ON A PROBLEM OF ERDOS AND HAJNAL

SHIMON GARTI, YAIR HAYUT, AND SAHARON SHELAH

ABSTRACT. We address a question of Erdős and Hajnal about the ordinary partition relation $\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_{\aleph_0})^2$. For $\theta = \mathrm{cf}(\lambda) < \lambda$, assuming $2^\lambda = \lambda^+$ they proved the negative relation $\lambda^+ \nrightarrow (\lambda^+, (3)_\theta)^2$ and asked whether the (local instance of) GCH is indispensable. We show that this negative relation is consistent with λ being strong limit and $2^\lambda > \lambda^+$. The result can be pushed down to $\aleph_\omega$.

2010 *Mathematics Subject Classification.* 03E02, 03E04, 03E55.

Key words and phrases. Infinite combinatorics, GCH, stick, tiltan, singular cardinals hypothesis, pcf theory.

The research was supported by Israel Science Foundation Grant no. 1838/19 and Grant no. 2320/23. This is publication 1249 (the 204th prime number) of the third author, publication 47 (the 15th prime number) of the first author and publication 37 (the 12th prime number) of the second author.

0. Introduction

Let $G = (V, E)$ be a graph of size λ. One may wonder whether there must be a monochromatic triangle under any edge coloring $c : E \to \theta$. The answer is trivially no, since the graph can be a set of isolated vertices with no edges at all, or a triangle-free graph. Thus in order to make the above question interesting, one should assume that there are many edges (and, in particular, many triangles) in the graph. One possible way to do it uses the following concept. A set of vertices $W \subseteq V$ is called *independent* if $[W]^2 \cap E = \varnothing$. If G is of size λ and there are no independent subsets of size λ in G, then there are many edges in the graph and the question makes more sense.

The above discussion can be formulated in the language of partition calculus, without mentioning graphs at all. The ordinary partition relation $\lambda \to (\kappa, (3)_\theta)^2$ says that for every coloring $c : [\lambda]^2 \to \theta$ there is either $A \in [\lambda]^\kappa$ so that $c''[A]^2 = \{0\}$, or $B \in [\lambda]^3$ and $\gamma \in (0, \theta)$ so that $c''[B]^2 = \{\gamma\}$. A particular interesting case is when $\kappa = \lambda$. In terms of graph theory, one can interpret the coloring as assigning zero to pairs of vertices with no edge, and some color $\gamma \in (0, \theta)$ to edges of a given graph. The positive relation $\lambda \to (\lambda, (3)_\theta)^2$ means that either there is an independent set of size λ, or a monochromatic triangle.

Erdős, Hajnal and Rado investigated this relation in [EHR65]. They established several results, and focused in particular on graphs whose size is a successor of a singular cardinal. A good account appears in the monograph [EHMR84], in which the following is phrased and proved:

Theorem 0.1. *Assume that λ is a singular cardinal and $2^\lambda = \lambda^+$. Then $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$.*

Actually, they proved something a bit stronger, see [EHMR84, Theorem 20.2]. A natural question is whether the assumption $2^\lambda = \lambda^+$ is removable. Let us indicate that if one forces $2^{\mathrm{cf}(\lambda)} \geq \lambda^+$ then a negative result obtains, as mentioned in [EHMR84]. Thus we shall assume from now on that $2^{\mathrm{cf}(\lambda)} < \lambda$, and in fact we shall force the negative relation while λ is a strong limit singular cardinal. The first case, in this context, is $\lambda = \aleph_\omega$. In a collection of unsolved problems [EH71, Problem 5], the pertinent question appears as follows:

Question 0.2. Can one prove without assuming GCH that the relation $\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_{\aleph_0})^2$ holds?

It appeared, again, in [EHMR84, Problem 20.1].[1] Despite the fact that powerful methods for dealing with successors of singular cardinals are available today, the problem is still open. A recent survey by Komjáth, [Kom25], describes the progress in every problem from the list of Erdős and Hajnal,

[1] In the monograph [EHMR84], the domain of the coloring is $\aleph_\omega^{\aleph_0}$. Under the assumption $2^{\aleph_\omega} = \aleph_{\omega+1}$, these two entities coincide, i.e. $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$.

and according to this survey no progress has been made with regard to this problem. In this paper we intend to show that $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ is consistent with $2^\lambda > \lambda^+$, where λ is singular and strong limit.

Our strategy is to replace the hypothesis $2^\lambda = \lambda^+$ by pcf arguments. More specifically, we obtain unbounded sequences of regular cardinals below λ that satisfy some relevant negative arrow relation, and we make sure that the true cofinality of these sequences is λ^+. These assumptions enable us to lift the negative relations below λ to λ^+, even if $2^\lambda > \lambda^+$.

This approach is rendered here twice. Firstly, we get the negative arrow relation $\lambda_i^+ \nrightarrow (\lambda_i^+, (3)_{\mathrm{cf}(\lambda_i)})^2$ where each λ_i is a successor of a singular cardinal, simply by assuming GCH below λ, and then we lift this relation to λ^+. The drawback here is that λ must be a limit of singular cardinals with the same cofinality. Thus $\aleph_\omega$ cannot be handled in this way. Secondly, we modify the negative arrow relation over each λ_i in such a way that it applies to regular cardinals even though these cardinals are not successors of singular cardinals. This strategy applies to small cardinals like the $\aleph_n$s, and consequently the negative arrow relation can be forced over $\aleph_\omega$.

