\langle\eta_{\alpha_b}:b<5\rangle$,
- $g_i(\alpha_j,\alpha_k)=g_i(\alpha_k,\alpha_j)=\big\{\rho_a:a<A\ \wedge\ j=j_a\ \wedge\ k=k_a\ \wedge\ i_a=i\big\}$ (for $i<\iota$ and $j<k<5$) and $\bar g=\langle g_i:i<\iota\rangle$.

It follows from Definition 3.6(iii) that $g_i(\alpha_j,\alpha_k)\in\mathcal{O}_i$.

We also let $M=10\cdot\iota$ and we fix a bijection $\varphi:[w]^2\times\iota\longrightarrow M$. Then for $j<k<5$ and $i<\iota$ we set $h_i(\alpha_j,\alpha_k)=h_i(\alpha_k,\alpha_j)=\varphi\big(\{\alpha_j,\alpha_k\},i\big)$. In this way we have defined $\bar{h}=\langle h_i:i<\iota\rangle$.

We put $r_m=n$ for $m<M$ and we let $t_m\subseteq {}^{n\geq}2$ be trees in which all terminal branches are of length n and such that

$$t_m\cap {}^n2=\bigcup\big\{\eta_\alpha+g_i(\alpha,\beta):(\alpha,\beta)\in w^{\langle 2\rangle}\text{ and } i<\iota \text{ and } h_i(\alpha,\beta)=m\big\}.$$

Finally, $\mathcal{M}$ is defined by clause $(*)_9$.

One easily verifies that $(w,n,\iota,M,\bar{\eta},\bar{t},\bar{r},\bar{h},\bar{g},\mathcal{M})\in\mathbb{P}$.

We see from the arguments above that $|\mathbb{P}|\geq\lambda$ and since there are only countably many elements p of $\mathbb{P}$ with $w^p=w$, we get $|\mathbb{P}|=\lambda$.

Clearly, $\leq$ is a partial order on $\mathbb{P}$.

(2) Let $p\in\mathbb{P}$, $\beta\in\lambda\setminus w^p$.

We will define a condition q in a manner similar to the construction in (1) above. Let $\alpha^-=\min(w^p)$ and $\alpha^+=\max(w^p)$.

Set $w^q=w^p\cup\{\beta\}$, $\iota^q=\iota^p$.

For $(\alpha_0,\alpha_1)\in(w^q)^{\langle 2\rangle}$ and $i<\iota^q$ pick $v(i,\alpha_0,\alpha_1)\in\mathcal{O}_i$ so that: for some ℓ, for all $i<\iota^q$ and $(\alpha_0,\alpha_1)\in(w^q)^{\langle 2\rangle}$ we have

- $\ell\big(v(i,\alpha_0,\alpha_1)\big)=\ell$,
- if $\alpha_0,\alpha_1\in w^p$ then $g^p_i(\alpha_0,\alpha_1)\prec_i v(i,\alpha_0,\alpha_1)=v(i,\alpha_1,\alpha_0)$,
- if $\alpha_0\in w^p$ then $g^p_i(\alpha^+,\alpha^-)\prec_i v(i,\alpha_0,\beta)=v(i,\beta,\alpha_0)$.

Fix an enumeration

$$\big\{(\sigma^a,i^a,\alpha^a_0,\alpha^a_1):a<A\big\}=\big\{(\sigma,i,\alpha_0,\alpha_1):\alpha_0<\alpha_1\text{ are from } w^q \text{ and } i<\iota^q\ \wedge\ \sigma\in v(i,\alpha_0,\alpha_1)\big\}.$$

Choose $n>\ell+|w^p|+1$ and a sequence $\langle\rho_a:a\leq A+|w^p|\rangle\subseteq {}^n2$ so that

- $\langle\rho_a{\restriction}[\ell,n):a\leq A+|w^p|\rangle$ is linearly independent in ${}^{[\ell,n)}2$ over $\mathbb{Z}_2$,
- $\sigma^a\vartriangleleft\rho_a$ for each $a<A$, and
- if $\alpha\in w^p$ is such that $|w^p\cap\alpha|=k$ then $\eta^p_\alpha\vartriangleleft\rho_{A+k}$.

Put

- $\eta^q_\beta=\rho_{A+|w^p|}$, and if $\alpha\in w^p$ is such that $|w^p\cap\alpha|=k$ then $\eta^q_\alpha=\rho_{A+k}$ and $\bar{\eta}^q=\langle\eta^q_\alpha:\alpha\in w^q\rangle$,
- $g^q_i(\alpha_0,\alpha_1)=g^q_i(\alpha_1,\alpha_0)=\big\{\rho_a:a<A\wedge i=i^a\wedge\alpha_0=\alpha^a_0\wedge\alpha_1=\alpha^a_1\big\}$ (for $i<\iota^q$ and $\alpha_0<\alpha_1$ from w^q) and $\bar{g}^q=\langle g^q_i:i<\iota^q\rangle$.

It follows from Definition 3.6(iii) that $g^q_i(\alpha_0,\alpha_1)\in\mathcal{O}_i$ and if $(\alpha_0,\alpha_1)\in(w^p)^{\langle 2\rangle}$ then $g^p_i(\alpha_0,\alpha_1)\prec_i g^q_i(\alpha_0,\alpha_1)$.

We also let $M^q=M^p+\iota^q\cdot|w^p|$ and we define mappings $h^q_i:(w^q)^{\langle 2\rangle}\longrightarrow M^q$ so that:

- if $(\alpha_0,\alpha_1)\in(w^p)^{\langle 2\rangle}$ and $i<\iota^q$, then $h^q_i(\alpha_0,\alpha_1)=h^p_i(\alpha_0,\alpha_1)$,
- if $\alpha\in w^p$ and $i<\iota^q$, then $h^q_i(\alpha,\beta)=h^q_i(\beta,\alpha)=M^p+|\alpha\cap w^p|\cdot\iota+i$.

In this way we have defined $\bar{h}^q=\langle h^q_i:i<\iota^q\rangle$.

We put $r^q_m=r^p_m$ for $m<M^p$ and $r^q_m=n$ for $M^p\leq m<M^q$. We let $t^q_m\subseteq {}^{n\geq}2$ be trees in which all terminal branches are of length n and such

that

$t^q_m \cap {}^n 2 = \bigcup \big\{\eta^q_\alpha + g^q_i(\alpha,\beta) : (\alpha,\beta) \in (w^q)^{\langle 2\rangle} \text{ and } i < \iota^q \text{ and } h^q_i(\alpha,\beta) = m\big\}$.

[Note that by our definitions above and by clause $(*)_7$ for p we have $t^p_m \cap {}^{n^p}2 = t^q_m \cap {}^{n^p}2$ for all $m < M^p$.] Naturally we also set $n^q = n$ and we define $\mathcal{M}^q$ by clause $(*)_9$.

We claim that $q = \big(w^q, n^q, \iota^q, M^q, \bar\eta^q, \bar t^q, \bar r^q, \bar h^q, \bar g^q, \mathcal{M}^q\big) \in \mathbb{P}$. Demands $(*)_1$–$(*)_9$ are pretty straightforward.

RE $(*)_{10}$ **:** To justify clause $(*)_{10}$, suppose that $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^q$, $\ell^{\mathfrak{d}_0} = \ell^{\mathfrak{d}_1} = \ell$, $\rho \in {}^\ell 2$ and $\mathbf{m} = \mathbf{m}^{\mathfrak{d}_0} = \mathbf{m}^{\mathfrak{d}_1} + \rho$, and consider the following two cases.

Case 1: $\beta \notin v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}$

If $\ell \le n^p$ then $r_{h^{\mathbf{m}}_i(\eta_{\alpha_0}\restriction\ell, \eta_{\alpha_1}\restriction\ell)} \le n^p$, so $h^{\mathbf{m}}_i(\eta_{\alpha_0}\restriction\ell, \eta_{\alpha_1}\restriction\ell) < M^p$ for all $(\alpha_0,\alpha_1) \in \big(v^{\mathfrak{d}_0}\big)^{\langle 2\rangle}$. Hence also $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^p$ and clause $(*)_{10}$ for p applies. If $\ell > n^p$ then the sequence $\langle \eta^q_\alpha\restriction\ell : \alpha \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}\rangle$ is linearly independent and

$$\{(\eta^q_\alpha\restriction\ell) + \rho : \alpha \in v^{\mathfrak{d}_0}\} = \{\eta^q_\alpha\restriction\ell : \alpha \in v^{\mathfrak{d}_1}\}.$$

Since $|v^{\mathfrak{d}_0}| \ge 5$ we immediately conclude $\rho = \mathbf{0}$, and therefore also $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$ (remember $\ell > n^p$).

