HOMOGENEOUS FORCING
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Abstract. Assume $\kappa = \kappa^{<\kappa}$ (usually $\aleph_0$ or an inaccessible).

We shall deal with iterated forcings preserving ${}^{\kappa>}\mathrm{Ord}$ and not collapsing cardinals along a linear order. The aim is to have homogeneous ones, so that for some natural ideals on ${}^{\kappa}2$, we get a model of $\mathsf{ZF} + \mathsf{DC}_\kappa$ + "modulo this ideal, every set is equivalent to a κ-Borel one."

The main application is improving the consistency result of Kellner and Shelah [KS11], and Horowitz and Shelah [HS] on saccharinity. But presently, we only have many automorphisms of the index set L and therefore of the iteration of iterands Q; we do not have homogeneity of Q, and we do not have automorphisms mapping names of Q-reals onto each other.

§ 0. Introduction

§ 0(A). **Aim.** We fix $\kappa = \kappa^{<\kappa}$ (maybe $\aleph_0$) and consider homogeneous iterations of $(<\kappa)$-complete forcing notions, with a version of κ^+-cc, preserving those properties.

To get homogeneity we intend to iterate along a linear order which is quite homogeneous (and therefore very much not well-ordered).

Ever since Solovay's celebrated work [Sol70], we know about the connection between the following two issues:

- $\bullet_1$ Forcing notions $\mathbb{P}$ with lots of automorphisms. E.g. for small $\mathbb{P}' \lessdot \mathbb{P}$ and two relevant $\mathbb{P}$-names $\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2$, generic for the same relevant forcing $\mathbb{Q}$ over $\mathbf{V}^{\mathbb{P}'}$, there is an automorphism of $\mathbb{P}$ over $\mathbb{P}'$ mapping $\underset{\sim}{\eta}_1$ to $\underset{\sim}{\eta}_2$.
- $\bullet_2$ Models of $\mathsf{ZF} + \mathsf{DC}+$ "every set of reals is equivalent to a Borel set modulo the null ideal (or other reasonable ideal)". (The relevant forcing $\mathbb{Q}$ was Random Real forcing for the null ideal — another prominent case: for the meagre ideal, Cohen forcing.)

Concerning the classical case of Lebesgue measurability, another formulation is "no non-measurable set is easily definable," formulated[1] in $\mathbf{L}[\mathbb{R}]$. See the history and more in [RS04], [RS06].

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References like e.g. [Sh:950, Th0.2$_{=\mathsf{Ly5}}$] mean that the internal label of Theorem 0.2 in Sh:950 is 'y5.' The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number 1257 on Saharon Shelah's list.

[1] That is, $\bullet_2$ holds for an inner model $\mathbf{L}[\mathcal{P}(\kappa)]^{\mathbf{V}}$ with $\mathbf{V} \models \mathsf{ZFC}$, so in $\mathbf{V}$ all 'reasonable' sets are 'measurable' for this ideal.

This applies to other ideals $\mathrm{id}(\mathbb{Q}, \underset{\sim}{\eta})$ for a definable forcing notion $\mathbb{Q}$ (mainly a ccc one) and a $\mathbb{Q}$-name $\underset{\sim}{\eta}$ of a real. Generally, it was not so easy to build such forcing notions: it required one to prove the existence of amalgamation in the relevant class of forcings. In Kellner-Shelah [KS11] it was suggested to look at so-called saccharine pairs $(\mathbb{Q}, \underset{\sim}{\eta})$, where $\mathbb{Q}$ is very non-homogeneous. (E.g. forcing with $\mathbb{Q}$ adds just one $(\mathbb{Q}, \underset{\sim}{\eta})$-generic, so we have few cases we need to build automorphisms for.)

Notation 0.1. 0) Given κ, the Borel sets are the smallest family of subsets of 2^κ containing all basic sets of the form $\{\nu \in 2^\kappa : \nu(\alpha) = i\}$ and closed under complements and union of $\leq\kappa$ many sets.

1) $\mathrm{id}_{<\partial}(\mathbb{Q}, \underset{\sim}{\eta})$ is the ideal consisting of the union of $<\partial$ many Borel sets $\mathbf{B}$ such that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin \mathbf{B}$".

2) Let $\mathrm{id}_{\leq\partial}(\mathbb{Q}, \underset{\sim}{\eta})$ be $\mathrm{id}_{<\partial^+}(\mathbb{Q}, \underset{\sim}{\eta})$.

3) $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ will denote ordinals; δ will be a limit ordinal if not stated otherwise.

4) $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

5) Recall that $\mathbb{L}_{\sigma,\sigma}$ is defined like first-order logic, but allowing $\bigwedge\limits_{i<\alpha} \varphi_i$ for $\alpha < \sigma$ and $(\exists \ldots x_i \ldots)_{i \in I}$ with I of cardinality $< \sigma$.

Comparing [KS11] to the older results (such as Solovay):

- $\bullet_{1.1}$ The forcing $\mathbb{Q}$ collapsed no cardinal, but was not ccc; this[2] we consider a drawback.
- $\bullet_{1.2}$ The model, as in those older results, does satisfy $\mathsf{ZF} + \mathsf{DC}$.
- $\bullet_{1.3}$ The iteration was along a homogeneous linear order.
- $\bullet_{1.4}$ We get only a somewhat weaker version of measurability, the ideal being $\mathrm{id}_{\leq\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$ instead of $\mathrm{id}_{<\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$.

Alternatively,

- $\bullet'_{1.4}$ Use $\mathrm{id}_{<\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta}) + X$, where X is the set $\{\underset{\sim}{\eta}[\mathbf{G}] : \mathbf{G} \subseteq \mathbb{Q}^{\mathbf{L}}$ is generic over $\mathbf{L}\}$.

The next step was Horowitz-Shelah [HS], where:

- $\bullet_{2.1}$ The forcing is ccc, which is a plus.
- $\bullet_{2.2}$ The model only satisfies ZF; we do not get DC or even $\mathsf{AC}_{\aleph_0}$ — not so good.
- $\bullet_{2.3}$ Again, the iteration is along a homogeneous linear order.
- $\bullet_{2.4}$ The ideal is again $\mathrm{id}_{\leq\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$ (or as in $\bullet'_{1.4}$ above).

Here (in 4.1) we regain both ccc (as in $\bullet_{2.1}$) as well as DC (as in $\bullet_{1.2}$). Moreover, we can demand $\mathsf{DC}_{\aleph_1}$ (or more — see §1) which is a significant plus.

We continue [She04b], [She], but do not rely on them. Instead of defining iterations we introduce them axiomatically and allow $\kappa > \aleph_0$ (in the support), but it suffices here to demand that the memory is a set, not an ideal. Unlike [She04b], the present paper does not address forcing $\mathfrak{a} > \mathfrak{d}$. Earlier continuations of [She04b] and [She] were the parallel papers, in preparation, with preliminary number F2009 and F2029 (and later, their descendants F2330 and F2329). There, as in [She04b],

[2] Note that Solovay uses Levy collapse of an inaccessible, but the later versions use ccc ones (mainly for the meagre ideal).

we sometimes replace the set $I_s^{\mathfrak{s}}$ (see 1.1) by an ideal (sometimes the whole power set) and use more general definable forcing notions.

In our iteration we are allowed to replace $\aleph_0$ by some $\kappa = \kappa^{<\kappa}$, so the forcing notions are $(<\kappa)$-complete κ^+-cc. But we need a forcing notion analogous to the one in [HS]: this will hopefully be done in a continuation (in preparation, preliminary number F2261).

§ 0(B). **Preliminaries.**

Hypothesis 0.2. 1) $\kappa = \kappa^{<\kappa}$ (mainly $\aleph_0$ or an inaccessible).

2) ∂ is a regular cardinal $> \kappa$.

3) D is a normal filter on κ^+ such that $S_\kappa^{\kappa^+} := \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\} \in D$.

Definition 0.3. Let $\mathbb{Q}$ be a forcing notion.

1) We say $\mathbb{Q}$ is a *strong κ-forcing* (or '$(\kappa, 1)$-forcing') <u>when</u>:

- (A) If $\kappa = \aleph_0$, then $\mathbb{Q}$ is Knaster (and hence ccc).
- (B) When $\kappa > \aleph_0$:
  - $\bullet_1$ $\mathbb{Q}$ satisfies $*^1_{\kappa,D}$ (which means a strong version of the κ^+-cc; see below in 0.3(4) and more in [She22, 0.2(B)(2)$_{a\texttt{=Lx2}}$]).
  - $\bullet_2$ $\mathbb{Q}$ is $(<\kappa)$-complete.
  - $\bullet_3$ Any increasing sequence of length $< \kappa$ has a lub.[3]

2) $\mathbb{Q}$ is a *weak* κ-forcing (or '$(\kappa, 2)$-forcing') <u>when</u>:

- (A) If $\kappa = \aleph_0$, then $\mathbb{Q}$ is a ccc forcing.
- (B) As in (1)(B).

3) Whenever we just write 'a κ-forcing,' we mean the strong version.

4) For D a normal filter on κ^+ containing $S_\kappa^{\kappa^+}$, we say the forcing notion $\mathbb{Q}$ satisfies $*^1_{\kappa,D}$ <u>when</u>:

$\kappa = \aleph_0$ and $\mathbb{Q}$ is ccc, <u>or</u> $\kappa > \aleph_0$ and

- $*_a$ Given a sequence $\langle p_i : i < \kappa^+\rangle$ of members of $\mathbb{P}$, there is a set[4] $C \in D$ and a regressive function $\mathbf{h}$ on C such that
$$\alpha, \beta \in C \wedge \mathbf{h}(\alpha) = \mathbf{h}(\beta) \Rightarrow \text{`}p_\alpha \text{ and } p_\beta \text{ have a lub.'}$$

Notation 0.4. 1) Here $\mathfrak{s}$ will denote a combinatorial template (that is, a member of $\mathbf{T}$ — see Definition 1.1).

2) Here $\mathbf{q}, \mathbf{r}, \mathbf{p}$ will denote ATIs (*abstract template iterations*); i.e. members of $\mathbf{Q}_{\mathrm{pre}}$ (the weakest version — see Definition 1.5).

3) L is a linear order (usually $L \subseteq L_{\mathfrak{s}}$) and $r, s, t \in L$.

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[3] It seems sufficient to just demand
- $\bullet_1'$ Instead of clause $(2)_a$ of [She22, 0.2(B)$_{\texttt{=Lx2}}$], we use the game of length ε of [She00] (with ε a limit ordinal $< \kappa$; the natural choice is $\varepsilon = \partial$).
- $\bullet_2'$ $\mathbb{Q}$ is strategically ζ-complete for every $\zeta < \kappa$.
- $\bullet_3'$ There exists some $\theta \in \mathrm{Reg}$ such that any increasing θ-sequence has a lub.

[4] Yes! Not just '$C \in D^+$;' see [She22].

L_+ is derived from L, with $\infty, t, t(+) \in L_+$ for $t \in L$. (See below in 1.1(2).)

4) $L_{\mathfrak{s}}$ or $L_{\mathbf{q}}$ will be the relevant linear order for $\mathfrak{s}$ or for $\mathbf{q}$, etc.

5) $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ denote forcing notions as in Definition 0.3 (which means quasi-orders).

§ 1. The frame

Definition 1.1. 0) Let $\mathbf{T}$ be the class of (∂,κ)-combinatorial templates (defined below), assuming $\partial = \mathrm{cf}(\partial) > \kappa$.

(∂ serves as an upper bound on the cardinality of some objects in the template: if there is no upper bound, we may write $\partial = \infty$ or we may omit it.)

1) A (∂,κ)-CT (a (∂,κ)-*combinatorial template*) $\mathfrak{s}$ consists of:

(a) A linear order $L_\mathfrak{s}$ (we could have used 'partial order'; it does not really matter for our purposes).
We may write $x \in \mathfrak{s}$ instead of $x \in L_\mathfrak{s}$, or $x <_\mathfrak{s} y$ instead of $x <_L y$.

(b) A sequence $\langle I^\mathfrak{s}_t : t \in L_\mathfrak{s}\rangle$, where $I_t = I^\mathfrak{s}_t \subseteq \{s \in L_\mathfrak{s} : s <_{L_\mathfrak{s}} t\}$ has cardinality $< \partial$.

2) For $\mathfrak{s} \in \mathbf{T}$, we add new objects $t(+)$ for all $t \in L_\mathfrak{s}$, as well as ∞, and define $L^+_\mathfrak{s}$, $L_{\mathfrak{s},x}$, $L^+_{\mathfrak{s},x}$, etc. as follows.

(a) $L^+_\mathfrak{s} := \{t, t(+) : t \in L_\mathfrak{s}\} \cup \{\infty\}$
(b) Naturally, $\langle t : t \in L_\mathfrak{s}\rangle\hat{\ }\langle t(+) : t \in L_\mathfrak{s}\rangle\hat{\ }\langle\infty\rangle$ is without repetition.
(c) $<_{L^+_\mathfrak{s}}$ is the closure, to a linear order, of the set

$$\big\{t < t(+) : t \in L_\mathfrak{s}\big\} \cup \big\{s(+) < t : s <_{L_\mathfrak{s}} t\big\} \cup \big\{t(+) < \infty : t \in L_\mathfrak{s}\big\}.$$

(d) For $t \in L^+_\mathfrak{s}$, let $L_{\mathfrak{s},t} := \{s \in L_\mathfrak{s} : s <_{L^+_\mathfrak{s}} t\}$ and $L^+_{\mathfrak{s},t} := \{s \in L^+_\mathfrak{s} : s <_{L^+_\mathfrak{s}} t\}$.

3) For $L \subseteq L_\mathfrak{s}$, we define $\mathfrak{s} \restriction L \in \mathbf{T}$ as follows.

- $\bullet_1$ $L_{\mathfrak{s}\restriction L} := L$
- $\bullet_2$ $I^{\mathfrak{s}\restriction L}_t := I^\mathfrak{s}_t \cap L$.

4) For $t \in L_\mathfrak{s}$, let $\mathfrak{s} \restriction t := \mathfrak{s} \restriction L_{\mathfrak{s},t}$.

5) We call $L \subseteq L_\mathfrak{s}$ *closed* (really, '$\mathfrak{s}$-closed') <u>when</u> $t \in L \Rightarrow I^\mathfrak{s}_t \subseteq L$. (E.g., if $L \trianglelefteq L_\mathfrak{s}$ is an end-extension of L).

6) We say $\mathfrak{s}$ is *closed* <u>when</u> $I^\mathfrak{s}_t$ is $\mathfrak{s}$-closed for every $t \in L_\mathfrak{s}$.

7) If $t \in L_\mathfrak{s}$ and $L \subseteq L_\mathfrak{s}$, we may abuse notation and write L_t in place of $L \cap L_{\mathfrak{s},t}$.

8) We say π is an *isomorphism from* $\mathfrak{s}_1$ *onto* $\mathfrak{s}_2$ (for $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbf{T}$) <u>when</u>

$$\pi : L_{\mathfrak{s}_1} \to L_{\mathfrak{s}_2}$$

is an order-preserving function mapping $I^{\mathfrak{s}_1}_t$ onto $I^{\mathfrak{s}_2}_{\pi(t)}$ for each $t \in L_{\mathfrak{s}_1}$.

