# UNIVERSAL GRAPHS BETWEEN A STRONG LIMIT SINGULAR AND ITS POWER

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ABSTRACT. The paper settles the problem of the consistency of the existence of a single universal graph between a strong limit singular and its power. Assuming that in a model of **GCH** $\kappa$ is supercompact and the cardinals $\theta < \kappa$, $\lambda > \kappa$ are regular, as an application of a more general method, we obtain a forcing extension in which $\mathrm{cf}(\kappa) = \theta$, the Singular Cardinal Hypothesis fails at $\kappa$ and there exists a universal graph at cardinality $\lambda \in (\kappa, 2^\kappa)$.

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References like [Shea, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version on Shelah's website is usually more updated than the one in the mathematical archive.

Annotated Content

## § 0. Introduction

### § 0(A). **Background.**

The existence of universal graphs at infinite cardinalities has received extensive investigation (where we mean that the graph $G$ is universal at cardinality $|G|$ if every graph of the same cardinality is isomorphic to some induced subgraph of $G$). According to the classical result [Rad64], the so called countable random graph is a universal graph at $\aleph_0$ (which is also unique, up to isomorphism). A classical result (which now follows as a standard induction argument) establishes the existence of a $\kappa^+$-saturated graph on the set $2^\kappa$ [CK73]. Consequently, there exists a graph on $2^\kappa$ into which every graph on $\kappa^+$ embeds (and we can replace $\kappa^+$, $2^\kappa$, $\kappa^+$-saturated with $\kappa$, $2^{<\kappa}$, $\kappa$-special). Therefore, assuming **GCH**, there exists a universal graph at every infinite cardinality. (However, concerning certain proper classes of graphs the situation is more intricate, even for the countable case, see [FK97], [Kom89], [KS95], [CS16], [KS19].) Regarding the problem of universal objects in more complex theories (i.e., beyond graphs) and the relevance of the present work in model theory, readers may consult the survey [She21] or earlier works such as [Dža05]. See also recent publications such as [She20] and [Sheb]. Another related question, the existence of universal Aronszajn trees has been extensively studied as well, see [Tod07], [DS21], and most recently [BNMV23].

However, without assuming **GCH**, it is generally much more challenging to construct universal objects. Furthermore, after adding $\kappa^{++}$ Cohen subsets to a regular $\kappa$, there are no universal graphs on $\kappa^+$, as shown in [KS92].

Regarding positive results, for regular cardinals $\kappa < \lambda$, there consistently exists a universal graph of size $\lambda$, while $2^\kappa > \lambda$ [She90]. Moreover, the argument presented in [She90] also provides a universal $\omega$-edge colored graph on $\omega_1$ assuming $\neg$**CH**. Features of this method will be used in this paper. However, a recent study [SS21] proved that assuming $\neg$**CH**, the existence of a universal graph on $\omega_1$ does not imply the existence of a universal $\omega$-edge colored graph on $\omega_1$. Furthermore, it should be noted that when considering specific classes of graphs, there are both negative [Koj98] and positive results [Mek90] for universal objects and weak universal families. (Given a class $\mathcal{K}_\lambda$ of models each of which is of cardinality $\lambda$, $\kappa < \lambda < 2^\kappa$, we say that the family $\mathcal{F} \subseteq \mathcal{K}_\lambda$ is a weak universal family for $\mathcal{K}_\lambda$ if every $G \in \mathcal{K}_\lambda$ embeds into some $G_* \in \mathcal{F}$, and $|\mathcal{F}| < 2^\kappa$). It is also consistent that there exists a singular $\kappa$, $2^\kappa > \kappa^+$, and there is no universal graph on $\kappa^+$ [FT10][Theorem 3.3] (and it follows from their proof that $\kappa$ is strong limit). For more consistency results in the absence of **GCH**, see [She93] and [DS04]. It is worth mentioning that dealing with the case $\lambda = \kappa^+$ was considerably easier in all the aforementioned cases.

In this paper, we investigate universal graphs in the interval between a strong limit singular cardinal and its power. The motivation for this question stems from the following observations. Recall that the cardinal exponentiation $2^{\aleph_0}$ can be quite large and at the same time relevant forcing axioms such as **MA** may hold. Similarly, for $\mu = \aleph_1 = 2^{\aleph_0}$, $2^\mu$ can be large, or for $\mu = \mu^{<\mu}$, parallel results hold for forcing notions that are, for example, $< \mu$-complete and satisfy a strong form of $\mu^+$-cc (the strong form is necessary, see [Shear]). On the other hand, much less is known for strong limit singular cardinals $\mu$, and thus the existence of universals serves as a central test problem for examining the consistency of forcing axioms at $\mu$.

In this paper, we continue the work of Džamonja-Shelah in [DS03], which demonstrated the consistency of the statement $(*)$ assuming the existence of a supercompact cardinal.

$(*)$ (a) $\mu$ is strong limit singular and $\mu^{++} < 2^{\mu}$,

(b) there is a graph $G_*$ of cardinality $\mu^{++}$ which is universal for graphs of cardinality $\mu^+$ (equivalently there is a sequence $\bar{G} = \langle G_\alpha : \alpha < \mu^{++}\rangle$ of graphs each of cardinality $\mu^+$, universal for the family of such graphs).

for the case $\mathrm{cf}(\mu) = \aleph_0$, and later Cummings-Džamonja-Magidor-Morgan-Shelah proved this for arbitrary cofinality in [CDM+17]. Earlier, Mekler-Shelah [MS89] had proved such consistency results replacing (b) with uniformization principles; also starting naturally with a supercompact cardinal. Later, $(*)$ was proved to be consistent for small singular $\mu$'s too, see [CDM16], [Dav17].

Our goal is to address the naturally arising problem by replacing weak universal families (in the sense of $(*)$(b)) with single universal objects and by considering $\lambda$ in the range of $(\mu, 2^\mu)$ instead of restricting it to $\mu^+$. Thus, we formulate the following assertions:

$(*)^+$ (a) $\mu$ is strong limit singular and $\mu^{++} < 2^{\mu}$,

(b) there is a universal graph $G_*$ in $\mu^+$, i.e. universal for graphs of cardinality $\mu^+$, $G_*$ itself is of cardinality $\mu^+$,

(b)$^+$ as (b), but changing $\mu^+$ for some cardinal in $(\mu, 2^\mu)$.

To initiate our proof, we consider a supercompact cardinal $\kappa$ as our starting point. We demonstrate, as part of a more general axiomatic framework, that a stronger version of a universal on $\lambda > \kappa$ (e.g., $\lambda = \kappa^+$) is sufficient to guarantee the existence of a universal graph on $\lambda$ even after forcing with a $\mathbb{P}$ that satisfies the axiomatic requirements. We first establish a general framework for the preparatory forcing, followed by the construction of a strong universal graph suited to the present framework, as in [She90]. (It is worth noting that certain large cardinal hypotheses are essential, as the failure of the Singular Cardinal Hypothesis itself implies the existence of an inner model with the Mitchell order $o(\kappa) = \kappa^{++}$ for a measurable cardinal $\kappa$; in fact, these are equiconsistent [Git91].)

The organization of the paper is as follows. In §1 we introduce the concept of $(\lambda, \kappa) - i$ $(i = 1, 2)$ systems, and in Claim 1.5 we prove that extending a ground model already admitting some strong version of universal using such a $(\lambda, \kappa) - i$ system results in a model with the desired universal object. In §2 we prove that Prikry forcing, Magidor forcing and Radin forcing give rise to a $(\lambda, \kappa) - 1$ system provided the relevant filters satisfy some reasonable directedness assumptions. In §3(A) we prepare the ground, in Claim 3.2 build the framework to force $(\lambda, \kappa) - 1$ systems using a supercompact cardinal. In §3(B) we construct a forcing for obtaining the strong universal, that fits in the framework in Claim 3.2.

In works in preparation we intend to replace graphs by more general classes; much of our work is not specific to graphs. Also for consistency of $(*)^+$ for a small singular $\mu$, e.g. $\mu = \aleph_\omega = \beth_\omega$ [PS].

§ 0(B). **Preliminaries.** We are interested in universal objects in the class of graphs, i.e. models of the first order language admitting no functions, only a single symmetric, nonreflexive binary relation. Under ordinals we always mean von Neumann ordinals, and for a set $X$ the symbol $|X|$ always refers to the smallest ordinal with the same cardinality. If $f$ is a mapping with $\mathrm{dom}(f) \supseteq X$, then $f``X = \{f(x) : x \in X\}$, i.e. the pointwise image of $X$. For a set $X$ the symbol $\mathscr{P}(X)$ denotes the power set of $X$, while if $\kappa$ is an ordinal we use the standard notation $[X]^\kappa$ for $\{Y \in \mathscr{P}(X) : |Y| = \kappa\}$, similarly for $[X]^{<\kappa}$, $[X]^{<\kappa}$, etc. By a sequence we mean a function on an ordinal, where for a sequence $\overline{s} = \langle s_\alpha : \alpha < \mathrm{dom}(\overline{s})\rangle$ the length of $\overline{s}$ (in symbols $\ell g(\overline{s})$) denotes $\mathrm{dom}(\overline{s})$. Moreover, for sequences $\overline{s}$, $\overline{t}$ let $\overline{s} \frown \overline{t}$ denote the natural concatenation (of length $\ell g(\overline{s}) + \ell g(\overline{t})$). For a set $X$, and ordinal $\alpha$ we use ${}^\alpha X = \{\overline{s} : \ell g(\overline{s}) = \alpha, \ \mathrm{ran}(\overline{s}) \subseteq X\}$, and for cardinals $\lambda$, $\kappa$ we use the symbol $\lambda^\kappa = |{}^\kappa\lambda|$ (that is, the least ordinal equivalent to it).

We call a set $T \subseteq {}^{<\alpha}X$ a tree (where $\alpha$ is an ordinal), if $T$ is downward closed, i.e. whenever $\overline{t} \in T$, $\gamma < \ell g(\overline{t})$, we have $\overline{t}{\restriction}\gamma \in T$. We call $\overline{t}$ a leaf, if there is no $\overline{s} \in T$ for which $t \subsetneq s$.

Regarding iterated forcing and quotient forcing we will mostly use the terminology of the survey [Bau76]. However we adhere to the following conventions.

**Convention 0.1.** Regarding forcing we follow the convention that "$p \le q$" means that $q$ is stronger, i.e. giving more information.

**Convention 0.2.** A notion of forcing $\mathbb{P}$ is $<\mu$-directed closed ($<\mu$-closed, resp.), if for any directed (increasing, resp.) system $\{p_\alpha : \alpha < \nu < \mu\}$ there exists a common upper bound $p_*$ in $\mathbb{P}$.

A filter $\mathcal{F} \subseteq \mathscr{P}(X)$ is $\kappa$-complete, if for each $\{F_\alpha : \alpha < \nu < \kappa\} \subseteq \mathcal{F}$ we have $\bigcap_{\alpha<\nu} F_\alpha \in \mathcal{F}$. A partial order $P$ is $< \mu$-directed, if for each $\{p_\alpha : \alpha < \nu < \mu\} \subseteq P$, there exists a common upper bound $p_* \in P$. (For example, if $\mathcal{F} \subseteq \mathscr{P}(X)$ is a $\kappa$-complete filter on $X$, then $\mathcal{F}$ is $< \kappa$-directed with respect to the relation $\supseteq$).

## § 1. The framework and deducing the consistency results

### § 1(A). **What We Do.**

In the present paper we introduce a general framework and apply it for the class of graphs.

We shall start with a large cardinal, such as a Laver indestructible supercompact, or with forcing a relative of it. We then have a two step forcing.

<u>First</u>, a forcing $\mathbb{P}$ with the following three properties:

(a) preserving the largeness of $\kappa$,
(b) moreover, in $\mathbf{V}^{\mathbb{P}}$
 there is a normal $\kappa$-complete filter $D$ on $\kappa$ such that $(D, {}^*\supseteq)$ is $\lambda^+$-directed for a suitable cardinal $\lambda < 2^\kappa$,
(c) preparing the ground for the results we like to have on $\lambda$, e.g. has a strong version of "there is a universal graph in $\lambda, \lambda < 2^\kappa$".

<u>Second</u>, a forcing $\mathbb{Q}$ (in $\mathbf{V}^{\mathbb{P}}$) such that:

(d) $\mathbb{Q}$ makes $\kappa$ singular,

(e) preserves $\kappa$ is strong limit and $2^\kappa$ large.

Thirdly,

(f) to get the desired property of $\lambda$, we use $\mathbb{Q}$ that fits in the framework in Definition 1.2 below,

(g) then prove the existence of a universal object using the framework

In §1(B) Definition 1.2 defines the family of $(\lambda, \kappa)$-systems fitting (f), then we deduce the existence of universal graphs in $\lambda$ (a case of (g)).

In §2 we shall prove that classical forcings for making $\kappa$ singular fit our framework, i.e. satisfy (d)-(g).

In §3 we shall deal with finding $\mathbb{P}$ as in (a),(b),(c), so we have to combine the specific forcing (say forcing a universal graph in $\lambda$, i.e. clause (c)) and guaranteeing the existence of e.g. a normal ultrafilter of which is $\lambda^+$-complete in a suitable sense (i.e. clause (b)).

## § 1(B). $(\lambda, \kappa)$-systems.

The following is standard, but we have to include these definitions in order to avoid ambiguity, thus clarify what we mean under $\kappa$-Borel sets.

**Definition 1.1.** Assume that $\mu$ is a cardinal, $Y$ is a set.

(1) We let $\mathscr{B}_\mu \subseteq \mathscr{P}({}^Y 2)$ denote the set of $\mu$-Borel subsets of ${}^Y 2$, i.e. $\mathscr{B}_\mu({}^Y 2) \subseteq \mathscr{P}({}^Y 2)$ is the smallest family that satisfies
   - for each function $f : \mathrm{dom}(f) \to 2$ with $\mathrm{dom}(f) \in [Y]^{<\aleph_0}$ the basic open set (wrt. the product topology)
$$[f] := \{g \in {}^Y 2 : g \supseteq f\} \in \mathscr{B}_\mu({}^Y 2),$$
   - whenever $\langle B_i : i \in \mu \rangle$ is a sequence with $(\forall i < \mu)\ B_i \in \mathscr{B}_\mu({}^Y 2)$, necessarily $\bigcup_{i \in \mu} B_i \in \mathscr{B}_\mu({}^Y 2)$,
   - $\forall B \in \mathscr{B}_\mu({}^Y 2)$: $({}^Y 2 \setminus B) \in \mathscr{B}_\mu({}^Y 2)$.

(2) we say that the tree
$$T \subseteq {}^{<\omega}\{\cup, \neg, [f] : f : \mathrm{dom}(f) \to 2,\ \mathrm{dom}(f) \in [Y]^{<\aleph_0}\}$$
is a code for a set in $\mathscr{B}_\mu({}^Y 2)$ (in symbols, $T \in \mathrm{code}_\mu(Y)$), if
   - $T \setminus \{\langle\rangle\}$ is nonempty, moreover, it has a stem $s \in T$ of length 1 (i.e. $\ell g(s) = 1$, and for each $t \in T$ with $\ell g(t) > 1$ $s \subseteq t$),
   - $T$ is well-founded, and
   - for each $t \in T \setminus \{\langle\rangle\}$ we have that
$$t \text{ is a leaf of } T \iff t(\ell g(t) - 1) = [f] \text{ for a partial function } f \text{ above},$$
   - for each $t \in T \setminus \{\langle\rangle\}$, if $t(\ell g(t) - 1) = \neg$, then neither does $T$ branch at $t$, nor is $t$ a leaf (that is, $\exists! t' \in T$, $\ell g(t') = \ell g(t) + 1$, $t \subsetneq t'$), and
   - for each $t \in T \setminus \{\langle\rangle\}$ with $t(\ell g(t) - 1) \neq \neg$, $t$ has at most $\mu$-many immediate successors, that is,
$$|\{s \in T : t \subsetneq s,\ \ell g(s) = \ell g(t) + 1\}| \leq \mu$$
(equivalently, $|T| \leq \mu$),

(3) we can define the evaluation $B_T$ for $T \in \mathrm{code}_\mu(Y)$ in the obvious fashion, by induction on the rank of $T$. If $T = \{\langle [f] \rangle\}$, then we let $B_T = [f]$. Otherwise, $T$ necessarily has a stem $s = \langle s(0) \rangle = \langle \cup \rangle$, or $s = \langle \neg \rangle$. For each $t \in T$, $\ell g(t) = 2$ we can naturally define the tree $T_t$ below $t$, i.e.

$$T_t = \{u : \langle s(0) \rangle \frown u \in T, \; s(0)\rangle \frown u \supseteq t\}.$$

Now if $s(0)$ is the symbol $\cup$, then we let

$$B_T = \bigcup_{t \in T, \ell g(t)=2} B_{T_t}.$$

Otherwise, if $s(0) = \neg$, then there exists a unique $t \in T$, $\ell g(t) = 2$, and we let

$$B_T = {}^Y 2 \setminus B_{T_t}.$$

(4) Using the natural identification between ${}^Y 2$, and $\mathscr{P}(Y)$, we can talk about $\mu$-Borel subsets of $\mathscr{P}(Y)$, $\mathscr{B}_\mu(\mathscr{P}(Y))$, and so about codes for $\mu$-Borel subsets of $\mathscr{P}(Y)$.

**Definition 1.2.**
1) We say $\mathbf{r}$ is a $(\lambda,\kappa) - 1$-system <u>when</u> $\mathbf{r} = (\mathbb{R}, \underset{\sim}{X}, \leq_{\rm pr}, \mathscr{S}) = (\mathbb{R}_\mathbf{r}, \underset{\sim}{X}_\mathbf{r}, \leq_{\mathbf{r},\rm pr}, \mathscr{S}_\mathbf{r})$ satisfies the following

- $(a)$ $\kappa$ is strongly inaccessible,
- $(b)$ $\lambda \in [\kappa^+, 2^\kappa)$,
- $(c)$ $\mathbb{R}$ is a forcing notion preserving "$\kappa$ is strong limit",
- $(d)$ $\underset{\sim}{X}$ is an $\mathbb{R}$-name of a subset of $\kappa$,
- $(e)$ $\leq_{\rm pr} \subseteq \leq_\mathbb{R}$ is a quasi-order,
- $(f)$ for each $p \in \mathbb{R}$ we have $\mathscr{S}_p \subseteq \{\bar{q} \in {}^\kappa\mathbb{R} : p \leq_{\rm pr} q_\varepsilon$ for every $\varepsilon < \kappa\}$,
- $(g)$ whenever $p \in \mathbb{R}$, $\underset{\sim}{\tau}$ are such that $p \Vdash$ "$\underset{\sim}{\tau} \in \{0,1\}$" (a truth value), <u>then</u>:
  - $(*)$ there are $\bar{q} \in \mathscr{S}_p, \bar{Y} = \langle Y_\varepsilon : \varepsilon < \kappa \rangle \in {}^\kappa V_\kappa$, $\bar{\mu} = \langle \mu_\varepsilon : \varepsilon < \kappa \rangle \in {}^\kappa\kappa$ and $\bar{T} = \langle T_\varepsilon : \varepsilon < \kappa \rangle$, where
    - $\bullet_1$ each $T_\varepsilon \in \mathbf{V}$ is a code for a $\mu_\varepsilon$-Borel set $B_\varepsilon \in \mathscr{B}_{\mu_\varepsilon}(\mathscr{P}(Y_\varepsilon))$ (in the sense of Definition 1.1 (2), (4)),
    - $\bullet_2$ $q_\varepsilon \Vdash$ "$\underset{\sim}{\tau} = 1 \iff \underset{\sim}{X} \cap Y_\varepsilon \in B_{T_\varepsilon}$";
- (h) for each $p \in \mathbb{R}$, and for each sequence $\langle \bar{q}_\alpha : \alpha < \lambda \rangle$ with $\forall \alpha < \lambda \; \bar{q}_\alpha \in \mathscr{S}_p$, there exists $q_* \in \mathbb{R}$ such that for every $\alpha < \lambda$ there exists $\varepsilon_\alpha < \kappa$ such that $q_{\alpha,\varepsilon_\alpha} \leq_\mathbb{R} q_*$.

2) We say $\mathbf{r}$ is a $(\lambda,\kappa) - 2$-system <u>when</u> above in clause (g) we restrict ourselves to $\underset{\sim}{\tau}$'s that are $\mathbb{R}_{\underset{\sim}{X}}$-names, where $\mathbb{R}_{\underset{\sim}{X}} \lessdot \mathbb{R}$ is the complete subforcing adding only $\underset{\sim}{X}[\mathbf{G}]$ (in other words, if $\mathbf{G} \subseteq \mathbb{R}$ is generic over $\mathbf{G}$, then letting $Z = \underset{\sim}{X}[\mathbf{G}]$, $\mathbf{V}[Z]$ is a $\mathbb{R}_{\underset{\sim}{X}}$-generic extension of $\mathbf{V}$);
2A) We may omit the 1 in "1-system", so that "$(\lambda,\kappa)$-system" is always meant as "$(\lambda,\kappa) - 1$-system".
3) We say $\mathbf{r}$ is nice <u>when</u> the forcing $\mathbb{R}_\mathbf{r}$ does not collapse any cardinal.

**Discussion 1.3.**
1) Here we only deal with the question "when is there a universal graph in the cardinal $\lambda$?".
2) Of course, in Definition 1.2, we are interested in the case $\Vdash_{\mathbb{R}_\mathbf{r}}$ "$\kappa$ is singular".

3) There are such $\mathbf{r}$'s: Prikry forcing, Magidor forcing, cases of Radin forcing, see Claim 2.1 and onwards. (In the specific case of Prikry forcing $(g)$ can be simplified, as $Y_\varepsilon$ will be an ordinal below $\kappa$, and the name $\underset{\sim}{\tau}$ will depend on the finite set in which the Prikry generic set meets the ordinal $Y_\varepsilon$.)

The following notion is necessary to phrase the framework for the main result (Claim 1.5).

**Definition 1.4.** Suppose that $\kappa$, $\lambda$ are cardinals.

1) We let $K_\kappa$ denote the class of edge colored graphs with the set of colors indexed by $\kappa$, so formally it is defined as follows. The model $M$ belongs to $K_\kappa$, <u>iff</u>
   (a) $M = (|M|, R^M_\varepsilon)_{\varepsilon<\kappa}$,
   (b) $R^M_\varepsilon$ is a symmetric irreflexive two-place relation on $|M|$,
   (c) $\langle R^M_\varepsilon : \varepsilon < \kappa\rangle$ is a partition of $\{(a, b) : a \neq b \in |M|\}$.
2) $(K_\kappa)_\lambda$ is the class of graphs in $K_\kappa$ that have $\lambda$-many vertices, i.e. for $M \in K_\kappa$ we have
$$M \in (K_\kappa)_\lambda \iff ||M|| = \lambda.$$

**Claim 1.5.**

1) *Assume that*
   $(i)$ $\iota \in \{1, 2\}$,
   $(ii)$ *$\kappa$, $\lambda$ are fixed cardinals,* $\kappa < \lambda < 2^\kappa$,
   $(iii)$ $\mathbf{r} \in \mathbf{V}$ *is a* $(\lambda, \kappa) - \iota$*-system, and let* $\mathbf{V}_\iota = \mathbf{V}^{\mathbb{R}_\mathbf{r}}$ *if* $\iota = 1$*;* $\mathbf{V}_\iota = \mathbf{V}[X_\mathbf{r}]$ *in case of* $\iota = 2$.
   $(iv)$ *there is a universal member of* $(K_\kappa)_\lambda$ *(in* $\mathbf{V}$*),*
   *Then*
$$\mathbf{V}_\iota \models \text{"there is a universal graph of cardinality } \lambda\text{"}$$
2) *Moreover, in general, if* $(i)$*-*$(iii)$ *hold, and*
   $(iv)^\chi$ *(in* $\mathbf{V}$*) there is a weak universal family of size* $\chi$ *in* $(K_\kappa)_\lambda$ *, i.e. a system* $\langle M_i : i < \chi\rangle$*, for which for each* $M \in (K_\kappa)_\lambda$ *there exists* $i_0 < \chi$ *such that* $M$ *can be embedded into* $M_{i_0}$ *(in the sense of* $K_\kappa$*),*
   *then*

$$\begin{aligned}\mathbf{V}_\iota \models\ & \exists\langle G_i : i < \chi\rangle : \\ & \circledcirc_1\ (\forall i < \chi)\ G_i \text{ is a graph on } \lambda, \\ & \circledcirc_2 \text{ and for every graph } G \text{ of size } \lambda \text{ there is } i_0 < \chi, \\ & \quad \text{s.t. } G \text{ can be embedded into } G_{i_0}.\end{aligned} \tag{1.1}$$

*Proof.* (Claim 1.5) First note that it suffices to prove 2), as 1) is just a special case with $\chi$ being equal to 1.

$(*)_1$ Let (in $\mathbf{V}$) $\langle (U_\vartheta, \xi_\vartheta, Z_\vartheta) : \vartheta < \kappa\rangle$ list
$$\begin{aligned}\{(U, \xi, Z) :\ & Z \in V_\kappa,\ \xi < \kappa \text{ is a cardinal}, \\ & U \text{ is a code for an } \xi - \text{Borel subset of } Z\},\end{aligned}$$

Assume that

$(*)_2$ there is a sequence $\bar{M} = \langle M_\delta : \delta < \chi\rangle$ in $(K_\kappa)_\lambda$ that forms a universal sequence for $(K_\kappa)_\lambda$ (in the universe $\mathbf{V}$, of course) i.e. $\bar{M}$ witnesses $(iv)^\chi$;

where $M_\delta = (\lambda, \dots, R^{M_\delta}_\varepsilon, \dots)_{\varepsilon<\kappa}$. It is enough to prove that $\mathbf{V}_\iota$ satisfies (1.1).

Now we define the sequence of $\mathbb{R}_{\mathbf{r}}$-names $\underset{\sim}{G}_\delta$ $(\delta < \chi)$ for graphs as follows.

$(*)_3$ (a) the set of nodes of $\underset{\sim}{G}_\delta$ is $\lambda$ (and so $R^{\underset{\sim}{G}_\delta} \subseteq \lambda \times \lambda$),

(b) for $\alpha \neq \beta < \lambda$ let the truth value of "$(\alpha,\beta) \in R^{\underset{\sim}{G}_\delta}$" is defined as follows. For the unique $\vartheta < \kappa$ with $(\alpha,\beta) \in R^{M_\delta}_\vartheta$ we demand

$$\mathbf{V}_\iota \models \ (\alpha,\beta) \in R^{\underset{\sim}{G}_\delta} \quad \Longleftrightarrow \quad \underset{\sim}{X} \cap Z_\vartheta \in B_{U_\vartheta}.$$

So clearly

$(*)_4$ for each $\delta < \chi$ $\underset{\sim}{G}_\delta$ is an $\mathbb{R}_{\mathbf{r}}$-name for a graph with set of nodes $\lambda$.

Hence it suffices to prove:

$(*)_5$ $\Vdash$ "$\mathbf{V}_\iota \models \langle \underset{\sim}{G}_\delta : \ \delta < \chi\rangle$ is a universal sequence in the class of graphs of size $\lambda$".

So why does $(*)_5$ hold? Assume

$(*)_{5.1}$ $p \Vdash$ "$\underset{\sim}{G}_* \in \mathbf{V}_\iota$ is a graph with set of nodes $\lambda$".

Let $\langle (\alpha_\gamma, \beta_\gamma) : \gamma < \lambda\rangle \in \mathbf{V}$ list the set of pairs $(\alpha,\beta)$ such that $\alpha < \beta < \lambda$. For each $\gamma < \lambda$ (considering the $\mathbb{R}_{\mathbf{r}}$-names $\underset{\sim}{\tau}_\gamma$ for the truth value of $(\alpha_\gamma,\beta_\gamma) \in R^{\underset{\sim}{G}_*}$) clause (g) of Definition 1.2 1) gives $\bar q_\gamma = \langle q_{\gamma,\varepsilon} : \ \varepsilon < \kappa\rangle \in \mathscr{S}_p$, $\bar\zeta_\gamma = \langle \zeta_{\gamma,\varepsilon} : \ \varepsilon < \kappa\rangle \in {}^\kappa\kappa$ and $\bar T_\gamma = \langle T_{\gamma,\varepsilon} : \varepsilon < \kappa\rangle$, $\langle Y_{\gamma,\varepsilon} : \ \varepsilon < \kappa\rangle$ such that for each $\gamma < \lambda$ and $\varepsilon < \kappa$

- $\bullet_1$ $T_{\gamma,\varepsilon}$ is a code for a $\zeta_{\gamma,\varepsilon}$-Borel subset of $\mathscr{P}(Y_{\gamma,\varepsilon})$ (in the sense of Definition 1.1 (2)-(4)))
- $\bullet_2$ $q_{\gamma,\varepsilon} \Vdash_{\mathbb{R}} \ (\alpha_\gamma,\beta_\gamma) \in R^{\underset{\sim}{G}_*} \ \Leftrightarrow \ \underset{\sim}{X} \cap Y_{\gamma,\varepsilon} \in B_{T_{\gamma,\varepsilon}}$.

Now by clause (h) of Definition 1.2 1), there are $q_* \in \mathbb{R}$, $\langle \varepsilon_\gamma = \varepsilon(\gamma) : \gamma < \lambda\rangle \in {}^\lambda\kappa$ such that:

- $\bullet_3$ $q_*$ is above $q_{\gamma,\varepsilon(\gamma)}$ for every $\gamma < \lambda$,

and recalling the enumeration from $(*)_1$, there exists $\langle \vartheta_\gamma = \vartheta(\gamma) : \ \gamma < \lambda\rangle \in {}^\lambda\kappa$ such that

- $\bullet_4$ $(T_{\gamma,\varepsilon(\gamma)}, \zeta_{\gamma,\varepsilon(\gamma)}, Z_{\gamma,\varepsilon(\gamma)}) = (U_{\vartheta(\gamma)}, \xi_{\vartheta(\gamma)}, Y_{\vartheta(\gamma)})$ holds for every $\gamma < \lambda$.

