ON THE WEAK BOREL CHROMATIC NUMBER AND CARDINAL INVARIANTS OF THE CONTINUUM

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Abstract. Following [Ges11], we study the uncountable Borel chromatic number of some notable graphs, viewing them as cardinal characteristics of the continuum. We prove that consistently, $\mathrm{cov}(\mathcal{M}) < \lambda_{\mathbf{0}} < \lambda_{\mathbf{1}} < \lambda_\infty < 2^{\aleph_0}$, where $\lambda_{\mathbf{0}}$ denotes the weak Borel chromatic number of the Kechris-Solecki-Todorčević graph $\mathbb{G}_0$, that is, the minimal cardinality of a $\mathbb{G}_0$-independent Borel covering of 2^ω, while $\lambda_{\mathbf{1}}$ and λ_∞ are the corresponding invariants of the Silver graph $\mathbb{G}_1$ and the simple graph associated with the Vitali equivalence relation E_0.

§ 0. Introduction

Borel graphs and their combinatorial properties have become a growing area of research in the last two decades and it has interesting connections with other areas such as the theory of graph limits, countable group actions, paradoxical decompositions, as well as ergodic theory.

The Borel chromatic number was studied and defined in [KST99][LM08]. For a graph $G = (X, E)$ on a Polish space X its Borel chromatic number $\chi_{\mathrm{B}}(G)$ is the least cardinal κ, such that for some Polish space Y there exists a Borel coloring $c : X \to Y$ of G with $|\mathrm{ran}(c)| = \kappa$ (i.e. for each $y \in Y$ the preimage $c^{-1}(y)$ is G-independent). It is clear by the theory of Polish spaces and the *Perfect Set Property* of analytic sets that this number is an element of the set $\{0, 1, 2, \dots\} \cup \{\aleph_0, 2^{\aleph_0}\}$.

The theory was extended by S. Geschke, who showed that if X is Polish, then for each closed graph $G = (X, E)$ without perfect cliques, as well as for each locally countable F_σ graph $G = (X, E)$ (i.e. each node has degree at most $\aleph_0$) there exists some ccc. forcing making the continuum large, while X can be covered by $\aleph_1$-many Borel (in fact, closed) G-independent sets [Ges11]. Later M. Gaspar- S. Geschke [GG22] have defined the weak Borel chromatic number of a fixed graph G as the least possible cardinal κ, such that the underlying space can be covered by κ-many G-independent Borel sets. Note that if either chromatic number is at most countable, then they coincide. (Here we remark that S. Geschke had defined the weak Borel chromatic number (of the graph $G = (X, E)$) as the smallest cardinal κ, such that there exists a coloring $c : X \to \kappa$ with Borel fibers (i.e. for each $\alpha < \kappa$

†The first author was supported by the Excellence Fellowship Program for International Postdoctoral Researchers of The Israel Academy of Sciences and Humanities, and by the National Research, Development and Innovation Office – NKFIH, grants no. 124749, 129211.

*The second author was supported by the Israel Science Foundation grant 1838/19.

Research of both authors partially supported by NSF grant no: DMS 1833363.

Paper 1226 on Shelah's list. References like [She, Th0.2=Ly5] mean the label of Th.0.2 is y5. The reader should note that the version on the second author's website is usually more updated than the one in the mathematical archive.

the preimage $c^{-1}(\alpha)$ is Borel) [Ges11]. Note that if for a fixed graph either variant of the weak Borel chromatic number is at most $\aleph_1$, then they coincide.)

In the celebrated paper of A. Kechris-S. Solecki- S. Todorčević [KST99] the graph $\mathbb{G}_0$ (with $X = 2^\omega$) was constructed, and proved to be minimal among analytic graphs of uncountable Borel chromatic number in the sense that for each simple graph $G = (Y, F)$, where Y is Polish, $E \in \mathbf{\Sigma}^1_1(Y^2)$ either $\chi_B(G) \leq \aleph_0$, or there exists a continuous homomorphism $f : 2^\omega \to Y$ from $\mathbb{G}_0$ to G (i.e. whenever $(x, x') \in E_{\mathbb{G}_0}$, then $(f(x), f(x')) \in F$ holds necessarily). This also implies that whenever $G = (Y, F)$ is an analytic graph on a Polish space with uncountable weak Borel chromatic number, it is at least as the weak Borel chromatic number of $\mathbb{G}_0$.

While the graph $\mathbb{G}_0$ is acyclic, so it can be colored by two colors, B. Miller showed that the measurable chromatic number of it is 3 [Mil08]. He asked whether anything can be said about the weak Borel chromatic number of $\mathbb{G}_0$ compared to other cardinal characteristics of the continuum. In [KST99] not only has the authors verified that $\chi_B(G) > \aleph_0$, but it also followed from their argument, that each $\mathbb{G}_0$-independent Baire-measurable set $S \subseteq 2^\omega$ must be meager. This immediately implies that $\mathrm{cov}(\mathcal{M})$ is a lower bound for the weak Borel chromatic number of $\mathbb{G}_0$ as well.

Due to M. Gaspar and S. Geschke, independently of this work, various Borel chromatic numbers of graphs were computed in models of set theory obtained by forcing with countable support iteration of uniform tree-forcing notions [GG22], or see further results by R. Banerjee, M. Gaspar [BG22]. Earlier F. Adams and J. Zapletal had studied cardinal invariants of closed graphs [AZ18]. Zapletal [Zap19] studied hypergraphable σ-ideals, namely, σ-ideals that are σ-generated by Borel anticliques in a fixed family of hypergraphs, proving also dichotomy theorems [Zap19, §4], highlighting the importance of the Silver, Vitali, and KST graphs.

§ 1. Preliminaries, notations

Under ordinals we always mean von Neumann ordinals, and for a set X the symbol $|X|$ always refers to the smallest ordinal with the same cardinality. For a set X the symbol $\mathscr{P}(X)$ denotes the power set of X, while if κ is an cardinal we use the standard notation $[X]^\kappa$ for $\{Y \in \mathscr{P}(X) : |Y| = \kappa\}$, similarly for $[X]^{<\kappa}$, $[X]^{\leq\kappa}$, etc. By a sequence we mean a function on an ordinal, where for a sequence $\bar{s} = \langle s_\alpha : \alpha < \mathrm{dom}(\bar{s})\rangle$ the length of $\bar{s}$ (in symbols $\ell g(\bar{s})$) denotes $\mathrm{dom}(\bar{s})$. We denote the empty sequence by $\langle\rangle$. Moreover, for sequences $\bar{s}$, $\bar{t}$, we let $\bar{s} \frown \bar{t}$ denote the natural concatenation of them (of length $\ell g(\bar{s}) + \ell g(\bar{t})$). For a set X, and ordinal α we use ${}^\alpha X = \{\bar{s} : \ell g(\bar{s}) = \alpha, \ \mathrm{ran}(\bar{s}) \subseteq X\}$, and for cardinals λ, κ we use the symbol $\lambda^\kappa = |{}^\kappa\lambda|$ (that is, the least ordinal equivalent to it).

For a finite sequence $\bar{s} \in {}^{\omega>}2$ the symbol $[\bar{s}]$ stands for the basic open set in 2^ω that $\bar{s}$ determines, i.e.

$$[\bar{s}] = \{x \in {}^\omega 2 : x \supseteq \bar{s}\}.$$

A tree T is a downward closed set consisting of finite sequences.

If $\varphi(n, x)$ is a formula, then $\forall^\infty n \varphi(n, x)$ is true, if for all but finitely many $n \in \omega$ $\varphi(n, x)$ is true, $\exists^\infty n$ stands for there exists infinitely many n, and we use the quantifier $\exists! y$ as "there exists a unique y". For $r, r' \in \omega^\omega$ under $r \leq^* r'$ we mean that $\forall^\infty n\ r_n \leq r'_n$.

Concerning forcing, $q \leq p$ means that q is stronger, and for a notion of forcing $\mathbb{P}$ the term $1_{\mathbb{P}}$ stands for the unique largest element of $\mathbb{P}$.

§ 2. The forcing construction

Let $\bar{\mathfrak{s}} = \langle \mathfrak{s}_n : n \in \omega \rangle$ be fixed, such that

(x$_1$) for each $n \in \omega$: $\mathfrak{s}_n \in {}^n\omega$,
(x$_2$) the set $\bigcup_{n\in\omega}[\mathfrak{s}_n]$ is dense in 2^ω.

Recall the definition of the graph $\mathbb{G}_0(\bar{s})$ on the Cantor space [KST99]:

Definition 2.1. The graph $\mathbb{G}_0(\bar{\mathfrak{s}})$ is defined as follows:

$$\mathbb{G}_0(\bar{\mathfrak{s}}) = \{(\mathfrak{s}_n \frown \langle 0 \rangle \frown x, \mathfrak{s}_n \frown \langle 1 \rangle \frown x) : n \in \omega, \ x \in 2^\omega\} \subseteq [2^\omega]^2.$$

Theorem 2.2. *([KST99]) For any sequence $\bar{t}$ satisfying (x$_1$), (x$_2$) the graph $\mathbb{G}_0(\bar{t}) \subseteq [2^\omega]^2$ is a closed acyclic graph such that whenever $H \subseteq 2^\omega$ has the Baire property and $\mathbb{G}_0(\bar{t})$-independent, then it must be meager.*

Moreover, if $G = (X, E)$ is an analytic graph on the Polish space X and $\chi_B(G) > \aleph_0$, then there exists a continuous map $f : 2^\omega \to X$, which is a homomorphism from $\mathbb{G}_0(\bar{t})$ into G.

From now on we will only write $\mathbb{G}_0$ instead of $\mathbb{G}_0(\bar{\mathfrak{s}})$. Note that the graph $\mathbb{G}_0(\bar{s})$ enjoys the expected properties if (x$_1$), (x$_2$) holds without any regard to the specific sequence $\bar{\mathfrak{s}}$, justifying the use of the terminology $\chi_{\mathrm{B}}(\mathbb{G}_0)$, $\chi_{\mathrm{wB}}(\mathbb{G}_0)$, $\mathrm{cov}(I_{\mathbb{G}_0})$.

Definition 2.3. The graph $\mathbb{G}_1$ is defined as follows:

$$\mathbb{G}_1 = \{(x, y) : x, y \in 2^\omega, \ \exists! n \in \omega \ x_n \neq y_n\} \subseteq [2^\omega]^2.$$

Definition 2.4. The Vitali relation E_0 is defined as follows:

$$E_0 = \{(x, y) : x \neq y \in 2^\omega, \ \forall^\infty n \in \omega \ x_n = y_n\} \subseteq [2^\omega]^2.$$

Note that this is not the standard definition of the Vitali relation, as we interpret it as a subset of $[2^\omega]^2$, while in the literature $E_0 \subseteq 2^\omega \times 2^\omega$ is an equivalence relation.

Definition 2.5. If X is a topological space, and G is a graph on it, then we let $I_G \subseteq \mathscr{P}(X)$ denote the σ-ideal generated by Borel G-independent sets.

Now we are ready to state our main theorem.

Theorem 2.6. *Assume* **CH***, and let $\lambda_{\mathbf{0}} \leq \lambda_{\mathbf{1}} \leq \lambda_\infty \leq \lambda_{\mathbb{S}} = \kappa$ be infinite cardinals such that $\lambda_\iota = \lambda_\iota^{\aleph_0}$ for each $\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\}$. Then in some cardinal preserving forcing extension we have*

$$\begin{aligned} \mathrm{cov}(\mathcal{M}) = \mathfrak{d} \quad &= \aleph_1, \\ \mathrm{cov}(I_{\mathbb{G}_0}) \quad &= \lambda_{\mathbf{0}}, \\ \mathrm{cov}(I_{\mathbb{G}_1}) \quad &= \lambda_{\mathbf{1}}, \\ \mathrm{cov}(I_{E_0}) \quad &= \lambda_\infty, \\ 2^{\aleph_0} \quad &= \kappa = \lambda_{\mathbb{S}}. \end{aligned}$$

Proof. We define our forcing posets in the following steps.

Definition 2.7.

(D1) For each n we let $C_n \subseteq 2^{<\omega}$ be a finite, non-empty set such that for each n
  (a) $x \neq y \in C_n \to x \not\subseteq y$ or $y \not\subseteq x$,
  (b) $C_{2n+1} = \{\langle 0 \rangle, \langle 1 \rangle\}$, and
  (c) for each $\bar{t} \in \prod_{i<2n} C_i$ there exists $\bar{t}' \in C_{2n}$ such that $\bar{t} \frown \bar{t}' = \mathfrak{s}_k$ (from (x$_1$)-(x$_2$)) for some $k \in \omega$.

(This can be achieved by induction, e.g. before constructing C_{2n} imposing the auxiliary demand that for some $r \in 2^\omega$ $\bar{t} \in C_{2n} \ \to \ r \notin [\bar{t}]$.)

(D2) Let $p \in \mathbb{P}^{\mathbf{0}}$, if
  (*i*) $p = \langle p_i : \ i \in \omega \rangle$, where $\forall i : \ \emptyset \neq p_i \subseteq C_i$, and
  (*ii*) $\exists^\infty j$: $p_{2j} = C_{2j} \wedge \ p_{2j+1} = C_{2j+1}$,
  with q stronger than p, (in symbols, $q \leq p$) iff $q_i \subseteq p_i$ for each i.

(D3) For $\mathbb{P}^{\mathbf{1}}$ just recall the definition of the Silver real forcing: we let $p \in \mathbb{P}^{\mathbf{1}}$, if
  (*i*) $p = \langle p_i : \ i \in \omega \rangle$, where $\forall i : \ \emptyset \neq p_i \subseteq \{0, 1\}$, and
  (*ii*) $\exists^\infty j$: $p_j = \{0, 1\}$,
  with q stronger than p, (in symbols, $q \leq p$) iff $q_i \subseteq p_i$ for each i.

(D4) We let $p \in \mathbb{P}^\infty$, if
  (*i*) $p = \langle p_i : \ i \in \omega \rangle$, where $\forall i \in \omega : \ \emptyset \neq p_i \subseteq \ {}^{k_{p_i}}2$ for some $k_{p_i} > 0$, and
  (*ii*) $\exists^\infty j$: $|p_j| = 2$,
  with q stronger than p, (in symbols, $q \leq p$) iff
  - there exists a strictly increasing infinite sequence $j_0 < j_1 < \dots$ of finite ordinals, for which
  $$(\forall i < \omega) : \ k_{q_i} = k_{p_{j_{i-1}+1}} + k_{p_{j_{i-1}+2}} + \cdots + k_{p_{j_i}},$$
  where under j_{-1} we mean -1,
  - for each $i < \omega$, $\bar{t} \in q_i$ there exists
  $$(\bar{t}^*_{j_{i-1}+1} \in p_{j_{i-1}+1}) \ \& \ (\bar{t}^*_{j_{i-1}+2} \in p_{j_{i-1}+2}) \ \& \dots \& \ (\bar{t}^*_{j_i} \in p_{j_i}),$$
  such that
  $$\bar{t} = \bar{t}^*_{j_{i-1}+1} {}^\frown t^*_{j_{i-1}+2} {}^\frown \cdots {}^\frown t^*_{j_i}.$$
  (So the biggest element p can be described as the condition satisfying for each j $p_j = \{\langle 0 \rangle, \langle 1 \rangle\}$.)

(D5) If $p \in \mathbb{P}^\iota$ ($\iota \in \{\mathbf{0}, \mathbf{1}, \infty\}$), $\bar{t} \in \prod_{j<i} p_j$ for some $i < \omega$, then let $p^{[\bar{t}]}$ denote the condition defined as $p^{[\bar{t}]}_j = \{t_j\}$ for $j < i$, and $p^{[\bar{t}]} \restriction [i, \omega) = p \restriction [i, \omega)$.

(D6) Moreover, let $\mathbb{P}^{-m}$ denote the subforcing $\{p \restriction [m, \omega) : \ p \in \mathbb{P}\}$ of $\mathbb{P}$ with the natural order.

Recall the definition of the Sacks forcing, and so let $\mathbb{P}^{\mathbb{S}} = \{T \subseteq 2^{<\omega} : \ T \text{ is a perfect tree}\}$ with $T \leq T'$ iff $T \subseteq T'$.

Definition 2.8.

(D7) For $n \in \omega$ we define the partial order $\leq_n$ on $\mathbb{P}^{\mathbf{0}}$ as $q \leq_n p$, iff
  - $q \leq p$, and
  - for some $\ell_0 < \ell_1 < \dots < \ell_{n-1} < \omega$ we have
  $$\forall j < n : \ (p_{2\ell_j} = q_{2\ell_j} = C_{2\ell_j}) \& (p_{2\ell_j+1} = q_{2\ell_j+1} = C_{2\ell_j+1}),$$
  and
  $$(\forall k < 2\ell_{n-1}) \ p_k = q_k.$$

(D8) For $n \in \omega$ we define the partial order $\leq_n$ on $\mathbb{P}^{\mathbf{1}}$ as $q \leq_n p$, iff
  - $q \leq p$, and
  - for some $l_0 < l_1 < \dots < l_{n-1} < \omega$ we have
  $$(\forall k < l_{n-1}) \ p_k = q_k.$$
  and
  $$\forall j < n : \ p_{l_j} = q_{l_j} = \{0, 1\},$$

(D9) For $n \in \omega$ we define the partial order $\leq_n$ on $\mathbb{P}^\infty$ as $q \leq_n p$, iff
- $q \leq p$, and
- for some $l_0 < l_1 < \dots l_{n-1} < \omega$ we have
$$\forall j < n: \ |p_{l_j}| = |q_{l_j}| = 2,$$
and
$$(\forall k < l_{n-1}) \ p_k = q_k,$$

(D10) For $q \in \mathbb{P}^{\mathbb{S}}$ (so $q = T_q \subseteq {}^{\omega>}2$ is a perfect tree) we define stem(q) to be the minimal branching node of q (i.e. q and stem(q) satisfy stem$(q) \frown \langle 0 \rangle$, stem$(q) \frown \langle 1 \rangle \in q$, but each proper initial segment of stem(q) has a unique immediate successor).

(D11) We define the partial order $\leq_n$ (for every $n \in \omega$) on $\mathbb{S}$ as
- $q \leq_0 p$, iff $q \leq p$,
- $q \leq_{n+1} p$, iff $q \leq p$, stem$(q) =$ stem(p), and for this common stem s:
$$q^{[s \frown \langle 0 \rangle]} \leq_n p^{[s \frown \langle 0 \rangle]}, \text{ and}$$
$$q^{[s \frown \langle 1 \rangle]} \leq_n p^{[s \frown \langle 1 \rangle]}.$$

A standard argument yields the following.

Observation 2.9. $\mathbb{P}^{\mathbf{0}}$, $\mathbb{P}^{\mathbf{1}}$, $\mathbb{P}^\infty$, $\mathbb{P}^{\mathbb{S}}$ *satisfy Baumgartner's Axiom A with the partial orders defined above in D*7) *and D*11), *in particular if we are given the sequence*
$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \dots \geq_{n-1} p_n \geq_n p_{n+1} \geq_{n+1} \dots,$$
then there exists a common lower bound p' w.r.t. $\geq$ (in fact even $p' \leq_n p_n$ can be assumed for each n).

(D12) For $I \subseteq \lambda_{\mathbf{0}}$ we let
$$\mathbb{Q}^{\mathbf{0}}_I = \{f \in {}^I(\mathbb{P}^{\mathbf{0}}): \ f(i) = 1_{\mathbb{P}^{\mathbf{0}}} \text{ for all, but countable } i's\}$$
be the countable support product of $\mathbb{P}$'s.

(D13) For $I \subseteq [\lambda_{\mathbf{0}}, \lambda_{\mathbf{1}})$ let
$$\mathbb{Q}^{\mathbf{1}}_I = \{f \in {}^I(\mathbb{P}^{\mathbf{1}}): \ f(i) = 1_{\mathbb{P}^{\mathbf{1}}} \text{ for all, but countable } i's\}$$
be the countable support product of $\mathbb{P}^{\mathbf{1}}$'s.

(D14) Similarly for $I \subseteq [\lambda_{\mathbf{1}}, \lambda_\infty)$ let
$$\mathbb{Q}^{\infty}_I = \{f \in {}^I(\mathbb{P}^{\infty}): \ f(i) = 1_{\mathbb{P}^{\infty}} \text{ for all, but countable } i's\}$$
be the countable support product of $\mathbb{P}^\infty$'s,

(D15) and for $I \subseteq [\lambda_\infty, \lambda_{\mathbb{S}})$ let
$$\mathbb{Q}^{\mathbb{S}}_I = \{f \in {}^I(\mathbb{P}^{\mathbb{S}}): \ f(i) = 1_{\mathbb{P}^{\mathbb{S}}} \text{ for all, but countable } i's\}$$
be the countable support product of $\mathbb{P}^{\mathbb{S}}$'s.