Definition 1.2. We define a two-place relation $\leq_\mathbf{T}$ (obviously a partial order) on the class of combinatorial templates by:

$$\mathfrak{s}_1 \leq_\mathbf{T} \mathfrak{s}_2 \text{ \underline{iff}}$$

(a) $L_{\mathfrak{s}_1} \subseteq L_{\mathfrak{s}_2}$ as linear orders.
(b) We use in $\mathfrak{s}_1$ and $\mathfrak{s}_2$ the same ∞ and $t(+)$ for all $t \in L_{\mathfrak{s}_1}$.
(c) If $t \in L_{\mathfrak{s}_1}$ then $I^{\mathfrak{s}_1}_t = I^{\mathfrak{s}_2}_t$ (hence $L_{\mathfrak{s}_1}$ is $\mathfrak{s}_2$-closed).

Claim 1.3. 1) $\leq_{\mathbf{T}}$ *is indeed a partial order on* $\mathbf{T}$.

2) *If* $\langle \mathfrak{s}_\varepsilon : \varepsilon < \delta \rangle$ *is* $\leq_{\mathbf{T}}$*-increasing* <u>then</u> $\bigcup_{\varepsilon<\delta} \mathfrak{s}_\varepsilon$ *(naturally defined) exists, is a* $\leq_{\mathbf{T}}$*-lub, and is unique.*

Proof. Easy. $\square_{1.3}$

Definition 1.4. $\mathbf{Q}^{\rm wk}_{\mathfrak{s}}$ is the class of weak $\mathfrak{s}$-ATIs (see below), and

$$\mathbf{Q}_{\rm wk} := \bigcup_{\mathfrak{s}\in\mathbf{T}} \mathbf{Q}^{\rm wk}_{\mathfrak{s}}.$$

(ATI stands for *abstract template iterations.*)

Definition 1.5. For $\mathfrak{s}$ a combinatorial template, we say $\mathbf{q}$ is a *weak* $\mathfrak{s}$*-ATI* <u>when</u> it consists of the objects

- $\mathfrak{s} \in \mathbf{T}$ (we write $L_{\mathbf{q}}$ and $L_{\mathbf{q},t}$ instead of $L_{\mathfrak{s}}$ and $L_{\mathfrak{s},t}$),
- a quasiorder $\mathbb{P}$
- for all $t \in L_{\mathbf{q}}$:
  - a "ground model" set S_t,
  - in case $\kappa > \aleph_0$, a "ground model" function $\mathbf{H}_t : {}^{\kappa>}(S_t) \to S_t$, and
  - names (for a suitable poset, see below) $\underset{\sim}{\mathbb{Q}}_t$ and $\underset{\sim}{\eta}_t$,

such that the following is satisfied:

(A) (a) $\mathbb{P}$ is a weak κ-forcing notion (as in Definition 0.3(2)).
 (b) If $p \in \mathbb{P}$ then p is a function with domain $\mathrm{dom}(p) \in [L_{\mathbf{q}}]^{<\kappa}$.

(B) For $t \in L^+_{\mathbf{q}}$, we define $\mathbb{P}_t := \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq L_{\mathbf{q},t}\}$ (with the order from $\mathbb{P}$), and require:
 (a) $\mathbb{P}_t$ is a weak κ-forcing, and
 (b) $\mathbb{P}_t \lessdot \mathbb{P}$ (a complete subforcing).
 So a $\mathbb{P}$-generic filter $\underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ canonically gives us, for each $s \in L^+_{\mathfrak{s}}$, a $\mathbb{P}_s$-generic (over $\mathbf{V}$) filter, which we call $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_s}$.

(C) For $t \in L_{\mathbf{q}}$,
 (a) $\underset{\sim}{\mathbb{Q}}_t$ is a $\mathbb{P}_t$-name,
 (b) $\mathbb{P}_t$ forces that $\underset{\sim}{\mathbb{Q}}_t$ is a weak κ-forcing with set of elements S_t.
 (c) $\underset{\sim}{\eta}_t$ is a $\mathbb{P}_{t(+)}$-name of a member of ${}^{S_t}2$, (which we may identify with the subset $\underset{\sim}{\eta}_t^{-1}(1)$ of S_t),
 (d) $\mathbb{P}_{t(+)}$ forces that $\underset{\sim}{\eta}_t^{-1}(1)$ is $\underset{\sim}{\mathbb{Q}}_t[\mathbf{G}_{\mathbb{P}_t}]$-generic over $\mathbf{V}[\mathbf{G}_{\mathbb{P}_t}]$.
 (e) We set $\bar{\underset{\sim}{\eta}} := \langle \underset{\sim}{\eta}_t : t \in L_{\mathbf{q}} \rangle$ (a $\mathbb{P}$-name).

(D) (a) We require that $p \in \mathbb{P}$ <u>iff</u>: p is a function with $\mathrm{dom}(p) \in [L_{\mathbf{q}}]^{<\kappa}$, and for $s \in \mathrm{dom}(p)$, $p(s)$ is a $\mathbb{P}_s$-name of a member of $\underset{\sim}{\mathbb{Q}}_s$ (i.e., of S_s) of the following specific form: $p(s) = \mathbf{B}(\ldots, \underset{\sim}{\eta}_{t^{p(s)}_j}(\varepsilon^{p(s)}_j), \ldots)_{j<j_{p(s)}}$, where
  - $\bullet_1$ $t^{p(s)}_j \in I_s$, $\varepsilon^{p(s)}_j \in S_{t_j}$ and $j_{p(s)} \leq \kappa$.
  - $\bullet_2$ $\mathbf{B}$ is a κ-Borel function[5] from ${}^{(j_{p(s)})}2$ to S_s such that the image has cardinality $\leq \kappa$. More concretely: There is (in V) a $S'_{p(s)} \in [S_s]^{\leq\kappa}$ such that the image of $\mathbf{B}$ is subset of S'.

 (b) If $\varepsilon, \zeta \in S_s$, <u>then</u> the we require that the truth value of $\varepsilon \leq_{\underset{\sim}{\mathbb{Q}}_s} \zeta$ is similarly defined by such a κ-Borel function $\mathbf{B}_{s,\varepsilon,\zeta}$ (this time, the possible values of $\mathbf{B}_{s,\varepsilon,\zeta}$ are the truth values 0 and 1).

[5] That is, a function where the pre-image of every element of S_s is a $\leq\kappa$-Borel set. (The point here is absoluteness.)

(E) (a) Note that a $\mathbb{P}_s$-generic filter lets us evaluate the $\mathbb{P}_{t_j^{p(s)}(+)}$-names $\underset{\sim}{\eta}_{t_j^{p(s)}}$, and therefore the value of the Borel function $p(s)$. This way we get a $\mathbb{P}_s$-name for the value, which we may write as $p(s)[\mathbf{G}_{\mathbb{P}_s}]$ or as $p(s)(\bar{\eta} \restriction s)$.
(b) We require that $\underset{\sim}{\eta}_t^{-1}(1) = \{p(t)[\mathbf{G}_{\mathbb{P}_t}] : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{t(+)}}\}$.
(c) XXXX So we know that $p \in \mathbf{G}_{\mathbb{P}}$ implies:
(∗) For all $t \in \mathrm{dom}(p)$, $\nu_t(p(t)(\bar{\nu} \restriction L_{\mathbf{q},t})) = 1$.
It is unclear whether the converse automatically holds, if not we probably require it, then prove it later when we construct the iteration?

(F) We require that $p \le q$ in $\mathbb{P}$ iff
(a) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$
(b) If $s \in \mathrm{dom}(p)$ then $q \restriction L_{\mathbf{q},s} \Vdash_{\mathbb{P}_s}$ '$p(s)[\mathbf{G}_{\mathbb{P}_s}] \le_{\underset{\sim}{\mathbb{Q}}_s} q(s)[\mathbf{G}_{\mathbb{P}_s}]$'.
(Note that for $p \in \mathbb{P}$ and $s \in L^+_{\mathbf{q}}$ we have $p \restriction L_{\mathbf{q},s}$ is in $\mathbb{P}_s$.)
Note that this is a requirement and *not* a definition, unlike the classical case.

(G) (a) Given $p \in \mathbb{P}$ and $s \in \mathrm{dom}(p)$, let $\mathrm{supp}(p(s))$ be the set of all coordinates used in the Borel function $p(s)$ (i.e., the $t_j^{p(s)}$), as well as those used in the Borel function $\mathbf{B}_{s,\varepsilon,\zeta}$ (calculating whether $\varepsilon \le_{\underset{\sim}{\mathbb{Q}}_s} \zeta$) for all ε, ζ in $S'_{p(s)}$. So $|\mathrm{supp}(p(s))| \le \kappa$.
(b) Set $\mathrm{supp}(p) := \mathrm{dom}(p) \cup \bigcup_{s \in \mathrm{dom}(p)} \mathrm{supp}(p(s)) \in [L_{\mathbf{q}}]^{\le\kappa}$.
(c) Note that $\mathrm{supp}(p) \subseteq L_{\mathbf{q},t}$ iff $\mathrm{dom}(p) \subseteq L_{\mathbf{q},t}$, i.e., iff $p \in \mathbb{P}_t$.
(d) (Generalizing $\mathbb{P}_s$ as the restriction to L_s:)
For $L \subseteq L_{\mathbf{q}}$ $\mathfrak{s}$-closed, we set $\mathbb{P}_L := \{p \in \mathbb{P} : \mathrm{supp}(p) \subseteq L\}$ (with the order of $\mathbb{P}$), and require
$\bullet_1$ $\mathbb{P}_L$ is a weak κ-forcing, and
$\bullet_2$ $\mathbb{P}_L \lessdot \mathbb{P}$.
$\bullet_3$ $\underset{\sim}{\bar{\eta}} \restriction L$ is a generic of $\mathbb{P}_L$. (Elaborate? XXXXX)
$\bullet_4$ Note that if L is closed, then so is L_s for any $s \in L^+_{\mathbf{q}}$, and therefore $\mathbb{P}_{L_s} \lessdot \mathbb{P}_L$ and $\mathbb{P}_{L_s} \lessdot \mathbb{P}_s$.

(H) If $\kappa > \aleph_0$ and $t \in L_{\mathfrak{s}}$, then there is

$$\mathbf{H}_t : {}^{\kappa>}(S_t) \to S_t$$

such that:
(a) $\Vdash_{\mathbb{P}_t}$ "if $\eta \in {}^{\kappa>}(S_t)$ is $\le_{\underset{\sim}{\mathbb{Q}}_t}$-increasing then $\mathbf{H}_t(\eta)$ is a lub of $\{\eta(i) : i < \ell g(\eta)\}$".
(b) If $\eta \in {}^2S_t$ and $\{\eta(0), \eta(1)\}$ has a $\le_{\mathbb{Q}_t}$-lub then $\mathbf{H}_t(\eta)$ is some lub.

(I) When dealing with different ATIs $\mathbf{q}$s, instead of $\mathbb{P}$, $\le$, $\mathbb{P}_t$, S_t, $\underset{\sim}{\mathbb{Q}}_t$, etc we may write $\mathbb{P}_{\mathbf{q}}$, $\le_{\mathbf{q}}$, $\mathbb{P}_{\mathbf{q},t}$, $S_{\mathbf{q},t}$, $\underset{\sim}{\mathbb{Q}}_{\mathbf{q},t}$ etc, to indicate that we mean the component of the respective $\mathbf{q}$.

Remark 1.6. 1) Recall that $L_{\mathbf{q}}$ is just a linear order and not necessarily a well-ordering. More concretely, we do not even exclude the possibility that there is an infinite sequence $(s_n)_{n\in\omega}$ with $s_{n+1} \in I_{s_n}$.

2) As a consequence: Given $L_{\mathbf{s}}$ and a sequence of (e.g., Definitions for) $\mathbb{Q}_s$, it is not clear whether there is an according iteration $\mathbb{P}$; nor whether it is unique.

(In contrast, the usual forcing iteration assumes that the index set is wellordered, and we always get a welldefined iteration from a sequence of iterands.)

3) But if $\mathfrak{s}$ is as in [She04b, §2], then it is unique.

Definition 1.7. 1) We define $\mathbf{Q}^{\rm st}_{\mathfrak{s}}$, $\mathbf{Q}_{\rm st}$, and say '*strong* ATI' when we replace "weak κ-forcing" by "strong κ-forcing" wherever it appears in 1.5.

2) We define $\mathbf{Q}_{\rm pre}$, $\mathbf{Q}^{\rm pre}_{\mathfrak{s}}$ as in Definition 1.5, replacing "weak κ-forcing" by "forcing" wherever it appears in 1.5.

3) Let $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$ be shorthand for $\mathbf{Q}_{\rm pre}, \mathbf{Q}_{\rm wk}$, and $\mathbf{Q}_{\rm st}$, respectively.

4) When we omit the subscripts, we mean 'weak.'

5) If $\mathbf{q} \in \mathbf{Q}_{\rm pre}$ and $L \subseteq L_{\mathbf{q}}$ is $\mathfrak{s}_{\mathbf{q}}$-closed, then $\mathbf{p} = \mathbf{q} \restriction L$ is defined by $\mathfrak{s}_{\mathbf{p}} := \mathfrak{s}_{\mathbf{q}} \restriction L$ and $\mathbb{P}_{\mathbf{p}} := \mathbb{P}_{\mathbf{q},L}$.

6) We define "π is an isomorphism from $\mathbf{q}$ onto $\mathbf{p}$" naturally.

7) We define $\mathbf{Q}^*_\ell$ (for $\ell = 0, 1, 2$, or pre, wk and st) as the class of $\mathbf{q} \in \mathbf{Q}_\ell$ such that

$$s \in L_{\mathbf{q}} \Rightarrow |S_{\mathbf{q},s}| < \partial.$$

(We shall only use ∂ starting with 2.4.)

Observation 1.8.

- *If $p \in \mathbb{P}$ and $L \subseteq \mathrm{dom}(p)$, then $p \restriction L \in \mathbb{P}$ and $p \restriction L \leq p$. If additionally L is closed, then $p \restriction L \in \mathbb{P}_L$.*
- *If $L \subseteq L_{\mathbf{q}}$ is closed, and $p \in \mathbb{P}_L$, σ a $\mathbb{P}_L$-name and $\varphi(x)$ a formula absolute between forcing extensions, then $p \Vdash_{\mathbb{P}_L} \varphi(\sigma)$ iff $p \Vdash_{\mathbb{P}} \varphi(\sigma)$.*
- *If $L \subseteq L_{\mathbf{q}}$ is closed, and p, q in $\mathbb{P}_L$, then $p \leq q$ iff (F) holds for $\mathbb{P}_L$, i.e., iff*
  - $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, *and*
  - *If $s \in \mathrm{dom}(p)$ then $q \restriction L_s \Vdash_{\mathbb{P}_{L_s}}$ '$p(s)[\mathbf{G}_{\mathbb{P}_{L_s}}] \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)[\mathbf{G}_{\mathbb{P}_{L_s}}]$'.*
- *If $p \leq q$ in $\mathbb{P}$ and $s \in L$, then $p \restriction L_s \in \mathbb{P}_s$ and $q \restriction L_s \leq p \restriction L_s$; and the same holds for $\mathbb{P}_L$ and $\mathbb{P}_{L,s}$ for L closed.*

Proof. Easy. □

Observation 1.9. *Let $\mathbf{q} \in \mathbf{Q}_{\rm pre}$.*

1) *If $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed, $p \in \mathbb{P}_{\mathbf{q}}$, and $p \restriction L \leq q$ in $\mathbb{P}_{\mathbf{q},L}$, then*

$$r := \big(p \restriction (\mathrm{dom}(p) \setminus L)\big) \cup q$$

is a lub of p and q.