Now we define the model $M_* \in (K_\kappa)_\lambda \cap \mathbf{V}$ as follows:

$(*)_{5.3}$ (a) $M_* = (\lambda, (R^{M_*}_\alpha)_{\alpha<\kappa})$, where

(b) for every $\vartheta \in \kappa$ we have

$$R^{M_*}_\vartheta = \{(\alpha_\gamma,\beta_\gamma) : \ (\gamma < \lambda) \ \wedge \ (\vartheta(\gamma) = \vartheta)\}.$$

Clearly

$(*)_{5.4}$ $M_* \in (K_\kappa)_\lambda$ (with the underlying set of nodes being $\lambda$), $M_*$ belongs to $\mathbf{V}$.

Now choose a suitable $\delta < \chi$ and a function $f$ so that:

$(*)_{5.5}$ $f : M^* \to M_\delta$ is an embedding, $f \in \mathbf{V}$

[which exists by $(*)_2$.] Finally it remains to check that

$(*)_{5.6}$ $q_* \Vdash$ "$f$ is an embedding of $\underset{\sim}{G}_*$ into $\underset{\sim}{G}_\delta$".

Recall that $q_* \geq q_{\gamma,\varepsilon(\gamma)}$ for each $\gamma < \lambda$ by $\bullet_3$. Fix $\gamma < \lambda$. Using $\bullet_2$ and $\bullet_4$ we get

$$q_{\gamma,\varepsilon(\gamma)} \leq q_* \Vdash \ (\alpha_\gamma, \beta_\gamma) \in R^{\underset{\sim}{G}_*} \ \Leftrightarrow \ \underset{\sim}{X} \cap Z_{\theta(\gamma)} \in B_{U_{\theta(\gamma)}}. \tag{1.2}$$

Also, note that by $(*)_{5.3}$ the color of the pair $(\alpha_\gamma, \beta_\gamma)$ in $M_*$ is $\vartheta(\gamma)$, i.e. $(\alpha_\gamma, \beta_\gamma) \in R^{M_*}_{\vartheta(\gamma)}$, and as $f : M^* \to M_\delta$ is an embedding, clearly

$$(f(\alpha_\gamma), f(\beta_\gamma)) \in R^{M_\delta}_{\vartheta(\gamma)}.$$

Recalling $(*)_3$, we obtain

$$\Vdash \ [(f(\alpha_\gamma), f(\beta_\gamma)) \in R^{\underset{\sim}{G}_\delta} \iff \underset{\sim}{X} \cap Z_{\vartheta(\gamma)} \in B_{U_{\vartheta(\gamma)}}]. \tag{1.3}$$

Finally, combining (1.2) and(1.3) we obtain

$$q_* \Vdash \ [(\alpha_\gamma, \beta_\gamma) \in R^{\underset{\sim}{G}_*} \iff (f(\alpha_\gamma), f(\beta_\gamma)) \in R^{\underset{\sim}{G}_\delta}],$$

as desired.

$\square_{\text{Claim1.5}}$

Naturally we can ask:

*Question* 1.6.
1) What can we say about universals in $(K_\kappa)_\lambda$?
2) An old open problem concerns the case of the theory of triangle free graphs [Mek90], and similarly it is open for $T_{\text{feq}}$ (equivalently $T_{\text{ceq}}$, see [Sheb]). On $T_{\text{feq}}$ we refer the reader to [She93], or [DS04], and on consistent instances of non-existence of universals in case of $T_{\text{ceq}}$ see [Sheb].
3) Moreover, what can we say about $(\text{Mod}_T, \prec)$ for $T$ simple? Or even $\text{NSOP}_2$? (of cardinality $< \kappa$). We have to be more careful because of, e.g. function symbols.

A work in preparation deals with 1.6 2), 3). Concerning 1.6 1) we have the following negative result (note that this does not reflect on Claim 1.5):

**Claim 1.7.** *Assume $\kappa$ is strong limit singular and $\kappa < \lambda < 2^\kappa$. Then in $(K_\kappa)_\lambda$ there is no universal member.*

*Proof.* By [She06, Thm 1.13 and 1.14 (2) on RGCH]

- $(*)_0$ there is a regular $\sigma \in (\text{cf}(\kappa), \kappa)$ such that $\lambda^{[\sigma,\kappa]} = \lambda$, i.e. there is $\mathscr{P}' \subseteq \{u \subseteq \lambda : |u| \leq \kappa\}$ of cardinality $\lambda$ such that every $u \subseteq \lambda$ of cardinality $\leq \kappa$ is the union $< \sigma$ members of $\mathscr{P}'$.

Therefore, as $\sigma = \text{cf}(\sigma) > \text{cf}(\kappa)$, replacing each $u \in \mathscr{P}'$ with a collection $u_\alpha \in [u]^{<\kappa}$ $(\alpha < \text{cf}(\kappa))$ satisfying $u = \bigcup_{\alpha<\text{cf}(\kappa)} u_\alpha$ we obtain

- $(*)_1$ there is $\mathscr{P} \subseteq \{u \subseteq \lambda : |u| < \kappa\}$ of cardinality $\lambda$ such that every $u \subseteq \lambda$ of cardinality $\leq \kappa$ is the union $< \sigma$ members of $\mathscr{P}$.

Fix $M_* \in (K_\kappa)_\lambda$ and we shall prove that it is not universal; without loss of generality the universe of $M_*$ is $\lambda$. Now for each $u \in \mathscr{P}$ and $\alpha < \lambda$ let

$$v(\alpha, u, M_*) = \{\varepsilon < \kappa : \text{ for some } \beta \in u \text{ we have } (\alpha, \beta) \in R^{M_*}_\varepsilon\},$$

so $v(\alpha, u, M_*) \subseteq \kappa$ has cardinality $< \kappa$. Let

$$\mathscr{P}_1 = \{w \in [v(\alpha, u, M_*)]^{\leq \text{cf}(\kappa)} : \ u \in \mathscr{P}, \alpha \in \lambda\},$$

so

- $(*)_2$ $\mathscr{P}_1 \subseteq [\kappa]^{\leq \text{cf}(\kappa)}$.

Now

$(*)_3$ $|\mathscr{P}_1| \le |\mathscr{P}| + 2^{<\kappa} \le \lambda < 2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.

Hence

$(*)_4$ we can find $v \subseteq \kappa$ of cardinality $\mathrm{cf}(\kappa)$ which is not in $\mathscr{P}_1$, moreover, $u \in \mathscr{P}_1 \Rightarrow |u \cap v| < \mathrm{cf}(\kappa)$,

which is justified by the following argument: Let $\langle v_\gamma : \gamma < 2^\kappa \rangle$ be a sequence of members of $[\kappa]^{\mathrm{cf}(\kappa)}$ with any two having intersection of cardinality $< \mathrm{cf}(\kappa)$, hence for every $u \in \mathscr{P}_1, \{\gamma < 2^\kappa : |u \cap v_\gamma| = \mathrm{cf}(\kappa)\}$ has cardinality $\le 2^{\mathrm{cf}(\kappa)} < \kappa$, so all but $\le \lambda$ of the $v_\gamma$'s are as required.

Now consider the following $N$:

$(*)_5$ (a) $N = (A \cup B, \ldots, R^N_\varepsilon, \ldots)_{\varepsilon<\kappa}$ belongs to $(K_\kappa)_{\sigma^{\mathrm{cf}(\kappa)}}$, where $|A| = \sigma$, $|B| = \sigma^{\mathrm{cf}(\kappa)}$, $A \cap B = \emptyset$,
(b) $R^N_\varepsilon \ne \emptyset$ <u>iff</u> $\varepsilon \in v$,
(c) letting $\langle \varepsilon_i : i < \mathrm{cf}(\kappa)\rangle$ list $v$ (from $(*)_4$), for every sequence $\bar\alpha = \langle \alpha_i : i < \mathrm{cf}(\kappa)\rangle$ in $A$ with no repetitions there is $\beta = \beta(\bar\alpha) \in B$ such that $(\alpha_i, \beta) \in R^N_{\varepsilon_i}$ for $i < \mathrm{cf}(\kappa)$.

Now if $g$ embeds $N$ into $M_*$ then since $|\mathrm{Rang}(g \restriction A)| = \sigma < \kappa$, by $(*)_1$ it is the case that for some $\{u_\varepsilon : \varepsilon < \partial < \sigma\} \subseteq \mathscr{P}$, we have $\mathrm{Rang}(g \restriction A) = \cup\{u_\varepsilon : \varepsilon < \partial\}$. Now as $|A| = \sigma = \mathrm{cf}(\sigma)$ but $\partial < \sigma$, there is $\varepsilon < \partial$ such that $|u_\varepsilon \cap \mathrm{Rang}(g \restriction A)| \ge \sigma \ge \mathrm{cf}(\kappa)$ so we can choose pairwise distinct $\alpha_i \in A$ $(i < \mathrm{cf}(\kappa))$ such that $\{g(\alpha_i) : i < \mathrm{cf}(\kappa)\} \subseteq u_\varepsilon$. Let $\beta = \beta(\bar\alpha) \in B$ given by $(*)_5(c)$. So $g(\beta)$ is well defined and we get an easy contradiction by $(*)_4$.

This shows that $N$ cannot be embedded into $M_*$, hence we are done. $\square_{1.7}$

*Remark* 1.8. In fact, the argument above could be modified so that it work with weaker assumptions: the conditions $\beth_\omega(\mathrm{cf}(\kappa)) < \kappa$, and $(\alpha < \kappa \ \rightarrow\ |\alpha|^{\mathrm{cf}(\kappa)} < \kappa)$ together are sufficient.

## § 2. Proving known forcings fit the framework

### § 2(A). **Near a Large Singular.**

Here we do not collapse cardinals, just change cofinalities.

**Claim 2.1.** *There is a nice $(\lambda, \kappa)$-system $\mathbf{r}$ such that $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$ when the following hold:*

*(A) (a) $\kappa < \lambda < 2^\kappa$ are cardinals,*
*(b) $D$ is a normal ultrafilter on $\kappa$,*
*(c) if $\mathscr{A} \subseteq D$ has cardinality $\le \lambda$, <u>then</u> for some $B \in D$ we have $(\forall A \in \mathscr{A})(B \subseteq A \mod [\kappa]^{<\kappa})$, (e.g. $D$ is generated by a $\subseteq^*_\kappa$-decreasing sequence of length of a regular cardinal $> \lambda$),*
*(d) $\mathbb{P}$ is the Prikry forcing for $D$ (so $\mathbb{P}$ changes the cofinality of $\kappa$ to $\aleph_0$ and adds no bounded subset of $\kappa$ and satisfies the $\kappa^+$-c.c).*

*Proof.* Recalling the definition of Prikry forcing for $D$:

$(*)_1$ (a) $p \in \mathbb{P}$ iff $p = (w, A) = (w_p, A_p)$, where $w_p \in [\kappa]^{<\aleph_0}$ and $A_p \in D$ and $[0, \max w_p] \cap A = \emptyset$,
(b) $p \leq_{\mathbb{P}} q$ iff $w_p \subseteq w_q \subseteq w_p \cup A_p$ and $A_p \supseteq A_q$.

We define the system $\mathbf{r}$ by letting:

$(*)_2$ (a) $\kappa_{\mathbf{r}} = \kappa$,
(b) $\lambda_{\mathbf{r}} = \lambda$,
(c) $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$,
(d) $\underaccent{\sim}{X}_{\mathbf{r}} =$ the Prikry generic sequence $= \cup\{w_p : p \in \mathbf{G}_{\mathbb{P}}\}$,
(e) $\leq_{\rm pr} = \leq_{\mathbf{r},{\rm pr}}$ is defined by $p \leq_{\rm pr} q$ iff $w_p = w_q \wedge A_p \supseteq A_q$ (and $p, q \in \mathbb{R}_{\mathbf{r}}$),
(f) for $p \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ let $\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \{\bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle$ and for some $B \in D$ we have $B \subseteq A_p$ and $\{A_{q_\varepsilon} : \varepsilon < \kappa\}$ list $\{A : A \subseteq A_p$ and $A \equiv B \mod [\kappa]^{<\kappa}\}$.

We still have to prove that $\mathbf{r}$ is as required, namely, that $\mathbf{r}$ satisfies conditions listed in Definition 1.2 1).

Now clauses (a)-(f) from Definition 1.2 1) hold trivially. For clause (g) fix $p$, $\underaccent{\sim}{\tau}$, with $p \Vdash_{\mathbb{P}}$ "$\tau \in \{0,1\}$". Recall the following well-known fact:

$(*)_3$ if $p \in \mathbb{P}$, $p \Vdash_{\mathbb{P}}$ "$\underaccent{\sim}{\tau} \in \{0,1\}$", then for some $A' \subseteq A_p$, $A' \in D$ we have: if $\alpha \in \kappa$ and $u \subseteq A_p \cap \alpha$ is finite then $(w_p \cup u, A' \setminus \alpha)$ forces a value for $\underaccent{\sim}{\tau}$.

[For the sake of completeness we prove $(*)_3$: by the Prikry-lemma, for each $s \in [A_p]^{<\aleph_0}$ there exists $A_s \subseteq A_p \setminus ((\max s) + 1)$, $A_s \in D$, such that $(w \cup s, A_s)$ decides the value of $\underaccent{\sim}{\tau}$. Now let $A'$ be the diagonal intersection of $A_s$'s ($s \in [A_p]^{<\aleph_0}$), pedantically $\Delta_{\alpha<\kappa}(\bigcap_{s \in [\alpha+1]^{<\aleph_0}} A_s)$, it is straightforward to check that $A'$ works.]

So given $p \in \mathbb{P}, \gamma$ and $\underaccent{\sim}{\tau}$ as in clause (g) from Definition 1.2, let $A' \subseteq A_p$ be as in $(*)_3$ and let $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle$ be defined by: $q_\varepsilon \in \mathbb{P}, w_{q_\varepsilon} = w_p$ and $\{A_{q_\varepsilon} : \varepsilon < \kappa\}$ list $\{A \subseteq A_p : A \equiv A' \mod [\kappa]^{<\kappa}\}$.

We still have to choose the $Y_\varepsilon, T_\varepsilon$. For each $\varepsilon$ choose $\zeta_\varepsilon \in A_{q_\varepsilon}$ such that $A_{q_\varepsilon} \setminus \zeta_\varepsilon = A' \setminus \zeta_\varepsilon$. Clause $(*)_3$ ensures that there is a function $f : [A_p \cap \zeta_\varepsilon]^{<\aleph_0} \to \{0,1\}$ in $\mathbf{V}$ such that $q_\varepsilon \Vdash \underaccent{\sim}{\tau} = f(\underaccent{\sim}{X} \cap \zeta_\varepsilon)$. This means we can let $Y_\varepsilon = \gamma_\varepsilon$, and choose a $\gamma_\varepsilon$-Borel code $T_\varepsilon$ such that whenever $w \in B_{T_\varepsilon}$ necessarily $w \in [\gamma_\varepsilon]^{<\aleph_0}$, and

$$q_\varepsilon \Vdash (\underaccent{\sim}{\tau} = 1) \iff (\underaccent{\sim}{X} \cap \zeta_\varepsilon) \in B_{T_\varepsilon}.$$

Lastly, for clause (h), assume $p \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ and $\bar{q} = \langle \bar{q}_\alpha : \alpha < \lambda\rangle$ satisfies $\bar{q}_\alpha \in \mathscr{S}_p$. So for each $\alpha < \lambda$ there exists $B_\alpha \subseteq A_p$ such that $\{A_{q_{\alpha,\varepsilon}} : \varepsilon < \kappa\}$ lists $\{A \in D : A \subseteq A_p,\ A \equiv B_\alpha \mod [\kappa]^{<\kappa}\}$, hence by clause (A)(c) of the assumption of the claim, there is $B \in D$, a subset of $A_p$ such that $B \subseteq B_\alpha \mod [\kappa]^{<\kappa}$ for each $\alpha \in \lambda$ and let $q_* = (w_p, B)$ so clearly $p \leq_{\rm pr} q_*$. Also for each $\alpha < \lambda$, for some $\zeta < \kappa$ we have $B \setminus \zeta \subseteq B_\alpha$. Finally, because $\bar{q}_\alpha \in \mathscr{S}_p$ we have that for some $\varepsilon < \kappa$ $A_{q_{\alpha,\varepsilon}} = (B_\alpha \setminus \zeta) \cup (A_p \cap \zeta) \supseteq B$ hence $q_{\alpha,\varepsilon} \leq q_*$.

We still have to prove that $\mathbf{r}$ is nice but as $\mathbb{P}$ satisfies the $\kappa^+$-c.c., and by the Prikry lemma this is obvious. $\square_{2.1}$

**Claim 2.2.** *There is a $(\lambda, \kappa) - 1$-system $\mathbb{R}_{\mathbf{r}}$ with $\mathbf{V}^{\mathbb{R}_{\mathbf{r}}} \models {\rm cf}(\kappa) = \theta$, when $(B)$ holds:*

*(B)* *(a)* $\theta = \mathrm{cf}(\theta) < \theta_* < \kappa < \lambda < 2^\kappa$,

*(b)* $\bar{D} = \langle D_i : i < \theta\rangle$ *is a sequence of normal ultrafilters on* $\kappa$, *increasing in Mitchell order, i.e.* $i < j \Rightarrow D_i \in \mathrm{MosCol}({}^\kappa\mathbf{V}/D_j)$,

*(c)* *each* $D_i$ *(*$i < \theta$*) is* $< \lambda^+$*-directed mod* $[\kappa]^{<\kappa}$, *i.e. satisfies the condition (A)(c) from Claim* 2.1.

*Moreover, the forcing* $\mathbb{R}_\mathbf{r}$ *changes the cofinality of* $\kappa$ *to* $\theta$, *preserves each cardinal and the function* $\mu \mapsto 2^\mu$, *satisfies the* $\kappa^+$*-c.c. Moreover, we can prescribe that in* $\mathbf{V}^\mathbb{P}$ *there is no new subset of* $\theta_*$.

*Proof.* Using [Kru07, Proposition 2.1], condition (b) implies the following.

**Subclaim 2.3.** *If* $\bar{D} = \langle D_i : i < \theta\rangle$ *is an increasing (w.r.t. the Mitchell order) sequence of normal ultrafilters on* $\kappa$, $\theta \le \kappa$, *then there exists a coherent sequence* $\langle \bar{U}_\varepsilon : \varepsilon < \kappa+1\rangle$, $\bar{U}_\varepsilon = \langle U_\varepsilon(\alpha) : \alpha < o^U(\varepsilon)\rangle$ *for some function* $o^U : \kappa+1 \to \kappa$ *such that* $\bar{D} = \bar{U}_\kappa$, *which means:*

$(\top)_a$ *for each* $\varepsilon \le \kappa$, $\alpha < o^U(\varepsilon)$ $U_\varepsilon(\alpha)$ *is an* $\varepsilon$*-complete normal ultrafilter on* $\varepsilon$,

$(\top)_b$ *moreover, for each* $\varepsilon \le \kappa$ *and* $\alpha < o^U(\varepsilon)$, *letting* $\mathbf{j}_{\varepsilon,\alpha} : \mathbf{V} \to \mathrm{MosCol}({}^\varepsilon\mathbf{V}/U_{\varepsilon,\alpha})$ *be the associated elementary embedding, we have*

$$(\mathbf{j}_{\varepsilon,\alpha}(\bar{U} \restriction \varepsilon))_\varepsilon = \langle U_\varepsilon(\beta) : \beta < \alpha\rangle,$$

$(\top)_c$ $\langle U_\kappa(\alpha) : \alpha < o^U(\kappa)\rangle = \langle D_\alpha : \alpha < \theta\rangle$.

Now we define the forcing $\mathbb{P}_{\overline{U}}$ to be the Magidor forcing associated with the sequence $\bar{D} = \bar{U}_\kappa = \langle U_\kappa(\alpha) : \alpha \le \theta\rangle$ , (see also [Mag78], or [Git10]), here we use the definition from [Git10, Definition 5.22]

**Definition 2.4.** Define $\mathbb{P}_{\overline{U}}$ to be the following (auxiliary) poset.

$(*_1)$ Let $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_\kappa\rangle\rangle \in \mathbb{P}_{\overline{U}}$, iff

(a) $A_\kappa \in \bigcap \bar{U}_\kappa = \bigcap_{\alpha<\theta} U_{\kappa,\alpha}$,

(b) each $d_j$ ($j \le n$) is of the form

- either $\langle \varepsilon, A_\varepsilon\rangle$ for some $\varepsilon < \kappa$, where $o^U(\varepsilon) > 0$, moreover,

$$A_\varepsilon \in \bigcap \bar{U}_\varepsilon = \bigcap_{\gamma < o^U(\varepsilon)} U_{\varepsilon,\gamma},$$

  (this case we define $\kappa(d_j) = \varepsilon$),
- or $d_j = \varepsilon$, when $o^U(\varepsilon) = 0$ (and we let $\kappa(d_j) = d_j = \varepsilon$).

(c) $\kappa(d_0) < \kappa(d_1) < \dots < \kappa(d_n) < \kappa(d_{n+1}) = \kappa$,

(d) moreover, for each $j \le n$, if $d_{j+1}$ is a pair, then $\kappa(d_j) < \min A_{\kappa(d_{j+1})}$.

$(*_2)$ We define

$$p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_\kappa\rangle\rangle \le q = \langle e_0, e_1, \dots, e_m, e_{m+1} = \langle \kappa, B_\kappa\rangle\rangle,$$

if

(a) $m \ge n$, and

(b) there exists a sequence $0 \le i_0 < i_1 < \dots < i_n < j_{n+1} = m+1$ such that for each $j \le n+1$ we have

- $\kappa(d_j) = \kappa(e_{i_j})$, and
- $B_{\kappa(d_j)} \subseteq A_{\kappa(d_j)}$,

(c) moreover, for each $k \le m$ not of the form $i_j$ ($j \le n+1$), if $i_+ = \min\{i_j : j \le n+1, i_j > k\}$, then

$$B_{\kappa(e_k)} \cup \{\kappa(e_k)\} \subseteq A_{\kappa(d_{i_+})}.$$

($*_3$) Now if we define the pairwise disjoint sets $Y_\alpha$ ($\alpha < \theta$) as

$$\delta \in Y_\alpha \iff o^U(\delta) = \alpha,$$

then

$$\{p \in \mathbb{P}_{\overline{U}} : \ p \geq \langle\langle \kappa, \bigcup_{\alpha<\theta} Y_\alpha\rangle\rangle\}$$

is the Magidor forcing changing the cofinality of $\kappa$ to $\max\{\omega, \mathrm{cf}(\theta)\}$.

**Definition 2.5.** We define $p \leq_* q$ to be true iff $p \leq q$ and $\ell g(p) = \ell g(q)$.

We define the system $\mathbf{r}$ by letting:

($*_4$) (a) $\kappa_{\mathbf{r}} = \kappa$,
(b) $\lambda_{\mathbf{r}} = \lambda$,
(c) $\mathbb{R}_{\mathbf{r}} = \{p \in \mathbb{P}_{\overline{U}} : \ p \geq \langle\langle \kappa, \bigcup_{\alpha<\theta} Y_\alpha\rangle\rangle\}$,
(d) let $\underset{\sim}{X}_{\mathbf{r}}$ be the generic sequence, i.e.

$$\underset{\sim}{X}_{\mathbf{r}} = \cup\{\{\kappa(d_j) : j < \ell g(p)\} : \ p = \langle d_0, d_1, \dots, d_{\ell g(p)-1}\rangle \in \mathbf{G}_{\mathbb{P}}\} \setminus \{\kappa\},$$

(e) $\leq_{\mathrm{pr}} = \leq_{\mathbf{r},\mathrm{pr}}$ is defined by $p \leq_{\mathrm{pr}} q$ iff $p \leq_* q$,
(f) for $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$, let

$$\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \left\{ \begin{array}{l} \bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle, \text{ where} \\ (\bullet_1) q_\varepsilon = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{q_\varepsilon,\kappa}\rangle\rangle, \text{ and} \\ \text{for some } B \in \bigcap \bar{U}_\kappa \text{ we have} \\ (\bullet_2) \ B \subseteq A_{p,\kappa}, \text{ and} \\ (\bullet_3) \ \{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \ \wedge \ A_* \equiv B \mod [\kappa]^{<\kappa}\} \end{array} \right\}.$$

It is known that $\underset{\sim}{X}$ is a club of $\kappa$ of order type $\theta$, moreover, if condition $\langle\langle\beta\rangle, \langle\kappa, A\rangle\rangle$ is in the generic filter (for some $\beta < \kappa$, $o^U(\beta) = 0$, then the forcing adds no new subset to $\beta$. Therefore (it is not difficult to see that) by $(\mathsf{T})_b$ the set $\{\beta < \kappa : o^U(\beta) = 0\} \in U_{\kappa,0}$, and so we can we can limit ourselves to the subposet consisting of conditions above $\langle\langle\beta\rangle, \langle\kappa, \bigcup_{\alpha<\theta} Y_\theta\rangle\rangle$ for some $\beta \geq \theta_*$. In order to finish the proof of Claim 2.2 it suffices to verify that the forcing defined in ($*_3$) is a $(\lambda, \kappa) - 1$-system.

**Subclaim 2.6.** *If $\langle \bar{U}_\varepsilon : \ \varepsilon < \kappa + 1\rangle$ is a coherent sequence, where the ultrafilters $\{U_\kappa(\alpha) : \ \alpha < o^{\bar{U}}(\kappa)\}$ are $< \lambda^+$-directed mod $[\kappa]^{<\kappa}$, then the forcing $\mathbb{P}_{\bar{U}}$ from Definition 2.4 is a $(\lambda, \kappa) - 1$-system.*

*Proof.* Now we have only to check the requirements of Definition 1.2 1). Recall the following properties of the Magidor forcing, see [Git10, Sec. 5.1 and 5.2].

**Fact 2.7.** *(Prikry Lemma) For each $p \in \mathbb{P}_{\overline{U}}$ and each formula $\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ there exists $q \geq_* p$, $q \parallel \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ (i.e. either $q \Vdash \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$, or $q \Vdash \neg\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$).*

*Notation* 2.8. If $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{P}_{\overline{U}}$, and $i \leq n+1$, then $q \restriction (\kappa(d_i) + 1)$ refers to the condition $\langle d_0, d_1, \dots, d_i\rangle$.

**Fact 2.9.** *Suppose that $\mathbf{G} \subseteq \mathbb{P}_{\overline{U}}$ is generic over $\mathbf{V}$, $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbf{G}$, $i \leq n+1$, $d_i = \langle \kappa(d_i), A_{\kappa(d_i)}\rangle$, then the filter $\mathbf{G} \restriction (\kappa(d_i)+1) := \{q \restriction (\kappa(d_i) + 1) : \ q \in \mathbf{G}\}$ is $\mathbf{V}$-generic over the Prikry forcing $\mathbb{P}_{\overline{U}\restriction(\kappa(d_i)+1)}$ associated with the coherent sequence $\langle \overline{U}_\delta = \langle U_\delta(\gamma) : \ \gamma < o^U(\delta)\rangle : \ \delta \leq \kappa(d_i)\rangle$.*

The Prikry Lemma and the subforcing $\mathbb{P}_{\overline{U}\restriction(\delta+1)}$ together give the following.

**Fact 2.10.** *For each* $\delta < \kappa$, $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \delta, A_{p,\delta} \rangle \rangle \in \mathbb{P}_{\overline{U} \restriction (\delta+1)}$ *and each formula* $\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ *there exists* $q \in \mathbb{P}_{\overline{U} \restriction (\delta+1)}$,

$$q \geq_* p \text{ (in the sense of } \mathbb{P}_{\overline{U} \restriction (\delta+1)}),$$

*such that for some* $A \in \bigcap \overline{U}_\kappa$ *we have*

$$q \frown \langle \kappa, A \rangle \parallel_{\mathbb{P}_{\overline{U}}} \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m).$$

**Lemma 2.11.** *Suppose that* $\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ *is a formula,* $\delta < \kappa$, $p = \langle d_0, d_1, \dots, d_n \rangle \in \mathbf{G} \restriction (\kappa(d_n) + 1)$, $\delta = \kappa(d_n)$, *the filter* $\mathbf{G} \subseteq \mathbb{P}_{\overline{U}}$ *is generic over* $\mathbf{V}$ *(so that* $p = p' \restriction (\delta + 1)$ *for some* $p' \in \mathbf{G}$, *and* $p' \Vdash \delta \in \underset{\sim}{X}_{\mathfrak{r}}$*).*

*Then there exists* $q \in \mathbf{G} \restriction (\delta+1)$, $\mathbb{P}_{\overline{U} \restriction (\delta+1)} \models q \geq p$, *such that for some* $A \in \bigcap \overline{U}_\kappa$ *letting* $q' = q \frown \langle \kappa, A \rangle$ *we have* $q' \in \mathbb{P}_{\overline{U}}$ *and*

$$q' = q \frown \langle \kappa, A \rangle \parallel_{\mathbb{P}_{\overline{U}}} \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m). \tag{2.1}$$

*Proof.* This is a standard density argument: First using Fact 2.9 $\mathbf{G} \restriction (\delta + 1) \subseteq \mathbb{P}_{\overline{U} \restriction (\delta+1)}$ is generic, and so by Fact 2.10 there exists $q \in \mathbf{G} \restriction (\delta + 1)$,

$$q \geq p \text{ (in the sense of } \mathbb{P}_{\overline{U} \restriction (\delta+1)}),$$

such that for some $A \in \bigcap \overline{U}_\kappa$ the condition $q \frown \langle \kappa, A \rangle \in \mathbb{P}_{\overline{U}}$ decides about $\sigma$. $\square$

Similarly to the case of Prikry forcing, this has the following consequence.