(D16) We let $\mathbb{Q}$ be the following countable support product:
$$\mathbb{Q} = \mathbb{Q}^0_{\lambda_{\mathbf{0}}} \times \mathbb{Q}^{\mathbf{1}}_{\lambda_{\mathbf{1}} \setminus \lambda_{\mathbf{0}}} \times \mathbb{Q}^\infty_{\lambda_\infty \setminus \lambda_{\mathbf{1}}} \times \mathbb{Q}^{\mathbb{S}}_{\lambda_{\mathbb{S}} \setminus \lambda_\infty}.$$

We have to check that the forcing $\mathbb{Q}$ is indeed cardinal preserving, forcing $\mathfrak{d}$ to be $\aleph_1$, the continuum to be $\kappa = \lambda_{\mathbb{S}}$, there exists a system of $\lambda_{\mathbf{0}}$-many $\mathbb{G}_0$-independent Borel sets covering 2^ω, but any system of smaller cardinality is not sufficient (and similarly for $\mathbb{G}_1$, and E_0). For this we will prove the following:

$(\circledast)_1$ $\mathbb{Q}$ is proper, and has the $\aleph_2$-cc,

$(\circledast)_2$ $\mathbb{Q}$ is ω^ω-bounding, i.e. for each $r \in {}^\omega\omega \cap V^{\mathbb{Q}}$ there exists $r' \in {}^\omega\omega \cap V$ such that $r' \geq^* r$,

$(\circledast)_3$ for each $r \in {}^\omega 2 \cap V^{\mathbb{Q}}$ there exists a
- tree $T_{\mathbf{0}} \in V^{\mathbb{Q}^{\mathbf{0}}_{\lambda_{\mathbf{0}}}} \cap \mathscr{P}(2^{<\omega})$ such that $r \in [T_{\mathbf{0}}]$, and $[T_{\mathbf{0}}]$ is $\mathbb{G}_0$-independent,
- tree $T_{\mathbf{1}} \in V^{\mathbb{Q}^{\mathbf{0}}_{\lambda_{\mathbf{0}}} \times \mathbb{Q}^{\mathbf{1}}_{\lambda_{\mathbf{1}} \setminus \lambda_{\mathbf{0}}}} \cap \mathscr{P}(2^{<\omega})$ such that $r \in [T_{\mathbf{1}}]$, and $[T_{\mathbf{1}}]$ is $\mathbb{G}_1$-independent,
- tree $T_\infty \in V^{\mathbb{Q}^{\mathbf{0}}_{\lambda_{\mathbf{0}}} \times \mathbb{Q}^{\mathbf{1}}_{\lambda_{\mathbf{1}} \setminus \lambda_{\mathbf{0}}} \times \mathbb{Q}^{\infty}_{\lambda_\infty \setminus \lambda_{\mathbf{1}}}} \cap \mathscr{P}(2^{<\omega})$ such that $r \in [T_\infty]$, and $[T_\infty]$ is E_0-independent,

$(\circledast)_4$ If $\alpha < \lambda_{\mathbb{S}}$, $T \in V^{\mathbb{Q} \restriction \lambda_{\mathbb{S}} \setminus \{\alpha\}} \cap \mathscr{P}({}^{\omega >}\{0,1\})$ is a tree, such that
- either $\alpha < \lambda_{\mathbf{0}}$, and $[T]$ is $\mathbb{G}_0$-independent,
- or $\alpha \in [\lambda_{\mathbf{0}}, \lambda_{\mathbf{1}})$, and $[T]$ is $\mathbb{G}_1$-independent,
- $\alpha \in [\lambda_{\mathbf{1}}, \lambda_\infty)$, and $[T]$ is E_0-independent,

then for r_α, the generic real given by the α'th coordinate ($\mathbb{Q} \restriction \{\alpha\}$), we have:
$$r_\alpha \notin [T].$$

Similar statements to $(\circledast)_3$ are proved independently in [GG22, §3] (for an extension adding a single real, and generalizing it to CS iterations), and see also [Zap19, Thm 3.47- Corollary 3.49], a more general result, albeit only for a single step extension, which is independent of both.

Observe that the properness of $\mathbb{Q}$ (together with our assumptions on the ground model) would imply that $(2^{\aleph_0})^{V^{\mathbb{Q} \restriction \lambda_\iota}} = \lambda_\iota$ ($\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\}$), and so $(2^{\aleph_0})^{V^{\mathbb{Q}}} = \kappa^{\aleph_0} = \kappa$ $(= \lambda_{\mathbb{S}})$, while it follows from $(\circledast)_3$ that $\mathrm{cov}(I_{\mathbb{G}_0}) \leq \lambda_{\mathbf{0}}$ (and the respective inequalities similarly hold for $\mathbb{G}_1$, and E_0).

By the $\aleph_2$-cc of $\mathbb{Q}$, if μ is uncountable, then each system of Borel sets of size μ is in $V^{\mathbb{Q} \restriction M}$ for some $M \in V$ of size at most $\mu \cdot \aleph_1 = \mu$.

Moreover, $(\circledast)_2$ clearly implies $\mathfrak{d}^{V^{\mathbb{Q}}} = \aleph_1$, and the inequality $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}$ holds in **ZFC** (since each compact set in $\mathbb{N}^{\mathbb{N}}$ is meager as well, or see [BJ95]). Finally, since each uncountable Borel subset of a Polish space is a continuous image of (a closed subspace of) $\mathbb{N}^{\mathbb{N}}$ [Kec12, Theorems 7.9, 13.1], each Borel set is the union of $\mathfrak{d}$-many compact sets. Thus, if a Polish space can be covered by $\aleph_1 \leq \mu$-many Borel sets, then we can replace each Borel set B with a system of $\aleph_1$-many compact sets $\langle K_\alpha : \alpha < \omega_1 \rangle$ with $B = \bigcup_{\alpha < \omega_1} K_\alpha$, so (in the extension) there exists $\langle K'_\alpha : \alpha < \mathrm{cov}(I_{\mathbb{G}_0}) \rangle$ with each K'_α compact, covering 2^ω. This together with $(\circledast)_4$ implies $\mathrm{cov}(I_{\mathbb{G}_0}) \geq \lambda_{\mathbf{0}}$, $\mathrm{cov}(I_{\mathbb{G}_1}) \geq \lambda_{\mathbf{1}}$, $\mathrm{cov}(I_{E_0}) \geq \lambda_\infty$. Therefore it is indeed enough to verify clauses $(\circledast)_1$- $(\circledast)_4$.

Claim 2.10. *$\mathbb{Q}$ has the $\aleph_2$-cc.*

Proof. Suppose that $\langle a_i : i < \omega_2 \rangle$ is an antichain. Since $|\mathrm{supp}(a_i)| \leq \aleph_0$ for $i < \omega_2$, by **CH** we can assume that $\{\mathrm{supp}(a_i) : i < \omega_2\}$ forms a Δ-system with kernel K. But $|\mathbb{P}^{\mathbf{0}}| = |\mathbb{P}^{\mathbf{1}}| = |\mathbb{P}^\infty| = |\mathbb{P}^{\mathbb{S}}| = |(2^{\aleph_0})^V| = \aleph_1$ (by Definition 2.7), so $|{}^K(\mathbb{P}^{\mathbf{0}} \cup \mathbb{P}^{\mathbf{1}} \cup \mathbb{P}^\infty \cup \mathbb{P}^{\mathbb{S}})| = \aleph_1$, we are done. $\square_{\text{Claim2.10}}$

Convention 2.11. By passing down to a dense subset of $\mathbb{P}^{\mathbf{0}}$, from now on we can assume that whenever $p \in \mathbb{P}^{\mathbf{0}}$, $k \in \omega$,
$$\neg(p_{2k} = C_{2k} \ \wedge \ p_{2k+1} = C_{2k+1}) \ \rightarrow \ |p_{2k}| = |p_{2k+1}| = 1.$$

Definition 2.12.

(D*1) if $p \in \mathbb{P}^{\mathbf{0}}$, $n \in \omega$, and d is the smallest integer for which for some $l_0 < l_1 < \cdots < l_{n-1} < d$ we have

$$\forall j < n : \ (p_{2l_j} = q_{2l_j} = C_{2l_j}) \& (p_{2l_j+1} = q_{2l_j+1} = C_{2l_j+1}),$$

then we let

$$T_n(p) = \prod_{j<2d} p_j,$$

and

$$T(p) = \bigcup_{n\in\omega} T_n(p),$$

(D*2) if $p \in \mathbb{P}^{\mathbf{0}}$, $\bar{u} \in \bigcup_{n\in\omega} \prod_{j<n} p_j$, (e.g. $\bar{u} \in T(p)$), then we define $p^{[\bar{u}]} = p^{(\bar{u})}$, to be a condition in $\mathbb{P}^0$ satisfying

- $p^{[\bar{u}]}_j = p^{(\bar{u})}_j = \{u_j\}$, if $j < \ell g(\bar{u})$,
- $p^{[\bar{u}]}_j = p^{(\bar{u})}_j = p_j$, if $j \geq \ell g(\bar{u})$.

(D*3) for $n \in \omega$, $p \in \mathbb{P}^{\mathbf{1}}$, if $d < \omega$ is the minimal natural number such that for some $l_0 < l_1 < \cdots < l_{n-1} < d$

$$\forall j < n : \ \left(p_{l_j} = \{0, 1\}\right),$$

then for each $\bar{v} = \langle v_0, v_1, \ldots, v_{n-1}\rangle \in {}^n2$ we define the sequence $\bar{\mathfrak{t}}_{\bar{v}}(p) = \langle \mathfrak{t}_{\bar{v}}(p)_j : j < d\rangle \in {}^d2$ as

- $\mathfrak{t}_{\bar{v}}(p)_{l_j} = v_j$,
- $\mathfrak{t}_{\bar{v}}(p)_k = a_k$, where $p_k = \{a_k\}$, $k \in (l_j, l_{j+1})$ for some $j < n-1$.

(D*4) if $p \in \mathbb{P}^{\mathbf{1}}$, $\bar{t} \in \bigcup_{n\in\omega} \prod_{j<n} p_j$, (e.g. $\bar{t} = \bar{\mathfrak{t}}_{\bar{u}}(p)$ for some $\bar{u} \in {}^{\omega>}2$), then we define $p^{[\bar{t}]}$, to be a condition in $\mathbb{P}^{\mathbf{1}}$ satisfying

- $p^{[\bar{t}]}_j = \{t_j\}$, if $j < \ell g(\bar{t})$,
- $p^{[\bar{t}]}_j = p_j$, if $j \geq \ell g(\bar{t})$.

Moreover, if $\bar{u} \in {}^n2$ for some n, then we let

$$p^{(\bar{u})} := p^{[\bar{\mathfrak{t}}_{\bar{u}}(p)]}.$$

(D*5) for $n \in \omega$, $p \in \mathbb{P}^{\infty}$ if $d < \omega$ is the minimal natural number such that for some $l_0 < l_1 < \cdots < l_{n-1} < d$

$$\forall j < n : \ \left(|p_{l_j}| = 2\right),$$

then for each $\bar{v} = \langle v_0, v_1, \ldots, v_{n-1}\rangle \in {}^n2$ we define the sequence $\bar{\bar{\mathfrak{t}}}_{\bar{v}}(p) = \langle \bar{\mathfrak{t}}_{\bar{v}}(p)_j : j < d\rangle \in {}^d2$ as

- $\bar{\mathfrak{t}}_{\bar{v}}(p)_{l_j} = \bar{t}^{p_k}_{l_j, w_j}$, where $p_{l_j} = \{\bar{t}^{p_k}_{l_j,0} <_{\text{lex}} \bar{t}^{p_k}_{l_j,1}\}$,
- $\bar{\mathfrak{t}}_{\bar{v}}(p)_k = \bar{t}^{p_k}$, where $p_k = \{\bar{t}^{p_k}\}$, $k \in (l_j, l_{j+1})$ for some $j < n-1$.

(D*6) if $p \in \mathbb{P}^{\infty}$, $\bar{\bar{t}} \in \bigcup_{n\in\omega} \prod_{j<n} p_j$, (e.g. $\bar{\bar{t}} \in T(p)$), then we define $p^{[\bar{\bar{t}}]}$, to be a condition in $\mathbb{P}^{\infty}$ satisfying

- $p^{[\bar{\bar{t}}]}_j = \{\bar{t}_j\}$, if $j < \ell g(\bar{\bar{t}})$,
- $p^{[\bar{\bar{t}}]}_j = p_j$, if $j \geq \ell g(\bar{\bar{t}})$.

Furthermore, if $\bar{w} \in {}^n2$ for some n, then we let

$$p^{(\bar{w})} := p^{[\bar{\bar{\mathfrak{t}}}_{\bar{w}}(p)]}.$$

(D*7) For $p \in \mathbb{P}^{\mathbb{S}}$, $\bar{s} \in {}^n2$ we define the node $\bar{\mathfrak{t}}_{\bar{s}}(p) \in p$ by induction on $\ell g(\bar{s})$ as follows. Let

$$\bar{\mathfrak{t}}_{\langle\rangle}(p) = \langle\rangle, \tag{2.1}$$

and

$$\begin{aligned} \bar{\mathfrak{t}}_{\bar{s}\frown\langle 0\rangle}(p) = &\quad \text{stem}\left(p^{[\bar{\mathfrak{t}}_{\bar{s}}(p)]}\right) \frown \langle 0\rangle, \\ \bar{\mathfrak{t}}_{\bar{s}\frown\langle 1\rangle}(p) = &\quad \text{stem}\left(p^{[\bar{\mathfrak{t}}_{\bar{s}}(p)]}\right) \frown \langle 1\rangle, \end{aligned} \tag{2.2}$$

(recall D10), for the stem of a tree T is the unique largest element in

$$\{t \in T:\ \forall t' \in T:\ t \subseteq t'\ \vee\ t' \subseteq t\}).$$

Note that $q \leq_n p$, iff $q \leq p$, and for each $s \in {}^n 2$ we have $\bar{\mathfrak{t}}_s(p) = \bar{\mathfrak{t}}_s(q)$.

(D*8) if $p \in \mathbb{P}^{\mathbb{S}}$ and $\bar{s} \in p$, then we define $p^{[\bar{s}]}$, to be the condition $p^{[\bar{s}]} = \{\bar{t} \in p : \bar{t} \supseteq \bar{s}\}$, and for arbitrary $\bar{s} \in {}^{\omega >}2$ we let

$$p^{(\bar{s})} = p^{[\bar{\mathfrak{t}}_{\bar{s}}(p)]}.$$

Observation 2.13. *If $p \in \mathbb{P}^{\infty}$, $i \in \omega$,*

$$p_0 = \{\bar{t}^p_0\}, p_1 = \{\bar{t}^p_1\}, p_{i-1} = \{\bar{t}^p_{i-1}\}, \tag{2.3}$$

and we consider the condition $p' \leq p$ defined as

$$\begin{aligned} &p'_0 = \{\bar{t}^{p'}_0 = \bar{t}^p_0 \frown \bar{t}^p_1 \frown \cdots \frown \bar{t}^p_{i-1}\}, \\ &p'_{j+1} = p_{j+i}\ (\textit{for } j \in \omega) \end{aligned}$$

then $p \Vdash_{\mathbb{P}^\infty} p' \in \underset{\sim}{\mathbf{G}}$.

Proof. Suppose that $p^* \leq p$, and $p^* \perp p'$. By further strengthening p^* w.l.o.g. we can assume that $|p^*_0| = 1$, and for the unique $\bar{t}^* \in p^*_0$ we have

$$\ell g(\bar{t}^*) > |\bar{t}^p_0| + |\bar{t}^p_1| + \cdots + |\bar{t}^p_{i-1}|.$$

But then this bound together with $p^* \leq p$ and (2.3) implies that

$$\bar{t}^* = (\bar{t}^p_0 \frown \bar{t}^p_1 \frown \cdots \frown \bar{t}^p_{i-1}) \frown t^p_i \frown \ldots t^p_{i+k} = \bar{t}^{p'}_0 \frown \bar{t}^p_i \frown \ldots \bar{t}^p_{i+k}$$

for some $\bar{t}^p_i \in p_i = p'_1$, $\bar{t}^p_{i+1} \in p_{i+1} = p'_2$, $\ldots$, $\bar{t}^p_{i+k} \in p_{i+k} = p'_{k+1}$, so not only is p' compatible with p^*, but $p^* \leq p'$.

$\square_{\text{Observation2.13}}$

Definition 2.14. We let $\bar{\bar{\ell}} \in \mathcal{L}$, iff $\bar{\bar{\ell}} = (\bar{\ell}^{\mathbf{0}}, \bar{\ell}^{\mathbf{1}}, \bar{\ell}^{\infty}, \bar{\ell}^{\mathbb{S}})$, where

- $\bar{\ell}^{\mathbb{S}} = \langle \ell^{\mathbb{S}}_i :\ i < \omega\rangle \in {}^{\omega}\omega$, with $\sum_{i<\omega} \ell^{\mathbb{S}}_i < \infty$,
- $\bar{\ell}^{\mathbf{0}} = \langle \ell^{\mathbf{0}}_i :\ i \in \omega\rangle \in {}^{\omega}\omega$, with $\sum_{i<\omega} \ell^{\mathbf{0}}_i < \infty$,
- $\bar{\ell}^{\mathbf{1}} = \langle \ell^{\mathbf{1}}_i :\ i \in \omega\rangle \in {}^{\omega}\omega$, with $\sum_{i<\omega} \ell^{\mathbf{1}}_i < \infty$,
- $\bar{\ell}^{\infty} = \langle \ell^{\infty}_i :\ i \in \omega\rangle \in {}^{\omega}\omega$, with $\sum_{i<\omega} \ell^{\infty}_i < \infty$,

Definition 2.15. Fix $\langle \varepsilon^{\mathbf{0}}_j :\ j \in \omega\rangle \in {}^{\omega}\lambda_{\mathbf{0}}$, $\langle \varepsilon^{\mathbf{1}}_j :\ j \in \omega\rangle \in {}^{\omega}[\lambda_{\mathbf{0}}, \lambda_{\mathbf{1}})$, $\langle \varepsilon^{\infty}_j :\ j \in \omega\rangle \in {}^{\omega}[\lambda_{\mathbf{1}}, \lambda_{\infty})$, $\langle \varepsilon^{\mathbb{S}}_j :\ j \in \omega\rangle \in {}^{\omega}[\lambda_{\infty}, \lambda_{\mathbb{S}})$. The following symbols depend on the particular fixed $\bar{\varepsilon}^{\iota}$'s, but we omit it as it will be always clear what those sequences are.

Now

($\blacktriangleright_1$) for $j \in \omega$, $n \in \omega$ let

$$T^j_n(q) = T_n(q(\varepsilon^{\mathbf{0}}_j))$$

and

$$T^j(q) = \bigcup_{n\in\omega} T^j_n(q),$$

($\blacktriangleright_2$) Suppose that $\bar{\bar{\ell}} \in \mathcal{L}$, $q \in \mathbb{Q}'$. Then $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(q)$, iff

- $\bar{s} \in {}^{\omega}({}^{<\omega}2)$ with $s_j \in {}^{\ell^{\mathbb{S}}_j}2$ $(j < \omega)$,

- $\bar{u} \in \prod_{j<\omega} T^j(q)$ with $u_j \in T^j_{\ell^{\mathbf{0}}_j}(q)$ $(j < \omega)$,
- $\bar{v} \in {}^{\omega}({}^{<\omega}2)$ with $v_j \in {}^{\ell^{\mathbf{1}}_j}2$ $(j < \omega)$,
- $\bar{w} \in {}^{\omega}({}^{<\omega}2)$ with $w_j \in {}^{\ell^{\infty}_j}2$ $(j < \omega)$.

$(\blacktriangleright_3)$ For each

$$\begin{array}{ll} \bar{u} = \langle u_j : j \in \omega\rangle & \in \prod_{j\in\omega} T^j(q)), \\ \bar{v} = \langle v_j : j \in \omega\rangle & \in \prod_{j\in\omega} ({}^{\omega>}2), \\ \bar{w} = \langle v_j : j \in \omega\rangle & \in \prod_{j\in\omega} ({}^{\omega>}2), \\ \bar{s} = \langle s_j : j \in \omega\rangle & \in \prod_{j\in\omega} ({}^{\omega>}2), \end{array}$$

we let $q^{(\bar{u},\bar{v},\bar{w},\bar{s})} \in \mathbb{Q}$ be defined as
- $q^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon^{\mathbf{0}}_j) = q(\varepsilon^{\mathbf{0}}_j)^{(u_j)}$ $(j \in \omega)$,
- $q^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon^{\mathbf{1}}_j) = q(\varepsilon^{\mathbf{1}}_j)^{(v_j)}$ $(j \in \omega)$,
- $q^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon^{\infty}_j) = q(\varepsilon^{\infty}_j)^{(w_j)}$ $(j \in \omega)$,
- $q^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon^{\mathbb{S}}_j) = q(\varepsilon^{\mathbb{S}}_j)^{(s_j)}$ $(j \in \omega)$,

Definition 2.16.

Observe that

$$\text{whenever } \bar{\bar{\ell}} \in \mathcal{L} : \ |\mathbf{seq}_{\bar{\bar{\ell}}}(q)| < \aleph_0, \tag{2.4}$$

since $\bar{\bar{\ell}}$ has a finite sum.