2) *For $\mathbf{q}$-closed L, we have $\mathbb{P}_{\mathbf{q},L} \models$ "$p \leq q$" iff[6] ($p, q \in \mathbb{P}_{\mathbf{q},L}$ and)*

- $\bullet_1$ $\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \subseteq L$
- $\bullet_2$ *If $s \in \mathrm{dom}(p)$ then for some $\mathbf{q}$-closed L_1 satisfying $I^{\mathbf{q}}_s \subseteq L_1 \subseteq L \cap L_{\mathbf{q},s}$, we have $q \restriction L_1 \Vdash_{\mathbb{P}_{L_1}}$ "$p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)$".*

3) *Like (2), but in $\bullet_2$ we replace "for some" with "for every."*

3A) *Like (2), but in $\bullet_2$ we demand $L_1 = I^{\mathbf{q}}_s$.*

4) *If $\mathbf{q}$ is closed, then in (2)$\bullet_2$ we can choose $L_1 = I^{\mathbf{q}}_s$.*

Proof. 1) Note

[6] Note that $\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \subseteq L$ does not imply $\mathrm{supp}(p) \subseteq \mathrm{supp}(q)$; we could add that demand, but have chosen not to.

$(*)_1$ $r \in \mathbb{P}_{\mathbf{q}}$.

[Why? First, r is a well-defined function. Second, $\mathrm{dom}(r) \in [L_{\mathbf{q}}]^{<\kappa}$, and third, for $s \in \mathrm{dom}(r)$, $r(s)$ is a Borel function as required.]

$(*)_2$ $\mathbb{P}_{\mathbf{q}} \models$ '$q \le r$'

As $r \restriction \mathrm{dom}(q) = q$, this is trivial.

$(*)_3$ $\mathbb{P}_{\mathbf{q}} \models$ '$p \le r$'

We have to check 1.5(F). Now (a) is trivial, as $\mathrm{dom}(p \restriction L) \subseteq \mathrm{dom}(q) \subseteq L$; as for (b), let $s \in \mathrm{dom}(p)$ and we have two possibilities to check: If $s \in \mathrm{dom}(p) \setminus L$, then again $r(s) = p(s)$, so this is clear. So assume that $s \in \mathrm{dom}(p) \cap L$. We have to show that $r \restriction L_{\mathbf{q},s}$ forces that $r(s) \ge p(s)$. But $r \restriction L_{\mathbf{q},s} \le q \restriction L_{\mathbf{q},s}$, and $r(s) = q(s)$, and $q \restriction L_{\mathbf{q},s}$ forces that $q(s) \ge p(s)$.

$(*)_4$ If $\mathbb{P}_{\mathbf{q}} \models$ "$p \le r' \wedge q \le r'$" then $\mathbb{P}_{\mathbf{q}} \models r \le r'$.

Easy as well.

2-4) Also straightforward. $\square_{1.9}$

Definition 1.10. 1) Let $\mathbf{q}_1 \le^{\mathrm{wk}}_{\mathbf{Q}} \mathbf{q}_2$ mean:

- (a) $\mathbf{q}_\ell$ is a weak $\mathfrak{s}_\ell$-ATI for $\ell = 1, 2$ (where $\mathfrak{s}_\ell = \mathfrak{s}_{\mathbf{q}_\ell}$; recall that $\mathbf{q}_\ell$ determines $\mathfrak{s}_\ell$).
- (b) $\mathfrak{s}_1 \le_{\mathbf{T}} \mathfrak{s}_2$
- (c) $\mathbb{P}_{\mathbf{q}_1} \lessdot \mathbb{P}_{\mathbf{q}_2}$, which implies $\mathbb{P}_{\mathbf{q}_1,t} \lessdot \mathbb{P}_{\mathbf{q}_2,t}$ for $t \in L_{\mathfrak{s}_1}$.
- (d) For $t \in L_{\mathfrak{s}_1}$, we have $S_{\mathbf{q}_2,t} = S_{\mathbf{q}_1,t}$ and $\underset{\sim}{\mathbb{Q}}^{\mathbf{q}_1}_t = \underset{\sim}{\mathbb{Q}}^{\mathbf{q}_2}_t$.
- (e) $\Vdash_{\mathbb{P}_{\mathbf{q}_2}}$ "$\underset{\sim}{\eta}^{\mathbf{q}_1}_t = \underset{\sim}{\eta}^{\mathbf{q}_2}_t$" for $t \in L_{\mathfrak{s}_1}$.

2) We define $\le^{\mathrm{pre}}_{\mathbf{Q}}$ as above, changing clause (a) to '$\mathbf{q}_\ell \in \mathbf{Q}_{\mathrm{pre}}$' and weakening clause (c) to '$\mathbb{P}_{\mathbf{q}_1} \subseteq \mathbb{P}_{\mathbf{q}_2}$.'

We define $\le_{\mathbf{Q}_2} := \le_{\mathbf{Q}} \restriction \mathbf{Q}_2$.

XXX what is $\mathbf{Q}_2$? Also, if you are not completely embedded, how can you formulate that it is forces that both $\mathbb{Q}_t$ are the same?

2A) If $\mathbf{r} \le^{\mathrm{pre}}_{\mathbf{Q}} \mathbf{q}$ and $p \in \mathbb{P}_{\mathbf{q}}$, then we define $q := p \restriction \mathbf{r}$ as follows:

- $\bullet_1$ $\mathrm{dom}(q) = \mathrm{dom}(p) \cap L_{\mathbf{r}}$
- $\bullet_2$ If $s \in \mathrm{dom}(q)$ then $q(s) = p(s)$ (recalling 1.2(b) and 1.5(D)(a)).

3) If $\langle \mathbf{q}_\alpha : \alpha < \delta \rangle$ is $\le_{\mathbf{Q}}$-increasing then "$\mathbf{q} := \bigcup\limits_{\alpha<\delta} \mathbf{q}_\alpha$" will mean the following:

- (a) $\mathbf{q} \in \mathbf{Q}$
- (b) $\mathfrak{s}_{\mathbf{q}} := \bigcup\limits_{\alpha<\delta} \mathfrak{s}_{\mathbf{q}_\alpha}$
- (c) $\mathbf{q}_\alpha \le_{\mathbf{Q}} \mathbf{q}$ for all $\alpha < \delta$.
- (d) **[Follows]** If $s \in L_{\mathbf{q}_\alpha}$ <u>then</u> $\underset{\sim}{\mathbb{Q}}^{\mathbf{q}}_s = \underset{\sim}{\mathbb{Q}}^{\mathbf{q}_\alpha}_s$ and $\underset{\sim}{\eta}^{\mathbf{q}}_s = \underset{\sim}{\eta}^{\mathbf{q}_\alpha}_s$.

4) We say $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha < \alpha_* \rangle$ is $\le_{\mathbf{Q}}$-*increasing continuous* <u>if</u> it is $\le_{\mathbf{Q}}$-increasing and $\mathbf{q}_\delta = \bigcup\limits_{\alpha<\delta} \mathbf{q}_\alpha$ for every limit $\delta < \alpha_*$.

Remark 1.11. 1) Note that in parts (3),(4) of Definition 1.10, for a given $\langle \mathbf{q}_\alpha : \alpha < \delta\rangle$, it is not *a priori* clear that such $\mathbf{q}$ exists — and even if it does, whether it is unique.

2) Regarding 1.10(1)(c), does "$\mathbb{P}_{\mathbf{q}_1} \lessdot \mathbb{P}_{\mathbf{q}_2}$" follow by 1.5(G)(d), as $L_{\mathfrak{s}_1}$ is $\mathbf{q}_2$-closed by Definition 1.2? This is not clear. (See 1.6(2).)

3) We can only show that given $\mathbf{q}$ and a $\mathbf{q}$-closed $L \subseteq L_{\mathbf{q}}$, we have $(\mathbf{q} \restriction L) \leq_{\mathbf{Q}} \mathbf{q}$.

Observation 1.12. 1) *Assume* $\mathbf{q}_1 \leq^{\rm pre}_{\mathbf{Q}} \mathbf{q}_2$.

- (A) *If* $p \in \mathbb{P}_{\mathbf{q}_1}$ *and* $q \in \mathbb{P}_{\mathbf{q}_2}$, *<u>then</u> we have* $(a) \Leftrightarrow (b)$, *where:*
  - (a) $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \leq q$"
  - (b) *If* $s \in \mathrm{dom}(p)$ *then* $s \in \mathrm{dom}(q) \wedge q \restriction L_{\mathbf{q}_1,s} \Vdash_{\mathbb{P}_{\mathbf{q}_1,s}}$ "$p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)$".
- (B) *If* $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \leq q$" *and* $s \in \mathrm{dom}(p) \cap L_{\mathbf{q}_1}$, *<u>then</u>*
$$q \restriction L_{\mathbf{q}_1,s} \Vdash_{\mathbb{P}_{\mathbf{q}_1,s}} \text{“}p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)\text{”}.$$
- (C) *Assume there exist* $L^1_1, L^2_1, L^1_2, L^2_2$ *such that:*
  - (a) $L^2_1 \lhd L^2_2 \trianglelefteq L_{\mathbf{q}_2}$
  - (b) $\bigwedge\limits_{\ell=1}^{2} [L^1_\ell = L^2_\ell \cap L_{\mathbf{q}_1}]$
  - (c) $p \in \mathbb{P}_{\mathbf{q}_2 \restriction L^2_1}$ *and* $q \in \mathbb{P}_{\mathbf{q}_1 \restriction L^1_2}$.
  - (d) $\mathbb{P}_{\mathbf{q}_2, L^2_1} \models q \restriction L^1_1 \leq p^+$.

  If in addition, $p^+ \in \mathbb{P}_{\mathbf{q}_2 \restriction L^1_1}$ *is* $\leq_{\mathbb{P}_{\mathbf{q}_2}}$*-above* $q \restriction L_{\mathbf{q}_1 \restriction L^1_1}$ *and* $p \restriction L_{\mathbf{q}_1 \restriction L^1_1}$, *<u>then</u>* $\{p, p^+, q\}$ *have a common upper bound in* $\mathbb{P}_{\mathbf{q}_2 \restriction L^2_2}$.

  XXX what does this mean? Please check all indices i, j, k *in* $\mathbf{q}_i$, L^j_k.

2) *If* $x \in L^+_{\mathfrak{s}}$ *then* $\mathfrak{s} \restriction L_x \in \mathbf{T}$ *and*
$$\mathbf{q} \in \mathbf{Q}_{\mathfrak{s}} \Rightarrow \mathbf{q} \restriction L_x \in \mathbf{Q}_{\mathfrak{s}_{\mathbf{q}} \restriction x}. \text{ (See 1.1(4) and 1.5(F)(d).)}$$

3) *Assume* $\mathbf{q}_1 \leq_{\mathbf{Q}} \mathbf{q}_2$.

Then

- (a) *If* $L \subseteq L_{\mathbf{q}_1}$ *<u>then</u>* L *is* $\mathbf{q}_1$*-closed iff* L *is* $\mathbf{q}_2$*-closed.*
- (b) *If* $L_1 \subseteq L_2$, L_1 *is* $\mathbf{q}_1$*-closed, and* L_2 *is* $\mathbf{q}_2$*-closed (so* $L_\iota \subseteq L_{\mathbf{q}_\iota}$ *for* $\iota = 1, 2$*) <u>then</u>*
  - $\bullet_1$ $\mathbb{P}_{\mathbf{q}_1, L_1} \lessdot \mathbb{P}_{\mathbf{q}_2, L_2}$
  - $\bullet_2$ *If* $p_\iota \in \mathbb{P}_{\mathbf{q}_\iota, L_\iota}$ *for* $\iota = 1, 2$, $p_1 = p_2 \restriction L_1$, *and* $\mathbb{P}_{\mathbf{q}_1, L_1} \models$ "$p_1 \leq q$", *<u>then</u>* p_2 *and* q *are compatible in* $\mathbb{P}_{\mathbf{q}_2, L_2}$.

Proof. 1A) First assume $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \leq q$" (i.e. clause (A)(a)). Then for every $s \in \mathrm{dom}(p)$, we have $s \in \mathrm{dom}(q)$ (by 1.5(D)(a) and 1.2) and
$$\Vdash_{\mathbb{P}_{\mathbf{q}_1,s}} \text{“}q \restriction L_{\mathbf{q}_1,s} \Vdash \text{‘}p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)\text{’”}$$
by 1.9(3A). Together we get clause (A)(b).

Now assume clause (A)(b). So $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and by 1.9(2) we get $\mathbb{P}_{\mathbf{q}_2} \models$ "$p \leq q$". (Note that closedness holds, so 1.9(2) applies.)

1B) Similar proof.

1C) Use the proof of 1.9(1).

2-3) Easy. $\square_{1.12}$

Claim 1.13. *If* $\langle \mathbf{q}_\alpha : \alpha < \delta \rangle$ *is a* $\leq_{\mathbf{Q}}$*-increasing continuous sequence of* (∂, κ)*-combinatorial templates (Note: when* $\kappa > \aleph_0$ *this does* *NOT* *mean that* $\langle \mathbb{P}_{\mathbf{q}_\alpha} : \alpha < \delta \rangle$ *is* $\subseteq$*-increasing continuous!) and* $\mathrm{cf}(\delta) \geq \kappa$*, then* $\bigcup_{\alpha<\delta} \mathbf{q}_\alpha$ *exists and is unique.*

Proof. Straightforward — anyhow, we shall use 2.1 for $\mathbf{Q} \in \{\mathbf{Q}_{\mathrm{wk}}, \mathbf{Q}_{\mathrm{st}}\}$. $\square_{1.13}$

§ 2. Unions

Claim 2.1. 1) *If* $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha < \delta \rangle$ *is* $\leq_{\mathbf{Q}_{\rm wk}}$*-increasing continuous (see* 1.10(4)*)* *then* $\mathbf{q}_\delta := \bigcup_{\alpha<\delta} \mathbf{q}_\alpha$ *exists and is unique, belongs to* $\mathbf{Q}_{\rm wk}$*, and* $\overline{\mathbf{q}}\hat{\ }\langle \mathbf{q}_\delta \rangle$ *is* $\leq_{\mathbf{Q}}$*-increasing continuous.*

2) *Similarly for* $\leq_{\mathbf{Q}_{\rm st}}$.

Remark 2.2. Note that this is not a repeat of 1.13, as we have dropped the assumption on $\mathrm{cf}(\delta)$.

Proof. 1) Let $\mathfrak{s}_\alpha := \mathfrak{s}_{\mathbf{q}_\alpha}$ and $L_\alpha := L_{\mathfrak{s}_\alpha}$ for $\alpha < \delta$.

