MODIFIED AECS FOR STRICTLY STABLE THEORIES SH:1238

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ABSTRACT. Good frames were suggested in [She09d] as the bare-bones parallel, in the context of AECs, to superstable (among elementary classes). Here we consider (μ, λ, κ)-frames as candidates for being (in the context of AECs) the correct parallel to the class of $|T|^+$-saturated models of a strictly stable (complete first-order) theory among elementary classes (we call them DAECs; *directed* AECs).

One thing we lose compared to the superstable case is that going up by induction on cardinals is problematic (because of stages of small cofinality). But this arises only when we try to lift such classes to higher cardinals. Also, we may assume (as a replacement) the existence of prime models over unions of increasing chains.

Annotated Content

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References like [Sh:950, Th0.2$_{=\mathrm{Ly5}}$] mean that the internal label of Theorem 0.2 is y5 in Sh:950. The reader should note that the version in my website is usually more up-to-date than the one in arXiv.

§ 0. Introduction

In this part we try to deal with classes like "$\aleph_1$-saturated models of a first-order theory T, and even strictly stable ones" rather than of "a model of T," but in the AEC framework. The parallel problem for "a model of T, even a superstable one" is the subject of [She09d], [She09e], and [Bal09].

In model theory, the first context was investigating the class of models Mod_T of a first-order theory T. However, not all interesting classes of models are like this — e.g.

(A) The *locally finite* models (i.e. every finitely generated submodel is finite). The case of locally finite groups is a natural one for group theorists (see [KW73]).

(B) The class of atomic models of a first-order countable complete theory T. I.e. if $M \in \mathrm{Mod}_T$ and $\bar{a} \in {}^n M$, then the type $p(\bar{x}) = \mathrm{tp}(\bar{a}, p, M)$ is *isolated.* That is, for some $\varphi(\bar{x}) \in M$, we have $T, \varphi(\bar{x}) \vdash p$. (See e.g. [BLS24].)

(C) The model of such theories omitting a type p, or models of $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$. (See e.g. [Kei70], [Kei71].)

(D) Other logics (see [Dic75], [Mak85]).

One way to address this was to use logics stronger than first-order. A related approach (which we continue here) is to use AECs.

AECs (*abstract elementary classes*) try to address such problems by forgetting the syntax (equivalently, the formulas) and just concentrate on the basic properties of $(\mathrm{Mod}_T, \prec)$ (the class of models of a first-order theory T, ordered by '. . . is an elementary submodel of . . .'). On this, see [She09d], [She09e], and [Bal09].

Much of the work was on investigating "categoricity in λ" (having a unique model of cardinality λ, up to isomorphism) and stability / superstability. However, the following prominent class falls outside of even that framework:

(E) Complete metric models, where there is a metric under which all relations and functions are closed (see e.g. [CK66] and the survey [Kei20]).

An example closer to the author's heart is

(F) The class of $\aleph_1$-saturated models of a first-order complete theory (see [She98]).

What do we do here? In §1 we suggest a solution to the analogy

Being $\aleph_1$-saturated inside an elementary class : elementary class $(= (\mathrm{Mod}_T, \prec))$
________? : AEC.

The suggested solutions are various versions of DAECs (*directed* abstract elementary classes). In Section 1, we deal with their basic properties and analogues of the lifting theorem, which says that if we have an AEC of some cardinality then we can define it in a higher one.

In §2 we suggest how to generalize stability in this context. In §3, we speculate about what we consider the major question: the so-called 'main gap.' (This remains open even for the $\aleph_1$-saturated models of a countable stable theory T.)

This was the middle part of [Sheb], which was was divided into three by editor request: the third part is [Shea]. The original full paper has existed (and to some extent, has been circulated) since 2002.

We thank the referee for his help in improving the presentation.

Notation 0.1. 1) Let $\lambda^{<\kappa} := \sum_{\sigma<\kappa} \lambda^\sigma$; inside subscripts we may use $\lambda[<\kappa]$.

2) λ, κ, μ will denote cardinals — infinite, if not stated otherwise.

3) $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi$ will denote ordinals.

4) $\mathfrak{k}$ will always be a 0-DAEC, and when we write 'DAEC' we mean 0-DAEC. If not stated otherwise, $(\mu, \lambda, \kappa) = (\mu_{\mathfrak{k}}, \lambda_{\mathfrak{k}}, \kappa_{\mathfrak{k}})$ (see Definition 1.2).
On $\mathfrak{k}_\lambda$, $\mathfrak{k}_{[\lambda,\mu]}$, see 1.18. We may write $\mathfrak{k}^*_\lambda$ instead of $(\mathfrak{k}_*)_\lambda$.

5) M, N are models from $K_{\mathfrak{k}}$.

6) I and J will denote partial orders — directed, if not stated otherwise.

7) We say I is κ-*directed* when every directed $J \subseteq I$ of cardinality $< \kappa$ has a $\leq_I$-upper bound.

We shall freely use the following fact.

Fact 0.2. *If* $\lambda = \lambda^{<\kappa}$ *then* $\chi \geq \lambda \Rightarrow (\chi^{<\kappa})^{<\kappa} = \chi^{<\kappa}$.

§ 1. Axiomatizing AECs without full continuity

§ 1(A). **DAEC.** Classes like "the $\aleph_1$-saturated models of a first-order T which is not superstable" do not fall under AEC — still, they are close; in particular, for the case when T is stable. Below we suggest a framework for generalizing them, as AECs generalize elementary classes. So for increasing sequences of short length the union is not necessarily in the class, but we have weaker demands. In the main case, as 'compensation,' we demand that prime models exist (in particular, over short increasing chains of models).

We shall lift a (μ, λ, κ)-DAEC to a $(\infty, \lambda, \kappa)$-DAEC (see below), so actually $\mathfrak{k}_\lambda$ will suffice. Now for the case of AECs, a central point was the replacement of 'superstable T' by 'good λ-frames;' they were used (e.g.) for investigating categoricity. In our case this is even more complicated, as their properties are not necessarily preserved by the lifting. (Not only e.g. the amalgamation property, but even the existence of a $\leq_{\mathfrak{k}}$-upper bound of a short sequence.)

This section generalizes [She09b, §1]; in some cases the differences are minor, whereas sometimes the differences are the whole point.

Convention 1.1. In this section, if not said otherwise, $\mathfrak{k}$ will denote a 1-DAEC (i.e. a directed AEC; see Definition 1.2). We may write DAEC (the D stands for directed).

Definition 1.2. Assume $\lambda < \mu$, $\lambda^{<\kappa} = \lambda$ (for notational simplicity), $\alpha < \mu \Rightarrow |\alpha|^{<\kappa} < \mu$, and κ is regular. (The case $\kappa > \aleph_0$ is our main interest.)

We say that $\mathfrak{k}$ is a (μ, λ, κ)-*1-DAEC* when $\boxplus$ and all the axioms in this definition hold.

(We may omit or add the '1' and '(μ, λ, κ)' by $\boxplus$(a) below; similarly in similar definitions. Instead of $\mu = \mu_1^+$, we may write $\leq \mu_1$.)

We write 0^+-DAEC when we omit **Ax.III**(a),(b) and **IV**(a),(b), and 0-DAEC if we also omit **Ax.VI**.

- $\boxplus$ [=**Ax.O**] $\mathfrak{k}$ consists of the objects in clauses (a)-(d), having the properties listed in (e)-(g).
  - (a) The cardinals $\mu = \mu_{\mathfrak{k}} = \mu(\mathfrak{k})$, $\lambda = \lambda_{\mathfrak{k}} = \lambda(\mathfrak{k})$ and $\kappa = \kappa_{\mathfrak{k}} = \kappa(\mathfrak{k})$, satisfying $\mu > \lambda = \lambda^{<\kappa} \geq \kappa = \mathrm{cf}(\kappa)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\kappa} < \mu$ (but possibly $\mu = \infty$).
  - (b) $\tau = \tau_{\mathfrak{k}}$, a vocabulary with each predicate and function symbol of arity[1] $\leq \lambda$.
  - (c) K, a class of τ-models.
  - (d) A two-place relation $\leq_{\mathfrak{k}}$ on K.
  - (e) If $M_1 \cong M_2$ then $M_1 \in K \Leftrightarrow M_2 \in K$.
  - (f) if $(N_1, M_1) \cong (N_2, M_2)$ <u>then</u> $M_1 \leq_{\mathfrak{k}} N_1 \Leftrightarrow M_2 \leq_{\mathfrak{k}} N_2$.
  - (g) Every $M \in K$ has cardinality $\lambda \leq \|M\| < \mu$.

Ax.I (a) $M \leq_{\mathfrak{k}} N \Rightarrow M \subseteq N$

(b) If $M \leq_{\mathfrak{k}} N$ and they both have the same universe, then $M = N$.

Ax.II $\leq_{\mathfrak{k}}$ is a partial order.

Ax.III Assume that $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathfrak{k}}$-increasing sequence and $\sum_{i<\delta} \|M_i\| < \mu$. <u>Then</u>:

[1] The reason for allowing infinite arity is explained in [Sheb]. But if we ignore "the type $p \in \mathcal{S}_{\mathfrak{k}}(M)$ is based on $A \subseteq M$," then we can restrict ourselves to finite arity.

(a) **Existence of limits**
There is $M \in K$ such that $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M$.

(b) **Existence of unions**
If $\mathrm{cf}(\delta) \geq \kappa$ then there is $M \in K$ such that $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M$ and $|M| = \bigcup_{i<\delta} |M_i|$. (Note that M may not necessarily be unique, as some members of $\tau_{\mathfrak{k}}$ may have infinite arity.)

Ax.IV **Weak uniqueness of limit** (= weak smoothness)
For $\langle M_i : i < \delta \rangle$ as above (in **Ax.III**),

(a) If $N_\ell \in K$ and $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} N_\ell$ for $\ell = 1, 2$, then there are $N \in K$ and N'_ℓ such that

$$i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} N'_\ell \leq_{\mathfrak{k}} N_\ell.$$

Furthermore, there are f_1, f_2 such that f_ℓ is a $\leq_{\mathfrak{k}}$-embedding of N'_ℓ into N for $\ell = 1, 2$ and $i < \delta \Rightarrow f_1 \restriction M_i = f_2 \restriction M_i$.

(b) If $\mathrm{cf}(\delta) \geq \kappa$, M is as in **Ax.III**(b), and $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} N$, then $M \leq_{\mathfrak{k}} N$. (This implies the uniqueness of M, and justifies writing '$M = \bigcup_{i<\delta} M_i$.')

Ax.V If $N_\ell \leq_{\mathfrak{k}} M$ for $\ell = 1, 2$ and $N_1 \subseteq N_2$ then $N_1 \leq_{\mathfrak{k}} N_2$.

Ax.VI **LST property**
If $A \subseteq N \in K$ and $|A| \leq \lambda$, then there is $M \leq_{\mathfrak{k}} N$ of cardinality λ such that $A \subseteq M$.

Remark 1.3. 1) There are some additional axioms listed in 1.4(6), but we shall mention them in any claim in which they are used. Note that the classes in 1.4(1)-(5) are defined in terms of them.

2) Note that in **Ax.III**(c)-(f), the demands $\sum_{s\in I} \|M_s\| < \mu$ and $\bigcup_{i<\delta} |M_i| < \mu$ are needed because otherwise (by **Ax.O**(g)) there would be no $\leq_{\mathfrak{k}}$-upper bound.

Definition 1.4. 1) We say $\mathfrak{k}$ is a 2-DAEC when it is a 0^+-DAEC and we add **Ax.III**(b),(d), **Ax.IV**(b),(d).

2) We say $\mathfrak{k}$ is a 3-DAEC when it is a 2-DAEC and satisfies **Ax.III**(a),(c), and **Ax.IV**(a),(c).

3) We say $\mathfrak{k}$ is a 4-DAEC when it is a 3-DAEC and we add **Ax.III**(e), **Ax.IV**(e).

4) We say $\mathfrak{k}$ is a 5-DAEC when it is a 3-DAEC and **Ax.III**(f) holds.

5) We say $\mathfrak{k}$ is a 6-DAEC when it is a 5-DAEC and we add **Ax.IV**(f).

6) The additional axioms are as follows:

Ax.III (c) If I is κ-directed, $\overline{M} = \langle M_s : s \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing,[2] and

$$\sum_{s\in I} \|M_s\| < \mu$$

then $\overline{M}$ has a $\leq_{\mathfrak{k}}$-upper bound M (i.e. $s \in I \Rightarrow M_s \leq_{\mathfrak{k}} M$).

[2] That is, $s \leq_I t \Rightarrow M_s \leq_{\mathfrak{k}} M_s$.

(d) **Union of directed systems**
If I is κ-directed, $|I| < \mu$, $\langle M_t : t \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing, and

$$\sum_{s \in I} \|M_s\| < \mu,$$

then there is one and only one M with universe $\bigcup_{s \in I} |M_s|$ such that $M_s \leq_{\mathfrak{k}} M$ for every $s \in I$. (We call it the $\leq_{\mathfrak{k}}$-union of $\langle M_t : t \in I \rangle$.)

(e) Like **Ax.III**(c), but I is just directed.

(f) If $\overline{M} = \langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing and

$$\sum_{i<\delta} \|M_i\| < \mu$$

then there is M which is $\leq_{\mathfrak{k}}$-prime over $\overline{M}$; i.e.
- If $N \in K_{\mathfrak{k}}$ and $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} N$ then there is a $\leq_{\mathfrak{k}}$-embedding of M into itself over $\bigcup_{i<\delta} |M_i|$.

Ax.IV (c) If I is κ-directed, $\overline{M} = \langle M_s : s \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing, and N_1, N_2 are $\leq_{\mathfrak{k}}$-upper bounds of $\overline{M}$, then for some N'_1, N'_2, f_1, f_2, N we have

$$s \in I \wedge \ell \in \{1,2\} \Rightarrow M_s \leq_{\mathfrak{k}} N'_\ell \leq_{\mathfrak{k}} N_\ell$$

and f_ℓ is a $\leq_{\mathfrak{k}}$-embedding of N'_ℓ into N which is the identity on M_s for every $s \in I$. (This is a weak form of uniqueness.)

(d) If I is a κ-directed partial order, $\overline{M} = \langle M_s : s \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing, $s \in I \Rightarrow M_s \leq_{\mathfrak{k}} M$, and $|M| = \bigcup_{s \in I} |M_s|$, then

$$\bigwedge_s [M_s \leq_{\mathfrak{k}} N] \Rightarrow M \leq_{\mathfrak{k}} N.$$

(e) Like **Ax.IV**(c), but I is just directed.

(f) If I is directed, $\overline{M} = \langle M_s : s \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing, and $\sum_{s \in I} \|M_s\| < \mu$, then there is M which is a $\leq_{\mathfrak{k}}$-prime over $\overline{M}$, defined as in **Ax.III**(f).

Ax.VII Amalgamation

If $M_0 \leq_{\mathfrak{k}} M_\ell$ for $\ell = 1, 2$, then there exists N such that $M_0 \leq_{\mathfrak{k}} N$ and there are f_1, f_2 such that f_ℓ is a $\leq_{\mathfrak{k}}$-embedding of M_ℓ into N over M_0 (for $\ell = 1, 2$).

Claim 1.5. *Assume*[3] $\mathfrak{k}$ *is a* 0*-DAEC.*

1) **Ax.III**(d) *implies* **Ax.III**(b) *and* **Ax.III**(c).

2) **Ax.III**(e) *implies* **Ax.III**(a) *and* **Ax.III**(c).

3) **Ax.IV**(d) *implies* **Ax.IV**(a)*,* **Ax.IV**(b)*, and* **Ax.IV**(c).

4) **Ax.IV**(e) *implies* **Ax.IV**(c).

5) **Ax.IV**(f) *implies* **Ax.III**(c)*,* **Ax.III**(f)*, and* **Ax.IV**(e).

6) *'If* $\mathfrak{k}$ *is a* j*-DAEC then it is an* i*-DAEC,' for*

$$(i,j) \in \big\{(0,0^+),(0^+,1),(0^+,2),(1,3),(2,3),(3,4),(4,5),(5,6)\big\}.$$

7) *If* $\mathfrak{k}$ *is an AEC then (letting* $(\mu,\lambda,\kappa) := (\infty, \mathrm{LST}_{\mathfrak{k}}, \aleph_0)$*) it is an* ι*-DAEC for all* $\iota \leq 6$.

8) **Ax.IV**(f) *implies* **Ax.III**(f)*, which implies* **Ax.III**(a).

9) *Assume* **Ax.VII**. *Then*

[3] By 1.1, it is not necessary to say this.

(*A*) **Ax.IV**(a) *is equivalent to*
"If $N_\ell \in K$ and $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} N_\ell$ for $\ell = 1, 2$, then there are $N \in K$ and f_1, f_2 such that f_ℓ is a $\leq_{\mathfrak{k}}$-embedding of N_ℓ into N for $\ell = 1, 2$ and $i < \delta \Rightarrow f_1 \restriction M_i = f_2 \restriction M_i$."

(*B*) *Similarly for* **Ax.IV**(c), **Ax.IV**(e).

10) $\mathfrak{k}$ *is an AEC with* $\mathrm{LST}_{\mathfrak{k}} = \lambda$ *when*
- $\mu_{\mathfrak{k}} = \infty$ *and* $\kappa_{\mathfrak{k}} = \aleph_0$.
- $\mathfrak{k}$ *is a* 2*-DAEC (or just a* 0^+*-DAEC which satisfies* **Ax.III**(b) *and* **Ax.IV**(b)*).*

Proof. Easy. $\square_{1.5}$

Example 1.6. As stated in 1.5(7), AECs with $\kappa_{\mathfrak{k}} = \aleph_0$ are an example. (See [She87a], [She09a], and more in [She09b].)

Example 1.7. The first order stable case.

Let T be a stable complete first order theory, and[4]

$$\kappa = \mathrm{cf}(\kappa) \geq \kappa_r(T) \in [\aleph_1, |T|^+].$$

(Equivalently, κ is the minimal regular uncountable cardinal such that $\lambda = \lambda^{<\kappa} \geq 2^{|T|} \Rightarrow$ 'T stable in λ'.) We shall define $\mathfrak{k} = \mathfrak{k}_T$:

⊞ (a) $K = K_{\mathfrak{k}}$ is the class of κ-saturated models of T. (Equivalently, $\mathbf{F}^a_\kappa$-saturated — see [She90, Ch.IV].)
(b) $\leq = \leq_{\mathfrak{k}}$ means "is an elementary submodel of."
(c) $(\mu_{\mathfrak{k}}, \lambda_{\mathfrak{k}}, \kappa_{\mathfrak{k}}) = (\infty, \lambda, \kappa)$, where λ is the first cardinal in which T is stable. (As $\kappa \geq \aleph_1$, we have $\lambda = |\mathbf{D}(T)|^{<\kappa}$.)