Definition 2.17. Assuming the sequences $\bar{\varepsilon}^\iota$ are as in Definition 2.15 ($\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\}$, and $\bar{\bar{\ell}} \in \mathcal{L}$, we let the partial order $\leq_{\bar{\bar{\ell}}}$ defined by

$$p \leq_{\bar{\bar{\ell}}} q \iff \begin{array}{l} \bullet_1\ (p \leq q) \wedge \\ \bullet_2\ \mathbf{seq}_{\bar{\bar{\ell}}}(p) = \mathbf{seq}_{\bar{\bar{\ell}}}(q) \wedge \\ \bullet_3\ \forall (\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(q) : \ p^{(\bar{u},\bar{v},\bar{w},\bar{s})} \leq q^{(\bar{u},\bar{v},\bar{w},\bar{s})}. \end{array}$$

Note the following easy corollaries of our definitions:

Observation 2.18. *Let $\bar{\varepsilon}^\iota$ $(\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\})$ be as in Definition 2.15, and $\bar{\bar{\ell}} \in \mathcal{L}$ be given.*

Then, if $p \geq q \in \mathbb{Q}$ holds, then $p \geq_{\bar{\bar{\ell}}} q$, iff
- *for each $j \in \omega$: $T^j_{\ell^{\mathbf{0}}_j}(p) = T^j_{\ell^{\mathbf{0}}_j}(q)$, and*
- *for each $j \in \omega$ and $u \in {}^{\ell^{\mathbf{1}}_j}2$ we have $\mathfrak{t}^{\mathbf{1},j}_u(p) = \mathfrak{t}^{\mathbf{1},j}_u(q)$, and*
- *for each $j \in \omega$ and $v \in {}^{\ell^{\infty}_j}2$: $\mathfrak{t}^{\infty,j}_v(p) = \mathfrak{t}^{\infty,j}_v(q)$, and*
- *for each $j \in \omega$ and $s \in {}^{\ell^{\mathbb{S}}_j}2$: $\mathfrak{t}^{\mathbb{S},j}_s(p) = \mathfrak{t}^{\mathbb{S},j}_s(q)$.*

Observation 2.19. *If $p, q \in \mathbb{Q}$, $\bar{\bar{\ell}} \in \mathcal{L}$, $\bar{\varepsilon}^\iota$ $(\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\})$ are as in Definition 2.15, $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(p)$, and $q \leq p^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ then for some $q_* \leq_{\bar{\bar{\ell}}} p$,*

$$q_*^{(\bar{u},\bar{v},\bar{w},\bar{s})} \Vdash q \in \mathbf{G},$$

even $q_*^{(\bar{u},\bar{v},\bar{w},\bar{s})} = q$, if in addition $p^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon^{\infty}_j) \geq_{\ell^{\infty}_j} q(\varepsilon^{\infty}_j)$ holds for each j.

Moreover, we can assume that whenever $m \in \omega$, and $s' \in {}^{\omega>}2$ is not comparable with s_m (i.e. $s' \not\subseteq s_m$, $s' \not\supseteq s_m$), then $q_*(\varepsilon^{\mathbf{0}}_m) \in \mathbb{P}^{\mathbf{0}}$ satisfies $q_*(\varepsilon^{\mathbf{0}}_m)^{(s')} = p(\varepsilon^{\mathbf{0}}_m)^{(s')}$.

Note that we cannot expect above $q_*^{(\bar{u},\bar{v},\bar{w},\bar{s})} = q$ to hold in general, since on coordinates of the form ε^{∞}_j possibly $q(\varepsilon^{\infty}_j)_0 = \{\bar{t}\}$, where $\bar{t} = (\bar{t}^*)^0 \frown (\bar{t}^*)^1$ with $(\bar{t}^*)^0 \in q_*^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon^{\infty}_j)_0$, and $(\bar{t}^*)^1 \in q_*^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon^{\infty}_j)_1$.

Observation 2.20. *If $p \geq q \in \mathbb{Q}$, $\bar{\bar{\ell}} \in \mathcal{L}$, $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(q)$, then $q^{(\bar{u},\bar{v},\bar{w},\bar{s})} \leq p^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}$ for some $(\bar{u}', \bar{v}', \bar{w}', \bar{s}') \in \mathbf{seq}_{\bar{\bar{\ell}}}(p)$.*

The next claim verifies the properness part of $(\circledast)_1$, and $(\circledast)_2$.

Claim 2.21. *Let $q \in \mathbb{Q}$, $D_0, D_1, \dots, D_i, \dots$ be a countable sequence of maximal antichains of $\mathbb{Q}$. Then for a suitable extension $q' \leq q$ we have that for each $i \in \omega$ q' is compatible with only finitely many elements of D_i.*

Proof. Assume that $q \in \mathbb{Q}$, and the D_j's are fixed. In what follows we will sketch a standard fusion argument for Baumgartner's Axiom A.

The following is a trivial application of Observation 2.9:

Observation 2.22. *Suppose that the sequence $\langle q_n : n \in \omega \rangle \in {}^{\omega}\mathbb{Q}$ is decreasing, the sequences $\bar{\varepsilon}^{\iota}$ ($\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\}$) are as in Definition 2.15, and for each k there is $\bar{\bar{\ell}}^k = ((\bar{\ell}^k)^{\mathbf{0}}, (\bar{\ell}^k)^{\mathbf{1}}, (\bar{\ell}^k)^{\infty}, (\bar{\ell}^k)^{\mathbb{S}}) \in \mathcal{L}$ such that*

- *for each $\alpha \in \bigcup_{n \in \omega} \mathrm{supp}(q_n)$ we have that $\alpha = \varepsilon^{\iota}_j$ for some ι and j, and*
$$\langle (\ell^k)^{\iota}_j : k \in \omega \rangle \text{ is nondecreasing, converging to } \infty,$$
- *$q_{n+1} \leq_{\bar{\bar{\ell}}^n} q_n$ holds for each n.*

Then there exists a common lower bound $q_\omega \in \mathbb{Q}$ of the sequence $\langle q_n : n \in \omega \rangle$. Moreover, there exists q_ω for which for each n $q_\omega \leq_{\bar{\bar{\ell}}^n} q_n$.

We will define the sequences $\langle q_i : i < \omega \rangle$, $\langle \varepsilon^{\mathbf{0}}_i : i \in \omega \rangle$, $\langle \varepsilon^{\mathbf{1}}_i : i \in \omega \rangle$, $\langle \varepsilon^{\infty}_i : i < \omega \rangle$ $\langle \varepsilon^{\mathbb{S}}_i : i < \omega \rangle$, $\langle \bar{\bar{\ell}}^i : i \in \omega \rangle$ satisfying the following:

($\boxtimes_1$) $q_0 = q$, and for each i we have $q_i \in \mathbb{Q}$,

($\boxtimes_2$) $\{\varepsilon^{\mathbf{0}}_i : i \in \omega\} \subseteq \lambda_{\mathbf{0}}$, $\{\varepsilon^{\mathbf{1}}_i : i \in \omega\} \subseteq \lambda_{\mathbf{1}} \setminus \lambda_{\mathbf{0}}$, $\{\varepsilon^{\infty}_i : i < \omega\} \subseteq \lambda_\infty \setminus \lambda_{\mathbf{1}}$, $\{\varepsilon^{\mathbb{S}}_i : i < \omega\} \subseteq \lambda_{\mathbb{S}} \setminus \lambda_\infty$,

($\boxtimes_3$) for each n $\mathrm{supp}(q_n) \subseteq \{\varepsilon^{\iota}_j : \iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\},\ j \in \omega\}$,

($\boxtimes_4$) for each $\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\}$ and $j \in \omega$ the sequence
$$\langle (\ell^n)^{\iota}_j : n \in \omega \rangle \text{ is nondecreasing, and tends to } \infty,$$

($\boxtimes_5$) $\forall n$ $q_n \geq_{\bar{\bar{\ell}}^n} q_{n+1}$,

($\boxtimes_6$) $\forall n$ the condition q_{n+1} is compatible with only finitely many conditions in D_n.

Provided that such sequences exist we can appeal to Observation 2.22, which will complete the proof of Claim 2.21.

We can clearly define a sequence of $\bar{\bar{\ell}}^i$'s as in Observation 2.22. Now by Observation 2.18, and some standard bookkeeping arguments it is easy to see that the entire induction can be done once we specify how to define the condition $q_{n+1} \in \mathbb{Q}$ from q_n and the adequate fragment of $\bar{\varepsilon}^{\iota}$'s. This q_{n+1} will satisfy that

($\blacktriangle_1$) $q_{n+1} \leq_{\bar{\bar{\ell}}^n} q_n$,

($\blacktriangle_2$) whenever $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(q_n)$, then $(q_{n+1})^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ is compatible with exactly one element of D_n.

For this

($\circledcirc_1$) let
$$M = |\mathbf{seq}_{\bar{\bar{\ell}}^n}(q_n)|,$$
and fix an enumeration
$$\langle (\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i) : i < M \rangle \tag{2.5}$$

of $\mathbf{seq}_{\bar{\ell}^n}(q_n)$.

Note that $(\blacktriangle_2)$ includes M-many different objectives, each one is corresponding to some $(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)$ from (2.5). So

$(\circledcirc_2)$ we construct the sequence $\langle q^*_i : i \leq M \rangle$ satisfying

$$q^*_0 = q_n \geq_{\bar{\ell}^n} q^*_1 \geq_{\bar{\ell}^n} \cdots \geq_{\bar{\ell}^n} q^*_M,$$

and

$$(\forall i < M): \ (q^*_{i+1})^{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)} \Vdash p_* \in \mathbf{G}, \ \text{ for some } p_* \in D_n,$$

thus q^*_M will work (i.e. $(\blacktriangle_1)$, $(\blacktriangle_2)$ hold).

Assuming that $i < M$ and q^*_i is defined, pick $q' \leq (q^*_i)^{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)}$, such that $q' \leq p_*$ for some $p_* \in D_n$. Let $q^*_{i+1} \leq_{\bar{\ell}^n} q^*_i$, $(q^*_{i+1})^{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)} \Vdash p_* \in \mathbf{G}$ (guaranteed by Observation 2.19). Clause $(\circledcirc_2)$ clearly holds, so we are done.

$\square_{\text{Claim2.21}}$

We can turn to the proof of $(\circledast)_3$:

Claim 2.23. *For the forcing $\mathbb{Q}$ defined above clause $(\circledast)_3$ holds.*

Proof. Fix a $\mathbb{Q}$-name $\underset{\sim}{z}$ with $q \Vdash_{\mathbb{Q}} \underset{\sim}{z} \in 2^\omega$. By Claim 2.21 (and a standard density argument) we can assume, that

$(\boxtimes_1)$ $\underset{\sim}{z}$ is a $\mathbb{Q}' = \prod_{\iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\}} \mathbb{Q}^\iota_{X_\iota}$-name for some $X_{\mathbf{0}} \in [\lambda_{\mathbf{0}}]^{\aleph_0}$, $X_{\mathbf{1}} \in [\lambda_{\mathbf{1}} \setminus \lambda_{\mathbf{0}}]^{\aleph_0}$, $X_\infty \in [\lambda_\infty \setminus \lambda_{\mathbf{1}}]^{\aleph_0}$, $X_{\mathbb{S}} \in [\lambda_{\mathbb{S}} \setminus \lambda_\infty]^{\aleph_0}$,

moreover, w.l.o.g.

$(\boxtimes_2)$ $\Vdash_{\mathbb{Q}'} \underset{\sim}{z} \notin V$.

$(\boxtimes_3)$ Fix enumerations $X_{\mathbf{0}} = \{\varepsilon^{\mathbf{0}}_j : j \in \omega\}$, $X_{\mathbf{1}} = \{\varepsilon^{\mathbf{1}}_j : j \in \omega\}$, $X_\infty = \{\varepsilon^\infty_j : j \in \omega\}$, $X_{\mathbb{S}} = \{\varepsilon^{\mathbb{S}}_j : j \in \omega\}$.

$(\boxtimes_4)$ If $\varepsilon \in X_{\mathbf{1}} \cup X_\infty \cup X_{\mathbb{S}}$, $q \in \mathbb{Q}'$, $\bar{s} \in {}^{\omega>}2$ we let $q^{\{\varepsilon\},(\bar{s})} \in \mathbb{Q}'$ be defined as

$$q^{\{\varepsilon\},(\bar{s})} \restriction X_{\mathbf{0}} \cup X_{\mathbf{1}} \cup X_\infty \cup X_{\mathbb{S}} \setminus \{\varepsilon\} = (q \restriction X_{\mathbf{0}} \cup X_{\mathbf{1}} \cup X_\infty \cup X_{\mathbb{S}} \setminus \{\varepsilon\}),$$
$$q^{\{\varepsilon\},(\bar{s})}(\varepsilon) = (q(\varepsilon)^{(\bar{s})}).$$

$(\boxtimes_5)$ If $\varepsilon \in X_{\mathbf{1}} \cup X_\infty \cup X_{\mathbb{S}}$, $q, p \in \mathbb{Q}'$, $n \in \omega$, then $q \leq_{\{\varepsilon\},n} p$, if $q \leq p$ and $q(\varepsilon) \leq_n p(\varepsilon)$.

We will again need the terminology introduced in Definition 2.15.

Definition 2.24. If $\Vdash_{\mathbb{Q}'} \underset{\sim}{z} \in 2^\omega$, $n \in \omega$, $\iota \in \{\mathbf{1}, \infty\}$, then we let $q \in D^{\iota,\mathbf{un}}_n(\underset{\sim}{z})$ (where we mean **un** as an abbreviation for "unique"), iff

(*i*) $q \in \mathbb{Q}'$,

(*ii*) there exist $k \in \omega$, and $i_0 \neq i_1 \in \{0, 1\}$ for which

$(ii)_1$ $q^{\{\varepsilon^\iota_n\},(\langle 0 \rangle)} \Vdash \underset{\sim}{z}_k = i_0$,

$(ii)_2$ $q^{\{\varepsilon^\iota_n\},(\langle 1 \rangle)} \Vdash \underset{\sim}{z}_k = i_1$, and

$(ii)_3$ whenever $q \geq_{\{\varepsilon^\iota_n\},1} r$, and $r^{\{\varepsilon^\iota_n\},\langle 0 \rangle}$ or $r^{\{\varepsilon^\iota_n\},\langle 1 \rangle}$ decides $\underset{\sim}{z}_j$ for some $j \neq k$ then so does r.

Definition 2.25. If $\Vdash_{\mathbb{Q}'} \underset{\sim}{z} \in 2^\omega$, $n \in \omega$, $\iota \in \{\mathbf{1}, \infty\}$, then we let $q \in D^{\iota,\mathbf{eq}}_n(\underset{\sim}{z})$, iff whenever $q \geq_{\{\varepsilon^\iota_n\},1} r$, and $r^{\{\varepsilon^\iota_n\},(\langle 0 \rangle)}$ or $r^{\{\varepsilon^\iota_n\},(\langle 1 \rangle)}$ decides $\underset{\sim}{z}_k$ for some $k \in \omega$ then so does r.

Definition 2.26. If $\Vdash_{\mathbb{Q}'} \underset{\sim}{z} \in 2^\omega$, $n \in \omega$, $\iota \in \{\mathbf{1}, \infty\}$, then we let $q \in D^{\iota,\mathbf{mul}}_n(\underset{\sim}{z})$, iff there is no $q' \leq_{\{\varepsilon^\iota_n\},1} q$ with $q' \in D^{\iota,\mathbf{un}}_n(\underset{\sim}{z}) \cup D^{\iota,\mathbf{eq}}_n(\underset{\sim}{z})$.

Note the following:

Fact 2.27. *If* $\Vdash_{\mathbb{Q}'} \underset{\sim}{z} \in 2^\omega$, $n \in \omega$, $\iota \in \{\mathbf{1}, \infty\}$, *then*

(1) $D_n^{\iota,\boldsymbol{un}}(\underset{\sim}{z}) \cup D_n^{\iota,\boldsymbol{mul}}(\underset{\sim}{z}) \cup D_n^{\iota,\boldsymbol{eq}}(\underset{\sim}{z})$ *is dense (in fact, even* $\leq_{\{\varepsilon_n^\iota\},1}$*-dense) in* $\mathbb{Q}'$.

(2) *if* $q \in D_n^{\iota,\boldsymbol{un}}(\underset{\sim}{z}))$ $(D_n^{\iota,\boldsymbol{mul}}(\underset{\sim}{z}),\ D_n^{\iota,\boldsymbol{eq}}(\underset{\sim}{z})$, *resp.), and* $q' \leq_{\{\varepsilon_n^\iota\},1} q$, *then* $q' \in D_n^{\iota,\boldsymbol{un}}(\underset{\sim}{z}))$ $(D_n^{\iota,\boldsymbol{mul}}(\underset{\sim}{z}),\ D_n^{\iota,\boldsymbol{eq}}(\underset{\sim}{z})$, *resp.).*

(3) $D_n^{\iota,\boldsymbol{un}}(\underset{\sim}{z}),\ D_n^{\iota,\boldsymbol{mul}}(\underset{\sim}{z}),\ D_n^{\iota,\boldsymbol{eq}}(\underset{\sim}{z})$ *are pairwise disjoint.*

The proof of the present claim is by clarifying Subclaims 2.28 and 2.33:

Claim 2.28. *Let* $q \in \mathbb{Q}'$, $\Vdash_{\mathbb{Q}'} \underset{\sim}{z} \in 2^\omega$, *and* $\bar{\bar{\ell}} = (\bar{\ell}^{\mathbf{0}}, \bar{\ell}^{\mathbf{1}}, \bar{\ell}^\infty, \bar{\ell}^{\mathbb{S}}) \in \mathcal{L}$ *be given, and let* $m \in \omega$ *be fixed. Then for some* $r \in \mathbb{Q}'$, $r \leq_{\bar{\bar{\ell}}} q$, *for each* $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(r)$ *one of the following holds:*

$\odot_1^m(r^{(\bar{u},\bar{v},\bar{w},\bar{s})})$: $r^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ *forces that* $\underset{\sim}{z}$ *does not depend on* $\{p(\varepsilon_m^{\mathbb{S}}) :\ p \in \mathbf{G}\}$, *i.e. there is no* $p \in \mathbb{Q}'$, $p \leq r^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ *for which there exists* $k \in \omega$ *and* $c \in \{0,1\}$:

$$\begin{array}{lll} p^{\{\varepsilon_m^{\mathbb{S}}\},(\langle 0\rangle)} & \Vdash_{\mathbb{Q}'} & \underset{\sim}{z}_k = c, \\ p^{\{\varepsilon_m^{\mathbb{S}}\},(\langle 1\rangle)} & \Vdash_{\mathbb{Q}'} & \underset{\sim}{z}_k = 1 - c. \end{array}$$

$\odot_2^m(r^{(\bar{u},\bar{v},\bar{w},\bar{s})})$: *for each* $p \in \mathbb{Q}'$, $p \leq r^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ *there exist* $q \leq p$, $k \in \omega$, *and* $c \in \{0,1\}$ *such that:*

$$\begin{array}{lll} q^{\{\varepsilon_m^{\mathbb{S}}\},(\langle 0\rangle)} & \Vdash_{\mathbb{Q}'} & \underset{\sim}{z}_k = c, \\ q^{\{\varepsilon_m^{\mathbb{S}}\},(\langle 1\rangle)} & \Vdash_{\mathbb{Q}'} & \underset{\sim}{z}_k = 1 - c. \end{array}$$

Proof. Observe that

($\blacksquare_1$) if $p \geq r \in \mathbb{Q}'$, then $\odot_1^m(p) \to \odot_1^m(r)$, and similarly, $\odot_2^m(p) \to \odot_2^m(r)$ for every $m \in \omega$.

Note that

($\blacksquare_2$) if for $p \in \mathbb{Q}$ there is no extension $p' \leq p$ with $\odot_1^m(p')$, then $\odot_2^m(p)$ holds (and conversely),

therefore,

($\blacksquare_3$) for $n \in \omega$ the set

$$D_\odot^m = \{p \in \mathbb{Q}' :\ \odot_1^m(p)\ \vee\ \odot_2^m(p)\}$$

is dense open (and the sets $\{p \in \mathbb{Q}' :\ \odot_1^m(p)\}$, $\{p \in \mathbb{Q}' :\ \odot_2^m(p)\}$ are open).

For later reference we remark the following corollary of Observation 2.20:

Observation 2.29. *If* $r \in \mathbb{Q}'$ *is given by Subclaim* 2.28 *(for a fixed* m *and* $\bar{\bar{\ell}} \in \mathcal{L}$*), and* $r \geq_{\bar{\bar{\ell}}} r'$, *then for each* $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(r)$ *we have* $(r')^{(\bar{u},\bar{v},\bar{w},\bar{s})} \in D_\odot^m$, *i.e. either* $\odot_1^m((r')^{(\bar{u},\bar{v},\bar{w},\bar{s})})$, *or* $\odot_2^m((r')^{(\bar{u},\bar{v},\bar{w},\bar{s})})$ *holds.*

Fact 2.30. *For every* $p \in \mathbb{Q}'$, $m \in \omega$, $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(p)$ *there exists* $p' \leq_{\bar{\bar{\ell}}} p$ *for which either* $\odot_1^m((p')^{(\bar{u},\bar{v},\bar{w},\bar{s})})$, *or* $\odot_2^m((p')^{(\bar{u},\bar{v},\bar{w},\bar{s})})$ *holds.*

Proof. Using ($\blacksquare_3$) choose $p'' \leq p^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ with $p'' \in D_\odot^m$. By an argument similar to that of Observation 2.13 we can assume that $p''(\varepsilon_j^\infty) \leq_{\ell_j^\infty} p^{(\bar{u},\bar{v},\bar{w},\bar{s})}(\varepsilon_j^\infty)$ for each j, so by Observation 2.19 there exists a condition $p' \leq_{\bar{\bar{\ell}}} p$ such that $(p')^{(\bar{u},\bar{v},\bar{w},\bar{s})} = p''$. $\square_{\text{Fact2.30}}$

Since $\leq_{\bar{\bar{\ell}}}$ is a partial order, enumerating the finite set

$$\{(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}}(q)\}$$

as $\{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i) : i < M\}$ we can choose a sequence

$$q_0 = q_0^* \geq_{\bar{\bar{\ell}}} \cdots \geq_{\bar{\bar{\ell}}} q_{M-1}^* \geq_{\bar{\bar{\ell}}} q_M^*$$

requiring $(q_{i+1}^*)^{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)} \in D^m_{\odot}$ $(i < M)$ (recall Observation 2.18). Thus $r = q_M^*$ works. $\square_{\text{Subclaim2.28}}$

Now we can turn back to the proof of Claim 2.23.