Note that $\mathfrak{s} = \mathfrak{s}_{\mathbf{q}} := \bigcup_{\alpha<\delta} \mathfrak{s}_\alpha$ is well defined (by 1.3),[7] but when $\mathrm{cf}(\delta) < \kappa$ we cannot choose $\mathbb{P}_{\mathbf{q}} := \bigcup_{\alpha<\delta} \mathbb{P}_{\mathbf{q}_\alpha}$. We have to choose $\mathbf{q} = \mathbf{q}_\delta$ as follows:

$(*)_1$ (a) $\mathfrak{s}_{\mathbf{q}} = \mathfrak{s}_\delta := \bigcup_{\alpha<\delta} \mathfrak{s}_\alpha$, and let $L_\delta := L_{\mathfrak{s}_\delta}$.

(b) $p \in \mathbb{P}_{\mathbf{q}}$ iff
- $\bullet_1$ $\mathrm{dom}(p) \in [L_{\mathfrak{s},\delta}]^{<\kappa}$
- $\bullet_2$ If $s \in \mathrm{dom}(p)$ then $p \restriction \{s\} \in \bigcup_{\alpha<\delta} \mathbb{P}_{\mathbf{q}_\alpha}$.

(Recall 1.5(D)(a).)

(c) '$p \leq_{\mathbb{P}_{\mathbf{q}}} q$' is defined by 1.9(2); that is, $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and

$$\big(\forall s \in \mathrm{dom}(p)\big)\big[q \restriction L_{\mathbf{q}_\beta} \Vdash_{\mathbb{P}_{\mathbf{q}_\beta}} \text{“}p(s) \leq_{\underset{\sim}{\mathbb{Q}}_s} q(s)\text{”}\big],$$

where $\beta = \beta(s) := \min\{\alpha < \delta : s \in L_\alpha\}$. (Recall 1.9(3A) and note that $I_{\mathfrak{s}_\delta,s} = I_{\mathfrak{s}_\beta,s}$.)

Let $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha \leq \delta \rangle$. Easily,

$(*)_2$ (a) $\alpha < \delta \Rightarrow \mathbb{P}_{\mathbf{q}_\alpha} \subseteq \mathbb{P}_{\mathbf{q}}$ (As partial orders, of course.)

(b) If $\beta < \delta$ and $L \subseteq L_\beta$ is $\mathfrak{s}_\delta$-closed, then $\mathbb{P}_{\mathbf{q},L} = \mathbb{P}_{\mathbf{q}_\beta,L}$.

(c) $L \subseteq L_\delta$ is $\mathbf{q}$-closed iff $L \cap L_\alpha$ is $\mathbf{q}_\alpha$-closed for every $\alpha < \delta$.

(d) If L is $\mathfrak{s}_\delta$-closed then $\mathbb{P}_{\mathbf{q},L}$ is defined from $\langle \mathbf{q}_\alpha \restriction (L \cap L_{\mathbf{q}_\alpha}) : \alpha < \delta \rangle$, as $\mathbf{q}_\delta$ was defined from $\langle \mathbf{q}_\alpha : \alpha < \delta \rangle$.

Why? Obvious, but we will elaborate.

Clause (a): Let $\alpha < \delta$.

First, if $p \in \mathbb{P}_{\mathbf{q}_\alpha}$, then by $(*)_{2.1}$+$(*)_{2.2}$ below we have $p \in \mathbb{P}_{\mathbf{q}_\delta}$.

$(*)_{2.1}$ $\mathrm{dom}(p) \subseteq L_{\mathbf{q}_\alpha}$ is of cardinality $< \kappa$, by 1.5(D)(a). Also, $L_\alpha \subseteq L_{\mathbf{q}_\delta}$ by $(*)_1$(a), so p satisfies $(*)_1$(b)$\bullet_1$.

$(*)_{2.2}$ If $s \in \mathrm{dom}(p)$ then $p \restriction \{s\} \in \mathbb{P}_{\mathbf{q}_\alpha}$ by 1.5(D)(a), hence $p \restriction \{s\} \in \mathbb{P}_{\mathbf{q}_\delta}$.

Second, assume $p, q \in \mathbb{P}_{\mathbf{q}_\alpha}$. Then

$$\mathbb{P}_{\mathbf{q}_\alpha} \models \text{“}p \leq q\text{”} \Rightarrow \mathbb{P}_{\mathbf{q}_\delta} \models \text{“}p \leq q\text{”}$$

by $(*)_2$(b) and 1.12(1)(B).

[7] Really, the linear order on $L_{\mathfrak{s}_{\mathbf{q}}}$ is

$$L_{\mathfrak{s}_{\mathbf{q}}} \models \text{“}s < t\text{”} \Leftrightarrow (\exists \alpha < \delta)[L_{\mathfrak{s}_\alpha} \models \text{“}s < t\text{”}],$$

recalling that $L_{\mathfrak{s}_\alpha}$ is increasing with α (as linear orders).

Clauses (b)-(d): Similarly.

So $(*)_2$ does indeed hold.

$(*)_3$ (a) $\alpha < \delta \Rightarrow \mathbb{P}_{\mathbf{q}_\alpha} \lessdot \mathbb{P}_{\mathbf{q}}$

(b) If $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed then $\mathbb{P}_{\mathbf{q},L} \lessdot \mathbb{P}_{\mathbf{q}}$.

(c) $\bar{\underset{\sim}{\eta}} = \langle \underset{\sim}{\eta}_s : s \in L_\delta \rangle$ is a generic for $\mathbb{P}_{\mathbf{q}_\delta}$.
(That is, $\Vdash_{\mathbb{P}_{\mathbf{q}_\delta}}$ "$\mathbf{V}[\bar{\underset{\sim}{\eta}}] = \mathbf{V}[\mathbf{G}_{\mathbb{P}_{\mathbf{q}_\delta}}]$".)

(d) If $L \subseteq L_\delta$ is $\mathfrak{s}$-closed then $\langle \underset{\sim}{\eta}_s : s \in L \rangle$ is a generic for $\mathbb{P}_{\mathbf{q}_\delta \restriction L}$.

(e) Clause 1.5(C)(d) holds.

To prove clause (a), let $p \in \mathbb{P}_{\mathbf{q}}$. By the assumptions, $\langle \mathfrak{s}_{\mathbf{q}_\beta} : \beta < \delta \rangle$ is increasing. So easily, recalling $(*)_1$(c), letting $p_\beta := p \restriction (\mathrm{dom}(p) \cap L_\beta)$ for $\beta \in [\alpha, \delta)$, we have

$(*)_\beta$ $\mathbb{P}_{\mathbf{q}_\alpha} \models$ "$p_\alpha \leq q$" $\Rightarrow p$ and q are compatible in $\mathbb{P}_{\mathbf{q}}$.

See 1.9(1). This is okay even for $p = p_\delta$ which are the union of $\big\langle p_\beta : \beta \in [\alpha, \delta) \big\rangle$.

So clause (a) holds. The proof of clause (b) is similar.

As for (c), let $\mathbf{G}_\delta \subseteq \mathbb{P}_{\mathbf{q}_\delta}$ be generic over $\mathbf{V}$. By clause (a), $\mathbf{G}_\alpha := \mathbf{G}_\delta \cap \mathbb{P}_{\mathbf{q}_\alpha}$ is a generic subset of $\mathbb{P}_{\mathbf{q}_\alpha}$ for $\alpha < \delta$. So $p \in \mathbf{G}_\delta \Rightarrow p \restriction L_\alpha \in \mathbf{G}_\alpha$, recalling $p \in \mathbb{P}_{\mathbf{q}_\delta} \Rightarrow p \restriction L_\delta \leq_{\mathbb{P}_{\mathbf{q}_\delta}} p$.

Also,

$$p \in \mathbb{P}_{\mathbf{q}_\delta} \wedge \bigwedge_{\alpha<\delta} \big[p \restriction L_\alpha \in \mathbf{G}_\alpha\big] \Rightarrow p \in \mathbf{G}_\delta$$

because $\mathbb{P}_{\mathbf{q}_\delta}$ is $(<\kappa)$-complete, and $\mathbb{P}_{\mathbf{q}_\delta} \models$ "$\bigwedge_{\alpha<\delta} [p \restriction L_\alpha \leq q]$" implies $\mathbb{P}_{\mathbf{q}_\delta} \models$ "$p \leq q$".

Together, $\langle \underset{\sim}{\eta}_s : s \in L_\alpha \rangle$ determines $\mathbf{G}_\alpha$ for $\alpha < \delta$ and $\langle \mathbf{G}_\alpha : \alpha < \delta \rangle$ determines $\mathbf{G}_\delta$, hence $\big\langle \underset{\sim}{\eta}_s : s \in \bigcup_{\alpha<\delta} L_\alpha \big\rangle$ determines $\mathbf{G}_\delta$.

So clause (c) holds. Clauses (d) and (e) are proved similarly.

Next,

$(*)_4$ If L is $\mathfrak{s}_\delta$-closed then $\mathbb{P}_{\mathbf{q}_\delta,L}$ is a weak κ-forcing.

Why? If $\kappa = \aleph_0$ then $\langle \mathbb{P}_{\mathbf{q}_\alpha, L \cap L_\alpha} : \alpha < \delta \rangle$ is a $\lessdot$-increasing continuous sequence of ccc forcing notions with union $\mathbb{P}_{\mathbf{q}_\delta,L}$, and so this is known. Therefore we assume $\kappa > \aleph_0$, and then prove that $\mathbb{P}_{\mathbf{q}_\delta,L}$ satisfies $*^1_{\kappa,D}$ for D and κ as in 0.3(4).

Let $\langle p_i : i < \kappa^+ \rangle \in {}^{\kappa^+}(\mathbb{P}_L)$ be given. First, let $u_i := \mathrm{dom}(p_i)$, so $u_i \in [L]^{<\kappa}$. As $\kappa = \kappa^{<\kappa}$, there are $C \in D$ and $\mathbf{h} : C \to C$ such that:

$(*)_{4.1}$ (a) $(\forall \alpha \in C)[\mathrm{cf}(\alpha) = \kappa]$

(b) $\mathbf{h}$ is a regressive function on C.

(c) If $\zeta \in \mathrm{rang}(\mathbf{h})$, then for some $v_\zeta \subseteq L$ we have

$$i \neq j \in C \wedge \mathbf{h}(i) = \mathbf{h}(j) = \zeta \Rightarrow u_i \cap u_j = v_\zeta.$$

[Why? This holds not by the Δ-system lemma, but by its proof (using Fodor's Lemma).]

$(*)_{4.2}$ (a) Without loss of generality, $\zeta \in \mathrm{rang}(\mathbf{h}) \Rightarrow \mathbf{h}^{-1}(\{\zeta\}) \in D^+$.

(b) For $s \in L_{\mathbf{q}_\delta}$, let $\alpha(s) := \min\{\alpha : s \in L_{\mathbf{q}_\alpha}\}$.

[Why? For clause (a), recall that D is a normal filter on κ^+.]

The proof of $(*)_4$ now splits into cases.

Case 1: $\mathrm{cf}(\delta) \leq \kappa$.

Without loss of generality $\delta \leq \kappa$, hence there is a function $\mathbf{g} : \kappa^+ \to \kappa \cap (\delta+1)$ such that $i < \kappa^+ \Rightarrow p_i \in \mathbb{P}_{\mathbf{q}_{\mathbf{g}(i)}}$. Without loss of generality $\mathbf{g}(i)$ is a limit ordinal. (Recall that we are presently assuming $\kappa = \mathrm{cf}(\kappa) > \aleph_0$).

Now, using $\mathbf{q}_\alpha \in \mathbf{Q}_{\mathrm{wk}}$ for $\alpha < \delta$, consider $\langle p_i \restriction L_{\mathbf{q}_\alpha} : i < \kappa^+ \rangle$. There are $C_\alpha \in D$ and $\mathbf{h}_\alpha$ (a regressive function on C_α) as follows from '$\mathbb{P}_{\mathbf{q}_\alpha}$ satisfies $*^1_{\kappa,D}$.'

Now, recalling $\kappa = \kappa^{<\kappa}$ and $(\forall \gamma \in C)[\mathrm{cf}(\gamma) = \kappa]$, we can find C_* and $\mathbf{h}_*$ such that

$(*)_{4.3}$ (a) $C_* \in D$ and
$$C_* \subseteq \big\{ j \in C \cap \bigcap_{\zeta<\kappa} C_\zeta : (\exists k \in C \cap j)\big[\mathbf{h}(j) = \mathbf{h}(k)\big] \big\}.$$

(b) $\mathbf{h}_*$ is a regressive function on C_*.

(c) If $j_1, j_2 \in C_*$, $\mathbf{h}_*(j_1) = \mathbf{h}_*(j_2)$, and $\mathbf{g}(j_1) = \mathbf{g}(j_2)$, then $\mathbf{h}(j_1) = \mathbf{h}(j_2)$ and $\zeta \leq \mathbf{g}(j_1) \Rightarrow \mathbf{h}_\zeta(j_1) = \mathbf{h}_\zeta(j_2)$.

[Why? Easy, but we elaborate.

Let $C^*_1 := \big\{ \zeta \in \bigcap_{\alpha<\delta} C_\alpha : \zeta \text{ a limit ordinal } < \kappa^+ \big\}$. So $C^*_1 \in D$, as D_α is a normal filter on κ^+ and every C_α belongs to D by our choices. As C^*_1 and C belong to the filter D, clearly $C^*_1 \cap C$ does as well.

As $\kappa = \kappa^{<\kappa}$, there is a one-to-one function $\mathrm{cd} : {}^{\kappa>}(\kappa^+) \to \kappa^+$ such that
$$\beta < \kappa^+ \wedge \eta \in {}^{\kappa>}(\beta+\kappa) \Rightarrow \mathrm{cd}(\eta) < \beta + \kappa.$$

Let
$$C^*_2 := \big\{ \zeta < \kappa^+ : \bar{\alpha} \in {}^{\kappa>}\zeta \Rightarrow \mathrm{cd}(\bar{\alpha}) < \zeta \big\};$$
it is a club of κ^+, hence $C_* := C^*_1 \cap C^*_2 \cap C \in D$.

Lastly, define the function $\mathbf{h}_*$ with domain C_* by
$$\zeta \mapsto \mathrm{cd}(\langle g(\zeta) \rangle \hat{}\, \langle \mathbf{h}_\alpha(\zeta) : \alpha < \mathbf{g}(\delta) \rangle).$$
It is easy to check that C_* and $\mathbf{h}_*$ are as desired.]

$(*)_{4.4}$ If $i, j \in C_*$ with $\mathbf{h}_*(i) = \mathbf{h}_*(j)$, then
$$(\forall \alpha < \delta)\big[\{p_i \restriction \alpha, p_j \restriction \alpha\} \text{ has a } \leq_{\mathbb{P}_{\mathbf{q}_\alpha}}\text{-lub}\big],$$
hence p_i and p_j have a $\leq_{\mathbb{P}_{\mathbf{q}_\delta}}$-lub.

[Why? Easy. (By 0.3(1)(B)$\bullet_3$.)]