There is another possible approach towards the same negative arrow relation. This approach is based on prediction principles from the tiltan family. Suppose that diamond holds at λ^+, in which case $2^\lambda = \lambda^+$. In many cases, diamond can be replaced by weaker prediction principles like tiltan or stick. Usually, these principles at λ^+ are consistent with $2^\lambda > \lambda^+$. We shall prove that $\stick(\lambda)$ yields the negative arrow relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$. However, we do not know whether $\stick(\lambda)$ is consistent with $2^\lambda > \lambda^+$ when λ is a strong limit singular cardinal. Recall that in $\stick(\lambda)$ the size of the guessing sets is λ, and this is the main problematic issue in our context. This point will be discussed in the relevant section.

The rest of the paper contains three additional sections. In the first section we solve the general problem using simple arguments of pcf theory. As mentioned above, this solution does not cover the case of $\aleph_\omega$. In the second section we modify the pcf arguments and show how to push down the result to $\aleph_\omega$. Finally, in the third section we show that a special version of stick at λ^+ yields the desired negative arrow relation.

Our notation is coherent with [EHMR84]. We shall use the Jerusalem forcing notation, namely we force upwards. A function $f : E \to \mathcal{P}(E)$ is a *set mapping* if $x \notin f(x)$ whenever $x \in E$. A subset $X \subseteq E$ is free for f iff $f(y) \cap X = \varnothing$ whenever $y \in X$. If $\kappa = \mathrm{cf}(\kappa) < \lambda$ then we let $S^\lambda_\kappa = \{\delta \in \lambda \mid \mathrm{cf}(\delta) = \kappa\}$. If $\aleph_0 < \mathrm{cf}(\lambda)$ then S^λ_κ is a stationary subset of λ.

We shall use the idea of indestructibility (at supercompact cardinals) as appeared in the seminal work of Laver, [Lav78]. It is shown there that a supercompact cardinal κ can be made indestructible under any generic extension by κ-directed-closed forcing notions. In our context, we need a modification of Laver's proof, applicable to strategically-closed forcing notions. This version will be phrased and proved below. For basic background concerning Prikry type forcings we refer to [Git10], and to the papers of Magidor [Mag77a] and [Mag77b] in which the basic method of Prikry forcing with interleaved collapses was introduced. We also refer to [Hay23] in this context. For background in pcf theory we suggest [AM10], [BM90] and [She94].

1. A negative relation from simple pcf arguments

In this section we suggest our first approach for proving the negative relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ where λ is a strong limit singular cardinal and $2^\lambda > \lambda^+$. The idea is to assume the negative relation at an unbounded sequence of cardinals below λ (by assuming GCH at these cardinals) and to obtain the negative relation at λ^+ by means of pcf theory. Recall that $\mathrm{tcf}(\prod_{i\in\theta} \lambda_i, J) = \kappa$ iff there is a J-cofinal and increasing sequence in the product $\prod_{i\in\theta} \lambda_i$, and κ is the minimal length of such a sequence. We commence with the combinatorial theorem, followed by a description of the ways to force the assumptions in this theorem.

Theorem 1.1. *Assume that:*

(a) $\mu > \mathrm{cf}(\mu) = \theta$.
(b) μ *is a strong limit cardinal.*
(c) $2^\mu > \mu^+$.
(d) $(\mu_i \mid i \in \theta)$ *is increasing and* $\mu = \bigcup_{i\in\theta} \mu_i$.
(e) $\mu_i > \mathrm{cf}(\mu_i) = \theta$ *for every* $i \in \theta$.
(f) μ_i *is a strong limit cardinal for every* $i \in \theta$.
(g) $2^{\mu_i} = \mu_i^+$ *for every* $i \in \theta$.
(h) $\mathrm{tcf}(\prod_{i\in\theta} \mu_i^+, J^{\mathrm{bd}}_\theta) = \mu^+$.

Then $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$.

Proof.
For every $i \in \theta$ let $c_i : [\mu_i^+]^2 \to \theta$ be a witness to the negative relation $\mu_i^+ \nrightarrow (\mu_i^+, (3)_\theta)^2$. This negative relation follows from assumption (g). Our goal is to define a coloring $c : [\mu^+]^2 \to \theta$ by combining the c_is together in such a way that the corresponding negative relation at μ^+ will follow.

We need two mathematical objects to define our coloring. The first is a scale $(f_\alpha \mid \alpha \in \mu^+)$ in the product $(\prod_{i\in\theta} \mu_i^+, J^{\mathrm{bd}}_\theta)$. The second is a system of functions $(h_i \mid i \in \theta)$ where $h_i \in {}^\theta\theta$ is injective and $h_i(0) = 0$ for each $i \in \theta$. Also, if $i < j < \theta$ then $rang(h_i) \cap rang(h_j) = \{0\}$. Suppose that $\alpha < \beta < \mu^+$. Let $i(\alpha, \beta)$ be the minimal $j \in \theta$ so that $f_\alpha(j) \neq f_\beta(j)$. Such an ordinal always exists since $f_\alpha <_{J^{\mathrm{bd}}_\theta} f_\beta$. We define the coloring $c : [\mu^+]^2 \to \theta$ as follows. Given $\alpha < \beta < \mu^+$ let $i = i(\alpha, \beta)$ and let $c(\alpha, \beta) = h_i(c_i(\{f_\alpha(i), f_\beta(i)\}))$.[2] Let us show that c exemplifies the negative relation $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$.