Now:

$(*)_1$ All the axioms mentioned in Definitions 1.2, 1.4 are satisfied.

[Why? See [She90, Ch.III-IV]. In particular, for '$\lambda = |\mathbf{D}(T)|^{<\kappa}$,' see [She98, Ch.III,§6].]

$(*)_2$ If we omit '$\kappa \geq \kappa_r(T)$,' we may lose **Ax.III**(f) and **Ax.IV**(e),(f).

Example 1.8. As in 1.7, but $\kappa = \aleph_0$ and we replace 'κ-saturated' by '$\mathbf{F}^a_\kappa$-saturated' (also called '$\aleph_\varepsilon$-saturated;' see [She90, Ch.IV] and [She87b]).
- $\mathfrak{k}$ satisfies every axiom mentioned in Definitions 1.2, 1.4.
- Moreover, it is an AEC.

Example 1.9. Existentially closed models.

Let T be a universal first order theory with the JEP, for transparency.[5] We shall define $\mathfrak{k}$:

⊞ (a) $K = K_{\mathfrak{k}}$ is the class of existentially closed models of T.
(b) $\leq = \leq_{\mathfrak{k}}$ means "is a submodel of."
(c) $(\mu_{\mathfrak{k}}, \lambda_{\mathfrak{k}}, \kappa_{\mathfrak{k}}) = (\infty, |T|, \aleph_0)$.

Now,

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[4] Actually, we can just demand $\mathrm{cf}(\kappa) \geq \kappa_r(T)$, but in Definition 1.2 we already specified κ to be regular.

[5] Otherwise the class of existentially closed models of T is divided into $\leq 2^{|T|}$ subclasses, each of them of this form.

- All the axioms mentioned in Definitions 1.2, 1.4 are satisfied.
  (In particular, **Ax.VII**: amalgamation.)
- Moreover, $\mathfrak{k}$ is an AEC.

Example 1.10. Let T be a universal first-order theory. We shall define $\mathfrak{k} = \mathfrak{k}^0_T$:

$\boxplus$ (a) $K = K_{\mathfrak{k}}$ is the class of models of T.
(b) $\leq = \leq_{\mathfrak{k}}$ means "is a submodel of."

Now $\mathfrak{k}^0_T$ is a 6-DAEC; that is, it satisfies all the axioms of 1.2, 1.4 except possibly **Ax.VII** — amalgamation.

Example 1.11. Let T be a universal first order theory with the JEP, for transparency.

Let $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and K_κ be the class of models M of T which are $(<\kappa)$-*existentially closed*, in the sense that if $A \subseteq M$ has cardinality $< \kappa$, $p(x)$ is a finitely satisfiable set of formulas of the form $\varphi(x, \bar{b})$ with $\bar{b} \subseteq A$, and $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$ is existential, then $p(x)$ is realized in M.

Assume T is stable, in the sense that for no existentially closed model M of T and no existential formula $\varphi(\bar{x}_n, \bar{y}_n)$, does φ linearly order any infinite subset of ${}^nM$. Hence we may assume every $M \in K_\kappa$ is stable in some $\lambda = \lambda^{<\kappa}$ (hence in every $\mu = \mu^{<\kappa} \geq 2^{|T|}$; see [She75]).

We define $\mathfrak{k}$ as follows:

$\boxplus$ (a) $K_{\mathfrak{k}} := K_\kappa$
(b) $\leq_{\mathfrak{k}}$ will be $\prec \restriction K_{\mathfrak{k}}$.
(c) $(\mu_{\mathfrak{k}}, \lambda_{\mathfrak{k}}, \kappa_{\mathfrak{k}}) = (\infty, \lambda, \kappa)$.

As in the previous examples, $\mathfrak{k}$ satisfies every axiom listed.

In [She75] this was called the "Kind III context;" recall that Kind II was for such T with amalgamation and JEP. More was done by Hrushovski.

Example 1.12. Metric Spaces

1) (a) We say that τ is a *metric vocabulary* if it has the distinguished 2-place predicates R_q (q a positive rational) and nmet,[6] where

$$\mathrm{nmet}(\tau) := \tau \setminus \{R_q : q \in \mathbb{Q}^+\}$$

and τ is finitary. (That is, each predicate and function symbol has finitely many places.)

(b) M is a *metric model* when its vocabulary τ_M is a metric vocabulary and there is a metric $\mathbf{d}_M(-,-)$ on M such that:
- $\bullet_1$ $\mathbf{d}_M(a, b) = \inf\{q \in \mathbb{Q}^+ : (a, b) \in R^M_q\}$
- $\bullet_2$ For any predicate $R \in \tau$, R^M is closed (under the topology induced by the metric).
- $\bullet_3$ For any function symbol F, F^M is a continuous function.
- $\bullet_4$ M is complete as a metric space.

(c) Without clause $\bullet_4$, we say M is an *almost metric* model.

(d) We say that the metric models M_1, M_2 are *topologically isomorphic* when there is a π such that
- $\bullet_1$ π is an isomorphism from $M_1 \restriction \mathrm{nmet}(\tau)$ onto $M_2 \restriction \mathrm{nmet}(\tau)$.

[6] nmet stands for 'non-metric.'

- $\bullet_2$ $\text{dist}_\pi(M_1, M_2) :=$

$$\sup\left\{\frac{\mathbf{d}_{M_2}\big(\pi(a),\pi(b)\big)}{\mathbf{d}_{M_1}(a,b)}, \frac{\mathbf{d}_{M_1}(a,b)}{\mathbf{d}_{M_2}\big(\pi(a),\pi(b)\big)} : a \neq b \in M_1\right\}$$

is finite.

(Note that this is the meaning of isomorphism for Banach space theorists; what we (model theorists) call isomorphism, they would call isometry.)

2) (a) We say $\mathfrak{k}$ is a *metric* AEC (or MAEC) when:
- $\bullet_1$ $\tau_{\mathfrak{k}}$ is a metric vocabulary.
- $\bullet_2$ $\mathfrak{k}$ is a DAEC with $\mu_{\mathfrak{k}} := \infty$, $\kappa := \aleph_1$, and $\lambda = \lambda^{\aleph_0}$ (and for convenience $|\tau_{\mathfrak{k}}| \leq \lambda$).
- $\bullet_3$ Each $M \in K_{\mathfrak{k}}$ is a metric model.
- $\bullet_4$ If I is a directed partial order and $\overline{M} = \langle M_s : s \in I\rangle$ is $\leq_{\mathfrak{k}}$-increasing, then the completion M of $\bigcup_{s\in I} M_s$, naturally defined, is a $\leq_{\mathfrak{k}}$-l.u.b. of $\overline{M}$.

(b) We say $\mathfrak{k}$ is an *almost metric* AEC when we omit the completeness demand in (1)(b), and add
- If N is the completion of $M \in K_{\mathfrak{k}}$ (so necessarily $N \in K_{\mathfrak{k}}$, $M \leq_{\mathfrak{k}} N$) then $M \subseteq M' \subseteq N \Rightarrow M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} N$.

3) (a) If $\mathfrak{k}$ is a metric AEC then all the axioms in Definitions 1.2, 1.4 hold.

(b) If $\mathfrak{k}$ is an almost metric AEC then

$$\text{comp}(\mathfrak{k}) := \mathfrak{k} \restriction \big\{M \in K_{\mathfrak{k}} : \big(|M|, \mathbf{d}_M\big) \text{ is complete}\big\}$$

is a metric AEC; also, $\mathfrak{k}$ itself is an AEC. In this case, "the completion of $M \in K_{\mathfrak{k}}$" is naturally defined.

(c) **The representation theorem**.
If $\mathfrak{k}$ is a metric AEC then for some τ_1, T_1, Γ, we have:
- $\bullet_1$ $\tau_1 \supseteq \tau_{\mathfrak{k}}$ and $|\tau_1| \leq \lambda_{\mathfrak{k}}$.
- $\bullet_2$ T_1 is a universal f.o. theory in $\mathbb{L}(\tau_1)$.
- $\bullet_3$ Γ is a set of $\mathbb{L}(\tau_1)$-types consisting of formulas (so they are m-types for some m). They may even just be quantifier-free formulas.
- $\bullet_4$ Every $M \in \text{EC}(T_1, \Gamma)$ is a weak metric model.
- $\bullet_5$ $K_{\mathfrak{k}} :=$

$$\big\{M : M \text{ is the completion of } M_1 \restriction \tau_{\mathfrak{k}} \text{ for some } M_1 \in \text{EC}(T_1, \Gamma)\big\}$$

- $\bullet_6$ $\leq_{\mathfrak{k}}$ is defined as

$$\big\{(M,N) : M \subseteq N, \text{ and there are } M_1 \subseteq N_1 \text{ from } \text{EC}(T_1,\Gamma) \text{ such that } M, N$$
$$\text{are the completions of } M_1 \restriction \tau_{\mathfrak{k}} \text{ and } N_1 \restriction \tau_{\mathfrak{k}}, \text{ respectively}\big\}$$

[Why is this true? As in the AEC case; see [She87a], [She09a].]

Regarding metric model theory and topological model theory, the field was started (in a more general frame) by Chang and Keisler in [CK66]; for an introduction and history, see a recent survey [Kei20].

Example 1.13. If $\mathfrak{k}$ is an ess-(μ,λ)-AEC (see [Sheb, §1]) then $\mathfrak{k}$ is a $(\mu, \lambda, \aleph_0)$-4-DAEC and satisfies all the axioms from 1.4 (except possibly **Ax.VII**, amalgamation).

Definition 1.14. We say $\langle M_i : i < \alpha\rangle$ is $\leq_{\mathfrak{k}}$-increasing $(\geq \kappa)$-continuous when it is $\leq_{\mathfrak{k}}$-increasing and $\delta < \alpha \wedge \mathrm{cf}(\delta) \geq \kappa \Rightarrow |M_\delta| = \bigcup\limits_{j<\delta} |M_j|$.

As an exercise, we consider directed systems with mappings.

Definition 1.15. 1) We say that $\overline{M} = \langle M_t, h_{t,s} : s,t \in I,\ s \leq_I t\rangle$ is a $\leq_{\mathfrak{k}}$-*directed system* when

- (A) I is a directed partial order.
- (B) If $s \leq_I t$ then $h_{t,s}$ is a $\leq_{\mathfrak{k}}$-embedding of M_s into M_t (that is, an isomorphism from M_s onto some $M' \leq_{\mathfrak{k}} M_t$).
- (C) If $t_0 \leq_I t_1 \leq_I t_2$ then $h_{t_2,t_0} = h_{t_2,t_1} \circ h_{t_1,t_0}$.

1A) We say that $\overline{M} = \langle M_t, h_{t,s} : s \leq_I t\rangle$ is a $\leq_{\mathfrak{k}}$-θ-directed system when in addition, I is θ-directed.

2) We may omit $h_{t,s}$ when $s \leq_I t \Rightarrow h_{t,s} = \mathrm{id}_{M_s}$, and write $\overline{M} = \langle M_t : t \in I\rangle$.

3) We say $(M, \bar{h})$ is a $\leq_{\mathfrak{k}}$-limit of $\overline{M}$ when $\bar{h} = \langle h_s : s \in I\rangle$, h_s is a $\leq_{\mathfrak{k}}$-embedding of M_s into M, and $s \leq_I t \Rightarrow h_s = h_t \circ h_{t,s}$.

4) We say $\overline{M} = \langle M_\alpha : \alpha < \alpha^*\rangle$ is $\leq_{\mathfrak{k}}$-semi-continuous when we assume **Ax.III**(f) from 1.4.
That is,

- (A) $\overline{M}$ is $\leq_{\mathfrak{k}}$-increasing.
- (B) If $\alpha < \alpha^*$ has cofinality $\geq \kappa$ then $M_\alpha = \bigcup\limits_{\beta<\alpha} M_\beta$.
- (C) If $\alpha < \alpha^*$ has cofinality $< \kappa$ then M_δ is $\leq_{\mathfrak{k}}$-prime over $\overline{M} \restriction \alpha$.

Observation 1.16. [$\mathfrak{k}$ ***is a DAEC.***]
1) *If $\overline{M} = \langle M_t, h_{t,s} : s \leq_I t\rangle$ is a $\leq_{\mathfrak{k}}$-directed system, then we can find a $\leq_{\mathfrak{k}}$-directed system $\langle M'_t : t \in I\rangle$ (so $s \leq_I t \Rightarrow M'_s \leq_{\mathfrak{k}} M'_t$) and $\bar{g} = \langle g_t : t \in I\rangle$ such that:*

- *(a) g_t is an isomorphism from M_t onto M'_t.*
- *(b) If $s \leq_I t$ then $g_s = g_t \circ h_{t,s}$.*

2) *So in the axioms* **III**(a),(b) *and* **IV**(a),(b) *from Definition* 1.2 *(as well as those of* 1.4*) we can use a $\leq_{\mathfrak{k}}$-directed system $\langle M_s, h_{t,s} : s \leq_I t\rangle$ with I as there (e.g. $I := (\delta, <)$).*

3) *If $(M, \bar{h})$ is prime over $\overline{M} = \langle M_t, h_{t,s} : s \leq_I t\rangle$ and $\chi := \sum\limits_{t \in I} \|M_t\|$, then*

$$\|M\| \leq \chi^{<\kappa}.$$

Proof. Straightforward; e.g. for part (4) we can use "$\mathfrak{k}$ has $(\chi^{<\kappa})$-LST" (i.e. Claim 1.17 below). $\square_{1.16}$

More serious is proving the LST theorem in our context. Recall that in the axioms (see **Ax.VI** in particular) we demand it only down to λ.

Claim 1.17. *Assume $\mathfrak{k}$ is a (μ, λ, κ)-2-DAEC (see Definition* 1.4(1)*).*
If $\lambda_{\mathfrak{k}} \leq \chi = \chi^{<\kappa} < \mu_{\mathfrak{k}}$, $A \subseteq N \in K_{\mathfrak{k}}$, and $|A| \leq \chi \leq \|N\|$, then there is $M \leq_{\mathfrak{k}} N$ of cardinality χ such that $A \subseteq M$. (That is, $\mathrm{LST}_{\mathfrak{k}}(\chi)$ *holds.)*

Proof. 1) As $\chi \leq \|N\|$:

- $(*)_0$ Without loss of generality $|A| = \chi$.

Let $\langle u_\alpha : \alpha < \alpha_*\rangle$ list $[A]^{<\kappa_{\mathfrak{k}}}$, and let I be the following partial order:

$(*)_1$ (a) The set of elements is $\{\alpha < \chi : (\forall \gamma < \alpha)[u_\alpha \not\subseteq u_\gamma]\}$.
(b) $\alpha \leq_I \beta$ iff ($\alpha, \beta \in I$ and) $u_\alpha \subseteq u_\beta$ (hence $\alpha \leq \beta$).

Easily,

$(*)_2$ (a) I is κ-directed.
(b) For every $\alpha < \alpha_*$, for some $\beta < \alpha_*$, we have $u_\alpha \subseteq u_\beta \wedge \beta \in I$.
(c) $\bigcup\limits_{\alpha \in I} u_\alpha = A$.

Now we choose M_α by induction on $\alpha < \chi$ such that

$(*)_3$ (a) $M_\alpha \leq_{\mathfrak{k}} N$
(b) $\|M_\alpha\| = \lambda_{\mathfrak{k}}$
(c) M_α includes $u_\alpha \cup \bigcup\limits_{\beta <_I \alpha} M_\beta$.

Note that

$$\big|\{\beta \in I : \beta <_I \alpha\}\big| \leq \big|\{u : u \subseteq u_\alpha\}\big| = 2^{|u_\alpha|} \leq 2^{<\kappa(\mathfrak{k})} \leq \lambda_{\mathfrak{k}},$$

and by the induction hypothesis $\beta < \alpha \Rightarrow \|M_\beta\| \leq \lambda_{\mathfrak{k}}$. Recall $|u_\alpha| < \kappa(\mathfrak{k}) \leq \lambda_{\mathfrak{k}}$ hence the set $\bigcup\limits_{\beta<\alpha} M_\beta \cup u_\alpha$ is a subset of N of cardinality $\leq \lambda$, hence by **Ax.VI** there exists M_α as required.

Having chosen $\langle M_\alpha : \alpha \in I\rangle$, clearly by **Ax.V** it is a $\leq_{\mathfrak{k}}$-increasing $(< \kappa)$-directed system; hence by **Ax.III**(d), $M := \bigcup\limits_{\alpha\in I} M_\alpha$ is well-defined with universe $\bigcup\limits_{\alpha \in I} |M_\alpha|$, and by **Ax.IV**(d) we have $M \leq_{\mathfrak{k}} N$.

Clearly $\|M\| \leq \sum\limits_{\alpha \in I} \|M_\alpha\| \leq |I| \cdot \lambda_{\mathfrak{k}} = \chi$, and by $(*)_2$(c) + $(*)_3$(c) we have

$$A \subseteq \bigcup_{\alpha<\chi} u_\alpha = \bigcup_{\alpha \in I} u_\alpha \subseteq \bigcup_{\alpha \in I} |M_\alpha| = M$$

and so M is as required.

2) Similarly. $\square_{1.17}$

Notation 1.18. 1) For $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$, let $K_\chi = K^{\mathfrak{k}}_\chi := \{M \in K_{\mathfrak{k}} : \|M\| = \chi\}$ and $K_{<\chi} := \bigcup\limits_{\mu<\chi} K_\mu$.

2) $\mathfrak{k}_\chi := (K_\chi, \leq_{\mathfrak{k}} \restriction K_\chi)$.

3) If $\lambda_{\mathfrak{k}} \leq \lambda_1 < \mu_1 \leq \mu_{\mathfrak{k}}$, $\lambda_1 = \lambda_1^{<\kappa}$, and $(\forall \alpha < \mu_1)\big[|\alpha|^{<\kappa} < \mu_1\big]$, <u>then</u> we define $K_{[\lambda_1,\mu_1)} = K^{\mathfrak{k}}_{[\lambda_1,\mu_1)}$ and $\mathfrak{k}_1 = \mathfrak{k}_{[\lambda_1,\mu_1)}$ similarly:

(A) $K_{\mathfrak{k}_1} := \big\{M \in K_{\mathfrak{k}} : \|M\| \in [\lambda_1, \mu_1)\big\}$
(B) $\leq_{\mathfrak{k}_1} := \leq_{\mathfrak{k}} \restriction K_{\mathfrak{k}_1}$
(C) $\lambda_{\mathfrak{k}_1} = \lambda_1$, $\mu_{\mathfrak{k}_1} = \mu_1$, $\kappa_{\mathfrak{k}_1} = \kappa_{\mathfrak{k}}$.

4) Let $\mathfrak{k}_{[\lambda_1,\mu_1]} := \mathfrak{k}_{[\lambda_1,\mu_1^+)}$.