Together we are done. That is, C_* and $\mathbf{h}_*$ are as required.

Case 2: $\mathrm{cf}(\delta) > \kappa^+$.

For some $\alpha < \delta$, $\{p_i : i < \kappa^+\} \subseteq \mathbb{P}_{\mathbf{q}_\alpha}$ so the conclusion is obvious.

Case 3: $\mathrm{cf}(\delta) = \kappa^+$.

Without loss of generality $\delta = \kappa^+$; hence

$(*)_{4.5}$ In clause $(*)_{4.1}$, without loss of generality, for each $\zeta \in \mathrm{rang}(\mathbf{h})$ and $i \in \mathbf{h}^{-1}(\{\zeta\})$, we have
- $v_\zeta \subseteq L_{\mathbf{q}_i}$ and $i < j \in C \Rightarrow p_i \in \mathbb{P}_{\mathbf{q}_j}$.
- C_* and $\mathbf{h}_*$ are as in $(*)_{4.3}$.

(Recall from $(*)_{4.1}$(c) that $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j)$ is constant for all $i, j \in \mathbf{h}^{-1}(\{\zeta\})$; we denoted this set by v_ζ.)

Now easily $i, j \in C_* \wedge \mathbf{h}_*(i) = \mathbf{h}_*(j) \Rightarrow$ "p_i and p_j are comparable."

So clearly we have proved $(*)_4$.

$(*)_5$ $\mathbf{q} \in \mathbf{Q}_{\mathrm{wk}}$

[Why? We have to check all clauses of Definition 1.5; this is straightforward by $(*)_1$–$(*)_4$.]

$(*)_6$ $\mathbf{q}_\alpha \leq_{\mathbf{Q}} \mathbf{q}_\delta$ for $\alpha < \delta$.

[Why? We should check Definition 1.10(1). Clause (a) holds by $(*)_5$. Clause (b) holds by $(*)_1$(a) (recalling $\mathbf{p} \leq_{\mathbf{Q}} \mathbf{q} \Rightarrow \mathfrak{s}_{\mathbf{p}} \leq_{\mathbf{T}} \mathfrak{s}_{\mathbf{q}}$ and 1.3(2)). Clause (c) is covered by $(*)_3$(a), and clauses (d) and (e) are obvious.]

$(*)_7$ $\mathbf{q}_\delta = \bigcup\limits_{\alpha<\delta} \mathbf{q}_\alpha$

[Why? We should check Definition 1.10(3):

Clause (a): $(\mathbf{q} \in \mathbf{Q})$

Holds by $(*)_5$.

Clause (b): $(\mathfrak{s}_{\mathbf{q}_\delta} = \bigcup\limits_{\alpha<\delta} \mathfrak{s}_{\mathbf{q}_\alpha})$

Holds by $(*)_1$(a), recalling $\mathbf{q}_\alpha \leq_{\mathbf{Q}} \mathbf{q}_\beta \Rightarrow \mathfrak{s}_\alpha \leq_{\mathbf{T}} \mathfrak{s}_\beta$ and Claim 1.3(2).

Clause (c): $(\mathbf{q}_\alpha \leq_{\mathbf{Q}} \mathbf{q})$

Holds by $(*)_6$.]

2) Similarly, as in the nontrivial case $\kappa = \aleph_0$, the Knaster condition is preserved by the union of $\lessdot$-increasing continuous chains.

So we are done proving 2.1. $\square_{2.1}$

Claim 2.3. 1) *We have '(A) implies (B),' where:*

(A) (a) $\mathbf{r} \in \mathbf{Q}_{\mathrm{st}}$

(b) $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{r}}$*-name of a strong* κ*-forcing.*

(b)$^+$ *Moreover, it is a* $\mathbb{P}_{\mathbf{r}\restriction L_0}$*-name, where* $L_0 \subseteq L \trianglelefteq L_{\mathbf{r}}$ *is* $\mathbf{r}$*-closed.*

(B) There are $\mathbf{q} \in \mathbf{Q}_{\mathrm{st}}$ *and* $t_* \in L_{\mathbf{q}} \setminus L_{\mathbf{r}}$ *such that*

(a) $\mathbf{r} \leq^{\mathrm{st}}_{\mathbf{Q}} \mathbf{q}$

(b) $L_{\mathbf{q}} = L + \{t_*\} + (L_{\mathbf{r}} \setminus L)$ *as linear orders.*

(c) $\underset{\sim}{\mathbb{Q}}_{\mathbf{q},t_*} = \underset{\sim}{\mathbb{Q}}$ *and* $I^{\mathbf{q}}_{t_*} = L_0$.

2) *Identical to part (1), but replacing 'strong' by 'weak' everywhere (so of interest only when* $\kappa = \aleph_0$*) and adding to the antecedent:*

(A)(c) L_0 *is* $\mathbf{r}$*-closed and* $\mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}$*, where* $\sigma = (2^\kappa)^+$*. (See* 0.1(5)*.)*

3) *In part* (2) *we can weaken* (A)(c) *to*

$(A)(c)'$ *If* $\kappa = \aleph_0$ *then* $\Vdash_{\mathbb{P}_{\mathbf{q},L_0}}$ "$\mathsf{MA}_{\aleph_1}$".

Proof. Easy. $\square_{2.3}$

Claim 2.4. 1) *For every* $\mathbf{r} \in \mathbf{Q}^*_{\rm st}$ *— that is, with*[8] $\partial = \mathrm{cf}(\partial) > \sup\limits_{t \in L_{\mathbf{r}}} |S_{\mathbf{r},t}|$ *such that* $(\forall \alpha < \partial)\big[|\alpha|^{2^\kappa} < \partial\big]$ *— there is a* $\mathbf{q} \in \mathbf{Q}^*_{\rm st}$ *such that:*

- $(A)^1_\partial$ (a) $\mathbf{r} \le_{\mathbf{Q}^*_{\rm st}} \mathbf{q}$
  - (b) $\|\mathbb{P}_{\mathbf{q}}\| \le \big\|\mathbb{P}_{\mathbf{r}}\big\|^{<\partial}$
  - (c) **[Follows]** $|S_{\mathbf{q},t}| < \partial$ *for all* $t \in L_{\mathbf{q}}$.
- $(B)^1_\partial$ (a) $\mathbf{q}$ *satisfies* $\mathrm{cf}(L_{\mathbf{q}}) \ge \partial$.
  - (b) *If* $t \in L_{\mathbf{q}}$ *then* $\mathrm{cf}(L_{\mathbf{q},t}) \ge \partial$.
  - (c) *If* $L \lhd L_{\mathbf{q}}$ *is of cofinality* $\ge \partial$*,* $L_0 \subseteq L$ *is* $\mathbf{q}$*-closed,* $\underaccent{\tilde}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{q},L_0}$*-name of a weak* κ*-forcing of cardinality* $< \partial$*, and*
  $$\kappa = \aleph_0 \Rightarrow \mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}$$
  (where $\sigma := (2^\kappa)^+$*)* <u>*then*</u>
    - *For some* $s \in L$*,* $\underaccent{\tilde}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{q},s}$*-name and*
    $$\Vdash_{\mathbb{P}_{\mathbf{q},s}} \text{"}\underaccent{\tilde}{\mathbb{Q}}_{\mathbf{q},s} \text{ and } \underaccent{\tilde}{\mathbb{Q}} \text{ are isomorphic"}.$$

2) *Similar to part* (1)*, but* $\mathbf{r}, \mathbf{q} \in \mathbf{Q}^*_{\rm wk}$*,* $(\forall \alpha < \partial)\big[|\alpha|^\kappa < \partial\big]$*, and*

- $(A)^2_\partial$ (a) $\mathbf{r} \le^{\rm wk}_{\mathbf{Q}} \mathbf{q}$
  - (b) *As above.*
- $(B)^2_\partial$ (a) *As above.*
  - (b) *As above.*
  - (c) *Like* $(B)^1_\partial(c)$*, but replacing 'weak* κ*-forcing' by 'strong* κ*-forcing' and omitting* $\mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}$.

3) *Like part* (1)*, but replacing*

$$\text{"}\kappa = \aleph_0 \Rightarrow \mathbb{P}_{\mathbf{r},L_0} \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_{\mathbf{r}}\text{"}$$

by $\Vdash_{\mathbb{P}_{\mathbf{r},L_0}}$ "$\mathsf{MA}_{\aleph_1}$".

(We shall call the resulting clauses $(A)^{0.5}_\partial$ *and* $(B)^{0.5}_\partial$*.)*

Proof. 1) We shall prove more. Let $\mathbf{Q}_*$ be the class of $\mathbf{q} \in \mathbf{Q}^*_{\rm st}$ satisfying $(A)^1_\partial$. (E.g. $\mathbf{r} \in \mathbf{Q}_*$.) Consider the statement

⊞ If $\mathbf{p} \in \mathbf{Q}_*$ then there exists $\mathbf{q} \in \mathbf{Q}_*$ such that:
  - (a) $\mathbf{p} \le_{\mathbf{Q}_2} \mathbf{q}$
  - (b) There is $t_* \in L_{\mathbf{q}}$ such that $(\forall s \in L_{\mathbf{p}})[s <_{L_{\mathbf{q}}} t_*]$.
  - (c) If $t \in L_{\mathbf{p}}$, $L_0 \subseteq L_{\mathbf{q},t}$ is $\mathbf{q}$-closed, and $\underaccent{\tilde}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{q},L_0}$-name of a weak κ-forcing of cardinality $< \partial$, then $\bullet_1$ or $\bullet_2$ holds, where

[8] If we omit "$\partial = \mathrm{cf}(\partial) > \sup\limits_{t \in L_{\mathbf{r}}} |S_{\mathbf{r},t}|$," then in 2.3 we need to expand by $S'_s \subseteq S_{\mathbf{q},s}$ of cardinality $< \partial$ for $s \in L$, and make further changes.

- $\bullet_1$ For some $s \in L_{\mathbf{q},t}$ we have
$$\Vdash_{\mathbb{P}_\mathbf{q}} \text{“}\underset{\sim}{\mathbb{Q}}_{\mathbf{q},s} \cong \underset{\sim}{\mathbb{Q}}\text{”}.$$
- $\bullet_2$ $\Vdash_{\mathbb{P}_\mathbf{q}}$ “$\underset{\sim}{\mathbb{Q}}$ is not ccc”.

We shall prove that $\boxplus$ is both true and sufficient, which is more than is needed to prove part (1).

Why $\boxplus$ is true:

Let

$$\mathcal{Y} := \big\{(t, L, \underset{\sim}{\mathbb{Q}}) : t \in L \cup \{\infty\},\ L \text{ a } \mathbf{p}\text{-closed subset of } L_{\mathbf{p},t} \text{ of cardinality} < \partial, \text{ and } \underset{\sim}{\mathbb{Q}} \text{ a } \mathbb{P}_{\mathbf{q},L}\text{-name of a forcing notion with set of elements an ordinal} < \partial\big\}.$$

[Is this well-defined? t is defined in terms of L and L is defined in terms of t.]

Easily, $|\mathcal{Y}| \le \|\mathbb{P}_\mathbf{p}\|^{<\partial}$, and we can find a sequence $\big\langle(t_\alpha, L_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha) : \alpha < |\mathcal{Y}|\big\rangle$ listing $\mathcal{Y}$.

Now we choose $\mathbf{p}_\alpha$ by induction on $\alpha \le |\mathcal{Y}|$ such that

$\oplus^1_\alpha$
- (a) $\mathbf{p}_\alpha \in \mathbf{Q}_*$
- (b) $\mathbf{p}_0 := \mathbf{p}$
- (c) $\langle \mathbf{p}_\beta : \beta \le \alpha\rangle$ is $\le_\mathbf{Q}$-increasing continuous.
- (d) If $\alpha = \beta + 1$, then
  - $\bullet_1$ If $\Vdash_{\mathbb{P}_{\mathbf{p}_\beta}}$ “$\underset{\sim}{\mathbb{Q}}_\beta$ is not a strong κ-forcing” then $\underset{\sim}{\mathbb{Q}}'_\beta := ({}^{\kappa>}2, \triangleleft)$, and if it is, then $\underset{\sim}{\mathbb{Q}}'_\beta := \underset{\sim}{\mathbb{Q}}_\beta$.
  - $\bullet_2$ For some s_β, $L_{\mathbf{p}_\alpha} \setminus L_{\mathbf{p}_\beta} = \{s_\beta\}$, $L_{\mathbf{p}_\beta, t_\beta} < s_\beta <_{L_{\mathbf{p}_\alpha}} t_\beta$, and $\underset{\sim}{\mathbb{Q}}_{\mathbf{p}_\alpha, s_\beta} = \underset{\sim}{\mathbb{Q}}'_\beta$.

Why can we carry the induction? The base case is covered by clause (b), and for α a limit ordinal we use Claim 2.1. For $\alpha \le |\mathcal{Y}|$ successor we use Claim 2.3 (with $\mathbf{p}_\beta$, L_α, $L_{\mathbf{p}_\beta, t_\beta}$, $\underset{\sim}{\mathbb{Q}}'_\alpha$, s_α, $\mathbf{p}_\alpha$ here standing in for $\mathbf{r}, L_0, L_1, \mathbb{Q}, t_*, \mathbf{q}$ there).

So $\boxplus$ does indeed hold.

Why $\boxplus$ is sufficient:

We choose $\mathbf{q}_\alpha$ by induction on $\alpha \le \partial$ such that

$\oplus^2_\alpha$
- (a) $\mathbf{q}_\alpha \in \mathbf{Q}_*$
- (b) $\mathbf{q}_0 := \mathbf{p}$
- (c) $\langle \mathbf{q}_\beta : \beta \le \alpha\rangle$ is $\le_\mathbf{Q}$-increasing continuous.
- (d) If $\alpha = \beta + 1$ then $\boxplus$ is satisfied, with $(\mathbf{q}_\beta, \mathbf{q}_\alpha)$ standing in for $(\mathbf{p}, \mathbf{q})$.

We can carry the induction, using $\boxplus$ for α a successor. Now,

$\oplus_3$ $\mathbf{q}_\partial$ is as required.

Why? We shall check 2.4(1)(A),(B).

Clauses (A)(a),(b): This means $\mathbf{q}_\partial \in \mathbf{Q}_*$, which holds by $\oplus^2_\partial$.

Clause (B)(a): This says $\mathrm{cf}(L_\mathbf{q}) \ge \partial$.

It holds because $\langle L_{\mathbf{q}_\alpha} : \alpha < \partial\rangle$ is increasing continuous and $L_{\mathbf{q}_\beta}$ is bounded in $L_{\mathbf{q}_{\beta+1}}$, by $\boxplus$(b) and $\oplus^2_\alpha$(d).

Clause (B)(b):

Similarly, using $\boxplus$(c) we can find $L_0 \subseteq L_{\mathbf{q}_\partial,t}$ as required, because

$$(\forall\alpha < \partial)\big[|\alpha|^{2^\kappa} < \partial\big],$$

because necessarily $L_0 \subseteq L_{\mathbf{q}_\beta}$ for some $\beta < \partial$, and by our choice of $\mathbf{q}_{\beta+1}$.

Clause (B)(c): Similarly to (B)(b).

So we are done proving part (1).