Definition 2.31. Fix a sequence $\langle \xi_n : n \in \omega \rangle$ that lists

$$X_{\mathbf{0}} \cup X_{\mathbf{1}} \cup X_\infty \cup X_{\mathbb{S}} = \{\varepsilon^\iota_j : \iota \in \{\mathbf{0}, \mathbf{1}, \infty, \mathbb{S}\}, j \in \omega\}$$

with each such element occurring infinitely many times (where the X^ι's are from $(\boxtimes_3)$). Then we define

1) the sequence $\langle \bar{\bar{\ell}}^n : n \in \omega \rangle$ so that
   - $\bar{\bar{\ell}}^n = ((\bar{\ell}^n)^{\mathbf{0}}, (\bar{\ell}^n)^{\mathbf{1}}, (\bar{\ell}^n)^\infty, (\bar{\ell}^n)^{\mathbb{S}}) \in \mathcal{L}$ for each n,
   - $\bar{\bar{\ell}}^0$ consists of constant zero sequences,
   - if $\xi_n = \varepsilon^\iota_m$, then we define $\bar{\bar{\ell}}^{n+1}$ so that

$$(\ell^{n+1})^{\iota'}_k = \begin{cases} (\bar{\ell}^n)^{\iota'}_k + 1, & \text{if } \iota' = \iota \ \wedge \ k = m \\ (\bar{\ell}^n)^{\iota'}_k, & \text{otherwise.} \end{cases}$$

2) for $q \in \mathbb{Q}'$, $n \in \omega$ and $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(q)$ we define the sequence

$$\bar{t}^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})} = \langle t_j^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})} : j < n \rangle$$

   inductively as follows: if $k < n$, $K = |\{j < k : \xi_k = \xi_j\}|$ and if $\xi_k =$
   - $= \varepsilon^{\mathbf{0}}_m$, then set $t_k^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})} = u_m(K)$,
   - $= \varepsilon^{\mathbf{1}}_m$, then set $t_k^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})} = v_m(K)$,
   - $= \varepsilon^{\infty}_m$, then set $t_k^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})} = w_m(K)$,
   - $= \varepsilon^{\mathbb{S}}_m$, then set $t_k^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})} = s_m(K)$,

3) for $q \in \mathbb{Q}'$ and the finite sequence $\bar{t}'$ we let

$$\mathbf{qp}(q, \bar{t}') = (\bar{u}_*, \bar{v}_*, \bar{w}_*, \bar{s}_*),$$

   if

$$\bar{t}' = \bar{t}^{(\bar{u}_*, \bar{v}_*, \bar{w}_*, \bar{s}_*)},$$

   (where $(\bar{u}_*, \bar{v}_*, \bar{w}_*, \bar{s}_*) \in \mathbf{seq}_{\bar{\bar{\ell}}^{|\bar{t}|}}(q)$, and $\bar{t}^{(\bar{u}_*, \bar{v}_*, \bar{w}_*, \bar{s}_*)}$ is defined as above),

4) and (for $q \in \mathbb{Q}'$), $(\bar{u}, \bar{v}, \bar{w}, \bar{s})$, $(\bar{u}', \bar{v}', \bar{w}', \bar{s}') \in \bigcup_{j<\omega} \mathbf{seq}_{\bar{\bar{\ell}}^j}(q)$ we define $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \sqsubseteq (\bar{u}', \bar{v}', \bar{w}', \bar{s}')$ naturally, i.e.

$$(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \sqsubseteq (\bar{u}', \bar{v}', \bar{w}', \bar{s}'), \text{ iff } \bar{t}^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})} \subseteq \bar{t}^{(\bar{u}', \bar{v}', \bar{w}', \bar{s}')},$$

   as well as

$$(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \sqsubset (\bar{u}', \bar{v}', \bar{w}', \bar{s}'),$$

$$\text{iff } ((\bar{u}, \bar{v}, \bar{w}, \bar{s}) \sqsubseteq (\bar{u}', \bar{v}', \bar{w}', \bar{s}') \ \wedge \ (\bar{u}, \bar{v}, \bar{w}, \bar{s}) \neq (\bar{u}', \bar{v}', \bar{w}', \bar{s}'),$$

5) for $(q \in \mathbb{Q}')$, $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(q)$, $k \leq n$ we let $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \restriction k$ to be the (unique) member $(\bar{u}', \bar{v}', \bar{w}', \bar{s}')$ of $\mathbf{seq}_{\bar{\bar{\ell}}^k}(q)$ for that

$$(\bar{u}', \bar{v}', \bar{w}', \bar{s}') \sqsubseteq (\bar{u}, \bar{v}, \bar{w}, \bar{s}).$$

Observation 2.32.

a) *If* $\bar{t}^*$ *and* $\bar{t}^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ *(with* $(\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\ell}\restriction|\bar{t}^*|}(q)$*) satisfies that whenever* k *is such that* $t_k \neq t_k^{(\bar{u},\bar{v},\bar{w},\bar{s})}$*, then*
$(*)_k$ $t_k \in \{0,1\}$ *and* ξ_k *is* not *of the form* $\varepsilon_m^{\mathbf{0}}$ *(for any* m*),*
then $\mathbf{qp}(q,\bar{t}^*)$ *is defined.*

b) *If*
$$(\bar{u},\bar{v},\bar{w},\bar{s}) \neq (\bar{u}',\bar{v}',\bar{w}',\bar{s}') \in \mathbf{seq}_{\bar{\ell}^n}(q)$$
are such that $\bar{t}^{(\bar{u},\bar{v},\bar{w},\bar{s})}$ *and* $\bar{t}^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}$ *differs on exactly one coordinate, the* k*'th for which* $(*)_k$ *holds, then there exists a condition* $p' \leq_{\{\xi_k\},1} p^{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction k}$ *such that*
$$\{(p')^{\{\xi_k\},(\langle 0\rangle)}, (p')^{\{\xi_k\},(\langle 1\rangle)}\} = \{p^{(\bar{u},\bar{v},\bar{w},\bar{s})}, p^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}\}.$$

Subclaim 2.33. *Let* r*,* $\underset{\sim}{z}$ *be as in Subclaim 2.28, and* $\langle \bar{\bar{\ell}}^n : n \in \omega\rangle$ *defined in Definition 2.31. Suppose that* $r \Vdash \underset{\sim}{z} \notin V$*. Then there exists a condition* $r_* \in \mathbb{Q}'$*,* $r \geq r_*$*, and*
$$\bar{y} = \left\langle y^{(\bar{u},\bar{v},\bar{w},\bar{s})} : n \in \omega,\ (\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(r_*)\right\rangle,$$
$$\bar{\mathbf{x}} = \left\langle \mathbf{x}_{(\bar{u},\bar{v},\bar{w},\bar{s})} : n \in \omega,\ (\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(r_*)\right\rangle,$$
such that for each n

$\varphi_a(r_*,\bar{y},\bar{\mathbf{x}})$*: for each* n *and* $(\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(r_*)$*:*
- $y^{(\bar{u},\bar{v},\bar{w},\bar{s})} \in {}^{\omega>}2$,
- $\mathbf{x}_{(\bar{u},\bar{v},\bar{w},\bar{s})} \in \{\mathbf{un},\mathbf{eq},\mathbf{mul}\}$,

$\varphi_b(r_*,\bar{y})$*: for each* $m < n$*:*
- $(\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^m}(r_*)$,
- $(\bar{u}',\bar{v}',\bar{w}',\bar{s}') \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(r_*)$,

we have
$$(\bar{u},\bar{v},\bar{w},\bar{s}) \sqsubset (\bar{u}',\bar{v}',\bar{w}',\bar{s}') \Rightarrow\ y^{(\bar{u},\bar{v},\bar{w},\bar{s})} \subsetneq y^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')},$$

$\varphi_c(r_*,\bar{y})$*: for each* n*,* $(\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(r_*)$*:*
$$(r_*)^{(\bar{u},\bar{v},\bar{w},\bar{s})} \Vdash \underset{\sim}{z} \in [y^{(\bar{u},\bar{v},\bar{w},\bar{s})}],$$

$\varphi_d(r_*,\bar{y},\bar{\mathbf{x}})$*: if* $(\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^{n+1}}(r_*)$*, then*
1) *if* $\xi_n = \varepsilon_m^{\mathbb{S}}$ *for some* m*, then* $\odot_1^m(r_*^{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n}) \vee \odot_2^m(r_*^{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n})$ *(from Subclaim 2.28), and*
$$\mathbf{x}_{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n} = \mathbf{eq} \iff \odot_1^m(r_*^{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n}),$$
$$\mathbf{x}_{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n} = \mathbf{mul} \iff \odot_2^m(r_*^{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n}),$$
2) *if* $\xi_n = \varepsilon_m^{\iota}$*, where* $\iota \in \{\mathbf{1},\infty\}$ *for some* m*, then for each* $(\bar{u},\bar{v},\bar{w},\bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^n}(r_*)$
   - $\mathbf{x}_{(\bar{u},\bar{v},\bar{w},\bar{s})} = \mathbf{eq}$*, iff* $r_*^{(\bar{u},\bar{v},\bar{w},\bar{s})} \in D_m^{\iota,\mathbf{eq}}(\underset{\sim}{z})$*,*

$\varphi_e(r_*,\bar{y},\bar{\mathbf{x}})$*: if* $(\bar{u},\bar{v},\bar{w},\bar{s}) \neq (\bar{u}',\bar{v}',\bar{w}',\bar{s}') \in \mathbf{seq}_{\bar{\bar{\ell}}^{n+1}}(r_*)$ *(for some* n*), are such that* $\bar{u} = \bar{u}'$*, then the following implications hold true:*
e1) *if* $\xi_n = \varepsilon_m^{\mathbb{S}}$ *for some* m*,* $\bar{s}_m \neq \bar{s}'_m$*, and* $\odot_2^m((r_*)^{(\bar{u},\bar{v},\bar{w},\bar{s})})$*, then there exists*
$$j \geq \ell g(y^{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n}, \ell g(y^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')\restriction n}),$$
such that $(j < \ell g(y^{(\bar{u},\bar{v},\bar{w},\bar{s})}, \ell g(y^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}))$ *and*
$$y_j^{(\bar{u},\bar{v},\bar{w},\bar{s})} \neq y_j^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}.$$

e2) *if* $\psi((\bar{u},\bar{v},\bar{w},\bar{s}),(\bar{u}',\bar{v}',\bar{w}',\bar{s}'))$, *under which we mean that* ($\bar{u}=\bar{u}'$, *and) for each* $k<n+1$ *either*
- $t_k^{(\bar{u},\bar{v},\bar{w},\bar{s})}=t_k^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}$, *or*
- $\mathbf{x}_{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction k-1}=\mathbf{eq}$,

then $y^{(\bar{u},\bar{v},\bar{w},\bar{s})}=y^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}$,

e3) *if* $\xi_n=\varepsilon^{\mathbf{1}}_m$ *or* ε^{∞}_m *for some* m, *and*
 (i) $\psi((\bar{u},\bar{v},\bar{w},\bar{s})\restriction n,(\bar{u}',\bar{v}',\bar{w}',\bar{s}')\restriction n)$, *but*
 (ii) $\mathbf{x}_{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n}\neq\mathbf{eq}$ *and* $t_n^{(\bar{u},\bar{v},\bar{w},\bar{s})}\neq t_n^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}$,
 (iii) *and* ξ_n *is of the form* $\varepsilon^{\mathbf{1}}_m$, *or* ε^{∞}_m *for some* m,

then
- *if* $\xi_n=\varepsilon^{\mathbf{1}}_m$, *then*
$$\exists i<\ell g(y^{(\bar{u},\bar{v},\bar{w},\bar{s})}),\ell g(y^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}):\\ y_i^{(\bar{u},\bar{v},\bar{w},\bar{s})}\neq y_i^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')},$$
moreover, if this i *is unique, then*
$$y^{(\bar{u},\bar{v},\bar{w},\bar{s})}\restriction i=y^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}\restriction i\neq\mathfrak{s}_i$$
(where $\bar{\mathfrak{s}}$ *is from* (x_1), (x_2)*),*
- *if* $\xi_n=\varepsilon^{\infty}_m$, *then*
$$\exists i<i'<\ell g(y^{(\bar{u},\bar{v},\bar{w},\bar{s})}),\ell g(y^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}):\\ y_i^{(\bar{u},\bar{v},\bar{w},\bar{s})}\neq y_i^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}\quad\text{and}\\ y_{i'}^{(\bar{u},\bar{v},\bar{w},\bar{s})}\neq y_{i'}^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')},$$

First we verify Claim 2.23 provided the extension r_* of r and the $y^{(\bar{u},\bar{v},\bar{w},\bar{s})}$'s given by Subclaim 2.33, i.e. satisfying $\varphi_a(r_*,\bar{y},\bar{\mathbf{x}})$–$\varphi_e(r_*,\bar{y},\bar{\mathbf{x}})$. First define
$$T_*=\{y^{(\bar{u},\bar{v},\bar{w},\bar{s})}:\ (\bar{u},\bar{v},\bar{w},\bar{s})\in\bigcup_{n\in\omega}\mathbf{seq}_{\bar{\bar{\ell}}^n}(r^*)\},$$
and note that
$$r_*\Vdash\underset{\sim}{z}\in[T_*].$$
As
$$\mathbb{Q}'\simeq\mathbb{Q}^{\mathbf{0}}_{X_\mathbf{0}}\times\mathbb{Q}^{\mathbf{1}}_{X_\mathbf{1}}\times\mathbb{Q}^{\infty}_{X_\infty}\times\mathbb{Q}^{\mathbb{S}}_{X_\mathbb{S}},$$
we can

(1) first add a $\mathbb{Q}^{\mathbf{0}}_{X_\mathbf{0}}$-generic filter $\mathbf{G}_{X_\mathbf{0}}$ to V with $r_*\restriction X_\mathbf{0}\in\mathbf{G}_{X_\mathbf{0}}$, and define $T_\mathbf{0}\in V[\mathbf{G}_{X_\mathbf{0}}]$, such that $[T_\mathbf{0}]$ is $\mathbb{G}_0(\bar{\mathfrak{s}})$-independent, and
$$V[\mathbf{G}_{X_\mathbf{0}}]\models\text{“}r_*\restriction(X_\mathbf{1}\cup X_\infty\cup X_\mathbb{S})\Vdash_{\mathbb{Q}'\restriction(X_\mathbf{1}\cup X_\infty\cup X_\mathbb{S})}\underset{\sim}{z}\in[T_\mathbf{0}]\text{”},\tag{2.6}$$
or

(2) add a $\mathbb{Q}^{\mathbf{0}}_{X_\mathbf{0}}\times\mathbb{Q}^{\mathbf{1}}_{X_\mathbf{1}}$-generic filter $\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}}$ to V with $r_*\restriction(X_\mathbf{0}\cup X_\mathbf{1})\in\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}}$, and define $T_\mathbf{1}\in V[\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}}]$, such that $[T_\mathbf{1}]$ is $\mathbb{G}_1$-independent, and
$$V[\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}}]\models\text{“}r_*\restriction(X_\infty\cup X_\mathbb{S})\Vdash_{\mathbb{Q}'\restriction(X_\infty\cup X_\mathbb{S})}\underset{\sim}{z}\in[T_\mathbf{1}]\text{”},\tag{2.7}$$
or

(3) add a $\mathbb{Q}^{\mathbf{0}}_{X_\mathbf{0}}\times\mathbb{Q}^{\mathbf{1}}_{X_\mathbf{1}}\times\mathbb{Q}^{\infty}_{X_\infty}$-generic filter $\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}\cup X_\infty}$ to V with $r_*\restriction(X_\mathbf{0}\cup X_\mathbf{1}\cup X_\infty)\in\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}\cup X_\infty}$, and define $T_\infty\in V[\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}\cup X_\infty}]$, such that $[T_\infty]$ is E_0-independent, and
$$V[\mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}\cup X_\infty}]\models\text{“}r_*\restriction(X_\mathbb{S})\Vdash_{\mathbb{Q}'\restriction X_\mathbb{S}}\underset{\sim}{z}\in[T_\infty]\text{”}.\tag{2.8}$$

So fix the mutually generic filters $\mathbf{G}_{X_\mathbf{0}}$, $\mathbf{G}_{X_\mathbf{1}}$, $\mathbf{G}_{X_\infty}$ (containing $r_*\restriction X_\mathbf{0}$, $r_*\restriction X_\mathbf{1}$ and $r_*\restriction X_\infty$), define $T_\mathbf{0} \in V[\mathbf{G}_{X_\mathbf{0}}]$, $T_\mathbf{1} \in V[\mathbf{G}_{X_\mathbf{0}} \times \mathbf{G}_{X_\mathbf{1}}]$, $T_\infty \in V[\mathbf{G}_{X_\mathbf{0}} \times \mathbf{G}_{X_\mathbf{1}} \times \mathbf{G}_{X_\infty}]$ as follows:

$$T_\mathbf{0} = \{y^{(\bar u,\bar v,\bar w,\bar s)} : (\bar u,\bar v,\bar w,\bar s) \in \bigcup_{n\in\omega} \mathbf{seq}_{\bar{\bar\ell}^n}(r_*) : r_*^{(\bar u,\bar v,\bar w,\bar s)}\restriction X_\mathbf{0} \in \mathbf{G}_{X_\mathbf{0}}\},$$

$$T_\mathbf{1} = \{y^{(\bar u,\bar v,\bar w,\bar s)} : (\bar u,\bar v,\bar w,\bar s) \in \bigcup_{n\in\omega} \mathbf{seq}_{\bar{\bar\ell}^n}(r_*) : r_*^{(\bar u,\bar v,\bar w,\bar s)}\restriction (X_\mathbf{0} \cup X_\mathbf{1}) \in \mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}}\},$$

$$T_\infty = \{y^{(\bar u,\bar v,\bar w,\bar s)} : (\bar u,\bar v,\bar w,\bar s) \in \bigcup_{n\in\omega} \mathbf{seq}_{\bar{\bar\ell}^n}(r_*) : r_*^{(\bar u,\bar v,\bar w,\bar s)}\restriction (X_\mathbf{0}\cup X_\mathbf{1}\cup X_\infty) \in \mathbf{G}_{X_\mathbf{0}\cup X_\mathbf{1}\cup X_\infty}\},$$

(Recalling Definition 2.7), for each fixed n

$$\{r_*^{(\bar u,\bar v,\bar w,\bar s)} : (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^n}(r_*)\} \text{ is predense below } r_*,$$

hence a standard density argument implies (2.6), (2.7), and (2.8). It remains to check that $[T_\mathbf{0}]$ ($[T_\mathbf{1}]$, $[T_\infty]$, resp.) is indeed $\mathbb{G}_0(\bar{\mathfrak{s}})$- ($\mathbb{G}_1$-, E_0-, resp.)-independent. For these one only needs to check the following assertions (using $\varphi_b(r^*,\bar y)$, $\varphi_e(r^*,\bar y)$ from Subclaim 2.33), which is left to the reader:

- For every branch $\langle b_i : i \in \omega\rangle$ in T_* there is an infinite sequence $\langle (\bar u^i,\bar v^i,\bar w^i,\bar s^i) : i \in \omega\rangle$, such that $(\bar u^i,\bar v^i,\bar w^i,\bar s^i) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_*)$ with
$$(\bar u^i,\bar v^i,\bar w^i,\bar s^i) \sqsubseteq (\bar u^{i+1},\bar v^{i+1},\bar w^{i+1},\bar s^{i+1}),$$
and $b_i = y^{(\bar u^i,\bar v^i,\bar w^i,\bar s^i)}$ (for each $i \in \omega$) (recall that $\mathbf{seq}_{\bar{\bar\ell}^i}(r_*)$ is finite by (2.4), and use Kőnig's theorem).
- If $\langle (\bar u^i,\bar v^i,\bar w^i,\bar s^i) : i \in \omega\rangle$ and $\langle ((\bar u')^i,(\bar v')^i,(\bar w')^i,(\bar s')^i) : i \in \omega\rangle$ are different, $\sqsubseteq$-increasing, $(\bar u^i,\bar v^i,\bar w^i,\bar s^i), ((\bar u')^i,(\bar v')^i,(\bar w')^i,(s')^i) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_*)$, and we have $\bar u^i = (\bar u')^i$ for each i, then at least one of the following holds:
  - $(\wr)_1$ for each n the premise $e2$) (from $\varphi_e(r_*,\bar y)$) holds, and thus
  $$y^{(\bar u^n,\bar v^n,\bar w^n,\bar s^n)} = y^{(\bar{(u')}^n,\bar{(v')}^n,\bar{(w')}^n,\bar{(s')}^n)}.$$
  - $(\wr)_2$ for some n the premise in $e1$) holds, so it holds infinitely many often (since $\{q \in \mathbb{Q}' : \odot_2^m(q)\}$ is open for arbitrary m) and so $\cup\{y^{(\bar u^n,\bar v^n,\bar w^n,\bar s^n)} : n \in \omega\}$ and $\cup\{y^{((\bar u')^n,(\bar v')^n,(\bar w')^n,(\bar s')^n)} : n \in \omega\}$ differ on infinitely many digits.
  - $(\wr)_3$ for some n the premise in $e3$) holds, and so $\cup\{y^{(\bar u^n,\bar v^n,\bar w^n,\bar s^n)} : n \in \omega\}$ and $\cup\{y^{((\bar u')^n,(\bar v')^n,(\bar w')^n,(\bar s')^n)} : n \in \omega\}$ are not connected in $\mathbb{G}_0(\bar{\mathfrak{s}})$.
- If $\langle (\bar u^i,\bar v^i,\bar w^i,\bar s^i) : i \in \omega\rangle$ and $\langle ((\bar u')^i,(\bar v')^i,(\bar w')^i,(\bar s')^i) : i \in \omega\rangle$ are different, $\sqsubseteq$-increasing, $(\bar u^i,\bar v^i,\bar w^i,\bar s^i), ((\bar u')^i,(\bar v')^i,(\bar w')^i,(\bar s')^i) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_*)$, and we have $\bar u^i = (\bar u')^i$, $\bar v^i = (\bar v')^i$ for each i, then either $(\wr)_1$ or $(\wr)_2$ holds, or
  - $(\wr)_3'$ for some n the premise in $e3$) holds,where necessarily $\xi_n = \varepsilon_m^\infty$ for some m (since $\bar v^i = (\bar v')^i$ for each i, in particular $v_m^{n+1} = (v')_m^{n+1}$) and so $\cup\{y^{(\bar u^n,\bar v^n,\bar w^n,\bar s^n)} : n \in \omega\}$ and $\cup\{y^{((\bar u')^n,(\bar v')^n,(\bar w')^n,(\bar s')^n)} : n \in \omega\}$ differ in at least two digits.
- If $\langle (\bar u^i,\bar v^i,\bar w^i,\bar s^i) : i \in \omega\rangle$ and $\langle ((\bar u')^i,(\bar v')^i,(\bar w')^i,(\bar s')^i) : i \in \omega\rangle$ are different, $\sqsubseteq$-increasing, $(\bar u^i,\bar v^i,\bar w^i,\bar s^i), ((\bar u')^i,(\bar v')^i,(\bar w')^i,(\bar s')^i) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_*)$, and we have $\bar u^i = (\bar u')^i$, $\bar v^i = (\bar v')^i$, $\bar w^i = (\bar w')^i$ for each i, then either $(\wr)_1$ or $(\wr)_2$ holds.