Case 2: $\beta \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}$

Say, $\beta \in v^{\mathfrak{d}_0}$. If $\alpha \in v^{\mathfrak{d}_0} \setminus \{\beta\}$, then $h^q_j(\alpha,\beta) \ge M^p$ for all $j < \iota$, and hence $r^q_{h^{\mathbf{m}}_i(\eta_\alpha\restriction\ell, \eta_\beta\restriction\ell)} = n^q$ (remember $(*)^{\rm d}_9$). Consequently, $\ell = n^q$. Since the sequence $\langle \eta^q_\alpha : \alpha \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}\rangle$ is linearly independent, like before we get $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$.

RE $(*)_{11}$ **:** Assume towards contradiction that for some $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^q$ we have:

- $v^{\mathfrak{d}}_0 \subseteq v^{\mathfrak{d}}_1$ and without loss of generality $|v^{\mathfrak{d}_1}| = |v^{\mathfrak{d}_0}| + 1$,
- $\alpha_0 \in v^{\mathfrak{d}_0}$, $|\alpha_0 \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0})$, $\mathrm{rk}(v^{\mathfrak{d}_0}) = -1$, and $\mathbf{m}^{\mathfrak{d}_0} \sqsubset \mathbf{m}^{\mathfrak{d}_1}$, and
- there is $\alpha_1 \in v^{\mathfrak{d}_1}$ such that $\eta^q_{\alpha_0}\restriction\ell^{\mathfrak{d}_0} = \eta^q_{\alpha_1}\restriction\ell^{\mathfrak{d}_0}$ but $\eta^q_{\alpha_0}\restriction\ell^{\mathfrak{d}_1} \ne \eta^q_{\alpha_1}\restriction\ell^{\mathfrak{d}_1}$.

Let $\ell_0 = \ell^{\mathfrak{d}_0}$, $\ell_1 = \ell^{\mathfrak{d}_1}$.

Suppose $\beta \in v^{\mathfrak{d}_0}$ and take $\beta' \in v^{\mathfrak{d}_0} \setminus \{\beta\}$. Then $h^q_j(\beta,\beta') \ge M^p$ for all $j < \iota$. Hence, for some $j < \iota$,

$$r^q_{h^{\mathbf{m}^{\mathfrak{d}_0}}_0(\eta_\beta\restriction\ell_0, \eta_{\beta'}\restriction\ell_0)} = r^q_{h^q_j(\beta,\beta')} = n^q = \ell_0 = \ell_1,$$

contradicting the last item in our assumptions.

If we had $v^{\mathfrak{d}_1} = v^{\mathfrak{d}_0} \cup \{\beta\}$, then considering a $\beta' \in v^{\mathfrak{d}_0} \setminus \{\alpha_0\}$ would give us

$$M^p > h^{\mathbf{m}^{\mathfrak{d}_0}}_0(\eta_{\alpha_0}\restriction\ell_0, \eta_{\beta'}\restriction\ell_0) = h^{\mathbf{m}^{\mathfrak{d}_1}}_0(\eta_\beta\restriction\ell_1, \eta_{\beta'}\restriction\ell_1) \ge M^p,$$

a contradiction.

Therefore the only remaining possibility is that $\beta \notin v^{\mathfrak{d}_1}$.

If $\ell_1 \le n^p$, then $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^p$ and clause $(*)_{11}$ for p gives us a contradiction. So assume $\ell_1 > n^p$. Since $\{\eta^q_\gamma\restriction n^p : \gamma \in v^{\mathfrak{d}_1}\}$ are all pairwise distinct, we conclude $\ell_0 < n^p$ and $\mathbf{m}^{\mathfrak{d}_0} \in \mathcal{M}^p$. We define $\mathbf{n} \in \mathbf{M}^n_{\bar t, \bar{\mathcal{O}}^6}$ by setting:

- $\ell^{\mathbf{n}} = n^p$, $u^{\mathbf{n}} = \{\eta^q_\gamma\restriction n^p : \gamma \in v^{\mathfrak{d}_1}\} = \{\eta^p_\gamma : \gamma \in v^{\mathfrak{d}_1}\}$, $\iota^{\mathbf{n}} = 6$,
  and for $(\gamma,\gamma') \in (v^{\mathfrak{d}_1})^{\langle 2\rangle}$ and $i < 6$:

- if $\{\gamma,\gamma'\}\neq\{\alpha_0,\alpha_1\}$, then
$$g^{\mathbf{n}}_i(\eta^p_\gamma,\eta^p_{\gamma'})=\{\sigma{\restriction}n^p\colon\sigma\in g^{\mathbf{m}^{\mathfrak{d}_1}}_i(\eta^q_\gamma{\restriction}\ell_1,\eta^q_{\gamma'}{\restriction}\ell_1)\}$$
and $h^{\mathbf{n}}_i(\eta^p_\gamma,\eta^p_{\gamma'})=h^{\mathbf{m}^{\mathfrak{d}_1}}_i(\eta^q_\gamma{\restriction}\ell_1,\eta^q_{\gamma'}{\restriction}\ell_1)$,
- if $\{\gamma,\gamma'\}=\{\alpha_0,\alpha_1\}$, then we fix distinct $\sigma_0,\ldots,\sigma_5\in\bigcup\limits_{j<\iota^q}g^p_j(\alpha_0,\alpha_1)$ (remember $(*)_6$ for p), and we let $g^{\mathbf{n}}_i(\eta^p_{\alpha_0},\eta^p_{\alpha_1})=g^{\mathbf{n}}_i(\eta^p_{\alpha_1},\eta^p_{\alpha_0})=\{\sigma_i\}$ and $h^{\mathbf{n}}_i(\eta^p_{\alpha_0},\eta^p_{\alpha_1})=h^{\mathbf{n}}_i(\eta^p_{\alpha_1},\eta^p_{\alpha_0})=m$ where $\eta^p_{\alpha_0}+\sigma_i,\eta_{\alpha_1}+\sigma_i\in t^p_m$ (for $i<6$).

Since $\mathbf{m}^{\mathfrak{d}_0}\sqsubset\mathbf{m}^{\mathfrak{d}_1}$, in the case when $\{\gamma,\gamma'\}\neq\{\alpha_0,\alpha_1\}$ we have
$$g^{\mathbf{m}^{\mathfrak{d}_0}}_i(\eta^p_\gamma{\restriction}\ell_0,\eta^p_{\gamma'}{\restriction}\ell_0)\prec_{\mathcal{O}^0}g^{\mathbf{m}^{\mathfrak{d}_1}}_i(\eta^p_\gamma{\restriction}\ell_1,\eta^p_{\gamma'}{\restriction}\ell_1)$$
and hence $g^{\mathbf{n}}_i(\eta^p_\gamma,\eta^p_{\gamma'})\cap g^{\mathbf{n}}_j(\eta^p_\gamma,\eta^p_{\gamma'})=\emptyset$ whenever $i<j<6$. Hence 3.10(c) is satisfied. Other cases and other conditions of 3.10 follow immediately by our choices, and hence
$$\mathbf{n}=(n^p,6,u^{\mathbf{n}},\bar{h}^{\mathbf{n}},\bar{g}^{\mathbf{n}})\in\mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}.$$
Moreover, $\mathbf{m}^{\mathfrak{d}_0}\sqsubset\mathbf{n}$ and $\mathfrak{d}_*=(n^p,v^{\mathfrak{d}_1},\mathbf{n})\in\mathcal{M}^p$. However, then $\mathfrak{d}_0,\mathfrak{d}_*$ contradict clause $(*)_{11}$ for p.

(3) Let $p\in\mathbb{P}$. Set $w^q=w^p$ and $\iota^q=\iota^p+1$. For $(\alpha_0,\alpha_1)\in(w^q)^{\langle 2\rangle}$ and $i<\iota^q$ we use Proposition 3.8 to pick $v(i,\alpha_0,\alpha_1)\in\mathcal{O}_i$ so that: for some ℓ, for all $i<\iota^q$ and $(\alpha_0,\alpha_1)\in(w^q)^{\langle 2\rangle}$ we have

- $\ell\big(v(i,\alpha_0,\alpha_1)\big)=\ell$,
- if $i<\iota^p$ then $g^p_i(\alpha_0,\alpha_1)\prec_i v(i,\alpha_0,\alpha_1)=v(i,\alpha_1,\alpha_0)$,
- for some $v\in\mathcal{O}_{\iota^p}$, $v\prec_{\iota^p}v(\iota^p,\alpha_0,\alpha_1)=v(\iota^p,\alpha_1,\alpha_0)$.

Fix an enumeration
$$\big\{(\sigma^a,i^a,\alpha^a_0,\alpha^a_1):a<A\big\}=\big\{(\sigma,i,\alpha_0,\alpha_1):\alpha_0<\alpha_1\text{ are from }w\text{ and }i<\iota^q\ \wedge\ \sigma\in v(i,\alpha_0,\alpha_1)\big\}.$$
Choose $n=n^q>\ell$ and a sequence $\langle\rho_a:a<A+|w^p|\rangle\subseteq{}^n2$ so that

- $\langle\rho_a{\restriction}[\ell,n):a<A+|w^q|\rangle$ is linearly independent in ${}^{[\ell,n)}2$ over $\mathbb{Z}_2$, and
- $\sigma^a\vartriangleleft\rho_a$ for each $a<A$, and
- if $\alpha\in w^q$ is such that $|w^q\cap\alpha|=k$ then $\eta^p_\alpha\vartriangleleft\rho_{A+k}$.