**Claim 2.12.** *For each* $p = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{p,\kappa} \rangle \rangle \in \mathbb{P}_{\overline{U}}$ *and* $\underset{\sim}{\tau}$ *(with* $p \Vdash \underset{\sim}{\tau} \in \{0, 1\}$*) there exists a set* $A' \in \bigcap \bar{U}_\kappa$, $A' \subseteq A_{p,\kappa}$, *such that the condition* $p' = \langle d_0, d_1, \dots, d_n, \langle \kappa, A' \rangle \rangle$ *satisfies the following:*

*Whenever* $\alpha \in A_{p,\kappa}$, $q = \langle e_0, e_1, \dots, e_m, \langle \kappa, A_{q,\kappa} \rangle \rangle \geq p' = \langle d_0, d_1, \dots, d_n, \langle \kappa, A' \rangle \rangle$ *are given with* $\kappa(e_m) \leq \alpha$, *and* $q$ *forces a value to* $\underset{\sim}{\tau}$, *then so does*

$$q' = \langle e_0, e_1, \dots, e_m, \langle \kappa, A' \cap (\alpha, \kappa) \rangle \rangle ,$$

*i.e.*

$$q' \parallel_{\mathbb{P}_{\overline{U}}} \text{“} \underset{\sim}{\tau} = 1 \text{”}.$$

*Proof.* For each $\alpha \in A_{p,\kappa}$ define $B_\alpha \subseteq A_{p,\kappa}$ so that whenever

$$q = \langle e_0, e_1, \dots, e_m, e_{m+1} = \langle \kappa, A_{q,\kappa} \rangle \rangle \geq p$$

(with $\kappa(e_0), \kappa(e_1), \dots, \kappa(e_m) \leq \alpha$) decides the value of $\underset{\sim}{\tau}$, then so does

$$q' = \langle e_0, e_1, \dots, e_{m+1} = \langle \kappa, B_\alpha \rangle \rangle.$$

This can be done easily: first for each possible $e_0, e_1, \dots, e_m$ choose a set $B_{e_0, e_1, \dots, e_m} \subseteq (\alpha, \kappa)$ with

$$\langle e_0, e_1, \dots, e_m, \langle \kappa, B_{e_0, e_1, \dots, e_m} \rangle \rangle \text{ deciding the value of } \underset{\sim}{\tau},$$

if such a $B_{e_0, e_1, \dots, e_m}$ exists, otherwise just let $B_{e_0, e_1, \dots, e_m} = A_{p,\kappa} \cap (\alpha, \kappa)$. Second, let $B_\alpha = \bigcap_{e_0, e_1, \dots, e_m} B_{e_0, e_1, \dots, e_m}$. Now it is easy to check that the diagonal intersection $A' = \Delta_{\alpha \in A_{p,\kappa}} B_\alpha \in \bigcap \bar{U}_\kappa$ works (note that the intersection of normal measures is a normal filter). $\square$

**Claim 2.13.** *For every* $p \in \mathbb{P}_{\overline{U}}$ *and* $\underset{\sim}{\tau}$, *if* $p \Vdash \underset{\sim}{\tau} \in \{0, 1\}$, *then we can choose* $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa \rangle \in \mathscr{P}_p$, $\langle \gamma_\varepsilon : \varepsilon < \kappa \rangle$, $\langle T_\varepsilon : \varepsilon < \kappa \rangle$, $\langle Y_\varepsilon : \varepsilon < \kappa \rangle$, *where each* $T_\varepsilon$ *is a code for a* $\gamma_\varepsilon$*-Borel subset of* $\mathscr{P}(Y_\varepsilon)$ *such that*

$$q_\varepsilon \Vdash \underset{\sim}{\tau} = 1 \iff (\underset{\sim}{X} \cap \gamma_\varepsilon) \in B_{T_\varepsilon}.$$

*Proof.* First if $p = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ are in the Lemma, let $A' = A'(p, \underset{\sim}{\tau}) \subseteq A_{p,\kappa}$ be given by Claim 2.12 and

($*_5$) let $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle \in \mathscr{S}_p$ be defined by: $q_\varepsilon \in \mathbb{P}, q_\varepsilon = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{q_\varepsilon,\kappa}\rangle\rangle$ where $\{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\}$ lists $\{A_* \subseteq A_{p,\kappa} : A_* \equiv A' \mod [\kappa]^{<\kappa}\}$.

We still have to choose $\gamma_\varepsilon, T_\varepsilon, Y_\varepsilon$. For each $\varepsilon$ choose $\zeta_\varepsilon \in A_{q_\varepsilon,\kappa} \setminus \kappa(d_n)$ such that

$$A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) = A' \setminus (\zeta_\varepsilon + 1). \tag{2.2}$$

Now we claim that $q_\varepsilon$ forces that $\underset{\sim}{\tau}$ only depends on $\mathbf{G} \restriction (\zeta_\varepsilon + 1)$ in the following sense:

**Subclaim 2.14.** *If $q_\varepsilon \in \mathbf{G}$, then for some $q^* \in \mathbf{G}$ with $q^* \geq q_\varepsilon$ and $\delta \leq \zeta_\varepsilon$,*

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1)\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”}.$$

*Proof.* First observe that if $q_\varepsilon \in \mathbf{G}$, then by genericity there is some $\delta \leq \zeta_\varepsilon$, and $q' \geq q_\varepsilon$, $q' \in \mathbf{G}$, such that

$$q' \Vdash \max(\underset{\sim}{X} \cap (\zeta_\varepsilon + 1) = \delta), \tag{2.3}$$

i.e.

$$q' = \langle e_0, e_1, \dots, e_m, e_{m+1} = \langle \kappa, A_{q'}\rangle\rangle, \tag{2.4}$$

and for some $k \leq m$ we have

$$[\kappa(e_k) = \delta)] \quad \wedge \quad [A_{q',\kappa(e_{k+1})} \cap (\zeta_\varepsilon + 1) = \emptyset]. \tag{2.5}$$

Now by Lemma 2.11 there is some $q^* \in \mathbf{G}$, $A^* \in \bigcap \overline{U}_\kappa$ with

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A^*\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”}, \tag{2.6}$$

w.l.o.g. $q^* \geq q' \geq q_\varepsilon$. But then by the construction of $A' = A(p, \underset{\sim}{\tau})$ we have

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A' \setminus (\delta + 1)\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”}. \tag{2.7}$$

Therefore, as $A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) = A' \setminus (\zeta_\varepsilon + 1)$ by (2.2) (and $\delta \leq \zeta_\varepsilon$ by (2.3)),

$$A' \setminus (\delta + 1) \subseteq A' \setminus (\zeta_\varepsilon + 1) = A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1),$$

thus

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A' \setminus (\delta + 1)\rangle \leq q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1)\rangle. \tag{2.8}$$

This means that by (2.7)

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1)\rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”},$$

so recalling that $q_\varepsilon \leq q^*$, and $q^* \in \mathbf{G}$, we are done. $\square_{\text{Subclaim2.14}}$

Now we claim that

$$q^* \geq q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1)\rangle. \tag{2.9}$$

To this end first recall, that

$$q^* = \langle d_0^*, d_1^*, \dots, d_\ell^*, d_{\ell+1}^* = \langle \kappa, A_{q^*}\rangle\rangle \geq q' \geq q_\varepsilon = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{q_\varepsilon,\kappa}\rangle\rangle,$$

where $\kappa(d_n) \leq \zeta_\varepsilon$ (by the choice of $\zeta_\varepsilon$), and $q'$ is from (2.10). Moreover, (2.5) implies that

$$q' = q' \restriction (\delta + 1) \frown \langle e_{k+1}, e_{k+2}, \dots, e_m, e_{m+1} = \langle \kappa, A_{q'}\rangle\rangle,$$

where

$$A_{q',\kappa(e_{k+1})} \cap (\zeta_\varepsilon + 1) = \emptyset.$$

Now by $q' \leq q^*$ necessarily (for some $j \leq \ell$) $\kappa(d^*_j) = \delta$, and

$$q^* = q^* \restriction (\delta+1) ^\frown \langle d^*_{j+1}, d^*_{j+2}, \dots, d^*_\ell, d^*_{\ell+1} = \langle \kappa, A_{q'} \rangle \rangle, \tag{2.10}$$

and

$$A_{q^*, \kappa(d^*_{j+1})} \cap (\zeta_\varepsilon + 1) = \emptyset. \tag{2.11}$$

Then one the one hand,

$$A^{**} := \bigcup_{i \in (j, \ell+1]} (A_{q^*, \kappa(e_i)} \cup \{\kappa(e_i)\}) \cap (\zeta_\varepsilon + 1) = \emptyset,$$

and on the other hand,

$$A^{**} \subseteq A_{q_\varepsilon, \kappa},$$

since $q^* \geq q_\varepsilon$, so $A^{**} \subseteq A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1)$, and recalling (2.10) we can conclude that (2.9) holds, indeed.

By Subclaim 2.14 $q_\varepsilon \in \mathbf{G}$ implies that there is always a $q^* \in \mathbf{G}$ and $\delta \leq \zeta_\varepsilon$ such that $q^* \restriction (\delta+1) ^\frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle$ decides the value of $\underset{\sim}{\tau}$, and by (2.9)

$$q^* \restriction (\delta+1) ^\frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle \in \mathbf{G}.$$

It is not difficult to check (using the definition of the partial order) that for every $q^{**} = \langle e_0, e_1, \dots, e_m \rangle \in \bigcup_{\delta \leq \zeta_\varepsilon} \mathbb{P}_{\overline{U} \restriction (\delta+1)}$

$$q^{**} \in \mathbf{G} \iff \left( \{\kappa(e_i) : i \leq m\} \subseteq \underset{\sim}{X} \cap (\zeta_\varepsilon + 1) \subseteq \{\kappa(e_i) :\ i \leq m\} \cup (\cup \{A_{q^{**},\kappa(e_i)} :\ i \leq m\}) \right).$$

Therefore, for any $q^{**} \geq q_\varepsilon$ with

$$q^{**} {}^\frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle \parallel \ \underset{\sim}{\tau} = 1,$$

fix the forced value $j_{q^{**}} \in \{0,1\}$:

$$q^{**} {}^\frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle \Vdash \ \underset{\sim}{\tau} = j_{q^{**}},$$

and fix the code $T_{q^{**}}$ for the $2^{\zeta_\varepsilon}$-Borel subset of $\mathscr{P}(\zeta_\varepsilon)$ with

$$q^{**} \in \mathbf{G} \iff \underset{\sim}{X} \cap (\zeta_\varepsilon + 1) \in B_{T_{q^{**}}}.$$

Finally, let $T_\varepsilon$ be the code for the $2^{\zeta_\varepsilon}$-Borel subset of $\mathscr{P}(\zeta_\varepsilon)$ defined as

$$B_{T_\varepsilon} = \cup \{B_{T_{q^{**}}} :\ q^{**} \geq q_\varepsilon,\ j_{q^{**}} = 1\}.$$

Then

$$q_\varepsilon \Vdash \ (\underset{\sim}{\tau} = 1) \iff ((X \cap \zeta_\varepsilon) \in B_{T_\varepsilon}),$$

and choosing $\gamma_\varepsilon = 2^{\zeta_\varepsilon}$, $Y_\varepsilon = \zeta_\varepsilon$ works, which completes the proof of Claum 2.13.

$\square_{\text{Claim2.13}}$

$\square_{\text{Subclaim2.6}}$

Finally it remains to verify clause (h) from Definition 1.2. Fix $p \in \mathbb{P}$ and $\bar{q}_\alpha = \langle q_{\alpha,\varepsilon} :\ \varepsilon < \kappa \rangle \in \mathscr{S}_p$ $(\alpha < \lambda)$. Now recall $((*_4))$ $((f))$, and let $A'_\alpha \in \bigcap \bar{U}_\kappa = \bigcap_{\beta < \theta} U_{\kappa,\beta}$ the set corresponding to the sequence $\bar{q}_\alpha$, i.e. (if $d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa} \rangle$ denote the components of $p$)

$$\begin{array}{l} \bar{q}_\alpha = \langle q_{\alpha,\varepsilon} : \varepsilon < \kappa \rangle \text{ where } q_{\alpha,\varepsilon} = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{q_{\alpha,\varepsilon},\kappa} \rangle \rangle \text{ and} \\ \{A_{q_{\alpha,\varepsilon},\kappa} :\ \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \text{ and } A_* = A'_\alpha \mod [\kappa]^{<\kappa}\}. \end{array} \tag{2.12}$$

Then for each fixed $\beta < \theta$ as $A'_\alpha \in U_{\kappa,\beta}$ $(\forall \alpha < \lambda)$, using (B) ((c)) there is a pseudointersection in $U_{\kappa,\beta}$, i.e. a set $B_\beta \in U_{\kappa,\beta}$ such that $B_\beta \subseteq A_{p,\kappa}$, and

($*_6$) for each $\alpha < \lambda$ $|B_\beta \setminus A'_\alpha| < \kappa$.

Now taking the union of these pseudointersections, clearly

($*_7$) $B_* = \bigcup_{\beta<\theta} B_\beta \in \bigcap \bar{U}_\kappa$.

Therefore ($*_6$) implies (recalling $\theta < \kappa$)

($*_8$) for each $\alpha < \lambda$: $|B_* \setminus A'_\alpha| < \kappa$, and we can infer that for some $\zeta_\alpha < \kappa$:

$$B_* \cap (\zeta_\alpha, \kappa) \subseteq A'_\alpha.$$

At this point we are ready to define $q_*$. We let $q_* = \langle d_0, d_1, \dots, d_n, \langle \kappa, B_* \rangle \rangle$, clearly $p \leq q_*$ as $B_* \subseteq A_{p,\kappa}$. Moreover, for any fixed $\alpha < \lambda$ by (2.12) there exists some $\varepsilon < \kappa$ with the property that

($*_9$) $A_{q_{\alpha,\varepsilon},\kappa} \cap (\zeta_\alpha, \kappa) = A'_\alpha \cap (\zeta_\alpha, \kappa) \supseteq B_* \cap (\zeta_\alpha, \kappa)$, and

($*_{10}$) $A_{q_{\alpha,\varepsilon},\kappa} \cap (\zeta_\alpha + 1) = B_* \cap (\zeta_\alpha + 1)$,

so $B_* \subseteq A_{q_{\alpha,\varepsilon},\kappa}$, thus concluding $q_{\alpha,\varepsilon} \leq_* q_*$.

$\square_{2.2}$

Next we will give another example of a $(\lambda, \kappa)$-system, the Radin forcing, provided the measure sequence satisfies a similar $< \lambda^+$-directedness condition.

**Definition 2.15.** In order to state the following claim we need to prepare and introduce the notions below.

(i) Let $\kappa$ be a cardinal $\mathbf{j} : \mathbf{V} \to \mathbf{M}$ be an elementary embedding (into a transitive inner model $\mathbf{M}$) with $\mathrm{crit}(\mathbf{j}) = \kappa$. We call the sequence $\overline{F} = \langle F(\alpha) : \alpha < \mathrm{dom}(\overline{F}) \rangle$ a $\mathbf{j}$-sequence of ultrafilters, if
 (a) $F(0) = \kappa$,
 (b) $F(\alpha) \subseteq \mathscr{P}(\mathbf{V}_\kappa)$ for every $\alpha < \mathrm{dom}(\overline{F})$,
 (c) and for each $0 < \alpha < \mathrm{dom}(\overline{F})$, $\forall X \subseteq \mathbf{V}_\kappa$: $[X \in F(\alpha)$ iff $(\overline{F} \restriction \alpha) \in \mathbf{j}(X)]$.

(ii) for each ultrafilter sequence $\overline{F}$ that is a $\mathbf{j}$-sequence witnessed by some suitable $\mathbf{j}$ we let $\kappa(\overline{F})$ denote the critical point of the witnessing $\mathbf{j}$, thus the $F_\alpha$'s are concentrated on $\mathbf{V}_{\kappa(\overline{F})}$. For each ordinal $\alpha$ we mean $\kappa(\alpha) = \alpha$.

(iii) for an ultrafilter sequence $\overline{F}$ that is a $\mathbf{j}$-sequence witnessed by some suitable $\mathbf{j}$ we reserve the notation $\bigcap \overline{F}$ for the intersection of all $F(\alpha)$'s but $F(0)$, i.e. :

$$\cap \overline{F} := \bigcap_{0<\alpha<\mathrm{dom}(\overline{F})} F_*(\alpha).$$

Therefore, for each $\alpha < \mathrm{dom}(\overline{F})$ $F(\alpha)$ is a $\kappa$-complete normal ultrafilter on $\mathbf{V}_\kappa$, where under normality we mean that for each sequence $\langle X_\beta : \beta < \kappa \rangle$ in $F(\alpha)$ the diagonal intersection

$$\Delta_{\beta<\kappa} X_\beta = \{\overline{f} : \forall \gamma < \kappa(\overline{f}) : \overline{f} \in X_\gamma\} \in F(\alpha).$$

We will work with ultrafilter sequences $\overline{F}_*$ according to that almost every element of $V_{\kappa(\overline{F}_*)}$ is itself an ultrafilter sequence, i.e. the $F_*(\alpha)'$s are concentrated on the following classes:

(iii) Let $A^{(n)}$ $(n \in \omega)$ be the following sequence of classes

$$A^{(0)} = \{\overline{F} : \overline{F} \text{ is a } \mathbf{j}\text{-sequence of ultrafilters for some } \mathbf{j} : \mathbf{V} \to \mathbf{M}\},$$

and

$$A^{(n+1)} = \{\overline{F} \in A^{(n)} : \forall \alpha \in \mathrm{dom}(\overline{F}) \setminus \{0\} \; V_{\kappa(\overline{F})} \cap A^{(n)} \in F(\alpha)\}.$$

Finally let

$$\mathbf{A} = \bigcap_{n\in\omega} A^{(n)}.$$

(iv) For any set $X \subseteq A^{(0)}$ and a set $I$ of ordinals let

$$X \restriction I = \{\overline{F} \in X : \; \kappa(\overline{F}) \in I\}.$$

**Claim 2.16.** *There is a $(\lambda, \kappa)$-system such that $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$ <u>when</u> the following hold:*

(C) (a) $\theta_* < \kappa < \lambda < 2^\kappa$,

(b) *$\overline{F}_*$ is an ultrafilter sequence consisting of $\kappa$-complete ultrafilters on $\mathbf{V}_\kappa$, $\overline{F}_* \in \mathbf{A}$.*

(c) *there exists $f : \kappa \to \kappa$ such that*

$$\{\overline{F} : \; \mathrm{dom}(\overline{F}) < f(\kappa(\overline{F}))\} \in \bigcap \overline{F}_* = \bigcap_{0<\alpha<\mathrm{dom}(\overline{F}_*)} F_*(\alpha),$$

*(i.e. when for a witnessing $\mathbf{j}$ for $\overline{F}_*$ the inequality $\mathbf{j}(f)(\kappa) \geq \mathrm{dom}(\overline{F}_*)$ holds, for instance this holds if $\mathrm{dom}(\overline{F}_*) \leq (2^{2^\kappa})^{\mathbf{M}}$),*

(d) *$\bigcap \overline{F}_* = \bigcap_{0<\alpha<\mathrm{dom}(\overline{F}_*)} F_*(\alpha)$ is $<\lambda^+$-directed in the following sense. For every sequence $\langle X_\alpha : \alpha < \lambda \rangle$ in $\bigcap \overline{F}_*$ there exists $X_* \in \bigcap \overline{F}_*$ such that*

$$\forall \alpha < \lambda \; \exists \beta < \kappa : \; X_* \restriction (\beta, \kappa) \subseteq X_\alpha.$$

(e) *$\mathbb{P} = \mathbb{P}_{\overline{F}_*}$ is the Radin forcing for $\overline{F}_*$ (see Definition 2.17 below), so preserves the function $\mu \mapsto 2^\mu$, moreover, we can prescribe that in $\mathbf{V}^{\mathbb{P}}$ there is no new subset of $\theta_*$, and $\mathbb{P}$ satisfies the $\kappa^+$-c.c.*

*Proof.* We will use the definition of the Radin forcing from [Git10, Definition 5.2]. Observe that the definition only depends on $\bigcap \overline{F}_*$.

**Definition 2.17.** [Git10, Definition 5.2] For an ultrafilter sequence $\overline{F}_* \in \mathbf{A}$ we define the Radin forcing $\mathbb{P}$ to be the collection of finite sequences of the form $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle$, where

($*_1$) (a) $A_{p,\kappa} \in \bigcap \overline{F}_* = \bigcap_{0<\alpha<\mathrm{dom}(\overline{F}_*)} F_*(\alpha)$, $A_{p,\kappa} \in \mathbf{A}$,

(b) each $d_j$ ($j \leq n$) is either of the form

- $\langle \overline{F}_{d_j}, A_{d_j} \rangle$ where $\overline{F}_{d_j} \in \mathbf{A}$, $A_{d_j} \subseteq \mathbf{A}$, moreover,

$$A_{d_j} \in \bigcap \overline{F}_{d_j} = \bigcap_{0<\gamma<\mathrm{dom}(\overline{F}_{d_j})} F_{d_j}(\gamma).$$

If $\varepsilon = \kappa(\overline{F}_{d_j})$ we may refer to $\langle \overline{F}_{d_j}, A_{d_j} \rangle$ as $\langle \overline{F}_{p,\varepsilon}, A_{p,\varepsilon} \rangle$, and we also define $\kappa(d_j) = \kappa(\overline{F}_{d_j})$.

- or $d_j = \varepsilon$ for some $\varepsilon < \kappa$ (when we let $\kappa(d_j) = \varepsilon$).

(c) $\kappa(d_0) < \kappa(d_1) < \cdots < \kappa(d_n) < \kappa(d_{n+1}) = \kappa$,

(d) moreover, for each $j \leq n$ if $d_{j+1}$ is a triplet, then $A_{p,\kappa(d_{j+1})} \cap V_{\kappa(d_j)} = \emptyset$.

($*_2$) For the sequences

$$p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle,$$
$$q = \langle e_0, e_1, \dots, e_n, e_{m+1} = \langle \overline{F}_*, A_{q,\kappa}\rangle\rangle$$

we let $p \leq q$, if
- (a) $m \geq n$, and
- (b) there exists a sequence $0 \leq i_0 < i_1 < \dots < i_n < j_{n+1} = m$ such that for each $j \leq n+1$ we have
  - $\kappa(d_j) = \kappa(e_{i_j})$,
  - and

  either $\overline{F}_{p,\kappa(d_j)} = \overline{F}_{q,\kappa(e_{i_j})}$ and $A_{q,\kappa(e_{i_j})} \subseteq A_{p,\kappa(d_j)}$,

  or $d_j = e_{i_j} = \kappa(d_j) = \kappa(e_{i_j})$,
- (c) moreover, for each $l \leq m$ not of the form $i_j$ $(j \leq n+1)$, if $i_l = \min\{i_j : j \leq n+1,\ i_j > l\}$, then

$$A_{q,\kappa(e_k)} \cup \{\overline{F}_{q,\kappa(e_k)}\} \subseteq A_{p,\kappa(d_l)}.$$

**Definition 2.18.** We define $p \leq_* q$ to be true iff $p \leq q$ and $\ell g(p) = \ell g(q)$.

We define the system $\mathbf{r}$ by letting:

($*_3$)
- (a) $\kappa_{\mathbf{r}} = \kappa$,
- (b) $\lambda_{\mathbf{r}} = \lambda$,
- (c) $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$,
- (d) let $\underset{\sim}{X}_{\mathbf{r}}$ be the generic sequence, i.e.

$$\underset{\sim}{X}_{\mathbf{r}} = \cup\{\{\kappa(d_j), \overline{F}_{p,\kappa(d_j)} : j < \ell g(p)\} : \ p = \langle d_0, d_1, \dots, d_{\ell g(p)-1}\rangle \in \mathbf{G}_{\mathbb{P}}\} \setminus \{\kappa\},$$

- (e) $\leq_{\rm pr} = \leq_{\mathbf{r},\rm pr}$ is defined by $p \leq_{\rm pr} q$ iff $p \leq_* q$,
- (f) for $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ let

$$\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \left\{ \begin{array}{l} \bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle, \text{ where} \\ (\bullet_1) q_\varepsilon = \langle d_0, d_1, \dots, d_n, \langle \overline{F}_*, A_{q_\varepsilon,\kappa}\rangle\rangle, \text{ and} \\ \text{for some } B \in \bigcap \overline{F}_* \text{ we have} \\ (\bullet_2)\ B \subseteq A_{p,\kappa}, \text{ and} \\ (\bullet_3)\ \{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \ \wedge \ A_* = B \mod [\kappa]^{<\kappa}\} \end{array} \right\}.$$

Now we check the requirements of Definition 1.2.

It is known that if a condition $\langle\langle\beta\rangle, \langle \overline{F}_*, A_\kappa\rangle\rangle$ is in the generic filter (for some $\beta < \kappa$) then the forcing adds no new subset of $\beta$. This implies that as $\bigcap \overline{F}_* \subseteq F_*(0)$, which is concentrated on the ordinals, i.e. on $\kappa$ itself, w. l. o. g. we can assume that $\langle \beta, \langle \overline{F}_*, A\rangle\rangle \in \mathbf{G}$ for some $\beta \geq \theta_*$.

Now we have only to check the requirements of Definition 1.2. Recall the following properties of the Radin forcing, see [Git10, Sec. 5.1].

**Fact 2.19.** *(Prikry Lemma) For each $p \in \mathbb{P}$ and each formula $\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ there exists $q \geq_* p$, $q \parallel \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ (i.e. either $q \Vdash \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$, or $q \Vdash \neg\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$).*

The following claims, which complete the proof of Claim 2.16 have the same proofs as in the case of Magidor forcing. In Claim 2.20 condition $(C)/(c)$ is essential for the argument.

**Claim 2.20.** *For each $p = \langle d_0, d_1, \ldots, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$) there exists a set $A' \in \bigcap \overline{F}_*$, $A' \subseteq A_{p,\kappa}$, such that whenever $q = \langle e_0, e_1, \ldots, e_m, \langle \overline{F}_*, A_{q,\kappa}\rangle\rangle \geq p' = \langle d_0, d_1, \ldots, d_n, \langle \overline{F}_*, A'\rangle\rangle$, $\alpha \geq \kappa(e_m)$ are given and $q$ forces a value for $\underset{\sim}{\tau}$, then so does*

$$q' = \left\langle e_0, e_1, \ldots, e_m, \langle \overline{F}_*, A' \restriction (\alpha, \kappa)\rangle \right\rangle.$$

**Claim 2.21.** *Suppose $p = \langle d_0, d_1, \ldots, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle \in \mathbb{P}_{\overline{F}_*}$, $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$), and $\alpha \geq \kappa(d_n)$. If $p \in \mathbf{G}$, $\mathbf{G} \subseteq \mathbb{P}_{\overline{F}_*}$ is generic over $\mathbf{V}$, then there exists*

$$q = \langle e_0, e_1, \ldots, e_{m+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle \in \mathbb{P}_{\overline{F}_*},$$

$$q \in \mathbf{G},$$

*where $\kappa(e_m) \leq \alpha$, $A_{q,\kappa} \cap \mathbf{V}_{\alpha+1} = \emptyset$, and there exists $A \subseteq A_{p,\kappa}$, $A \in \bigcap \overline{F}_*$, such that*

$$q \restriction (\kappa(e_m) + 1) \frown \langle \overline{F}_*, A\rangle \parallel \ \underset{\sim}{\tau} = 1.$$

Claims 2.20, 2.21 implies the following.

**Claim 2.22.** *For each $p = \langle d_0, d_1, \ldots, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa}\rangle\rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$) there exists a set $A' \in \bigcap \overline{F}_*$, $A' \subseteq A_{p,\kappa}$, such that whenever $\alpha < \kappa$, and*

$$p' = p \restriction (\alpha + 1) \frown \langle \overline{F}_*, A_{p,\kappa} \restriction (\alpha + 1) \cup A' \restriction (\alpha, \kappa)\rangle\rangle \in \mathbf{G},$$

*$\mathbf{G} \subseteq \mathbb{P}_{\overline{F}_*}$ is a generic filter, then there exists $q \in \mathbf{G}$, $q$ is of the form*

$$q = q \restriction (\alpha + 1) \frown \langle \overline{F}_*, A' \restriction (\alpha, \kappa)\rangle\rangle,$$

*and*

$$q \parallel \ \underset{\sim}{\tau} = 1.$$

**Claim 2.23.** *Suppose that $p \in \mathbb{P}$ and $\underset{\sim}{\tau}$. If $p \Vdash \tau \in \{0,1\}$, then there exists $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle \in \mathscr{S}_p$, $\langle \gamma_\varepsilon : \varepsilon < \kappa\rangle \in {}^\kappa\kappa$, $\langle Y_\varepsilon : \varepsilon < \kappa\rangle \in {}^\kappa\mathbf{V}_\kappa$, $\langle T_\varepsilon : \varepsilon < \kappa\rangle$, such that each $T_\varepsilon$ is a code for a $\gamma_\varepsilon$-Borel subset of $\mathscr{P}(Y_\varepsilon)$, and*

$$q_\varepsilon \Vdash \ (\underset{\sim}{\tau} = 1) \iff (X \cap Y_\varepsilon) \in B_{T_\varepsilon}.$$

$\square_{\text{Claim2.16}}$

## § 3. The preparatory forcing

§ 3(A). **The general framework.** This subsection is devoted to the preparatory forcing, in Claim 3.2 we provide a general framework to force a $(\lambda, \kappa) - 1$ system.