Note that the assertions above are absolute between transitive models.

Proof. (Subclaim 2.33) Similarly to that in the proof of Claim 2.21

($\blacklozenge_1$) we are going to define the sequences

$$\begin{array}{l}\langle r_i :\ i < \omega\rangle,\\ \langle y^{(\bar u,\bar v,\bar w,\bar s)} :\ (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_i), i \in \omega\rangle\\ \langle \mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)} :\ (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_i), i \in \omega\rangle\end{array}$$

satisfying the requirements of the following scheme:

($\boxtimes_1$) $r_0 = r$, and for each i $r_i \in \mathbb{Q}'$,

($\boxtimes_2$) $\forall i$ $r_i \geq_{\bar{\bar\ell}^i} r_{i+1}$,

($\boxtimes_3$) for each i:

- $\varphi_a(r_i, \bar y^{*(i)}, \bar{\mathbf{x}}^{*(i-1)})$,
- $\varphi_b(r_i, \bar y^{*(i)})$,
- $\varphi_c(r_i, \bar y^{*(i)})$,
- $\varphi_d(r_i, \bar y^{*(i)}, \bar{\mathbf{x}}^{*(i-1)})$,
- $\varphi_e(r_i, \bar y^{*(i)}, \bar{\mathbf{x}}^{*(i-1)})$,

where $\bar y^{*(i)}$ is a restriction of the sequence $\bar y$ defined as

$$\bar y^{*(i)} = \langle y^{(\bar u,\bar v,\bar w,\bar s)} :\ (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^k}(r_i), k \leq i\rangle,$$

and

$$\bar{\mathbf{x}}^{*(i-1)} = \langle \mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)} :\ (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^k}(r_i),\ k \in i\rangle.$$

Again, once we have constructed the r_i's and $y^{(\bar u,\bar v,\bar w,\bar s)}$'s, we can let r_* be a common lower bound of the r_i's such that for each i $r_i \geq_{\bar{\bar\ell}^i} r_*$ holds. Then for each $i < j$ $r_i \geq_{\bar{\bar\ell}^i} r_j \geq_{\bar{\bar\ell}^i} r_*$, so by Observation 2.18 for each $j > i$:

$$\mathbf{seq}_{\bar{\bar\ell}^i}(r_i) = \mathbf{seq}_{\bar{\bar\ell}^i}(r_j) = \mathbf{seq}_{\bar{\bar\ell}^i}(r_*),$$

and for each $(\bar u,\bar v,\bar w,\bar s) \in$ $\mathbf{seq}_{\bar{\bar\ell}^i}(r_i)$ we have

$$r_i^{(\bar u,\bar v,\bar w,\bar s)} \geq r_j^{(\bar u,\bar v,\bar w,\bar s)} \geq r_*^{(\bar u,\bar v,\bar w,\bar s)}.$$

Also note that by Observation 2.29 (and recalling ($\blacksquare_3$)) if $\xi_i = \varepsilon^{\mathbb{S}}_m$ for some m, and $(\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_{i+1})$, then

$$\begin{array}{l}\odot_1^m(r_{i+1}^{(\bar u,\bar v,\bar w,\bar s)}) \iff \odot_1^m(r_*^{(\bar u,\bar v,\bar w,\bar s)}),\\ \odot_2^m(r_{i+1}^{(\bar u,\bar v,\bar w,\bar s)}) \iff \odot_2^m(r_*^{(\bar u,\bar v,\bar w,\bar s)}).\end{array}$$

Similarly, for each i, if $\xi_i = \varepsilon^\iota_m$ for some m, where $\iota \in \{\mathbf{1}, \infty\}$, and $(\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^i}(r_{i+1})$, then $r_{i+1}^{(\bar u,\bar v,\bar w,\bar s)} \in D_m^{\iota,\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)}}(\underset{\sim}{z})$ (by (($\boxtimes_3$)), recalling the definition of $\varphi_d(.,.)$ in Subclaim 2.33). So we obtain (by Observation 2.37), that

$$\forall r \leq_{\bar{\bar\ell}^{i+1}} r_{i+1} :\quad r \in D_m^{\iota,\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)}}(\underset{\sim}{z}).$$

This clearly implies that if for each i and $g \in \{a,b,c,d,e\}$ we have $\varphi_g(r_i, \bar y^{*(i)}, \bar{\mathbf{x}}^{*(i-1)})$ (or $\varphi_g(r_i, \bar y^{*(i)})$) holds, then for each $g \in \{a,b,c,d,e\}$ $\varphi_g(r_*, \bar y, \bar{\mathbf{x}})$ (or $\varphi_g(r_*, \bar y)$) holds, too, as desired.

($\blacklozenge_2$) So suppose that we have set $r_0 = r$, and we have already defined

$$r_0 \geq_{\bar{\bar\ell}^0} r_1 \geq_{\bar{\bar\ell}^1} r_2 \geq_{\bar{\bar\ell}^2} \cdots \geq_{\bar{\bar\ell}^{n-1}} r_n$$

satisfying ($\boxtimes_1$)-($\boxtimes_3$).

Depending on the value of ξ_n we do the following.

($\blacklozenge_3$)(i) Case i: $\xi_n = \varepsilon^{\mathbf{0}}_m$ for some m:

Let $N = |\mathbf{seq}_{\bar{\bar{\ell}}^{n+1}}(r_n)|$, and fix an enumeration $\langle(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j) : j < N\rangle$. Then defining the $\leq_{\bar{\bar{\ell}}^{n+1}}$-decreasing sequence $\langle p_j : j \leq M\rangle$ with $p_0 = r_n$, and $p_{j+1}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)}$ deciding the value of $\underset{\sim}{z}_k$ (recalling the first part of Observation 2.19), where $k = \ell g(y^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)\restriction n})$, setting $r_{n+1} = p_N$ works.

$(\blacklozenge_3)(ii)$ Case ii: $\xi_n = \varepsilon_m^{\mathbb{S}}$ (for some m).

$(\blacklozenge_3)(ii)_1$ Let $M = |\mathbf{seq}_{\bar{\bar{\ell}}^n}(r_n)|$, and fix an enumeration $\langle(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j) : j < M\rangle$. Then (again by Observation 2.19) defining the $\leq_{\bar{\bar{\ell}}^n}$-decreasing sequence $\langle p_j : j \leq M\rangle$ with $p_0 = r_n$, and

$$\odot_1^m(p_{j+1}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)}) \vee \odot_2^m(p_{j+1}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})$$

and set $r_n^* = p_M$ (so easily

$$\mathbf{seq}_{\bar{\bar{\ell}}^n}(r_n) = \mathbf{seq}_{\bar{\bar{\ell}}^n}(r_n^*), \tag{2.9}$$

and

$$\forall j < M : \odot_1^m((r_n^*)^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)}) \vee \odot_2^m((r_n^*)^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})). \tag{2.10}$$

$(\blacklozenge_3)(ii)_2$ Now let

$$Y = \{\langle(\bar{u}, \bar{v}, \bar{w}, \bar{s}), (\bar{u}', \bar{v}', \bar{w}', \bar{s}')\rangle : \quad (\bar{u}, \bar{v}, \bar{w}, \bar{s}), (\bar{u}', \bar{v}', \bar{w}', \bar{s}') \in \mathbf{seq}_{\bar{\bar{\ell}}^{n+1}}(r_n^*), \; \bar{u} = \bar{u}'\}|,$$

$$N = |Y|,$$

and fix the enumeration

$$\big\langle\langle(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j), ((\bar{u}')^j, (\bar{v}')^j, (\bar{w}')^j, (\bar{s}')^j)\rangle : j < N\big\rangle \text{ of } Y.$$

We are going to construct

- the sequence
$$q_0 = r_n^* \geq_{\bar{\bar{\ell}}^{n+1}} q_1 \geq_{\bar{\bar{\ell}}^{n+1}} q_2 \geq_{\bar{\bar{\ell}}^{n+1}} \cdots \geq_{\bar{\bar{\ell}}^{n+1}} q_N,$$
- and for each $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^{n+1}}(r_n^*)$
$$x_0^{(\bar{u},\bar{v},\bar{w},\bar{s})} \subseteq x_1^{(\bar{u},\bar{v},\bar{w},\bar{s})} \subseteq \cdots \subseteq x_N^{(\bar{u},\bar{v},\bar{w},\bar{s})}$$

  with
  - (a) $x_k^{(\bar{u},\bar{v},\bar{w},\bar{s})} \in {}^{\omega>}2$ $(k \leq N)$,
  - (b) $x_0^{(\bar{u},\bar{v},\bar{w},\bar{s})} = y^{(\bar{u},\bar{v},\bar{w},\bar{s})\restriction n}$,
  - (c) and $q_k^{(\bar{u},\bar{v},\bar{w},\bar{s})} \Vdash \underset{\sim}{z} \in [x_k^{(\bar{u},\bar{v},\bar{w},\bar{s})}]$,
  - (d) and for each $k < N$
  $$x_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)} \subsetneq x_{k+1}^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)},$$
  $$x_k^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)} \subsetneq x_{k+1}^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)},$$
  - (e) for each $k < N$ if $s_m^k \neq (s')_m^k$ and $\odot_2^m((p_k)^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)})$ (from Subclaim 2.28) hold, then for some
  $$j \geq \ell g(x_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}), \ell g(x_k^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)})$$
  $$x_{k+1}^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}(j) \neq x_{k+1}^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)}(j).$$

Before constructing the q_k's $(k \leq N)$ and $x_k^{(\bar{u},\bar{v},\bar{w},\bar{s})}$'s $((\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^{n+1}}(r_n^*))$

$(\blacklozenge_3)(ii)_3$ we clarify why setting

$r_{n+1} = q_N,$
$y^{(\bar u,\bar v,\bar w,\bar s)} =$ the maximal element in $\{x \in {}^{\omega>}2 : q_N^{(\bar u,\bar v,\bar w,\bar s)} \Vdash x \in \underset{\sim}{z}\} (\supseteq x_N^{(\bar u,\bar v,\bar w,\bar s)})$

works for our purposes (i.e. satisfying $(\boxtimes_2)$, $(\boxtimes_3)$):

First recall that $r_{n+1} \le r_0 \Vdash \underset{\sim}{z} \notin V$, so the maximal elements above do really exist, and so the $y^{(\bar u,\bar v,\bar w,\bar s)}$'s are well defined finite sequences.

First, (d) clearly implies $\varphi_b(r_{n+1}, \bar y^{*(n+1)})$. Second, as $p_0 = r_n \ge_{\bar{\bar\ell}^n} p_M = r_n^* \ge_{\bar{\bar\ell}^{n+1}} q_N = r_{n+1}$ recalling Observation 2.18 clearly $\mathbf{seq}_{\bar{\bar\ell}^n}(r_n^*) = \mathbf{seq}_{\bar{\bar\ell}^n}(r_{n+1})$ holds. Recalling $(\blacklozenge_3)(ii)_1$ for each $j < M$ either

$$\odot_1^m((p_{j+1})^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)}), \text{ or } \odot_2^m((p_{j+1})^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})$$

(where $(\bar u^j, \bar v^j, \bar w^j, \bar s^j)$ is meant as the j'th entry on the list in $(\blacklozenge_3)(ii)_1$). Now (by $p_k \ge_{\bar{\bar\ell}^n} p_M = r_n^* \ge_{\bar{\bar\ell}^n} r_{n+1}$) clearly either

$$\odot_1^m((r_{n+1})^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})), \text{ or } \odot_2^m((r_{n+1})^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})$$

holds, and $\varphi_d(r_{n+1}, \bar y^{*(n+1)})$ follows. Finally, for $\varphi_e(r_{n+1}, \bar y^{*(n+1)}$ we need to check clause $e2$), i.e. when there is no reason for $y^{(\bar u,\bar v,\bar w,\bar s)}$ and $y^{(\bar u',\bar v',\bar w',\bar s')}$ to be different, then these two are the same, which follows from the next claim (applying to $r_{n+1} = q_N$), so $(\boxtimes_3)$ holds, indeed:

Subclaim 2.34. *Assume that* $p \in \mathbb{Q}'$, $(\bar u, \bar v, \bar w, \bar s), (\bar u', \bar v', \bar w', \bar s') \in \mathbf{seq}_{\bar{\bar\ell}^{n+1}}(p)$, *such that* $\bar u = \bar u'$, *and whenever* $k \le n$

- *if* $\xi_k = \varepsilon_m^\iota$, *for some* $\iota \in \{\mathbf{1}, \infty\}$ *and* $m \in \omega$, *then* $p^{(\bar u,\bar v,\bar w,\bar s)\restriction k} \in D_m^{\iota,\mathbf{eq}}(\underset{\sim}{z})$,
- *if* $\xi_k = \varepsilon_m^{\mathbb{S}}$, *for some* $m \in \omega$, *then* $\odot_1^m(p^{(\bar u,\bar v,\bar w,\bar s)\restriction k})$,

Then for every $x \in {}^{\omega>}2$

$$p^{(\bar u,\bar v,\bar w,\bar s)} \Vdash \underset{\sim}{z} \in [x] \iff p^{(\bar u',\bar v',\bar w',\bar s')} \Vdash \underset{\sim}{z} \in [x].$$

Proof. Fix $x \in {}^{\omega>}2$, and let

- $(\bar u_*^0, \bar v_*^0, \bar w_*^0, \bar s_*^0) = (\bar u', \bar v', \bar w', \bar s') \in \mathbf{seq}_{\bar{\bar\ell}^{n+1}}(p)$,
- $t^0 = t^{(\bar u_*^0,\bar v_*^0,\bar w_*^0,\bar s_*^0)}$,
- $(\bar u_*^{n+1}, \bar v_*^{n+1}, \bar w_*^{n+1}, \bar s_*^{n+1}) = (\bar u, \bar v, \bar w, \bar s) \in \mathbf{seq}_{\bar{\bar\ell}^{n+1}}(p)$,
- $t^{n+1} = t^{(\bar u_*^{n+1},\bar v_*^{n+1},\bar w_*^{n+1},\bar s_*^{n+1})}$,

as defined in 2) from Definition 2.31. Now set

$$\bar t^k = \bar t^{(\bar u_*^{n+1},\bar v_*^{n+1},\bar w_*^{n+1},\bar s_*^{n+1})}\restriction[0,k) \cup (\bar t^{(\bar u_*^0,\bar v_*^0,\bar w_*^0,\bar s_*^0)}\restriction[k, n+1) \quad (k \le n+1),$$

and let

$$(\bar u_*^k, \bar v_*^k, \bar w_*^k, \bar s_*^k) = \mathbf{qp}(p, \bar t^k)$$

(which exists by clause a) from Observation 2.32). Let $p_k = p^{(\bar u_*^k,\bar v_*^k,\bar w_*^k,\bar s_*^k)}$ for $k \le n+1$, observe that $p^{(\bar u,\bar v,\bar w,\bar s)} = p_{n+1}$, $p^{(\bar u',\bar v',\bar w',\bar s')} = p_0$. We claim that

$$\text{for each } k \le n: \ p_k \Vdash \underset{\sim}{z} \in [x] \iff p_{k+1} \Vdash \underset{\sim}{z} \in [x], \tag{2.11}$$

which will complete the proof of Subclaim 2.34.

Fix $k \le n+1$, observe that $\bar t^k$ and $\bar t^{k+1}$ differ by at most a single digit, and if $t_k^k \ne t_k^{k+1}$, then $\xi_k = \varepsilon_m^\iota$ for some $\iota \in \{\mathbf{1}, \infty, \mathbb{S}\}$, and m, by the assumptions of the subclaim. Therefore, clause b) from Observation 2.32 implies that there exists $p'_k \leq_{\{\varepsilon^\iota_m\},1} p^{(\bar{u}^k_*,\bar{v}^k_*,\bar{w}^k_*,\bar{s}^k_*)\restriction k}$, and

$$\{(p'_k)^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}, (p'_k)^{\{\varepsilon^\iota_m\},(\langle 1\rangle)}\} = \{p_k, p_{k+1}\}.$$

But by our construction $(\bar{u}^k_*, \bar{v}^k_*, \bar{w}^k_*, \bar{s}^k_*)\restriction k = (\bar{u}, \bar{v}, \bar{w}, \bar{s})\restriction k$, so by the conditions of the subclaim $p^{(\bar{u}^k_*,\bar{v}^k_*,\bar{w}^k_*,\bar{s}^k_*)\restriction k} \in D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})$. This yields that $p'_k \in D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})$ by 2 from Fact 2.27. Hence by the definition of being in $D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})$ we obtain

$$(p'_k)^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} \Vdash \underset{\sim}{z} \in [x] \iff (p'_k)^{\{\varepsilon^\iota_m\},(\langle 1\rangle)\}} \Vdash \underset{\sim}{z} \in [x],$$

and so (2.11) holds, indeed.

$\square_{\text{Subclaim2.34}}$

Now it only remains to construct the sequence promised in $(\blacklozenge_3)(ii)_2$. Assume $k < N$, and $r^*_n \geq_{\bar{\bar{\ell}}^{n+1}} q_k$ has been already chosen. Recall that $m \in \omega$ is defined so that $\xi_n = \varepsilon^{\mathbb{S}}_m$. The properties of r^*_n imply that

$$\odot^m_1(q_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)\restriction n}) \ \vee \ \odot^m_2(q_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)\restriction n}),$$

so

$$\odot^m_1(q_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}) \ \vee \ \odot^m_2(q_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}).$$

First suppose that $\odot^m_1(q_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)})$. Then it suffices to set $q_{k+1} \leq_{\bar{\bar{\ell}}^{n+1}} q_k$, such that both $q_{k+1}^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}$, and $q_{k+1}^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)}$) decide the first

$$\ell g(x_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}) + \ell g(x_k^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)}) + 1$$

digits of $\underset{\sim}{z}$, ensuring that the relevant parts in (a)-(e) from $(\blacklozenge_3)(ii)_2$ hold.

So we can turn to the case of $\odot^m_2(q_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)})$. By this property, there exists $q_* \leq q_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}$, and $j \in \omega$, such that $q_*^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 0\rangle)} \Vdash \underset{\sim}{z}_j = i_0$, $q_*^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 1\rangle)} \Vdash \underset{\sim}{z}_j = i_1$ for some $i_0 \neq i_1$ (w.l.o.g. we can assume that

$$j \geq \ell g(x_k^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)}) + \ell g(x_k^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)}) + 1).$$

W.l.o.g. $q_*(\varepsilon^\infty_l)^{(w^k_l)} = q_k(\varepsilon^\infty_l)^{(w^k_l)}$ for each $l \in \omega$, so by Observation 2.19 for some $q^*_k \leq_{\bar{\bar{\ell}}^{n+1}} q_k$ we have $(q^*_k)^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)} = q_*$, and so

$$((q^*_k)^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 0\rangle)} \Vdash \underset{\sim}{z}_j = i_0, \tag{2.12}$$

$$((q^*_k)^{(\bar{u}^k,\bar{v}^k,\bar{w}^k,\bar{s}^k)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 1\rangle)} \Vdash \underset{\sim}{z}_j = i_1. \tag{2.13}$$

For a suitable extension $q_+ \leq (q^*_k)^{((\bar{u}')^k,(\bar{v}')^k,(\bar{w}')^k,(\bar{s}')^k)}$

$$q_+ \Vdash \underset{\sim}{z}_j = i_* \text{ for some } i_* \in \{0,1\}. \tag{2.14}$$

We need the following simple fact, which uses only that if p is a Sacks condition, then when replacing p by $q \leq_1 p$, then $p^{(\langle 0\rangle)}$ and $p^{(\langle 1\rangle)}$ can be dealt with independently.