(ℵ) Assume that $A \in [\mu^+]^{\mu^+}$. For every $i \in \theta$ let $A_i = \{f_\alpha(i) \mid \alpha \in A\}$. Set $X = \{i \in \theta \mid \mu_i^+ = \bigcup A_i\}$, and notice that $X = \theta \bmod J^{\mathrm{bd}}_\theta$. Fix $i \in X$. For every $\varepsilon \in \mu_i^+$ we choose $\alpha_\varepsilon \in A$ such that $f_{\alpha_\varepsilon}(i) \geq \varepsilon$. Since $|\prod_{j\in i} \mu_j^+| < \mu_i^+$ (this inequality follows from (f)), there are a set $B_i \in [\mu_i^+]^{\mu_i^+}$ and a fixed element $g \in \prod_{j\in i} \mu_j^+$ such that if $\varepsilon < \zeta$ are taken from B_i then $\alpha_\varepsilon < \alpha_\zeta$ and $f_{\alpha_\varepsilon} \restriction i = g$ for every

[2] The fact that $i = i(\alpha, \beta)$ implies that $\{f_\alpha(i), f_\beta(i)\}$ is a pair of ordinals.

$\varepsilon \in B_i$. Since c_i witnesses the negative relation $\mu_i^+ \nrightarrow (\mu_i^+, (3)_\theta)^2$, one can choose $\varepsilon, \zeta \in B_i$ such that $\varepsilon < \zeta$ and $c_i(\varepsilon, \zeta) \neq 0$. But then $c(\alpha_\varepsilon, \alpha_\zeta) \neq 0$, so the proof of the first case is accomplished.

($\beth$) Assume that $\alpha < \beta < \gamma < \mu^+$. If $i(\alpha, \beta) \neq i(\alpha, \gamma)$ or $i(\alpha, \beta) \neq i(\beta, \gamma)$ or $i(\alpha, \gamma) \neq i(\beta, \gamma)$ then $\{\alpha, \beta, \gamma\}$ cannot be c-monochromatic with any color $\xi > 0$ since for $i \neq j$ one has $rang(h_i) \cap rang(h_j) = \{0\}$ and by the definition of c. If $i(\alpha, \beta) = i(\alpha, \gamma) = i(\beta, \gamma) = i$ then $c \restriction [\{\alpha, \beta, \gamma\}]^2 = \{\xi\}$ with $\xi > 0$ implies $c_i \restriction [\{f_\alpha(i), f_\beta(i), f_\gamma(i)\}]^2 = \{\xi\}$, since h_i is injective. But this is impossible by the choice of c_i, so we are done.

$\square_{1.1}$

A corollary to the above theorem gives an answer to the question of Erdős and Hajnal. Within the proof of this corollary we force with $\mathbb{Q}_{\bar{\mu}}$ from [GS12, Definition 2.3]. For being self-contained, we unfold the definition of this forcing notion.

Let μ be supercompact, and let $\bar{\mu} = (\mu_i \mid i \in \mu)$ be an increasing sequence of regular cardinals so that $2^{|i|} < \mu_i$ for every $i \in \mu$. A condition $p \in \mathbb{Q}_{\bar{\mu}}$ is a pair $(\eta, f) = (\eta^p, f^p)$ such that $\ell g(\eta) \in \mu$ and $\eta \in \prod_{i \in \ell g(\eta)} \mu_i$. We refer to η as the stem of p. Also, $f \in \prod_{i \in \mu} \mu_i$ and $\eta \triangleleft f$. If $p, q \in \mathbb{Q}_{\bar{\mu}}$ then $p \leq q$ iff $\eta^p \trianglelefteq \eta^q$ and $f^p(j) \leq f^q(j)$ for every $j \in \mu$.

Intuitively, $\mathbb{Q}_{\bar{\mu}}$ adds a function $h \in \prod_{i \in \mu} \mu_i$ which dominates every old function in this product. If $2^\mu = \mu^+$ in the ground model then $\mathbb{Q}_{\bar{\mu}}$ is μ^+-cc. Also, $\mathbb{Q}_{\bar{\mu}}$ is $(< \mu)$-strategically-closed. Hence one can iterate $\mathbb{Q}_{\bar{\mu}}$ and preserve cardinals. If $\theta = \mathrm{cf}(\theta) > \mu$ is the length of the iteration then the generic functions added at each step form a scale. Moreover, upon singularizing μ either by Prikry or by Magidor forcing one preserves the properties of this scale, thus forcing $\mathrm{tcf}(\prod_{i \in \mathrm{cf}(\mu)} \mu_i, J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}) = \theta$ in the generic extension.

Corollary 1.2. *Assuming the existence of large cardinals in the ground model, one can force $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$ with $2^\mu > \mu^+$ and μ is a strong limit cardinal.*

Proof.
Our goal is to force the assumptions of Theorem 1.1. Let μ be a supercompact cardinal and fix a regular cardinal $\aleph_0 \leq \theta \in \mu$. We may assume that μ is Laver-indestructible, and GCH holds above μ. Let $(\mu_i \mid i \in \mu)$ be an increasing sequence of singular cardinals so that $\mathrm{cf}(\mu_i) = \theta$ for every $i \in \mu$ and $\mu = \bigcup_{i \in \mu} \mu_i$. We may assume that $2^{\mu_i} = \mu_i^+ < \mu_{i+1}$ for every $i \in \mu$.

We force with $\mathbb{Q}_{\bar{\mu}}$ followed by Magidor forcing to make $\theta = \mathrm{cf}(\mu)$ to obtain the assumption $\mathrm{tcf}(\prod_{i \in \theta} \mu_i^+, J^{\mathrm{bd}}_\theta) = \mu^+$. If $\theta = \aleph_0$ then one can simply use Prikry forcing. Thus, the length of the iteration should be an ordinal of cofinality μ^+. We increase 2^μ to any desired point (this can be done by choosing the length of the iteration to be in the desired cardinality). Notice that $2^{\mu_i} = \mu_i^+$ remains true, as $\mathbb{Q}_{\bar{\mu}}$ is χ-strategically-closed for every $\chi \in \mu$ and the component of Prikry or Magidor forcing also preserves this fact.