First we are going to define a variant of Mathias forcing, for which we need to recall the notations from Definition 2.15 (ii), (iv), so if $I \subseteq \kappa$, $A \subseteq V_\kappa$, then

$$A \restriction I = \{x \in A : \kappa(x) \in I\},$$

where $\kappa(\alpha) = \alpha$ if $\alpha$ is an ordinal, $\kappa(\overline{F}) = \text{crit}(\mathbf{j})$ for the elementary embedding $\mathbf{j}$ if $\overline{F}$ is a $\mathbf{j}$-sequence (and for every other $x$, we can let $\kappa(x) = -1$). Using this convention we will have Mathias forcing for filters in the context of Radin forcing, too, not only filters concentrated on $\kappa$.

**Definition 3.1.** For $D$ a $< \kappa$-centered system (i.e. generating a $\kappa$-complete filter $D^*$) on $\cup D \subseteq \mathbf{V}_\kappa$ (so $D^* \subseteq \mathscr{P}(\cup D)$) we let $\mathbb{Q} = \mathbb{Q}_D$ be the following forcing notion:

(A) $p \in \mathbb{Q}$ iff
  (a) $p = (w, A) = (w_p, A_p)$, and for some $\sigma_p < \kappa$ we have
  (b) $w_p \subseteq \mathbf{V}_\kappa$, $w_p = w_p \restriction [0, \sigma_p)$ (so $w_p \in \mathbf{V}_\kappa$ holds, too)
  (c) $A_p \subseteq \cup D$, $A_p \in D^*$ and $A_p = A_p \restriction [\sigma_p, \kappa)$.

(B) $\mathbb{Q} \models p \le q$ iff
  (a) $p, q \in \mathbb{Q}$,
  (b) $w_p \subseteq w_q \subseteq w_p \cup A_p$,
  (c) $A_p \supseteq A_q$,

(C) $\underset{\sim}{w} = \cup\{w_p : p \in \underset{\sim}{\mathbf{G}}\}$.

**Claim 3.2.** *If (A) and (B) hold, then so does (C), where:*

*(A)* $\mathbf{v} = (\mathbf{V}_0, \kappa, \mathbf{h}, \mathbf{p}, \mathbf{G}_\kappa, \mathbf{V}_1)$ *satisfies:*
  *(a)* $\mathbf{V}_0$ *is a universe of set theory,*
  *(b) in* $\mathbf{V}_0$ $\kappa$ *is supercompact and* $\mathbf{h} : \kappa \to \mathscr{H}(\kappa)$ *is a Laver diamond,*
  *(c)* $\mathbf{p}$ *is the Easton support iteration* $\langle \mathbb{P}_{\mathbf{p},\alpha}, \underset{\sim}{\mathbb{Q}}_{\mathbf{p},\beta} : \alpha \le \kappa, \beta < \kappa\rangle = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \le \kappa, \beta < \kappa\rangle$ *built as specified Definition* 3.4 $(\bullet)_I$-$(\bullet)_{II}$*, and* $(\bullet)_a$-$(\bullet)_b$ *using* $\mathbf{h}$ *(essentially as in Laver* [Lav78]*) and let* $\mathbb{P}_\mathbf{p} = \mathbb{P}_{\mathbf{p},\kappa}$ *(hence for* $\alpha < \kappa$ *also* $\mathbb{P}^0_\alpha \in V_\kappa^{\mathbf{V}_0}$*),*
  *(d)* $\mathbf{G}_\kappa = \mathbf{G}_{\mathbf{p},\kappa} \subseteq \mathbb{P}_\mathbf{p}$ *is generic over* $\mathbf{V}_0$ *and* $\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_\kappa]$.

*(B)* *(a)* $\kappa < \lambda < \chi = \chi^\lambda$ *(in* $\mathbf{V}_0$*, of course),*
  *(b)* $\mathbb{P}^1_\chi = \langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \le \chi, \beta < \chi\rangle \in \mathbf{V}_1$ *is an iteration with* $< \kappa$ *support such that* $\mathbb{P}^1_\chi$ *is* $\lambda^+$*-c.c. and* $<\kappa$*-directed closed, preserving cardinals,*
  *(c) for each* $\alpha < \chi$

$$\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \le \chi.$$

  *(d) for the set* $S^* \subseteq \chi$ *there is a system* $\langle \underset{\sim}{D}_\delta : \delta \in S^*\rangle \in \mathbf{V}_1$*,* $\underset{\sim}{D}_\delta$ *is a* $\mathbb{P}^1_\delta$*-name of a subset of* $\mathscr{P}^{\mathbf{V}_1^{\mathbb{P}^1_\delta}}(V_\kappa)$*, and if*

(3.1) $$\mathbf{V}_1^{\mathbb{P}^1_\delta} \models \begin{array}{l} \underset{\sim}{D}_\delta \text{ generates a } \kappa\text{-complete filter, satisfying} \\ (\forall \alpha < \kappa)\ |(\cup \underset{\sim}{D}_\delta) \restriction \alpha| < \kappa \end{array}$$

  *then the forcing* $\underset{\sim}{\mathbb{Q}}^1_\delta$*,* $\delta \in S^*$ *is of the form* $Q_{\underset{\sim}{D}_\delta}$*, the forcing from Definition* 3.1*. Moreover, we assume that each* $D \in [\mathscr{P}^{\mathbf{V}_1^{\mathbb{P}^1_\chi}}(V_\kappa)]^{\le\lambda}$ *that satisfies* (3.1) *appears as a* $D_\delta$ *for some* $\delta \in S^*$*, i.e.*

  *(#)* $\mathbf{V}_1^{\mathbb{P}^1_\chi} \models \forall D \in [\mathscr{P}(V_\kappa)]^{\le\lambda}$ :
  [*if* $D$ *generates a* $< \kappa$*-complete filter, and*

$$\forall \alpha < \kappa : \ |(\cup D) \restriction \alpha| < \kappa,$$

  *then* $(D = D_\delta$ *for some* $\delta \in S^*)$.]

*(C) in* $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ *we have* $2^\kappa$ *is* $\chi$*, and the following.*
  *(a) There is a* $\kappa$*-complete normal ultrafilter* $U$*, which is* $< \lambda^+$*-directed mod* $[\kappa]^{<\kappa}$.

(b) *(Setting for Magidor forcing:) There is a sequence* $\bar{U} = \langle U_i : i < \kappa \rangle$ *of normal ultrafilters on* $\kappa$*, strictly increasing in the Mitchell order, i.e.* $i < j \Rightarrow U_i \in \text{MosCol}({}^\kappa(\mathbf{V}^{\mathbb{P}^1_\chi})/U_j)$*, such that each* $U_i$ *is* $< \lambda^+$*-directed mod* $[\kappa]^{<\kappa}$*.*

(c) *(Setting for Radin forcing:) For any* $\Upsilon \geq \kappa$ *and* $\eta$ *there is a* $\kappa$*-complete fine normal ultrafilter* $W$ *on* $[\Upsilon]^{<\kappa}$ *such that for the elementary embedding* $\mathbf{j}_W$ *of* $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ *with critical point* $\kappa$ *we have (letting* $\overline{U}$ *denote the measure sequence associated to* $\mathbf{j}_W$*):*

($\star$) *for every* $\sigma \leq \min(\text{dom}(\overline{U}, \eta))$ *if the filter* $\bigcap(\overline{U} \restriction \sigma) = \bigcap_{\gamma<\sigma} U_\gamma$ *concentrates on a set* $X \subseteq V_\kappa$ *with* $(\forall \alpha < \kappa)\ |X \restriction \alpha| < \kappa$*, then* $\bigcap(\overline{U} \restriction \sigma)$ *is* $< \lambda^+$*-directed in the following sense: Whenever* $\langle A_i : i < \lambda \rangle$ $(\forall i < \lambda\ A_i \in \bigcap(\overline{U} \restriction \sigma))$ *is given, there exists* $A_* \in \bigcap(\overline{U} \restriction \sigma)$ *such that*

$$\forall i \in \lambda\ \exists \delta_i < \kappa :\ A_* \restriction [\delta_i, \kappa) \subseteq A_i. \tag{3.2}$$

*In particular* $\kappa$ *is supercompact.*

*Remark* 3.3. This continues Džamonja-Shelah [DS03].

*Proof.* First we have to construct the iteration $\mathbb{P}^0$ using the Laver function $\mathbf{h} : \kappa \to \mathscr{H}(\kappa) \in \mathbf{V}_0$. The construction $\mathbb{P}^0 = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta :\ \alpha \leq \kappa, \beta < \kappa \rangle$ goes by induction, we follow [Lav78], only with a slight technical modification which we will need in the proof of $(C)$((b)).

Let $\mathbf{h}$ be as in [Lav78] (i.e.

$(\bullet)_1$ for each $\lambda \geq \kappa$, $x \in \mathscr{H}(\lambda^+)$ there exists a $\kappa$-complete fine normal ultrafilter $U$ on $[\lambda]^{<\kappa}$ such that for the associated elementary embedding $\mathbf{j}_U$

$$\mathbf{j}_U(\mathbf{h})(\kappa) = x).$$

**Definition 3.4.** We define $\mathbb{P}^0 = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta :\ \alpha \leq \kappa, \beta < \kappa \rangle$ and $\langle \mu_\alpha :\ \alpha < \kappa \rangle$ by induction. If $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta :\ \alpha < \gamma, \beta < \gamma \rangle$ are already defined, then

$(\bullet)_I$ if $\gamma$ is strongly inaccessible then $\mathbb{P}^0_\gamma$ is the direct limit (i.e. we use bounded support),

$(\bullet)_{II}$ otherwise let $\mathbb{P}^0_\gamma$ be the inverse limit of $\mathbb{P}^0_\beta$'s $(\beta < \gamma)$ (i.e. for a function $p$ with $\text{dom}(f) = \gamma$ $p \in \mathbb{P}^0_\gamma$ iff $(\forall \beta < \gamma)\ p \restriction \beta \in \mathbb{P}^0_\beta$). .

Second,

$(\bullet)_a$ if $\sup\{\mu_\alpha : \alpha < \gamma\} \leq \gamma$, and $\gamma$ is strongly inaccessible, moreover, $\mathbf{h}(\gamma)$ happens to be of the form $\langle \underset{\sim}{Q}_*, \mu_*, \underset{\sim}{U} \rangle$, where $\underset{\sim}{Q}_*$ is a $\mathbb{P}^0_\gamma$-name for a $<\gamma$-directed closed notion of forcing, $\mu_*$ is an ordinal, $\underset{\sim}{U}$ is a (possibly trivial) $\mathbb{P}^0_\gamma$-name, then let

$$\underset{\sim}{\mathbb{Q}}^0_\gamma = \underset{\sim}{\mathbb{Q}}_*, \quad \mu_\gamma = \mu_*.$$

$(\bullet)_b$ In the remaining case let $\underset{\sim}{\mathbb{Q}}^0_\gamma$ be the trivial forcing, $\mu_\gamma = \gamma$.

Recall $\mathbf{G}^0_\kappa \subseteq \mathbb{P}^0_\kappa$ is generic over $\mathbf{V}_0$ so that $\mathbf{V}_0[\mathbf{G}^0_\kappa] = \mathbf{V}_1$, and let $\mathbf{G}^1_\chi \subseteq \mathbb{P}^1_\chi$ be generic over $\mathbf{V}_1$, let $\mathbf{V}_2 = \mathbf{V}_1[\mathbf{G}^1_\chi] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$. Note that as $|\mathbb{P}^0_\kappa| = \kappa$ and $\kappa < \lambda$, $(B)$(a) implies that

$(\bowtie)_1$ $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}^0_\kappa] \models\ \chi^\lambda = \chi^{\lambda \cdot \kappa} = \chi$, thus $\text{cf}(\chi) > \lambda$ is preserved, too.

Since $\kappa$ is strongly inaccessible, and $\mathbb{P}^0$ is an Easton support iteration, where $\mathbb{Q}^0_\beta$ is $< \alpha$-closed for $\alpha < \beta$, and for stationarily many $\alpha$'s $|\mathbb{P}^0_\alpha| = \alpha$ (actually for each strongly inaccessible cardinal $\alpha$), by standard arguments

$(\bowtie)_2$ $\mathbb{P}^0_\kappa$ has the $\kappa$-cc (so forcing with it preserves the regularity of $\kappa$),

moreover

$(\bowtie)_3$ $\mathbb{P}^0_\kappa$ preserves $\kappa$ to be strongly inaccessible.

Also note that as $\mathbb{P}^1_\chi$ is $< \kappa$-closed

$(\bowtie)_4$ $V_\kappa^{\mathbf{V}_2} = V_\kappa^{\mathbf{V}_1}$, and $\mathbf{V}_2 \models$ "$\kappa$ is still strongly inaccessible."

First observe that because of our cardinal arithmetic assumptions $\chi^\kappa \leq \chi^\lambda = \chi$ in $(B)$(a), and as $|\mathbb{P}^0_\kappa| = \kappa$, not only do we have $(\bowtie)_1$ $(\chi^\lambda)^{\mathbf{V}_1} = \chi^{\lambda\cdot\kappa} = \chi$, but by an easy induction (and by the $\lambda^+$-cc) $|\underset{\sim}{\mathbb{P}}^1_\chi|^{\mathbf{V}_1} = \chi$, so

$(\bowtie)_5$ $|\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi| = \chi$ up to equivalence (and so obviously $\chi^+$-cc).

Recalling $\chi^\lambda = \chi$ again, clearly

$(\bowtie)_6$ $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models 2^\chi = (2^\chi)^{\mathbf{V}_0}$,
$(\bowtie)_7$ $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models 2^\kappa = \chi$.

**Definition 3.5.** We have to introduce the following objects.

- $(\bullet)_2$ Let $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$ be an elementary embedding with critical point $\kappa$ such that $(\mathbf{j}(\mathbf{h}))(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \check{\emptyset} \rangle$ ($\check{\emptyset} = \emptyset$ is the canonical name for the empty set) and $\mathbf{j}(\kappa) > \chi$, ${}^\chi\mathbf{M} \subseteq \mathbf{M}$,
- $(\bullet)_3$ Let $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \mathbf{j}(\kappa), \beta < \mathbf{j}(\kappa) \rangle = \mathbf{j}(\langle \mathbb{P}^0_\alpha, \mathbb{Q}^0_\beta : \alpha \leq \kappa, \beta < \kappa \rangle)$ so $\underset{\sim}{\mathbb{Q}}^0_\kappa = \underset{\sim}{\mathbb{P}}^1_\chi$, and
- $(\bullet)_4$ let $\underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)} = \mathbf{j}(\underset{\sim}{\mathbb{P}}^1_\chi)$, i.e.

  (a $\mathbb{P}^0_{\mathbf{j}(\kappa)}$-name for a $< \mathbf{j}(\kappa)$-directed closed notion of forcing)$^{\mathbf{M}}$.

  (Recall that $\underset{\sim}{\mathbb{P}}^1_\chi$ is a $\mathbb{P}^0_\kappa$-name for the iteration $\langle \underset{\sim}{\mathbb{P}}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi \rangle \in \mathbf{V}_0^{\mathbb{P}^0_\kappa}$.)

Similarly to $(\bowtie)_4$, recalling ${}^\chi M \subseteq M$,

$(\bowtie)_8$ $V_\kappa^{\mathbf{M}[\mathbf{G}^0_{\kappa+1}]} = V_\kappa^{\mathbf{M}[\mathbf{G}^0_\kappa]} = V_\kappa^{\mathbf{V}_2}$, and ($\kappa$ is strongly inaccessible)$^{\mathbf{M}[\mathbf{G}^0_{\kappa+1}]}$.

From now on we will identify $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ with the $(\kappa+1)$-step iteration $\mathbb{P}^0_{\kappa+1}$, and also

$(\bowtie)_9$ $\mathbf{G}^0_{\kappa+1} = \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$ is a generic subset of $\mathbb{P}^0_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ (over $\mathbf{V}_0$).

**Remark 3.6.** Having completed the requirements of Claim 3.2 we remark that given a scheme for an iteration fitting all our assumptions except perhaps $((B))$(d), it is easy to adapt it to have $(\#)$ using $\chi^\lambda = \chi$ $(\bowtie)_1$.

Now we can prove the statements in 3.2(C).
<u>Case 1</u>: First we verify 3.2(C)(a).
We would like to find an appropriate $\kappa$-complete ultrafilter in $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$, for which we will use the basic trick: using the elementary embedding $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$, then extending $\mathbf{V}_0$ with $\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$, and extending $\mathbf{M}$ with $\mathbf{G}^0_{\kappa+1} (= \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi)$, and finding a single condition in $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)} / \mathbf{G}^0_{\kappa+1}$ compatible with $\{\mathbf{j}(p \restriction \{\kappa\}) = \mathbf{j}(p) \restriction \{\mathbf{j}(\kappa)\} : p \in \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi\}$ giving us sufficient information (just as if there existed some

lifting $\tilde{\mathbf{j}} : \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \to \mathbf{M}[\mathbf{H}^0_{\mathbf{j}(\kappa)} * \mathbf{H}'_{\mathbf{j}(\chi)}])$ of $\mathbf{j}$ extending it). (Here the quotient $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}/\mathbf{G}^0_{\kappa+1}$ is formally

$$\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}/\mathbf{G}^0_{\kappa+1} = \{(p\restriction(\kappa,\mathbf{j}(\kappa)),\underset{\sim}{q}) :\ (p,\underset{\sim}{q}) \in \mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}\},$$

and

$$\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}/\mathbf{G}^0_{\kappa+1} \models\ (p\restriction(\kappa,\mathbf{j}(\kappa)),q) \leq (p'\restriction(\kappa,\mathbf{j}(\kappa)),q'),$$

if there exists $p_* \in \mathbf{G}^0_{\kappa+1}$ such that

$$\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)} \models\ (p_* \frown p\restriction(\kappa,\mathbf{j}(\kappa)),\underset{\sim}{q}) \leq (p_* \frown p'\restriction(\kappa,\mathbf{j}(\kappa)),\underset{\sim}{q}').)$$

We will need the following facts.

**Fact 3.7.** *The filter $\mathbf{G}^0_{\kappa+1}$ is generic over $\mathbf{M}$ as well, and the forcing notions $\mathbb{P}^0_{j(\kappa)}/\mathbf{G}^0_{\kappa+1}$ and $(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ $(\gamma \leq \mathbf{j}(\chi))$ are well defined and $< \chi^+$-directed closed in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.*

*Proof.* Note that $\mathbf{G}^0_{\kappa+1}$ is generic, as $\mathbb{P}^0_{\kappa+1} \subseteq \mathbf{M} \subseteq \mathbf{V}_0$.

For the second assertion we first recall that a pair $(p,\underset{\sim}{q}) \in (\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}}(\chi))/\mathbf{G}^0_{\kappa+1}$ iff $p = p_0 \restriction (\kappa,\mathbf{j}(\kappa))$ for some $p_0 \in \mathbb{P}^0_{\mathbf{j}(\kappa)}$, and $(\Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)}} \underset{\sim}{q} \in \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)})^{\mathbf{M}}$. We only have to refer to the construction of the iteration Definition 3.4 i.e. recall that

- $(i)$ $\Vdash_{\mathbb{P}^0_\kappa}$ "$\underset{\sim}{\mathbb{P}}^1_\chi$ is a $< \kappa$-support iteration of $< \kappa$-directed closed forcing notions", and
- $(ii)$ for each $\alpha \leq \beta < \kappa$ we have that $\Vdash_{\mathbb{P}^0_\beta}$ "$\underset{\sim}{\mathbb{Q}}^0_\beta$ is $< \beta$-directed closed", and is the trivial forcing if $\beta < \sup\{\mu_\varrho :\ \varrho < \beta\}$ (in particular, if $\beta < \sup\{\mu_\varrho :\ \varrho < \alpha\}$),
- $(iii)$ for each $\alpha < \beta < \kappa$, where $\beta$ is limit and $\mathrm{cf}(\beta) < \mu_\alpha$ the iteration $\mathbb{P}^0_\beta$ is the inverse limit of $\mathbb{P}^0_\delta$'s $(\delta < \beta)$.

So using [Bau78, Thm. 5.5], for each $\alpha < \beta < \kappa$ the quotient $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi)/\mathbf{G}^0_\alpha$ (of the $\kappa+1$-long iteration $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi = \mathbb{P}^0_{\kappa+1}$) is $< \beta$-directed closed in $\mathbf{V}_0[\mathbf{G}^0_\alpha]$ provided $\beta \leq \sup\{\mu_\varrho :\ \varrho < \alpha\}$, and $\mathbb{P}^0_\alpha$ has the $\beta$-cc. (In typical applications $\underset{\sim}{\mathbb{Q}}^0_\alpha$ is the trivial forcing.) Thus by elementarity (letting $\alpha = \kappa+1$, $\beta = \chi^+ = \mu_\kappa$, recalling $\mathbb{P}^0_{\kappa+1}$ has the $\chi^+$-cc by $(\bowtie)_5$, and $(\chi^+)^{\mathbf{M}} = \chi^+$ by ${}^\chi\mathbf{M} \subseteq \mathbf{M}$):

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models\ \text{"}(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}}(\chi))/\mathbf{G}^0_{\kappa+1} \text{ is } < \chi^+\text{-directed closed."}$$

$\square$

**Fact 3.8.** $\mathbf{V}_1 \models$ *"$(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ is $< \chi^+$-directed closed."*

Fact 3.8 follows from the fact below.

**Fact 3.9.** $\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models {}^\chi\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.

*Proof.* For, pick a name $\underset{\sim}{f}$ for a function $\underset{\sim}{f} : \chi \to \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$, and observe that w.l.o.g. we can assume that $\underset{\sim}{f} : \chi \to \mathrm{ORD}$, i.e. for each $\alpha < \chi$, $f(\alpha)$ is an ordinal, in particular $\mathrm{ran}(\underset{\sim}{f}) \subseteq \mathbf{M}$. Now for each $\alpha$ there exists a maximal antichain $A_\alpha = \{a^\alpha_i :\ i < |A_\alpha|\} \subseteq \mathbb{P}^0_{\kappa+1}$, and $\{x^\alpha_i :\ i < |A_\alpha|\} \subseteq \mathbf{M}$, s.t. $a^\alpha_i \Vdash \underset{\sim}{f}(\alpha) = x^\alpha_i$. As $\mathbb{P}^0_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ is of power $\chi$, we have $|A_\alpha| \leq \chi$ trivially, therefore as $\mathbf{M}$ is closed under sequences of length $\chi$ ($(\bullet)_2$, Definition 3.5) $\langle (x^\alpha_i, a^\alpha_i) :\ \alpha < \chi,\ i < |A_\alpha|\rangle \in M$, which means that there is indeed a name $\underset{\sim}{g} \in \mathbf{M}$, such that $\Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi} \underset{\sim}{f} = \underset{\sim}{g}$. $\square_{\text{Fact3.9}}$

**Definition 3.10.** (In $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$) for $\zeta \in S^*$ we let

(1) $\varepsilon_\zeta \in \mathbf{V}[\mathbf{G}^0_\kappa * \underset{\sim}{\mathbf{G}}^1_{\zeta+1}]$ denote the generic subset of $V^{\mathbf{V}_1}_\kappa$ (or just $\kappa$) given by $\mathbb{Q}^1_\zeta$, i.e.

$$\Vdash_{\mathbb{P}^0_\kappa * \mathbb{P}^1_{\zeta+1}} \ \varepsilon_\zeta = \cup\{\underset{\sim}{\varepsilon} : \ \exists \underset{\sim}{A} : (\underset{\sim}{\varepsilon}, \underset{\sim}{A}) \in \mathbf{G}_{\mathbb{Q}^1_\zeta}\}$$

(after identifying $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1} = \mathbb{P}^0_\kappa * (\underset{\sim}{\mathbb{P}}^1_\zeta * \underset{\sim}{\mathbb{Q}}^1_\zeta)$ with $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta) * \underset{\sim}{\mathbb{Q}}^1_\zeta$).

(2) Define $\mathscr{N}_\zeta$ to be a set of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-names of subsets of $V_\kappa$ containing exactly one name from each equivalence class, i.e. no $\underset{\sim}{A} \neq \underset{\sim}{B} \in \mathscr{N}_\zeta$ satisfy $\Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta} \underset{\sim}{A} = \underset{\sim}{B}$, but each set in the extension is represented.

Observe that (as $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta) * \underset{\sim}{\mathbb{Q}}^1_\zeta, \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta \in \mathbf{M}$) we can assume that

$(\bowtie)_{10}$ $\mathscr{N}_\zeta \subseteq \mathbf{M}$,

and as $|V^{\mathbf{V}_2}_\kappa| = \kappa$, and by the $\lambda^+$-cc $(B)$b

$(\bowtie)_{11}$ $|\mathscr{N}_\zeta| \leq |\mathbb{P}^0_\kappa * \mathbb{P}^1_\zeta|^\lambda = \chi$,

so by ${}^\chi\mathbf{M} \subseteq \mathbf{M}$:

$(\bowtie)_{12}$ $\mathscr{N}_\zeta \in \mathbf{M}$, and $\mathbf{j} \restriction \mathscr{N}_\zeta \in \mathbf{M}$.

(3) Using the notation

$$\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} = \{\underset{\sim}{A} \in \mathscr{N}_\zeta : \ (\underset{\sim}{\varepsilon}, \underset{\sim}{A}) \in \mathbf{G}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} \text{ for some } \underset{\sim}{\varepsilon}\},$$

note that $\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} \in \mathbf{M}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\zeta+1}]$ (so $\underset{\sim}{\mathscr{A}}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}$ is a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1}$-name for a set of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-names). Now similarly

$$\mathbf{j}``\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} = \{\mathbf{j}(\underset{\sim}{A}) : \ \underset{\sim}{A} \in \mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\} \in \mathbf{M}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\zeta+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$$

is a set of $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$-names, and each of which collection corresponds to a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta / \mathbf{G}^0_{\kappa+1}$-name, we can define the $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)} / \mathbf{G}^0_{\kappa+1}$-name $\underset{\sim}{A}'_{\mathbf{j}(\zeta)} \in \mathbf{M}$ for a subset of $V_{\mathbf{j}(\kappa)}$ so that

$$(\text{in } \mathbf{M}[\mathbf{G}^0_{\kappa+1}] :) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\zeta)} / \mathbf{G}^0_{\kappa+1}} \ \underset{\sim}{A}'_{\mathbf{j}(\zeta)} = \cap\{\mathbf{j}(\underset{\sim}{A}) : \ \underset{\sim}{A} \in \mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\}.$$

**Claim 3.11.** *There is a sequence $\langle q_\zeta : \ \zeta \leq \chi\rangle \in \mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$ such that:*

$(*)_{1.1}$ *(a)* $q_\zeta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)}) / \mathbf{G}^0_{\kappa+1}$, *and if $\varepsilon < \zeta \leq \chi$, then $q_\varepsilon \leq q_\zeta$,*

*(b)* $q_\zeta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\zeta)}) / \mathbf{G}^0_{\kappa+1}$ *(i.e. $q_\zeta \restriction \mathbf{j}(\kappa) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)}} q_\zeta(\mathbf{j}(\kappa)) \in \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$),*

*(c) whenever $p \in \mathbf{G}^0_{\kappa+1} \cap (\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta)$ then*

$$(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)}) / \mathbf{G}^0_{\kappa+1} \models \mathbf{j}(p) \leq q_\zeta$$

*(i.e. $\mathbf{j}(p) \leq q_\zeta$ in the order of the quotient forcing $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)}) / \mathbf{G}^0_{\kappa+1}$),*

*(d) whenever $\underset{\sim}{A}$ is a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-name of a subset of $\kappa$ (so $\mathbf{j}(\underset{\sim}{A})$ is a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$-name for a subset of $\mathbf{j}(\kappa)$) then for $\kappa \in$ORD*

$$q_\zeta \parallel_{(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}) / \mathbf{G}^0_{\kappa+1}} \ \kappa \in \mathbf{j}(\underset{\sim}{A}).$$

*(e) if $\zeta \in S^*$ (from (#) of d) then we have the following: If $D_\zeta := \underset{\sim}{D}_\zeta[\mathbf{G}^1_\zeta]$ generates a $\kappa$-complete filter on $V_\kappa$ (in $\mathbf{V}_1[\mathbf{G}^1_\zeta] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\zeta]$) <u>then</u> (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$ in the poset $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)} / \mathbf{G}^0_{\kappa+1}$)*

$$(q_{\zeta+1}(\mathbf{j}(\kappa)))\,(\mathbf{j}(\zeta)) \geq \left(\varepsilon_\zeta \cup \left(\underset{\sim}{A}'_{\mathbf{j}(\zeta)} \restriction \{\kappa\}\right), \underset{\sim}{A}'_{\mathbf{j}(\zeta)} \restriction (\kappa+1, \mathbf{j}(\kappa))\right). \tag{3.3}$$

*(In this generality this will be relevant for the proof (c). For $D_\zeta$'s for which $D_\zeta \subseteq \mathscr{P}(\kappa)$ it is enough to ensure that if for each $A \in D_\zeta$ we have $\kappa \in \mathbf{j}(A)$ (forced by $q_\zeta$), then $q_{\zeta+1} \Vdash \kappa \in \tilde{\mathbf{j}}(\varepsilon_\zeta)$.)*

*Proof.* Working in $\mathbf{V}_2 = \mathbf{V}_0[\mathbf{G}_{\kappa+1}]$ we can define the $q_\eta$'s ($\eta \le \chi$, $q_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$) by induction on $\eta$. Assume that $q_\xi$'s ($\xi < \eta$) are chosen and (a) – (e) hold. First we choose $q'_\xi$ satisfying (a), (c), (e) which we will then further strengthen to get $q_\xi \ge q'_\xi$.

Recalling Fact 3.8, let $q'_0 \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(0)})/\mathbf{G}^0_{\kappa+1} = \mathbb{P}^0_{\mathbf{j}(\kappa)}/\mathbf{G}^0_{\kappa+1}$ be the empty condition.