Put

- if $\alpha\in w^q$ is such that $|w^q\cap\alpha|=k$ then $\eta^q_\alpha=\rho_{A+k}$ and $\bar{\eta}^q=\langle\eta^q_\alpha:\alpha\in w^q\rangle$,
- $g^q_i(\alpha_0,\alpha_1)=g^q_i(\alpha_1,\alpha_0)=\big\{\rho_a:a<A\wedge i=i^a\wedge\alpha_0=\alpha^a_0\wedge\alpha_1=\alpha^a_1\big\}$ (for $i<\iota^q$ and $\alpha_0<\alpha_1$ from w^q) and $\bar{g}^q=\langle g^q_i:i<\iota^q\rangle$.

It follows from Definition 3.6(iii) that $g^q_i(\alpha_0,\alpha_1)\in\mathcal{O}_i$ and if $(\alpha_0,\alpha_1)\in(w^p)^{\langle 2\rangle}$ then $g^p_i(\alpha_0,\alpha_1)\prec_i q^q_i(\alpha_0,\alpha_1)$.

We also let $M^q=M^p+|[w^q]^2|$ and we fix a bijection $\psi:[w^q]^2\longrightarrow[M^p,M^q)$. Then we define mappings $h^q_i:(w^q)^{\langle 2\rangle}\longrightarrow M^q$ so that for $\alpha_0<\alpha_1$ from w^q we have

- if $i<\iota^q$, then $h^q_i(\alpha_0,\alpha_1)=h^q_i(\alpha_1,\alpha_0)=h^p_i(\alpha_0,\alpha_1)$,

- $h^q_{\iota^p}(\alpha_0,\alpha_1) = h^q_{\iota^p}(\alpha_1,\alpha_0) = \psi(\{\alpha_0,\alpha_1\})$.

This way we defined $\bar{h}^q = \langle h^q_i : i < \iota^q\rangle$.

We put $r^q_m = r^p_m$ for $m < M^p$ and $r^q_m = n$ for $M^p \le m < M^q$. We let $t^q_m \subseteq {}^{n\ge}2$ be trees in which all terminal branches are of length n and such that

$$t^q_m \cap {}^n2 = \bigcup \big\{\eta^q_\alpha + g^q_i(\alpha,\beta) : (\alpha,\beta) \in (w^q)^{\langle 2\rangle} \text{ and } i < \iota^q \text{ and } h^q_i(\alpha,\beta) = m\big\}.$$

[Note that by our definitions above and by clause $(*)_7$ for p we have $t^p_m \cap {}^{n^p}2 = t^q_m \cap {}^{n^p}2$ for all $m < M^p$.] We define $\mathcal{M}^q$ by clause $(*)_9$. Like previously, one easily verifies that $q = \big(w^q, n^q, \iota^q, M^q, \bar\eta^q, \bar t^q, \bar r^q, \bar h^q, \bar g^q, \mathcal{M}^q\big) \in \mathbb{P}$. [The crucial point is that if $\mathfrak{d} \in \mathcal{M}^q$, $\eta,\nu \in u^{\mathbf{m}^{\mathfrak{d}}}$ and $h^{\mathbf{m}^{\mathfrak{d}}}_i(\eta,\nu) \ge M^p$, then $\ell^{\mathfrak{d}} = n^q$.] □

Claim 4.4.2. *The forcing notion $\mathbb{P}$ has the Knaster property.*

Proof of the Claim. Suppose that $\langle p_\xi : \xi < \omega_1\rangle$ is a sequence of pairwise distinct conditions from $\mathbb{P}$ and let

$$p_\xi = \big(w_\xi, n_\xi, \iota_\xi, M_\xi, \bar\eta_\xi, \bar t_\xi, \bar r_\xi, \bar h_\xi, \bar g_\xi, \mathcal{M}_\xi\big)$$

where $\bar\eta_\xi = \langle \eta^\xi_\alpha : \alpha \in w_\xi\rangle$, $\bar t_\xi = \langle t^\xi_m : m < M_\xi\rangle$, $\bar r_\xi = \langle r^\xi_m : m < M_\xi\rangle$, and $\bar h_\xi = \langle h^\xi_i : i < \iota_\xi\rangle$, $\bar g_\xi = \langle g^\xi_i : i < \iota_\xi\rangle$. By a standard Δ–system cleaning procedure we may find an uncountable set $A \subseteq \omega_1$ such that the following demands $(*)_{12}$–$(*)_{15}$ are satisfied.

$(*)_{12}$ $\{w_\xi : \xi \in A\}$ forms a Δ–system with the kernel w^*.

$(*)_{13}$ If $\xi,\varsigma \in A$, then $|w_\xi| = |w_\varsigma|$, $n_\xi = n_\varsigma$, $\iota_\xi = \iota_\varsigma$, $M_\xi = M_\varsigma$, and $t^\xi_m = t^\varsigma_m$ and $r^\xi_m = r^\varsigma_m$ (for $m < M_\xi$).

$(*)_{14}$ If $\xi < \varsigma$ are from A and $\pi : w_\xi \longrightarrow w_\varsigma$ is the order isomorphism, then
  (a) $\pi(\alpha) = \alpha$ for $\alpha \in w^* = w_\xi \cap w_\varsigma$,
  (b) if $\emptyset \ne v \subseteq w_\xi$, then $\mathrm{rk}(v) = \mathrm{rk}(\pi[v])$, $\zeta(v) = \zeta(\pi[v])$ and $k(v) = k(\pi[v])$,
  (c) $\eta^\xi_\alpha = \eta^\varsigma_{\pi(\alpha)}$ (for $\alpha \in w_\xi$),
  (d) $g^\xi_i(\alpha,\beta) = g^\zeta_i(\pi(\alpha),\pi(\beta))$ and $h^\xi_i(\alpha,\beta) = h^\zeta_i(\pi(\alpha),\pi(\beta))$ for $(\alpha,\beta) \in (w_\xi)^{\langle 2\rangle}$ and $i < \iota_\xi$, and

$(*)_{15}$ $\mathcal{M}_\xi = \mathcal{M}_\varsigma$ (this actually follows from the previous demands).

Note that then also

$(*)_{16}$ if $\xi \in A$, $v \subseteq w^*$ and $\delta \in w_\xi \setminus w^*$ are such that $\mathrm{rk}\big(v \cup \{\delta\}\big) = -1$, then $k\big(v \cup \{\delta\}\big) \ne |\delta \cap v|$.

[Why? Suppose $\mathrm{rk}\big(v \cup \{\delta\}\big) = -1$ and $k = k\big(v \cup \{\delta\}\big) = |\delta \cap v|$, $j = j\big(v \cup \{\delta\}\big)$. For $\varsigma \in A$ let $\pi_\varsigma : w_\xi \longrightarrow w_\varsigma$ be the order isomorphism and let $\delta_\varsigma = \pi_\varsigma(\delta)$. By $(*)_{14}$ we know that $k = k\big(v \cup \{\delta_\varsigma\}\big) = |\delta_\varsigma \cap v|$ and $j = j\big(v \cup \{\delta_\varsigma\}\big)$. Therefore, letting $v \cup \{\delta\} = \{a_0,\ldots,a_{n-1}\}$ be the increasing enumeration, for every $\varsigma \in A$ we have $\mathbb{M} \models R_{n,j}[a_0,\ldots,a_{k-1},\delta_\varsigma,a_{k+1},\ldots,a_{n-1}]$. Hence the set

$$\{b < \lambda : \mathbb{M} \models R_{n,j}[a_0,\ldots,a_{k_1},b,a_{k+1},\ldots,a_{n-1}]\}$$

is uncountable, contradicting $(\circledast)_{\rm e}$ from the beginning of the proof of the theorem.]

We will show that for distinct ξ,ς from A the conditions p_ξ,p_ς are compatible. So let $\xi,\varsigma\in A$, $\xi<\varsigma$ and let $\pi:w_\xi\longrightarrow w_\varsigma$ be the order isomorphism. We will define $q=\big(w,n,\iota,M,\bar\eta,\bar t,\bar r,\bar h,\bar g,\mathcal{M}\big)$ where $\bar\eta=\langle\eta_\alpha:\alpha\in w\rangle$, $\bar t=\langle t_m:m<M\rangle$, $\bar r=\langle r_m:m<M\rangle$, and $\bar h=\langle h_i:i<\iota\rangle$, $\bar g=\langle g_i:i<\iota\rangle$.

We set

$(*)_{17}$ $\iota=\iota_\xi$ and $w=w_\xi\cup w_\varsigma$.