Fact 2.35. *If $r \in \mathbb{Q}'$, $(\bar{u},\bar{v},\bar{w},\bar{s}), (\bar{u}',\bar{v}',\bar{w}',\bar{s}') \in \mathbf{seq}_{n+1}(r)$, $r_+ \leq r^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')}$ (where we demand also $r_+(\varepsilon^\infty_l)^{(w'_l)} \leq r(\varepsilon^\infty_l)^{(w'_l)}$ for each $l \in \omega$), and $s_m \neq s'_m$, then there exists $r' \leq_{\bar{\bar{\ell}}} r$ such that*

- $(r')^{(\bar{u}',\bar{v}',\bar{w}',\bar{s}')} = r_+$,

- $r'(\varepsilon^{\mathbb{S}}_m)^{(s_m)} = r(\varepsilon^{\mathbb{S}}_m)^{(s_m)}$, *in particular*

$$((r')^{(\bar u,\bar v,\bar w,\bar s)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 0\rangle)} \leq (r^{(\bar u,\bar v,\bar w,\bar s)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 0\rangle)},$$

$$((r')^{(\bar u,\bar v,\bar w,\bar s)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 1\rangle)} \leq (r^{(\bar u,\bar v,\bar w,\bar s)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 1\rangle)}.$$

Applying the fact to q^*_k and q_+ yields the condition q^{**}_k with

$$(q^{**}_k)^{((\bar u')^k,(\bar v')^k,(\bar w')^k,(\bar s')^k)} = q_+, \tag{2.15}$$

and so

$$(q^{**}_k)^{((\bar u')^k,(\bar v')^k,(\bar w')^k,(\bar s')^k)} \Vdash \underset{\sim}{z}_j = i_*. \tag{2.16}$$

Furthermore, the fact gives us that

$$(q^{**}_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)} \leq_{\{\varepsilon^{\mathbb{S}}_m\},1} (q^*_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)},$$

so by (2.12) and (2.13)

$$((q^{**}_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 0\rangle)} \Vdash \underset{\sim}{z}_j = i_0, \tag{2.17}$$

$$((q^{**}_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 1\rangle)} \Vdash \underset{\sim}{z}_j = i_1. \tag{2.18}$$

Now since $i_0 \neq i_1$ either $i_0 \neq i_*$, or $i_1 \neq i_*$, w.l.o.g. we can assume that

$$i_0 \neq i_*. \tag{2.19}$$

Finally, appealing to Observation 2.19 again, there exists $q^{***}_k \leq_{\bar{\bar\ell}^{n+1}} q^{**}_k$, such that

$$(q^{***}_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)} = ((q^{**}_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)})^{\{\varepsilon^{\mathbb{S}}_m\},(\langle 0\rangle)},$$

so

$$(q^{***}_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)} \Vdash \underset{\sim}{z}_j \neq i_*. \tag{2.20}$$

This, together with (2.16) shows that setting $q_{k+1} = q^{***}_k$ works, since by possibly replacing q^{***}_k with a $\leq_{\bar{\bar\ell}^{n+1}}$-extension w.l.o.g. both $(q^{***}_k)^{(\bar u^k,\bar v^k,\bar w^k,\bar s^k)}$ and $(q^{***}_k)^{((\bar u')^k,(\bar v')^k,(\bar w')^k,(\bar s')^k)}$ decide $\underset{\sim}{z}\restriction[0,j]$.

$(\blacklozenge_3)(iii)$ Case *iii*: $\xi_n = \varepsilon^\iota_m$ for some $\iota \in \{\mathbf{1},\infty\}$ (and m).

Lemma 2.36. *Let $q_* \in \mathbb{Q}'$, $\Vdash_{\mathbb{Q}'} \underset{\sim}{z} \in 2^\omega$, $n \in \omega$ be given, suppose that $\iota \in \{\mathbf{1},\infty\}$, $m \in \omega$ are such that $\xi_n = \varepsilon^\iota_m$. Then there exists*

(i) $r_* \leq_{\bar{\bar\ell}^n} q_*$
(ii) $\bar{\mathbf{x}} = \langle \mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)} : (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^n}(q_*)\rangle$
(iii) $\bar x = \langle x^{(\bar u,\bar v,\bar w,\bar s)} : (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^{n+1}}(r_*)\rangle$

satisfying the following.

($\boxtimes$) *for each $(\bar u,\bar v,\bar w,\bar s)$ we have*
- $x^{(\bar u,\bar v,\bar w,\bar s)} \in {}^{<\omega}2$, *and*
- $\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)\restriction n} \in \{\mathbf{mul},\mathbf{eq},\mathbf{un}\}$,

such that $r_*^{((\bar u,\bar v,\bar w,\bar s)\restriction n)} \in D^{\iota,\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)\restriction n}}_m(\underset{\sim}{z})$, and whenever $(\bar u',\bar v',\bar w',\bar s') \neq (\bar u,\bar v,\bar w,\bar s)$, and $\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)\restriction n} \neq \mathbf{eq}$, then either

$(\boxtimes)_{(a)}$ $\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)\restriction n} = \mathbf{un}$, *and then* $\iota = \mathbf{1}$ *holds, when for the unique* k *satisfying*

$$k < \min(\ell g(x^{(\bar u,\bar v,\bar w,\bar s)}), \ell g(x^{(\bar u',\bar v',\bar w',\bar s')}))$$
$$x_k^{(\bar u,\bar v,\bar w,\bar s)} \neq x_k^{(\bar u',\bar v',\bar w',\bar s')},$$

we have

$$x^{(\bar u,\bar v,\bar w,\bar s)}\restriction k = x^{(\bar u,\bar v,\bar w,\bar s)}\restriction k \neq \mathfrak{s}_k$$

(where $\bar{\mathfrak{s}}$ *is from* (x_1)*-*(x_2)*),*

$(\boxtimes)_{(b)}$ *or* $\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)\restriction n} = \mathbf{mul}$, *and*

$$k_0 < k_1 < \min(\ell g(x^{(\bar u,\bar v,\bar w,\bar s)}), \ell g(x^{(\bar u',\bar v',\bar w',\bar s')})$$
$$x_{k_0}^{(\bar u,\bar v,\bar w,\bar s)} \neq x_{k_0}^{(\bar u',\bar v',\bar w',\bar s')},$$
$$x_{k_1}^{(\bar u,\bar v,\bar w,\bar s)} \neq x_{k_1}^{(\bar u',\bar v',\bar w',\bar s')}.$$

Observation 2.37. *If* n, ι, m *are as in the lemma, and* r' *satisfies*

$$\forall (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^n}(r') : \ (r')^{(\bar u,\bar v,\bar w,\bar s)} \in D_m^{\iota,\mathbf{x}^{(\bar u,\bar v,\bar w,\bar s)}}(\underset{\sim}{z}),$$

then the same statements holds for any $r'' \leq_{\bar{\bar\ell}^{n+1}} r'$.

Before proving the lemma note that

$(\blacklozenge_3)(iii)_1$ applying to $q_* = r_n$ it yields the desired condition $r_{n+1} \leq_{\bar{\bar\ell}^n} r_n$, $\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)}$'s $((\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^n}(r_{n+1})$, and setting

- $y^{(\bar u,\bar v,\bar w,\bar s)} =$ the maximal element in

$$\{x \in {}^{\omega>}2 : \ r_{n+1}^{(\bar u,\bar v,\bar w,\bar s)} \Vdash x \in \underset{\sim}{z}\}(\supseteq x_N^{(\bar u,\bar v,\bar w,\bar s)})$$

the requirements in $(\boxtimes_1)$-$(\boxtimes_3)$ are clearly satisfied: just use the same argument as after $(\blacklozenge_3)(ii)_3$, therefore finishing the case $(\blacklozenge_3)(iii)$, and the induction in $(\blacklozenge_1)$, too.

Proof. (Lemma 2.36) We are going to construct r_* regardless of the specific value of $\iota \in \{\mathbf{1}, \infty\}$. We remark that although for $\iota = \infty$ a simpler argument would also suffice, as the case of $\iota = \mathbf{1}$ itself needs a slightly more involved (and painful) reasoning, it is easier to handle the two together. First

$(\blacktriangle)_1$ we choose an enumeration $\langle (\bar u^j,\bar v^j,\bar w^j,\bar s^j) : j < M\rangle$ of all the possible quadruples $(\bar u,\bar v,\bar w,\bar s)$ from the set $\mathbf{seq}_{\bar{\bar\ell}^n}(q_*)$.

We need the following.

$(\blacktriangle)_2$ We are going to define $q_{**} \leq_{\bar{\bar\ell}^n} q_*$, as well as the sequence

$$\bar{\mathbf{x}} = \langle \mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)} : \ (\bar u,\bar v,\bar w,\bar s) \in \mathbf{seq}_{\bar{\bar\ell}^n}(q_*)\rangle,$$

with $\mathbf{x}_{(\bar u,\bar v,\bar w,\bar s)} \in \{\mathbf{un},\mathbf{mul},\mathbf{eq}\}$ satisfying the following. (For $j < M$ writing sometimes $\mathbf{x}_j$ instead of $\mathbf{x}_{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)}$) we would like the sequences to have the properties as follows:

(a) $\bigcap_{j<M}\{q \leq_{\bar{\bar\ell}^n} q_{**} : \ q^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)} \in D_m^{\iota,\mathbf{x}_j}(\underset{\sim}{z})\}$ is $\leq_{\bar{\bar\ell}^n}$-dense below q_{**},

(b) moreover, whenever $\mathbf{x}_i = \mathbf{mul}$ for some $i < M$, then for every $p \leq_{\bar{\bar\ell}^n} q_{**}$

$$p \in \bigcap_{j<i}\{q \leq_{\bar{\bar\ell}^n} q_{**} : \ q^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)} \in D_m^{\iota,\mathbf{x}_j}(\underset{\sim}{z})\}$$
$$\Rightarrow$$
$$p^{(\bar u^i,\bar v^i,\bar w^i,\bar s^i)} \in D_m^{\iota,\mathbf{mul}}(\underset{\sim}{z}).$$

Subclaim 2.38. *Suppose that ι, m, n, q_* are as in the lemma. Then there exist $q_{**} \leq_{\bar{\bar{\ell}}^n} q_*$ and $\bar{\mathbf{x}}$ satisfying the requirements in $(\blacktriangle)_2$.*

Proof. First we define q_0, $\mathbf{x}_0$ as follows.

$(\top)_1$ Set the auxiliary variable $q_0 = q_*$.

- $\bullet_1$ First, suppose that the set
$$\{q \leq_{\bar{\bar{\ell}}^n} q_0 : \ q^{(\bar{u}^0, \bar{v}^0, \bar{w}^0, \bar{s}^0)} \in D^{\iota, \mathbf{un}}_m(\underset{\sim}{z})\}$$
is $\leq_{\bar{\bar{\ell}}^n}$-dense below q_0, in which case set $\mathbf{x}_0 = \mathbf{un}$, $q_1 = q_0$.
- $\bullet_2$ Otherwise, define $q_0' \leq_{\bar{\bar{\ell}}^n} q_0$ so that there is no $q \leq_{\bar{\bar{\ell}}^n} q_0'$ with $q^{(\bar{u}^0, \bar{v}^0, \bar{w}^0, \bar{s}^0)} \in D^{\iota, \mathbf{un}}_m(\underset{\sim}{z})$.
- $\bullet_3$ Second, if the set
$$\{q \leq_{\bar{\bar{\ell}}^n} q_0' : \ q^{(\bar{u}^0, \bar{v}^0, \bar{w}^0, \bar{s}^0)} \in D^{\iota, \mathbf{eq}}_m(\underset{\sim}{z})\}$$
is $\leq_{\bar{\bar{\ell}}^n}$-dense below q_0', then we let $\mathbf{x}_0 = \mathbf{eq}$, and $q_1 = q_0'$.
- $\bullet_4$ If it is not the case, then there is $q_0'' \leq_{\bar{\bar{\ell}}^n} q_0'$ for which there is no $q \leq_{\bar{\bar{\ell}}^n} q_0''$ with $q^{(\bar{u}^0, \bar{v}^0, \bar{w}^0, \bar{s}^0)} \in D^{\iota, \mathbf{eq}}_m(\underset{\sim}{z})$.

Then

$(\top)_2$ set $\mathbf{x}_0 = \mathbf{mul}$, and $q_1 = q_0''$,

and observe that by Definition 2.26

$(\top)_3$ for each $q \leq_{\bar{\bar{\ell}}^n} q_0'' = q_1$ we have $q^{(\bar{u}^0, \bar{v}^0, \bar{w}^0, \bar{s}^0)} \in D^{\iota, \mathbf{mul}}_m(\underset{\sim}{z})$.

$(\top)_4$ This way we are going to define

- the $\leq_{\bar{\bar{\ell}}^n}$-decreasing sequence
$$q_* = q^0 \geq_{\bar{\bar{\ell}}^n} q^1 \geq_{\bar{\bar{\ell}}^n} \cdots \geq_{\bar{\bar{\ell}}^n} q^M,$$
- together with the sequence $\bar{\mathbf{x}}$

by induction on j such that for each $i \leq M$

$(\top)_4^{(a)}$ $\bigcap_{j<i} \{q \leq_{\bar{\bar{\ell}}^n} q_i : \ q^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D^{\iota, \mathbf{x}_j}_m(\underset{\sim}{z})\}$ is $\leq_{\bar{\bar{\ell}}^n}$-dense below q_i,

$(\top)_4^{(b)}$ and if $\mathbf{x}_{i-1} = \mathbf{mul}$, then for arbitrary $p \leq_{\bar{\bar{\ell}}^n} q_i$:

$$p \in \bigcap_{j<i-1} \{q \leq_{\bar{\bar{\ell}}^n} q_i : \ q^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D^{\iota, \mathbf{x}_j}_m(\underset{\sim}{z})\}$$
$$\Rightarrow$$
$$p^{(\bar{u}^{i-1}, \bar{v}^{i-1}, \bar{w}^{i-1}, \bar{s}^{i-1})} \in D^{\iota, \mathbf{mul}}_m(\underset{\sim}{z}).$$

Note that q_1 and $\mathbf{x}_0$ clearly satisfy the demands if $\mathbf{x}_0 \in \{\mathbf{un}, \mathbf{eq}\}$, and also when $\mathbf{x}_0 = \mathbf{mul}$, for which recall $(\top)_2$, $(\top)_3$.

$(\top)_5$ Suppose that $0 < i < M$, and q_i, and the $\mathbf{x}_j$'s are already defined for $j < i$.

- $\bullet_1$ Set
$$D^* = \bigcap_{j<i} \{q \leq_{\bar{\bar{\ell}}^n} q_i : \ q^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D^{\iota, \mathbf{x}_j}_m(\underset{\sim}{z})\},$$
which is $\leq_{\bar{\bar{\ell}}^n}$-dense below q_i.
- $\bullet_2$ Note that
$$q \leq_{\bar{\bar{\ell}}^{n+1}} p \iff (\forall j < M) q^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \leq_{\{\varepsilon^\iota_m\},1} p^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)}).$$
Using this observation (recalling 2 from Fact 2.27) we obtain
$$(\forall q', q \in \mathbb{Q}') : \ (q' \leq_{\bar{\bar{\ell}}^{n+1}} q, \ q \in D^*) \ \rightarrow \ (q' \in D^*). \tag{2.21}$$

Again, following the pattern of the definition of q_1 and $\mathbf{x}_0$,

$\bullet_3$ if
$$\{q \in D^* : \ q \leq_{\bar{\bar{\ell}}^n} q_i, \ q^{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)} \in D_m^{\iota, \mathbf{un}}(\underset{\sim}{z})\}$$
is $\leq \bar{\bar{\ell}}^n$-dense below q_i, then we set $\mathbf{x}_i = \mathbf{un}$, and $q_{i+1} = q_i$,

$\bullet_4$ otherwise let $q_i' \leq_{\bar{\bar{\ell}}^n} q_i$ be such that for no $q \leq_{\bar{\bar{\ell}}^n} q_i'$ do we have $q^{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)} \in D_m^{\iota, \mathbf{un}}(\underset{\sim}{z})$.

$\bullet_5$ If the set
$$\{q \in D^* : \ q \leq_{\bar{\bar{\ell}}^n} q_i' : \ q^{(\bar{u}^i, \bar{v}^i, \bar{w}^i, \bar{s}^i)} \in D_m^{\iota, \mathbf{eq}}(\underset{\sim}{z})\}$$
is $\leq_{\bar{\bar{\ell}}^n}$-dense below q_i', then we let $\mathbf{x}_i = \mathbf{eq}$, and $q_{i+1} = q_i'$.

$\bullet_6$ If it is not the case, then there is $q_i'' \leq_{\bar{\bar{\ell}}^n} q_i'$, such that
$$\forall q \in D^* : \ (q \leq_{\bar{\bar{\ell}}^n} q_i'') \ \to \ (q \notin D_m^{\iota, \mathbf{un}}(\underset{\sim}{z}) \cup D_m^{\iota, \mathbf{eq}}(\underset{\sim}{z})).$$
But then by (2.21), $\bullet_1$ and by Definition 2.26 we have that
$$\forall q \in D^* : \ (q \leq_{\bar{\bar{\ell}}^n} q_i'') \ \to \ (q \in D_m^{\iota, \mathbf{mul}}(\underset{\sim}{z})),$$
and letting $q_{i+1} = q_i''$, $\mathbf{x}_i = \mathbf{mul}$ we are done, $(\mathsf{T})_4^{(a)}$, $(\mathsf{T})_4^{(b)}$ hold for $i_* = i + 1$.

Finally, letting $q_{**} = q_M$, it is easy to check that q_{**}, $\bar{\mathbf{x}}$ are as desired. $\square_{\text{Subclaim2.38}}$

Subclaim 2.39. *Suppose that $\iota \in \{\mathbf{1}, \infty\}$, l, m, n are as in Lemma 2.36, $\bar{\mathbf{x}}$, q_{**} given by Subclaim 2.38 (i.e. satisfying the requirements in $(\blacktriangle)_2$, w.r.t. the fixed enumeration of $\mathbf{seq}_{\bar{\bar{\ell}}^n}(q_*) = \mathbf{seq}_{\bar{\bar{\ell}}^n}(q_{**})$). Then there exists a sequence*
$$q_{**} \geq_{\bar{\bar{\ell}}^n} q_0 \geq_{\bar{\bar{\ell}}^n} q_1 \geq_{\bar{\bar{\ell}}^n} \cdots \geq_{\bar{\bar{\ell}}^n} q_l$$
such that

(1) $(\forall k \leq l)\ q_k \in \bigcap_{j<M} \{q \leq_{\bar{\bar{\ell}}^n} q_{**} : \ q^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D_m^{\iota, \mathbf{x}_j}(\underset{\sim}{z})\}$,

(2) for each $k < l$, $j < M$:
$$q_{k+1}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \leq (q_k^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\langle 0 \rangle)},$$

(3) and for each $j < M$, if $\mathbf{x}_j = \mathbf{un}$, then there exist $i_0^j < i_1^j < \cdots < i_l^j$ (and $c_k^j \in \{0, 1\}$), such that
$$\begin{aligned} \forall k \leq l : \quad & (q_k^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\langle 0 \rangle)} \quad \Vdash \underset{\sim}{z}_{i_k^j} = c_k^j, \\ & (q_k^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\langle 1 \rangle)} \quad \Vdash \underset{\sim}{z}_{i_k^j} = 1 - c_k^j, \end{aligned}$$

Proof. By $(\blacktriangle)_2$ we can choose $p_0 \leq_{\bar{\bar{\ell}}^n} q_{**}$ with
$$p_0 \in D_* := \bigcap_{j<M} \{p \leq_{\bar{\bar{\ell}}^n} q_{**} : \ p^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D_m^{\iota, \mathbf{x}_j}(\underset{\sim}{z})\}. \tag{2.22}$$

This means that

$(\ddagger)_1$ whenever $j < M$ is such that $\mathbf{x}_j = \mathbf{un}$, then for some $i_0^j \in \omega$, $c_0^j \in \{0, 1\}$:
$$\begin{aligned} (p_0^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\langle 0 \rangle)} \quad & \Vdash \underset{\sim}{z}_{i_0^j} = c_0^j, \\ (p_0^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\langle 1 \rangle)} \quad & \Vdash \underset{\sim}{z}_{i_0^j} = 1 - c_0^j. \end{aligned}$$

Now for each such fixed j there is $p' \leq_{\{\varepsilon_m^\iota\}, 1} q_0^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)}$ so that both $(p')^{\{\varepsilon_m^\iota\}, (\langle 0 \rangle)}$ and $(p')^{\{\varepsilon_m^\iota\}, (\langle 1 \rangle)}$ decide $\underset{\sim}{z} \restriction [0, i_0^j)$ (in fact $q^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D_m^{\iota, \mathbf{un}}(\underset{\sim}{z})$ implies that then p' decides that initial segment). Therefore,

$(\ddagger)_2$ there exists $q_0 \le_{\bar{\bar{\ell}}^{n+1}} p_0$, for which

$$\forall j < M: \ q_0^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \parallel \underset{\sim}{z}\restriction[0,i_0^j), \tag{2.23}$$

and (automatically by (2.22) $q_0 \le_{\bar{\bar{\ell}}^{n+1}} p_0$)

$$q_0 \in D_*. \tag{2.24}$$

Now let $p_1 \le_{\bar{\bar{\ell}}^n} q_0$ be such that

$$\text{for each } j < M: \ p_1^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \le (q_0^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} \tag{2.25}$$

(in particular, $p_1 \not\le_{\bar{\bar{\ell}}^{n+1}} q_0$), and $p_1 \in D_*$. Similarly to $(\ddagger)_1$ and $(\ddagger)_2$

$(\ddagger)_3$ whenever $j < M$ is such that $\mathbf{x}_j = \mathbf{un}$, then for some $i_1^j \in \omega$, $c_1^j \in \{0,1\}$:

$$\begin{aligned}(p_1^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} &\Vdash \underset{\sim}{z}_{i_1^j} = c_1^j,\\ (p_1^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 1\rangle)} &\Vdash \underset{\sim}{z}_{i_1^j} = 1 - c_1^j,\end{aligned}$$

$(\ddagger)_4$ there exists $q_1 \le_{\bar{\bar{\ell}}^{n+1}} p_1$, for which

$$\forall j < M: \ q_1^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \parallel \underset{\sim}{z}\restriction[0,i_1^j), \tag{2.26}$$

and (automatically by $q_1 \le_{\bar{\bar{\ell}}^{n+1}} p_1$)

$$q_1 \in D_*. \tag{2.27}$$

Observe that by (2.25) and (2.23)

$(\ddagger)_5$ for each j: $i_0^j < i_1^j$.