For $\eta$ limit we choose $q'_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}``\eta})/\mathbf{G}^0_{\kappa+1}$ to be an upper bound of the increasing sequence $\langle q_\xi : \xi < \eta\rangle$ satisfying (c). Now it follows from standard arguments that $q'_\eta$ satisfies (c), even if $\mathbb{P}^0_\kappa * \mathbb{P}^1_\eta$ is bigger than the direct limit of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$'s ($\xi < \eta$), (in the case $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\eta = \bigcup_{\xi<\eta} \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$ it is automatic), but for completeness we elaborate:

If $p \in \mathbf{G}^0_{\kappa+1}$ is fixed, $p \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$, then for each $\xi < \eta$ let $p_\xi \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi \subseteq \mathbb{P}^0_{\kappa+1}$ be such that $p{\restriction}\kappa \Vdash_{\mathbb{P}^0_\kappa} p(\kappa){\restriction}\xi = p_\xi(\kappa)$. Now if $\mathrm{cf}(\eta) < \kappa$, then $\sup\{\mathbf{j}(\xi) : \xi < \eta\} = \mathbf{j}(\eta)$, and so $\mathbf{j}(p)(\mathbf{j}(\kappa))$ is the least upper bound for the system $\{\mathbf{j}(p_\xi(\kappa)) = \mathbf{j}(p_\xi)(\mathbf{j}(\kappa)) : \xi < \eta\}$, and

$$(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1} \models \mathbf{j}(p_\xi) \le q_\xi$$

by our hypothesis. If $\mathrm{cf}(\eta) \ge \kappa$, then by the $\kappa$-cc of $\mathbb{P}^0_\kappa$ $(\bowtie)_2$ there exists a $\xi < \eta$ such that

$$\Vdash_{\mathbb{P}^0_\kappa} p(\kappa) = p(\kappa){\restriction}\xi,$$

and so $p \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$ (remember, $\mathbb{P}^1_\chi$ is a $<\kappa$ support iteration). This in turn implies

$$(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1} \models \mathbf{j}(p) \le q_\xi \le q'_\eta.$$

If $\eta = \xi + 1$ is a successor and

- if $\xi \notin S^*$,

then using simply the $<(2^\chi)^+$-directed closedness of $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}$ (by Fact 3.8) define $q'_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi+1)})/\mathbf{G}^0_{\kappa+1}$ to be an upper bound of $q_\xi \in \mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi)}$ and the set $\{\mathbf{j}(p) : p \in (\mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1}) \cap \mathbf{G}^0_{\kappa+1}\}$.

Otherwise,

- if $\xi \in S^*$,

(where $\eta = \xi+1$) then recall that by the definition of $\underset{\sim}{\mathbb{Q}}^1_{\xi+1}$ each $p \in (\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{(\xi+1)})$ the coordinate $(p(\kappa))(\xi+1)$ is a $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi)$-name for a pair $(\varepsilon, A)$ with $\varepsilon = \varepsilon \restriction (0,\gamma)$ for some $\gamma < \kappa$, and where $A \subseteq V_\kappa^{\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\xi]}$, $A = A \restriction [\gamma, \kappa)$. Note that $\underset{\sim}{D}_\xi$ generates a $<\kappa$-closed filter on $V_\kappa$, therefore $\mathbf{j}(\underset{\sim}{D}_\xi)$ generates a $<\mathbf{j}(\kappa)$-closed filter on $V_{\mathbf{j}(\kappa)}$. We claim that

(3.4)

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi+1)}/\mathbf{G}^0_{\kappa+1} \models q'_\xi{}^\frown\left(\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}\right) \ge \mathbf{j}(p) \text{ whenever } p \in \mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1} \cap \mathbf{G}^0_{\kappa+1},$$

where $q'_\xi{}^\frown\left(\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}\right)$ denotes the condition that agrees with $q'_\xi$ on coordinates below $\mathbf{j}(\xi)$, and $(\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)})$ at $\mathbf{j}(\xi)$. Note that by our hypothesis it suffices to check that

$$\forall p \in \mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1} \cap \mathbf{G}^0_{\kappa+1} : \quad (\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}) \ge \mathbf{j}(p)(\mathbf{j}(\xi)).$$

But a contradiction may only arise if for some $x = \mathbf{j}(x) \in V_\kappa^{\mathbf{V}_1}$ it were the case that

$$\mathbf{j}(p) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi)+1}} \mathbf{j}(x) \in \mathbf{j}(\varepsilon_\xi)(= \mathbf{j}(\varepsilon)_{\mathbf{j}(\xi)}),$$

equivalently,

$$p \Vdash_{\mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1}} x \in \varepsilon_\xi,$$

while

$$q'_\xi \frown (\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi+1)}/\mathbf{G}^0_{\kappa+1}} x = \mathbf{j}(x) \notin \mathbf{j}(\varepsilon)_{\mathbf{j}(\xi)}$$

(or the other way around). However, this is impossible as clearly $x \in \varepsilon_\xi$ by $p \in \mathbf{G}^0_{\kappa+1}$ and the very definition of $\varepsilon_\xi$, and by the fact that $q'_\xi \frown (\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)})$ forces $\mathbf{j}(\varepsilon)_{\mathbf{j}(\xi)} \restriction \kappa = \varepsilon_\xi$.

Having the claim established we can choose $q'_{\xi+1}$ so that $q'_{\xi+1}(\mathbf{j}(\kappa))(\mathbf{j}(\xi))$ satisfies (3.3) (with $\zeta = \xi$), hence (e) as well.

Finally, for (d), first note that we can assume $\underset{\sim}{A} \in \mathscr{N}_\eta$, so there are at most $\chi$-many such names. Now choosing an increasing sequence of conditions $\langle q''_\gamma : \gamma < \chi \rangle$ in $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$ with $q''_0 = q'_\eta$, we can decide for each name $\underset{\sim}{X}$ the statement $\kappa \in \mathbf{j}(\underset{\sim}{X})$. So using the $< \chi^+$-directed closedness of $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$ in $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$ (Fact 3.8), we can choose $q_\eta$ to be an upper bound of the sequence $\langle q''_\gamma : \gamma < \chi \rangle$, yielding (d) as desired.

Finally, $q_\chi$ is defined to be an upper bound of the $q_\eta$'s ($\eta < \chi$).

$\square_{\text{Claim3.11}}$

**Fact 3.12.** *By the definition of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$, and the way $q_\chi$ was constructed, we have:*

$(\bowtie)_{13}$ *For each $\delta \in S^*$, if $D_\delta$ generates a $\kappa$-complete ultrafilter on $V_\kappa$, then*

$$\Vdash_{\mathbb{P}^0_{\kappa+1}} \forall \underset{\sim}{A} \in \underset{\sim}{D}_\delta\ \exists \alpha < \kappa \text{ s.t. } (\underset{\sim}{\varepsilon}_\delta \restriction (\alpha, \kappa) \subseteq \underset{\sim}{A}),$$

$(\bowtie)_{14}$ *moreover, (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$) by e*

$$q_\chi \Vdash_{(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}} \forall d \left( \kappa(d) = \kappa \ \wedge\ d \in \bigcap_{\underset{\sim}{A} \in \underset{\sim}{D}_\delta} \mathbf{j}(\underset{\sim}{A}) \right) \to (d \in \mathbf{j}(\underset{\sim}{\varepsilon}_\delta))$$

*(where $\kappa(d)$ is defined in Definition 2.15 (ii)).*

$(\bowtie)_{15}$ *If $\delta \in S^*$, then $\varepsilon_\delta$ is a pseudointersection of $D_\delta$.*

$\mathbf{j}$ *and $q_\chi$ defines the normal ultrafilter*

$(\bullet)_5$ $D^\bullet = \{\underset{\sim}{A}[\mathbf{G}^0_{\kappa+1}] : \Vdash_{\mathbb{P}^0_{\kappa+1}} \underset{\sim}{A} \subseteq \kappa,\ q_\chi \Vdash \text{“}\kappa \in \mathbf{j}(\underset{\sim}{A})\text{”}\} \subseteq \mathscr{P}(\kappa),$

$(\bowtie)_{16}$ *and if $D_\delta \subseteq D^\bullet$, then $\varepsilon_\delta \in D^\bullet$.*

This together with (#) complete the proof of $((C))((\text{a}))$.

<u>Case 2</u>: For 3.2(C)(b) we proceed as follows. In $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ we have to find a sequence $\bar{U} = \langle U_\alpha :\ \alpha < \kappa \rangle$ of normal measures on $\kappa$ increasing in the Mitchell order, such that each $U_\alpha$ satisfies our closedness properties, namely, whenever $\langle X_\nu : \nu < \lambda \rangle$ is a sequence in $U_\alpha$, there exists $X' \in U_\alpha$, $|X' \setminus X_\nu| < \kappa$ for each $\nu < \lambda$. Let $U_0$ be the normal ultrafilter provided by appealing to $((C))((\text{a}))$ which we have already proved.

Working in $\mathbf{V}_1[\mathbf{G}^1_\chi] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$ we will construct the sequence by induction, so fixing $\alpha < \kappa$, we assume that $U_\beta$'s are already defined for $\beta < \alpha$. So we

$(\bullet)_6$ let $\underset{\sim}{\overline{U}}$ be a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi = \mathbb{P}^0_{\kappa+1}$-name for $\langle U_\beta : \beta < \alpha \rangle \in \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$, where $1_{\mathbb{P}^0_{\kappa+1}}$ forces that $\underset{\sim}{\overline{U}} = \langle \underset{\sim}{U}_\beta : \beta < \alpha \rangle$ is an increasing sequence of $\kappa$-complete normal ultrafilters w.r.t. the Mitchell-order of length $\alpha$, each $\underset{\sim}{U}_\beta$ is $<\lambda^+$-directed modulo $[\kappa]^{<\kappa}$.

and fix an elementary embedding $\mathbf{j}_* : \mathbf{V}_0 \to \mathbf{M}_*$ with critical point $\kappa$, ${}^\chi\mathbf{M}_* \subseteq \mathbf{M}_*$ with

$$\mathbf{j}_*(\mathbf{h})(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \underset{\sim}{U} \rangle \tag{3.5}$$

(recall the definition of $\mathbf{h}$ $(\bullet)_1$, this is possible). We are going to define a normal ultrafilter $U_\alpha$ associated with $\mathbf{j}_*$, above the $U_\beta$'s w.r.t. the Mitchell-order.

Defining $\underset{\sim}{\mathbb{P}}'_* = \mathbf{j}_*(\underset{\sim}{\mathbb{P}}^1)$, and letting $(\mathbb{P}^0_*)_{\mathbf{j}_*(\kappa)} = \mathbf{j}_*(\mathbb{P}^0_\kappa)$, observe that by the definition of $\mathbb{P}^0_\kappa$ (Definition 3.4)

$$\mathbf{j}_*(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi) = (\mathbb{P}^0_*)_{j_*(\kappa)} * (\underset{\sim}{\mathbb{P}}'_*)_{\mathbf{j}_*(\chi)},$$

and

$$(\mathbb{P}^0_*)_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi.$$

Now our fixed $\mathbf{G}^0_{\kappa+1} \subseteq \mathbb{P}^0_{\kappa+1}$ is generic over $\mathbf{V}_0$ and also over $\mathbf{M}_*$.

With a slight abuse of notation (in the proof of Case 2 from now on, in order to avoid notational awkwardness) we will refer to $(\mathbb{P}^0_*)_{\mathbf{j}_*(\kappa)}$ as $\mathbb{P}^0_{\mathbf{j}_*(\kappa)}$, and to $(\underset{\sim}{\mathbb{P}}'_*)_{\mathbf{j}_*(\chi)}$ as $\mathbb{P}'_{\mathbf{j}_*(\chi)}$; moreover, observe that all the preceding facts and claims hold in this setting, we only used that $\mathbf{j}(\mathbf{h}(\kappa)) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \underset{\sim}{x} \rangle$ for some name $\underset{\sim}{x}$, which obviously holds for $\mathbf{j}_*$ as well (where $\underset{\sim}{x}$ is not arbitrary anymore). In this new setting we appeal to Claim 3.11, obtaining the condition $q^*_\chi \in \mathbb{P}^0_{\mathbf{j}_*(\kappa)+1}/\mathbf{G}^0_{\kappa+1}$, and the $\kappa$-complete normal ultrafilter

$$D^\bullet_* = \{\underset{\sim}{A}[\mathbf{G}^0_{\kappa+1}] : \mathbf{M}_*[\mathbf{G}^0_{\kappa+1}] \models \text{“}q^*_\chi \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1}} \kappa \in \mathbf{j}_*(\underset{\sim}{A})\text{”}\} \tag{3.6}$$

(which is a $\kappa$-complete normal ultrafilter over $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$, belonging to $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$) and $<\lambda^+$-directed w.r.t. $\supseteq^*$. We only need to prove the following claim, implying that the filter $D^\bullet_*$ dominates $\{U_\beta : \beta < \alpha\}$ w.r.t. the Mitchell order:

**Claim 3.13.** *For each $\beta < \alpha$ there exists a sequence $\langle W_\gamma : \gamma < \kappa \rangle \in \mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$, where*

- *for $D^\bullet_*$-many $\gamma < \kappa$ the set $W_\gamma$ is an ultrafilter over $\gamma$,*
- *for each $X \in \mathscr{P}(\kappa) \cap \mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$*

$$X \in U_\beta \iff \{\gamma < \kappa : (X \cap \gamma) \in W_\gamma\} \in D^\bullet_*.$$

*Proof.* Using (reinterpreting) (3.5)

$$\left\{ \begin{array}{ll} \gamma < \kappa : & \mathbf{h}(\gamma) = \langle \underset{\sim}{x}_\gamma, \mu_\gamma, \underset{\sim}{y}_\gamma \rangle, \text{ where } \underset{\sim}{y}_\alpha \text{ is a } \mathbb{P}^0_{\gamma+1}\text{-name} \\ & \text{for a sequence of subsets of } \mathscr{P}(\gamma) \text{ of length } \alpha), \\ & \underset{\sim}{x}_\alpha = \underset{\sim}{\mathbb{Q}}^0_\alpha, \end{array} \right\} \in D^\bullet_* \cap \mathbf{V}_0.$$

Now suppose that $\beta < \alpha$ is fixed. Since $\underset{\sim}{x}_\gamma$ is name for a sequence of length $\alpha$, we can easily get a name for its $\beta$'th coordinate. This way, we can fix $Y \in D^\bullet_* \cap \mathbf{V}_0$, and the sequence $\langle \underset{\sim}{W}_\gamma : \gamma < \kappa \rangle$ such that

($\blacktriangle_1$) for each $\gamma \in Y$, $\underset{\sim}{W}_\gamma$ is a $\mathbb{P}^0_{\gamma+1}$-name for a subset of $\mathscr{P}(\gamma)$ (the $\beta$'th coordinate of $\underset{\sim}{x}_\gamma$), and

($\blacktriangle_2$) $\mathbf{j}_*(\langle \underset{\sim}{W}_\gamma : \gamma < \kappa \rangle)(\kappa) = \underset{\sim}{U}_\beta$.

In what follows, we will prove that the natural candidate $W_\gamma = \underset{\sim}{W}_\gamma[\mathbf{G}^0_{\kappa+1}]$ $(\gamma < \kappa)$ works (utilizing standard arguments, so a reader familiar with this kind of proofs can jump to Case 3).

For a fixed $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{X} \in \mathbf{V}_0$ (for a subset of $\kappa$) define the $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{Z}_X \in \mathbf{V}_0$ as follows.

$$1_{\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi} \Vdash \underset{\sim}{Z}_X = \{\gamma < \kappa : \underset{\sim}{X} \restriction \gamma \in \underset{\sim}{W}_\gamma\}, \tag{3.7}$$

We only have to verify that

$$\underset{\sim}{X}[\mathbf{G}^0_{\kappa+1}] \in \underset{\sim}{U}_\beta[\mathbf{G}^0_{\kappa+1}] \text{ iff } \underset{\sim}{Z}_X[\mathbf{G}^0_{\kappa+1}] \in D^\bullet_*. \tag{3.8}$$

But the latter is defined (by (3.6)) as

$$\begin{gathered} \underset{\sim}{Z}_X[\mathbf{G}^0_{\kappa+1}] \in D^\bullet_*, \\ \Updownarrow \\ (\text{in } \mathbf{M}_*[\mathbf{G}^0_{\kappa+1}])\ q^*_\chi \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1}} \kappa \in \mathbf{j}_*(\underset{\sim}{Z}_X), \end{gathered}$$

Therefore, as $\mathbf{j}_*(\underset{\sim}{W})(\kappa) = \underset{\sim}{U}_\beta$ by (3.7), and

$$\mathbf{M}_*[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models (q^*_\chi \Vdash \kappa \in \mathbf{j}_*(\underset{\sim}{Z}_X) \iff q^*_\chi \Vdash \mathbf{j}_*(\underset{\sim}{X}) \restriction \kappa \in \underset{\sim}{U}_\beta)$$

(since $\mathbf{j}_*(\langle \underset{\sim}{W}_\gamma : \gamma < \kappa\rangle)(\kappa) = \underset{\sim}{U}_\beta$), we observe that in order to get (3.8) it suffices to show the following

$$\underset{\sim}{X}[\mathbf{G}^0_{\kappa+1}] \in \underset{\sim}{U}_\beta[\mathbf{G}^0_{\kappa+1}] \text{ iff } q^*_\chi \Vdash \mathbf{j}_*(\underset{\sim}{X}) \restriction \kappa \in \underset{\sim}{U}_\beta. \tag{3.9}$$

But then by the elementarity of $\mathbf{j}_*$ (and $\mathrm{crit}(\mathbf{j}_*) = \kappa$)

$$\forall \alpha < \kappa, \forall p \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi : \quad p \Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi} \check{\alpha} \in \underset{\sim}{X} \iff \mathbf{j}_*(p) \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}_*(\chi)}} \check{\alpha} \in \mathbf{j}_*(\underset{\sim}{X}),$$

and recalling $p \in \mathbf{G}^0_{\kappa+1}$ implies $q^*_\chi \geq \mathbf{j}_*(p)$ in the quotient forcing $\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1})$ we get that

$(*)_1$ (in $\mathbf{M}_*[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$) the condition $q^*_\chi$ forces $\mathbf{j}_*(\underset{\sim}{X}) \restriction \kappa$ to be equal to $\underset{\sim}{X}[\mathbf{G}^0_{\kappa+1}]$.

This yields (3.9), completing the proof of Case 2.

□

Case 3: For 3.2(C)(c). We fix $\Upsilon > \kappa$, and $\eta$, and we would like to define the $\kappa$-complete fine normal ultrafilter $W$ on $[\Upsilon]^{<\kappa}$ that satisfies $\star$ from (c). First we redefine the elementary embedding $\mathbf{j}$ from Definition 3.5 (as well as $\mathbb{P}^0_{\mathbf{j}(\kappa)}$, $\underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)}$):

**Definition 3.14.**

- $(\bullet)_1$ Let $\rho = |2^{(\Upsilon \cdot \chi)^\kappa} + \eta|$, and
- $(\bullet)_2$ define $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$ to be an elementary embedding with critical point $\kappa$ such that $(\mathbf{j}(\mathbf{h}))(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \rho^+, \check{\emptyset}\rangle$ ($\check{\emptyset} = \emptyset$ is the canonical name for the empty set) and $\mathbf{j}(\kappa) > \rho$, ${}^\rho\mathbf{M} \subseteq \mathbf{M}$,
- $(\bullet)_3$ Let $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \mathbf{j}(\kappa), \beta < \mathbf{j}(\kappa)\rangle = \mathbf{j}(\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa\rangle)$ so $\underset{\sim}{\mathbb{Q}}^0_\kappa = \underset{\sim}{\mathbb{P}}^1_\chi$, and let $\underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)} = \mathbf{j}(\underset{\sim}{\mathbb{P}}^1_\chi)$.

Similarly as in Facts 3.7, 3.8, 3.9 we can get the following.

**Fact 3.15.** *The filter $\mathbf{G}^0_{\kappa+1}$ is generic over $\mathbf{M}$ as well, and the forcing notions $\mathbb{P}^0_{j(\kappa)}/\mathbf{G}^0_{\kappa+1}$ and $(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ ($\gamma \leq \mathbf{j}(\chi)$) are well defined and $< |2^\Upsilon + \eta|^+$-directed closed in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.*

**Fact 3.16.** $\mathbf{V}_1 \models$ *"$(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ is $< |2^\Upsilon + \eta|^+$-directed closed."*

Fact 3.8 follows from the fact below.

**Fact 3.17.** $\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models \ {}^{2^\Upsilon+\eta}\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.

Using this new $\mathbf{j}$, we will extract the ultrafilter $W \subseteq \mathscr{P}([\Upsilon]^{<\kappa})$ (in the sense of $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$), and the sequence of ultrafilters $\overline{U}$ as well from the information provided by $\mathbf{G}^0_{\kappa+1} = \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$, and $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$ (given by Claim 3.11), and then we will prove that it is indeed a measure sequence corresponding to the elementary embedding $\mathbf{j}_W$. Obviously,

($\circledcirc_1$) $\mathbf{j}(\kappa) > \chi$, ${}^\chi M \subseteq M$.

Observe that Claim 3.11 is true in this setting as well, and let the master condition $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$ be given by it. First we claim that by possibly extending $q_\chi$, we can assume that

($\circledcirc_2$) For each $A \in \mathscr{P}([\Upsilon]^{<\kappa}) \cap \mathbf{V}_2$ the condition $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}$ decides about (the truth value of) "($\mathbf{j}$"$\Upsilon \in \mathbf{j}(A)$)" (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$).

To this end we first count the possible $A$'s. Recall that $\mathbb{P}^1_\chi$ is $<\kappa$-closed $(((B))/$ (b)

$$[\chi]^{<\kappa} \cap \mathbf{V}_2 = [\chi]^{<\kappa} \cap \mathbf{V}_1 = [\chi]^{<\kappa} \cap \mathbf{V}_0[\mathbf{G}^0_\kappa],$$

and as $|\mathbb{P}^0_\kappa| = \kappa$,

$$|[\Upsilon]^{<\kappa} \cap \mathbf{V}_2| \leq (\Upsilon \cdot \chi)^\kappa. \tag{3.10}$$

Second, as $|\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi| = \chi$, we have

$$\mathbf{V}_2 = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models \mathscr{P}([\chi]^{<\kappa})| \leq (2^{(\chi\cdot\Upsilon)^\kappa})^{\mathbf{V}_0} \leq \rho. \tag{3.11}$$

Now using Fact 3.8 we can extend $q_\chi$ to another condition $q_*$ in (at most) $\rho$-many steps (in $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\zeta)})/\mathbf{G}^0_{\kappa+1}$) so that

($\circledcirc_3$) for each name $\underset{\sim}{A}$ for a subset of $[\chi]^{<\kappa}$

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \ q_* \parallel \ \mathbf{j}\text{“}\Upsilon \in \underset{\sim}{A},$$

and so (by possibly replacing $q_\chi$ by $q_*$) ($\circledcirc_2$) holds, indeed. Now we can define the $\kappa$-complete, fine, normal ultrafilter

$$W = \{\underset{\sim}{A}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\kappa] \in [\Upsilon]^{<\kappa} : \ q_\chi \Vdash \mathbf{j}\text{“}\Upsilon \in \mathbf{j}(\underset{\sim}{A})\} \in \mathbf{V}_2, \tag{3.12}$$

Now let $\mathbf{j}_W : \mathbf{V}_2 \to \mathbf{M}_W = \mathrm{Mos}({}^{[\Upsilon]^{<\kappa}}\mathbf{V}_2/W)$ be the corresponding elementary embedding, and let $\overline{U} = \langle U_\alpha : \ \alpha < \mathrm{dom}(\overline{U})\rangle$ be the ultrafilter sequence of maximal length associated to $\mathbf{j}_W$, that is, the following holds in $\mathbf{V}_2$.

($\boxminus_1$) $U_0 = \kappa$, and for each $\alpha \in \mathrm{dom}(\overline{U})$, $\alpha > 0$ the set $U_\alpha \subseteq \mathscr{P}(V_\kappa)$ is a $\kappa$-complete normal ultrafilter satisfying

$$\forall A \subseteq V_\kappa : \ A \in U_\alpha \iff U \restriction \alpha \in \mathbf{j}_W(A)$$

(therefore for each $\alpha < \mathrm{dom}(\overline{U})$, $U \restriction \alpha \in \mathbf{M}_W$),

($\boxminus_2$) $\overline{U} \notin \mathbf{M}_W$.

The following two claims complete the proof of 3.2($(C)$)((c)) as we study the ultrafilter sequence $\overline{U}\restriction(\min(\mathrm{dom}(\overline{U}), \eta)$.

**Claim 3.18.** *For every ultrafilter sequence $\overline{F} \in \mathbf{M}_W$ with $\kappa(\overline{F}) = \kappa$ there exists a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-name $\underset{\sim}{\overline{F}}' \in \mathbf{M}$ for an ultrafilter sequence with $\kappa(\overline{F}') = \kappa$ such that for each name $\underset{\sim}{A}$ for a subset of $V_\kappa^{\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]}$ we have*

$$\overline{F} \in \mathbf{j}_W(\underset{\sim}{A}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]) \iff \mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models q_* \Vdash \underset{\sim}{\overline{F}}' \in \mathbf{j}(\underset{\sim}{A}).$$

**Claim 3.19.** *Suppose that $\sigma \leq \min(\mathrm{dom}(\overline{U}, \eta))$, and assume $\{\underset{\sim}{\overline{F}_i} : i < \sigma\} \subseteq \mathbf{M}$ is a set of ($\mathbb{P}_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-names for) ultrafilter sequences with $\kappa(\underset{\sim}{F}_i) = \kappa$ ($i < \sigma$).*

*If the filter*

$$F_* = \bigcap_{i<\sigma} \{A \subseteq V_\kappa^{\mathbf{V}_2} : \ q_\chi \Vdash \underset{\sim}{\overline{F}_i} \in \mathbf{j}(A)\}$$

*satisfies $(\forall \alpha < \kappa) : |\cup F_* \restriction \alpha| < \kappa$, then $F_*$ is $< \lambda^+$-directed in the sense that for any system $\langle X_\alpha : \alpha < \lambda\rangle$ in $F_*$ there is a set $X' \in F_*$ s.t. for each $\alpha < \lambda$ there exists $\delta < \kappa$ with $X' \restriction [\delta, \kappa) \subseteq X_\alpha$.*

*Proof.* (Claim 3.18) Instead of factoring through our elementary embeddings (after forcing) we provide a direct calculation. Fix the ultrafilter sequence $\overline{F} \in \mathbf{M}_W$, and pick a function $f \in \mathbf{V}_2$, $\mathrm{dom}(f) = [\Upsilon]^{<\kappa}$, $\mathbf{j}_W(f)(\mathbf{j}_W``\Upsilon) = \overline{F}$, where we can assume that

$$\forall x \in \mathrm{dom}(f) \ f(x) \text{ is an u.f. sequence with } \kappa(f(x)) = \mathrm{otp}(\kappa \cap x). \tag{3.13}$$

Now we can fix a $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{f} \in \mathbf{V}_0$ of $f$, such that $1_{\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi}$ forces (3.13). Now as $\underset{\sim}{f} \in \mathbf{V}_0$ is a $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name for a function with $\mathrm{dom}(f) = [\Upsilon]^{<\kappa}$, by elementarity $\mathbf{j}(\underset{\sim}{f})$ is a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-name for a function with domain $[\mathbf{j}(\Upsilon)]^{<\mathbf{j}(\kappa)}$. Now, as $\mathbf{j}``\Upsilon \in \mathbf{M} \cap [\mathbf{j}(\Upsilon)]^{\leq \rho}$, there is a name $\underset{\sim}{F}' \in \mathbf{M}$ such that

$$\mathbf{M} \models \ \Vdash_{\mathbb{P}^1_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}} \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) = \underset{\sim}{F}'. \tag{3.14}$$

It only remains to check that for each $X \subseteq V_\kappa^{\mathbf{V}_2}$ the conditions "$F \in \mathbf{j}_W(X)$" and "$q_* \Vdash \underset{\sim}{F}' \in \mathbf{j}(X)$" are equivalent. More precisely, we prove the following.

($\circ$) For every fixed $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{X}$ for a subset of $V_\kappa^{\mathbf{V}_2}$

$$F \in \mathbf{j}_W(\underset{\sim}{X}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]) \iff q_* \Vdash \underset{\sim}{F}' \in \mathbf{j}(\underset{\sim}{X}).$$

As $F = \mathbf{j}_W(f)(\mathbf{j}_W``\Upsilon)$ we can reformulate the lhs. of the statement as

$$\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models \ \{y \in [\Upsilon]^{<\kappa} : \ f(y) \in X\} \in W,$$

i.e. for some $p \in \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$

$$p \Vdash_{\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi} \{y \in [\Upsilon]^{<\kappa} : \ \underset{\sim}{f}(y) \in \underset{\sim}{X}\} \in \underset{\sim}{W}.$$

Now for the the $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underset{\sim}{C} := \{y \in [\Upsilon]^{<\kappa} : \ \underset{\sim}{f}(y) \in \underset{\sim}{X}\}$ we have (by ($\circledcirc_2$) and (3.12))

$$\underset{\sim}{C}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \in W \iff q_* \Vdash \mathbf{j}``\Upsilon \in \mathbf{j}(\underset{\sim}{C}). \tag{3.15}$$

(Recall that $q_*$ decides this by ($\circledcirc_3$) as $\underset{\sim}{C}$ is a name for a subset of $[\Upsilon]^{<\kappa}$.) Now

$$(\Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}\chi}}) \, \mathbf{j}(\underset{\sim}{C}) = \mathbf{j}(\{y \in [\Upsilon]^{<\kappa} : \ \underset{\sim}{f}(y) \in \underset{\sim}{X}\}) = (\{y \in [\mathbf{j}(\Upsilon)]^{<\mathbf{j}(\kappa)} : \ \underset{\sim}{\mathbf{j}}(f)(y) \in \mathbf{j}(\underset{\sim}{X})\}),$$

so the rhs. of (3.15) is equivalent to

$$q_* \Vdash \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) \in \mathbf{j}(\underset{\sim}{X}), \tag{3.16}$$

so recalling the definition of $\underset{\sim}{F}'$, $\Vdash \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) = \underset{\sim}{F}'$ by (3.14) (3.16) is clearly equivalent to $q_* \Vdash \underset{\sim}{F}' \in \mathbf{j}(\underset{\sim}{X})$, therefore ($\circ$) holds, as desired. $\square_{\text{Claim3.18}}$

*Proof.* Fix $\langle \underset{\sim}{F}'_i : i < \sigma\rangle$ as in the Claim 3.18.