Definition 1.19. The embedding $f : N \to M$ is called a $\mathfrak{k}$-embedding or a $\leq_{\mathfrak{k}}$-embedding <u>when</u> its range is the universe of a model $N' \leq_{\mathfrak{k}} M$ (so $f : N \to N'$ is an isomorphism, hence it is onto).

Claim 1.20. [$\mathfrak{k}$ ***is a*** 0^+***-DAEC (hence satisfies* Ax.V, Ax.VI*).***]

1) *For every $N \in K$ there is a $\kappa_{\mathfrak{k}}$-directed partial order I of cardinality $\leq \|N\|^{<\kappa_{\mathfrak{k}}} < \mu$ and $\overline{M} = \langle M_t : t \in I\rangle$ such that*

- $t \in I \Rightarrow M_t \leq_{\mathfrak{k}} N$
- $\|M_t\| \leq \mathrm{LST}(\mathfrak{k}) = \lambda_{\mathfrak{k}}$
- $I \models$ "$s < t$" $\Rightarrow M_s \leq_{\mathfrak{k}} M_t$,
- $N = \bigcup_{t \in I} M_t$.

1A) *If in part* (1) *we weaken the demand on I to simply be directed, then we can choose $|I| \leq \|N\|$.*

2) *For every $N_1 \leq_{\mathfrak{k}} N_2$, we can find $\langle M^\ell_t : t \in I^* \rangle$ as in part* (1) *for N_ℓ such that $I_1 \subseteq I_2$ and $t \in I_1 \Rightarrow M^2_t = M^1_t$.*

Proof. 1,1A) As in the proof of 1.17 (but here $\langle u_\alpha : \alpha < \|N\|^{<\kappa_{\mathfrak{k}}} \rangle$ lists $\big[|N|\big]^{<\kappa_{\mathfrak{k}}}$).
2) Similarly. $\square_{1.20}$

Claim 1.21. *Assume $\mathfrak{k}$ is a 2-DAEC, $\lambda_{\mathfrak{k}} \leq \lambda_1 = \lambda_1^{<\kappa} < \mu_1 \leq \mu_{\mathfrak{k}}$, and*

$$(\forall \alpha < \mu_1)\big[|\alpha|^{<\kappa} < \mu_1\big].$$

1) *Then $\mathfrak{k}^*_1 := \mathfrak{k}_{[\lambda_1, \mu_1)}$ (as defined in 1.18(3)) is a $(\lambda_1, \mu_1, \kappa_{\mathfrak{k}})$-2-DAEC.*

2) *Each of the axioms satisfied by $\mathfrak{k}$ is also satisfied by $\mathfrak{k}_1$.*

Proof. Easy. E.g.,

- For **Ax.VI**, use 1.17(1).
- For **Ax.IV**(f), if N is $\mathfrak{k}$-prime over $\langle M_s : s \in I \rangle$ with $M_s \in K_{\mathfrak{k}}$ and $\chi := \sum_{s \in I} \|M_s\| < \mu$, then $\chi^{<\kappa_{\mathfrak{k}}} < \mu_1$ and there is $N_1 \leq_{\mathfrak{k}} N$ of cardinality χ which includes $\bigcup_{s \in I} |M_s|$. But the definition of '$\mathfrak{k}$-prime' gives us $\|N\| \leq \|N_1\| < \mu_1$.

$\square_{1.21}$

Claim 1.22. [***Assume $\mathfrak{k}$ is a*** 1-***DAEC.***[7]]

1) *If $\mathfrak{k}$ satisfies* **Ax.IV**(e) *and* **Ax.VII** *(amalgamation), then it satisfies* **Ax.III**(e), *provided that $\mu_{\mathfrak{k}}$ is regular or at least that the relevant I has cardinality $< \mathrm{cf}(\mu_{\mathfrak{k}})$.*

2) *If* **Ax.III**(d) *and* **IV**(d) *hold, we can waive '$\mu_{\mathfrak{k}}$ is regular.'*

Proof. 1) We prove this by induction on $\theta = |I|$.
Let $\chi := \lambda + \theta + \sum_{s \in I} \|M_s\|$, which is in the interval $[\lambda, \mu)$.

Case 1: I is finite.
So there is $t^* \in I$ such that $t \in I \Rightarrow t \leq_I t^*$, so this is trivial.

Case 2: I is countable.
So we can find a sequence $\langle t_n : n < \omega \rangle$ such that $t_n \in I$, $t_n \leq_I t_{n+1}$, and $s \in I \Rightarrow \bigvee_{n<\omega} [s \leq_I t_n]$. Now we can apply **Ax.III**(b) to $\langle M_{t_n} : n < \omega \rangle$.

Case 3: I uncountable.
First, we can find an increasing continuous sequence $\big\langle I_\alpha : \alpha < |I| \big\rangle$ such that $I_\alpha \subseteq I$ is directed of cardinality $\leq |\alpha| + \aleph_0$ and $I_{|I|} := I = \bigcup_{\alpha < |I|} I_\alpha$.

Second, by the induction hypothesis, for each $\alpha < |I|$ we choose N_α and $\bar{h}^\alpha = \langle h_{\alpha,t} : t \in I_\alpha \rangle$ such that:

(A) $N_\alpha \in K_{\mathfrak{k}}$

[7] Or just a 0^+-DAEC satisfying **Ax.III**(a).

(B) $h_{\alpha,t}$ is a $\leq_{\mathfrak{k}}$-embedding of M_t into N_α.
(C) If $s <_I t$ are in I_α then $h_{\alpha,s} = h_{\alpha,t} \circ h_{t,s}$.
(D) If $\beta < \alpha$ then $N_\beta \leq_{\mathfrak{k}} N_\alpha$ and $t \in I_\beta \Rightarrow h_{\alpha,t} = h_{\beta,t}$.

For $\alpha = 0$ use the induction hypothesis.

For α a limit ordinal, by **Ax.III**(a) there is $N_\alpha \in K_{\mathfrak{k}}$, a $\leq_{\mathfrak{k}}$-upper bound of $\langle N_\beta : \beta < \alpha \rangle$. We know N_α is as required; because $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$, there are no new h_t-s. (Well, we have to check $\sum_{\beta<\alpha} \|N_\beta\| < \mu_{\mathfrak{k}}$, but as we assume $\mu_{\mathfrak{k}}$ is regular – or at least $\mathrm{cf}(\mu_{\mathfrak{k}}) > |I|$ – this holds.)

For $\alpha = \beta + 1$, by the induction hypothesis there is $(N'_\alpha, \bar{g}^\alpha)$ which is a limit of $\langle M_s, h_{t,s} : s \leq_{I_\alpha} t \rangle$. Now apply **Ax.IV**(e)+**Ax.VII**, recalling 1.5(9): well, apply the directed system version with $\langle M_s, h_{t,s} : s \leq_{I_\beta} t \rangle$, $(N'_\alpha, \bar{g}_\alpha)$, $(N_\beta, \langle h_s : s \in I_\beta \rangle)$ here standing for $\overline{M}, N_1, N_2$ there.

So there are N_α, f^α_s (with $s \in I_\beta$) such that $N_\beta \leq_{\mathfrak{k}} N_\alpha$ and $s \in I_\beta \Rightarrow f^\alpha_s \circ g_s = h_s$. Lastly, for $s \in I_\alpha \setminus I_\beta$ we choose $h_s = f^\alpha_s \circ g_s$, so we are clearly done.

2) Similarly, noting that in the last case, without loss of generality the result has cardinality $\leq \chi$ by 1.17(2) or 1.21. $\square_{1.22}$

§ 1(B). **Basic Notions.** As in [She09b, §1], we now recall the definition of orbital types. (Note that it is natural to look at (orbital) types only over models which are amalgamation bases, recalling that **Ax.VII** says that this will hold for every $M \in K_{\mathfrak{k}}$.)

In this subsection, the reader may concentrate on the case where $\mathfrak{k}$ is a 2-DAEC with amalgamation.

Definition 1.23. Assume $\mathfrak{k}$ is a 0-DAEC.

1) For $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$ and $M \in K_\chi$, we define $\mathcal{S}(M)$ as

$$\{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{k}} N \in K_{\leq \chi^{<\kappa}} \text{ and } a \in N\},$$

where $\mathrm{ortp}(a, M, N) = (M, N, a)/\mathscr{E}_M$, where $\mathscr{E}_M$ is the transitive closure of $\mathscr{E}^{\mathrm{at}}_M$, and the two-place relation $\mathscr{E}^{\mathrm{at}}_M$ is defined as follows.

⊛ $(M, N_1, a_1)\ \mathscr{E}^{\mathrm{at}}_M\ (M, N_2, a_2)$ <u>iff</u>:
  (a) $M \leq_{\mathfrak{k}} N_\ell$ and $a_\ell \in N_\ell$ for $\ell = 1, 2$.
  (b) $\|M\| \leq \|N_\ell\| \leq \chi^{<\kappa}$ for $\ell = 1, 2$.
  (c) There exists an $N \in K_{\leq \chi^{<\kappa}}$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell : N_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \restriction M = \mathrm{id}_M = f_2 \restriction M$ and $f_1(a_1) = f_2(a_2)$.

2) We say "a realizes p in N" for $a \in N$ and $p \in \mathcal{S}(M)$ <u>when</u> (letting $\chi = \|M\|$) for some N' we have $M \leq_{\mathfrak{k}} N' \leq_{\mathfrak{k}} N$, $a \in N'$, and $p = \mathrm{ortp}(a, M, N')$. So necessarily $M, N' \in K_{\leq \chi^{<\kappa}}$, but possibly $N \notin K_{\leq \chi^{<\kappa}}$.

3) We say "a_2 strongly realizes $(M, N^1, a_1)/\mathscr{E}^{\mathrm{at}}_M$ in[8] N" <u>when</u> for some N^2 we have $M \leq_{\mathfrak{k}} N^2 \leq_{\mathfrak{k}} N$ and $a_2 \in N^2$ and $(M, N^1, a_1)\ \mathscr{E}^{\mathrm{at}}_M\ (M, N^2, a_2)$.

4) We say M_0 is a $\leq_{\mathfrak{k}[\chi_0,\chi_1)}$-amalgamation base if this holds in $\mathfrak{k}_{[\chi_0,\chi_1)}$; see below.

4A) We say $M_0 \in K_{\mathfrak{k}}$ is an *amalgamation base* (or $\leq_{\mathfrak{k}}$-amalgamation base) <u>when</u>: for every $M_1, M_2 \in K_{\mathfrak{k}}$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in K^{\mathfrak{k}}_\lambda$ and $\leq_{\mathfrak{k}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.

5) We say $\mathfrak{k}$ is *stable in* χ <u>when</u>:

[8] Note that $\mathscr{E}^{\mathrm{at}}_M$ is not an equivalence relation, in general (although $\mathscr{E}_M$ certainly is).

(a) $\lambda_{\mathfrak{k}} \leq \chi < \mu_{\mathfrak{k}}$
(b) $M \in K_\chi \Rightarrow |\mathcal{S}(M)| \leq \chi$
(c) $\chi = \chi^{<\kappa}$
(d) $\mathfrak{k}_\chi$ has amalgamation.

6) We say $p = q \restriction M$ if $p \in \mathcal{S}(M)$, $q \in \mathcal{S}(N)$, $M \leq_{\mathfrak{k}} N$, and for some N^+ such that $N \leq_{\mathfrak{k}} N^+$ and $a \in N^+$ we have $p = \mathrm{ortp}(a, M, N^+)$ and $q = \mathrm{ortp}(a, N, N^+)$. Note that $p \restriction M$ is well defined if $M \leq_{\mathfrak{k}} N$ and $p \in \mathcal{S}(N)$.

7) For finite m, for $M \leq_{\mathfrak{k}} N$ and $\bar{a} \in {}^m N$, we can define $\mathrm{ortp}(\bar{a}, N, N)$ and $\mathcal{S}^m(M)$ similarly, and let $\mathcal{S}^{<\omega}(M) = \bigcup_{m<\omega} \mathcal{S}^m(M)$. (But we shall not use this in any essential way, hence we choose $\mathcal{S}(M) = \mathcal{S}^1(M)$.)

Remark 1.24. 1) The reader may wonder: why is $\mathcal{E}^{\mathrm{at}}_M$ not necessarily an equivalence relation? Consider the following example. $\mathfrak{k}$ will be defined as follows:

$(*)_1$ (a) $\tau_{\mathfrak{k}} := \{R\}$, where R is a two-place predicate.
(b) $K_{\mathfrak{k}}$ is the class of $\tau_{\mathfrak{k}}$-models M such that $\big(|M|, R^M\big)$ is a directed graph with no (directed) cycle of length ≤ 4.
(c) $\leq_{\mathfrak{k}}$ will be $\subseteq \restriction K_{\mathfrak{k}}$.

Now (recalling 1.10):

$(*)_2$ $\mathfrak{k}$ is a $(\infty, \aleph_0, \aleph_0)$-6-DAEC.

$(*)_3$ We can find M, N_0, N_1, N_2 from $K_{\mathfrak{k}}$ and $c, a_0, a_1, a_2, b_0, b_1, b_2$ such that
(a) $|M| = \{c\}$ and $R^M = \varnothing$.
(b) $M \subseteq N_\ell$ and $|N_\ell| = \{c, a_\ell, b_\ell\}$ for $\ell = 0, 1, 2$.
(c) $R^{N_0} = \varnothing$
(d) $R^{N_1} = \big\{(c, b_1), (b_1, a_1)\big\}$
(e) $R^{N_2} = \big\{(a_2, b_2), (b_2, c)\big\}$

It is easy to check that

$(*)_4$ (a) $M \leq_{\mathfrak{k}} N_\ell$ for $\ell \leq 2$, and they are all indeed members of $K_{\mathfrak{k}}$.
(b) For $\ell = 1, 2$, there are N^+_ℓ such that $N_0 \leq_{\mathfrak{k}} N^+_\ell$ and we have $\leq_{\mathfrak{k}}$-embeddings f_ℓ of N_ℓ into N^+_ℓ over M.
(c) Hence the pairs $\big((M, N_\ell, a_\ell), (M, N_0, a_0)\big)$ belong to $\mathcal{E}^{\mathrm{at}}_M$.
(d) Hence the pair $\big((M, N_1, a_1), (M, N_2, a_2)\big)$ belongs to $\mathcal{E}_M$.

However,

$(*)_5$ There are no N, g_1 and g_2 such that $M \leq_{\mathfrak{k}} N$, g_ℓ embeds N_ℓ into N over M for $\ell \in \{1, 2\}$, and $g_1(a_1) = g_2(a_2)$. (Necessarily, $g_1(b_1) \neq g_2(b_2)$.)

[Why? If so, then $\big\langle c, g_1(b_1), g_1(a_1) = g_2(a_2), g_2(b_2)\big\rangle$ is a cycle of length 4 in the directed graph of $\big(|M|, R^M\big)$.]

Therefore the pair $\big((M, N_1, a_1), (M, N_2, a_2)\big)$ is not in $\mathcal{E}^{\mathrm{at}}_M$, and so (recalling $(*)_4$(c),(d)) $\mathcal{E}^{\mathrm{at}}_M$ is not an equivalence relation.

2) However, if the 0^+-DAEC $\mathfrak{k}$ satisfies **Ax.VII** (amalgamation) then $\mathcal{E}^{\mathrm{at}}_M$ is an equivalence relation for all $M \in K_{\mathfrak{k}}$. (This is easy to see; more details can be found in [She09b, §1].)

3) What is the meaning of 'orbital type' when $\mathfrak{k}$ fails amalgamation? We might ask what it would be if we 'force' the class to have it, in a suitable closure of $\mathfrak{k}$.

E.g. let $\tau_{\mathfrak{k}}$ have predicates only, and $\leq_{\mathfrak{k}} := \subseteq \restriction K_{\mathfrak{k}}$. Define $\mathfrak{k}_1$ as follows:

(A) $M \in K_{\mathfrak{k}_1}$ iff M is a $\tau_{\mathfrak{k}}$-model of cardinality $< \mu_{\mathfrak{k}}$ and there is a sequence $\langle M_s : s \in I \rangle$ such that
  (a) $M_s \subseteq M$
  (b) $|M| = \bigcup_{s \in I} |M_s|$
  (c) $R^M = \bigcup_{s \in I} R^{M_s}$ for $R \in \tau_{\mathfrak{k}}$.

(B) $\leq_{\mathfrak{k}_1}$ will be $\subseteq \restriction K_{\mathfrak{k}_1}$.

4) We may replace 1.23(5)(c) by
  (c)′ $\chi \in \mathrm{Car}_{\mathfrak{k}}$ (which means $\chi = \chi^{<\kappa}$, or at least that the conclusion of 1.17 holds).

If so, then 1.25(1) below, we change the default value of χ to $\mathrm{rnd}_{\mathfrak{k}}(\|N\|)$ (where $\mathrm{rnd}_{\mathfrak{k}}(\theta) := \min(\mathrm{Car}_{\mathfrak{k}} \setminus \theta)$) so it is $\leq \|N\|^{<\kappa(\mathfrak{k})}$. (Similarly in 1.26(1).)

Definition 1.25. 1) We say N is *χ-universal above or over M* when $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$, $M \in K_{\leq \chi}$, $M \leq_{\mathfrak{k}} N$, and for every M' with $M \leq_{\mathfrak{k}} M' \in K^{\mathfrak{k}}_{\chi}$, there is a $\leq_{\mathfrak{k}}$-embedding of M' into N over M. If we omit χ we mean $\|N\|^{<\kappa(\mathfrak{k})}$; clearly this implies that M is a $\leq_{\mathfrak{k}[\chi_0,\chi_1]}$-amalgamation base, where $\chi_0 = \|M\|$ and $\chi_1 = \|N\|^{<\kappa}$.

2) $K^3_{\mathfrak{k}} := \big\{(M,N,a) : M \leq_{\mathfrak{k}} N,\ a \in N \setminus M$ and $M, N \in K_{\mathfrak{k}}\big\}$, with the partial order $\leq \, = \, \leq_{\mathfrak{k}}$ defined by $(M,N,a) \leq (M',N',a')$ iff $a = a'$, $N \leq_{\mathfrak{k}} N'$, and $M \leq_{\mathfrak{k}} M'$ (which follows).

3) We say (M,N,a) is minimal if $(M,N,a) \leq (M',N_\ell,a) \in K^3_{\mathfrak{k}}$ for $\ell = 1,2$ implies $\mathrm{ortp}(a,M',N_1) = \mathrm{ortp}(a,M',N_2)$ and moreover, $(M',N_1,a)\ \mathscr{E}^{\mathrm{at}}_{\lambda}\ (M',N_2,a)$ (this is not needed if every $M' \in K_\lambda$ is an amalgamation basis).

4) $K^{3,\mathfrak{k}}_{\chi}$ is defined similarly, using $\mathfrak{k}_{[\chi,\mathrm{rnd}_{\mathfrak{k}}(\chi)]}$.

Generalizing superlimit, we have more than one reasonable choice.