2) Repeat the proof of part (1), but this time we choose $\mathbf{Q}_* := \mathbf{Q}^*_{\rm wk}$.

3) Straightforward. $\square_{2.4}$

Definition 2.5. We say $\mathbf{q}$ is *strongly* $(<\partial)$*-homogeneous* when

- If $L_\ell \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed and of cardinality $< \partial$ for $\ell = 1, 2$, and π_1 is an isomorphism from L_1 onto L_2 mapping $\mathbf{q} \restriction L_1$ to $\mathbf{q} \restriction L_2$, then there is an automorphism π_2 of $L_{\mathbf{q}}$ extending π_1 and mapping $\mathbf{q}$ to itself. Hence it induces an automorphism $\hat{\pi}_2$ of $\mathbb{P}_{\mathbf{q}}$ (e.g. mapping $\underset{\sim}{\eta}_t$ to $\underset{\sim}{\eta}_{\pi_2(t)}$).

Claim 2.6. 1) *If* $\mathbf{q} \in \mathbf{Q}_\ell$ *for* $\ell \in \{1, 2\}$ *and* $L \subseteq L_{\mathbf{q}}$ *is* $\mathbf{q}$*-closed, then* $\mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\mathbf{q},L}$ *is a* (κ, ℓ)*-forcing. (See* 0.3.*)*

2) $(\mathbf{Q}_{\rm st}, \leq_{\mathbf{Q}_{\rm st}})$ *satisfies amalgamation.*

3) *For* $\kappa = \aleph_0$, $\mathbf{Q}_1$ *satisfies a weak version of amalgamation:*[9]

(∗) *If* $\mathbf{q}_0 \in \mathbf{Q}_1$, $\mathbf{q}_0 \leq^{\rm wk}_{\mathbf{Q}} \mathbf{q}_\ell$ *for* $\ell = 1, 2$, $L_{\mathbf{q}_1} \cap L_{\mathbf{q}_2} = L_{\mathbf{q}_0}$, *and* $\Vdash_{\mathbb{P}_{\mathbf{q}_0}}$ "$\mathsf{MA}_{\aleph_1}$" *then there is a* $\mathbf{q}_3 \in \mathbf{Q}_1$ *such that* $\mathbf{q}_\ell \leq \mathbf{q}_3$ *for* $\ell = 0, 1, 2$.

4) *In* (3)(∗) *above, we may replace* $\Vdash_{\mathbb{P}_{\mathbf{q}_0}}$ "$\mathsf{MA}_{\aleph_1}$" *with the demand* "$\mathbf{q}_0 \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbf{q}_1$," *where* $\sigma := (2^{\aleph_1})^+$.

Proof. 1) **Case 1:** $\kappa > \aleph_0$ (so the choice of ℓ is immaterial).

Proving "$\mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\mathbf{q},L}$ is $(<\kappa)$-complete" is easy when $\kappa > \aleph_0$, and the existence of least upper bounds follows as well. So it suffices to do the following:

$\boxplus$ (a) Assume $p_* \Vdash_{\mathbb{P}_{\mathbf{q},L}}$ "$\underset{\sim}{q}_\alpha \in \mathbb{P}_{\mathbf{q}}/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{q},L}}$ for $\alpha < \kappa^+$".
(b) Now find $p_{**} \in \mathbb{P}_{\mathbf{q},L}$ above p_* and $\mathbb{P}_{\mathbf{q},L}$-names $\underset{\sim}{C}$, $\underset{\sim}{h}$ as required in $*^1_{\kappa,D}$.

Now

$(*)_1$ For each $\alpha < \kappa^+$, we can choose $\langle p_{\alpha,\iota}, q_{\alpha,\iota} : \iota < \iota(\alpha) \leq \kappa\rangle$ such that:[10]
(a) For $\iota < \iota(\alpha)$, $p_{\alpha,\iota} \in \mathbb{P}_{\mathbf{q},L}$ is above p_*, and
$$p_{\alpha,\iota} \Vdash_{\mathbb{P}_{\mathbf{q},L}} \text{“}\underset{\sim}{q}_\alpha = q_{\alpha,\iota}\text{”}.$$
(b) Without loss of generality, $\mathbb{P}_{\mathbf{q},L} \models$ '$(q^*_{\alpha,\iota} \restriction L) \leq p_{\alpha,\iota}$' for $\iota < \iota(\alpha)$.
(c) Therefore, $r_{\alpha,\iota} := p_{\alpha,\iota} \cup (q_{\alpha,\iota} \restriction (L_{\mathbf{q}} \setminus L))$ is a $\leq_{\mathbb{P}_{\mathbf{q}}}$-lub of p_α and $q_{\alpha,\iota}$.

[9] For $\kappa > \aleph_0$ this is not interesting, and is already covered by 2.10(1).

[10] Ignoring the trivial case, we can assume $\iota(\alpha) := \kappa$.

(d) $\langle p_{\alpha,\iota} : \iota < \kappa \rangle$ is a maximal antichain of $\mathbb{P}_{\mathbf{q},L}$.

[Why? Because $\mathbb{P}_{\mathbf{q},L}$ satisfies the κ^+-cc.]

Next,[11]

$(*)_2$ There are C_ι, h_ι, and $\bar{u}_\iota$ (for $\iota < \kappa$) such that
(a) $C_\iota \in D$
(b) h_ι is a pressing-down function on C_ι.
(c) $\bar{u}_\iota = \big\langle u^\iota_\zeta : \zeta \in \mathrm{rang}(h_\iota) \big\rangle$
(d) If $\zeta \in \mathrm{rang}(h_\iota)$ then
$\bullet_1$ The set $S^\iota_\zeta := h_\iota^{-1}(\{\zeta\})$ belongs to D^+.
$\bullet_2$ $\langle \mathrm{dom}(r_{\alpha,\iota}) : \alpha \in S^\iota_\zeta \rangle$ is a Δ-system with heart u^ι_ζ.
(e) If $\alpha, \beta \in C_\iota$ and $h_\iota(\alpha) = h_\iota(\beta)$, then $q_{\alpha,\iota}$ and $q_{\beta,\iota}$ have a lub.

$(*)_3$ (a) Without loss of generality, C_ι is constant in ι; call this set C.
(b) Without loss of generality, $\langle \mathrm{rang}(h_\iota) : \iota < \kappa \rangle$ is a sequence of pairwise disjoint sets.
(c) Let $\underset{\sim}{j}$ be a $\mathbb{P}_{\mathbf{q},L}$-name of a function $\kappa^+ \to \kappa$ such that for $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q},L}$ generic over $\mathbf{V}$, we have

$$\underset{\sim}{j}(\alpha)[\mathbf{G}] = \iota \text{ iff } p_{\alpha,\iota} \in \mathbf{G}.$$

[Why? Straightforward.]

$(*)_4$ Let $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q},L}$ be generic over $\mathbf{V}$.
(a) Let $j_\bullet := \underset{\sim}{j}[\mathbf{G}]$. We have $C \in D$ (so $j_\bullet$ is a function from C into κ).
(b) $h_\bullet : C \to \kappa^+$ will be defined as $\alpha \mapsto h_{j_\bullet(\alpha)}(\alpha)$.

Now,

$(*)_5$ (a) $h_\bullet$ is regressive.
(b) If $\alpha, \beta \in C$ with $h_\bullet(\alpha) = h_\bullet(\beta)$, then
$\bullet_1$ $j_\bullet(\alpha) = j_\bullet(\beta)$
$\bullet_2$ $q_{\alpha, j_\bullet(\alpha)}$ and $q_{\beta, j_\bullet(\beta)}$ have a lub in $\mathbb{P}_{\mathbf{q}}$ (and hence in $\mathbb{P}_{\mathbf{q}}[\mathbf{G}]$), noting that $\underset{\sim}{q}_\alpha[\mathbf{G}] = q_{\alpha, j_\bullet(\alpha)}$ and $\underset{\sim}{q}_\beta[\mathbf{G}] = q_{\beta, j_\bullet(\beta)}$.

[Also straightforward.]

This finishes the proof of Case 1 (that is, $\kappa > \aleph_0$).

Case 2: $\kappa = \aleph_0$ and $\ell = 1$.

Well-known.

Case 3: $\kappa = \aleph_0$ and $\ell = 2$.

Like Case 1, but simpler.

2) So assume

$(*)_0$ for $\ell = 0, 1, 2$,
(a) $\mathbf{q}_\ell \in \mathbf{Q}_2$
(b) $\mathbf{q}_0 \leq_{\mathbf{Q}_2} \mathbf{q}_\ell$
(c) $L_{\mathbf{q}_1} \cap L_{\mathbf{q}_2} = L_{\mathbf{q}_0}$ for transparency.

[11] See details in the proof of $(*)_4$ in the proof of 2.1(1).

$(*)_1$ Let L be a linear order with set of elements $L_{\mathbf{q}_1} \cup L_{\mathbf{q}_2}$, and $L_{\mathbf{q}_\ell} \subseteq L$ as linear orders.

$(*)_2$ We define $\mathfrak{s} \in \mathbf{T}$ such that $L_\mathfrak{s} = L$ and $I_{\mathfrak{s},t} = I_{\mathfrak{s}_{\mathbf{q}_\ell},t}$ for $t \in L_{\mathbf{q}_\ell}$.

$(*)_3$ We define $\mathbf{q} \in \mathbf{Q}^2_\mathfrak{s}$ above $\mathbf{q}_\ell$ (for $\ell \leq 2$) naturally.

We have to prove that $\mathbf{q} \in \mathbf{Q}_2$. Being $(<\kappa)$-complete (with $\kappa > \aleph_0$) is easy; satisfying $*^1_{\kappa,D}$ or Knaster is a consequence of 2.6(1).

3) Like part (1), but easier.

4) The point here is proving the implication '(A) $\Rightarrow$ (B),' where

(A) (a) $\mathbb{P}_0 \lessdot \mathbb{P}_\ell$ (for $\ell = 1,2$) are ccc forcing notions.
(b) $\mathbb{P}_0 \prec_{\mathbb{L}_{\sigma,\sigma}} \mathbb{P}_1$

(B) $\mathbb{P} := \mathbb{P}_1 *_{\mathbb{P}_0} \mathbb{P}_2$ is ccc.

Why does this hold?

Assume $(p_{\alpha,1}, p_{\alpha,2}) \in \mathbb{P}_1 *_{\mathbb{P}_0} \mathbb{P}_2$ for $\alpha < \omega_1$, and we need to prove that for some $\alpha < \beta < \alpha_0$, $(p_{\alpha,1}, p_{\alpha,2})$ and $(p_{\beta,1}, p_{\beta,2})$ have a common upper bound in $\mathbb{P}_1 *_{\mathbb{P}_0} \mathbb{P}_2$.

[α_0 isn't defined or used anywhere.]

Let $q_\alpha \in \mathbb{P}_0$ force '$p_{\alpha,1} \in \mathbb{P}_1 / \underset{\sim}{\mathbf{G}}_{\mathbb{P}_0}$' and '$p_{\alpha,2} \in \mathbb{P}_2 / \underset{\sim}{\mathbf{G}}_{\mathbb{P}_0}$.' For $\alpha, \beta < \omega_1$, let $\langle q_{\alpha,\beta,i} : i < \iota(\alpha,\beta) \leq \omega \rangle$ be a maximal antichain of $\mathbb{P}_0$ such that each $q_{\alpha,\beta,i}$ forces a truth value to "$p_{\alpha,\ell}$ and $q_{\beta,\ell}$ are compatible in $\mathbb{P}_\ell / \underset{\sim}{\mathbf{G}}_{\mathbb{P}_0}$", for $\ell = 1, 2$.

Now, finding a sequence $\langle p'_{\alpha,1} : \alpha < \omega_1 \rangle \in {}^{\omega_1}\mathbb{P}_0$ similar enough to $\langle p_{\alpha,1} : \alpha < \omega_1 \rangle$ over
$$\big\{ q_\alpha : \alpha < \omega_1 \big\} \cup \big\{ q_{\alpha,\beta,i} : \alpha, \beta < \omega_1,\ i < \iota(\alpha,\beta) \big\}$$
will contradict "$\mathbb{P}_2$ satisfies the ccc."

Let us elaborate on what we mean by 'similar enough.'

$(*)_1$ If $\alpha < \omega_1$ then q_α and $p'_{\alpha,1}$ are compatible in $\mathbb{P}_0$.

$(*)_2$ For $\alpha < \beta < \omega_1$ and $i < \iota(\alpha,\beta)$, we have '(a) $\Rightarrow$ (b),' where
(a) There is no $r \in \mathbb{P}_1$ such that $p_{1,\alpha} \leq_{\mathbb{P}_1} r$, $p_{1,\beta} \leq_{\mathbb{P}_1} r$, $q_{\alpha,\beta,i} \leq_{\mathbb{P}_1} r$, and
$$q_{\alpha,\beta,i} \Vdash_{\mathbb{P}_1} \text{"}p_{\alpha,2} \text{ and } p_{\beta,2} \text{ are compatible in } \mathbb{P}_2/\mathbb{P}_1\text{"}.$$
(b) There is no $r \in \mathbb{P}_0$ such that $p_{1,\alpha} \leq_{\mathbb{P}_1} r$, $p_{1,\beta} \leq_{\mathbb{P}_1} r$, $q_{\alpha,\beta,i} \leq_{\mathbb{P}_1} r$, and
$$q_{\alpha,\beta,i} \Vdash_{\mathbb{P}_1} \text{"}p'_{\alpha,2} \text{ and } p'_{\beta,2} \text{ are compatible in } \mathbb{P}_2/\mathbb{P}_1\text{"}.$$

Now for $\alpha < \omega_1$, let $p^+_{\alpha,1} \in \mathbb{P}_0$ be a common upper bound for $p'_{\alpha,1}$ and q_α. Hence the conditions $p^+_{\alpha,1}$ and $p_{\alpha,2}$ are compatible in $\mathbb{P}_2$, and let $p^+_{\alpha,2} \in \mathbb{P}_2$ be such a common upper bound. As $\mathbb{P}_2$ satisfies the ccc, there are $\alpha < \beta < \omega_1$ such that $p^+_{\alpha,2}$ and $p^+_{\beta,2}$ have a common upper bound — call it $r_{\alpha,\beta} \in \mathbb{P}_2$. Therefore $r_{\alpha,\beta}$ is an upper bound of $\{p'_{\alpha,1}, p'_{\alpha,2}, p_{\alpha,2}, p_{\beta,2}\}$.

We know $q_{\alpha,\beta,i} \leq_{\mathbb{P}_2} r$ for some $i < \iota(\alpha,\beta)$, so necessarily $q_{\alpha,\beta,i} \in \mathbb{P}_0$ forces that $p_{\alpha,2}$ and $p_{\beta,2}$ have a common upper bound in $\mathbb{P}_2$ and that $p'_{\alpha,1}$ and $p'_{\beta,1}$ have a common upper bound in $\mathbb{P}_2$ (hence in $\mathbb{P}_0$).