Similarly to the arguments in previous claims, we first pick

$$\langle v(i,\alpha_0,\alpha_1):(\alpha_0,\alpha_1)\in w^{\langle 2\rangle}\ \wedge\ i<\iota\rangle$$

and an ℓ such that for all $i<\iota$ and $(\alpha_0,\alpha_1)\in w^{\langle 2\rangle}$ we have

- $v(i,\alpha_0,\alpha_1)=v(i,\alpha_1,\alpha_0)\in\mathcal{O}_i$, $\ell\big(v(i,\alpha_0,\alpha_1)\big)=\ell$,
- if $\alpha_0,\alpha_1\in w_\xi$ then $g^\xi_i(\alpha_0,\alpha_1)\prec_i v(i,\alpha_0,\alpha_1)=v(i,\alpha_1,\alpha_0)$, and
- if $\alpha_0,\alpha_1\in w_\varsigma$ then $g^\varsigma_i(\alpha_0,\alpha_1)\prec_i v(i,\alpha_0,\alpha_1)=v(i,\alpha_1,\alpha_0)$.

Then we fix an enumeration

$$\big\{(\sigma^a,i^a,\alpha^a_0,\alpha^a_1):a<A\big\}=\big\{(\sigma,i,\alpha_0,\alpha_1):\alpha_0<\alpha_1\text{ are from }w\text{ and }\\ i<\iota\ \wedge\ \sigma\in v(i,\alpha_0,\alpha_1)\big\}$$

and we choose $n>\ell$ and $\langle\rho_a:a<A+|w|\rangle\subseteq{}^n2$ so that

- $\langle\rho_a{\restriction}[\ell,n):a<A+|w|\rangle$ is linearly independent in ${}^{[\ell,n)}2$ over $\mathbb{Z}_2$, and
- $\sigma^a\vartriangleleft\rho_a$ for each $a<A$, and
- if $\alpha\in w_\xi$ is such that $|w_\xi\cap\alpha|=k$ then $\eta^\xi_\alpha\vartriangleleft\rho_{A+k}$,
- if $\alpha\in w_\varsigma\setminus w_\xi$ is such that $|(w_\varsigma\setminus w_\xi)\cap\alpha|=k$ then $\eta^\varsigma_\alpha\vartriangleleft\rho_{A+|w_\xi|+k}$.

Put

$(*)_{18}$ n is the one chosen right above,

$(*)_{19}$ $\bar\eta=\langle\eta_\alpha:\alpha\in w\rangle$, where
- if $\alpha\in w_\xi$ is such that $|w_\xi\cap\alpha|=k$ then $\eta_\alpha=\rho_{A+k}$,
- if $\alpha\in w_\varsigma\setminus w_\xi$ is such that $|(w_\varsigma\setminus w_\xi)\cap\alpha|=k$ then $\eta_\alpha=\rho_{A+|w_\xi|+k}$,

$(*)_{20}$ $\bar g=\langle g_i:i<\iota\rangle$, where for $i<\iota$ and $\alpha_0<\alpha_1$ from w we put

$$g_i(\alpha_0,\alpha_1)=g_i(\alpha_1,\alpha_0)=\big\{\rho_a:a<A\ \wedge\ i=i^a\ \wedge\ \alpha_0=\alpha^a_0\ \wedge\ \alpha_1=\alpha^a_1\big\}.$$

As before, by 3.6(iii), we know that $g_i(\alpha_0,\alpha_1)\in\mathcal{O}_i$ and if $(\alpha_0,\alpha_1)\in(w_\xi)^{\langle 2\rangle}$ then $g^\xi_i(\alpha_0,\alpha_1)\prec_i g_i(\alpha_0,\alpha_1)$ and similarly for ς in place of ξ.

Let

$(*)_{21}$ $M=M_\xi+|w_\xi\setminus w_\varsigma|^2$

and let $\psi:(w_\xi\setminus w_\varsigma)\times(w_\varsigma\setminus w_\xi)\longrightarrow[M_\xi,M)$ be a bijection. Then we define

$(*)_{22}$ $\bar h=\langle h_i:i<\iota\rangle$, where mappings $h_i:w^{\langle 2\rangle}\longrightarrow M$ are such that for distinct $\alpha_0,\alpha_1\in w$ and $i<\iota$ we have
- $h_i(\alpha_0,\alpha_1)=h_i(\alpha_1,\alpha_0)$,
- if $\alpha_0,\alpha_1\in w_\xi$, then $h_i(\alpha_1,\alpha_0)=h^\xi_i(\alpha_1,\alpha_0)$,
- if $\alpha_0,\alpha_1\in w_\varsigma$, then $h_i(\alpha_1,\alpha_0)=h^\varsigma_i(\alpha_1,\alpha_0)$,
- if $\alpha_0\in w_\xi\setminus w_\varsigma$ and $\alpha_1\in w_\varsigma\setminus w_\xi$, then $h_i(\alpha_1,\alpha_0)=\psi(\alpha_0,\alpha_1)$.

$(*)_{23}$ $\bar{t} = \langle t_m : m < M \rangle$, where $t_m \subseteq {}^{n \geq}2$ are trees in which all terminal branches are of length n (see $(*)_{18}$) and such that

$$t_m \cap {}^n 2 = \bigcup \big\{ \eta_\alpha + g_i(\alpha,\beta) : (\alpha,\beta) \in w^{\langle 2 \rangle} \text{ and } i < \iota \text{ and } h_i(\alpha,\beta) = m \big\},$$

$(*)_{24}$ $\bar{r} = \langle r_m : m < M \rangle$, where $r_m = r^\xi_m$ for $m < M_\xi$, $r_m = n$ if $M_\xi \leq m < M$.

$(*)_{25}$ $\mathcal{M}$ is defined by $(*)_9$ (for the objects introduced in $(*)_{17}$–$(*)_{24}$).

In clauses $(*)_{17}$–$(*)_{25}$ we defined all the ingredients of

$$q = \big(w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M} \big).$$

We still need to argue that $q \in \mathbb{P}$ (after this it will be obvious that it is a condition stronger than both p_ξ and p_ς).

It is pretty straightforward that q satisfies demands $(*)_1$–$(*)_9$.

RE $(*)_{10}$: To justify clause $(*)_{10}$, suppose that $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}$, $\ell^{\mathfrak{d}_0} = \ell^{\mathfrak{d}_1} = \ell$ and $\rho \in {}^\ell 2$ and $\mathbf{m}^{\mathfrak{d}_1} = \mathbf{m}^{\mathfrak{d}_0} + \rho$, and consider the following three cases.

Case 1: $v^{\mathfrak{d}_0} \subseteq w_\xi$

Then for each $(\delta,\varepsilon) \in (v^{\mathfrak{d}_0})^{\langle 2 \rangle}$ and $i < \iota$ we have $h_i(\delta,\varepsilon) < M_\xi$, and consequently $\operatorname{rng}\big(h^{\mathbf{m}^{\mathfrak{d}_0}}_j\big) \subseteq M_\xi$ (for $j < 6$). Hence also $\operatorname{rng}\big(h^{\mathbf{m}^{\mathfrak{d}_1}}_j\big) \subseteq M_\xi$ (for $j < 6$). But looking at $(*)_{22}$ (and remembering $(*)_9^{\rm d}$) we now conclude $h_i(\delta,\varepsilon) < M_\xi$ for $(\delta,\varepsilon) \in (v^{\mathfrak{d}_1})^{\langle 2 \rangle}$ and $i < \iota$. Consequently, either $v^{\mathfrak{d}_1} \subseteq w_\xi$ or $v^{\mathfrak{d}_1} \subseteq w_\varsigma$.

If $v^{\mathfrak{d}_1} \subseteq w_\xi$ and $\ell \leq n_\xi$, then $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}_\xi$ and clause $(*)_{10}$ for p_ξ can be used to get the desired conclusion.

If $v^{\mathfrak{d}_1} \subseteq w_\xi$ and $\ell > n_\xi$, then $\{\eta_\alpha {\restriction} \ell : \alpha \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}\}$ is linearly independent and hence $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$.

If $v^{\mathfrak{d}_1} \subseteq w_\varsigma$ and $\ell \leq n_\xi$, then consider $v = \pi^{-1}[v^{\mathfrak{d}_1}] \subseteq w_\xi$ and $\mathfrak{d} = (\ell, v, \mathbf{m}^{\mathfrak{d}_1})$. Clearly, $\mathfrak{d} \in \mathcal{M}_\xi$ and we may use $(*)_{10}$ for p_ξ to conclude that $\operatorname{rk}(v) = \operatorname{rk}(v^{\mathfrak{d}_0})$, $\zeta(v) = \zeta(v^{\mathfrak{d}_0})$, $k(v) = k(v^{\mathfrak{d}_0})$, and if $\alpha \in v^{\mathfrak{d}_0}$, $\beta \in v$ are such that $|\alpha \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0}) = k(v) = |\beta \cap v|$, then $(\eta_\alpha {\restriction} \ell) + \rho = \eta_\beta {\restriction} \ell$. Now we use the properties $(*)_{14}$(b,c) of π to get a similar assertions with $v^{\mathfrak{d}_1}$ in place of v.