We only have to recall how we constructed $q_\chi$, which ensures the existence of the desired pseudointersection. Fix a sequence $\langle X_\alpha : \alpha < \lambda\rangle$ in the filter $F_*$. Now let $D' = \{X_\alpha : \alpha < \lambda\}$, which is equal to $D_\zeta$ for some $\zeta < \chi$ by (#) from our assumptions $(B)$/d. Now by our assumptions

$$(\forall i < \sigma)\ (\forall X \in D_\zeta)\ q_\chi \Vdash \underset{\sim}{F}_i \in \mathbf{j}(X),$$

so since $A'_{\mathbf{j}(\zeta)}$ is the name for the intersection of the $\mathbf{j}(\underset{\sim}{A})$'s, where $\underset{\sim}{A}$ runs over the $< \kappa$-complete filter generated by $D_\zeta$ (Definition 3.10) $A'_{\mathbf{j}(\zeta)}$

$$(\forall i < \sigma)\ q_\chi \Vdash \underset{\sim}{F}_i \in A'_{\mathbf{j}(\zeta)}.$$

Finally, recalling Definition 3.10 and (3.3) from Claim 3.11 we get that for the generic sequence $\varepsilon_\zeta$ (which is a pseudointersection of the $D' = D_\zeta$)

$$q_{\zeta+1} \Vdash \mathbf{j}(\varepsilon_\zeta) \restriction (\kappa+1) = \varepsilon_\zeta \cup (A'_{\mathbf{j}(\zeta)} \restriction [\kappa, \kappa+1)),$$

which means

$$(\forall i < \sigma)\ \ q_\chi \Vdash \overline{\underset{\sim}{F}_i} \in \mathbf{j}(\varepsilon_\zeta),$$

and we are done.

$\square_{\text{Claim3.19}}$

$\square_{\text{Lemma3.2}}$

**§ 3(B). The preliminary forcing for obtaining $(\kappa, \lambda) - 1$ systems together with a universal in $(K_\kappa)_\lambda$.**

This subsection deals with the application of Claim 3.2, we show that it is possible to force a universal object in $(K_\kappa)_\lambda$ with a notion of forcing satisfying requirements from Claim 3.2.

**Conclusion 3.20.** *Assume*

- $\kappa$ *is supercompact,*
- $\kappa < \lambda < \chi = \chi^\lambda$,
- $\lambda$ *is regular,*
- $(\forall\theta)(\theta \in \text{Card} \wedge \kappa \le \theta < \lambda \Rightarrow 2^\theta = \theta^+)$ , *and*
- $\sigma = \text{cf}(\sigma) < \kappa$.

*Then for some forcing extension* $\mathbf{V}^{\mathbb{P}}$ *preserving cardinals* $\ge \kappa$ *and cofinalities* $> \kappa$ *and* $\le \sigma$, *we have that in* $\mathbf{V}^{\mathbb{P}}$:

(1) $2^\kappa = \chi$,
(2) $\kappa$ *is a strong limit singular of cofinality* $\sigma$,
(3) *and there is a universal graph in cardinality* $\lambda$.

*Proof.* We shall use 1.2, but we have to justify it. That is, we need a forcing fitting in the scheme in Claim 3.2 with $\mathbf{V}_0 = \mathbf{V}$, specifying the $(< \kappa)$-directed-complete iteration $\mathbb{P}^1_\chi = \langle \mathbb{P}^1_\alpha, \mathbb{Q}^1_\beta : \alpha \le \chi, \beta < \chi\rangle \in \mathbf{V}_1 = \mathbf{V}^{\mathbb{P}^0_\kappa}$ in which we are free to choose $\underset{\sim}{\mathbb{Q}}_\beta$'s on $\beta$'s outside $S^* \subseteq \chi$. (And then conclusion $(C)$/ (a) or (b) with Claim 1.5 together with Claim 2.1 or 2.2 will give the desired consistency result.) Our task is to construct (in $\mathbf{V}_1$) a suitable iteration $\mathbb{P}^1_\chi$, and to check that $\mathbb{P}^1_\chi$

$(\mathsf{T})_1$ is $<\kappa$-directed closed,
$(\mathsf{T})_2$ is of cardinality $\chi$ (up to equivalence),
$(\mathsf{T})_3$ has the $\lambda^+$-c.c.,

$(\mathsf{T})_4$ does not collapse any cardinals, and

$(\mathsf{T})_5$ $\mathbf{V}_1 \models \Vdash_{\mathbb{P}^1_\chi}$ "there is a universal graph in $(K_\kappa)_\lambda$",

$(\mathsf{T})_6$ and we can choose $S^* \in [\chi \setminus \{0,1\}]^\chi$, $S^* \in \mathbf{V}_1$, $|\chi \setminus S^*| = \chi$, and the $\mathbb{P}^1_\delta$-names $\underset{\sim}{D}_\delta$ $(\delta \in S^*)$ satisfying $((B))$(d) from Claim 3.2.

We will do the same as in [She90], we define (in $\mathbf{V}_1$)

(1) $\mathbb{Q}^1_0$ to be the forcing of $\chi$-many stationary sets of $\lambda$, any two intersecting in a set of size smaller than $\kappa$,
(2) $\underset{\sim}{\mathbb{Q}}^1_\beta$ for $\beta \in \chi \setminus (S^* \cup \{0\})$ the main iteration from [She90] just with $\kappa$-many colors (i.e. in the class $K_\kappa$ instead of simple graphs, which is just equivalent to $K_2$): forcing a generic random graph, and the embeddings into it with $< \kappa$-support partial functions.

We need to check that the iteration is indeed $\lambda^+$-cc, which will be ensured by showing that (in $\mathbf{V}_1$) $\mathbb{Q}^1_0$ is $\lambda^+$-cc, and in $(\mathbf{V}_1)^{\mathbb{Q}^1_0}$ the iteration of $\mathbb{Q}^1_\alpha$'s $(0 < \alpha < \chi)$, i.e. $\mathbb{P}^1_\chi / \mathbf{G}^1_1$ has the $\kappa^+$-cc .
First for future reference we have to remark that by the construction of $\mathbb{P}^0_\kappa$

$(*)_1$ in $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{P}^0_\kappa}$ $\kappa$ is still strongly inaccessible (as we noted in $(\bowtie)_3$). As $|\mathbb{P}^0_\kappa| = \kappa$ our cardinal arithmetic assumptions above $\kappa$ are also preserved.

Working in $\mathbf{V}_1$, the next lemma concerns the first step $\mathbb{Q}^1_0$ which we can define to be $Q(\lambda, \chi, \kappa)$ as in [Bau76, Sec. 6.], see below $(b)$ in Definition 3.22.

**Lemma 3.21.** *In $\mathbf{V}_1$ there exists a forcing poset $\mathbb{Q}^1_0$ that is $<\kappa$-directed closed, of power $\chi$, having the $\lambda^+$-cc, preserving cardinals from $(\kappa, \lambda]$, and*

$\mathbf{V}_1^{\mathbb{Q}^1_0} \models \exists \{S_\alpha : \alpha < \chi\} \subseteq \mathscr{P}(\lambda)$, *a system of stationary sets s.t.* $\forall \alpha < \beta < \chi : |S_\alpha \cap S_\beta| < \kappa$.

*Proof.*

**Definition 3.22.** First we define the following auxiliary posets.

$(a)$ For a regular cardinal $\mu$ we let $Q'(\lambda, \chi, \mu)$ be the set of functions $f$ satisfying
  $(i)$ $\mathrm{dom}(f) \in [\chi]^{<\mu}$,
  $(ii)$ for each $\alpha \in \mathrm{dom}(f)$ $f(\alpha) \in [\lambda]^{<\mu}$,
  with $f \leq g$, iff
  $(iii)$ $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$,
  $(iv)$ $\forall \alpha \in \mathrm{dom}(f)$: $f(\alpha) \subseteq g(\alpha)$,
  $(v)$ for each $\alpha \neq \beta \in \mathrm{dom}(f)$: $f(\alpha) \cap f(\beta) = g(\alpha) \cap g(\beta)$.

$(b)$ Let $Q(\lambda, \chi, \kappa) \subseteq \prod_{\mu \in \mathrm{Reg} \cap [\kappa, \lambda]} Q'(\lambda, \chi, \mu)$ be the collection of the following functions $f$
  $(i)$ $(\forall \mu < \nu \in \mathrm{Reg} \cap [\kappa, \lambda])$, $(\forall \alpha \in \mathrm{dom}(f_\mu))$: $f_\mu(\alpha) \subseteq f_\nu(\alpha)$
  with the pointwise ordering inherited from the full product $\prod_{\mu \in \mathrm{Reg} \cap [\kappa, \lambda]} Q'(\lambda, \chi, \mu)$.

**Definition 3.23.** We let $\mathbb{Q}^1_0 = Q(\lambda, \chi, \kappa) \in \mathbf{V}_1$.

For later reference we note the following. Recall that $\chi^\lambda = \chi$ holds by our assumptions.

**Observation 3.24.** *For each $\mu \in \mathrm{Reg} \cap [\kappa, \lambda]$ $|Q'(\lambda, \chi, \mu)| \leq \chi^{<\mu} \cdot \lambda^{<\mu} = \chi$. Therefore $|\mathbb{Q}^1_0| = \chi$.*

By [Bau76, Lemma 6.3], recalling $(\sigma \in \mathrm{Card} \cap [\kappa, \lambda)) \to (2^\sigma = \sigma^+)$ by our premises, so $\lambda^{<\lambda} = \lambda$ we have the following.

**Claim 3.25.** *$Q(\lambda, \chi, \kappa)$ is $\lambda^+$-cc, $<\kappa$-directed closed, preserving cofinalities and cardinals.*

Clearly

$(\ddagger)_1$ every directed subset of power less than $\kappa$ in $\mathbb{Q}^1_0 = Q(\lambda, \chi, \kappa)$ has a least upper bound.

Now obviously, in $\mathbf{V}_1^{\mathbb{Q}^1_0}$

$(\ddagger)_2$ the generic subsets $S_\alpha$ $(\alpha < \chi)$ defined by $\Vdash_{\mathbb{Q}^1_0} \underset{\sim}{S}_\alpha = \cup\{f_\kappa(\alpha) : f \in \underset{\sim}{\mathbf{G}}\}$ form a $\kappa$-almost disjoint system, i.e. if $\alpha < \beta$, then $\Vdash |\underset{\sim}{S}_\alpha \cap \underset{\sim}{S}_\beta| < \kappa$,

we only need to verify that

$(\ddagger)_3$ for each $\alpha < \chi$ the subset

$$S_\alpha \text{ is a stationary subset of } \lambda,$$

which is a standard argument, but for the sake of completeness we elaborate. (In fact, recalling [Bau76, Lemmas 6.3-6.5.] with the aid of the following it is easy to argue that $(S_\alpha \cap E^\lambda_{\geq\kappa})$ i.e. restricting $S_\alpha$ to points of cofinality at least $\kappa$ is stationary.)

**Claim 3.26.** [Bau76, Lemmas 6.3-6.5.] *The notion of forcing $Q(\lambda, \chi, \kappa)$ is equivalent to the two-step iteration $Q(\lambda, \chi, \kappa^+) * \underset{\sim}{Q}'(\lambda, \chi, \kappa, \underset{\sim}{F})$ where*

$$\mathbf{V}_1^{Q(\lambda,\chi,\kappa^+)} \models \quad \begin{array}{l} \bullet\ F_\alpha\ (\alpha \in \chi) \text{ is the generic sequence in } [\lambda]^\lambda (\text{given by } Q(\lambda, \chi, \kappa^+)), \\ \bullet\ Q'(\lambda, \chi, \kappa, F) \subseteq Q'(\lambda, \chi, \kappa) \text{ defined by} \\ \quad [f \in Q'(\lambda, \chi, \kappa, \underset{\sim}{F}) \iff \forall \alpha \in \mathrm{dom}(f)\ f(\alpha) \subseteq F_\alpha]. \end{array}$$

*Moreover, $Q(\lambda, \chi, \kappa^+)$ is $<\kappa^+$-closed, (in $\mathbf{V}_1^{Q(\lambda,\chi,\kappa^+)}$), and $Q'(\lambda, \chi, \kappa, F)$ has the $\kappa^+ - cc$).*

Looking at the definition of the forcing $Q(\lambda, \chi, \kappa)$, if we are given a condition $p$, and a $Q(\lambda, \chi, \kappa)$-name $\underset{\sim}{C}_*$ for a club set in $\lambda$, first recall that $Q(\lambda, \chi, \kappa)$ is $<\kappa$-closed (Claim 3.26), in particular $< \omega_1$-closed, as $\kappa$ is strongly inaccessible. We can define an increasing sequence $p^j$ $(j < \omega)$ in $Q(\lambda, \chi, \kappa)$ with $p^0 = p$, and an increasing sequence of ordinals $\varrho_j$ $(j < \kappa)$ satisfying $p^j \Vdash \varrho_j \in \underset{\sim}{C}_*$, and if $j < k$, then $\sup \cup\{p^j_\lambda(\beta) : \beta \in \mathrm{dom}(p^j_\lambda)\} < \varrho_k$. This is possible, as $|\mathrm{dom}(p_j)| < \lambda$, as well as $|p^j_\lambda(\beta)| < \lambda$, and $\lambda$ is regular. Then clearly any upper bound of the $p^j$'s forces $\varrho_\omega := \sup\{\varrho_j : j < \omega\} \in \underset{\sim}{C}_*$, but as the least upper bound does not say anything about the statements $\varrho_\omega \in \underset{\sim}{S}_\beta$ $(\beta < \chi)$ we can extend it to a condition $p'$ with $\varrho_\omega \in p'_\mu(\alpha)$ for each $\mu \in \mathrm{Reg} \cap [\kappa, \lambda]$ (thus $p' \Vdash \varrho_\omega \in \underset{\sim}{S}_\alpha \cap \underset{\sim}{C}_*$). This completes the proof of Lemma 3.21. $\square_{\text{Lemma3.21}}$

As $\mathbb{Q}^1_0$ as already defined in Definition 3.23 we can define the iteration $\langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi \rangle$ for which we have to choose a suitable $S^*$.

**Definition 3.27.** We let $0, 1 \notin S^* \subseteq \chi$ be such that $|S^*| = \chi$, $|\chi \setminus S^*| = \chi$.

**Definition 3.28.** We let $\langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi \rangle$ be the following $< \kappa$-support iteration. The definition of the $\mathbb{P}^1_\beta$-name $\underset{\sim}{\mathbb{Q}}^1_\beta$ goes by induction on $\beta$ as follows, distinguishing three cases. But first

⊛ we have to remark that in steps with $\beta \in S^*$ we will only assume that $\underset{\sim}{D}_\beta$ is a $\mathbb{P}^1_\beta$-name for a system of subsets if $V_\kappa^{\mathbf{V}_1}$, where

$$\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{D}_\beta \in [\mathscr{P}(V_\kappa^{\mathbf{V}_1})]^{\leq\lambda},$$

first we will deduce some properties of $\mathbb{P}^1_\chi$ based on only this weak assumption up until the end of the proof of Lemmas 3.35 and 3.34 and then we will verify that the $\underset{\sim}{D}_\beta$'s ($\beta \in S^*$) can be suitably chosen (during the inductive process of defining the iteration $\mathbb{P}^1_\chi$) so that the iteration fulfills all our remaining demands from $(\top)_1$-$(\top)_6$. This will be a standard bookkeeping argument.

⊛ Similarly, for steps in $\chi \setminus S^* \setminus \{0,1\}$ up until the end of the proof of Lemmas 3.35 and 3.34 we only assume that $\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{M}_\beta \in (K_\kappa)_\lambda$, i.e. is a $\mathbb{P}^1_\beta$-name for a $\kappa$-colored graph on $\lambda$.

- For every $M = \langle |M|, R^M_\alpha : \alpha < \kappa\rangle \in (K_\kappa)_\lambda$ we will use the notation $c_M : [\lambda]^2 \to \kappa$ denoting the color of the edge between $i$ and $j$, i.e.

$$c_M(i,j) = \alpha \iff (i,j) \in R^M_\alpha.$$

Case (1): $\beta = 1$.

Let $\mathbb{Q}^1_1 \in \mathbf{V}_1^{\mathbb{Q}^1_0}$ be the forcing for obtaining a random $\kappa$-colored graph on $\lambda$ with conditions of power $< \kappa$, i.e. $q \in \mathbb{Q}^1_1$ iff

(i) $q \subseteq \{[i\ R_\gamma\ j] : i \neq j < \lambda,\ \gamma < \kappa\}$,

(ii) $\forall i \neq j < \lambda$ we have

$$([i\ R_\gamma\ j], [i\ R_{\gamma'}\ j] \in q) \longrightarrow (\gamma = \gamma'),$$

(iii) $|q| < \kappa$,

with the usual ordering. Then

$(\diamond)_1$ the generic object $\underset{\sim}{M}_* = \langle \lambda, \underset{\sim}{R}^{M_*}_\alpha : \alpha < \kappa\rangle$ satisfies

$$\Vdash_{\mathbb{P}^1_2} \langle \underset{\sim}{R}^{M_*}_\alpha : \alpha < \kappa\rangle \text{ is a partition of } [\lambda]^2.$$

Case (2): $\beta \in \chi \setminus S^* \setminus \{0,1\}$.

In order to define $\mathbb{Q}^1_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ (formally a $\mathbb{P}^1_\beta$-name $\underset{\sim}{\mathbb{Q}}^1_\beta \in \mathbf{V}_1$) we first need to work in $\mathbf{V}'_1 = \mathbf{V}_1^{\mathbb{P}^1_1} (= \mathbf{V}_1^{\mathbb{Q}^1_0})$ as preparation. Let $\Upsilon$ be a large enough regular cardinal, and define the continuous increasing chain $\overline{N}_\beta = \langle N_{\beta,\gamma} : \gamma < \lambda\rangle \in \mathbf{V}'_1$ so that

- $\beta$, $\mathbb{P}^1_\beta$, $\langle \overline{N}_\gamma : \gamma \in \beta \setminus S^* \setminus \{0,1\}\rangle$, $\mathbf{G}^1_1 \in N_{\beta,0}$ (the generic filter over $\mathbb{P}^1_1$),
- $\kappa + 1 \subseteq N_{\beta,0}$,
- for each $\gamma < \lambda$:
  - $(\bullet)_a$ $N_{\beta,\gamma} \prec \left(\mathscr{H}(\Upsilon)^{\mathbf{V}'_1}, \in\right)$,
  - $(\bullet)_b$ $|N_{\beta,\gamma}| < \lambda$,
  - $(\bullet)_c$ $N_{\beta,\gamma} \cap \lambda$ is an initial segment of $\lambda$
  - $(\bullet)_d$ $N_{\beta,\gamma} \cap \lambda < N_{\beta,\gamma+1} \cap \lambda$,
  - $(\bullet)_e$ for $\varepsilon < \lambda$ limit $N_{\beta,\varepsilon} = \bigcup_{\gamma<\varepsilon} N_{\beta,\gamma}$,

and

$(\diamond)_2$ let $\xi_\beta(\gamma) = N_{\beta,\gamma} \cap \lambda$ ($\gamma < \lambda$).

So the set $\{\xi_\beta(\gamma) : \gamma < \lambda\}$ is a club subset of $\lambda$, and as $S_\beta$ is stationary (Lemma 3.21) the set $C_\beta = \mathrm{cl}(S_\beta \cap \{\xi_\beta(\gamma) : \gamma < \lambda\})$ (i.e. the smallest closed set containing $S_\beta \cap \{\xi_\beta(\gamma) : \gamma < \lambda\}$) is a club. Therefore the system $\langle N_{\beta,\gamma} : \gamma < \lambda \wedge \xi_\beta(\gamma) \in C_\beta\rangle$ (after reparametrizing) clearly satisfies $(\bullet)_a$-$(\bullet)_e$, hence we can assume that

$(\diamond)_3$ $\{\xi_\beta(\gamma+1) : \gamma \in \lambda\} \subseteq S_\beta$,

and we let

$(\diamond)_4$ $N^*_\beta = \{\xi_\beta(\gamma) : \gamma \in \lambda\}$.

For later reference we remark that the $\kappa$-almost disjointness of the $S_\alpha$'s and $(\diamond)_3$ together implies

$(\diamond)_5$ if $\beta \neq \alpha < \chi$ then $|\{\xi_\beta(\delta+1) : \delta \in \lambda\} \cap \{\xi_\alpha(\delta+1) : \delta \in \lambda\}| < \kappa$.

Now the forcing $\mathbb{Q}^1_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ is defined so that it shall give an embedding $f_\beta$ of the $\kappa$-colored graph $M_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ into $M_*$, formally defined by

$(\diamond)_6$ $q \in \mathbb{Q}^1_\beta$, iff

- $(i)$ $q$ is a set of elementary conditions of the following form
  - $[f_\beta(i) = j]$, where $j \in \{\xi_\beta(\nu+1) : \kappa i \leq \nu < \kappa(i+1)\}$ (so necessarily $i < j$),
  - $[j \notin \mathrm{ran}(f_\beta)]$ for some $j < \lambda$,

  (this is necessary for the $\kappa$-cc),
- $(ii)$ the collection $q$ corresponds to a partial injection, and free of any explicitly contradictory subset of terms, by which we mean that
  - $(a)$ there are no $i, j \in \lambda$ s.t. $[f_\beta(i) = j]$, $[j \notin \mathrm{dom}(f_\beta)] \in q$,
  - $(b)$ there are no $i, j_0 \neq j_1 \in \lambda$ s.t. $[f_\beta(i) = j_0]$, $[f_\beta(i) = j_1] \in q$,
  - $(c)$ there are no $[f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in q$ s.t. $c_{M_\beta}(i_0, i_1) \neq c_{M_*}(j_0, j_1)$.

  Note that $f_\beta$'s are automatically injective by $(i)$.
- $(iii)$ $|q| < \kappa$.

Case (3): $\beta \in S^*$.

As $\underset{\sim}{D}_\beta$ is a $\mathbb{P}^1_\beta$-name for a system of subsets of $V_\kappa^{\mathbf{V}_1}$, if additionally for each $\alpha < \kappa$ $|(\cup \underset{\sim}{D}_\beta) \restriction \alpha| < \kappa$ holds (and if $\underset{\sim}{D}_\beta$ generates a proper $\kappa$-complete filter), then we define $\mathbb{Q}^1_\beta$ to be the Mathias forcing $\mathbb{Q}_{D_\beta}$ from Definition 3.1, otherwise we can let $\mathbb{Q}^1_\beta$ to be the trivial forcing. Note that this requirement ensures that

$(\diamond)_7$ if $(w, A) \in \mathbb{Q}^1_\beta$, then $|w| < \kappa$.

This completes Definition 3.28.

Now it is straightforward to check that each $\mathbb{Q}^1_\alpha$ is (forced to be) $< \kappa$-directed closed, so $\mathbb{P}^1_\chi$ is a $< \kappa$-support iteration of $< \kappa$-directed closed posets, $\mathbb{P}^1_\chi$ itself is $< \kappa$-directed closed by [Bau78, Thm 2.7]. (In particular it does not add any new sequence of length $< \kappa$.) Since forcing $M_*$ goes by $< \kappa$-approximations ($\Vdash_{\mathbb{P}^1_1} (\underset{\sim}{q} \in \underset{\sim}{\mathbb{Q}}^1_1) \to (|\underset{\sim}{q}| < \kappa)$, we have:

**Observation 3.29.** *For each $\beta \in \chi \setminus S^* \setminus \{0, 1\}$ forcing with $\mathbb{Q}^1_\beta$ over $\mathbf{V}_1^{\mathbb{P}^1_\beta}$ adds an embedding $f_\beta : M_\beta \to M_*$.*

We already saw that $\mathbb{P}^1_1 = \mathbb{Q}^1_0$ is $\lambda^+$-cc (Lemma 3.21), now we prove that in $\mathbf{V}_1^{\mathbb{P}^1_1}$ the quotient forcing $\mathbb{P}^1_\chi / \mathbf{G}^1_1$ has the $\kappa^+$-cc (no matter how we choose the $\mathbb{P}^1_\beta$-name $\underset{\sim}{D}_\beta$, or $\underset{\sim}{M}_\beta$, at first only assumed to satisfy $\circledast$ for $2 \leq \beta < \chi$), after which not only will the $\lambda^+$-ccness of $\mathbb{P}^1_1 * (\mathbb{P}^1_\chi/\underset{\sim}{\mathbf{G}}^1_1)$ follow (and of $\mathbb{P}^1_\chi$, too), but some easy calculation will be sufficient for ensuring $(\top)_2$-$(\top)_6$. In order to prove the antichain condition we will need some technical preparation, the same way as in [She90]. Recalling that each $\mathbb{P}^1_\alpha$ is $< \kappa$-closed (and $(\diamond)_7$) is straightforward to prove (by induction on $\alpha$) that

$(*)_2$ The set

$$D^\bullet_\alpha = \{p \in \mathbb{P}^1_\alpha :\ \forall \gamma \in \mathrm{dom}(p) \quad \begin{array}{ll} (\beta \in S^*) \ \to \ [\ \exists w_\gamma \in \mathbf{V}_1 \text{ s.t. } & \Vdash_{\mathbb{P}^1_\gamma} p(\gamma) = (\check{w}_\gamma, \underset{\sim}{A}_\gamma)\ ] \\ (\beta \notin S^*) \ \to \ [\ \exists s_\gamma \in \mathbf{V}_1 \text{ s.t. } & \Vdash_{\mathbb{P}^1_\gamma} p(\gamma) = \check{s}_\gamma \}\ ] \end{array}$$

is a dense subset of $\mathbb{P}^1_\alpha$.

$(*)_3$ Therefore, in the quotient forcing $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$ (as defined in [Bau78], or see below) the set

$$D^0_\alpha = \{p \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 :\ \exists q_0 \in \mathbf{G}^1_1 :\ \langle q_0 \rangle \cup p \in D^\bullet_\alpha\} \ \in \mathbf{V}'_1$$

is dense (where $\mathbb{P}^1_\alpha/\mathbf{G}^1_1 = \{p \restriction (\mathrm{dom}(p) \setminus \{0\}) :\ p \in \mathbb{P}^1_\alpha\} \in \mathbf{V}'_1$, and $p \leq_{\mathbb{P}^1_\alpha/\mathbf{G}^1_1} q$, iff for some $r_0 \in \mathbf{G}^1_1 \subseteq \mathbb{P}^1_1$: $\langle r_0 \rangle \cup p \leq_{\mathbb{P}^1_\alpha} \langle r_0 \rangle \cup q$).

$(*)_4$ With a slight abuse of notation (in order to avoid further notational awkwardness) we will identify each condition $p \in D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$ with the function on the same domain, but for each $\gamma \in \mathrm{dom}(p)$

- if $\beta \in S^*$ then writing $p(\beta) = (w, \underset{\sim}{A})$ (instead of some $\mathbb{P}^1_\beta$-name satisfying $\langle q_0 \rangle \cup p \restriction \beta \Vdash_{\mathbb{P}^1_\beta} p(\beta) = (\check{w}, \underset{\sim}{A})$ for some $q_0 \in \mathbf{G}^1_1$),
- if $\beta \notin S^*$, $\beta > 0$, then writing $p(\beta) = s$, where $s$ is a set of symbols as in Case (1), (2) in Definition 3.28 (instead of $\langle q_0 \rangle \cup p \restriction \beta \Vdash_{\mathbb{P}^1_\beta} p(\beta) = \check{s}$ for some $q_0 \in \mathbf{G}^1_1$).

Note that (as $\mathbb{P}^1_1$ is $< \kappa$-closed and $D^0_\alpha \subseteq \mathbf{V}_1$)

$(*)_5$ for any $\alpha \leq \chi$, and increasing sequence $\bar{p} = \langle p_\zeta :\ \zeta < \varepsilon < \kappa \rangle$ in $D^0_\alpha$ if $\bar{p} \in \mathbf{V}'_1$, then $\bar{p}$ has a least upper bound in $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$, which we will denote by $\lim_{\zeta<\varepsilon} p_\zeta$, and this limit is in $D^0_\alpha$. For the sake of completeness we include the formal definition of $\lim_{\zeta<\varepsilon} p_\zeta$. The limit of $\bar{p} = \langle p_\zeta :\ \zeta < \varepsilon < \kappa \rangle$ is the function $p^*$, where
  (a) $\mathrm{dom}(p^*) = \bigcup_{\zeta<\varepsilon} \mathrm{dom}(p_\zeta)$,
  (b) for $\beta \in S^* \cap \mathrm{dom}(p^*)$ $p^*(\beta) = (\bigcup_{\zeta<\varepsilon} w_{p_\zeta(\beta)}, \underset{\sim}{A}_\beta)$, where $p_\zeta(\beta) = (w_{p_\zeta(\beta)}, A_{p_\zeta(\beta)})$, and $\underset{\sim}{A}_\beta$ is the $\mathbb{P}^1_\beta$-name defined so that $\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{A}_\beta = \bigcap_{\zeta<\varepsilon} \underset{\sim}{A}_{p_\zeta(\beta)}$ holds,
  (c) for $\beta \in \chi \setminus S^*$, $\beta > 0$, set $p^*(\beta) = \bigcup_{\zeta<\varepsilon} p_\zeta(\beta)$.