But this implies
$$q_{\alpha,\beta,i} \Vdash \text{"}p_{\alpha,1} \text{ and } p_{\beta,1} \text{ have a common upper bound in } \mathbb{P}_2\text{"}.$$
All together, by the definition of $\mathbb{P} := \mathbb{P}_1 *_{\mathbb{P}_0} \mathbb{P}_2$, the conditions $(p_{\alpha,1}, p_{\alpha,2})$ and $(p_{\beta,1}, p_{\beta,2})$ are compatible in $\mathbb{P}$, finishing the proof. $\square_{2.6}$

Claim 2.7. 1) *Assume* $\mathbf{p} \in \mathbf{Q}_2^*$, L_ℓ *is a* $\mathbf{p}$*-closed subset of* $L_{\mathbf{p}}$ *(for* $\ell = 1,2$*), and* $\pi : L_1 \to L_2$ *is an isomorphism which induces an isomorphism* $\hat{\pi} : \mathbb{P}_{\mathbf{p},L_1} \to \mathbb{P}_{\mathbf{p},L_2}$.

Then we can find $\mathbf{q}$, π_1, L_1^+, L_2^+ *such that*

- (a) $\mathbf{p} \le_{\mathbf{Q}_2} \mathbf{q} \in \mathbf{Q}_2^*$
- (b) *For* $\ell = 1,2$, $L_\ell \subseteq L_\ell^+ \subseteq L_{\mathbf{q}}$, L_ℓ^+ *is* $\mathbf{q}$*-closed, and* $L_{\mathbf{p}} \subseteq L_1^+$.
- (c) $\pi_1 \supseteq \pi$ *is an isomorphism from* L_1^+ *onto* L_2^+ *which induces an isomorphism* $\hat{\pi}_1 : \mathbb{P}_{\mathbf{q},L_1^+} \to \mathbb{P}_{\mathbf{q},L_2^+}$.

2) *'If* (A) *then* (B)*,' where*

- (A)
  - (a) $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha \le \delta_* \rangle \subseteq \mathbf{Q}_2^*$ *is* $\le_{\mathbf{Q}}$*-increasing continuous.*
  - (b) $\langle \alpha_\varepsilon = \alpha(\varepsilon) : \varepsilon < \zeta \rangle$ *is an increasing continuous sequence of ordinals with limit* δ_*.
  - (c) $L^1_{\alpha(\varepsilon)}$ *and* $L^2_{\alpha(\varepsilon)}$ *are* $\mathbf{q}_{\alpha(\varepsilon)}$*-closed subsets of* $L_{\alpha(\varepsilon)}$.
  - (d) $\pi_\varepsilon : L^1_{\alpha(\varepsilon)} \to L^2_{\alpha(\varepsilon)}$ *is order-preserving and onto.*
  - (e) π_ε *induces an isomorphism from* $\mathbf{q}_{\alpha(\varepsilon)} \restriction L^1_{\alpha(\varepsilon)}$ *onto* $\mathbf{q}_{\alpha(\varepsilon)} \restriction L^2_{\alpha(\varepsilon)}$.
  - (f) $L^1_{\alpha(\varepsilon)}, L^2_{\alpha(\varepsilon)}, \pi_\varepsilon$ *are increasing continuously with* ε.
  - (g) *For* $\ell = 1,2$, *if* $L_{\mathbf{q}_{\alpha(\varepsilon)}} \not\subseteq L^\ell_{\alpha(\varepsilon)+1}$ *then* $L_{\mathbf{q}_{\alpha(\varepsilon)+1}} \subseteq L^\ell_{\alpha(\varepsilon)+2}$.
- (B) $\pi := \bigcup_{\varepsilon<\zeta} \pi_\varepsilon$ *is an automorphism of* $\mathbf{q}_{\delta_*}$.

Proof. 1) By 2.6(2).

2) Easy. $\square_{2.7}$

Definition 2.8. 1) For $\ell = 1,2$, we say $\mathbf{q}$ is (∂,ℓ)-*saturated* when it satisfies $2.4(\ell)(\mathrm{B})^\ell_\partial$.

2) We say $\overline{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha < \alpha_* \rangle$ is (∂,ℓ)-saturated <u>when</u>:

- (a) $\overline{\mathbf{q}}$ is $\le_{\mathbf{Q}_\ell}$-increasing continuous, recalling 1.7(3) and 1.10(2).
- (b) $\mathbf{q}_\alpha$ is (∂,ℓ)-saturated for $\alpha < \alpha_*$ non-limit.

Remark 2.9. Recall 1.7(3), so e.g. we denote $\mathbf{Q}_{\mathrm{st}}$ and $\mathbf{Q}_{\mathrm{wk}}$ by $\mathbf{Q}_1, \mathbf{Q}_2$, respectively. We may replace them by other classes.

Claim 2.10. 1) *If* $\lambda = \lambda^{<\partial}$ *and* $\partial = \mathrm{cf}(\partial) > \kappa$ *(recalling* $\mathbf{Q}^*_{\mathrm{st}}$ *is from* 1.7(7) *and* $\lambda, \partial, \kappa$ *are from Hypothesis* 0.2*) then there is a* $\mathbf{q} \in \mathbf{Q}_{\mathrm{st}}$ *such that*

- (a) $L_{\mathbf{q}}$ *and* $\mathbb{P}_{\mathbf{q}}$ *have cardinality* λ.
- (b) $\mathbf{q}$ *is strongly homogeneous.*
- (c) $\mathbf{q}$ *is* $(\partial,2)$*-saturated.*

2) *We can combine part* (1) *with* 2.6(3)*; that is, if* $\partial = \mathrm{cf}(\partial) > \kappa = \aleph_0$ *and* $\lambda = \lambda^{<\partial}$, <u>*then*</u> *there exists a* $\mathbf{q} \in \mathbf{Q}^{\mathrm{wk}}_{\partial,\kappa}$ *such that*

- (a) $L_{\mathbf{q}}$ *has cardinality* λ.
- (b) $\mathbf{q}$ *is weakly homogeneous, when we restrict ourselves to an* $L \subseteq L_{\mathbf{q}}$ *such that* $\Vdash_{\mathbb{P}_{\mathbf{q},L}}$ "$\mathsf{MA}_{\aleph_1}$".
- (c) $\mathbf{q}$ *is* $(\partial,1)$*-saturated.*

3) *Similarly for the* $\prec_{\mathbb{L}_{\sigma,\sigma}}$*-version.*

Proof. 1) By 2.7.

2,3) Easy as well. $\square_{2.10}$

§ 3. More on the iteration

Definition 3.1. 1) For $\iota \le 5$, we say $\mathbb{Q}$ is a (κ, ι)-forcing <u>when</u>

- (A) (a) If $\iota = 0$ it is a forcing.
  - (b) If $\iota = 1$ it is a weak κ-forcing.
  - (c) If $\iota = 2$ then it is a strong κ-forcing.
- (B) If $\iota = 3$ then $\mathbb{Q} = (Q, \le, \mathrm{tr}) = (\mathbb{Q}, \le_{\mathbb{Q}}, \mathrm{tr}_{\mathbb{Q}})$ satisfies the following.
  - (a) It is a strong κ-forcing.
  - (b) $\mathrm{tr}_{\mathbb{Q}}$ is a function $\mathbb{Q} \to \mathcal{H}(\kappa)$.
  - (c) $\partial(-)$ is a function with domain $\mathrm{rang}(\mathrm{tr}_{\mathbb{Q}})$.
  - (d) For each $x \in \mathrm{rang}(\mathrm{tr})$, for some $\partial(x) = \partial_{\mathbb{Q}}(x) \in [2, \kappa]$, any $< 1 + \partial(x)$ members of $\{p \in \mathbb{Q} : \mathrm{tr}(p) = x\}$ have a common upper bound.
- (C) If $\iota = 4$ then as in (B), but we add
  - (d) If $\sigma < \kappa$ then $\{p \in \mathbb{Q} : \partial(\mathrm{tr}(p)) \ge \sigma\}$ is dense.
- (D) If $\iota = 5$ then as in (B), but $\partial(x) = \kappa$ for every $x \in \mathrm{rang}(\mathrm{tr}_{\mathbb{Q}})$.

2) For $\iota \le 5$, let $\mathbf{Q}_\iota$ be the class of $\mathbf{q}$ such that[12]

- (A) $\mathbf{q} \in \mathbf{Q}_{\mathrm{pre}}$
- (B) If $t \in L_{\mathbf{q}}$ then $\Vdash_{\mathbb{P}_{\mathbf{q},t}}$ "$\underset{\sim}{\mathbb{Q}}_t$ is an ι-forcing", and if $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed then $\mathbb{P}_{\mathbf{q},L}$ is a (κ, ι)-forcing.
- (C) If $\iota = 3, 4, 5$ then
  - $\bullet_1$ If $p \in \mathbb{P}_{\mathbf{q}}$ and $s \in \mathrm{dom}(p)$, then $\mathrm{tr}_{\mathbb{Q}_s}(p(s))$ is an object, not just a name.
  - $\bullet_2$ If $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed then $\mathbb{P}_{\mathbf{q},L}$ is a $(\kappa, 2)$-forcing.
- (D) If $\iota = 4$ then in addition to $\bullet_1$ and $\bullet_2$,
  - $\bullet_3$ If $\sigma < \kappa$ and $L \subseteq L_{\mathbf{q}}$ is $\mathbf{q}$-closed, then
  $$\big\{p \in \mathbb{P}_{\mathbf{q}} : \big(\forall s \in \mathrm{dom}(p)\big)\big[\partial_{\mathbb{Q}_s}(p(s)) \ge \sigma\big]\big\}$$
  is dense in $\mathbb{P}_{\mathbf{q},L}$.

3) For $\iota \le 5$, let $\mathbb{Q}^\iota_{\partial,\theta}$ be the class of $\mathbf{q} \in \mathbf{Q}_\iota$ such that
$$t \in L_{\mathbf{q}} \Rightarrow |I_{\mathbf{q},t}| < \partial$$
and $\mathbf{q}$ is strongly $(<\theta)$-homogeneous.

Claim 3.2. 1) *For $\iota = 0, 1, 2$, the definition of $\mathbf{Q}_\iota$ in 3.1(2) agrees with the one in 1.7.*

2) *For $\iota = 3, 4, 5$, we can repeat the work done for $\iota = 2$ (i.e. $\mathbf{Q}_2$) in §1-2.*

Proof. 1) Easy to check.

2) Repeating previous proofs, using Definition 3.1. $\square_{3.2}$

Definition 3.3. If clause (A) holds, then we define $\mathbb{P}_{\bar{s}}$ as in clause (B), where:

[12] We may just demand that for $\mathbf{q}$-closed L, we have that
$$\{p \in \mathbb{P}_{\mathbf{q},L} : s \in \mathrm{dom}(p) \Rightarrow \mathrm{tr}_{\mathbb{Q}_s}(p(s)) \text{ is an object}\}$$
is dense. In this case, if $\kappa > \aleph_0$ then this follows.

(A) (a) $\mathbf{q} \in \mathbf{Q}_1$ and $\kappa = \aleph_0$.

(b) $\bar{s} = \langle s_i : i < \alpha \rangle \in {}^{\alpha}(L_{\mathbf{q}})$ and $u_i \subseteq \alpha$ for $i < \alpha$.

(c) $L_{\mathbf{q}} \models$ "$s_i < s_j$" for $i < j < \alpha$.

(d) $u_i := \{j < i : s_j \in I_{\mathbf{q},s_i}\}$

(e) $\mathbb{Q}_{\mathbf{q},s_i}$ is definable from $\bar{\underset{\sim}{\eta}}_i = \langle \eta_{s_j} : j \in u_i \rangle$ (say we have a definition $\bar{\varphi}_{i,\bar{\eta}}$ for any $\bar{\eta} \in X_i := \prod_{\varepsilon \in u_i} {}^{S_\varepsilon}2$, where $S_\varepsilon := S_{\mathbf{q},s_\varepsilon}$).

(B) $\mathbb{P}_{\bar{s}} := \mathbb{P}_{\mathbf{q}} \restriction L$, where

$$L := \big\{p \in \mathbb{P}_{\mathbf{q}} : \mathrm{dom}(p) \subseteq \{s_i : i < \alpha\}, \text{ and if } s_i \in \mathrm{dom}(p) \text{ then } \mathrm{supp}(p(s_i)) \subseteq \{s_j : j \in u_i\}\big\}.$$

Claim 3.4. 1) *For $\kappa = \aleph_0$ and $\mathbf{q}, n, \bar{s}$, X_i (for $i < \alpha$) as in* 3.3(A)(e), *we have $\mathbb{P}_{\mathbf{q},\bar{s}} \lessdot \mathbb{P}_{\mathbf{q}}$ <u>when</u>*

$\boxplus_1$ *If $i < \alpha$ <u>then</u> the demand on $\mathbb{Q}_{\bar{\varphi}_i,\bar{\underset{\sim}{\eta}}}$ holds absolutely (i.e. even after forcing by any κ-forcing).*

$\boxplus_2$ *Assuming $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$ is generic over $\mathbf{V}$ and $\bar{\eta} = \langle \underset{\sim}{\eta}_t[\mathbf{G}] : t \in L_{\mathbf{q}} \rangle$, we have: <u>if</u> $\mathbf{V}[\langle \eta_{s_j} : j \in u_i \rangle] \models$ "$\mathcal{J}$ is a maximal antichain of $\mathbb{Q}[\langle \eta_{s_j} : j \in u_i \rangle]$" <u>then</u> $\mathbf{V}[\bar{\eta} \restriction L_{\mathbf{q},s_i}] \models$ "$\mathcal{J}$ is a maximal antichain of $\mathbb{Q}[\bar{\eta} \restriction L_{\mathbf{q},s_i}]$" for $i < \alpha$.*

2) $\mathbb{Q}^2_{\mathbf{n}}$ *from* [HS, Defs. 2,4,5] *satisfies the criteria above. Moreover, so does any Suslin ccc forcing (see* [JS88]*).*

3) *Similarly to parts* (1), (2) *for $\bar{s} = \langle s_\alpha : \alpha < \alpha_* \rangle$, where $s_\alpha \in L_{\mathbf{q}}$ is $<_{\mathbf{q}}$-increasing.*

Proof. 1,2) By (3).

3) Straightforward by induction on α_*. $\square_{3.4}$

§ 4. A consequence

We prove the result promised in the introduction, continuing Kellner-Shelah [KS11] and Horowitz-Shelah [HS].

Theorem 4.1. *Let* $\kappa = \aleph_0$, $\partial = (2^{\aleph_0})^+$ *(or just* $\partial = \partial^{\aleph_0} = \mathrm{cf}(\partial)$, $\partial > 2^{\aleph_0}$ *for simplicity), and* $\partial \leq \theta \leq \lambda = \lambda^{<\theta}$.