If $v^{\mathfrak{d}_1} \subseteq w_\varsigma$ and $\ell > n_\xi$, then we consider $v = \pi^{-1}[v^{\mathfrak{d}_1}] \subseteq w_\xi$ and use the linear independence of $\{\eta_\alpha {\restriction} \ell : \alpha \in v^{\mathfrak{d}_0} \cup v\}$ to conclude that $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v = \pi^{-1}\big[v^{\mathfrak{d}_1}\big]$. Finally we use the properties $(*)_{14}$(b,c) of π to get the desired assertions.

Case 2: $v^{\mathfrak{d}_0} \subseteq w_\varsigma$

Same as the previous case, just interchanging ξ and ς.

Case 3: $v^{\mathfrak{d}_0} \setminus w_\xi \neq \emptyset \neq v^{\mathfrak{d}_0} \setminus w_\varsigma$

Then for some $(\delta,\varepsilon) \in (v^{\mathfrak{d}_0})^{\langle 2 \rangle}$ we have $h_i(\delta,\varepsilon) \geq M_\xi$ for all $i < \iota$, so necessarily $\ell = n$. Now, the linear independence of $\bar{\eta}$ implies $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$ and the desired conclusion follows.

RE $(*)_{11}$: Let us prove clause $(*)_{11}$ now. Suppose that $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}$, $\delta \in v^{\mathfrak{d}_0}$, $|\delta \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0})$, $\operatorname{rk}(v^{\mathfrak{d}_0}) = -1$, and $v^{\mathfrak{d}_0} \subseteq v^{\mathfrak{d}_1}$ and $\mathbf{m}^{\mathfrak{d}_0} \sqsubset \mathbf{m}^{\mathfrak{d}_1}$. Assume towards contradiction that there is an $\varepsilon \in v^{\mathfrak{d}_1}$ such that

$(*)_{26}$ $\eta_\varepsilon{\restriction}\ell^{\mathfrak{d}_1}\neq\eta_\delta{\restriction}\ell^{\mathfrak{d}_1}$ but $\eta_\varepsilon{\restriction}\ell^{\mathfrak{d}_0}=\eta_\delta{\restriction}\ell^{\mathfrak{d}_0}$.

Without loss of generality $v^{\mathfrak{d}_1}=v^{\mathfrak{d}_0}\cup\{\varepsilon\}$. Since we must have $\ell^{\mathfrak{d}_0}<n$, for no $\alpha,\beta\in v^{\mathfrak{d}_0}$ can we have $(\forall i<\iota)(h_i(\alpha,\beta)\geq M_\xi)$. Therefore either $v^{\mathfrak{d}_0}\subseteq w_\xi$ or $v^{\mathfrak{d}_0}\subseteq w_\varsigma$. By the symmetry, we may assume $v^{\mathfrak{d}_0}\subseteq w_\xi$. Note that

$(*)_{27}$ if $(\alpha,\beta)\in(v^{\mathfrak{d}_1})^{\langle 2\rangle}\setminus\{(\varepsilon,\delta),(\delta,\varepsilon)\}$ then $h_i(\alpha,\beta)<M_\xi$ for all $i<\iota$.

Now, if $v^{\mathfrak{d}_1}\subseteq w_\xi$ and $\ell^{\mathfrak{d}_1}\leq n_\xi$, then $\mathfrak{d}_0,\mathfrak{d}_1\in\mathcal{M}_\xi$ and they contradict clause $(*)_{11}$ for p_ξ. Let us consider the possibility that $v^{\mathfrak{d}_1}\subseteq w_\xi$ but $\ell^{\mathfrak{d}_1}>n_\xi$. Since $\eta_\varepsilon{\restriction}\ell^{\mathfrak{d}_0}=\eta_\delta{\restriction}\ell^{\mathfrak{d}_0}$ but (as $\varepsilon,\delta\in w_\xi$) $\eta_\varepsilon{\restriction}n_\xi\neq\eta_\delta{\restriction}n_\xi$, we also have $\ell^{\mathfrak{d}_0}<n_\xi$. Define $\mathbf{n}\in\mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ by:

- $\ell^{\mathbf{n}}=n_\xi$, $u^{\mathbf{n}}=\{\eta_\gamma{\restriction}n_\xi:\gamma\in v^{\mathfrak{d}_1}\}$ (note $\eta_\varepsilon{\restriction}n_\xi\neq\eta_\delta{\restriction}n_\xi$), $\iota^{\mathbf{n}}=6$, and for $(\gamma,\gamma')\in(v^{\mathfrak{d}_1})^{\langle 2\rangle}$ and $i<6$:
- if $\{\gamma,\gamma'\}\neq\{\varepsilon,\delta\}$, then
$$g_i^{\mathbf{n}}(\eta_\gamma{\restriction}n_\xi,\eta_{\gamma'}{\restriction}n_\xi)=\{\sigma{\restriction}n_\xi:\sigma\in g_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma{\restriction}\ell^{\mathfrak{d}_1},\eta_{\gamma'}{\restriction}\ell^{\mathfrak{d}_1})\}$$
and $h_i^{\mathbf{n}}(\eta_\gamma{\restriction}n_\xi,\eta_{\gamma'}{\restriction}n_\xi)=h_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma{\restriction}\ell^{\mathfrak{d}_1},\eta_{\gamma'}{\restriction}\ell^{\mathfrak{d}_1})$, and
- for $\{\gamma,\gamma'\}=\{\delta,\varepsilon\}$ we fix any distinct $\sigma_0,\ldots,\sigma_5\in\bigcup\limits_{j<\iota}g_j^\xi(\delta,\varepsilon)$ and we let $g_i^{\mathbf{n}}(\eta_\delta{\restriction}n_\xi,\eta_\varepsilon{\restriction}n_\xi)=g_i^{\mathbf{n}}(\eta_\varepsilon{\restriction}n_\xi,\eta_\delta{\restriction}n_\xi)=\{\sigma_i\}$ and $h_i^{\mathbf{n}}(\eta_\delta{\restriction}n_\xi,\eta_\varepsilon{\restriction}n_\xi)=h_i^{\mathbf{n}}(\eta_\varepsilon{\restriction}n_\xi,\eta_\delta{\restriction}n_\xi)=m$ where $(\eta_\delta{\restriction}n_\xi)+\sigma_i,(\eta_\varepsilon{\restriction}n_\xi)+\sigma_i\in t^\xi_m$ (for $i<6$).

Since $\mathbf{m}^{\mathfrak{d}_0}\sqsubset\mathbf{m}^{\mathfrak{d}_1}$, in the case when $\{\gamma,\gamma'\}\neq\{\delta,\varepsilon\}$ we have
$$g_i^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\gamma{\restriction}\ell^{\mathfrak{d}_0},\eta_{\gamma'}{\restriction}\ell^{\mathfrak{d}_0})\prec_{\mathcal{O}^0}g_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma{\restriction}\ell^{\mathfrak{d}_1},\eta_{\gamma'}{\restriction}\ell^{\mathfrak{d}_1}),$$
and hence $g_i^{\mathbf{n}}(\eta_\gamma{\restriction}n_\xi,\eta_{\gamma'}{\restriction}n_\xi)\cap g_j^{\mathbf{n}}(\eta_\gamma{\restriction}n_\xi,\eta_{\gamma'}{\restriction}n_\xi)=\emptyset$ whenever $i<j<6$. Hence 3.10(c) is satisfied. Other cases and other conditions of 3.10 follow immediately by our choices, and hence
$$\mathbf{n}=(\ell^{\mathbf{n}},6,u^{\mathbf{n}},\bar{h}^{\mathbf{n}},\bar{g}^{\mathbf{n}})\in\mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}.$$
Moreover, $\mathbf{m}^{\mathfrak{d}_0}\sqsubset\mathbf{n}$ and $\mathfrak{d}_*=(n_\xi,v^{\mathfrak{d}_1},\mathbf{n})\in\mathcal{M}_\xi$. However, then $\mathfrak{d}_0,\mathfrak{d}_*$ contradict clause $(*)_{11}$ for p_ξ.

Consequently, $v^{\mathfrak{d}_1}\setminus w_\xi\neq\emptyset$, so necessarily $\varepsilon\notin w^*$.