Thus the assumptions of Theorem 1.1 hold in the generic extension, and the corollary follows.

$\square_{1.2}$

It seems that the above method cannot be applied to $\aleph_\omega$. The main point is that our singular cardinal μ of cofinality θ should be a limit of singular cardinals with the same cofinality. Thus, the negative colorings along the way are always with the same number of colors (namely, θ) and hence one can produce a coloring over the cardinal μ^+ with θ-many colors. Since there are no singular cardinals below $\aleph_\omega$ at all, the above proof is inapplicable as it is to this case. However, $\aleph_{\omega^2}$ seems suitable for this pattern of proof. Indeed, the cofinality of $\aleph_{\omega^2}$ is ω and it is a limit of singular cardinals of countable cofinality.

Theorem 1.3. *Assuming the existence of large cardinals in the ground model, one can force $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$ with $2^\mu > \mu^+$ and μ is a strong limit cardinal, where $\mu = \aleph_{\omega^2}$.*

Proof.
Let μ be a strong cardinal and let $\lambda \geq \mu^{++}$ be a regular cardinal. Let E be a (μ, λ)-extender and let $\jmath : V \to M \cong \mathrm{Ult}(V, E)$ be the canonical embedding, where $M \supseteq V_{\mu^{++}}$. We assume GCH in the ground model. In order to force the above statement at μ we use the Extender-based Prikry forcing, and in order to obtain the negative relation at $\mu = \aleph_{\omega^2}$ we use the same forcing with interleaved collapses.

Let G be V-generic for this forcing notion. Notice that μ is a strong limit cardinal in $V[G]$, and $2^\mu = \mu^{++}$. Likewise, μ is a singular cardinal of countable cofinality in the generic extension, and GCH still holds below μ in $V[G]$. We can add the collapses to make $\mu = \aleph_{\omega^2}$ in $V[G]$.

Let $(\rho_n \mid n \in \omega)$ be the Prikry forcing added through the (unique) normal ultrafilter of E. It is known that $\mathrm{tcf}(\prod_{n\in\omega} \mu_n^+, J_\omega^{\mathrm{bd}}) = \mu^{++}$ in the generic extension. Moreover, up to a modification of a proper initial segment, this is the only sequence with true cofinality μ^{++} in this product. Hence, if $(\mu_n \mid n \in \omega)$ is an increasing sequence of singular cardinals with countable cofinality such that $\mu = \bigcup_{n\in\omega} \mu_n$ then $\mathrm{tcf}(\prod_{n\in\omega} \mu_n^+, J_\omega^{\mathrm{bd}}) = \mu^+$ in $V[G]$. For these facts we refer to [Hay23]. We see that all the assumptions of Theorem 1.1 hold, and therefore $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$. In the setting of the Extender-based Prikry forcing with interleaved collapses we can make $\mu = \aleph_{\omega^2}$ in $V[G]$. This is the first infinite cardinal which can be represented as a limit of a sequence $(\mu_n \mid n \in \omega)$ as above, so the proof is accomplished.

$\square_{1.3}$

We conclude with a couple of open problems. Maybe the most interesting problem which issues from our study is whether the negative relation holds in ZFC. We believe that the positive relation $\lambda^+ \to (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ is consistent, but we do not know how to prove this:

Question 1.4. Is it consistent that λ is a strong limit singular cardinal and $\lambda^+ \to (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$? Is it forceable at $\lambda = \aleph_\omega$?

It follows from the proofs in this section that a positive answer to the above question must be forced by adding a lot of bounded subsets to λ. In any attempt to force a positive relation one has to eliminate GCH at every unbounded sequence of cardinals in λ whose true cofinality is λ^+.

2. More pcf and the case of $\aleph_\omega$

In this section we prove that the negative relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ is consistent with λ being a strong limit singular cardinal and $2^\lambda > \lambda^+$ even if λ is not a limit of singular cardinals. In particular, this setting is forceable at $\lambda = \aleph_\omega$.

The basic idea is similar to that of the proof of Theorem 1.1 in the previous section. Namely, from negative arrow relations on a sequence of cardinals below λ one obtains the negative arrow relation at λ^+, provided that the true cofinality of the sequence is λ^+. Thus, pcf theory enables us to lift combinatorial properties below λ to λ^+. However, in order to incorporate $\aleph_\omega$ into this framework one has to prove relevant statements about regular cardinals (e.g., the $\aleph_n$s). For those cardinals we refine the negative partition relation to a weaker negative partition relation, relative to certain filters. Using an appropriate pcf structure that respects those filters, we will obtain the parallel of Theorem 1.1.

So, first we deal with obtaining those weaker instances of the negative partition relation from local instances of the generalized continuum hypothesis. This is the content of the first proposition of this section.