**Definition 3.30.** In $\mathbf{V}'_1$ for each $\alpha \leq \chi$, $\delta \leq \lambda$, for each condition $p \in D^0_\alpha$ we define $p^{[\delta]}$ to be the function with $\mathrm{dom}(p^{[\delta]}) = \mathrm{dom}(p)$, and

(a) if $1 \in \mathrm{dom}(p)$, then $p^{[\delta]}(1) = \{[i\ R_\gamma\ j] \in p(1) :\ i, j < \delta\}$,
(b) for $1 < \beta \in \mathrm{dom}(p) \cap S^*$ we let $p^{[\delta]}(\beta) = p(\beta)$,
(c) otherwise (for $1 < \beta \in \mathrm{dom}(p) \setminus S^*$) we let

$$p^{[\delta]}(\beta) = \begin{array}{c} \{[f_\beta(i) = j] \in p(\beta) :\ i, j < \max\{\xi_\beta(\gamma) :\ \gamma < \lambda,\ \xi_\beta(\gamma) \leq \delta\}\} \\ \cup \\ \{[j \notin \mathrm{ran}(f_\beta)] \in p(\beta) :\ j < \max\{\xi_\beta(\gamma) :\ \gamma < \lambda,\ \xi_\beta(\gamma) \leq \delta\}\}. \end{array}$$

Observe that, because of each $p$ and each $p(\beta)$ ($\beta \in \mathrm{dom}(p)$) has support of size $< \kappa$, and $\lambda > \kappa$ is regular,

$(*)_6$ for each $\alpha \le \chi$, $p \in D^0_\alpha \subseteq (\mathbb{P}^1_\alpha/\mathbf{G}^1_1)$ we have $p^{[\delta]} = p$ for every large enough $\delta$, and

$(*)_7$ clearly $p^{[\delta]} \restriction \beta = (p \restriction \beta)^{[\delta]}$ (for $\beta < \chi$).

Note that for $p \in D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$ the reduced function $p^{[\delta]}$ is in $\mathbf{V}'_1$ (even in $\mathbf{V}_1$), but is not necessarily a condition in $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$. Nevertheless,

$(*)_8$ for $p \le q \in D^0_\alpha$ with $p^{[\delta]}, q^{[\delta]} \in D^0_\alpha$ (i.e. if they are conditions in $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$) we obviously have $p^{[\delta]} \le q^{[\delta]}$.

It is straightforward to check the following (by induction on $\alpha$).

**Observation 3.31.** *For each $\alpha \le \chi$, $p \in D^0_\alpha$ and $\delta < \lambda$*

*a)* $p^{[\delta]}$ *is an actual condition (i.e. belongs to* $D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$*), iff for every* $\beta \in \mathrm{dom}(p)$

- $p^{[\delta]} \restriction \beta \in \mathbb{P}^1_\beta$, *and*
- *(letting* $\delta^-_\beta = \max(N^*_\beta \cap (\delta+1))$

$$\begin{array}{l}(\forall i_0, j_0, i_1, j_1)\ \ \text{if } [f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in p(\beta),\ \text{then}: \\ j_0, j_1 < \delta^-_\beta \ \longrightarrow\ p^{[\delta]} \restriction \beta \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_0, i_1) = c_{M_*}(j_0, j_1).\end{array} \tag{3.17}$$

*b) In particular, for limit* $\alpha$

$$p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 \iff \Big[(\text{for cofinally many } \varepsilon < \alpha):\ p^{[\delta]} \restriction \varepsilon \in \mathbb{P}^1_\varepsilon)\Big],$$

*c) while for* $\alpha = \beta + 1$

$$p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 \iff p^{[\delta]} \restriction \beta \in \mathbb{P}^1_\beta/\mathbf{G}^1_1 \text{ and (3.17) holds.}$$

The following notion and lemma is of central importance.

**Definition 3.32.** In $\mathbf{V}_1^{\mathbb{P}^1_1}$ for $\alpha \le \chi$ define

$$D^*_\alpha = \{p \in D^0_\alpha :\ (\forall \delta < \lambda)\ \ p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1\}.$$

Having Observation 3.31 in our mind it is easy to check the following.

$(*)_9$ Whenever $\langle p_\zeta : \zeta < \varepsilon < \kappa\rangle$ is an increasing sequence in $D^*_\alpha$, then $\lim_{\zeta<\varepsilon} p_\zeta \in D^*_\alpha$.

This leads to the statements about how $p \in D^*_\alpha$ and $p \restriction \beta \in D^*_\beta$ $(\beta < \alpha)$ relate to each other.

**Observation 3.33.** *For each $\alpha \le \chi$, $p \in D^0_\alpha$*

*a)* $p \in D^*_\alpha$, *iff for every* $\beta \in \mathrm{dom}(p)$ *and for every* $\delta < \lambda$

$$p \restriction \beta \in D^*_\beta,$$

*and (letting* $\delta^-_\beta = \max(N^*_\beta \cap (\delta+1))$

$$\begin{array}{l}(\forall i_0, j_0, i_1, j_1)\ \ \text{if } [f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in p(\beta),\ \text{then:} \\ j_0, j_1 < \delta^-_\beta \ \longrightarrow\ p^{[\delta]} \restriction \beta \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_0, i_1) = c_{M_*}(j_0, j_1).\end{array} \tag{3.18}$$

*b) In particular, for limit* $\alpha$

$$p \in D^*_\alpha \iff (\text{for cofinally many } \varepsilon < \alpha):\ p \restriction \varepsilon \in D^*_\varepsilon,$$

*c) while for* $\alpha = \beta + 1$

$$p \in D^*_\alpha \iff [p \restriction \beta \in D^*_\beta] \text{ and [for each } \delta < \lambda \text{ (3.18) holds for } \beta.]$$

We are ready to state the two lemmas on which the correctness of the entire construction depends. Lemma 3.35 makes it possible to enumerate and embed all possible graphs on $\lambda$ into $M_*$, which can be proved relying on Lemma 3.34.

**Lemma 3.34.** *For $\alpha \leq \chi$*

$(\blacksquare)^1_\alpha$
$$\mathbf{V}_1^{\mathbb{P}^1_1} \models D^*_\alpha \text{ is dense in } \mathbb{P}^1_\alpha/\mathbf{G}^1_1.$$

**Lemma 3.35.** *For every $\alpha \leq \chi$*

$(\blacksquare)^2_\alpha$
$$\mathbf{V}_1^{\mathbb{P}^1_1} \models \ \mathbb{P}^1_\alpha/\mathbf{G}^1_1 \text{ has the } \kappa^+\text{-cc.}$$

*Proof.* We proceed by induction, and prove Lemmas 3.34 and 3.35 simultaneously: More exactly we prove Lemma 3.34 for $\alpha$ provided that both Lemmas holds for $\beta$'s less than $\alpha$, and we verify the $\kappa^+$-cc property for $\mathbb{P}^1_\alpha$ assuming that $D^*_\alpha$ is a dense subset of $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$. For $\alpha \leq 2$ (when $\mathbb{P}^1_2/\mathbf{G}^1_1$ is essentially the forcing $\mathbb{Q}^1_1$ of the random graph Case (1) of Definition 3.28) the statement $(\blacksquare)^1_\alpha$ clearly holds, moreover, $(\blacksquare)^2_\alpha$ holds recalling $\kappa^{<\kappa} = \kappa$.

Suppose we know that for each $\varepsilon < \alpha$ $(\blacksquare)^1_\alpha$ and $(\blacksquare)^2_\alpha$ hold . Assume first that $\alpha$ is limit. If $\mathrm{cf}(\alpha) \geq \kappa$, then $\mathbb{P}^1_\alpha = \bigcup_{\varepsilon<\alpha} \mathbb{P}^1_\varepsilon$, $D^*_\alpha = \bigcup_{\varepsilon<\alpha} D^*_\varepsilon$, so the latter is dense, we are done.

Second, if $\alpha$ is limit, but $\mathrm{cf}(\alpha) < \kappa$, then let $\langle \eta_\theta : \ \theta < \mathrm{cf}(\alpha)\rangle$ be a continuous increasing sequence with limit $\alpha$, let $p_{-1} \in D^0_\alpha$ be arbitrary. We will choose the increasing sequence $\langle p_\theta : \ \theta < \mathrm{cf}(\alpha)\rangle$ in $D^0_\alpha$ with $p_0 \geq p_{-1}$, and $p_\theta \restriction \eta_\theta \in D^*_{\eta_\theta}$. This would suffice as for each $\theta < \mathrm{cf}(\kappa)$ the sequence $p_\varrho \restriction \eta_\theta$ ($\varrho < \mathrm{cf}(\alpha)$) is eventually in $D^*_{\eta_\theta}$, so for $p^* = \lim_{\varrho<\mathrm{cf}(\alpha)} p_\varrho$ using $(*)_9$ we have $p^* \restriction \eta_\theta \in D^*_{\eta_\theta}$, leading to
$$(\forall \theta < \mathrm{cf}(\alpha)) \ p^* \restriction \eta_\theta \in D^*_{\eta_\theta},$$
so by *b*) we are done. For the construction of the $p_\theta$'s, as $D^0_\alpha$ and $D^*_{\eta_\theta}$'s are $< \kappa$-closed we only have to ensure that $p_\theta \in D^0_\alpha$ can be chosen so that not only $p_\theta \geq p_\varrho$ ($\varrho < \theta$), but $p_\theta \restriction \eta_\theta \in D^*_{\eta_\theta}$. Now applying the induction hypothesis, we can find $p^*_\theta \in D^*_{\eta_\theta}$ such that it extends $(\lim_{\varrho<\theta} p_\varrho) \restriction \eta_\theta$ (in $\mathbb{P}^1_{\eta_\theta}/\mathbf{G}^1_1$). Finally, let $p_\theta$ be the least upper bound of $p^*_\theta$ and $(\lim_{\varrho<\theta} p_\varrho)$ (in fact for $\theta$ limit we did not even have to appeal to the induction hypothesis if $\bar{\eta}$ is continuous).

Third, if $\alpha = \beta + 1$, let $p_{-1} \in D^0_\alpha$ be arbitrary and we will extend $p_{-1} \restriction \beta$ to $p^* \in D^*_\beta$ (using $(\blacksquare)^1_\beta$) in such a way that the right hand side of Observation 3.33 *c*) holds for $p = p^* \cup \langle p_{-1}(\beta)\rangle$ (so that $p \in D^*_\alpha$, $p \geq p_{-1}$).

For this, let $\{j_\theta : \ \theta < \nu\}$ enumerate $\{j < \lambda : \ [f_\beta(i) = j] \in p_{-1}(\beta) \text{ for some } i < \lambda\}$ in increasing order, and we can fix the system $\{i_\theta : \ \theta < \nu\}$ so that

$(\odot)_1$ $\{i_\theta : \ \theta < \nu\}$ is such that for each $\theta$ $[f_\beta(i_\theta) = j_\theta] \in p_{-1}(\beta)$.

Note that by Definition 3.28/Case (2)/ (*i*)

$(\odot)_2$ for each $\theta$: $i_\theta < j_\theta$,

and also we can choose $\gamma_\theta$ for each $\theta < \nu$ such that $\xi_\beta(\gamma_\theta) = j_\theta$, thus

$(\odot)_3$ we have
$$\{j < \lambda : \ \exists i < \lambda \ [f_\beta(i) = j] \in p_{-1}(\beta)\} = \{j_\theta : \ \theta < \nu\} = \{\xi_\beta(\gamma_\theta) : \ \theta < \nu\}.$$

Now we construct the increasing sequence $\langle p_\theta : \ \theta < \nu\rangle$ in $D^*_\beta$ with the properties

($\alpha$) $p_{-1} \restriction \beta \leq p_0$,

($\beta$) for each $\theta < \nu$, for each $\varepsilon_0 < \varepsilon_1 < \theta$

$$p_\theta^{[\xi_\beta(\gamma_{\varepsilon_1}+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_{\varepsilon_0}, i_{\varepsilon_1}) = c_{M_*}(j_{\varepsilon_0}, j_{\varepsilon_1}).$$

This clearly suffices, as we can let $p^* = \lim_{\theta<\nu} p_\theta \in D^*_\beta$, and then $p = p^* \cup \langle p_{-1}(\beta)\rangle$ belongs to $D^*_\alpha$, $(\blacksquare)^1_\alpha$ follows, indeed. (To see that the condition $p$ belongs to $D^*_\alpha$, recall $j_{\varepsilon_1} = \xi_\beta(\gamma_{\varepsilon_1})$ so $\xi_\beta(\gamma_{\varepsilon_1}+1)$ is the minimal $\delta < \lambda$ with $p^{[\xi_\beta(\gamma_{\varepsilon_1}+1)]}(\beta)$ containing the symbol $[f_\beta(i_{\varepsilon_1}) = j_{\varepsilon_1}]$, therefore by Observation 3.33 $c$) we are done.)

Appealing to the induction hypothesis, let $p_0 \in D^*_\beta$, $p_0 \geq p_{-1}$. Using the $< \kappa$-closedness of $D^*_\beta$ ($(*)_9$) it is enough to deal with the successor case, that is, for each $\theta$ choose $p_{\theta+1}$ such that $p_{\theta+1}^{[\xi_\beta(\gamma_\theta+1)]}$ forces that the partial function $i_\varepsilon \mapsto j_\varepsilon$ ($\varepsilon \leq \theta$) is an embedding of $\underset{\sim}{M}_\beta \restriction \{i_\varepsilon : \varepsilon \leq \theta\}$ into $\underset{\sim}{M}_* \restriction \{j_\varepsilon : \varepsilon \leq \theta\}$. Using again $(*)_9$

$(\odot)_6$ it suffices to show that for each $\varepsilon < \theta$ and $q \geq p_{-1} \restriction \beta$, where $q \in D^*_\beta$, there exists $q' \in D^*_\beta$, $q' \geq q$

$$q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta).$$

We will see that this follows from the following (formally) more general lemma, stated here for later reference.

**Lemma 3.36.** *For every $\beta \leq \chi$, $q \in D^*_\beta$, $\delta < \lambda$, $i'$, $i'' < \max(N^*_\beta \cap (\delta+1))$ there exists $q' \in D^*_\beta$, $q' \geq q$ such that*

$$q'^{[\delta]} \text{ forces a value to } c_{M_\beta}(i', i'').$$

*Moreover, if $q$ satisfies*

$$(3.19)\qquad \begin{array}{l}(\forall \gamma \in \mathrm{dom}(q) \setminus S^*)(\forall i, j):\\ \big[([f_\beta(i) = j)] \in q(\gamma) \setminus q^{[\delta]}(\gamma)) \longrightarrow (j = \max(N^*_\gamma \cap (\delta+1)) \ \wedge j < \delta)\big]\\ \textit{and } (q(1) = q^{[\delta]}(1))\end{array}$$

*(hence $\delta \notin N^*_\gamma$ for $\gamma \in \mathrm{dom}(q) \setminus S^*$), then there exists $q'$ for which additionally:*

$$(\forall \gamma \in \mathrm{dom}(q') \setminus S^*): \quad q'(\gamma) \setminus q'^{[\delta]}(\gamma) = q(\gamma) \setminus q^{[\delta]}(\gamma)\ .$$

(Here we remark that lemma is for every $\beta$, and uses the $\kappa^+$-cc property of $\mathbb{P}^1_\beta/\mathbf{G}^1_1$, but we will only apply it to our fixed $\beta$, for proving $(\odot)_6$, that is, to complete the proof of $((\blacksquare)^1_\beta \wedge (\blacksquare)^1_\beta) \to (\blacksquare)^1_\alpha$.)

*Proof.* (Lemma 3.36) So fix $q \in D^*_\beta$, let $\varrho$ be chosen so that $\xi_\beta(\varrho) = \max(N^*_\beta \cap (\delta+1))$, so $i', i'' < \xi_\beta(\varrho) \leq \delta$, and recall that for the model $N_{\beta,\varrho} \prec (\mathscr{H}^{\mathbf{V}'_1}(\Upsilon), \in)$ we know that $i', i'', \underset{\sim}{M}_\beta, \mathbb{P}^1_\beta, \mathbf{G}^1_1 \in N_{\beta,\varrho}$ (and thus $\mathbb{P}^1_\beta/\mathbf{G}^1_1 \in N_{\beta,\varrho}$). So we can find $A \in N_{\beta,\varrho}$ such that $A$ is a maximal antichain in $D^*_\beta \subseteq \mathbb{P}^1_\beta/\mathbf{G}^1_1$, each $p \in A$ decides the value of $c_{M_\beta}(i', i'')$. But as $\mathbb{P}^1_\beta/\mathbf{G}^1_1$ has the $\kappa^+$-cc, and $\kappa + 1 \subseteq N_{\beta,\varrho}$ we have that $A \subseteq N_{\beta,\varrho}$.

So

$(\boxplus)_1$ let $q' \in D^*_\beta$ be a common upper bound of $q$ and some $q'' \in A$.

We have to argue that not only $q' \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i', i'') = c_*$ (for some $c_* < \kappa$) but

$$q'^{[\delta]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i', i'') = c_*. \tag{3.20}$$

For (3.20) it is enough to prove that $q''^{[\delta]} = q''$, because then $q'^{[\delta]} \geq q''^{[\delta]} = q''$ (by $(*)_8$), which decides $\underset{\sim}{c}_{M_\beta}(i', i'')$, yielding (3.20), as we wanted. But as $q'' \in N_{\beta,\varrho}$, and $\lambda \cap N_{\beta,\varrho} = \xi_\beta(\varrho) \leq \delta$, we have $\mathrm{dom}(q'') \subseteq N_{\beta,\varrho}$. Now for each $\zeta \in \mathrm{dom}(q'') \setminus S^* \setminus \{0,1\}$ we have $\langle N_{\zeta,\iota} : \iota < \lambda \rangle \in N_{\beta,\varrho}$ (recall Case (2) from Definition 3.28), so $\xi_\beta(\varrho)$ is an accumulation point of the $\xi_\zeta(\iota)$'s. Hence we get that

$(\boxplus)_2$ for each $\zeta \in \mathrm{dom}(q'') \setminus S^* \setminus \{0,1\}$ $\xi_\beta(\varrho) = \xi_\zeta(\iota)$ for some $\iota < \lambda$ (in fact, for $\iota = \xi_\beta(\varrho)$),

so $q''^{[\xi_\beta(\varrho)]} = q''^{[\delta]} = q''$, we are done.

Finally, for the moreover part, if $\gamma \in \mathrm{dom}(q) \setminus S^*$, let $\delta^-_\gamma = \max(N_\gamma \cap (\delta + 1)))$, and define $i^-_\gamma$ to be the unique ordinal s.t.

$$[f_\gamma(i^-_\gamma) = \delta^-_\gamma] \in q(\gamma) \tag{3.21}$$

(if there exists). Note that our conditions on $q$ imply that if $i^-_\gamma$ is defined, then $i^-_\gamma < \delta^-_\gamma$, and by our conditions (3.19)

$$\delta^-_\gamma < \delta.$$

Now by induction and by the first part define $q'' \geq q$ such that for every $\gamma \in \mathrm{dom}(q'') \setminus S^*$ with $i^-_\gamma$ defined

$$([f_\gamma(i) = j] \in q''^{[\delta]}(\gamma)) \;\rightarrow\; q''^{[\delta]} \restriction \gamma \text{ decides the value } c_{\underset{\sim}{M}_\gamma}(i, i^-_\gamma),$$

and

$$([f_\gamma(i) = j] \in q''^{[\delta]}(\gamma)) \;\rightarrow\; q''^{[\delta]}(1) \text{ decides the value } c_{\underset{\sim}{M}_*}(j, \delta^-_\gamma)$$

(in fact this latter follows from $j, \delta^-_\gamma < \delta$ and (3.21)). Now clearly $q''^{[\delta]} \geq q^{[\delta]}$, and we can define the condition $q'$ to be the least upper bound of $q''^{[\delta]}$ and $q$ (which is just adding symbols $[f_\gamma(i^-_\gamma) = \delta^-_\gamma] \in q(\gamma)$): this is possible, as for every $\gamma$ with $i^-_\gamma$ defined we have that $q''^{[\delta]} \restriction \gamma$ forces that $q''^{[\delta]}(\gamma) \cup \{[f_\gamma(i^-_\gamma) = \delta^-_\gamma]\}$ is indeed a partial embedding.

$\square_{\mathrm{Lemma 3.36}}$

Turning back to the statement from $(\odot)_6$, as $j_\varepsilon < j_\theta = \xi_\beta(\gamma_\theta) < \xi_\beta(\gamma_\theta + 1)$ we also have $i_\varepsilon, i_\theta < \xi_\beta(\gamma_\theta)$ (thus obviously $i_\varepsilon, i_\theta < \xi_\beta(\gamma_\theta + 1)$). Apply the lemma with $\delta = \xi_\beta(\gamma_\theta + 1)$, $i' = i_\varepsilon$, $i'' = i_\theta$,

$(\odot)_7$ let $q' \in D^*_\beta$ be given by the lemma, so

$$q' \Vdash_{\mathbb{P}^1_\beta / \mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta) \tag{3.22}$$

(which is obvious, as

$(\odot)_8$ $q' \geq p_{-1} \restriction \beta$, and $p_{-1}$ is a proper condition in $D^0_\alpha$ with $[f_\beta(i_\theta) = j_\theta]$, $[f_\beta(i_\varepsilon) = j_\varepsilon] \in p_{-1}(\beta)$, hence $q' \frown \langle p_{-1}(\beta) \rangle$, too).

It remains to argue that

$$q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta / \mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta). \tag{3.23}$$

But $q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta / \mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_*$ (for some $c_* < \kappa$) and if $[j_\varepsilon \; R_{c_*} \; j_\theta] \notin q'^{[\xi_\beta(\gamma_\theta+1)]}(1)$ (so does not belong to $q'(1)$), then adding $[j_\varepsilon \; R_{c_*+1} \; j_\theta]$ to the first coordinate of $q'$ would lead to a contradiction with (3.22). This verifies that assuming the induction hypotheses for $\beta$, the assertion $(\blacksquare)^1_\alpha$ holds, i.e. the set $D^*_{\beta+1} = D^*_\alpha$ is dense in $\mathbb{P}^1_\alpha / \mathbf{G}^1_1$.

Now assuming that $D^*_\alpha$ is dense we are ready to prove that $\mathbb{P}^1_\alpha / \mathbf{G}^1_1$ has the $\kappa^+$-cc. So let $\langle p_\gamma : \; \gamma < \kappa^+ \rangle$ be an antichain in $D^*_\alpha$. By extending each $p_\gamma$

$(\odot)_9$ we can assume that for each $\gamma < \kappa^+$

(*i*) for each $\beta' \in \mathrm{dom}(p_\gamma)$, for each $i_0, i_1, j_0 < j_1$ with $[f_{\beta'}(i_0) = j_0]$, $[f_{\beta'}(i_1) = j_1] \in p_\gamma(\beta')$ the condition $p^{[j_1]} \restriction \beta'$ decides the value $c_{\underset{\sim}{M}_{\beta'}}(i_0, i_1)$,

(*ii*) for each $\gamma < \kappa^+$ the condition $p_\gamma(1)$ is a complete graph on some set $L_\gamma$ with its edges colored, i.e.

$$L_\gamma = \{i < \lambda : \exists i' < \lambda \, \exists \varepsilon < \kappa \, [i \, R_\varepsilon \, i'] \in p_\gamma(1)\},$$
$$\text{so } (\forall i, j \in L_\gamma) \, (\exists \delta < \kappa) : \, [i \, R_\delta \, j] \in p_\gamma(1).$$

(*iii*) for each $\gamma < \kappa^+$ and $\beta' \neq \beta'' \in \mathrm{dom}(p_\gamma) \setminus S^* \setminus \{0, 1\}$ we have

$$\{\xi_{\beta'}(\rho + 1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho + 1) : \rho < \lambda\} \subseteq L_\gamma$$

(recall that $|\{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\}| < \kappa$ by $(\diamond)_5$),

(*iv*) for each $\gamma < \kappa^+$ and $\beta' \in \mathrm{dom}(p_\gamma) \setminus S^* \setminus \{0, 1\}$, for each $j < \lambda$ if either $[j \notin \mathrm{ran}(f_{\beta'})] \in p_\gamma(\beta')$, or $[f_{\beta'}(i) = j] \in p_\gamma(\beta')$ (for some $i < \lambda$), then $j \in L_\gamma$,

(*v*) for each $\gamma < \kappa^+$, $\beta' \in \mathrm{dom}(p_\gamma)$ and $j < \lambda$, if $j \in L_\gamma$, then

$$(j \in \{\xi_{\beta'}(\rho + 1) : \rho < \lambda\}) \Rightarrow \begin{cases} \text{either} & [j \notin \mathrm{ran}(f_{\beta'})] \in p_\gamma(\beta') \\ \text{or (for some } i) & [f_{\beta'}(i) = j] \in p_\gamma(\beta'), \end{cases}$$

(*vi*) the set $L_\gamma \subseteq \lambda$ is closed, of limit order type,

[This is possible, a simple induction using Lemma 3.36, and the fact

$$[f_\beta(i) = j] \in p_\gamma(\beta) \ \rightarrow \ j \in N^*_\beta$$

(and $(*)_9$) yield that there is $p'_\gamma \geq p_\gamma$ in $D^*_\alpha$, with $(p'_\gamma \restriction \beta)^{[j_1]}$ determining the value $c_{\underset{\sim}{M}_\beta}(i_0, i_1)$ whenever $[f_{\beta_0}(i_0) = j_0] \in p_\gamma(\beta_0)$, $[f_{\beta_1}(i_1) = j_1] \in p_\gamma(\beta_1)$ (for some $j_0 < j_1$, or if either of the $i$'s belongs to the universe of $p_\gamma(1)$). Now repeating this $\omega$-many times we get a condition satisfying (*i*). Then we can obtain an even stronger condition satisfying (*ii*)-(*vi*) by only adding symbols of the form $[j \notin \mathrm{ran}(f_{\beta'})]$ at coordinates $1 < \beta' \in \chi \setminus S^*$ and extending also $p'_\gamma(1)$.]

As $\kappa$ is strongly inaccessible in $\mathbf{V}_1$ (by $(*)_1$), and in $\mathbf{V}_1^{\mathbb{P}^1_1}$ (as $\mathbb{P}^1_1$ is $< \kappa$-closed), we can apply the delta system lemma, so w.l.o.g. $\langle \mathrm{dom}(p_\gamma) : \gamma < \kappa^+ \rangle$ forms a delta system. By applying the delta system lemma again we can assume that for each $\beta' \in \cap\{\mathrm{dom}(p_\gamma) : \gamma < \kappa\} \setminus S^*$ each of the following systems of sets forms a delta system:

- $L_\gamma \ (\gamma < \kappa^+)$ ,
- $I_\gamma(\beta') = \left\{ \begin{array}{ll} i : & [f_{\beta'}(i) = j] \in p_\gamma(\beta') \vee \exists j \in [\xi_{\beta'}(\kappa i), \xi_{\beta'}(\kappa(i+1))) \\ & [j \notin \mathrm{ran}(f_\beta)] \in p_\gamma(\beta') \end{array} \right\} (\gamma < \kappa^+)$.

Therefore (recalling that each $i < \lambda$ has $\kappa$-many possible images) there are $\xi \neq \zeta < \kappa^+$, such that $p_\xi$ and $p_\zeta$ has no explicitly contradictory terms on the coordinates concerning the $\kappa$-colored graphs, and agreeing in the first part of the condition on the coordinates dedicated to Mathias forcing, under which we mean the following (w.l.o.g. we can assume that $\xi = 0$, $\zeta = 1$):

$(\odot)_{10}$ for each $i, j \in L_0(1) \cap L_1(1)$ there exists some $\varepsilon < \kappa$ s.t. $[i \, R_\varepsilon \, j] \in p_0(1) \cap p_1(1)$,

$(\odot)_{11}$ for $\beta' \in \chi \setminus S^* \setminus \{0, 1\}$ (if $\beta' \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$) the set $p_0(\beta') \cup p_1(\beta')$ determines a partial injection from a subset of $\lambda$ to a subset of $\lambda$, i.e. satisfies (*ii*) (*a*), (*b*) (from Definition 3.28 Case (2)),

$(\odot)_{12}$ for $\beta \in S^* \cap \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$ $p_0(\beta) = (w_\beta, \underaccent{\tilde}{A}_{0,\beta})$, $p_1(\beta) = (w_\beta, \underaccent{\tilde}{A}_{1,\beta})$ for some $w_\beta \in [V_\kappa^{\mathbf{V}_1}]^{<\kappa}$, and $\mathbb{P}^1_\beta$-names $\underaccent{\tilde}{A}_{0.\beta}$, $\underaccent{\tilde}{A}_{1,\beta}$.

Now $p_0$ and $p_1$ appear as good candidates for a compatible pair in our supposed antichain, but we cannot take just the upper bound coordinate wise, as for coordinates $\beta' > 1$ outside $S^*$ it will not necessarily force that $p_0(\beta') \cup p_1(\beta')$ is an embedding of $\underaccent{\tilde}{M}_{\beta'}$ to $\underaccent{\tilde}{M}_*$. Although it is not immediate, the following claim shows that we can construct a common upper bound, which will complete the proof of $(\blacksquare)^2_\alpha$ for $\alpha$.