Suppose $|v^{\mathfrak{d}_0}\setminus w^*|\geq 2$, say $\alpha_0,\alpha_1\in v^{\mathfrak{d}_0}\setminus w^*$. Then $h_i(\varepsilon,\alpha_0),h_i(\varepsilon,\alpha_1)\geq M_\xi$ for all $i<\iota$. But $\mathbf{m}^{\mathfrak{d}_0}\sqsubset\mathbf{m}^{\mathfrak{d}_1}$ implies that for $\alpha\in v^{\mathfrak{d}_0}\setminus\{\delta\}$ we have
$$h_0^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\varepsilon{\restriction}\ell^{\mathfrak{d}_1},\eta_\alpha{\restriction}\ell^{\mathfrak{d}_1})=h_0^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\delta{\restriction}\ell^{\mathfrak{d}_0},\eta_\alpha{\restriction}\ell^{\mathfrak{d}_0})<M_\xi,$$
so we arrive at a contradiction.

If we had $v^{\mathfrak{d}_0}\subseteq w^*$, then $v^{\mathfrak{d}_1}\subseteq w_\varsigma$ and we could repeat the earlier arguments with ς in place of ξ to get a contradiction. Thus the only possibility left is that $|v^{\mathfrak{d}_0}\setminus w^*|=1$. Let $\{\alpha\}=v^{\mathfrak{d}_0}\setminus w^*$. If $\alpha\neq\delta$, then $h_0^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\alpha{\restriction}\ell^{\mathfrak{d}_1},\eta_\varepsilon{\restriction}\ell^{\mathfrak{d}_1})=h_0^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\alpha{\restriction}\ell^{\mathfrak{d}_0},\eta_\varepsilon{\restriction}\ell^{\mathfrak{d}_0})<M_\xi$ gives a contradiction like before. Therefore, $v^{\mathfrak{d}_0}=(v^{\mathfrak{d}_0}\cap w^*)\cup\{\delta\}$. But now our assumptions on $v^{\mathfrak{d}_0},\delta$ contradict $(*)_{16}$. □

Claim 4.4.3. *Assume $p=\big(w,n,\iota,M,\bar{\eta},\bar{t},\bar{h},\bar{g},\mathcal{M}\big)\in\mathbb{P}$. If $\mathbf{m}\in\mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ is such that $\ell^{\mathbf{m}}=n$ and $|u^{\mathbf{m}}|\geq 5$, then for some $\rho\in{}^n2$ and $v\subseteq w$ we have $\big(n,v,(\mathbf{m}+\rho)\big)\in\mathcal{M}$.*

Proof of the Claim. Let $\mathbf{m} \in \mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ be such that $\ell^{\mathbf{m}} = n$. Suppose $(\eta, \nu) \in \left(u^{\mathbf{m}}\right)^{\langle 2\rangle}$.

Let $g^{\mathbf{m}}_j(\eta,\nu) = \{\sigma_j\}$ for $j < 6$. Then σ_js are pairwise distinct, and if $\eta + \sigma_i = \nu + \sigma_j$ then

$$\eta + \sigma_k, \nu + \sigma_k \notin \{\eta + \sigma_i, \nu + \sigma_i\} = \{\eta + \sigma_j, \nu + \sigma_j\}$$

whenever $k \notin \{i, j\}$. Hence we may pick $j_0 < j_1 < j_2 < 6$ such that

$$\eta + \sigma_{j_0}, \nu + \sigma_{j_0}, \eta + \sigma_{j_1}, \nu + \sigma_{j_1}, \eta + \sigma_{j_2}, \nu + \sigma_{j_2}$$

are all pairwise distinct. Just to simplify notation let us assume that $j_0 = 0$, $j_1 = 1$ and $j_2 = 2$.

For each $j < 3$ we have $\eta + \sigma_j, \nu + \sigma_j \in \bigcup_{m<M} t_m$. By clause $(*)_7$ there are $(\alpha_j, \beta_j), (\alpha'_j, \beta'_j) \in w^{\langle 2\rangle}$ and $\rho_j \in \bigcup\limits_{i<\iota} g_i(\alpha_j, \beta_j)$ and $\rho'_j \in \bigcup\limits_{i<\iota} g_i(\alpha'_j, \beta'_j)$ such that $\eta + \sigma_j = \eta_{\alpha_j} + \rho_j$ and $\nu + \sigma_j = \eta_{\alpha'_j} + \rho'_j$ for $j < 3$.Then $\eta + \nu = \eta_{\alpha_j} + \eta_{\alpha'_j} + \rho_j + \rho'_j$ for all $j < 3$. We will consider 3 cases, and the first two of them will be shown to be impossible.

Case 1: $\eta_{\alpha_j} = \eta_{\alpha'_j}$ for some $j < 3$.
Then, by the linear independence demanded in $(*)_7$, $\eta_{\alpha_j} = \eta_{\alpha'_j}$ for all $j < 3$ and $\{\rho_0, \rho'_0\} = \{\rho_1, \rho'_1\} = \{\rho_2, \rho'_2\}$. But $g_i(\alpha, \beta)$'s are disjoint, so each $\rho \in \bigcup\{g_i(\alpha,\beta) : (\alpha,\beta) \in w^{\langle 2\rangle} \ \wedge \ i < \iota\}$ uniquely determines α, β such that $\eta_\alpha + \rho, \eta_\beta + \rho \in \bigcup\limits_{m<M} t_m$. Therefore, $|\{\alpha_0, \alpha_1, \alpha_2\}| \leq 2$ in the current case. Since $\eta + \sigma_j, \nu + \sigma_j$ are all pairwise distinct (for $j < 3$), this gives an immediate contradiction.

Case 2: $\eta_{\alpha_j} \neq \eta_{\alpha'_j}$ and $\rho_j \neq \rho'_j$ for some (equivalently: all) $j < 3$.
Then $\{\eta_{\alpha_0}, \eta_{\alpha'_0}\} = \{\eta_{\alpha_1}, \eta_{\alpha'_1}\} = \{\eta_{\alpha_2}, \eta_{\alpha'_2}\}$ and $\{\rho_0, \rho'_0\} = \{\rho_1, \rho'_1\} = \{\rho_2, \rho'_2\}$. However, this again contradicts $\eta + \sigma_j, \nu + \sigma_j$ being pairwise distinct.

Thus the only possible case is the following:
Case 3: $\eta_{\alpha_j} \neq \eta_{\alpha'_j}$ and $\rho_j = \rho'_j$ for all $j < 3$.
Then $\eta + \nu = \eta_{\alpha_0} + \eta_{\alpha'_0}$.

Consequently we have shown that

$$u^{\mathbf{m}} + u^{\mathbf{m}} \subseteq \{\eta_\alpha + \eta_\beta : \alpha, \beta \in w\}.$$

By Lemma 4.3 for some ρ we have $u^{\mathbf{m}} + \rho \subseteq \{\eta_\alpha : \alpha \in w\}$. Let $v = \{\alpha \in w : \eta_\alpha \in u^{\mathbf{m}} + \rho\}$. Let us argue that $\big(n, v, (\mathbf{m} + \rho)\big) \in \mathcal{M}$: demands $(*)^{\rm a}_9$–$(*)^{\rm c}_9$ are immediate consequences of our choices above. Let us verify $(*)^{\rm d}_9$.

Suppose that $(\alpha, \beta) \in v^{\langle 2\rangle}$ and $i < 6$. Let $\eta = \eta_\alpha + \rho, \nu = \eta_\beta + \rho$ (so they are in $u^{\mathbf{m}}$) and let $\{\sigma_i\} = g^{\mathbf{m}}_i(\eta, \nu)$. Then $\eta + \sigma_i, \nu + \sigma_i \in \bigcup\limits_{m<M} t_m$, so we may choose $(\alpha', \beta'), (\alpha'', \beta'') \in w^{\langle 2\rangle}$ and $j', j'' < \iota$ and $\rho' \in g_{j'}(\alpha', \beta')$ and $\rho'' \in g_{j''}(\alpha'', \beta'')$ such that $\eta + \sigma_i = \eta_{\alpha'} + \rho'$ and $\nu + \sigma_i = \eta_{\alpha''} + \rho''$. Then

$$\eta_\alpha + \eta_\beta = \eta + \nu = \eta_{\alpha'} + \eta_{\alpha''} + \rho' + \rho''.$$

By the linear independence stated in $(*)_8$ we get $\rho' = \rho''$ and $\{\eta_{\alpha'}, \eta_{\alpha''}\} = \{\eta_\alpha, \eta_\beta\}$. Consequently also $\{\alpha, \beta\} = \{\alpha', \alpha''\}$ and $\{\alpha', \beta'\} = \{\alpha'', \beta''\}$ and

$j'=j''$. Since $\alpha\neq\beta$ we get $\alpha'\neq\alpha''$ and thus $\alpha'=\beta''$, $\alpha''=\beta'$. Consequently, $\{\alpha'',\beta''\}=\{\alpha',\beta'\}=\{\alpha',\alpha''\}=\{\alpha,\beta\}$. Hence $\eta+\sigma_i=\eta_{\alpha'}+\rho'\in t_{h_{j'}(\alpha,\beta)}=t_{h_{j'}(\beta,\alpha)}$ and $\nu+\sigma_i=\eta_{\alpha''}+\rho'\in t_{h_{j'}(\alpha,\beta)}=t_{h_{j'}(\beta,\alpha)}$. Therefore, $h^{\mathbf{m}+\rho}_i(\eta_\alpha,\eta_\beta)=h^{\mathbf{m}}_i(\eta,\nu)=h_{j'}(\alpha,\beta)=h_{j'}(\beta,\alpha)$. □

Define $\mathbb{P}$–names $\underset{\sim}{T}_m$ and $\underset{\sim}{\eta}_\alpha$ (for $m<\omega$ and $\alpha<\lambda$) by
$\Vdash_{\mathbb{P}}$ " $\underset{\sim}{T}_m=\bigcup\{t^p_m:p\in\underset{\sim}{G}_{\mathbb{P}}\ \wedge\ m<M^p\}$ ", and
$\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\eta}_\alpha=\bigcup\{\eta^p_\alpha:p\in\underset{\sim}{G}_{\mathbb{P}}\ \wedge\ \alpha\in w^p\}$ ".