Claim 2.1. *Assume that $\kappa < \lambda$ are regular cardinals and $\lambda = \lambda^{<\kappa}$. Then there exists a pair $(c, \mathscr{D})$ such that:*

- *(a)* $c : [\lambda]^2 \to \{0, 1\}$.
- *(b)* *$\mathscr{D}$ is a κ-complete (proper) filter over λ.*
- *(c)* *If $A \subseteq \lambda$ and $c \restriction [A]^2$ is constantly zero then $A = \varnothing$ mod $\mathscr{D}$.*
- *(d)* *If $B = \{\alpha, \beta, \gamma\} \in [\lambda]^3$ then $0 \in c''[B]^2$.*

Proof.
Enumerate the elements of $[\lambda]^{<\kappa}$ by $(u_\xi \mid \xi \in \lambda)$, where $u_0 = \varnothing$ and each $u \in [\lambda]^{<\kappa}$ appears λ-many times in the enumeration. We shall define c as $\bigcup_{\alpha\in\lambda} c_\alpha$, so we define c_α and an ordinal ξ_α by induction on $\alpha \in \lambda$ as follows. For $\alpha = 0$ let $c_\alpha = \varnothing$ and if $0 < \alpha$ is a limit ordinal then $c_\alpha = \bigcup\{c_\beta : \beta \in \alpha\}$.

If $\alpha = \beta + 1$ then we define c_α and ξ_β (so ξ_β is picked at the $(\beta+1)$th stage). Let ξ_β be the minimal $\xi \in \lambda$ so that $\xi \notin \{\xi_\gamma \mid \gamma \in \beta\} \cup \{0\}$ and $c_\beta \restriction [u_\xi]^2$ is constantly zero, if there is such an ordinal. If not, let $\xi_\beta = 0$. Now if $\eta < \zeta < \beta$ then let $c_\alpha(\{\eta, \zeta\}) = c_\beta(\{\eta, \zeta\})$. For every $\gamma \in \beta$ let $c_\alpha(\{\gamma, \beta\}) = 1$ if $\gamma \in u_{\xi_\beta}$ and let $c_\alpha(\{\gamma, \beta\}) = 0$ if $\gamma \notin u_{\xi_\beta}$. Thus c_α extends c_β, and the new values are determined according to the membership in u_{ξ_β}. Let $c = \bigcup_{\alpha\in\lambda} c_\alpha$.

Let $\mathcal{I}$ be the ideal that is κ-generated by the 0-monochromatic subsets of λ under c, and the bounded subsets of λ. Formally, for every $A \subseteq \lambda$ let $A \in \mathcal{I}$ iff there are $\zeta \in \kappa$ and $A_\varepsilon \subseteq \lambda$ for every $\varepsilon \in \zeta$ so that $c''[A_\varepsilon]^2 = \{0\}$ for each $\varepsilon \in \zeta$ and $A - \bigcup\{A_\varepsilon \mid \varepsilon \in \zeta\} \in [\lambda]^{<\lambda}$. Let $\mathscr{D}$ be the dual filter, namely $\{\lambda - A \mid A \in \mathcal{I}\}$. Clearly, $\mathscr{D}$ is a κ-complete filter over λ. We must prove, however, that $\mathscr{D}$ is a proper filter (or, in other words, that $\lambda \notin \mathcal{I}$).

Assume towards contradiction that $\lambda \in \mathcal{I}$. Fix $\zeta \in \kappa$ and $A_\varepsilon \subseteq \lambda$ for every $\varepsilon \in \zeta$ such that $c \restriction [A_\varepsilon]^2$ is constantly zero for each $\varepsilon \in \zeta$ and $B = \lambda - \bigcup_{\varepsilon\in\zeta} A_\varepsilon \in [\lambda]^{<\lambda}$. Since λ is regular, B is bounded in λ. We may assume, without loss of generality, that $|A_\varepsilon| = \lambda$ for every $\varepsilon \in \zeta$, since B can be augmented by adding every A_ε of size less than λ to B.

We choose a sequence of ordinals $(\beta_{\alpha\varepsilon} \mid \alpha \in \lambda, \varepsilon \in \zeta)$ with no repetitions such that $\beta_{\alpha\zeta} \in A_\varepsilon$ for every $\alpha \in \lambda, \varepsilon \in \zeta$. The choice is possible since $|A_\varepsilon| = \lambda$ for every $\varepsilon \in \zeta$. For every $\alpha \in \lambda$ let $V_\alpha = \bigcup\{u_{\xi_{\beta_{\alpha\varepsilon}}} \mid \varepsilon \in \zeta\}$ and let $W_\alpha = \{\gamma \in \lambda \mid \exists\varepsilon \in \zeta, \beta_{\gamma\varepsilon} \in V_\alpha\}$. Notice that $V_\alpha \in [\lambda]^{<\kappa}$ since κ is regular, and hence $W_\alpha \in [\lambda]^{<\kappa}$ as well. Apply Hajnal's free subset theorem to the collection $\{W_\alpha \mid \alpha \in \lambda\}$ and let $Y \in [\lambda]^\lambda$ be free. That is, if $\{\alpha, \beta\} \subseteq Y$ then $\alpha \notin W_\beta$.

Let $f : \zeta \to Y$ be an increasing function satisfying $B \cap rang(f) = \varnothing$. Let $u = \{\beta_{f(\varepsilon)\varepsilon} \mid \varepsilon \in \zeta\}$. Observe that $u \in [\lambda]^{<\kappa}$ and $c \restriction [u]^2$ is constantly zero. Indeed, if $\varepsilon' < \varepsilon < \zeta$ and $c(\beta_{f(\varepsilon')\varepsilon'}, \beta_{f(\varepsilon)\varepsilon}) = 1$ then $\beta_{f(\varepsilon')\varepsilon'} \in u_{\xi_{\beta_{f(\varepsilon)\varepsilon}}} \subseteq V_{f(\varepsilon)}$. But then $f(\varepsilon') \in W_{f(\varepsilon)}$ by definition, and we know that $f(\varepsilon') \notin W_{f(\varepsilon)}$ as both $f(\varepsilon')$ and $f(\varepsilon)$ belong to Y.