Let $\mathbf{n} \in \mathbf{N}$ *be* special, *in the sense of* [HS, Definitions 2,4] *(and so* $T_{\mathbf{n}}$ *is a finite-branching subtree of* ${}^{\omega>}\omega$ *as defined there). Let* $(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$ *be as in* [HS, Def. 5], *except that we restrict ourselves to the (dense) subset of* $p \in \mathbb{Q}^2_{\mathbf{n}}$ *such that for some* $m \ll \ell g(\mathrm{tr}_{p(\alpha)})$,

$$\nu \in p(\alpha) \Rightarrow \mathrm{nor}(\mathrm{suc}_{p_{\overline{w}}}(\nu)) \geq 1 + \frac{1}{m}$$

(as done in the proof of [HS, Claim 21]*).*

Then there is a $\mathbf{q} \in \mathbf{Q}^2_{\partial,\theta}$ *such that:*

- (a) $L_{\mathbf{q}}$ *has cardinality* λ, $\mathrm{cf}(L_{\mathbf{q}}) = \mathrm{cf}(\lambda)$, *and* $t \in L_{\mathbf{q}} \Rightarrow |I_{\mathbf{q},t}| < \lambda$.
- (b) *For every* $t \in L_{\mathbf{q}}$, $\mathbb{Q}_{\mathbf{q},t} = \mathbb{Q}^2_{\mathbf{n}}[\mathbf{V}^{\bar{\underset{\sim}{\eta}} \restriction I_t}]$, *so* $\underset{\sim}{\eta}_t \in \lim T_{\mathbf{n}}$ *is* $\underset{\sim}{\eta}^2_{\mathbf{n}}$ *(recalling* [HS] *— that is,* 3.4(2)*).*
- (c) $\mathbf{q}$ *is strongly* $(<\theta)$*-homogeneous (see* 2.5*).*
- (d) *Letting* $\mathbf{V}_0 = \mathbf{V}$, $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$, *and* $\mathbf{V}_1 = \mathrm{HOD}\big(\{\bar{\underset{\sim}{\eta}} \restriction u : u \in [L_{\mathbf{q}}]^{<\theta}\}\big)$*:*
  - (α) $\mathbf{V}_1 \models \mathsf{ZF} + \mathsf{DC}_{<\theta}$
  - (β) *In* $\mathbf{V}_1$, *modulo the ideal*

    $$J = J_{\mathbf{n},<\theta} := \mathrm{id}_{<\theta}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}}),$$

    we have:
    - $\bullet_1$ $\lim(T_{\mathbf{n}}) \equiv \{\underset{\sim}{\eta}_t : t \in L_{\mathbf{q}}\} \mod J$
    - $\bullet_2$ *Every subset of* $\lim(T_{\mathbf{n}})$ *is equivalent to a Borel set modulo* J.

Remark 4.2. 1) The difference from the results in [HS] is that there we do not have "$\mathbf{V}_1$ satisfies $\mathsf{AC}_{\aleph_0}$" (to say nothing of DC), whereas here we have DC (even $\mathsf{DC}_{<\theta}$, with $\theta > \aleph_1$).[13]

2) In $\mathrm{id}_{<\theta}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$, is the '$<\theta$' necessary? ([HS, Def. 18] uses $\mathrm{id}_{\leq\aleph_1}$, in our notation.) That is, can we use $\mathrm{id}_{\leq\aleph_0}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$?

For this we have to use "amoeba for $\mathbb{Q}_{\mathbf{n}}$," hence we have to prove stronger amalgamation (which is far from clear). But see 4.5 below.

Proof. Let $\mathbf{Q}_{\mathbf{n}}$ be the set of $\mathbf{q} \in \mathbf{Q}$ which satisfy 4.1(b). Now we can replace $\mathbf{Q}$ by $\mathbf{Q}_{\mathbf{n}}$ in 2.6, and we rely on 4.3, 4.4, and 4.5 below. $\square_{4.1}$

Claim 4.3. *For* $\mathbf{q}$ *as in* 4.1,

$$\Vdash_{\mathbb{P}_{\mathbf{q}}} \text{"}\textit{if } \eta \in \lim(T_{\mathbf{n}}) \textit{ is } (\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})\textit{-generic over } \mathbf{V} \textit{ then } \eta \in \{\underset{\sim}{\eta}_s : s \in L_{\mathbf{q}}\}\text{"}.$$

Proof. We continue [HS, p.15, Claim 21] (but there it sufficed to consider iterations of finite length).

So assume

$(*)_1$ $p_* \Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$\underset{\sim}{\eta} \in \lim(T_{\mathbf{n}})$".

[13] As wrongly stated in [JS93], for the ideal of meagre sets.

$(*)_2$ For $n < \omega$, let $\bar{p}_n := \langle p_{n,\ell} : \ell < \omega \rangle$ be a maximal antichain of $\mathbb{P}_{\mathbf{q}}$ such that $p_{n,\ell} \Vdash \underset{\sim}{\eta} \restriction n = \nu_{n,\ell}$.

Let $L_* := \bigcup_{n,\ell<\omega} \mathrm{supp}(p_{n,\ell}) \cup \mathrm{supp}(p_*)$; it is a countable subset of $L_{\mathbf{q}}$.

$(*)_3$ (a) For $\eta \in T_{\mathbf{n}}$, define:

$$W_{\mathbf{n},\eta} := \big\{ w \subseteq \mathrm{suc}_{T_{\mathbf{n}}}(\eta) : \mathrm{nor}^{\mathbf{n}}_\eta(w) \geq 2 \big\}.$$

(b) For $n < \omega$ define $\Lambda_n := \{\eta \in T_{\mathbf{n}} : \ell g(\eta) < n\}$, so $T_{\mathbf{n}} = \bigcup_{n<\omega} \Lambda_n$.

(c) Define

- $\bullet_1$ $S_n := \{\overline{w} = \langle w_\eta : \eta \in \Lambda_n \rangle : w_\eta \in W_{\mathbf{n},\eta}\}$ for $n < \omega$.
- $\bullet_2$ $S := \bigcup_{n<\omega} S_n$
- $\bullet_3$ $(S, \trianglelefteq)$ is a tree with ω levels such that each level is finite.
- $\bullet_4$ $\lim(S) = \{\overline{w} = \langle w_\eta : \eta \in T_{\mathbf{n}} \rangle : \overline{w} \restriction \Lambda_n \in S_n$ for every $n\}$.

(d) For $\overline{w} \in \lim(S)$ let

$$\mathbf{B}_{\overline{w}} := \{\rho \in \lim(T_{\mathbf{n}}) : \text{for every } n \text{ large enough, } \rho \restriction (n+1) \in w_{\rho \restriction n}\}.$$

$(*)_4$ So $\mathbf{B}_{\overline{w}} = \bigcup_{m<\omega} \mathbf{B}_{\overline{w},m}$, where

$$\mathbf{B}_{\overline{w},m} := \big\{\rho \in \lim(T_{\mathbf{n}}) : (\forall n \geq m)[\rho \restriction (n+1) \in w_{\rho\restriction n}]\big\}$$

is a closed subset of $\lim(T_{\mathbf{n}})$.

As proved there,

$(*)_5$ For $\iota = 1, 2$, $\Vdash_{\mathbb{Q}^\iota_{\mathbf{n}}}$ "$\underset{\sim}{\eta}^\iota_{\mathbf{n}} \in B_{\overline{w}}$" for every $\overline{w} \in \lim(S)^{\mathbf{V}}$".

Hence as in [HS],

⊞ By $(*)_1$, it suffices to prove $p_* \nVdash_{\mathbb{P}_{\mathbf{q}}}$ "$\underset{\sim}{\eta} \in \mathbf{B}_{\overline{w}}$ for some $\overline{w} \in \lim(S)^{\mathbf{V}}$".

Toward contradiction, assume

$$\Vdash_{\mathbb{P}_{\mathbf{q}}} \text{"}\underset{\sim}{\eta} \text{ is generic for } (\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}}) \text{ over } \mathbf{V}\text{"},$$

<u>or</u> we just choose $\langle p_{\overline{w}} : \overline{w} \in \lim(S) \rangle$ such that $p_* \leq p_{\overline{w}}$ and $p_{\overline{w}} \Vdash \underset{\sim}{\eta} \in \mathbf{B}_{\overline{w}}$. Note that for $r \in \mathrm{dom}(p_{\overline{w}})$, $\mathrm{tr}(p_{\overline{w}}(r))$ is an object (not just a $\mathbb{P}_{\mathbf{q},s}$-name) because $\mathbf{q} \in \mathbf{Q}^2_{\partial,\kappa}$. We continue as there. $\square_{4.3}$

Claim 4.4. 1) *Forcing with* $\mathbb{Q}^2_{\mathbf{n}}$ *adds a Cohen real.*

2) *If* $\mathbb{Q}$ *adds a Cohen real then* $\Vdash_{\mathbb{Q}}$ "$(\lim T_{\mathbf{n}})^{\mathbf{V}} \in \mathrm{id}_{\leq\aleph_0}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$".

Proof. See [HS, Claim 19]. $\square_{4.4}$

Claim 4.5. *In the conclusion of Claim 4.1, we can replace* $\mathrm{id}_{<\partial}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$ *by the ideal* $J' := \mathrm{id}_{\leq\aleph_0}(\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}}) + Y$, *where in* $\mathbf{V}_1$ *we define*

$$Y := \bigcup \big\{\mathbf{B}^{\mathbf{V}_1} : \mathbf{B} \text{ is a Borel subset of } \mathbf{T_n} \text{ defined in } \mathbf{V}_0 \text{ such that } \Vdash_{\mathbb{Q}^2_{\mathbf{n}}} \text{"}\underset{\sim}{\eta}^2_{\mathbf{n}} \notin \mathbf{B}\text{"}\big\}.$$

Proof. The same proof as in 4.1; that is, in clause (d)(β) we use the ideal J' above instead of $J_{\mathbf{n},<\partial}$. $\square_{4.5}$

* * *

Definition 4.6. 1) Let Φ_κ be the set of pairs $(\bar\varphi, \underset{\sim}{\nu})$ such that

(a) $\bar\varphi$ is a definition of a κ^+-cc forcing notion $\mathbb{Q}_i = \mathbb{Q}_{\bar\varphi,i}$ in $\mathcal{H}(\kappa^+)$ from a parameter $c_i \in {}^\kappa\mathcal{H}(\kappa)$.

(b) $\Vdash_{\mathbb{Q}_{\bar\varphi,i}}$ "$\underset{\sim}{\nu} \in {}^\kappa\mathcal{H}(\kappa)$"; naturally the generic, but this is not necessary.

(c) Moreover, any κ-forcing preserves the properties of (a) and (b). Furthermore, the properties

$$\text{“}p \in \mathbb{Q}_{\bar\varphi,i}, \quad p \le_{\mathbb{Q}_{\bar\varphi,i}} q, \quad \langle p_\varepsilon : \varepsilon < \varepsilon_* \rangle \text{ is a } \mathbb{Q}_{\bar\varphi,i}\text{-MAC”}$$

will be absolute between $\mathbf{V}^{\mathbb{P}_1}$ and $\mathbf{V}^{\mathbb{P}_2}$, where $\mathbb{P}_\ell := \mathbb{P}_{\mathbf{q}_\ell}$, $\mathbf{q}_1 \le_{\mathbf{Q}} \mathbf{q}_2$, and $c_i \in \mathbf{V}[\mathbb{P}_{\mathbf{q}_1}]$.

(A $\mathbb{Q}$-*MAC* is a maximal antichain of the forcing notion $\mathbb{Q}$.)

2) For $(\bar\varphi, \underset{\sim}{\nu}) \in \Phi_\kappa$ and $\partial > \kappa$, we define the ideal $\mathrm{id}(\bar\varphi, \underset{\sim}{\nu})$ on $\mathcal{P}({}^\kappa\mathcal{H}(\kappa))$ as usual.

Claim 4.7. *Assume $\lambda = \lambda^{<\partial}$ and $\partial = \mathrm{cf}(\partial) > 2^\kappa$. Then there is $\mathbf{q}$ such that*

(A) $\mathbf{q} \in \mathbf{Q}_{\partial,\kappa}$, $L_\mathbf{q}$ has cardinality λ, and $\mathrm{cf}(L_\mathbf{q}) = \mathrm{cf}(\lambda)$.

(B) For every $t \in L_\mathbf{q}$ there are $(\bar\varphi_t, \underset{\sim}{\nu}) \in \Phi_\kappa$ and $\underset{\sim}{c}_t$ (a $\mathbb{P}_{\mathbf{q},L_t}$-name of a member of ${}^\kappa\mathcal{H}(\kappa)$) such that $\mathbb{Q}_{\mathbf{q},t} = (\underset{\sim}{\mathbb{Q}}_{\bar\varphi_t,\underset{\sim}{c}_t})^{\mathbf{V}[\eta]}$, and let $\underset{\sim}{\nu}_t$ be chosen naturally.

(C) For every $\underset{\sim}{c}$ (a $\mathbb{P}_\mathbf{q}$-name of a member of ${}^\kappa\mathcal{H}(\kappa)$), letting $X := \{t \in L_\mathbf{q} : (\bar\varphi_t, \underset{\sim}{c}_t) = (\bar\varphi, \underset{\sim}{c})\}$ and $Y := \{\underset{\sim}{\nu}_t : t \in X\}$, we have

(a) $\Vdash_{\mathbb{P}_\mathbf{q}} Y \notin \mathrm{id}_{<\theta}(\mathbf{Q}_{\varphi_{\underset{\sim}{c}}}, \underset{\sim}{\nu})$

(b) Let $\mathbf{V}_0 := \mathbf{V}$, $\mathbf{V}_2 := \mathbf{V}^{\mathbb{P}_\mathbf{q}}$, and

$$\mathbf{V}_1 := \mathrm{HOD}^{\mathbf{V}_2}\big(\{\bar\eta \restriction L : L \in [L_\mathbf{q}]^{<\theta}\}, \{Y\}, \mathbf{V}\big).$$

Then $\mathbf{V}_1$ is a model of $\mathsf{ZF} + \mathsf{DC}_{<\theta}$ + "every $Z \subseteq Y \subseteq {}^\kappa\mathcal{H}(\kappa)$ is equal to a κ-Borel set modulo the ideal generated by

$$\mathrm{id}_{<\theta}(\underset{\sim}{\mathbb{Q}}_{\bar\varphi,\underset{\sim}{c}}, \underset{\sim}{\nu}) \cup \{{}^\kappa\mathcal{H}(\kappa) \setminus Y\} \cup \{{}^\kappa\mathcal{H}(\kappa)^{\mathbf{V}[\underset{\sim}{\bar\eta} \restriction L_t]} : t \in L_\mathbf{q}\}\text{''}.$$

(c) If $(\mathbb{Q}_{\bar\varphi,\underset{\sim}{c}}, \underset{\sim}{\nu})$ does not commute with itself (see below) then we can use the ideal $\mathrm{id}_{<\theta}(\underset{\sim}{\mathbb{Q}}_{\bar\varphi,\underset{\sim}{c}}, \underset{\sim}{\nu}) \cup \{{}^\kappa\mathcal{H}(\kappa) \setminus Y\}$.

(d) If we restrict the parameter $\underset{\sim}{c}_t$ to be from $\mathbf{V}$, we can use $\mathbf{V}_1$ for all $(\bar\varphi, c)$.

Remark 4.8. In 4.7(C)(c) the assumption is very weak. It fails for Cohen reals and Random reals. By [She94], [She04a], among ccc Suslin forcings $\mathbb{Q}$ (see [JS88]) if $\mathbb{Q}$ is not bounding then only Cohen forcings do not commute with themselves.

Probably among the bounding ones, 'Random real' is the only one.

Proof. Straightforward. $\square_{4.7}$

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