Definition 1.26. 1) For $\ell = 1,2$ and $\chi = \chi^{<\kappa} \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$, we say $M^* \in K^{\mathfrak{k}}_{\chi}$ is superlimit$_\ell$ (or $(\chi, \geq \kappa)$-superlimit$_\ell$) when:[9]

(a) It is universal (i.e. every $M \in K^{\mathfrak{k}}_{\chi}$ can be properly $\leq_{\mathfrak{k}}$-embedded into M^*).
(b) M^* is an amalgamation base in $\mathfrak{k}_\chi$.
(c) There exists $N \in K^{\mathfrak{k}}_{\chi}$ such that $M^* <_{\mathfrak{k}} N$.
(d) **Case 1**: $\ell = 1$.[10] If $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing $(\geq \kappa)$-continuous, $\mathrm{cf}(\delta) \geq \kappa$, $\delta < \chi^+$, and $i < \delta \Rightarrow M_i \cong M^*$, then $M_\delta \cong M^*$.
Case 2: $\ell = 2$.[11] If I is a $(<\kappa)$-directed partial order of cardinality $\leq \chi$, $\langle M_t : t \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing, and $t \in I \Rightarrow M_t \cong M^*$, then $\bigcup_{t \in I} M_t \cong M^*$.

2) We say M is *χ-saturated above θ* when $\|M\| \geq \chi > \theta \geq \mathrm{LST}(\mathfrak{k})$ and

$$N \leq_{\mathfrak{k}} M \wedge \|N\| \in [\theta,\chi) \Rightarrow \big(\forall p \in \mathcal{S}_{\mathfrak{k}}(N)\big)\big[p \text{ is strongly realized in } M\big].$$

3) Let "M is χ^+-saturated" mean that M is χ^+-saturated above χ. Let

$$K(\chi^+\text{-saturated}) := \{M \in K : M \text{ is } \chi^+\text{-saturated}\}$$

4) When we say "M is saturated," we mean "M is $\|M\|$-saturated above some $\theta < \|M\|$."

[9] We may omit ℓ in the case $\ell = 2$.
[10] Usually used when **Ax.III**(b) holds.
[11] Usually used when **Ax.III**(d) holds.

Definition 1.27. Assume $\mathfrak{k}$ is a 1-DAEC.

1) We say N is (χ, σ)-*brimmed* over M when we can find a $\leq_{\mathfrak{k}}$-increasing sequence $\langle M_i : i < \sigma \rangle$ with $M_i \in K_\chi$ and $M_0 := M$, where M_{i+1} is $\leq_{\mathfrak{k}}$-universal over M_i and $\bigcup_{i<\sigma} M_i = N$. (Usually $\text{cf}(\sigma) \geq \kappa$.)

We say N is (χ, σ)-brimmed over A if $A \subseteq N \in K_\chi$ and we can find $\langle M_i : i < \sigma \rangle$ as above such that $A \subseteq M_0$; if $A = \varnothing$ we may omit "over A."

2) We say N is $(\chi, *)$-brimmed over M if it is (χ, σ)-brimmed over M for every $\sigma \in [\kappa, \chi)$. We say N is $(\chi, *)$-brimmed if N is $(\chi, *)$-brimmed over M for some M.

3) If $\alpha < \chi^+$, let "N is (χ, α)-brimmed over M" mean $M \leq_{\mathfrak{k}} N$ are from K_χ and $\text{cf}(\alpha) \geq \kappa \Rightarrow N$ is $(\chi, \text{cf}(\alpha))$-brimmed over M.

Recall

Claim 1.28. *Assume $\mathfrak{k}$ is a 1-DAEC satisfying* **Ax.VII**.

1) *If $\mathfrak{k}$ is stable in χ and $\sigma = \text{cf}(\sigma)$ (so $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$), then for every $M \in K^{\mathfrak{k}}_\chi$, there is an $N \in K^{\mathfrak{k}}_\chi$ universal over M which is (χ, σ)-brimmed over M.*[12]

2) *If N_ℓ is (χ, θ)-brimmed over M for $\ell = 1, 2$ and $\kappa \leq \theta = \text{cf}(\theta) \leq \chi^+$, then N_1 and N_2 are isomorphic over M.*

3) *If M_2 is (χ, θ)-brimmed over M_1 and $M_0 \leq_{\mathfrak{s}} M_1$, then M_2 is (χ, θ)-brimmed over M_0.*

Proof. 1) Straightforward. For universality, recall that saturated implies universal by [She87c]; this is repeated in [She09b].

2,3) As in [She09b]. $\square_{1.28}$

* * *

§ 1(C). **Lifting such classes to higher cardinals.** Here we deal with lifting; there are two aspects. First, if $\mathfrak{k}^1, \mathfrak{k}^2$ agree in λ they agree in every higher cardinal. Second, given $\mathfrak{k}$ we can find $\mathfrak{k}_1$ with $\mu_{\mathfrak{k}_1} = \infty$ and $(\mathfrak{k}_1)_\lambda = \mathfrak{k}_\lambda$.

Theorem 1.29. 1) *If $\mathfrak{k}_\ell$ is a (μ, λ, κ)-2-DAEC for $\ell = 1, 2$ and $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$, then $\mathfrak{k}_1 = \mathfrak{k}_2$, provided that the clause below holds.*

$(*)$ *$\tau_{\mathfrak{k}_1} = \tau_{\mathfrak{k}_2}$ has arity $\leq \kappa$, or just κ_1 and κ_2 have the same notion of unions.*

2) *If $\mathfrak{k}_\ell$ is a $(\mu_\ell, \lambda, \kappa)$-2-DAEC for $\ell = 1, 2$, $\mu_1 \leq \mu_2$, $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$, and $(*)$ above holds, then $\mathfrak{k}_1 = \mathfrak{k}_2[\lambda, \mu_1)$.*

Proof. 1) Note

$\boxplus_1$ If $N \in K_{\mathfrak{k}_1}$ then $N \in K_{\mathfrak{k}_2}$.

[Why? By 1.20(1), there is a κ-directed partial order I and a $\leq_{\mathfrak{k}_1}$-increasing sequence $\overline{M} = \langle M_s : s \in I \rangle$ of members of $K_{\mathfrak{k}_1}$ with cardinality λ, with union N. Hence $\overline{M}$ is also $\leq_{\mathfrak{k}_2}$-increasing.

As $\mathfrak{k}_2$ satisfies **Ax.III**(d), we know that $N = \bigcup_{s \in I} M_s \in K_{\mathfrak{k}_2}$ and $s \in I \Rightarrow M_s \leq_{\mathfrak{k}_2} N$. So $N \in K_{\mathfrak{k}_2}$ as promised.]

$\boxplus_2$ $K_{\mathfrak{k}_2} \subseteq K_{\mathfrak{k}_1}$.

[Why? By symmetry.]

$\boxplus_3$ If $M \leq_{\mathfrak{k}_1} N$ then $M \leq_{\mathfrak{k}_2} N$ (and so by symmetry, $\leq_{\mathfrak{k}_1} = \leq_{\mathfrak{k}_2}$).

[12] Hence it is S^χ_σ-limit: see [She09a], but this is not used here.

[Why? Similarly to the proof of $\boxplus_1$, using 1.20(2) and **Ax.IV**(d).]
Together we are done.

2) Similarly. $\square_{1.29}$

Theorem 1.30. The lifting-up Theorem
Assume $\mathfrak{k}_\lambda$ is a $(\lambda^+, \lambda, \kappa)$-2-DAEC.

[Notation:] *Let I be a κ-directed partial order and $\langle M_s : s \in I\rangle$ be a $\leq_{\mathfrak{k}}$-increasing sequence of $\tau_{\mathfrak{k}_\lambda}$-models.*

If $J \in [I]^{\leq\lambda}$ is κ-directed, then we define M_J as the union of $\langle M_t : t \in J\rangle$ (in the sense of **Ax.III**(d) *from Definition* 1.4*).*

Note that for every $J_1 \subseteq J_2$, we have $M_{J_1} \leq_{\mathfrak{k}_\lambda} M_{J_2}$ by **Ax.IV**(d)*. (This is clause $\odot$ below).*

The pair $\mathfrak{k}' = (K', \leq_{\mathfrak{k}'})$ defined below is an $(\infty, \lambda, \kappa)$-2-DAEC, and $\mathfrak{k}'_\lambda = \mathfrak{k}_\lambda$.

- (A) *K' is the class of $\tau_{\mathfrak{k}_\lambda}$-models M such that for some I and $\overline{M} = \langle M_s : s \in I\rangle$, we have:*
  - (a) *I is a κ-directed partial order.*
  - (b) *$M_s \in K_\lambda$*
  - (c) *$I \models$ "$s < t$" $\Rightarrow M_s \leq_{\mathfrak{k}_\lambda} M_t$*
  - (d) *$M = \bigcup\{M_J : J \subseteq I$ is κ-directed of cardinality $\leq \lambda\}$*
    (Note that this is well-defined even if $\tau_{\mathfrak{k}}$ has symbols of arity λ.)
- (A)′ *We call such $\langle M_s : s \in I\rangle$ a* witness *for $M \in K'$, and we call it* reasonable *if $|I| \leq \|M\|^{<\kappa}$.*
- (B) *$M \leq_{\mathfrak{k}'} N$ <u>iff</u> for some $I, J, \overline{M}$ we have:*
  - (a) *J is a κ-directed partial order,*
  - (b) *$I \subseteq J$ is κ-directed,*
  - (c) *$\overline{M} = \langle M_s : s \in J\rangle$ and is $\leq_{\mathfrak{k}_\lambda}$-increasing,*
  - (d) *$\langle M_s : s \in J\rangle$ is a witness for $N \in K'$,*
  - (e) *$\langle M_s : s \in I\rangle$ is a witness for $M \in K'$.*
- (B)′ *We call such $I, \langle M_s : s \in J\rangle$ witnesses for $M \leq_{\mathfrak{k}'} N$, or say $\big(I, J, \langle M_s : s \in J\rangle\big)$* witnesses *$M \leq_{\mathfrak{k}'} N$.*

Proof. Let us check the axioms one by one.

Ax.O(a)-(g): K' is a class of $\tau_{\mathfrak{k}_\lambda}$-models, $\leq_{\mathfrak{k}'}$ a two-place relation on K, K' and $\leq_{\mathfrak{k}'}$ are closed under isomorphisms, and $M \in K' \Rightarrow \|M\| \geq \lambda$, etc.
[Why? Trivially.]

Ax.I(a): If $M \leq_{\mathfrak{k}'} N$ then $M \subseteq N$.
[Why? As all members of $\tau_{\mathfrak{k}}$ have arity $\leq \lambda$, and

$$\big\{J \subseteq I : J \text{ is } \kappa\text{-directed and } |J| \leq \lambda\big\}$$

is λ^+-directed (recalling part (0)), clearly this holds.]

Ax.I(b): If $M \leq_{\mathfrak{k}'} N$ and they have the same universe, then $M = N$.
Let $\big(I, J, \langle M_s : s \in I\rangle\big)$ witness $M \leq_{\mathfrak{k}'} N$. As M and N have the same universe, the set

$$\mathcal{J}_* := \big\{J' \in [J]^{\leq\lambda} : \bigcup_{s \in J'} |M_s| = \bigcup_{s \in J' \cap I} |M_s|\big\}$$

is κ-directed and cofinal in $\big([J]^{\leq\lambda}, \subseteq\big)$. Furthermore,

$$J' \in \mathcal{J}_* \Rightarrow M_{J'} = M_{J' \cap I}$$

because $\mathfrak{k}_\lambda$ satisfies **Ax.I**(b). So clearly $M = N$.

Ax.II(a): $M_0 \le_{\mathfrak{k}'} M_1 \le_{\mathfrak{k}'} M_2$ implies $M_0 \le_{\mathfrak{k}'} M_2$ and $M \in K' \Rightarrow M \le_{\mathfrak{k}'} M$.
Why? The second phrase is trivial. For the first phrase, for $\ell \in \{1,2\}$, let $I_\ell \subseteq J_\ell$ be κ-directed partial orders and let $\overline{M}^\ell = \langle M^\ell_s : s \in J_\ell \rangle$ witness $M_{\ell-1} \le_{\mathfrak{k}'} M_\ell$.

Before proceeding, let us prove two small sub-lemmas.

- $\odot$ As mentioned above, if $S_1 \subseteq S_2 \subseteq I$ are $(<\kappa)$-directed of cardinality $\le \lambda$, then

$$M_{S_1} \le_{\mathfrak{k}_\lambda} M_{S_2}.$$

[Why? By **Ax.IV**(d).]

- $\boxdot$ If I is a κ-directed partial order, $\langle M^\ell_t : t \in I \rangle$ is a $\le_{\mathfrak{k}_\lambda}$-increasing sequence witnessing $M_\ell \in K'$ for $\ell = 1,2$, and $t \in I \Rightarrow M^1_t \le_{\mathfrak{k}_\lambda} M^2_t$, then $M_1 \le_{\mathfrak{k}} M_2$.

[Why? Let I_1 be the partial order with set of elements $I \times \{1\}$, ordered by

$$(s,1) \le_{I_1} (t,1) \Leftrightarrow s \le_I t.$$

Let I_2 be the partial order with set of elements $I \times \{1,2\}$ ordered by

$$(s_1,\ell_1) \le_{I_2} (s_2,\ell_2) \Leftrightarrow s_1 \le_I s_2 \wedge \ell_1 \le \ell_2.$$

Clearly $I_1 \subseteq I_2$ are both κ-directed.

Let $M_{(s,1)} := M^1_s$ and $M_{(s,2)} := M^2_s$, so clearly $\overline{M} = \langle M_t : t \in I_2 \rangle$ is a $\le_{\mathfrak{k}_\lambda}$-increasing, I-directed sequence witnessing $M_2 \in K'$. Lastly, $(I_1, I_2, \overline{M})$ witnesses $M_1 \le_{\mathfrak{k}'} M_2$, so we have proved $\boxdot$.]

Now we return to verifying **Ax.II**(a). Without loss of generality, J_1 and J_2 are disjoint. Let $\chi := \big(|J_1| + |J_2|\big)^{<\kappa}$ (so $\lambda \le \chi < \mu_{\mathfrak{k}} = \infty$) and let

$$\begin{aligned}\mathscr{U} := \Big\{ u \in [J_1 \cup J_2]^{\le\lambda} : \ & u \cap I_\ell \text{ is } \kappa\text{-directed under } \le_{I_\ell} \text{ and} \\ & u \cap J_\ell \text{ is } \kappa\text{-directed under } \le_{J_\ell} \text{ for } \ell = 1,2, \\ & \text{and } \bigcup\big\{|M^2_s| : s \in u \cap I_2\big\} = \bigcup\big\{|M^1_t| : t \in u \cap J_1\big\}\Big\}.\end{aligned}$$

Let $\langle u_\alpha : \alpha < \alpha^* \rangle$ list $\mathscr{U}$, and we define a partial order I as follows:

(a)′ Its set of elements is $\big\{\alpha < \alpha^* : (\forall \gamma < \alpha)[u_\gamma \not\subseteq u_\alpha]\big\}$.

(b)′ For $\alpha, \beta \in I$, $\alpha \le_I \beta$ iff $u_\alpha \subseteq u_\beta$.

Note that the set I may have cardinality $\big(\sum\limits_{i<\delta} \|M_i\|\big)^{<\kappa}$, which may be $> \lambda$.

As in the proof of 1.17, I is κ-directed.

For $\ell = 0,1,2$ and $\alpha \in I$, let $M_{\ell,\alpha}$ be

(A) The $\le_{\mathfrak{k}}$-union of $\langle M^0_t : t \in u_\alpha \cap I_1 \rangle$ if $\ell = 0$.

(B) The $\le_{\mathfrak{k}}$-union of the $\le_{\mathfrak{k}_\lambda}$-directed sequence $\langle M^1_t : t \in J_1 \rangle$ when $\ell = 1$.

(C) The $\le_{\mathfrak{k}}$-union of the $\le_{\mathfrak{k}_\lambda}$-directed sequence $\langle M^2_t : t \in J_2 \rangle$ when $\ell = 2$.

Now,

$(*)_1$ If $\ell = 0,1,2$ and $\alpha \le_I \beta$ then $M^\ell_\alpha \le_{\mathfrak{k}_\lambda} M^\ell_\beta$.

$(*)_2$ If $\alpha \in I$ then $M^0_\alpha \le_{\mathfrak{k}_\lambda} M^1_\alpha \le_{\mathfrak{k}_\lambda} M^2_\alpha$.

$(*)_3$ $\langle M_{\ell,\alpha} : \alpha \in I \rangle$ is a witness for $M_\ell \in K'$.

$(*)_4$ $M_{0,\alpha} \le_{\mathfrak{k}_\lambda} M_{2,\alpha}$ for $\alpha \in I$.

Together by $\boxdot$ we get that $M_0 \leq_{\mathfrak{k}'} M_2$ as required, and **Ax.II**(a) does indeed hold.

Ax.III(b): So we are given a $\leq_{\mathfrak{k}'}$-increasing sequence $\langle M_i : i < \delta\rangle$ (with $\mathrm{cf}(\delta) \geq \kappa$), and we need to show that $M = \bigcup\limits_{i<\delta} M_i$, as required, does exist.

Let $(I_{i,j}, J_{i,j}, \overline{M}^{i,j})$ witness $M_i \leq_{\mathfrak{k}'} M_j$ when $i \leq j < \delta$, and without loss of generality $\langle J_{i,j} : i < j < \delta\rangle$ are pairwise disjoint. Let $\mathscr{U}$ be the family of sets u such that for some $v \in [\delta]^{\leq\lambda}$,

(A) $v \subseteq \delta$ has cardinality $\leq \lambda$ and has order type of cofinality $\geq \kappa$.
(B) $u \subseteq \bigcup\{J_{i,j} : i < j$ are from $v\}$ has cardinality $\leq \lambda$.
(C) For $i \leq j$ from v, the set $u \cap J_{i,j}$ is κ-directed under $\leq_{J_{i,j}}$ and $u \cap I_{i,j}$ is κ-directed under $\leq_{I_{i,j}}$.
(D) If $i \leq j \leq k$ are from v then
$$\bigcup\big\{M^{i,j}_s : s \in u \cap J_{i,j}\big\} = \bigcup\big\{M^{j,k}_s : s \in u \cap I_{j,k}\big\}.$$
(E) If $i \leq k$ and $j \leq k$ are from v then
$$\bigcup\big\{M^{i,k}_s : s \in u \cap J_{i,k}\big\} = \bigcup\big\{M^{j,k}_s : s \in u \cap J_{j,k}\big\}.$$

Let the rest of the proof be as in the proof of **Ax.II**(a).

Ax.IV(b):
Similar, but $\mathscr{U} := \{u \subseteq I : u$ has cardinality $\leq \lambda$ and is κ-directed$\}$.