Recall that u appears λ many times in our enumeration of the elements of $[\lambda]^{<\kappa}$. At some point there will be an ordinal ξ_α so that $u = u_{\xi_\alpha}$ and ξ_α is the first ordinal for which $c \restriction u_{\xi_\alpha} = \{0\}$, since the previous ξ_γ's will be exhausted. From this argument it follows that ξ_α can be arbitrarily large, so we choose ξ_α so that $\alpha > \bigcup B$. Since $\alpha \notin B$, there is some $\varepsilon \in \zeta$ such that $\alpha \in A_\varepsilon$. By definition, $\beta_{f(\varepsilon)\varepsilon} \in A_\varepsilon$ as well, thus $c(\{\alpha, \beta_{f(\varepsilon)\varepsilon}\}) = 0$ as $c \restriction [A_\varepsilon]^2$ is constantly zero. However, $\beta_{f(\varepsilon)\varepsilon} \in u = u_{\xi_\alpha}$, so $c(\{\alpha, \beta_{f(\varepsilon)\varepsilon}\}) = 1$, a contradiction.

$\square_{2.1}$

The above claim is exactly what we need in order to lift negative partition relations to the negative arrow relation over a successor of a singular cardinal. We emphasize that the coloring c may possess a 0-monochromatic set A of size λ. Only the weaker statement $A = \varnothing \bmod \mathscr{D}$ is guaranteed. But this will be sufficient as shown in the following.

Theorem 2.2. *Suppose that:*

- (A) (א) $\lambda > \mathrm{cf}(\lambda) = \theta$.
  - (ב) $\lambda = \bigcup_{i\in\theta} \kappa_i = \bigcup_{i\in\theta} \lambda_i$.
  - (ג) $\kappa_i = \mathrm{cf}(\kappa_i) < \lambda_i = \mathrm{cf}(\lambda_i)$ *and* $\lambda_i^{<\kappa_i} = \lambda_i$ *for every* $i \in \theta$.
  - (ד) $\prod_{i<j} \lambda_i < \kappa_j$ *for every* $j \in \theta$.
- (B) (א) $\mathscr{D}_i$ *is a* κ_i*-complete proper filter over* λ_i *for every* $i \in \theta$.
  - (ב) $c_i : [\lambda_i]^2 \to \{0,1\}$ *for every* $i \in \theta$.
  - (ג) *If* $A \subseteq \lambda_i$ *and* $c_i''[A]^2 = \{0\}$ *then* $A = \varnothing \bmod \mathscr{D}_i$.
  - (ד) *If* $\alpha < \beta < \gamma < \lambda_i$ *and* $\varepsilon = c_i(\{\alpha,\beta\}) = c_i(\{\alpha,\gamma\}) = c_i(\{\beta,\gamma\})$ *then* $\varepsilon = 0$.
- (C) (א) $\bar\eta = (\eta_\alpha \mid \alpha \in \lambda^+) \subseteq \prod_{i\in\theta} \lambda_i$.
  - (ב) *If* $\alpha < \beta < \lambda^+$ *then* $\eta_\alpha \neq \eta_\beta$.

(ℶ) *If* $A_i \in \mathscr{D}_i$ *for every* $i \in \theta$ *then there is* $\alpha_0 \in \lambda^+$ *such that for every* $\alpha_0 \leq \alpha \in \lambda^+$ *one can find* $i_\alpha \in \theta$ *so that* $\eta_\alpha(i) \in A_i$ *whenever* $i_\alpha \leq i \in \theta$.

Then $\lambda^+ \nrightarrow (\lambda^+, (3)_\theta)^2$.

Proof.
We define $c : [\lambda^+]^2 \to \theta \times \{0,1\}$ as follows. For every $\alpha < \beta < \lambda^+$ we let $i_{\alpha\beta} = \ell g(\eta_\alpha \cap \eta_\beta) \in \theta$, and $\varepsilon_{\alpha\beta} = c_{i_{\alpha\beta}}(\{\eta_\alpha(i), \eta_\beta(i)\})$. Now if $0 < \varepsilon_{\alpha\beta}$ then we define $c(\{\alpha,\beta\}) = (i_{\alpha\beta}, \varepsilon_{\alpha\beta})$. Otherwise, we set $c(\{\alpha,\beta\}) = (0,0)$.

We claim that c witnesses the negative arrow relation $\lambda^+ \nrightarrow (\lambda^+, (3)_\theta)^2$. To show this, we must prove two propositions. Firstly, let us show that if $\varepsilon > 0$ then the graph $(\lambda, \{\{\alpha,\beta\} \mid c(\{\alpha,\beta\}) = (i,\varepsilon)\})$ has no triples. Indeed, if $c(\{\alpha,\beta\}) = c(\{\alpha,\gamma\}) = c(\{\beta,\gamma\}) = (i,\varepsilon)$ then the triple $\{\eta_\alpha(i), \eta_\beta(i), \eta_\gamma(i)\}$ is ε-monochromatic under c_i, contradicting (B)(ℷ) as $\varepsilon > 0$. Secondly, we argue that if $A \in [\lambda]^\lambda$ then $c \restriction [A]^2$ is not constantly $(0,0)$.