Following this pattern, we can define the sequence by induction on $k \le l$.

$\square_{\text{Subclaim2.39}}$

Subclaim 2.40. *Suppose that $\iota \in \{\mathbf{1},\infty\}$, n, m, are as in Lemma 2.36, $\bar{\mathbf{x}}$, q_{**} given by Subclaim 2.38 (i.e. satisfying the requirements in $(\blacktriangle)_2$, w.r.t. the fixed enumeration of $\mathbf{seq}_{\bar{\bar{\ell}}^n}(q_*) = \mathbf{seq}_{\bar{\bar{\ell}}^n}(q_{**})$). Suppose that*

$$q_{**} \ge_{\bar{\bar{\ell}}^n} q_0 \ge_{\bar{\bar{\ell}}^n} q_1 \ge_{\bar{\bar{\ell}}^n} \cdots \ge_{\bar{\bar{\ell}}^n} q_l$$

such that the q_k's are given by Subclaim 2.39, so

$$(\forall k \le l) \ q_k \in \bigcap_{j<M} \{q \le_{\bar{\bar{\ell}}^n} q_{**} : \ q^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \in D_m^{\iota,\mathbf{x}_j}(\underset{\sim}{z})\}, \tag{2.28}$$

moreover,

$$\text{for each } k < l: \ \forall j < M: \ q_{k+1}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \le (q_k^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} \tag{2.29}$$

(in particular, $q_k \not\ge_{\bar{\bar{\ell}}^{n+1}} q_{k+1}$).

Then,

(1) there exists $q_{***} \le_{\bar{\bar{\ell}}^n} q_0$ (in fact, even $q_{***} \le_{\bar{\bar{\ell}}^{n+1}} q_0$), for which

$$\forall j < M, \ \forall \bar{t} \in {}^{k\ge}2: \ (q_{***}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t})} \in D_m^{\iota,\mathbf{x}_j}(\underset{\sim}{z})\},$$

and if j is such that $\mathbf{x}_j = \mathbf{un}$, then for the sequence $i_0^j < i_1^j < \cdots < i_l^j$ from Subclaim 2.39:

$$\forall \bar{t} \in {}^{k\ge}2 \ \exists c, z^*: \ \begin{aligned}(q_{***}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t}^\frown\langle 0\rangle)} &\Vdash \underset{\sim}{z}_{i_k^j} = c,\\ (q_{***}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t}^\frown\langle 1\rangle)} &\Vdash \underset{\sim}{z}_{i_k^j} = 1 - c,\\ (q_{***}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t})} &\Vdash \underset{\sim}{z}\restriction[0,i_k^j) = z^*.\end{aligned} \tag{2.30}$$

(2) *Moreover, if for each k "$=$" holds in (2.29), then q_{***} can be chosen to be q_0.*

Proof. Condition (2.29) implies that we can define the condition $q^*_{l-1} \leq_{\bar{\bar{\ell}}^{n+1}} q_{l-1}$ so that

$$(\forall j < M): \ q_l^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} = (q^*_{l-1})^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}$$

$(\circledcirc)_1$ note that replacing q_{l-1} with q^*_{l-1} still preserves $q_{l-1} \geq_{\bar{\bar{\ell}}^n} q_l$, but $q_{l-1} \not\geq_{\bar{\bar{\ell}}^* n+1} q_l$, that is, $q^*_{l-1} \geq_{\bar{\bar{\ell}}^n} q^*_l$, but $q^*_{l-1} \not\geq_{\bar{\bar{\ell}}^{n+1}} q^*_l$. Similarly, for $k = l-1$ (2.28) holds recalling that $D^{\iota,\mathbf{y}}_m$'s are closed under $\leq_{\{\varepsilon^\iota_m\},1}$-extensions if $\mathbf{y} \in \{\mathbf{un}, \mathbf{eq}, \mathbf{mul}\}$ (2 from Fact 2.27).

Doing this by downward induction on $k = l-1, l-2, \ldots, 0$,

$(\circledcirc)_2$ replacing q_k by $q^*_k \leq_{\bar{\bar{\ell}}^{n+1}} q_k$ when necessary w.l.o.g. we can assume that

$$(\forall k < l)(\forall j < M): \ q_{k+1}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} = ((q_k)^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)},$$

and introducing the

$(\circledcirc)_3$ notation $\vec{0}^k$ for the constant 0 sequence of length k, i.e. $\vec{0}^1 = \langle 0\rangle$, $\vec{0}^{k+1} = \vec{0}^k \frown \langle 0\rangle$,

$$(\forall k < l)(\forall j < M): \ q_k^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} = ((q_0)^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\vec{0}^k)}.$$

Now

$(\circledcirc)_4$ we claim that (assuming $(\circledcirc)_2$) choosing $q_{***} = q_0$ works.

$(\circledcirc)_5$ Fix $j < M$, we are going to prove that

$$(\forall \bar{t} \in {}^{l\geq}2): \ (q_0^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t})} \in D^{\iota,\mathbf{x}_j}_m(\underset{\sim}{z}),$$

$(\circledcirc)_6$ First we argue $(\circledcirc)_5$ for j's such that $\mathbf{x}_j \in \{\mathbf{un}, \mathbf{eq}\}$, and prove (2.30) as well.

So fix j with $\mathbf{x}_j \in \{\mathbf{un}, \mathbf{eq}\}$. If $\mathbf{x}_j = \mathbf{un}$, then for each $k \leq l$ for the natural number i^j_k from Subclaim 2.39 (and for some c_0, c_1)

$$\begin{aligned}(q_k^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} &\Vdash \ \underset{\sim}{z}_{i^j_k} = c_0,\\ (q_k^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 1\rangle)} &\Vdash \ \underset{\sim}{z}_{i^j_k} = c_1,\\ &\text{where } c_0 \neq c_1,\end{aligned} \tag{2.31}$$

where we also have

$$i^j_0 < i^j_1 < \cdots < i^j_l.$$

(For convenience, if $\mathbf{x}_j = \mathbf{eq}$, then we let $i^j_k = -1$ for each $k \leq l$.) Observe that the fact that $(q_k^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} \in D^{\iota,\mathbf{x}_j}_m(\underset{\sim}{z})$, where $\mathbf{x}_j \in \{\mathbf{un}, \mathbf{eq}\}$ implies that

$$\begin{aligned}&\text{whenever } p \leq_{\{\varepsilon^\iota_m\},1} q_k^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)}), \text{ and } a \in \{0,1\}:\\ &(i \neq i^j_k) \to \ [(p^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} \Vdash \underset{\sim}{z}_i = a \iff p^{\{\varepsilon^\iota_m\},(\langle 1\rangle)} \Vdash \underset{\sim}{z}_i = a].\end{aligned} \tag{2.32}$$

$(\circledcirc)_7$ Now for any $\bar{t} \in {}^{n\geq}2$, if $\mathbf{x}_j = \mathbf{un}$, then set $i^j_* = i^j_*(|\bar{t}|) = i^j_{|\bar{t}|}$, otherwise if $\mathbf{x}_j = \mathbf{eq}$, then set $i^j_* = -1$. It suffices to show that

- $\circ_1$ whenever $i \in \omega$, $i \neq i^j_*$ and $r \leq_{\{\varepsilon^\iota_m\},1} q_0^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t})}$ are such that $r^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}$, or $r^{\{\varepsilon^\iota_m\},(\langle 1\rangle)}$ decides the value of $\underset{\sim}{z}_i$, then so does r, and

$\circ_2$ if $\mathbf{x}_j = \mathbf{un}$, and so $i^j_* \geq 0$, then

$(q_0^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 0\rangle)}$, and $(q_0^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 1\rangle)}$

force different values to $\underset{\sim}{z}_{i^j_*}$.

We fix $k \leq l$, and argue $\circ_1$ and $\circ_2$ simultaneously for each $\bar t \in {}^k2$. Let $r \leq_{\{\varepsilon^m\},k+1} q_0^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)}$, (so

$$\forall \bar t \in {}^k2: \quad \begin{array}{l} r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 0\rangle)} \leq (q_0^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 0\rangle)} \\ r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 1\rangle)} \leq (q_0^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 1\rangle)}) \end{array} \tag{2.33}$$

by symmetry it is enough to show that

(2.34)
$$\forall i \neq i^j_k,\ \forall \bar t \in {}^k2: \quad (r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 0\rangle)} \Vdash \underset{\sim}{z}_i = a) \quad \Leftrightarrow \quad (r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 1\rangle)} \Vdash \underset{\sim}{z}_i = a),$$

and

$$\forall \bar t \in {}^k2: \quad (r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 0\rangle)} \Vdash \underset{\sim}{z}_{i^j_*} = a) \quad \Leftrightarrow \quad (r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 1\rangle)} \Vdash \underset{\sim}{z}_{i^j_*} = 1-a). \tag{2.35}$$

$(\circledcirc)^{(2)}_7$ We claim that for any $\bar t^* \in {}^{\omega>}2$, $d \leq k$ and $a \in \{0,1\}$, if $i \neq i^j_d$, then

$$r^{\{\varepsilon^\iota_m\},(\vec 0^d{}^\frown\langle 1\rangle^\frown \bar t^*)} \Vdash \underset{\sim}{z}_i = a \iff r^{\{\varepsilon^\iota_m\},(\vec 0^d{}^\frown\langle 0\rangle^\frown \bar t^*)} \Vdash \underset{\sim}{z}_i = a, \tag{2.36}$$

and

$$r^{\{\varepsilon^\iota_m\},(\vec 0^d{}^\frown\langle 1\rangle^\frown \bar t^*)} \Vdash \underset{\sim}{z}_{i^j_d} = a \iff r^{\{\varepsilon^\iota_m\},(\vec 0^d{}^\frown\langle 0\rangle^\frown \bar t^*)} \Vdash \underset{\sim}{z}_{i^j_d} = 1-a, \tag{2.37}$$

Before arguing $(\circledcirc)^{(2)}_7$ first we note that it would finish the proof of $\circ_1$ and $\circ_2$: For any $\bar t \in {}^k2$ and $i \in \omega$ (applying $\circ_1$ $2\cdot|\{b<k:\ t_b=1\}|$-many times) we obtain that

$$\forall a \in \{0,1\}: \ r^{\{\varepsilon^\iota_m\},(\vec 0^k{}^\frown\langle 0\rangle)} \Vdash \underset{\sim}{z}_{i^j_l} = a \iff r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 0\rangle)} \Vdash \underset{\sim}{z}_{i^j_l} = f^{t_l}(a),$$

and

$$\forall a \in \{0,1\}: \ r^{\{\varepsilon^\iota_m\},(\vec 0^k{}^\frown\langle 1\rangle)} \Vdash \underset{\sim}{z}_{i^j_l} = a \iff r^{\{\varepsilon^\iota_m\},(\bar t^\frown\langle 1\rangle)} \Vdash \underset{\sim}{z}_{i^j_l} = f^{t_l}(a),$$

where $f(a) = 1-a$, $f^0 = f^2 = f\circ f = \mathrm{id}$, and we mean 0 under t_l, when $l \geq k$. But then by $(\circledcirc)_2$, (2.33) we have $r^{\{\varepsilon^\iota_m\},(\vec 0^k)} \leq_{\{\varepsilon^\iota_m\},1} (q_k^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})$, which means

$$r^{\{\varepsilon^\iota_m\},(\vec 0^k{}^\frown\langle 0\rangle)} \leq (q_k^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}),$$

$$r^{\{\varepsilon^\iota_m\},(\vec 0^k{}^\frown\langle 1\rangle)} \leq (q_k^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\langle 1\rangle)}).$$

Now if $i = i^j_*$, then this together with (2.31) and (2.36) from $(\circledcirc)^{(2)}_7$ imply (2.35). Similarly, for $i \neq i^j_*$ (2.32) and (2.37) from $(\circledcirc)^{(2)}_7$ imply (2.34). Hence it remains to verify $(\circledcirc)^{(2)}_7$.

But (2.33) and $(\circledcirc)_2$ imply that

$$r^{\{\varepsilon^\iota_m\},(\vec 0^d{}^\frown\langle 0\rangle^\frown \bar t^*)} \leq (q_d^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\langle 0\rangle)},$$

$$r^{\{\varepsilon^\iota_m\},(\vec 0^d{}^\frown\langle 1\rangle^\frown \bar t^*)} \leq (q_d^{(\bar u^j,\bar v^j,\bar w^j,\bar s^j)})^{\{\varepsilon^\iota_m\},(\langle 1\rangle},$$

and clearly if $i = i^j_d$, then (2.31) implies (2.36), while for $i \neq i^j_d$, then (2.37) follows from (2.32).

$(\circledcirc)_8$ Now assuming that we have $(\circledcirc)_6$, we prove that (by induction on j):

$$\forall j < M,\ \forall \bar{t} \in {}^{n\geq}2:\ \mathbf{x}_m = \mathbf{mul}\ \rightarrow\ (q_0^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t})} \in D^{\iota,\mathbf{x}_m}_m(\underset{\sim}{z}).$$

Assume that $j_* < M$ is such that $\mathbf{x}_{j_*} = \mathbf{mul}$,

$$\forall j < j_*,\ \forall \bar{t} \in {}^{l\geq}2:\ (q_0^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t})} \in D^{\iota,\mathbf{x}_j}_m(\underset{\sim}{z}). \tag{2.38}$$

Fix $\bar{t} \in {}^{l\geq}2$, and suppose on the contrary, that

$$(q_0^{(\bar{u}^{j_*},\bar{v}^{j_*},\bar{w}^{j_*},\bar{s}^{j_*})})^{\{\varepsilon^\iota_m\},(\bar{t})} \notin D^{\iota,\mathbf{mul}}_m(\underset{\sim}{z}).$$

Recalling Definition 2.26 for some $p \leq_{\{\varepsilon^\iota_m\},1} (q_0^{(\bar{u}^{j_*},\bar{v}^{j_*},\bar{w}^{j_*},\bar{s}^{j_*})})^{\{\varepsilon^\iota_m\},(\bar{t})}$ we have $p \in D^{\iota,\mathbf{un}}_m(\underset{\sim}{z}) \cup D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})$. Then there is a condition $p' \leq_{\bar{\bar{\ell}}^n} q_0$, for which

$$(p')^{(\bar{u}^{j_*},\bar{v}^{j_*},\bar{w}^{j_*},\bar{s}^{j_*})}\ \in\ (D^{\iota,\mathbf{un}}_m(\underset{\sim}{z}) \cup D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})), \tag{2.39}$$

and

$$\forall j < M:\ (p')^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \leq_{\{\varepsilon^\iota_m\},1} (q_0^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)})^{\{\varepsilon^\iota_m\},(\bar{t})},$$

so

$$(p')^{(\bar{u}^{j_*},\bar{v}^{j_*},\bar{w}^{j_*},\bar{s}^{j_*})} \in D^{\iota,\mathbf{un}}_m(\underset{\sim}{z}) \cup D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z}), \tag{2.40}$$

and recalling 2 from Fact 2.27 we could infer from 2.38 that

$$\forall j < j_*:\ ((p')^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)}) \in D^{\iota,\mathbf{x}_j}_m(\underset{\sim}{z}). \tag{2.41}$$

But now, since $p' \leq_{\bar{\bar{\ell}}^n} q_0 \leq_{\bar{\bar{\ell}}^n} q_{**}$, $\mathbf{x}_{j_*} = \mathbf{mul}$, necessarily

$$(p')^{(\bar{u}^{j_*},\bar{v}^{j_*},\bar{w}^{j_*},\bar{s}^{j_*})} \in D^{\iota,\mathbf{mul}}_m(\underset{\sim}{z}),$$

contradicting (2.39) (as $D^{\iota,\mathbf{un}}_m(\underset{\sim}{z})$, $D^{\iota,\mathbf{mul}}_m(\underset{\sim}{z})$, $D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})$ are pairwise disjoint by obvious reasons (3). $\square_{\text{fascl}}$

Subclaim 2.41. *Suppose that $\iota \in \{\mathbf{1},\infty\}$, let the condition q be in $D^{\iota,\mathbf{mul}}_m(\underset{\sim}{z})$. Then some $q' \leq_{\{\varepsilon^\iota_m\},1} q$ satisfies the following:*

There exist $i_* \neq i_{**} \in \omega$, $c_*, c_{**} \in \{0,1\}$, such that

$$\begin{array}{llll} (q')^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} \Vdash & \underset{\sim}{z}_{i_*} = c_*, & (q')^{\{\varepsilon^\iota_m\},(\langle 1\rangle)} \Vdash & \underset{\sim}{z}_{i_*} = 1 - c_* \\ (q')^{\{\varepsilon^\iota_m\},(\langle 0\rangle)} \Vdash & \underset{\sim}{z}_{i_{**}} = c_{**}, & (q')^{\{\varepsilon^\iota_m\},(\langle 1\rangle)} \Vdash & \underset{\sim}{z}_{i_{**}} = 1 - c_{**}. \end{array}$$

Moreover, q' can be chosen so that both $(q')^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}$ and $(q')^{\{\varepsilon^\iota_m\},(\langle 1\rangle)}$ decide the first $\max(i_*, i_{**}) + 1$*-many digits of $\underset{\sim}{z}$.*

Proof. For each $i \in \omega$ there exists $q_+ \leq_{\{\varepsilon^\iota_m\},1} q$, such that both $q_+^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}$ and $q_+^{\{\varepsilon^\iota_m\},(\langle 1\rangle)}$ decide $\underset{\sim}{z}_i$. Since $q \notin D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})$, for some $i_* \in \omega$ and $q_+ \leq_{\{\varepsilon^\iota_m\},1} q$ the conditions $q_+^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}$ and $q_+^{\{\varepsilon^\iota_m\},(\langle 1\rangle)}$ decide about $\underset{\sim}{z}_{i_*}$ differently. Since $q_+ \notin D^{\iota,\mathbf{eq}}_m(\underset{\sim}{z})$ (again by $q_+ \leq_{\{\varepsilon^\iota_m\},1} q$ and Definition 2.26) there exists $i_{**} \neq i_*$, such that $q' \leq_{\{\varepsilon^\iota_m\},1} q_+$, and $(q')^{\{\varepsilon^\iota_m\},(\langle 0\rangle)}$ and $(q')^{\{\varepsilon^\iota_m\},(\langle 1\rangle)}$ force different values to $\underset{\sim}{z}_{i_{**}}$. $\square_{\text{Subclaim2.41}}$

Subclaim 2.42. *Assume that $\iota \in \{\mathbf{1},\infty\}$, n, m, q_* are as in Lemma 2.36, q_{**}, $\bar{\mathbf{x}}^\iota$ are as in $(\blacktriangle)_2$, moreover, there is no $j < M$ for which $\mathbf{x}_j =$***un***. Then if $r_{**} \leq_{\bar{\bar{\ell}}^n} q_{**}$ is such that*

$$r_{**} \in \bigcap_{j<M} \{q \leq_{\bar{\bar{\ell}}^n} q_{**}:\ q^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \in D^{\iota,\mathbf{x}_j}_m(\underset{\sim}{z})\},$$

then there is an $r_* \leq \bar{\bar{\ell}}^{n+1} r_{**}$ that satisfies the requirements in $(\boxtimes)$ of Lemma 2.36.