Ax.III(d):
Recall that we are assuming $\mathfrak{k}$ satisfies **Ax.III**(d). Similar proof.

Ax.IV(d):
This also follows from our assumption that $\mathfrak{k}$ satisfies **Ax.IV**(d).

Ax.V: Assume $N_0 \leq_{\mathfrak{k}'} M$ and $N_1 \leq_{\mathfrak{k}'} M$, with $N_0 \subseteq N_1$. We must prove
$$N_0 \leq_{\mathfrak{k}'} N_1.$$

[Why? Let $\big(I_\ell, J_\ell, \langle M^\ell_s : s \in J_\ell\rangle\big)$ witness $N_\ell \leq_{\mathfrak{k}} M$ for $\ell = 0, 1$; without loss of generality J_0, J_1 are disjoint.

Let
$$\mathscr{U} := \Big\{u \subseteq J_0 \cup J_1 : |u| \leq \lambda,\ u \cap J_\ell \text{ and } u \cap I_\ell \text{ are } \kappa\text{-directed for } \ell = 0, 1, \text{ and } \bigcup_{s \in u \cap J_0} |M^0_s| = \bigcup_{t \in u \cap J_1} |M^0_t|\Big\}.$$

For $u \in \mathscr{U}$, let

- $M_u = M \restriction \bigcup\{M^\ell_s : s \in u \cap J_\ell\}$ for $i = 0, 1$.
- $N_{\ell,u} = N_\ell \restriction \{M^\ell_s : s \in u \cap I_\ell\}$.

Let

(∗) (a) $(\mathscr{U}, \subseteq)$ is κ-directed.
(b) $N_{\ell,u} \leq_{\mathfrak{k}} M$
(c) $M_{\ell,u} \leq_{\mathfrak{k}} M_{\ell,v}$ when $u \subseteq v$ are from $\mathscr{U}$ and $\ell = 0, 1$.
(d) $M_{0,u} \leq_{\mathfrak{k}} M_{1,u}$
(e) $N_\ell = \bigcup\limits_{u \in \mathscr{U}} N_{\ell,u}$

By $\boxdot$ above we are done.

Ax.VI: $\mathrm{LST}(\mathfrak{k}') = \lambda$.

[Why? Let $M \in K'$, $A \subseteq M$, $|A| + \lambda \leq \chi < \|M\|$, and let $\langle M_s : s \in I\rangle$ witness $M \in K'$. Without loss of generality $|A| = \chi^{<\kappa}$. Now choose a directed $I \subseteq J$ of cardinality $\leq |A| = \chi^{<\kappa}$ such that $A \subseteq M' := \bigcup_{s\in I} M_s$, and so $(I, J, \langle M_s : s \in J\rangle)$ witnesses $M' \leq_{\mathfrak{k}'} M$. So as $A \subseteq M'$ and $\|M'\| \leq |A| + \mu$, we are done.] $\square_{1.30}$

Also, if two such DAECs have some cardinal in common then we can put them together.

Claim 1.31. *Let $\iota \in \{0, 1, 2, 4\}$, assume $\lambda_1 < \lambda_2 < \lambda_3$, and*

(a) *$\mathfrak{k}^1$ is an $(\lambda_2^+, \lambda_1, \kappa)$-2-DAEC and $K^1 = K_{\mathfrak{k}^1}$.*
(b) *$\mathfrak{k}^2$ is a $(\lambda_3, \lambda_2, \kappa)$-2-DAEC.*
(c) *$K^{\mathfrak{k}^1}_{\lambda_2} = K^{\mathfrak{k}^2}_{\lambda_2}$ and $\leq_{\mathfrak{k}^2}\restriction K^{\mathfrak{k}^2}_{\lambda_2} = \leq_{\mathfrak{k}^1}\restriction K^{\mathfrak{k}^1}_{\lambda_2}$.*
(d) *We define $\mathfrak{k}$ as follows: $K_{\mathfrak{k}} = K_{\mathfrak{k}^1} \cup K_{\mathfrak{k}^2}$, $M \leq_{\mathfrak{k}} N$ iff $M \leq_{\mathfrak{k}^1} N$ or $M \leq_{\mathfrak{k}^2} N$ or for some M', $M \leq_{\mathfrak{k}^1} M' \leq_{\mathfrak{k}^2} N$.*

<u>Then</u> *$\mathfrak{k}$ is an $(\lambda_3, \lambda_1, \kappa)$-2-DAEC.*

Proof. Straightforward. E.g.:

Ax.III(d): Assume I is a κ-directed system, $\langle M_s : s \in I\rangle$ is a $\leq_{\mathfrak{k}}$-increasing sequence, and $\sum_{s\in I} \|M_s\| < \lambda_3$.

If $\|M_s\| \geq \lambda_2$ for some s, use $\langle M_t : s \leq t \in I\rangle$ and clause (b) of the assumption. If $\bigcup_{s\in I} M_s$ has cardinality $\leq \lambda_2$, use clause (a) in the assumption. If neither one of them holds, recall $\lambda_2 = \lambda_2^{<\kappa}$ by clause (b) of the assumption, and let

$$\mathscr{U} := \big\{u \subseteq I : |u| \leq \lambda_2,\ u \text{ is } \kappa\text{-directed (in } I\text{), and } \bigcup_{s\in u} M_s \text{ has cardinality } \lambda\big\}.$$

Easily, $(\mathscr{U}, \subseteq)$ is λ_2^+-directed. For $u \in J$, let M_u be the $\leq_{\mathfrak{s}}$-union of $\langle M_s : s \in u\rangle$. Now by clause (a) of the assumption

$(*)_1$ $M_u \in K^{\mathfrak{k}^1}_{\lambda_2} = K^{\mathfrak{k}^2}_{\lambda_2}$
$(*)_2$ If $u_1 \subseteq v$ are from $\mathscr{U}$ then $M_u \leq_{\mathfrak{k}^1} M_v$, $M_u \leq_{\mathfrak{k}^2} M_v$.

Now use clause (b) of the assumption.

Ax.IV(d): Similar proof.

Axiom V: We shall freely use

$(*)$ $\mathfrak{k}^2_{\lambda_2} = \mathfrak{k}^1_{\lambda_2} = \mathfrak{k}_{\lambda_2}$

So assume $N_0 \leq_{\mathfrak{k}} M$, $N_1 \leq_{\mathfrak{k}} M$, $N_0 \subseteq N_1$.

Now if $\|N_0\| \geq \lambda_2$ use assumption (b), so we can assume $\|N_0\| < \lambda_2$. If $\|M\| \leq \lambda_2$ we can use assumption (a), so assume $\|M\| > \lambda_2$; by the definition of $\leq_{\mathfrak{k}}$ there is $M'_0 \in K^{\mathfrak{k}^1}_{\lambda_2} = K^{\mathfrak{k}^2}_{\lambda_2}$ such that $N_0 \leq_{\mathfrak{k}^1} M'_0 \leq_{\mathfrak{k}^2} M$.

First, assume $\|N_1\| \leq \lambda_2$, so we can find $M'_1 \in K^{\mathfrak{k}^1}_{\lambda_2}$ such that $N_1 \leq_{\mathfrak{k}^1} M'_1 \leq_{\mathfrak{k}^2} M$. [Why? If $N_1 \in K^{\mathfrak{k}^1}_{<\lambda_2}$ by the definition of $\leq_{\mathfrak{k}}$, and if $N_1 \in K^{\mathfrak{k}^1}_{\lambda_2}$ just choose $M'_1 = N_1$.]

Now by assumption (b), we can find $M'' \in K^{\mathfrak{k}^1}_{\lambda_2}$ such that $M'_0 \cup M'_1 \subseteq M'' \leq_{\mathfrak{k}} M$, hence by assumption (b) (i.e. **Ax.V** for $\mathfrak{k}^2$) we have $M'_0 \leq_{\mathfrak{k}} M''$, $M'_1 \leq_{\mathfrak{k}} M''$. As $N_0 \leq_{\mathfrak{k}} M'_0 \leq_{\mathfrak{k}} M'' \in K^{\mathfrak{k}}_{\leq\lambda_2}$ by assumption (a) we have $N_0 \leq_{\mathfrak{k}} M''$, and similarly we have $N_1 \leq_{\mathfrak{k}} M''$. So $N_0 \subseteq N_1$, $N_0 \leq_{\mathfrak{k}} M''$, and $N_1 \leq_{\mathfrak{k}} M'$, so by assumption (b) we have $N_0 \leq_{\mathfrak{k}} N_1$.

So we are left with the case '$\|N_1\| > \lambda$.' By assumption (b) there is $N_1' \in K_{\lambda_2}$ such that $N_0 \subseteq N_1' \leq_{\mathfrak{k}^2} N_2$. Also by assumption (b), we have $N_1' \leq_{\mathfrak{k}} M$, so by the previous paragraph we get $N_0 \leq_{\mathfrak{k}} N_1'$; together with the previous sentence we have $N_0 \leq_{\mathfrak{k}^1} N_1' \leq_{\mathfrak{k}^2} N_1$, so by the definition of $\leq_{\mathfrak{k}}$ we are done. $\square_{1.31}$

Definition 1.32. If $M \in K_\chi$ is $(\chi, \geq \kappa)$-superlimit$_1$, let

$$K_\chi^{[M]} = K_\chi[M] := \{N \in K_\chi : N \cong M\}$$

and $\mathfrak{k}_\chi^{[M]} := (K_\chi^{[M]}, \leq_{\mathfrak{k}} \restriction K_\chi^{[M]})$. We define $\mathfrak{k}^{[M]}$ as the $\mathfrak{k}'$ we get in 1.30(1), with $(\mathfrak{k}_\chi^{[M]}, \mathfrak{k}^{[M]})$ here standing in for $(\mathfrak{k}_\lambda, \mathfrak{k}')$ there.

Claim 1.33. *Assume $\mathfrak{k}$ is an (μ, λ, κ)-0-DAEC and $\chi \in [\lambda, \mu)$.*

If $M \in K_\chi$ is $(\chi, \geq \kappa)$-superlimit$_1$ (see Definition 1.26) then

- $\bullet_1$ *$\mathfrak{k}_\chi^{[M]}$ is a (χ^+, χ, κ)-0^+-DAEC.*
- $\bullet_2$ *In addition, $\mathfrak{k}_\chi^{[M]}$ satisfies* **Ax.III**(b) *and* **Ax.IV**(b).
- $\bullet_3$ *For each of the following axioms, if $\mathfrak{k}_\chi$ satisfies it then so does $\mathfrak{k}_\chi^{[M]}$:* **Ax.II**(a), **Ax.III**(c), **Ax.III**(e), **Ax.IV**(a).
- $\bullet_4$ *The same is true for* **Ax.IV**(d), *provided that M is $(\chi, \geq \kappa)$-superlimit$_2$.*
- $\bullet_5$ *If M is $(\chi, \geq \kappa)$-superlimit$_2$ then $\mathfrak{k}_\chi^{[M]}$ satisfies* **Ax.III**(d) *and* **Ax.IV**(d).

Proof. **Ax.O:** Easy.

Ax.I: As $\leq_{\mathfrak{k}_\chi[M]} := \leq_{\mathfrak{k}} \restriction K_\chi^{[M]}$, this is obvious.

Ax.II(a): Easy.

Ax.III(a)**:** Holds whenever $\mathfrak{k}_\chi$ satisfies it.
[Why? Let $\langle M_\alpha : \alpha < \delta < \chi^+\rangle$ by $\leq_{\mathfrak{k}_\chi[M]}$-increasing. As $\mathfrak{k}_\chi$ satisfies this axiom, there is $N \in K_\chi^{\mathfrak{k}}$ such that $\alpha < \delta \Rightarrow M_\alpha \leq_{\mathfrak{k}} N$. By the definition of superlimit (see 1.26(1)(a)), without loss of generality we can say $N \cong M$, and so N is as required.]

Ax.II(b)**:** Holds by Case 1 in 1.26(1)(d).

Ax.IV(a)**:** Holds whenever $\mathfrak{k}_\chi$ satisfies it.
Recalling 1.5(9) and **Ax.VI**, this is easy.

Ax.IV(b)**:** Follows by Case 1 in 1.26(1)(c).

Ax.III(c)**:** Holds whenever $\mathfrak{k}_\chi$ satisfies it.
[Why? Given a $\leq_{\mathfrak{k}_\chi[M]}$-increasing sequence $\langle M_s : s \in I\rangle$ with I κ-directed of cardinality $\leq \chi$, by our assumption there is $N \in K_\chi^{[M]}$ such that $s \in I \Rightarrow M_s \leq_{\mathfrak{k}} N$. By 1.26(1)(a), without loss of generality we have $N \cong M$, so N is as required.]

Ax.III(e)**:** Holds whenever $\mathfrak{k}_\chi$ satisfies it.
[Why? Similarly to the proof of **Ax.III**(c).]

Ax.IV(e)**:** Holds whenever $\mathfrak{k}_\chi$ satisfies it.
[Similar to earlier proof.]

Ax.V: Obvious, by the definition of $\leq_{\mathfrak{k}_\chi[M]}$.

Ax.VI: Trivial.

Ax.VII: [Amalgamation]
Holds by clause (1)(b) of Definition 1.26. $\square_{1.33}$

§ 2. PR FRAMES

Whereas in §1 we generalized AECs (abstract elementary classes), in §2 we try to generalize the concept of a 'good λ-frame' from [She09b, §1].

Now '$\mathfrak{s}$ is a good λ-frame' is trying to say "this class looks like superstable first-order theory," but we restrict ourselves to one cardinal and (as in the case of AECs) forget the formulas. Here we will try to get the analogue to $\aleph_1$-saturated models.

Convention 2.1. In this section, $\mathfrak{s}$ will denote a good (μ, λ, κ)-ι-frame and $\mathfrak{k} = \mathfrak{k}_{\mathfrak{s}}$ (see 2.2).

Below, the main case is $\iota = 4$.

Note that there is no connection between the iotas on the ι-frames here and the index on the ι-DAECs in §1.

Definition 2.2. Here $\iota = 0, 1, 2, 3, 4$. We say that $\mathfrak{s}$ is a *good* (μ, λ, κ)-ι-*frame* <u>when</u> $\mathfrak{s}$ consists of the following objects satisfying the following conditions: μ, λ, κ and[13]

(A) $\mathfrak{k} = \mathfrak{k}_{\mathfrak{s}}$ is a (μ, λ, κ)-4-DAEC (see 1.4(5)), and $\chi \in [\lambda, \mu) \Rightarrow \mathrm{LST}(\chi^{<\kappa})$ (by 1.17).

Below, we may write $\mathfrak{s}$ instead of $\mathfrak{k}$.

(B) $\mathfrak{k}$ has a $(\lambda, \geq \kappa)$-superlimit$_1$ model M^* — i.e.:
   - (a) $M^* \in K^{\mathfrak{s}}_\lambda$
   - (b) If $M_1 \in K^{\mathfrak{s}}_\lambda$ <u>then</u> for some $M_2 \in K^{\mathfrak{s}}_\lambda$, $M_1 <_{\mathfrak{s}} M_2$ and M_2 is isomorphic to M^*.
   - (c) If $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing, $\delta < \lambda^+$, $\mathrm{cf}(\delta) \geq \kappa$, and
   $$i < \delta \Rightarrow M_i \cong M$$
   <u>then</u> $\bigcup_{i<\delta} M_i$ is isomorphic to M^*.
   - (d) M^* is an amalgamation base in $\mathfrak{k}_{\mathfrak{s}}$.

(C)
   - (a) $\mathfrak{k}$ has the amalgamation property, the JEP (joint embedding property), and has no $\leq_{\mathfrak{k}}$-maximal member.
   - (b) If $\iota \geq 1$ then $\mathfrak{k}$ has primes over chains (i.e. **Ax.III**(f)).
   - (c) If $\iota \geq 4$ then $\mathfrak{k}$ has primes over $\leq_{\mathfrak{s}}$-directed sequences (i.e. **Ax.IV**(f)).

(D)
   - (a) $\mathcal{S}^{\mathrm{bs}} = \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}$ (the class of basic types for $\mathfrak{k}_{\mathfrak{s}}$) is included in $\bigcup_{M \in K_{\mathfrak{s}}} \mathcal{S}(M)$ and is closed under isomorphisms including automorphisms. For $M \in K_\lambda$, let $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) := \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}} \cap \mathcal{S}(M)$; there is no harm in allowing types of finite sequences.
   - (b) If $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, <u>then</u> p is non-algebraic (i.e., not realized by any $a \in M$).
   - (c) **Density**:
   If $M \leq_{\mathfrak{k}} N$ are from $K_{\mathfrak{s}}$ and $M \neq N$, <u>then</u> for some $a \in N \setminus M$ we have $\mathrm{ortp}(a, M, N) \in \mathcal{S}^{\mathrm{bs}}$. The intention in [She09b] was that examples would be minimal types[14] and regular types for superstable theories (but here this does not help).
   - (d) **bs-Stability**: $\mathcal{S}^{\mathrm{bs}}(M)$ has cardinality $\leq \|M\|^{<\kappa}$ for $M \in K_{\mathfrak{s}}$.

[13] So we should write $\mu_{\mathfrak{s}}, \lambda_{\mathfrak{s}}, \kappa_{\mathfrak{s}}$, but we may ignore them when defining $\mathfrak{s}$.

[14] On existence of minimal orbital types, see [She01] (and the revised version [She09c]).

(E) (a) $\bigcup = \bigcup_{\mathfrak{s}}$ is a four place relation called *non-forking*, with

$$\bigcup(M_0, M_1, a, M_3)$$

implying $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3$ are from $K_{\mathfrak{s}}$, $a \in M_3 \backslash M_1$, $\text{ortp}(a, M_0, M_3) \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(M_0)$, and $\text{ortp}(a, M_1, M_3) \in \mathcal{S}^{\text{bs}}(M_1)$. Also, $\bigcup$ is preserved under isomorphisms.

We may also write $M_1 \overset{M_3}{\underset{M_0}{\bigcup}} a$, and consider that $M_0 = M_1 \leq_{\mathfrak{k}} M_3$ are both in K_λ. We may state $M_1 \overset{M_3}{\underset{M_0}{\bigcup}} a$ as "$\text{ortp}(a, M_1, M_3)$ does not fork over M_0 (inside M_3)." (This is justified by clause (b) below.)

[Explanation: The intention is to axiomatize non-forking of types, but we allow ourselves to deal only with basic types. Note that in [She01] (i.e. [She09c]) we know something on minimal types but other types are something else.]

(b) **Monotonicity:**

$$\text{If } M_0 \leq_{\mathfrak{k}} M'_0 \leq_{\mathfrak{k}} M'_1 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3 \leq_{\mathfrak{k}} M'_3$$

and $M_1 \cup \{a\} \subseteq M''_3 \leq_{\mathfrak{k}} M'_3$, with all of them in K_λ, <u>then</u>

$$\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup(M'_0, M'_1, a, M'_3) \Leftrightarrow \bigcup(M'_0, M'_1, a, M''_3)$$

<u>so</u> it is legitimate to just say "$\text{ortp}(a, M_1, M_3)$ does not fork over M_0."