To see this, assume towards contradiction that $A \in [\lambda]^\lambda$ is a counterexample. For every $i \in \theta$ let $A_i = \{\eta_\alpha(i) \mid \alpha \in A\} \subseteq \lambda_i$. We distinguish two cases. In the first case, $A_i \in \mathscr{D}_i^+$ for some $i \in \theta$. Fix such i and choose $\alpha_\beta \in A$ for every $\beta \in A_i$ so that $\eta_{\alpha_\beta}(i) = \beta$. For every $\nu \in \prod_{j<i} \lambda_j$ let $A_{i\nu} = \{\beta \in A_i \mid \eta_{\alpha_\beta} \restriction i = \nu\}$. Thus $A_i = \bigcup\{A_{i\nu} \mid \nu \in \prod_{j<i} \lambda_j\}$. Since $\mathscr{D}_i$ is κ_i-complete and $\prod_{j<i} \lambda_j < \kappa_i$, for some $\nu \in \prod_{j<i} \lambda_j$ one has $A_{i\nu} \in \mathscr{D}_i^+$. By assumption (B)(ℶ), there are $\beta, \beta' \in A_{i\nu}$ so that $\beta \neq \beta'$ and $c_i(\{\beta,\beta'\}) = \varepsilon > 0$. By definition, $c(\{\alpha_\beta, \alpha_{\beta'}\}) = (i,\varepsilon) \neq (0,0)$. But $\alpha_\beta, \alpha_{\beta'} \in A$, so $c(\{\alpha_\beta, \alpha_{\beta'}\}) = (0,0)$, a contradiction.

In the second case, $A_i = \varnothing \bmod \mathscr{D}_i$ for every $i \in \theta$. Define $B = \{\alpha \in \lambda \mid (\forall^* i < \theta)(\eta_\alpha(i) \in A_i)\}$. From (C)(ℶ) we see that $B \in [\lambda]^{<\lambda}$, as $\lambda_i - A_i \in \mathscr{D}_i$ for every $i \in \theta$ according to the second case. Now if $\alpha \in A$ then, by definition, $\eta_\alpha(i) \in A_i$ for every $i \in \theta$. Hence $A \subseteq B$ and consequently $|A| < \lambda$, a contradiction.

$\square_{2.2}$

In order to utilize the above theorem one has to show that the assumptions there hold (or can be forced) under the relevant circumstances. The assumptions of (A) are easily satisfied for an appropriate choice of cardinals when λ is a strong limit singular cardinal. The assumptions of (B) follow from Claim 2.1. Thus the only challenge is (C), and this is our next goal.

Lemma 2.3. *Let* λ *be a strongly inaccessible cardinal. Let* $R \subseteq \lambda$ *be an unbounded set of regular cardinals. For every* $\zeta \in R$ *let* $\mathscr{D}_\zeta$ *be a* ζ*-complete filter over* ζ^+. *There is a* $(<\lambda)$*-strategically-closed* λ^+*-cc forcing notion* $\mathbb{D}$, *for which the following hold in the generic extension by* $\mathbb{D}$:

(א) $(f_\alpha \mid \alpha \in \lambda^+)$ *is a scale in* $\prod_{\zeta \in R} \mathscr{D}_\zeta$.
(ב) *For every* $(A_\zeta \mid \zeta \in R) \in \prod_{\zeta \in R} \mathscr{D}_\zeta$ *there is* $\alpha_0 \in \lambda^+$ *such that for each* $\alpha_0 \leq \alpha \in \lambda^+$ *there exists* $i_\alpha \in \lambda$ *such that if* $i_\alpha \leq \zeta \in \lambda$ *then* $f_\alpha(\zeta) \in A_\zeta$.

Proof.
Let $(\mathbb{D}_\alpha, \underset{\sim}{\mathbb{H}}_\beta \mid \beta \in \lambda^+, \alpha \leq \lambda^+)$ be a $(<\lambda)$-support iteration of length λ^+, where $\underset{\sim}{\mathbb{H}}_\beta$ is (a name of) a forcing notion defined in the generic extension by $\mathbb{D}_\beta$ as follows. A condition $p \in \mathbb{H}_\beta$ is a pair $(s, \bar{A}) = (s_p, \bar{A}_p)$, where s is a function with $\mathrm{dom}(s) = R \cap \xi_s$ for some $\xi_s \in \lambda$ and $\bar{A}$ is a sequence of sets $(A_\zeta \mid \zeta \in R - \mathrm{dom}(s))$. For every $\zeta \in \mathrm{dom}(s)$ one has $s(\zeta) \in \zeta^+$ and for every $\zeta \in R - \mathrm{dom}(s)$ one has $A_\zeta \in \mathscr{D}_\zeta$. If $p, q \in \mathbb{H}_\beta$ then $p \leq q$ iff $s_p \trianglelefteq s_q, A^q_\zeta \subseteq A^p_\zeta$ whenever $\zeta \in R - \mathrm{dom}(s_q)$ and if $\zeta \in \mathrm{dom}(s_q) - \mathrm{dom}(s_p)$ then $s_q(\zeta) \in A^p_\zeta$. Thus $\mathbb{H}_\beta$ approximates a function in $\prod R$ in a Hechlerish style. We shall say that s_p is *the stem* of p and $\bar{A}_p$ is *the pure part* of p.