Proof. Note that all the requirements except $(\boxtimes)_{(b)}$ hold for r_{**}, and so for any $r_* \leq \bar{\bar{\ell}}^{n+1} r_{**}$ too by Observation 2.19. Now we only have to appeal to Subclaim 2.41 $|\{j < M : \mathbf{x}_j = \mathbf{mul}\}|$-many times. $\square_{\text{Subclaim2.42}}$

Subclaim 2.43. *Assume that $\iota = \infty$, n, m, q_* are as in Lemma 2.36, $q_{**} \leq_{\bar{\bar{\ell}}^n} q_*$, $\bar{\mathbf{x}}$ are as in $(\blacktriangle)_2$, and suppose that there exists $j < M$ with $\mathbf{x}_j = \mathbf{un}$. Then there exists $r_* \leq_{\bar{\bar{\ell}}^n} q_{**}$, and $\bar{\mathbf{x}}'$ that satisfy $(\boxtimes)$ of Lemma 2.36 (with some $\bar{x}$), where*

$$\begin{array}{lll} \forall j < M : & \mathbf{x}_j \in \{\mathbf{mul}, \mathbf{eq}\} & \to \ \mathbf{x}_j = \mathbf{x}'_j, \\ & \mathbf{x}_j = \mathbf{un} & \to \ \mathbf{x}'_j = \mathbf{mul}. \end{array}$$

Proof. Proceed first similarly to the proof of Subclaim 2.42, and appeal to Subclaim 2.41 $|\{j < M : \mathbf{x}_j = \mathbf{mul}\}|$-many times, and so for some $p_* \leq_{\bar{\bar{\ell}}^n} q_{**}$ we have that

$\boxdot_{(a)}$ $\forall j < M : \ p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D_m^{\iota, \mathbf{x}_j}(\underset{\sim}{z})$,

$\boxdot_{(b)}$ if $\mathbf{x}_j = \mathbf{mul}$, then for some $i_j^* \neq i_j^{**}$ the conditions

$$(p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{(\{\varepsilon_m^\infty\}, (\langle 0 \rangle))}, (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{(\{\varepsilon_m^\infty\}, (\langle 1 \rangle))}$$

decide differently about $\underset{\sim}{z}_{i_j^*}$, as well as about $\underset{\sim}{z}_{i_j^{**}}$. Moreover, both conditions decide $\underset{\sim}{z} \restriction [0, \max(i_j^*, i_j^{**}) + 1]$.

Now pick $p_{**} \leq_{\bar{\bar{\ell}}^n} p_*$, so that

$\odot_{(a)}$ $\forall j < M : \ p_{**}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \leq (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{(\{\varepsilon_m^\infty\}, (\langle 0 \rangle)}$,

$\odot_{(b)}$ and $\forall j < M : \ p_{**}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D_m^{\iota, \mathbf{x}_j}(\underset{\sim}{z})$.

So (by replacing p_* with a $\leq_{\bar{\bar{\ell}}^{n+1}}$-extension of it) w.l.o.g. we can assume that

$$\forall j < M : \ p_{**}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} = (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{(\{\varepsilon_m^\infty\}, (\langle 0 \rangle)}. \tag{2.42}$$

Define $r_* \leq_{\bar{\bar{\ell}}^n} p_*$ so that

$$\forall j < M : \ (r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle 0 \rangle)} = (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle 0, 0 \rangle)}, \tag{2.43}$$

and similarly,

$$\forall j < M : \ (r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle 1 \rangle)} = (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle 1, 1 \rangle)}. \tag{2.44}$$

Now if $j < M$ is such that $\mathbf{x}_j = \mathbf{mul}$, then for $a \in \{0, 1\}$

$$(r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle a \rangle)} \leq (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle a \rangle)},$$

so $(r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle a \rangle)}$ $(a \in \{0, 1\})$ decide differently about $\underset{\sim}{z}_{i_j^*}$ and $\underset{\sim}{z}_{i_j^{**}}$ (by $(\boxdot_{(b)})$).

If $j < M$ is such that $\mathbf{x}_j = \mathbf{eq}$, then it follows from (2.42), $\boxdot_{(a)}$, $\odot_{(b)}$ and 2 from Subclaim 2.40, that

$$\forall t_0 \in \ \{0, 1\} : \ (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle t_0 \rangle)} \in D_m^{\iota, \mathbf{eq}}(\underset{\sim}{z}),$$

but then a similar straightforward calculation shows that for no i and $p' \leq_{\{\varepsilon_m^\infty\}, 2} p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)}$, no $\bar{t}_0 \neq \bar{t}_1 \in {}^2\{0, 1\}$ do $(p')^{\{\varepsilon_m^\infty\}, (\bar{t}_0)}$ and $(p')^{\{\varepsilon_m^\infty\}, (\bar{t}_1)}$ decide differently about $\underset{\sim}{z}_i$. This clearly implies that $r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)}) \in D_m^{\iota, \mathbf{eq}}(\underset{\sim}{z})$.

Finally, if j is such that $\mathbf{x}_j = \mathbf{un}$, then we argue that $r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D_m^{\iota, \mathbf{mul}}(\underset{\sim}{z})$. Again, (2.42), $\boxdot_{(a)}$, $\odot_{(b)}$ and 2 from Subclaim 2.40 together imply that

$$\forall t_0 \in \ \{0, 1\} : \ (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle t_0 \rangle)} \in D_m^{\iota, \mathbf{un}}(\underset{\sim}{z}),$$

and similarly to the argument in $(\circledcirc)_6$ in Subclaim 2.40 there are $i_0 < i_1$ and $c_0, c_1 \in \{0,1\}$, such that

$$\forall \bar{t} = \langle t_0, t_1 \rangle \in {}^2\{0,1\} : (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle t_0, t_1 \rangle)} \Vdash \underset{\sim}{z}_{i_0} = f^{t_0}(c_0) \ \wedge \underset{\sim}{z}_{i_0} = f^{t_1}(c_1),$$

(where $f(c) = 1 - c$, $f^0 = \mathrm{id}$). From this we obtain

$$\begin{array}{lcl} (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle 0,0 \rangle)} & \Vdash & \underset{\sim}{z}_{i_0} = c_0 \ \wedge \underset{\sim}{z}_{i_1} = c_1, \\ (p_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\infty\}, (\langle 1,1 \rangle)} & \Vdash & \underset{\sim}{z}_{i_0} = 1 - c_0 \ \wedge \underset{\sim}{z}_{i_1} = 1 - c_1, \end{array}$$

so recalling (2.43), (2.44) clearly $r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} \in D_m^{\infty, \mathbf{mul}}(\underset{\sim}{z})$. (And since we can always $\leq_{\bar{\bar{\ell}}^{n+1}}$-extend r_* so that $(r_*^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon^\infty\}, (\langle a \rangle)}$ $(a \in \{0,1\})$ decides $\underset{\sim}{z} \restriction [0, \max(i_0, i_1) + 1)$, which gives the $x^{(\bar{u}, \bar{v}, \bar{w}, \bar{s})}$'s for the two $(\bar{u}, \bar{v}, \bar{w}, \bar{s}) \in \mathbf{seq}_{\bar{\bar{\ell}}^{n+1}}(r_*)$ for which

$$(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j) \sqsubseteq (\bar{u}, \bar{v}, \bar{w}, \bar{s}).)$$

$\square_{\text{Subclaim2.43}}$

Subclaim 2.44. *Assume that $\iota = \mathbf{1}$, q_*, $\underset{\sim}{z}$, n, m are as in Lemma 2.36, $q_{**} \leq_{\bar{\bar{\ell}}^n} q_*$, $\bar{\mathbf{x}}$ are as in $(\blacktriangle)_2$, $l \in \omega$, moreover,*

$$0 < |\{j < M : \mathbf{x}_j = \mathbf{un}\}| < 2^l. \tag{2.45}$$

Suppose that $q_{***} \leq_{\bar{\bar{\ell}}^n} q_{**}$ is given by applying Subclaims 2.39 and 2.40 to q_{**} and l. Then there exists $\bar{t} \in {}^l2$ for which some $r_* \leq_{\bar{\bar{\ell}}^{n+1}} r_{\bar{t}}$ satisfies the requirements in $(\boxtimes)$, (where $r_{\bar{t}}$ is defined by the equality

$$\forall j < M : \ r_{\bar{t}}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)} = (q_{***}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\bar{t})}.$$

Proof. Fixing $j < M$ so that $\mathbf{x}_j = \mathbf{un}$,

$$\forall \bar{t} \in {}^l2 : \ (q_{***}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\bar{t})}) \in D_m^{\mathbf{1}, \mathbf{un}}(\underset{\sim}{z}),$$

and by (2.30)

(1) there are natural numbers

$$i_0^j < i_1^j < \cdots < i_l^j,$$

such that

$$\forall \bar{t} \in {}^l2, \ \forall a \in \{0,1\} : \ (q_{***}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\bar{t}^\frown \langle a \rangle)} \parallel \underset{\sim}{z} \restriction [0, i_l^j + 1).$$

- $\forall \bar{t} \in {}^l2, \ \exists \bar{z}_{\bar{t}}^j \in {}^{i_l^j}2 : \ (q_{***}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\bar{t})}) \Vdash \bar{z}_{\bar{t}}^j \subseteq \underset{\sim}{z}$,
- if $\bar{t} \in {}^{l \geq}2, \ i \neq i_{|\bar{t}|}^j$, $r \leq_{\{\varepsilon_m^{\mathbf{1}}\}, 1} (q_{***}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\bar{t})})$, then

$$r^{\{\varepsilon_m^\iota\}, (\bar{t}^\frown \langle 0 \rangle)} \Vdash \underset{\sim}{z}_i = a \iff r^{\{\varepsilon_m^\iota\}, (\bar{t}^\frown \langle 1 \rangle)} \Vdash \underset{\sim}{z}_i = a \ (\forall a \in \{0,1\}),$$

- if $\bar{t} \in {}^{l \geq}2$, $r \leq_{\{\varepsilon_m^{\mathbf{1}}\}, 1} (q_{***}^{(\bar{u}^j, \bar{v}^j, \bar{w}^j, \bar{s}^j)})^{\{\varepsilon_m^\iota\}, (\bar{t})})$, $a \in \{0,1\}$, then

$$r^{\{\varepsilon_m^\iota\}, (\bar{t}^\frown \langle 0 \rangle)} \Vdash \underset{\sim}{z}_{i_{|\bar{t}|}^j} = a \iff r^{\{\varepsilon_m^\iota\}, (\bar{t}^\frown \langle 1 \rangle)} \Vdash \underset{\sim}{z}_{i_{|\bar{t}|}^j} = 1 - a.$$

(2) Observe that if $j < M$, $\bar{t} \neq \bar{t}'$, then $\bar{z}_{\bar{t}}^j \neq \bar{z}_{\bar{t}'}^j$.

Now for $(\boxtimes)$

(3) it suffices to choose $\bar{t} \in {}^{l}2$, and set $r_* = r_{\bar{t}}$ so that whenever $j < M$ is such that $\mathbf{x}_j = \mathbf{un}$, then

$$\bar{z}^j_{\bar{t}} \neq \mathfrak{s}_{i^j_l}, \tag{2.46}$$

which is shown by the following: since for any j with $\mathbf{x}_j = \mathbf{eq}$ obviously $r_{\bar{t}}^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \in D_m^{\mathbf{1},\mathbf{eq}}$, while if $\mathbf{x}_j = \mathbf{mul}$, then we can replace r_* with some $r_{**} \leq_{\{\varepsilon^{\mathbf{1}}_m\},1} r_* = r_{\bar{t}}$ given by Subclaim 2.41, preserving

$$r_{**} \in \bigcap_{j<M} \{p \in \mathbb{Q}' : \ p^{(\bar{u}^j,\bar{v}^j,\bar{w}^j,\bar{s}^j)} \in D_m^{\iota,\mathbf{x}_j}(\underset{\sim}{z})\}.$$

But for each $j < M$ with $\mathbf{x}_j = \mathbf{un}$ there is at most one $\bar{t} \in {}^{i^j_l}2$ that does not satisfy (2.46), so by (2.45) there exists a sequence $\bar{t}$ that is suitable for our demands.

$\square_{\text{Subclaim2.44}}$

$\square_{\text{Lemma2.36}}$

$\square_{\text{Subclaim2.33}}$

It is only left to argue $(\circledast)_4$, that will complete the proof of Theorem 2.6.
So fix

- $\alpha < \lambda_{\mathbf{0}}$, and a $\mathbb{G}_{\mathbf{0}}(\bar{\mathfrak{s}})$-independent tree
$$T_{\mathbf{0}} \in V_{\mathbf{0}} \cap \mathscr{P}({}^{\omega>}2),$$
where $V_{\mathbf{0}} = V^{\mathbb{Q}^{\mathbf{0}}_{\lambda_{\mathbf{0}}\setminus\{\alpha\}} \times \mathbb{Q}^{\mathbf{1}} \times \mathbb{Q}^{\infty} \times \mathbb{Q}^{\mathbb{S}}}$,
- $\beta \in [\lambda_{\mathbf{0}}, \lambda_{\mathbf{1}})$, and a $\mathbb{G}_{\mathbf{1}}$-independent tree
$$T_{\mathbf{1}} \in V_{\mathbf{1}} \cap \mathscr{P}({}^{\omega>}2),$$
where $V_{\mathbf{1}} = V^{\mathbb{Q}^{\mathbf{0}} \times \mathbb{Q}^{\mathbf{1}}_{\lambda_{\mathbf{1}}\setminus\lambda_{\mathbf{0}}\setminus\{\beta\}} \times \mathbb{Q}^{\infty} \times \mathbb{Q}^{\mathbb{S}}}$,
- $\gamma \in [\lambda_{\mathbf{1}}, \lambda_{\infty})$, and an E_0-independent tree
$$T_{\infty} \in V_{\infty} \cap \mathscr{P}({}^{\omega>}2),$$
where $V_{\infty} = V^{\mathbb{Q}^{\mathbf{0}} \times \mathbb{Q}^{\mathbf{1}} \times \mathbb{Q}^{\infty}_{\lambda_{\infty}\setminus\lambda_{\mathbf{1}}\setminus\{\gamma\}} \times \mathbb{Q}^{\mathbb{S}}}$,

and we shall check that the generic real in question is not in $[T_\iota]$ ($\iota \in \{\mathbf{0}, \mathbf{1}, \infty\}$). Assume on the contrary (i.e. $\neg\ (\circledast)_4$), let $p_\iota \in \mathbb{P}^\iota$ be such that

$$V_\iota \models \text{“}p \Vdash_{\mathbb{P}^\iota} \underset{\sim}{r} \in [T_\iota]\text{”}, \tag{2.47}$$

where $\underset{\sim}{r}$ is the generic real given by $\mathbb{P}^\iota$ (note that $\mathbb{P}^\iota \in V$, and so we have to carefully manipulate p when working in V_ι as there are more reals in that model than in V).

By Definition 2.7 and $p \in \mathbb{P}^\iota$

- if $\iota = \mathbf{0}$, then there exists $j \in \omega$ with $p_{2j} = C_{2j}$, $p_{2j+1} = C_{2j+1}$. W.l.o.g. we can assume that $|p_0| = |p_1| = \cdots = |p_{2j-1}| = 1$, and if $p_i = \{t_i\}$ $(i < 2j)$, then let
$$\bar{t}^* = \bar{t}_0 {}^\frown \bar{t}_1 {}^\frown \cdots {}^\frown \bar{t}_{2j-1},$$
- if $\iota = \mathbf{1}$, then there exists $j \in \omega$ with $p_j = \{0, 1\}$, and $|p_0| = |p_1| = \cdots = |p_{2j-1}| = 1$, and if $p_i = \{a_i\}$ $(i < j)$, then let
$$\bar{t}^* = \langle a_i : \ i < j\rangle,$$

- if $\iota = \infty$, then there exists $j \in \omega$ with

$$p_j = \{\bar{t}'_j, \bar{t}''_j\}, \tag{2.48}$$

  and $|p_0| = |p_1| = \cdots = |p_{2j-1}| = 1$, and if $p_i = \{\bar{t}_i\}$ $(i < j)$, then let

$$\bar{t}^* = \bar{t}_0 \frown \bar{t}_1 \frown \cdots \frown \bar{t}_{j-1},$$

Now

- if $\iota = \mathbf{0}$, then using D1) pick $t_{2j} \in p_{2j}$ so that

$$\bar{t}^{**} := \bar{t}^* \frown \bar{t}_{2j} = \mathfrak{s}_k$$

  for some $k \in \omega$. Letting $p' \in \mathbb{P}^{\mathbf{0}}$ denote a condition for which $p' \leq p$, $p'_{2j} = \{t_{2j}\}$,
- if $\iota = \mathbf{0}$, or ∞, then

$$\bar{t}^{**} := \bar{t}^*,$$

  and let $p' = p$.

Clearly

$$p' \Vdash \underset{\sim}{r} \in [\bar{t}_{**}],$$

so we can assume that $[t_{**}] \cap [T_\iota] \neq \emptyset$. Consider

- the sets

$$T_{\bar{t}_{**}(0)} = \{\bar{t} \in {}^{\omega\geq}2 : \bar{t}_{**} \frown \langle 0 \rangle \frown \bar{t} \in T\},$$

  and

$$T_{\bar{t}_{**}(1)} = \{\bar{t} \in {}^{\omega\geq}2 : \bar{t}_{**} \frown \langle 1 \rangle \frown \bar{t} \in T\},$$

  if $\iota = \mathbf{0}$ or $\mathbf{1}$,
- while if $\iota = \infty$, then let

$$T_{\bar{t}_{**}(0)} = \{\bar{t} \in {}^{\omega\geq}2 : \bar{t}_{**} \frown \bar{t}'_j \frown \bar{t} \in T\},$$

  and

$$T_{\bar{t}_{**}(1)} = \{t \in {}^{\omega\geq}2 : \bar{t}_{**} \frown \bar{t}''_j \frown \bar{t} \in T\},$$

  (where $p'_j = \{\bar{t}'_j, \bar{t}''_j\}$ recalling (2.48)).

Now as $[T_{\mathbf{0}}]$ ($[T_{\mathbf{1}}]$, $[T_\infty]$, resp.) is $\mathbb{G}_{\mathbf{0}}(\bar{\mathfrak{s}})$-independent ($\mathbb{G}_{\mathbf{1}}$-, E_0-independent, resp.) compact set for which $t_{**} \in T$, there must be $k \in \omega$ such that the sets $T_{\bar{t}_{**}(0)} \cap {}^k2$, and $T_{\bar{t}_{**}(1)} \cap {}^k2$ are disjoint.

Now by further extending p' if necessary we can assume that

- (if $\iota = \mathbf{0}$) $|p'_{2j+2}| = |p'_{2j+3}| = \cdots = |p'_{2j+k+1}| = 1$, and if $p'_{2j+2+i} = \{\bar{t}_{2j+2+i}\}$ $(i < k)$, then the sequence

$$\bar{t}_{***} = \bar{t}_{2j+2} \frown \bar{t}_{2j+3} \frown \cdots \frown \bar{t}_{2j+k+1} \in {}^{\omega\geq}2$$

  is obviously of length $\geq k$.
- (if $\iota = \mathbf{1}$) $|p'_{j+2}| = |p'_{j+3}| = \cdots = |p'_{j+k+1}| = 1$, and if $p'_{j+2+i} = \{a_{j+2+i}\}$ $(i < k)$, then the sequence

$$\bar{t}_{***} = \langle a_{j+2+i} : i < k \rangle \in {}^{\omega\geq}2$$

  is of length k,

- (if $\iota = \infty$) $|p'_{j+2}| = |p'_{j+3}| = \cdots = |p'_{j+k+1}| = 1$, and if $p'_{2j+2+i} = \{\bar{t}_{2j+2+i}\}$ $(i < k)$, then the sequence

$$\bar{t}_{***} = \bar{t}_{2j+2} \frown \bar{t}_{2j+3} \frown \cdots \frown \bar{t}_{2j+k+1} \in {}^{\omega\geq}2$$

is obviously of length $\geq k$.

If $\iota \in \{\mathbf{0}, \mathbf{1}\}$, then let $a \in \{0,1\}$ be such that $t_{***} \restriction k \in T_{\bar{t}_{**}(a)}$. Our observation above means that $t_{***} \restriction k \notin T_{\bar{t}_{**}(1-a)}$, thus

$$\bar{t}_{**} \frown \langle 1-a \rangle \frown \bar{t}_{***} \notin T_\iota. \tag{2.49}$$

Extend p' to $p'' \in \mathbb{Q}$ such that $p''_{2j+1} = \{\langle 1-a \rangle\}$ (if $\iota = \mathbf{0}$), or $.p''_{j+1} = \{1-a\}$ (if $\iota = \mathbf{1}$), and then

$$p'' \Vdash \underset{\sim}{r} \in [\bar{t}_{**} \frown \langle 1-a \rangle \frown \bar{t}_{***}]$$

which together with (2.49) contradicts (2.47). We can also reach the same contradiction in the case $\iota = \infty$, just working with $p_j = \{\bar{t}'_j, \bar{t}''_j\}$ instead of $\{0,1\}$.

$\square_{\text{Claim2.23}}$

Problem 2.45. Is it true, that in the final model there is a partition of the Cantor space into $\lambda_{\mathbf{0}}$-many $\mathbb{G}_0(\bar{\mathfrak{s}})$-independent Borel sets (while the other assertions from Theorem 2.6 still hold)? Is it consistent that there is a partition of 2^ω into λ-many $\mathbb{G}_0(\bar{\mathfrak{s}})$-independent Borel sets, where $\mathrm{cov}(\mathcal{M}) < \lambda < 2^{\aleph_0}$, and less than λ-many (or just not more than $\mathrm{cov}(\mathcal{M})$-many) $\mathbb{G}_0(\bar{\mathfrak{s}})$-independent Borel sets do not cover 2^ω? What can we say about the corresponding invariant of $\mathbb{G}_1$, or E_0?

Problem 2.46. Define the graph

$$\mathbb{G}_n = \{(x,y) \in [{}^\omega 2]^2 : |\{j \in \omega : x_j \neq y_j\}| \leq n\}$$

for $n \in \omega$ fixed. Can we separate $\mathrm{cov}(I_{\mathbb{G}_n})$ and $\mathrm{cov}(I_{\mathbb{G}_{n+1}})$? Can we separate infinitely many $\mathrm{cov}(T_{\mathbb{G}_j})$'s?

§ 3. Acknowledgement

The first named author is grateful for M. Gaspar for some valueable discussions. We thank Shimon Garti for the helpful suggestions and remarks, which improved the paper.

$\square_{\text{Theorem2.6}}$

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