[Explanation: non-forking is preserved by decreasing the type, increasing the basis (i.e. the set over which it does not fork) and increasing or decreasing the model inside which all this occurs. The same holds for stable theories, only here we restrict ourselves to "legitimate" types.]

(c) **Local Character:**

Case 1: $\iota = 1, 2, 3$.
If $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-semi-continuous, $p \in \mathcal{S}^{\text{bs}}(M_\delta)$, and $\text{cf}(\delta) \geq \kappa$ <u>then</u> for every $\alpha < \delta$ large enough, p does not fork over M_α.

Case 2: $\iota = 4$.
If I is a κ-directed partial order, $\overline{M} = \langle M_t : t \in I \rangle$ is a $\leq_{\mathfrak{s}}$-directed system, M is its $\leq_{\mathfrak{k}}$-union, $M \leq_{\mathfrak{s}} N$, and $\text{ortp}(a, M, N) \in \mathcal{S}^{\text{bs}}(M_\delta)$ <u>then</u> for every $s \in I$ large enough $\text{ortp}(a, M, N)$ does not fork over M_s.

Case 3: $\iota = 0$.
Like Case 1, using a $(\geq \kappa)$-continuous $\overline{M}$.

[Explanation — This is a replacement for the following first-order case: assuming $\kappa \geq \kappa_r(T)$, if $p \in \mathcal{S}(A)$ then there is a $B \subseteq A$ of cardinality $< \kappa$ such that p does not fork over B.]

(d) **Transitivity**:
If $M_0 \leq_{\mathfrak{k}} M'_0 \leq_{\mathfrak{k}} M''_0 \leq_{\mathfrak{k}} M_3$ and $a \in M_3$ and $\text{ortp}(a, M''_0, M_3)$ does not fork over M'_0 and $\text{ortp}(a, M'_0, M_3)$ does not fork over M_0 (all models are in K_λ, of course, and necessarily the three relevant types are in $\mathcal{S}^{\text{bs}}$), <u>then</u> $\text{ortp}(a, M''_0, M_3)$ does not fork over M_0.

(e) **Uniqueness**:
If $p, q \in \mathcal{S}^{\text{bs}}(M_1)$ do not fork over $M_0 \leq_{\mathfrak{k}} M_1$ (all in $K_{\mathfrak{s}}$) and $p \upharpoonright M_0 = q \upharpoonright M_0$ <u>then</u> $p = q$.

(f) **Symmetry**:

Case 1: $\iota \geq 2$.
If $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$ and $(M_0, M_\ell, a_\ell) \in K^{3,\text{pr}}_{\mathfrak{s}}$ (see clause (j) and Definition 2.3 below) for $\ell = 1, 2$, <u>then</u> $\text{ortp}_{\mathfrak{s}}(a_2, M_1, M_3)$ does not fork over M_0 <u>iff</u> $\text{ortp}_{\mathfrak{s}}(a_1, M_2, M_3)$ does not fork over M_0.

Case 2: $\iota = 0, 1$.
If $M_0 \leq_{\mathfrak{k}} M_3$ are in $\mathfrak{k}_\lambda$, and for $\ell = 1, 2$ we have $a_\ell \in M_3$ and $\text{ortp}(a_\ell, M_0, M_3) \in \mathcal{S}^{\text{bs}}(M_0)$, <u>then</u> the following are equivalent:

- (α) There are M_1, M_3' in $K_{\mathfrak{s}}$ such that $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{K}} M_3'$, $a_1 \in M_1$, $M_3 \leq_{\mathfrak{k}} M_3'$, and $\text{ortp}(a_2, M_1, M_3')$ does not fork over M_0.
- (β) There are M_2, M_3' in K_λ such that $M_0 \leq_{\mathfrak{k}} M_2 \leq_{\mathfrak{k}} M_3'$, $a_2 \in M_2$, $M_3 \leq_{\mathfrak{k}} M_3'$ and $\text{ortp}(a_1, M_2, M_3')$ does not fork over M_0.

[Explanation: this is a replacement to "$\text{ortp}(a_1, M_0 \cup \{a_2\}, M_3)$ forks over M_0 iff $\text{ortp}(a_2, M_0 \cup \{a_1\}, M_3)$ forks over M_0," which is not well defined in our context.]

(g) **Existence**:
If $M \leq_{\mathfrak{s}} N$ and $p \in \mathcal{S}^{\text{bs}}(M)$ <u>then</u> there is $q \in \mathcal{S}^{\text{bs}}(N)$ which is a non-forking extension of p.

(h) **Continuity**:

Case 1: $\iota = 1, 2, 3$.
If $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing and $\leq_{\mathfrak{s}}$-semi-continuous, $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ (which holds if $\text{cf}(\delta) \geq \kappa$), $p \in \mathcal{S}_{\mathfrak{s}}(M_\delta)$, and $p \restriction M_\alpha$ does not fork over M_0 for $\alpha < \delta$ <u>then</u> $p \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(M_\delta)$ and it does not fork over M_0.

Case 2: $\iota = 4$.
Similarly, but for $\overline{M} = \langle M_t : t \in I \rangle$, I directed, and $M = \bigcup_{t \in I} M_t$ is a $\leq_{\mathfrak{s}}$-upper bound of $\overline{M}$.

Case 3: $\iota = 0$.
Like Case 1, with $\overline{M}$ being $(\geq \kappa)$-continuous.

(i) **Strong continuity**:

Case 1: $\iota = 1, 2$.
We have that $\text{ortp}(b, M_\delta, M_{\delta+1})$ does not fork over M_0 <u>when</u>:

- $\bullet_1$ $\overline{M} = \langle M_i : i \leq \delta + 1 \rangle$ is $\leq_{\mathfrak{s}}$-increasing.
- $\bullet_2$ M_δ is prime over $\overline{M} \restriction \delta$.
- $\bullet_3$ $b \in M_{\delta+1} \setminus M_\delta$
- $\bullet_4$ $\text{ortp}(b, M_i, M_{\delta+1})$ does not fork over M_0 for $i < \delta$.

Case 2: $\iota = 3, 4$.
We have that $\text{ortp}(b, N_0, N_1)$ does not fork over M_0 <u>when</u>:

- $\bullet_1$ $\overline{M} = \langle M_s : s \in I \rangle$ is $\leq_{\mathfrak{s}}$-increasing, I a partial order with $0 \in I$ minimal.
- $\bullet_2$ N_0 is prime over $\overline{M}$.
- $\bullet_3$ $b \in N_1 \setminus N_0$, where $N_0 \leq_{\mathfrak{s}} N_1$.
- $\bullet_4$ $\text{ortp}(b, M_s, N_1)$ does not fork over M_0 for all $s \in I$.

(j) **Existence of Primes**
If $\iota \geq 1$, $\mathfrak{s}$ has $K^{3,\text{pr}}_{\mathfrak{s}}$-primes (see 2.3 below).

(k) If $p \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(N)$ <u>then</u> p does not fork over M for some $M \leq_{\mathfrak{s}} N$ from K_λ.

Definition 2.3. 0) $K^{3,\mathrm{bs}}_{\mathfrak{s}} :=$

$$\big\{(M,N,a) : M \le_{\mathfrak{s}} N,\ a \in N,\ \text{and } \mathrm{ortp}_{\mathfrak{s}}(a,M,N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)\big\}$$

1) $K^{3,\mathrm{pr}}_{\mathfrak{s}} :=$

$$\big\{(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}} : \text{if } M \le N',\ a' \in N',\ \mathrm{ortp}_{\mathfrak{s}}(a',M,N') = \mathrm{ortp}(a,M,N) \text{ \underline{then} there is a } \le_{\mathfrak{k}}\text{-embedding of } N \text{ into } N' \text{ extending } \mathrm{id}_M \text{ and mapping } a \text{ to } a'\big\}$$

2) $\mathfrak{k}_{\mathfrak{s}}$ has $K^{3,\mathrm{pr}}_{\mathfrak{s}}$-primes <u>if</u> for every $M \in K_{\mathfrak{s}}$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ there exists a pair (N,a) such that $(M,N,a) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ and $\mathrm{ortp}_{\mathfrak{s}}(a,M,N) = p$.

Observation 2.4. 1) *If $\mathfrak{s}$ is a good ι-frame <u>then</u> it is a good j-frame, for all $0 \le \iota \le j \le 4$.*

2) *If $\iota = 4$ then clause* (E)(k) *of* 2.2 *follows from* (E)(h).

Proof. 1) Clauses (C) and (E)(c),(i),(j) in Definition 2.2 are all obvious.

Clause (E)(f):

We shall prove that if $\mathfrak{s}$ is a good j-frame and $0 \le \iota < 2 \le j$, <u>then</u> the demand in Case 2 of clause (E)(f) holds.

So we are given that $M_0 \le_{\mathfrak{k}} M_3$ are in $\mathfrak{k}_\lambda$, and (for $\ell = 1,2$) we have $a_\ell \in M_3$ and $\mathrm{ortp}(a_\ell, M_0, M_3) \in \mathcal{S}^{\mathrm{bs}}(M_0)$. Our job is to prove $(\alpha) \Leftrightarrow (\beta)$ as above, and by symmetry it suffices to prove $(\alpha) \Rightarrow (\beta)$.

So we assume

- (α) There are M_1, M_3' in $K_{\mathfrak{s}}$ such that $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{K}} M_3'$, $a_1 \in M_1$, $M_3 \le_{\mathfrak{k}} M_3'$, and $\mathrm{ortp}(a_2, M_1, M_3')$ does not fork over M_0.

Let $M_2 \le_{\mathfrak{k}} M_3$ be such that $(M_0, M_2, a_2) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ (this exists by clause (E)(j)) and similarly let $M_1' \le_{\mathfrak{k}} M_1$ be such that $(M_0, M_1', a_1) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.

By the present assumption and monotonicity (i.e. clause (E)(h)) $\mathrm{ortp}(a_2, M_1', M_3')$ does not fork over M_0, and neither does $\mathrm{ortp}(a_2, M_1', M_3)$ (over M_0).

By Case 1 of (E)(f), as $\mathfrak{s}$ is a good ι-frame, $\mathrm{ortp}(a_1, M_2, M_3)$ does not fork over M_0. Therefore (M_1', M_2) witnesses that clause (β) of (E)(f) Case 2 holds.

2) Easy, by Case 2 of clause (E)(h). $\square_{2.4}$

Claim 2.5. 1) *If $\langle M_i : i < \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing, $\sum\limits_{i<\delta} \|M_i\| < \mu$, $p_i \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ does not fork over M_0 for $i < \delta$, and $i < j \Rightarrow p_j \restriction M_i = p_i$, <u>then</u>:*

- (a) *We can find M_δ such that $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} M_\delta$.*
- (b) *For any such M_δ, we can find $p_\delta \in \mathcal{S}_{\mathfrak{s}}(M_\delta)$ such that $\bigwedge\limits_{i<\delta} [p_\delta \restriction M_i = p_i]$ and p_δ does not fork over M_0.*
- (c) *In clause (b), p_δ is unique.*
- (d) *If $\ell \ge \kappa \wedge \mathrm{cf}(\delta) \ge \kappa$, we can add '$M = \bigcup\limits_{\alpha<\delta} M_\alpha$.'*

2) *Similarly for $\overline{M} = \langle M_t : t \in I\rangle$ with I directed.*

Proof. 1) First, choose M_δ by 2.2, Clause (A). Second, choose $p_\delta \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$, a non-forking extension of p_0, which exists by Axiom (g) of 2.2(E). Now $p_\delta \restriction M_i \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ does not fork over M_0 by 2.2(E)(b) and it extends p_0, so it is equal to p_i by (E)(e). Third, p_δ is unique by (E)(e).

2) Should be clear as well. (Just replace the role of M_0 by M_{s_0} for some minimal $s_0 \in I$.) $\square_{2.5}$

Definition 2.6. 1) Assume $M_\ell \leq_{\mathfrak{s}} N$ and $p_\ell \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\ell)$ for $\ell = 1, 2$. We say that p_1, p_2 are parallel <u>when</u> some $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ is a non-forking extension of p_ℓ for $\ell = 1, 2$.

2) We say $\mathfrak{s}$ is *type-full* when $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathcal{S}^{\mathrm{na}}_{\mathfrak{k}_{\mathfrak{s}}}(M)$ for $M \in K_{\mathfrak{s}}$, where

$$\mathcal{S}^{\mathrm{na}}_{\mathfrak{k}}(M) := \big\{p \in \mathcal{S}_{\mathfrak{k}}(M) : p \text{ is not algebraic}\big\}$$

(That is, p is not realized by any $a \in M$. 'na' stands for *non-algebraic.*)

3) We say $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is based on $\bar{\mathbf{a}}$ when:

- (A) $\bar{\mathbf{a}}$ is a sequence of elements of M.
- (B) If $M \leq_{\mathfrak{s}} N$, $q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ is a non-forking extension of p, and π is an automorphism of N over $\bar{\mathbf{a}}$ then $\pi(q) = q$. (See [Sheb] for how we can guarantee the existence of such $\bar{\mathbf{a}} \in {}^{\lambda}M$, and even $\bar{\mathbf{a}} \in {}^{1}M$.)

3A) Similarly for $p \in \mathcal{S}^{\varepsilon}_{\mathfrak{s}}(M)$; similarly for part (4).

4) We say $\mathfrak{s}$ is *θ-based* when in clause 2.6(3) above there is such $\bar{\mathbf{a}} \in {}^{\theta>}M$.

We will show naturally that the older cases apply.

Example 2.7. Assume T is first-order complete and superstable. We define $\mathfrak{s} = \mathfrak{s}_T$ as follows.

$\boxplus$ $(\mu, \lambda, \kappa) := (\infty, |T|, \aleph_0)$
- (a) $K_{\mathfrak{s}} = K_{\mathfrak{k}_{\mathfrak{s}}}$ is the class of models of T.
- (b) $\leq_{\mathfrak{s}} = \leq_{\mathfrak{k}_{\mathfrak{s}}}$ means 'is an elementary submodel of.'
- (c) Pedantically, $\mathcal{S}_{\mathfrak{s}}(M)$ is

$$\big\{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{s}} N,\ a \in N \setminus M, \text{ and } \mathrm{tp}(a, M, N) \text{ is a regular type}\big\}.$$

Essentially, the reader should think of this as

$$\big\{\mathrm{tp}(a, M, N) : M \prec N,\ a \in N \setminus M, \text{ and } \mathrm{tp}(a, M, N) \text{ is regular}\big\}.$$

- (d) $\bigcup\limits_{\mathfrak{s}} := \big\{(M_0, M_1, a, M_3) : M_0 \prec M_1 \prec M_3$ are from $K_{\mathfrak{s}},\ a \in M_3 \setminus M_1,$ and $\mathrm{tp}(a, M_1, M_3)$ is regular and does not fork over $M_0\big\}$. (See [She90, Ch.III].)

Now,

- $\mathfrak{s}$ is a good 4-frame.

Concerning (weak) orthogonality and dominating (see Definitions 2.10, 2.13 below), they are as in [She90, Ch.V, §1-2] and have the 'nice' properties from 2.16-2.20, and much more. (See more in [She90, Ch.V, §3-4].)

Example 2.8. Assume T is first-order complete and strictly stable, and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$. We define $\mathfrak{s}$ as follows.

$\boxplus$ (a) $K_{\mathfrak{s}}$ is the class of κ-saturated (equivalently, $\mathbf{F}^a_\kappa$-saturated) models of T.

(b) $\leq_{\mathfrak{s}}$ is defined as $\prec\restriction K_{\mathfrak{s}}$ (as in 2.5).

(c) Pedantically, $\mathcal{S}_{\mathfrak{s}}(M)$ is

$$\big\{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{s}} N,\ a \in N \setminus M,\ \text{and } \mathrm{tp}(a, M, N) \text{ is a regular type}\big\}.$$

Essentially, the reader should think of this as

$$\big\{\mathrm{tp}(a, M, N) : M \prec N \text{ and } a \in N \setminus M,\ \text{and } \mathrm{tp}(a, M, N) \text{ is regular}\big\}.$$

(d) $\bigcup\limits_{\mathfrak{s}} := \big\{(M_0, M_1, a, M_3) : M_0 \prec M_1 \prec M_3$ are from $K_{\mathfrak{s}}$, $a \in M_3 \setminus M_1$, and $\mathrm{tp}(a, M_1, M_3)$ is regular and does not fork over $M_0\big\}$.
(See [She90, Ch.III].)

Now,

- $\mathfrak{s}$ is a good 4-frame, and even *strongly good* (see 2.13 below).

The analogues of Definitions 2.10, 2.13 appear in [She90, Ch.V, §1-2] with the basic properties from 2.13, 2.16, and more from [She90, Ch.V, §5] and [She91].
(For more, see Hernandez [Her92] and [Shea].)

Example 2.9. In [She09f], we have good 4-frames with $(\mu, \lambda, \kappa) = (\lambda^+, \lambda, \aleph_0)$, and get results similar to [She90, Ch.V, §1-3].

Definition 2.10. [Assume $\iota \geq 3$.]

1) Assume $p_1, p_2 \in \mathcal{S}^{\mathrm{bs}}(M)$. We say p_1, p_2 are *weakly orthogonal* (and denote it $p_1 \underset{\mathrm{wk}}{\perp} p_2$) when the following implication holds: if $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$, $(M_0, M_\ell, a_\ell) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$, and $\mathrm{ortp}_{\mathfrak{s}}(a_\ell, M_0, M_\ell) = p_\ell$ for $\ell = 1, 2$, then $\mathrm{ortp}_{\mathfrak{s}}(a_2, M_1, M_3)$ does not fork over M_0 (this is symmetric by Axiom (f) of 2.2(E), as $\iota \geq 3$).

2) We say p_1, p_2 are orthogonal (denoted $p_1 \perp p_2$) when: if $M \leq_{\mathfrak{s}} M_2$, $M_1 \leq_{\mathfrak{s}} M_2$ and $q_\ell \in \mathcal{S}^{\mathrm{bs}}(M_2)$ is a non-forking extension of p_ℓ and q_ℓ does not fork over M_1 then $q_1 \underset{\mathrm{wk}}{\perp} q_2$.

3) We say that $\{a_t : t \in I\}$ is independent in (M_0, M_1, M_2) when:

(A) $a_t \in M_2 \setminus M_1$
(B) $\mathrm{ortp}_{\mathfrak{s}}(a_t, M_1, M_2)$ does not fork over M_0.
(C) There is a sequence $\langle t(\alpha) : \alpha < \alpha_* \rangle$ listing I with no repetitions, and a $\leq_{\mathfrak{s}}$-increasing sequence $\langle M_{1,\alpha} : \alpha \leq \alpha_* + 1 \rangle$ with $M_1 \leq_{\mathfrak{s}} M_{1,0}$ and $M_2 \leq M_{1,\alpha_*+1}$ such that $a_{t(\alpha)} \in M_{1,\alpha+1}$ and $\mathrm{ortp}_{\mathfrak{s}}(a_{t(\alpha)}, M_{1,\alpha}, M_{1,\alpha+1})$ does not fork over M_0.