Claim 4.4.4. (1) *For each $m<\omega$ and $\alpha<\lambda$,*
$\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\eta}_\alpha\in{}^\omega 2$ *and* $\underset{\sim}{T}_m\subseteq{}^{\omega>}2$ *is a tree without terminal nodes* ".

(2) *For all $\alpha<\beta<\lambda$ we have*

$$\Vdash_{\mathbb{P}}\text{ “ }\big(\underset{\sim}{\eta}_\alpha+\bigcup_{m<\omega}\lim(\underset{\sim}{T}_m)\big)\cap\big(\underset{\sim}{\eta}_\beta+\bigcup_{m<\omega}\lim(\underset{\sim}{T}_m)\big)\textit{ is }\bar{\mathcal{O}}\textit{–large ”.}$$

(3) $\Vdash_{\mathbb{P}}$ " $\bigcup_{m<\omega}\lim(\underset{\sim}{T}_m)$ *is a* $\bar{\mathcal{O}}^6$–**npots** *set* ".

Proof of the Claim. (1, 2) By Claim 4.4.1 (and the definition of the order in $\mathbb{P}$).

(3) Let $G\subseteq\mathbb{P}$ be a generic filter over $\mathbf{V}$ and let us work in $\mathbf{V}[G]$. Let $\bar{T}=\langle(\underset{\sim}{T}_m)^G:m<\omega\rangle$.

Suppose towards contradiction that $B=\bigcup\limits_{m<\omega}\lim\big((\underset{\sim}{T}_m)^G\big)$ is an $\bar{\mathcal{O}}^6$–**pots** set. Then, by Proposition 3.16, $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T})=\infty$. Using Lemma 3.15(5), by induction on $j<\omega$ we choose $\mathbf{m}_j,\mathbf{m}^*_j\in\mathbf{M}_{\bar{T},\bar{\mathcal{O}}^6}$ and $p_j\in G$ such that

(i) $\mathrm{ndrk}_{\bar{\mathcal{O}}^6}(\mathbf{m}_j)\geq\omega_1$, $|u^{\mathbf{m}_j}|>5$ and $\mathbf{m}_j\sqsubset\mathbf{m}^*_j\sqsubset\mathbf{m}_{j+1}$,
(ii) for each $\nu\in u^{\mathbf{m}^*_j}$ the set $\{\eta\in u^{\mathbf{m}_{j+1}}:\nu\vartriangleleft\eta\}$ has at least two elements, and
(iii) $p_j\leq p_{j+1}$, $\ell^{\mathbf{m}_j}<\ell^{\mathbf{m}^*_j}=n^{p_j}<\ell^{\mathbf{m}_{j+1}}$ and $\mathrm{rng}(h^{\mathbf{m}_j}_i)\subseteq M^{p_j}$ for all $i<6$, and
(iv) $|\{\eta{\restriction}n^{p_j}:\eta\in u^{\mathbf{m}_{j+1}}\}|=|u^{\mathbf{m}_j}|=|u^{\mathbf{m}^*_j}|$.

To carry out the construction we proceed as follows. Suppose we have determined $\mathbf{m}_j$ so that $\mathrm{ndrk}_{\bar{\mathcal{O}}^6}(\mathbf{m}_j)\geq\omega_1$. Using densities given in Claim 4.4.1, we find $p_j\in G$ stronger than p_{j-1} and such that $n^{p_j}>\ell^{\mathbf{m}_j}$ and $\mathrm{rng}(h^{\mathbf{m}_j}_i)\subseteq M^{p_j}$ (for $i<6$). Next we choose $\mathbf{n}$ such that $\mathbf{m}_j\sqsubset\mathbf{n}$, $\mathrm{ndrk}_{\bar{\mathcal{O}}^6}(\mathbf{n})\geq\omega_1$, and $\ell^{\mathbf{n}}>n^{p_j}$. Using Lemma 3.15(8) (for a $u'\subseteq u^{\mathbf{n}}$ such that $\{\eta{\restriction}\ell^{\mathbf{m}_j}:\eta\in u'\}=u^{\mathbf{m}_j}$, $|u'|=|u^{\mathbf{m}_j}|$) we may also demand that $|u^{\mathbf{n}}|=|u^{\mathbf{m}_j}|$. Now we let

- $\ell=n^{p_j}$, $u=\{\eta{\restriction}\ell:\eta\in u^{\mathbf{n}}\}$,
- $\bar{h}=\langle h_i:i<6\rangle$, where for $i<6$ and $(\eta,\nu)\in\big(u^{\mathbf{n}}\big)^{\langle 2\rangle}$
  $h_i(\eta{\restriction}\ell,\nu{\restriction}\ell)=h^{\mathbf{n}}_i(\eta,\nu)=h^{\mathbf{m}_j}_i(\eta{\restriction}\ell^{\mathbf{m}_j},\nu{\restriction}\ell^{\mathbf{m}_j})$,
- $\bar{g}=\langle g_i:i<6\rangle$, where for $i<6$ and $(\eta,\nu)\in\big(u^{\mathbf{n}}\big)^{\langle 2\rangle}$
  $g_i(\eta{\restriction}\ell,\nu{\restriction}\ell)=\big\{\rho{\restriction}\ell:\rho\in g^{\mathbf{n}}_i(\eta,\nu)\big\}$.

Clearly, $\mathbf{m}^*_j=(\ell,6,u,\bar{h},\bar{g})\in\mathbf{M}_{\bar{T},\bar{\mathcal{O}}^6}$ and $\mathbf{m}_j\sqsubset\mathbf{m}^*_j$. Finally use Lemma 3.15(5) to pick $\mathbf{m}_{j+1}\sqsupset\mathbf{n}$ such that $\mathrm{ndrk}(\mathbf{m}_{j+1})\geq\omega_1$ and condition (ii) is satisfied. Note that $\mathbf{m}^*_j\sqsubset\mathbf{m}_{j+1}$.

Then, by (iii)+(iv), $\mathbf{m}_j, \mathbf{m}^*_j \in \mathbf{M}^{n^{p_j}}_{\bar{t}^{p_j},\bar{\mathcal{O}}^6}$. It follows from Claim 4.4.3 that for some $w_j \subseteq w^{p_j}$ and $\rho_j \in {}^{n^{p_j}}2$ we have $(n^{p_j}, w_j, \mathbf{m}^*_j + \rho_j) \in \mathcal{M}^{p_j}$.

Fix j for a moment and consider $(n^{p_j}, w_j, \mathbf{m}^*_j + \rho_j) \in \mathcal{M}^{p_j} \subseteq \mathcal{M}^{p_{j+1}}$ and $(n^{p_{j+1}}, w_{j+1}, \mathbf{m}^*_{j+1} + \rho_{j+1}) \in \mathcal{M}^{p_{j+1}}$. (Note that since $(n^{p_j}, w_j, \mathbf{m}^*_j + \rho_j) \in \mathcal{M}^{p_j}$, we know that $r_{h_i^{\mathbf{m}^*_j}(\eta,\nu)} \leq n^{p_j}$ for all $i < 6$, $(\eta,\nu) \in u^{\mathbf{m}^*_j}$.) Since $(\mathbf{m}^*_j + (\rho_{j+1}{\restriction}n^{p_j})) \sqsubset (\mathbf{m}^*_{j+1} + \rho_{j+1})$, we may choose $w^*_j \subseteq w_{j+1}$ such that $(n^{p_j}, w^*_j, \mathbf{m}^*_j + (\rho_{j+1}{\restriction}n^{p_j})) \in \mathcal{M}^{p_{j+1}}$. Since $(\mathbf{m}^*_j + \rho_j) + (\rho_j + \rho_{j+1}{\restriction}n^{p_j}) = \mathbf{m}^*_j + (\rho_{j+1}{\restriction}n^{p_j})$, we may use clause $(*)_{10}$ for p_{j+1} to conclude that $\mathrm{rk}(w^*_j) = \mathrm{rk}(w_j)$.