**Claim 3.37.** *There exists a condition $q \in D^*_\alpha$ extending both $p_0$ and $p_1$.*

*Proof.* $(\bullet)_1$ By adding symbols of the form $[j \notin \mathrm{ran}(f_\beta)]$ to $p_0(\beta)$, $p_1(\beta)$ we can assume the following (not harming $(\odot)_{11}$)

$(\bullet)_{1a}$ for $1 < \beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ if $[f_\beta(i) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$ holds for no $i$ then $[j_\theta \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cap p_1(\beta)$,

$(\bullet)_{1b}$ whenever $\beta' \neq \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$, $j^* \in \{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\} \cap j_\varrho$ and there is no $i$ with $[f_{\beta'}(i) = j^*] \in p_0(\beta') \cup p_0(\beta')$ then $[j^* \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cap p_1(\beta')$,

$(\bullet)_2$ Let $\{j_\varepsilon : \varepsilon < \varrho\}$ be a continuous increasing sequence for which,

$(\bullet)_{2a}$ whenever $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$, and $j$ is such that either $[j \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta')$, or $[f_{\beta'}(i) = j] \in p_0(\beta') \cup p_1(\beta')$ for some $i$, then $j = j_\theta$ for some $\theta < \varrho$. (Therefore, $L_0 \cup L_1 = \{j : [j\ R_\nu\ j'] \in p_0(1) \cup p_1(1)$ for some $j' < \lambda,\ \nu < \kappa\} \subseteq \{j_\theta : \theta < \varrho\}$.)

let $j_\varrho = \sup\{j_\theta : \theta < \varrho\}$, let $j_{\varrho+1}$ be an ordinal which is bigger than $\min(N^*_{\beta'} \setminus j_\varrho)$ for any $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$.

$(\bullet)_3$ We construct the increasing sequence $\langle q_\varepsilon : \varepsilon < \varrho+2\rangle$ in $D^*_\alpha$ satisfying

$$q_\varepsilon^{[j_\varepsilon]} \geq p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]},$$

$(\bullet)_4$ and also we require that for each $\varepsilon < \varrho$ the strict inequality $q_\varepsilon(\beta') \gneq q_\varepsilon^{[j_\varepsilon]}(\beta')$ is possible if and only if $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus \{1\}$ and $(\delta_\varepsilon^{\beta'})^- = \max(N^*_{\beta'} \cap (j_\varepsilon + 1)) < j_\varepsilon$ hold, and then for each such $\beta'$ the difference

$$q_\varepsilon(\beta') \setminus q_\varepsilon^{[j_\varepsilon]}(\beta') = \begin{cases} \{[f_{\beta'}(i) = (\delta_\varepsilon^{\beta'})^-]\}, & \text{if} \quad [f_{\beta'}(i) = (\delta_\varepsilon^{\beta'})^-] \in p_0(\beta') \cup p_1(\beta'), \\ \{\left[(\delta_\varepsilon^{\beta'})^- \notin \mathrm{ran}(f_{\beta'})\right]\}, & \text{if} \quad \left[(\delta_\varepsilon^{\beta'})^- \notin \mathrm{ran}(f_{\beta'})\right] \in p_0(\beta') \cup p_1(\beta'), \end{cases}$$

While otherwise, if neither $[(\delta_\varepsilon^{\beta'})^- \notin \mathrm{ran}(f_{\beta'})]$ belongs to $p_0(\beta') \cup p_1(\beta')$ nor is there an $i$ with $[f_{\beta'}(i) = (\delta_\varepsilon^{\beta'})^-] \in p_0(\beta') \cup p_1(\beta')$, then $q_\varepsilon(\beta') = q_\varepsilon^{[j_\varepsilon]}(\beta')$. (Since for the generic embedding $f_{\beta'}$ $\mathrm{ran}(f_{\beta'}) \subseteq N^*_{\beta'}$ must hold, roughly speaking $q_\varepsilon$ contains all the information from $p_0$ and $p_1$ strictly below $j_\varepsilon$.)

Now we claim that provided the sequence $\langle q_\varepsilon : \varepsilon < \varrho+2\rangle$ exists there is a common upper bound of $p_0$ and $p_1$.

**Observation 3.38.** *$q_{\varrho+1}$ is an upper bound of $p_0$ and $p_1$.*

**Claim 3.39.** *There exists a sequence $\langle q_\varepsilon : \varepsilon < \varrho+2\rangle$ satisfying $(\bullet)_3$, $(\bullet)_4$.*

*Proof.* We define $q_0$ to be the upper bound of $p_0^{[j_0]}$ and $p_1^{[j_0]}$ to satisfy $(\bullet)_{1a}$, $(\bullet)_{1b}$: For $\beta' \in S^*$ if $p_0(\beta') = (w_{\beta'}, \underaccent{\tilde}{A}_{0,\beta'})$, $p_1(\beta') = (w_{\beta'}, \underaccent{\tilde}{A}_{1,\beta'})$ then we let $s_0(\beta') = (w, \underaccent{\tilde}{B}_{\beta'})$ (where $\underaccent{\tilde}{B}_{\beta'}$ is the $\mathbb{P}^1_{\beta'}$-name satisfying $\Vdash_{\mathbb{P}^1_{\beta'}} \underaccent{\tilde}{B}_{\beta'} = \underaccent{\tilde}{A}_{0,\beta'} \cap \underaccent{\tilde}{A}_{1,\beta'}$). Because of $q_0 = q_0^{[j_\varepsilon]}$ (by $(\bullet)_3$), and recalling $(\odot)_9/(iv)$ for $\gamma = 0, 1$, $q_0(1)$ can only be the empty condition. Furthermore, for $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$, $\beta' > 1$ we let

$(\triangle)_1$ $q_0(\beta') = \{[j \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta') : \ j < j_0 \ \wedge \ j \le \sup(N^*_{\beta'} \cap j_0)\}$.

So $q_0$, $q_0^+ \in D^0_\alpha$ in fact belong to $D^*_\alpha$, and we obviously have $(\bullet)_3$, $(\bullet)_4$.

Now suppose that $q_\theta$'s are already defined for $\theta < \varepsilon$, and we shall construct $q_\varepsilon$, but we need to deal with limit and successor $\varepsilon$'s differently.

Case A: $\varepsilon$ is limit.

Let $s_\varepsilon = \lim_{\theta<\varepsilon} q_\theta \in D^*_\alpha$, we argue that we can choose a suitable extension of $s_\varepsilon$ to be $q_\varepsilon$. For $q_\varepsilon$ we only extend $s_\varepsilon$ on coordinates $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus (\{1\} \cup S^*)$. So fix such a $\beta'$. First, if $j_\varepsilon \notin N^*_{\beta'}$ (hence $N^*_{\beta'}$ is bounded in $j_\varepsilon$) then we let $q_\varepsilon(\beta') = s_\varepsilon(\beta')$. Second, if $j_\varepsilon \in N^*_{\beta'}$, and it is an accumulation point of $N^*_{\beta'}$, then again we do nothing, we just let $q_\varepsilon(\beta') = s_\varepsilon(\beta')$. But if $j_\varepsilon$ is a successor of $(j^{\beta'}_\varepsilon)^- = \max(N^*_{\beta'} \cap j_\varepsilon)$ in $N^*_{\beta'}$, then first note that

$(\triangle)_2$ $p_0^{[j_\varepsilon]}(\beta') \cup p_1^{[j_\varepsilon]}(\beta') \subseteq p_0^{[(j^{\beta'}_\varepsilon)^-]}(\beta') \cup p_1^{[(j^{\beta'}_\varepsilon)^-]}(\beta') \cup \{[j_\theta \notin \mathrm{ran}(f_{\beta'})] : \ j_\theta \ge (j^{\beta'}_\varepsilon)^-\} \cup \{[f_{\beta'}(i) = (j^{\beta'}_\varepsilon)^-] : \ i < (j^{\beta'}_\varepsilon)^-\}$

(in fact $j$'s between two consecutive element of $N^*_{\beta'}$ are irrelevant in terms of the forcing and the embedding $f_{\beta'}$). Moreover, as $\varepsilon$ is limit (and $\langle j_\theta : \ \theta < \varrho + 2\rangle$ is closed by $(\bullet)_2$) there is $\theta \in \varepsilon$ with $j_\theta \in ((j^{\beta'}_\varepsilon)^-, j_\varepsilon)$, and by $(\bullet)_3$, $(\bullet)_4$ we have (for such $\theta$)

$(\triangle)_3$ $q_\theta(\beta') \subseteq s_\varepsilon(\beta') \subseteq s_\varepsilon^{[(j^{\beta'}_\varepsilon)^-]}(\beta') \cup \{[(j^{\beta'}_\varepsilon)^- \notin \mathrm{ran}(f_{\beta'})], [f_{\beta'}(i) = (j^{\beta'}_\varepsilon)^-] : \ i < (j^{\beta'}_\varepsilon)^-\}$.

Again

$(\triangle)_4$ $s_\varepsilon(\beta') \supseteq p_0^{[(j^{\beta'}_\varepsilon)^-]}(\beta') \cup p_1^{[(j^{\beta'}_\varepsilon)^-]}(\beta')$, and

$(\triangle)_5$ $s_\varepsilon(\beta') \supseteq (p_0(\beta') \cup p_1(\beta')) \cap \Big\{[(j^{\beta'}_\varepsilon)^- \notin \mathrm{ran}(f_{\beta'})], [f_{\beta'}(i) = (j^{\beta'}_\varepsilon)^-] : \ i < (j^{\beta'}_\varepsilon)^-\Big\}$.

so there is no problem adding $\{[j_\theta \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta') : \ (j^{\beta'}_\varepsilon)^- < j_\theta < j_\varepsilon\}$ to $s_\varepsilon(\beta')$ obtaining $q_\varepsilon(\beta')$. In each of the cases it is also easy to check $(\bullet)_4$.

Case B: $\varepsilon = \theta + 1$.

We summarize first which symbols the $q_\varepsilon(\beta')$'s ($\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$) would have to include in order for $q_\varepsilon$ to satisfy $q_\varepsilon^{[j_\varepsilon]} \ge p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]}$, and $(\bullet)_4$. Of course only the case $\beta' \notin S^*$ is relevant.

$(\triangle)_6$ for $\beta' = 1$ the set to cover is

$$p_0^{[j_\varepsilon]}(1) \cup p_1^{[j_\varepsilon]}(1) \setminus q_\theta(1) = \{[j_\theta \ R_\tau \ j] \in p_0(0) \cup p_1(0) : \ j < j_\theta, \ \tau < \kappa\}. \tag{3.24}$$

By $(\bullet)_{2a}$

$(\triangle)_7$ for $1 < \beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$ the set $q_\varepsilon(\beta')$ has to include the set

$$\{[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta') : \ i \in \lambda\} \tag{3.25}$$

(which is actually either a singleton, or the empty set) and

$$\{[j \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta') : \ j \in \Big((\delta^{\beta'}_\theta)^-, \delta^{\beta'}_\varepsilon)^-\Big] \cup \{j_\theta\} \setminus \{j_\varepsilon\} \tag{3.26}$$

(where $(\delta^{\beta'}_\theta)^- = \sup(N^*_{\beta'} \cap (j_\theta + 1))$, $(\delta^{\beta'}_\varepsilon)^- = \sup(N^*_{\beta'} \cap (j_\varepsilon + 1))$, possibly $(\delta^{\beta'}_\theta)^- = (\delta^{\beta'}_\varepsilon)^- \le j_\theta$). Recall that if $[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ for some $i$, then necessarily $j_\theta \in N^*_{\beta'}$, hence $(\delta^{\beta'}_\theta)^- = j_\theta$.

First we are going to extend $q_\theta$ to a condition $q_\theta^+$ with $q_\theta^+(1)$ including the set in (3.24), and for $\beta' \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1) \setminus S^*$ the condition $q_\theta^+(\beta')$ including the set in (3.25).

**Subclaim 3.40.** *There exists $q_\theta^+ \geq q_\theta$ in $D^*_\alpha$ with*

$(*)_a$ $q_\theta^+(1) \supseteq \{[j_\theta \ R_\tau \ j] \in p_0(0) \cup p_1(0) : \ j < j_\theta, \ \tau < \kappa\}$,
$(*)_b$ *for each* $0 < \beta' \notin S^*$

$$q_\theta^+(\beta') \ni [j_\theta \notin \operatorname{ran}(f_{\beta'})], \ \textit{if } [j_\theta \notin \operatorname{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta'),$$
$$q_\theta^+(\beta') \supseteq \{[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta') : \ i < j_\theta\},$$

*while*

$(*)_c$ $q_\theta^+(1) \subseteq q_\theta^{+[j_\theta]}(1) \cup \{[j \ R_\nu \ j_\theta] : \ j < j_\theta, \ \nu < \kappa\}$,
$(*)_d$ *and for each* $1 < \beta' \notin S^*$

$$q_\theta^+(\beta') \subseteq q_\theta^{+[(j_\theta^{\beta'})^-]}(\beta') \cup \{[f_{\beta'}(i) = j_\theta] : \ i < j_\theta\} \cup \{[j_\theta \notin \operatorname{ran}(f_{\beta'})]\}.$$

Assuming the subclaim (which guarantees that $q_\theta^+$ satisfies $(\bullet)_4$) we only have to add symbols of the form $[j \notin \operatorname{ran}(f_{\beta'})]$ (sets in (3.26)) to the $q_\theta^+(\beta')$'s to obtain the condition $q_{\theta+1} = q_\varepsilon$ satisfying $(\bullet)_3$ and $(\bullet)_4$, therefore Subclaim 3.40 will finish the proof of Claim 3.39

*Proof.* (Subclaim 3.40)

$(\blacktriangle)_1$ For each fixed $\beta'$ where $\beta' \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1)$ with $[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ for some $i$ let $i_\theta^{\beta'}$ denote this unique $i$.

Now observe that

$(\blacktriangle)_2$ for each $\beta'$ with $i_\theta^{\beta'}$ defined, for each $j' < j_\theta$ with $[f_{\beta'}(i') = j'] \in q_\theta(\beta')$ for some $i'$ note that $i' < j' \leq (\delta_\theta^{\beta'})^- = j_\theta$ and $i_\theta^{\beta'} < (\delta_\theta^{\beta'})^- = j_\theta$, so we can apply Lemma 3.36, and thus each condition $q$ in $D^*_\alpha$ can be extended to $q' \in D^*_\alpha$ with $q'^{[j_\theta]}$ deciding the color $c_{\underset{\sim}{M}_{\beta'}}(i', i_\theta^{\beta'})$.

So enumerating all possible pairs $(\beta', i')$ (that are as in $(\blacktriangle)_2$) and recalling $(*)_9$ we infer that

$(\blacktriangle)_3$ for some $q^* \geq q_\theta$ the condition $q^{*[j_\theta]} \restriction \beta' \in D^*_\alpha$ decides the color $c_{\underset{\sim}{M}_{\beta'}}(i', i_\theta^{\beta'})$ for all such pairs from $\{(\beta', i') : \ \beta' \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1), \ \exists j \ [f_{\beta'}(i') = j] \in q_\theta\}$,
$(\blacktriangle)_4$ repeat this for pairs in $\{(\beta', i') : \ \exists j \ [f_{\beta'}(i') = j] \in q^{*[j_\theta]}\}$, and let $q^{**} \in D^*$ be the condition obtained after countable many such steps,

so

$(\blacktriangle)_5$ the condition $q^{**} \in D^*_\alpha$, $q^{**} \geq q_\theta$ with $q^{**[j_\theta]} \restriction \beta'$ deciding the color $c_{\underset{\sim}{M}_{\beta'}}(i', i_\theta^{\beta'})$ for all $(\beta', i') \in \{(\beta', i') : \ \beta' \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1), \ \exists j \ [f_{\beta'}(i') = j] \in q^{**[j_\theta]}(\beta')$,

Finally recall that by $(\bullet)_4$ $q_\theta(1) = q_\theta^{[j_\theta]}(1)$, and for each $\beta' \in \operatorname{dom}(q_\theta) \setminus S^*$ then $q_\theta(\beta') \setminus q_\theta^{[j_\theta]}(\beta')$ can only be non-empty if $\beta' \in \operatorname{dom}(p_0) \cup \operatorname{dom}(p_1)$ (and if it is indeed non-empty then it is a singleton $[j_\theta \notin \operatorname{ran}(f_{\beta'})]$ or $[f_{\beta'}(i) = j_\theta]$, where $(\delta_\theta^{\beta'})^- < j_\theta$).

$(\blacktriangle)_6$ This means that after possibly replacing $q_\theta^{**}(\beta')$ by $q^{**[j_\theta]}(\beta') \cup q_\theta(\beta')$ using $(\blacktriangle)_5$ it is easy to see that we get a condition $q^{**} \in D^*_\alpha$ (which still satisfies both $(\bullet)_4$ and $(\blacktriangle)_5$).

Now we are at the position to construct the desired $q_\theta^+$ as an extension of $q^{**}$. (In order to include the symbols listed in $(*)_a$, and $(*)_b$ for $\beta'$'s with $(\delta_\theta^{\beta'})^- = j_\theta$, but constructing a proper condition in $D_\alpha^*$), our task is to determine the color $\nu(j^*, j_\theta) = c_{M_*}(j^*, j_\theta)$ (i.e. add $[j^* \ R_{\nu(j^*,j_\theta)} \ j_\theta]$ to $q^{**}(1)$) for each $j^*$ and $\beta'$ such that

- $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$,
- and for some $i^*$ $[f_{\beta'}(i^*) = j^*] \in q^{**[j_\theta]}(\beta')$,

so that $\nu(j^*, j_\theta) = c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'})$ (this latter value is the color forced by $q^{**[j_\theta]} \restriction \beta'$ by $(\blacktriangle)_5$). Then adding also the symbols $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ will work.

So fix a pair $j^*, j_\theta$ as above. Now we will make use of the preparations $(\odot)_9$ and $(\bullet)_1$ and show that there are no contradicting demands concerning the value of $\nu(j^*, j_\theta)$. We distinguish the following cases.

Case (1): for some $\nu^* < \kappa$ we have $[j^* \ R_{\nu^*} \ j_\theta] \in p_0(1) \cup p_1(1)$.

Then necessarily $j^* = j_\eta$ for some $\eta < \theta$, and the only option is to

$$\text{put } [j_\eta \ R_{\nu^*} \ j_\theta] \in q_\varepsilon^+(1), \tag{3.27}$$

i.e. define $\nu(j_\eta, j_\theta) = \nu^*$. Note that this implies $j_\eta, j_\theta \in L_0$. Pick an arbitrary $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ satisfying $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ and for some $i^*$ $[f_{\beta'}(i^*) = j_\eta] \in q^{**}(\beta')$.

If $\beta' \in \mathrm{dom}(p_0)$, then by $(\odot)_9/(v)$, which implies that both $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta], [f_{\beta'}(i^*) = j_\eta] \in p_0(\beta')$, so by $(\odot)_9/(i)$ $p_0^{[j_\theta]} \restriction \beta'$ forces a value to $c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'})$. Hence, $q^{**[j_\theta]} \restriction \beta' \geq q_\theta^{[j_\theta]} \restriction \beta' \geq p_0^{[j_\theta]} \restriction \beta'$ forces the same value for $c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'})$ (by our hypothesis on $q_\theta$ $(\bullet)_3$), which is $\nu^*$.

Otherwise, assume that $\beta' \notin \mathrm{dom}(p_0)$ (so necessarily $\beta' \in \mathrm{dom}(p_1)$ and $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_1(\beta')$, and $j_\theta \in L_1$). Then again (by our construction and $(\bullet)_1/(\bullet)_{1a}$) the only way that $[f_{\beta'}(i^*) = j_\eta] \in q_\theta$ can happen for some $i^*$ is when $[f_{\beta'}(i^*) = j_\eta] \in p_1(\beta')$, but then $(\odot)_9/\ (iv)$ implies that $j_\eta \in L_1$, so $[j_\eta \ R_{\nu^*} \ j_\theta] \in p_1(\beta')$ is a member of $p_1(\beta')$, too, and then we can proceed as in the case above (i.e. arguing that $p_1^{[j_\theta]} \restriction \beta' \Vdash c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'}) = \nu^*$).

Case (2): for no $\nu^* < \kappa$ do we have $[j^* \ R_{\nu^*} \ j_\varepsilon] \in p_0(1) \cup p_1(1)$.

Case (2$A$): $j^* = j_\eta$ for some $\eta < \theta$ (so by $(ii)$ necessarily $|\{j_\eta, j_\theta\} \cap (L_0 \setminus L_1)| = |\{j_\eta, j_\theta\} \cap (L_1 \setminus L_0)| = 1$).

We can assume, that $j_\eta \in L_0 \setminus L_1$, $j_\theta \in L_1 \setminus L_0$. This means that

$(\blacktriangle)_7$ for no $\beta'$ does there exist $i$ such that $[f_{\beta'}(i) = j_\eta] \in p_1(\beta')$, and similarly, $[f_{\beta'}(i) = j_\theta] \in p_0(\beta')$ is impossible

by our assumption $(\odot)_9/(iv)$ on $p_0$ and $p_1$. So by $(\bullet)_1/(\bullet)_{1a}$ $[f_{\beta'}(i) = j_\eta] \in q_\theta(\beta')$ is only possible for any $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ if $[f_{\beta'}(i) = j_\eta] \in p_0(\beta') \cup p_1(\beta')$, so this case necessarily $[f_{\beta'}(i) = j_\eta] \in p_0(\beta')$. Summing up, for each $\beta'$ with the prospective $q_\theta^+$ forcing $j_\eta \in L_0 \setminus L_1$, $j_\theta \in L_1 \setminus L_0$ to be in the range of $f_{\beta'}$ the only possibility is that

$$[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_1(\beta'), \text{ and} \tag{3.28}$$

$$\text{for some } i^* \ [f_{\beta'}(i^*) = j_\eta] \in p_0(\beta'). \tag{3.29}$$

Now we argue that at most one such $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ may exist (then by $(\blacktriangle)_5$ we can put $[j^* \ R_{\nu^*} \ j_\varepsilon] \in q^+_\theta(\beta')$ with $\nu^* < \kappa$ defined by $q^{**[j_\theta]} \restriction \beta' \Vdash c_{\underset{\sim}{M}_{\beta'}}(i^*, i^{\beta'}_\theta) = \nu^*$, and we are done).

So assume on the contrary, let $\beta' \neq \beta''$ be such that (3.28) (3.29) hold. Then clearly $\beta', \beta'' \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$, and $j_\theta, j_\eta \in \{\xi_{\beta'}(\rho+1) : \ \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\}$, then by our assumption (on all the $p_\gamma$'s) $(\odot)_9/(iii)$ contradicts $(\blacktriangle)_7$.

Case $(2B)$: $j^*$ is not of the form $j_\theta$ for any $\theta < \varepsilon$.

This case we argue that at most one $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ could exist with $[f_{\beta'}(i^{\beta'}_\theta) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ satisfying that for some $i^*$ $[f_{\beta'}(i^*) = j^*] \in q^{**}(\beta')$. (Then again by $(\blacktriangle)_5$ we can put $[j^* \ R_{\nu^*} \ j_\theta] \in q^+_\theta(\beta')$ with $\nu^* < \kappa$, $q^{**[j_\theta]} \restriction \beta' \Vdash c_{\underset{\sim}{M}_{\beta'}}(i^*, i^{\beta'}_\theta) = \nu^*$.)

So if there are $\beta' \neq \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ with

- $[f_{\beta'}(i^*) = j^*] \in q_\theta(\beta')$ for some $i^*$,
- $[f_{\beta''}(i^{**}) = j^*] \in q_\theta(\beta'')$ for some $i^{**}$,
- $[f_{\beta'}(i^{\beta'}_\theta) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$,
- $[f_{\beta''}(i^{\beta''}_\theta) = j_\theta] \in p_0(\beta'') \cup p_1(\beta'')$,

then again as in Case $(2A)$ we can get to an easy contradiction (i.e. $\beta', \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$, and $j^* \in \{\xi_{\beta'}(\rho+1) : \ \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \ \rho < \lambda\}$, hence $(\bullet)_1/(\bullet)_{1b}$ implies $[j^* \notin \mathrm{ran}(f_\beta)] \in p_0(\beta') \cap p_1(\beta')$, similarly for $\beta''$. Now recall $q^{**} \geq q_\theta$ and $(\bullet)_4$).

$\square_{\text{Subclaim3.40}}$

$\square_{\text{Claim3.39}}$

$\square_{\text{Claim3.37}}$

$\square_{\text{Lemmas3.34and3.35}}$

Having proven that $\mathbb{P}^1_\chi$ (and each $\mathbb{P}^1_\alpha$, $\alpha \leq \chi$) is the composition of a $\lambda^+$-cc and a $\kappa^+$-cc forcing, so itself $\lambda^+$-cc, we have $(\top)_3$. Moreover, recall Claim 3.25 and that $\mathbb{Q}^1_0 = Q(\lambda, \chi, \kappa)$, so $\mathbb{Q}^1_0$ does not collapse any cardinal, while $\mathbb{P}^1_\chi/\mathbf{G}^1_1$ is $\kappa^+$-cc, $<\kappa$-closed, so $\mathbb{P}^1_\chi$ being the composition of the forcings preserving cardinals itself does not collapse any cardinal, we get $(\top)_4$. An easy calculation yields the following.

**Claim 3.41.** *For each $\alpha < \chi$ we have $\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi$. Therefore, up to equivalence $\mathbb{P}^1_\chi$ is of power $\chi$.*

*Proof.* For $\mathbb{P}^1_1 = \mathbb{Q}^1_0$ we already know $|\mathbb{Q}^1_1|$ by Observation (3.24). We have to prove the two statements simultaneously by induction on $\alpha$. As $\mathbb{P}^1_\chi$ is a $<\kappa$-support iteration, and $\chi^{<\kappa} \leq \chi^\lambda = \chi$, by our premises it is enough to prove for the successor case. So for each $\alpha < \chi$ it is enough to show that $\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi$. For $\alpha = 1$ as $\mathbb{Q}^1_1$ is a forcing of a $\kappa$-colored random graph on $\lambda$ with conditions of size $<\kappa$ we get that $|\mathbb{Q}^1_1| = \lambda^{<\kappa} \leq \chi$ (in fact $|\mathbb{Q}^1_1| = \lambda$).

For $\alpha$ with $1 < \alpha \notin S^*$ (so Definition 3.28 Case (2)). Again, each condition in $\mathbb{Q}^1_\alpha$ can be coded by a partial function of size $<\kappa$ on $\lambda$ to $\lambda+1$, so $|\mathbb{Q}^1_\alpha| = \lambda^{<\kappa} \leq \chi$.

Finally, for $\alpha \in S^*$ (Definition 3.28 Case (3)), $\mathbb{Q}^1_\alpha = \mathbb{Q}_{D_\alpha}$ is the Mathias type forcing from Definition 3.1, where $D_\alpha$ is a system of subsets of $V_\kappa^{\mathbf{V}_1}$ generating a

$\kappa$-complete filter, so $|\mathbb{Q}^1_\alpha| \le (2^{|V_\kappa|})^{\mathbf{V}_1^{\mathbb{P}^1_\alpha}} = (2^\kappa)^{\mathbf{V}_1^{\mathbb{P}^1_\alpha}} \le \chi$ (because $|\mathbb{P}^1_\alpha| = \chi$, $\mathbb{P}^1_\alpha$ is $\lambda^+$-cc, and we assumed $(\chi^\lambda)^{\mathbf{V}_1} = \chi$).

$\square_{\text{Lemma3.41}}$

So now we are ready to complete the definition of $\mathbb{P}^1_\chi$ by prescribing the names $\underset{\sim}{D}_\delta$ ($\delta \in S^*$) and $\underset{\sim}{M}_\delta$ ($1 < \delta \notin S^*$), which are standard easy bookkeeping arguments (using $|\mathbb{P}^1_\chi| = \chi$ and the $\lambda^+$-cc), but for the sake of completeness we elaborate. This will prove $(\top)_5$ and $(\top)_6$, so complete the proof of Conclusion 3.20.

**Claim 3.42.** *The system of $\underset{\sim}{D}_\delta$'s can be chosen so that for every $\mathbb{P}^1_\chi$-name $\underset{\sim}{D}$ with $\mathbf{V}_1 \Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{D} \in [\mathscr{P}(V_\kappa)]^{\le\lambda}$ there exists a $\delta \in S^*$, such that for the $\mathbb{P}^1_\delta$-name $\underset{\sim}{D}_\delta$ we have $\Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{D} = \underset{\sim}{D}_\delta$*

*Proof.* It is obvious that by $\chi^\lambda = \chi$ (so $\mathrm{cf}(\chi) > \lambda$) and the $\lambda^+$-cc for every such $\underset{\sim}{D}$ there is a nice $\mathbb{P}^1_\delta$-name for some $\delta < \chi$. As forcing with the $< \kappa$-closed $\mathbb{P}^1_\chi$ does not add new elements to $V_\kappa$ we get that for each $\delta$ there are $\chi^{\kappa\cdot\lambda} = \chi$-many such nice names. Also, as $|S^*| = \chi$ we can partition $S^* = \bigcup_{\alpha<\chi} S^*_\alpha$ with $S^*_\alpha \cap \alpha = \emptyset$, $|S^*_\alpha| = \chi$, we can let $\langle \underset{\sim}{D}_\delta : \delta \in S^*_\alpha \rangle$ list the nice names for subsets of $\mathscr{P}(V_\kappa)$. $\square_{\text{Claim3.42}}$

A similar calculation yields the following.

**Claim 3.43.** *The system of $\underset{\sim}{M}_\delta$'s can be chosen so that for every $\mathbb{P}^1_\chi$-name for a $\kappa$-colored graph $\underset{\sim}{M}$ on $\lambda$ there exists a $1 < \delta \notin S^*$, such that for the $\mathbb{P}^1_\delta$-name $\underset{\sim}{M}_\delta$ we have $\Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{M} = \underset{\sim}{M}_\delta$.*

*Proof.* Easy. $\square_{\text{Claim3.43}}$

$\square_{3.20}$

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