4) Let $(M, N, \mathbf{J}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ if $M \leq_{\mathfrak{s}} N$ and $\mathbf{J}$ is independent in (M, N).

5) Let $(M, N, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$ if:

(A) $M \leq_{\mathfrak{s}} N$
(B) $\mathbf{J}$ is independent in (M, N).
(C) If $M \leq_{\mathfrak{s}} N'$ and h is a one-to-one function from $\mathbf{J}$ into N' such that

$$(\forall a \in \mathbf{J})\big[\mathrm{ortp}(a, M, N) = \mathrm{ortp}(h(a), M, N)\big]$$

and $(M, N', \mathrm{rang}(h)) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$, then there is a $\leq_{\mathfrak{s}}$-embedding g of N into N' over M extending h.

Remark 2.11. We can now imitate relations of the axioms (as in [She09b, §2]), and basic properties of the notions introduced in 2.10.

Claim 2.12. [Assume $\iota \geq 1$.]

1) $(M, M, \varnothing) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$ *and* $(M, N, \varnothing) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *when* $M \leq_{\mathfrak{s}} N$.

2) $(M, N, a) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ *iff* $(M, N, \{a\}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$.

3) *We have* $(M, N, \mathbf{J}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *when*

- $\bullet_1$ $\overline{M} = \langle M_\alpha : \alpha \leq \alpha_* \rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing.*
- $\bullet_2$ $\mathbf{J}$ *is the disjoint union of* $\langle \mathbf{J}_\alpha : \alpha \leq \alpha_* \rangle$.
- $\bullet_3$ $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$
- $\bullet_4$ $\mathrm{ortp}(a, M_\alpha, M_{\alpha+1})$ *does not fork over* M_0 *(so it is* $\in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\alpha)$*) for every* $a \in \mathbf{J}_\alpha$.

4) *We have* $(M, N, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$ *when*

- $\bullet_1$ $\overline{M} = \langle M_\alpha : \alpha \leq \alpha_* \rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing semi-continuous.*
- $\bullet_2$ $\mathbf{J}$ *is the disjoint union of* $\langle \mathbf{J}_\alpha : \alpha \leq \alpha_* \rangle$.
- $\bullet_3$ $(M_\alpha, M_{\alpha+1}, \mathbf{J}_\alpha) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$
- $\bullet_4$ $\mathrm{ortp}(a, M_\alpha, M_{\alpha+1})$ *does not fork over* M_0 *(so it is* $\in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\alpha)$*) for every* $a \in \mathbf{J}_\alpha$.

5) *Assume* $(M, N.\mathbf{J}), \langle M_\alpha : \alpha \leq \alpha_* \rangle, \langle \mathbf{J}_\alpha : \alpha < \alpha_* \rangle$ *are as in part* (3). *If* $(M, N, \mathbf{J}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *then there is*

$$\overline{M}' = \langle M'_\alpha : \alpha \leq \alpha_* \rangle$$

as in part (4) *such that* $M'_\alpha \leq_{\mathfrak{s}} M_\alpha$ *for* $\alpha \leq \alpha_*$, *and so* $(M, M_{\alpha_*}, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$ *and* $M'_{\alpha_*} \leq_{\mathfrak{s}} N$.

6) *In part* (3), *if* $\beta_* \leq \alpha_*$ *then*

$$\big(M, M_{\beta_*}, \bigcup_{\beta < \beta_*} \mathbf{J}_\beta\big) \in K^{3,\mathrm{bs}}_{\mathfrak{s}} \text{ and } \big(M_{\beta_*}, M_{\alpha_*}, \bigcup_{\alpha \in [\beta_*, \alpha_*)} \mathbf{J}_\beta\big) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}.$$

7) *In part* (4), *if* $\beta_* \leq \alpha_*$ *then*

$$\big(M, M_{\beta_*}, \bigcup_{\beta < \beta_*} \mathbf{J}_\beta\big) \in K^{3,\mathrm{qr}}_{\mathfrak{s}} \text{ and } \big(M_{\beta_*}, M_{\alpha_*}, \bigcup_{\alpha \in [\beta_*, \alpha_*)} \mathbf{J}_\beta\big) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}.$$

Proof. Straightforward.

E.g. for part (5), let $\langle a_\gamma : \gamma < \gamma_* \rangle$ list $\mathbf{J}$, $\langle \beta_i : i \leq \alpha_* \rangle$ be increasing continuous, $\big\langle a_\gamma : \gamma \in [\beta_i, \beta_{i+1}) \big\rangle$ list $\mathbf{J}_i$, such that $\beta_0 = 0$ and $\beta_{\alpha_*} = \gamma_*$, and define $f : \gamma_* \to \alpha_*$ such that $f(\gamma) = i$ when $\gamma \in [\beta_i, \beta_{i+1})$.

We shall choose $M'_\alpha \leq_{\mathfrak{s}} M_{f(\gamma)}$ by induction on $\gamma \leq \gamma_*$ such that

- (A) $M'_0 \coloneqq M_0$
- (B) If $\gamma = \beta + 1$ then $(M'_\beta, M'_\gamma, a_\beta) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.
- (C) If $\gamma \leq \gamma_*$ is a limit ordinal, then M'_γ is $\leq_{\mathfrak{s}}$-prime over $\langle M_\beta : \beta < \gamma \rangle$.

Why can we carry the induction?

For $\alpha = 0$: Already given.

For $\alpha = \beta + 1$:
We know that $\mathrm{ortp}(a_\beta, M_\beta, M_\alpha) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\beta)$ does not fork over M_0 (by 2.12(3)$\bullet_3$).

For α limit:
Use 2.2(E)(i). $\square_{2.12}$

Definition 2.13. 1) We say p is *strongly dominated* by $\{p_t : t \in I\}$ (and write $p \leq_{\rm st} \{p_t : t \in I\}$)[15] <u>when</u>:

(A) $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$ and $p_t \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_t)$.

(B) If $N^+ \leq_{\mathfrak{s}} N^*$, $a_t \in N^*$, ${\rm ortp}(a_t, N^+, N^*) \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N^+)$ is parallel to p_t and $p' \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N^+)$ is parallel to p (see Definition 2.6), and $\{a_t : t \in I\}$ is independent in (N^+, N^*), <u>then</u> some $a \in N^*$ realizes p'.

2) We say p is *weakly* dominated by $\{p_t : t \in I\}$ (and write $p \leq_{\rm wk} \{p_t : t \in I\}$) <u>when</u> for some set J and function h from J onto I (not necessarily one-to-one) we have $p \leq_{\rm st} \{p_{h(t)} : t \in J\}$.

3) Let 'dominated' mean strongly dominated.

4) Let $\iota \geq 3$. We say $\mathfrak{s}$ is a *strongly good* ι-frame <u>when</u>

(A) It is a good ι-frame.

(B) If $\mathbf{J}$ is the disjoint union of $\mathbf{J}_1$ and $\mathbf{J}_2$, $(M, N, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$, $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N$, and $(M, M_1, \mathbf{J}) \in K^{3,\rm qr}_{\mathfrak{s}}$, <u>then</u> $(M_1, N, \mathbf{J}_2) \in K^{3,\rm bs}_{\mathfrak{s}}$ and ${\rm ortp}(a, M_1, N)$ does not fork over M for all $a \in \mathbf{J}_2$.

Hypothesis 2.14. For the rest of this section, assume $\mathfrak{s}$ is a strong ι-frame with $\iota \geq 1$.

Claim 2.15. [Assume $\iota \geq 1$.]

1) *If* $(M, N, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$ *and* $\bar{a}^* = \langle a^*_\beta : \beta < \beta_* \rangle$ *lists* $\mathbf{J}$ *with no repetitions,* <u>*then*</u> *we can find* $\overline{M} = \langle M_\beta : \beta \leq \beta_* \rangle$ *such that* $(M, N, \bar{a}, \overline{M})$ *is as in* 2.12(4) *(so* $(M, M_{\beta_*}, \mathbf{J}) \in K^{3,\rm qr}_{\mathfrak{s}}$*).*

2) $(M, N, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$ <u>*iff*</u> *for every finite* $\mathbf{I} \subseteq \mathbf{J}$ *we have* $(M, N, \mathbf{I}) \in K^{3,\rm bs}_{\mathfrak{s}}$*.*

3) *In* 2.13(4)(B)*, it is enough to consider the case where* $\mathbf{J}_2$ *is a singleton.*

4) *Clause* 2.2(E)(f) *holds, and so* $\mathfrak{s}$ *is a good* 2*-frame.*

Proof. 1) We shall choose M_β by induction on $\beta \leq \beta_*$ such that

$(*)_\beta$ (a) $\langle M_\gamma : \gamma \leq \beta \rangle$ is $\leq_{\mathfrak{s}}$-increasing semi-continuous.

(b) $M_0 := M$

(c) $(M_\gamma, M_{\gamma+1}, a^*_\gamma) \in K^{3,\rm pr}_{\mathfrak{s}}$ for all $\gamma < \beta$.

(d) $\big(M_0, M_\beta, \{a^*_\gamma : \gamma < \beta\}\big) \in K^{3,\rm qr}_{\mathfrak{s}}$.

Now $\beta = 0$ is handled; for β limit we use 2.12, and for $\beta = \gamma + 1$ we use 2.13(4)(B) with $(\mathbf{J}_1, \mathbf{J}_2) = \big(\{a^*_\alpha : \alpha < \gamma\}, \{a^*_\gamma\}\big)$, recalling 2.12(7).

2) We prove this by induction on $\theta := |\mathbf{J}|$. It is obvious for θ finite, and for θ infinite we let $\langle a_\alpha : \alpha < \theta \rangle$ list $\mathbf{J}$ without repetition, and work as in the proof of part (1).

3) By the proof of part (1).

4) Easy. $\square_{2.15}$

[15] This set may contain repetitions, so pedantically, we should use a sequence and write $p \leq_{\rm st} \langle p_t : t \in I \rangle$.

Claim 2.16. 1) *If p is strongly dominated by $\{p_t : t \in I\}$ <u>then</u> p is weakly dominated by $\{p_t : t \in I\}$.*

2) *Assume $\mathfrak{s}$ is type-full. If p is strongly dominated by $\{p_t : t \in I\}$, <u>then</u> for some $J \subseteq I$ of cardinality $< \kappa_{\mathfrak{s}}$, p is strongly dominated by $\{p_t : t \in J\}$.*

3) *Assume $\mathfrak{s}$ is type-full. p is weakly dominated by $\{p_t : t \in I\}$ <u>iff</u> for some $\langle i_t : t \in I\rangle$, p is strongly dominated by $\big\{p'_s : s \in \{(t,i) : t \in I,\ i < i_t\}\big\}$, where $p'_{(t,i)} = p_t$ and $i_t < \kappa_{\mathfrak{s}}$ for each $t \in I$.*

4) *In Definition 2.13(2), without loss of generality $(\forall s \in I)\big[\,\big|h^{-1}(\{s\})\big| < \kappa\big]$.*

5) *Preservation by parallelism — that is:*

- (A) *p_1 is strongly dominated by $\{p^1_t : t \in I\}$ <u>iff</u> p_2 is strongly dominated by $\{p^2_t : t \in I\}$, assuming*
  - (a) *$p_\ell \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N^\ell)$, $p^\ell_t \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N^\ell_t)$, $N^\ell_t \leq_{\mathfrak{s}} N^+ \in K_{\mathfrak{s}}$, and $N^\ell \leq_{\mathfrak{s}} N^+$, for $\ell = 1, 2$.*
  - (b) *p_1 and p_2 are parallel.*
  - (c) *p^1_t and p^2_t are parallel for all $t \in I$.*
- (B) *Similarly, replacing 'strongly dominated' by 'weakly dominated.'*

Proof. 1) Easy; in 2.15(1) we choose $J \coloneqq I$ and h the identity on I.

2) It suffices to prove '(A) $\Rightarrow$ (B)', where

- (A) $(M, N, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$, $\langle a_t : t \in I\rangle$ lists $\mathbf{J}$ with no repetitions, $\mathrm{ortp}(a_t, M, N)$ is parallel to p_t for $t \in I$, and $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is parallel to p.
- (B) There exists $I_* \subseteq I$ of cardinality $< \kappa$ and $N_* \leq_{\mathfrak{s}} N$ such that
$$(M, N_*, \{a_t : t \in I_*\}) \in K^{3,\rm qr}_{\mathfrak{s}}$$
and p is realized in N_*.

Without loss of generality
$$(M, N, \{a_t : t \in I\}) \in K^{3,\rm qr}_{\mathfrak{s}}$$
By the assumption, some $b \in N$ realizes q. We will try to choose $(I_\varepsilon, M_\varepsilon)$ by induction on $\varepsilon < \kappa$ such that

$(*)$
- (a) $\langle I_\zeta : \zeta \leq \varepsilon\rangle$ are pairwise disjoint subsets of I of cardinality $< \kappa$.
- (b) $\langle M_\zeta : \zeta \leq \varepsilon\rangle$ is $\leq_{\mathfrak{s}}$-increasing and semi-continuous, with $M_0 \coloneqq M$.
- (c) $(M_\zeta, M_{\zeta+1}, \{a_t : t \in I_{\zeta+1}\}) \in K^{3,\rm qr}_{\mathfrak{s}}$ for all $\zeta < \varepsilon$.
- (d) For all $\zeta < \varepsilon$, $\mathrm{tp}(b, M_{\zeta+1}, N)$ forks over M_ζ.

Of course, if $b \in M_\varepsilon$ we are done, and if we succeed in carrying the induction there is $M_\kappa \coloneqq \bigcup\limits_{\varepsilon<\kappa} M_\varepsilon \leq_{\mathfrak{s}} N$ by **Ax.III**(b)+**Ax.IV**(b).

Now we get a contradiction to clause (E)(i) of Definition 2.2.

3) By 2.16(2) and the definition.

4) By 2.16(4).

5) Easy. $\square_{2.16}$

Claim 2.17. *Assume $\mathfrak{s}$ is type-full. If $(M, N, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$ and $b \in N$, then there exist $\mathbf{I} \subseteq \mathbf{J}$ and M_1 such that:*

- (A) $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N$
- (B) $|\mathbf{I}| < \kappa_{\mathfrak{s}}$
- (C) $b \in M_1$

(D) $(M, M_1, \mathbf{I}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$

Proof. Without loss of generality $b \notin \mathbf{J}$. We try to choose N_i (and if possible, $\mathbf{I}_i$) by induction on $i \le \kappa$ such that

$(*)$ (a) $N_i \le_{\mathfrak{s}} N$ and $\mathbf{I}_i \subseteq \mathbf{J} \setminus \bigcup_{j<i} \mathbf{I}_j$, with $|\mathbf{I}_i| < \kappa$.

(b) If $j < i$ then $N_j \le_{\mathfrak{s}} N_i$ and $(N_j, N_{j+1}, \mathbf{I}_j) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$.

(c) $N_0 := M$

(d) If i is a limit ordinal then N_i is $\le_{\mathfrak{s}}$-prime over $\langle N_j : j < i\rangle$.

(e) If $i = j+1$ and N_j has already been defined with $b \notin N_j$, and there is $\mathbf{I} \subseteq \mathbf{J} \setminus \bigcup_{\ell<j} \mathbf{I}_\ell$ of cardinality $< \kappa$ (or simply finite) such that

$$\big(N_j, N, \mathbf{I} \cup \{b\}\big) \notin K^{3,\mathrm{bs}}_{\mathfrak{s}}$$

<u>then</u> we can choose such $\mathbf{I}$ as our $\mathbf{I}_j$ and choose $N_i \le_{\mathfrak{s}} N$ such that $(N_j, N_i, \mathbf{I}_j) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.

(f) **[Follows:]** $\big(N_i, N, \mathbf{J} \setminus \bigcup_{j<i} \mathbf{I}_j\big) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$.

If we carry the induction for all $i < \kappa$ we get a contradiction (see 2.2(E)(c)), so for some $i_* < \kappa$ we will hit a point where N_{i_*} is well defined, but $\mathbf{I}_{i_*}$ is not.

We prove, by induction on $\theta \le |\mathbf{J}|$, that if $\mathbf{I} \subseteq \mathbf{J}' := \mathbf{J} \setminus \bigcup_{j<i_*} \mathbf{I}_j$ has cardinality θ then $\big(N_{i_*}, N, \mathbf{I} \cup \{b\}\big) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$. So, using Case 1 of Definition 2.2(E)(i), we are finished. $\square_{2.17}$

Claim 2.18. 1) *If $p \le_{\mathrm{wk}} \{p_i : i < i^*\}$ and $i < i^* \Rightarrow q \perp p_i$ then $q \perp p$ (see Definition 2.3(3)).*

2) *Assume $\mathfrak{s}$ is type-full. If $p \le_{\mathrm{wk}} \{p_i : i < i^*\}$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>then</u> $p \not\perp p_i$ for some $i < i^*$.*

3) *If $p \le_{\mathrm{st}} \{p_i : i < \alpha\}$ then $p \le_{\mathrm{st}} \{p_i : i < \alpha,\ p_i \not\perp p\}$ (see Definition 2.13).*

Proof. 1) By induction on i^*: for i^* limit we use 2.2(E)(i), and for i^* successor use $q \perp p_{i^*-1}$.

2) By part (1) and 2.16(3).

3) Easy. $\square_{2.18}$

Claim 2.19. *Assume $\mathfrak{s}$ is type-full.*
If $\chi = \chi^{<\kappa} \in [\lambda, \mu)$, the following is impossible:

(a) *$\langle M_i : i < \chi^+\rangle$ is $\le_{\mathfrak{s}}$-increasing $\le_{\mathfrak{s}}$-semi-continuous.*

(b) *$\langle N_i : i < \chi^+\rangle$ is $\le_{\mathfrak{s}}$-increasing, $\le_{\mathfrak{s}}$-semi-continuous.*

(c) *$M_i \le_{\mathfrak{s}} N_i \in K_{\le \chi}$*

(d) *For some stationary $S \subseteq \{\delta < \chi^+ : \mathrm{cf}(\gamma) \ge \kappa\}$, for every $i \in S$,*
- *There is $a_i \in M_{i+1} \setminus M_i$ such that $\mathrm{ortp}(a_i, N_i, N_{i+1})$ is not the non-forking extension of $\mathrm{ortp}(a_i, M_i, M_{i+1}) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$.*

Proof. For some club E of χ^+, we have

$$i \in E \wedge j \in [i, \chi^+) \Rightarrow N_i \cap M_j = M_i.$$

For each $i \in S \cap E$, by 2.2(E)(c), there is a $j_i < i$ such that $\mathrm{ortp}(a_i, M_i, M_{i+1})$ does not fork over M_{j_i}. By clause (E)(i) of 2.2, for some $j \in [j_i, i)$, we have that $\mathrm{ortp}(a_i, N_j, N_{i+1})$ is not the non-forking extension of $\mathrm{ortp}(a_i, M_{j_i}, M_{i+1})$, so without loss of generality this holds for $j = j_i$.