Condition (ii) of the choice of $\mathbf{m}_{j+1}$ implies that

$$(\forall\gamma \in w^*_j)(\exists\delta \in w_{j+1} \setminus w^*_j)(\eta^{p_{j+1}}_\gamma{\restriction}n^{p_j} = \eta^{p_{j+1}}_\delta{\restriction}n^{p_j}).$$

Let $\delta(\gamma)$ be the smallest $\delta \in w_{j+1} \setminus w^*_j$ with the above property and let $w^*_j(\gamma) = (w^*_j\setminus\{\gamma\})\cup\{\delta(\gamma)\}$. Then, for $\gamma \in w^*_j$, $(n^{p_j}, w^*_j(\gamma), \mathbf{m}^*_j + (\rho_{j+1}{\restriction}n^{p_j})) \in \mathcal{M}^{p_{j+1}}$ and therefore, by clause $(*)_{10}$ for p_{j+1}, we get that for each $\gamma \in w_j$:

$$\mathrm{rk}(w^*_j(\gamma)) = \mathrm{rk}(w^*_j), \quad \zeta(w^*_j(\gamma)) = \zeta(w^*_j), \quad \text{and} \quad k(w^*_j(\gamma)) = k(w^*_j).$$

Let $n = |w^*_j|$, $\zeta = \zeta(w^*_j)$, $k = k(w^*_j)$, and let $w^*_j = \{\alpha_0, \ldots, \alpha_k, \ldots, \alpha_{n-1}\}$ be the increasing enumeration. Let $\alpha^*_k = \delta(\alpha_k)$. Then clause $(*)_{10}$ also gives that $w^*_j(\alpha_k) = \{\alpha_0, \ldots, \alpha_{k-1}, \alpha^*_k, \alpha_{k+1}, \ldots, \alpha_{n-1}\}$ is the increasing enumeration. Now,

$$\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n-1}] \qquad \text{and}$$
$$\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha^*_k, \alpha_{k+1}, \ldots, \alpha_{n-1}],$$

and consequently if $\mathrm{rk}(w^*_j) \geq 0$, then

$$\mathrm{rk}(w_{j+1}) \leq \mathrm{rk}(w^*_j \cup \{\alpha^*_k\}) < \mathrm{rk}(w^*_j) = \mathrm{rk}(w_j)$$

(remember $(\circledast)_{\mathrm{d}}$ from the very beginning of the proof of the Theorem).

Now, unfixing j, it follows from the above considerations that for some $j_0 < \omega$ we must have:

(a) $\mathrm{rk}(w^*_{j_0}) = -1$, and
(b) $(n^{p_{j_0}}, w^*_{j_0}, \mathbf{m}^*_{j_0} + (\rho_{j_0+1}{\restriction}n^{p_{j_0}})), (n^{p_{j_0+1}}, w_{j_0+1}, \mathbf{m}^*_{j_0+1} + \rho_{j_0+1}) \in \mathcal{M}^{p_{j_0+1}}$,
(c) for each $\nu \in u^{\mathbf{m}^*_{j_0}}$ the set $\{\eta \in u^{\mathbf{m}^*_{j_0+1}} : \nu \vartriangleleft \eta\}$ has at least two elements.

However, this contradicts clause $(*)_{11}$ (for p_{j_0+1}). □

□

5. Conclusions and Questions

Corollary 5.1. *Assume* $\mathrm{NPr}_{\omega_1}(\lambda)$ *and* $\lambda = \lambda^{\aleph_0} < \mu = \mu^{\aleph_0}$.

(1) *Let* $\bar{\mathcal{O}}$ *be a nice indexed base. Then there is a ccc forcing notion* $\mathbb{Q}$ *of size* μ *forcing that* $2^{\aleph_0} = \mu$ *and*
- *there is a* Σ^0_2 *set* $B \subseteq {}^\omega 2$ *which has* λ *many pairwise* $\bar{\mathcal{O}}$*–nondisjoint translates but does not have* λ^+ *many pairwise* $\bar{\mathcal{O}}^6$*–nondisjoint translates.*

(2) *In particular, there is a ccc forcing notion* $\mathbb{Q}'$ *of size* μ *forcing that* $2^{\aleph_0} = \mu$ *and for some* Σ^0_2 *set* $B \subseteq {}^\omega 2$*:*
- *there are pairwise distinct* $\langle \eta_\xi : \xi < \lambda \rangle$ *such that* $(B+\eta_\xi) \cap (B+\eta_\zeta)$ *is uncountable for each* $\xi, \zeta < \lambda$*, but*
- *for any set* $A \subseteq {}^\omega 2$ *of size* λ^+ *there are* $x, y \in A$ *such that* $|(B+x) \cap (B+y)| < 6$*.*

Proof. (1) Let $\mathbb{P}$ be the forcing notion given by Theorem 4.4 and let $\mathbb{Q} = \mathbb{P} * \mathbb{C}_\mu$. The set B added by $\mathbb{P}$ is a $\bar{\mathcal{O}}^6$–**npots**–set in $\mathbf{V}^{\mathbb{P}}$, so by Proposition 3.16 we got $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T}) < \infty$. The rank $\mathrm{ndrk}^{\bar{T}}_{\bar{\mathcal{O}}^6}$ is absolute, so in $\mathbf{V}^{\mathbb{Q}}$ we still have $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T}) < \infty$ and thus B is a $\bar{\mathcal{O}}^6$–**npots**–set in $\mathbf{V}^{\mathbb{Q}}$. By 3.5(3) this set cannot have λ^+ pairwise $\bar{\mathcal{O}}^6$–nondisjoint translates, but it does have λ many pairwise $\bar{\mathcal{O}}$–nondisjoint translates (by absoluteness). □

Corollary 5.2. *Assume* **MA** *and* $\aleph_\alpha < \mathfrak{c}$*,* $\alpha < \omega_1$*.*

(1) *Let* $\bar{\mathcal{O}}$ *be a nice indexed base. Then there exists a* Σ^0_2 $\bar{\mathcal{O}}^6$*–***npots***–set* $B \subseteq {}^\omega 2$ *which has* $\aleph_\alpha$ *many pairwise* $\bar{\mathcal{O}}$*–nondisjoint translations.*

(2) *In particular, there exists a* Σ^0_2 *set* $B \subseteq {}^\omega 2$ *such that*
- *for some pairwise distinct* $\langle \eta_\xi : \xi < \aleph_\alpha \rangle \subseteq {}^\omega 2$ *the intersections* $(B+\eta_\xi) \cap (B+\eta_\zeta)$ *are uncountable for each* $\xi, \zeta < \aleph_\alpha$*, but*
- *for every perfect set* $P \subseteq {}^\omega 2$ *there are* $x, y \in P$ *such that* $|(B+x) \cap (B+y)| < 6$*.*

Proof. Standard consequence of the proof of Theorem 4.4, using the fact that "B is a $\bar{\mathcal{O}}^6$–**npots**–set" is sufficiently absolute by Proposition 3.16. □

Problem 5.3. (1) Can one differentiate between various nice $\bar{\mathcal{O}}$ in the context of our results? In particular:

(2) Is it consistent that for some nice $\bar{\mathcal{O}}$ there is a Σ^0_2 $\bar{\mathcal{O}}$–**npots**–set which has $\aleph_\alpha$ many pairwise $\bar{\mathcal{O}}$–nondisjoint translations, but for some other nice $\bar{\mathcal{O}}^*$ every Σ^0_2 set with $\aleph_\alpha$ many pairwise $\bar{\mathcal{O}}^*$–nondisjoint translations is automatically $\bar{\mathcal{O}}^*$–**pots** ?

(3) Is it consistent that there is a Σ^0_2 set $B \subseteq {}^\omega 2$ which is has $\aleph_\alpha$ many pairwise $\bar{\mathcal{O}}^{\rm per}$–nondisjoint translations, is $\bar{\mathcal{O}}^{\rm per}$–**npots**, but is also $\bar{\mathcal{O}}^6$–**pots**?

Problem 5.4. (1) Consider the forcing notion $\mathbb{P}$ given by Theorem 4.4 for $\bar{\mathcal{O}}^{\rm per}$. In the forcing extension by $\mathbb{P}$, the ranks $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T})$ and $\mathrm{NDRK}_{\bar{\mathcal{O}}^{\rm per}}(\bar{T})$ are both countable. Are they equal? What are their values?

(2) Does there exist a sequence of trees $\bar{T}^*$ (as in Assumptions 3.1) for which the ranks $\mathrm{NDRK}_{\bar{\mathcal{O}}^{\rm per}}(\bar{T})$ and $\mathrm{NDRK}_{\bar{\mathcal{O}}^\iota}(\bar{T})$ are different (for some/all ι)?

(3) Generalize the construction of [5] to arbitrary nice $\bar{\mathcal{O}}$.

(4) Generalize the result of the present paper to the context of arbitrary perfect Abelian Polish groups.

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