By Fodor's Lemma, for some $j_* < j$, the set $S' := \{i \in S \cap E : j_i = j_*\}$ is stationary. So $\{b_i : i \in S'\}$ is independent in $\big(\bigcup_j M_j, M_{j+1}\big)$. By part (3) we are done.

Also, there is a sequence $\langle M_{j_*,\varepsilon} : \varepsilon \leq \varepsilon_* \leq \kappa\rangle$ which is $\varepsilon_{\mathfrak{s}}$-increasing continuous, with $M_{j_*,0} = M_{j_*}$, $M_{j_*,\varepsilon} = N_{j_*}$, and $(M_{j_*,\varepsilon}, M_{j_*,\varepsilon+1}, c_\varepsilon) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$. Now we can choose $\zeta_\varepsilon < \chi^+$ by induction on $\varepsilon < \varepsilon*$, increasing continuous, such that

$$\big\{a_i : i \in [\zeta_i, \chi^+)\big\}$$

is independent in $(M_{j_*,\varepsilon}, \bigcup_j N_j)$ and $\mathrm{ortp}(a_i, M_{j_*,\varepsilon}, N_{i+1})$ does not fork over M_{j_*} for $i \in [\zeta, \chi^+)$ — an easy contradiction. The induction works for $\varepsilon = 0$ trivially, for ε limit by 2.16(6), and for $\varepsilon = \xi + 1$ we use 2.17. $\square_{2.19}$

Claim 2.20. *If $p, p_i \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ for $i < \kappa_{\mathfrak{s}}$ and $i < j \Rightarrow p_i \perp p_j$, then $p \perp p_i$ for every $i < \kappa$ large enough.*

Proof. Follows from 2.19. (See more in [Shea, 1.6=Lj20].) $\square_{2.20}$

Remark 2.21. This is used in [BS18, 2.4=Lj35].

§ 3. Thoughts on the main gap

Here we address two problems: type theory (i.e. dimension, orthogonality, etc. as in [She98, Ch.V]) for strictly stable classes, and the main gap concerning somewhat saturated models. The hope always was that advances in the first will help the second.

Concerning the first-order case, work was started in [She90, Ch.V] (particularly §5) and [She91] and was advanced much further in Hernandez [Her92]; but this was not enough for the main gap for somewhat saturated models.

In the third part [Shea] we shall deal with the type dimension in a general framework.

* * *

The motivation for [She90] was

Remark 3.1 (**The Main Gap thesis**). This tells us that

1) For a reasonable class K of models, we have a dichotomy: <u>either</u> the class of models in a large enough cardinal is hopelessly complicated, or we have a structure theorem.

2) The original interpretation depends too much on cardinal arithmetic. It was as follows: we have many non-isomorphic models, so 'maximally complicated' means $I(\lambda, \kappa) = 2^\lambda$, <u>but</u> under strong violation of GCH

$$\aleph_\alpha \geq \aleph_\omega \Rightarrow 2^{|\alpha|} = 2^{\aleph_\alpha}.$$

However, some very uncomplicated theories have $2^{|\alpha|+\aleph_0}$ in $\aleph_\alpha$ (e.g. the theory of one equivalence relation).

3) So we had better assume GCH, or change the question to "can the isomorphism type of M code a stationary set?"

The main gap for $\aleph_1$-saturated models of a countable first order theory is open. *A priori*, it has looked easier than the one for models (which was preferred, being "the original question") because of the existence of prime models over any set $A \subseteq M$, but is still open. (The problem for uncountable first-order $|T|^+$-saturated models is as well).

Why doesn't the proof in [She90, Ch.XII] work? What's missing is, in $\mathfrak{C}^{\rm eq}$,

- ⊛ If $M_0 \prec M_1 \prec M_2$ are $\aleph_1$-saturated, $a \in M_2 \setminus M_1$, and $(a/M_1) \not\perp M_0$, <u>then</u> for some $b \in M_2 \setminus M_1$ we have $b \underset{M_0}{\bigcup} M_1$.

The central case is when a/M_1 is orthogonal to q if $q \perp M_0$.

Possible Approach 1: For T being first order countable, stable NDOP (even shallow) can try to understand types. See [LS06].

Possible Approach 2: We use the context dealt with in this paper. We are poorer in knowledge on the class <u>but</u> we have a richer $\mathfrak{C}^{\rm eq}$, so we may prove ⊛ even if it fails for T in the elementary case (this is a connection between [Sheb] and this work).

Possible Approach 3: We start with the context here. If things are not OK, we define such a derived DAEC; this was done in [She09g] and [She09b]. It may have non-structure properties — enough to get the maximal number of models up to isomorphism. If not, we arrive to a finer $\mathfrak{k}$, but still a case of our context. Similarly in limit. If we succeed enough times we shall prove that all is OK.

Possible Approach 4: Now we have a maximal non-forking tree $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ inside a somewhat saturated model; for [She90], e.g. $\|M_\eta\| \leq \lambda$, the models are λ^+-saturated, but we use content from here. If M is prime over $\bigcup_{\eta\in\mathcal{T}} M_\eta$ we are done, but maybe there is a residue. This appears in the following way: for $\eta \in \mathcal{T}$ and $p \in \mathcal{S}^{\rm bs}(M_\eta)$, the dimension of p is not exhausted by

$$\{a_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in \mathcal{T} \text{ and } (a_{\eta\hat{}\langle\alpha\rangle}/M_\eta) \not\perp p\}$$

but the lost part is not infinite! This imposes $\leq \lambda$ unary functions from $\mathcal{T}$ to $\mathcal{T}$. Now it seems to us that the question of whether this possible non-exhaustion can arise[16] is not a good dividing line, as though its negation is informative it is not clear whether it has any consequence. However, there are two candidates for dividing lines (actually, their disjunction seems to be what we want).

(A_*) We can find M, $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ as above and $\eta_* \in \mathcal{T}$ with $\ell g(\eta) = 2$, $\nu_* \in \mathcal{T}$ with $\ell g(\nu_*) = 1$, $\eta_* \restriction 1 \neq \nu_*$, and $p \in \mathcal{S}^{\rm bs}(M_{\eta_*})$, $p \perp M_{\eta\restriction 1}$ with a residue as above such that we need M_{ν_*} to explicate it.

More explicitly,

$(*)'$ If $M' \leq_{\mathfrak{s}} M$ is prime over $\bigcup_{\eta\in\mathcal{T}} M_\eta$ and we can find $a_{\eta_*,\nu_*} \in M \setminus M'$ such that $\mathrm{ortp}\big(\mathscr{C}(a_{\eta^*,\nu^*}, M'), \bigcup_{\eta\in\mathcal{T}} M_\eta\big)$ marks (M_{η_*}, M_{ν_*}).

Even in $(*)'$ we have to say more in order to succeed in using it.

From $(*)'$ we can prove a non-structure result: on $\mathcal{T}$ we can code any two-place relation R on $\{\eta \in \mathcal{T} : \ell g(\eta) = 1,\ M_\eta, M_{\eta_*\restriction 1}$ are isomorphic over $M_{\langle\ \rangle}\}$ which is of the form

$$\eta_1 \ R \ \eta_2 \Leftrightarrow (\exists \nu) \bigwedge_\ell \big[\text{there is } \eta' \text{ with } \eta_\ell \triangleleft \eta' \in \mathcal{T} \text{ and } \ell g(\eta') = 2,$$
$$\nu \in T \text{ with } \ell g(\nu) = 1, \text{ and there is } a_{\eta',\nu} \text{ as above}\big].$$

More complicated is the case

$(B)_*$ We can fix M and $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ as above, and find $\eta_*, \nu, \nu_* \in \mathcal{T}$ with $\ell g(\eta_*) = \ell g(\nu) = \ell g(\nu_*) = 1$ such that $(\eta_*, \nu), (\eta_*, \nu_*)$ are as above.

But whereas for (A_*) we have to make both η_* and ν_* not redundant in (B_*), in order to get non-structure we have to use a case of (B_*) which is not a "fake;" e.g. we cannot replace (M_{η_*}, a_{η_*}) by two such pairs.

That is, the "faker" is a case where we can find $M'_{\eta_*}, M''_{\eta_*}$ such that:[17]

- $\mathrm{NF}(M_{\langle\ \rangle}, M'_{\eta_*}, M''_{\eta_*}, M_{\eta_*})$
- M_{η_*} is prime over $M'_{\eta_*} \cup M''_{\eta_*}$.
- Only (M'_{η_*}, M_ν) and $(M''_{\eta_*}, M_{\nu_*})$ relate.

(C_*) If both (A_*) and (B_*), in the right formulation, do not appear then

  (α) **A good possibility**
  We can prove a structure theory: for M, $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ as above; that is, on each $\mathrm{suc}_{\mathcal{T}}(\eta)$ we have a two-place relation, but it is very simple: you have to glue some together or expand the set of successors by a tree structure.

If this fails, we may fall back to approach (3).

We may consider (see [She08], [PS18]):

[16] Essentially: there is a non-algebraic $p \in (M^\perp)^\perp$ which does not 1-dominate any $q \in \mathcal{S}(M)$.
[17] NF stands for *non-forking*; see [PS].

Question 3.2. 1) For an AEC $\mathfrak{k}$, when does the theory of a model in the logic $\mathscr{L} := $ "$\mathbb{L}_{\infty,\kappa}[\mathfrak{k}]$ enriched by dimension quantifiers" characterize models of $\mathfrak{k}$ up to isomorphism? (Similarly, also enriching by game quantifiers of length $\leq \kappa$.)

2) Prove the main gap theorem in the version: "if $\mathfrak{s}$ is n-beautiful then the main gap holds for $K_{\lambda^{+n}}$." (See [She09f, §12].) In particular, if $\mathfrak{s}$ has NDOP, then every $M \in K_{\lambda^{+n}}$ is prime over some non-forking tree of $\leq_{\mathfrak{k}_\mathfrak{s}}$-submodels $\langle M_\eta : \eta \in \mathcal{T}\rangle$, where each M_η is of cardinality $\leq \lambda$ and $\mathcal{T} \subseteq {}^{\omega>}(\lambda^{+n})$. If $\mathfrak{s}$ is shallow then the tree has depth at most $\mathrm{Depth}(\mathfrak{s}) < \lambda^+$, and we can draw a conclusion on the number of models.

Discussion 3.3. Assume stability in $\lambda_\mathfrak{s}$.

Let $M_0 \in K_\mathfrak{s}$, $\lambda_\mathfrak{s}^+$-saturated, at least for the time being.

1) Assume

$\boxplus_1$ $N_0 \leq_\mathfrak{s} N_1 \leq_\mathfrak{s} M$, $N_\ell \in K^\mathfrak{s}_\lambda$, $a \in N_0$, and $(N_0, N_1, a) \in K^{3,\mathrm{pr}}_\mathfrak{s}$.

We choose $(N^+_{1,i}, N_{1,i}, \mathbf{I}_i)$ and also, if possible, (M_1, a_i) by induction on $i \leq \lambda^+_\mathfrak{s}$ such that

- $(*)$ (a) $N_{0,i} \leq_\mathfrak{s} N_{1,i} \leq_\mathfrak{s} N^+_{1,i} \leq_\mathfrak{s} M$
  - (b) $\mathbf{I}_i \subseteq \{c \in M : \mathrm{ortp}(c, N_{1,i}, M_0) \perp N_0\}$ is independent in $(N_{1,i}, N^+_{1,i}, M)$ and minimal.
  - (c) $\langle N_j : j \leq i\rangle$ is $\leq_\mathfrak{s}$-semi-continuous; also, $\langle N^+_j : j \leq i\rangle$ is as well.
  - (d) If $i = j+1$ then $N^+_{1,i}$ is $\leq_\mathfrak{s}$-universal over $N^+_{1,j}$ and $(N_0, N_{1,i}, a) \in K^{3,\mathrm{pr}}_\mathfrak{s}$.
  - (e) If $j < i$ then $\mathbf{I}_j \setminus (N_i \cap \mathbf{I}_j) \subseteq \mathbf{I}_i$.
  - (f) If possible:
    - (α) $N_i \leq_\mathfrak{s} M^+_i \leq_\mathfrak{s} M$
    - (β) $(\mathbf{I}_i \setminus M_i)$ is independent in (M_i, M).
    - (γ) $a_i \in M \setminus (\mathbf{I}_i)$
    - (δ) $\mathrm{ortp}(a_i, M^*_1, M) \in \mathcal{S}^{\mathrm{bs}}_\mathfrak{s}(N^+_i)$ is $\perp N_i$.
    - (ε) $N^*_i \leq N_{1,i+1}$
  - (g) If $i = j + 1$ and there are (b, N^+_*, N_{**}) such that $b \in N^+_{1,j} \setminus N_{1,j}$,
$$N_{1,i} \leq_\mathfrak{s} N_* \leq_\mathfrak{s} N_{**} \in K^\mathfrak{s}_{\lambda_\mathfrak{s}},$$
$N^+_{1,i} \leq_\mathfrak{s} N_{**}$, and $\mathrm{ortp}_\mathfrak{s}(b, N_*, N_{**})$ forks over $N_{1,j}$ then, for some $b \in N^+_{1,j} \setminus N_{1,j}$, the type $\mathrm{ortp}_\mathfrak{s}(N_{1,i}, N^+_{1,i})$ forks over $N_{1,j}$.

There is no problem to carry the induction.

$\boxplus_2$ The following subsets of $\lambda^+_\mathfrak{s}$ are not stationary — say, disjoint to the club C:
- $S := \{i < \lambda^+_\mathfrak{s} : \mathrm{cf}(i) \geq \kappa_\mathfrak{s}$ and (M_i, a_i) is well defined$\}$
- $S_2 := \big\{i : \mathrm{cf}(i) \geq \kappa_\mathfrak{s}$ and for some $b \in N^+_{1,i},\ \mathrm{tp}(b, N_{1,i}, N^+_{1,i}) = N_0\big\}$.

2) Similarly, without (N_0, a) (and hence without "$\perp N_0$;" it's just simpler).

Definition 3.4. We say $(\overline{N}, \bar{a}, \bar{I})$ is a decreasing pair for M when for some n:

- (A) $\overline{N} = \langle N_\ell : \ell \leq n\rangle$ is $\leq_\mathfrak{s}$-increasing.
- (B) $N_\ell \leq_\mathfrak{s} M$, $N_\ell \in K^\mathfrak{s}_{\lambda_\mathfrak{s}}$
- (C) $\bar{a} = \langle a_\ell : \ell < n\rangle$
- (D) $(N_\ell, N_{i+1}, a_\ell) \in K^{3,\mathrm{pr}}_\mathfrak{s}$
- (E) $\bar{\mathbf{I}} = \langle \mathbf{I}_\ell : \ell \leq n\rangle$

(F) $\mathbf{I}_\ell$ is independent in (N_ℓ, M).

(G) $\mathbf{I}_\ell \subseteq \{c \in M : \text{ortp}(c, N_\ell, M) \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(N_\ell) \text{ is } \perp N_k \text{ if } k < \ell\}$

(H) If $N_\ell \leq_{\mathfrak{s}} N \leq_{\mathfrak{s}} M$, $b \in M \setminus N_0 \setminus \mathbf{I}_\ell$, and $\text{ortp}(b, N, M)$ is $\not\perp N_\ell$ but is orthogonal to N_k for $k < \ell$ then b depends on $\mathbf{I}_\ell$ in (N_ℓ, M).

Attempt to prove decomposition

We assume dimensional continuity to prove decomposition. If we would like to get rid of "M is $\lambda^+_{\mathfrak{s}}$-saturated", we must assume we have a somewhat weaker version $\mathfrak{s}_*$ of $\mathfrak{s}$ where $\lambda_{\mathfrak{s}_*} < \lambda_{\mathfrak{s}}$ and $\langle N_{0,i} : i < \lambda_{\mathfrak{s}}\rangle$ does $\leq_{\mathfrak{s}_*}$-represent N_0, and work with that. Assuming CH, $|T| = \aleph_0$ is fine. Without dimensional discontinuity, we will call 'nice' any $(\bar{N}, \bar{a}, \bar{\mathbf{I}})$ of length $\leq \kappa_{\mathfrak{s}}$!

* * *

Definition 3.5. We say $\mathbf{d} = (I, N, \bar{a}, \bar{\mathbf{I}}) = (I_{\mathbf{d}}, \overline{N}_d, \bar{a}_{\mathbf{d}}, \bar{\mathbf{I}}_{\mathbf{d}})$ is a partial decomposition of when:

⊞ (a) $I \subseteq {}^{\omega>}\text{Ord}$ is closed under initial segments.

(b) $\overline{N} = \langle N_\eta : \eta \in I\rangle$ (so $N_\eta = N_{\mathbf{d},\eta}$).

(c) $\bar{a} = \big\langle a_\eta : \eta \in I \setminus \{\langle\ \rangle\}\big\rangle$ (so $a_\eta = a_{\mathbf{d},\eta}$).

(d) $\bar{\mathbf{I}} = \langle \mathbf{I}_\eta = \mathbf{I}_{\mathbf{d},\eta} : \eta \in I\rangle$

(e) If $\eta \in I$ then

$$\big(\big\langle N_{\eta\restriction\ell} : \ell \leq \ell g(\eta)\big\rangle, \big\langle \bar{a}_{\eta\restriction(\ell+1)} : \ell < \ell g(\eta)\big\rangle, \big\langle \mathbf{I}_{\eta\restriction\ell} : \ell \leq \ell g(\eta)\big\rangle\big)$$

is nice in M.

(f) If $\eta \in I$ then $\langle a_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in I\rangle$ is a sequence of members of $\mathbf{I}_\eta$ with no repetitions.

Definition 3.6. Let $\leq_\mu$ be the following two-place relation on the set of decompositions of M:

$\bar{\mathbf{d}}_1 \leq_M \mathbf{d}_2$ iff

(A) $I_{\mathbf{d}_1} \subseteq \mathbf{I}_{d_1}$

(B) $\overline{N}_{\mathbf{d}_1} = \overline{N}_{\mathbf{d}_2} \restriction I_{d_1}$

(C) $\bar{a}_{\mathbf{d}_1} = \bar{a}_{\mathbf{d}_2} \restriction (I_{\mathbf{d}_1} \setminus \{< j\})$

(D) $\bar{\mathbf{I}}_{\mathbf{d}_1} = \bar{\mathbf{I}}_{d_2} \restriction I_{\mathbf{d}_1}$.

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