Observe that $\underset{\sim}{\mathbb{H}}_\beta$ is (forced to be) $(<\lambda)$-strategically-closed. In fact, if $\gamma \in \lambda$ then there exists a γ^+-directed-closed dense open set of conditions, that is, $T_\gamma = \{p \in \mathbb{H}_\beta \mid \bigcup \mathrm{dom}(s_p) > \gamma\}$. To see this, suppose that $\{p_i \mid i < \gamma\}$ is directed. Define $s_p = \bigcup_{i<\gamma} s_{p_i}$ and for every $\zeta \in R - \mathrm{dom}(s_p)$ let $A^p_\zeta = \bigcap_{i<\gamma} A^{p_i}_\zeta$. Notice that $A^p_\zeta \in \mathscr{D}_\zeta$ since $\zeta > \gamma$ and $\mathscr{D}_\zeta$ is ζ-complete. Now $p = (s_p, \bar{A}_p)$ is an upper bound for $\{p_i \mid i < \gamma\}$. It is easy to see that $\mathbb{H}_\beta$ is λ-centered. Indeed, if $p, q \in \mathbb{H}_\beta$ and $s_p = s_q$ then $p \parallel q$. Since the number of stems is λ (recall that λ is strongly inaccessible) we conclude that $\mathbb{H}_\beta$ is λ-centered.

Let $H \subseteq \mathbb{H}_\beta$ be generic. Define $f_\beta = \bigcup\{s \mid (s, \bar{A}) \in H\}$. By the directness of H and simple density arguments, f_β is a function, $\mathrm{dom}(f_\beta) = R$ and $f_\beta(\zeta) \in \zeta^+$ for every $\zeta \in R$. Let $\mathbb{D} = \mathbb{D}_{\lambda^+}$. Observe that $\mathbb{D}$ is $(<\lambda)$-strategically-closed and λ^+-cc. Indeed, $\mathbb{D}$ is a $(<\lambda)$-support iteration and each component in the iteration is $(<\lambda)$-strategically-closed and λ-centered. Fix a V-generic set $G \subseteq \mathbb{D}$. Let us show that the statement of the lemma holds in $V[G]$.

As noted above, $(f_\alpha \mid \alpha \in \lambda^+) \subseteq \prod R$. Since $\mathbb{D}$ is $(<\lambda)$-strategically-closed, no new bounded subsets of λ are introduced in $V[G]$. However, new *sequences* of measure-one sets (of length λ) are introduced. Fix such a sequence $\bar{A} = (\underset{\sim}{A}_\zeta \mid \zeta \in R) \in \prod_{\zeta \in R} \mathscr{D}_\zeta \cap V[G]$. Since $\mathbb{D}$ is λ^+-cc, there must be some $\gamma \in \lambda^+$ so that $\bar{A} \in V^{\mathbb{D}_\gamma}$. We claim that if $\gamma^+ \leq \beta \in \lambda^+$ then there exists $i_\beta \in \lambda$ such that $f_\beta(\zeta) \in \underset{\sim}{A}_\zeta$ whenever $i_\beta \leq \zeta \in \lambda$. If we prove this statement then the proof of the lemma will be accomplished.

To see this, let p be an arbitrary condition that forces $\bar{A}$ to be a sequence of measure-one sets. Fix $\beta \in [\gamma^+, \lambda^+)$. Define $q \in \mathbb{D}$ so that $p \leq q, \beta \in \mathrm{dom}(q)$ and let $q(\beta) = (t_q, \bar{B}_q)$. We require that $q \upharpoonright \beta \Vdash B^q_\xi \subseteq \underset{\sim}{A}_\xi$ for every $\xi \in \mathrm{dom}(\bar{B}_q)$. This is possible as $\underset{\sim}{A}_\xi$ is a $\mathbb{D}_\gamma$-name and hence also a $\mathbb{D}_\beta$-name. Finally, let $i_\beta = \bigcup \mathrm{dom}(t_q)$. If $i_\beta \leq \zeta \in \lambda$ then $q \Vdash f_\beta(\zeta) \in B^q_\zeta \subseteq \underset{\sim}{A}_\zeta$, so as p was arbitrary we are done.

$\square_{2.3}$

Remark that for densely many conditions $p \in \mathbb{D}$, for every $\alpha \in \mathrm{dom}(p)$, the stem of the condition $p(\alpha)$ is a canonical name of a ground model function. Hence we may assume, without loss of generality, that this is true for every condition in $\mathbb{D}$.

By the previous lemma, we can force our pcf assumption using a $(<\lambda)$-strategically-closed forcing notion. We would like, therefore, to prove that a supercompact cardinal λ will remain supercompact after such a forcing. This will be instrumental since we will singularize this cardinal with various kinds of Prikry-type forcings. Thus we have to show that a supercompact cardinal λ can be indestructible under such forcings. We shall follow the ideas of Laver, with slight little changes, in order to force this property over a supercompact cardinal in the ground model.

As our forcing is not sufficiently directed closed, we need a weaker property that would imply the existence of a master condition for lifting certain supercompact embeddings.

Lemma 2.4. *Let R and $\mathbb{D}$ be as above, and let $\zeta \in R$. Suppose that $F \subseteq \mathbb{D}$ is a directed set of conditions, $|F| < \zeta$, and for every $\alpha \in \bigcup\{\mathrm{dom}(p) \mid p \in F\}$ one has $\bigcup\{\mathrm{dom}(s_{p(\alpha)}) \mid p \in F\} \geq \zeta$. Then there exists a master condition $m \in \mathbb{D}$ so that $p \leq m$ for every $p \in F$.*

Proof.
We commence with letting $\mathrm{dom}(m) = \bigcup\{\mathrm{dom}(p) \mid p \in F\}$. For every $\alpha \in \mathrm{dom}(m)$ let $s_{m(\alpha)} = \bigcup\{s_{p(\alpha)} \mid \alpha \in F\}$ and let $\bar{A}_\xi^{m(\alpha)} = \bigcap\{\bar{A}_\xi^{p(\alpha)} \mid p \in F\}$, for every $\xi \in R - \bigcup \mathrm{dom}(s_{m(\alpha)})$. We prove by induction on $\delta \in \lambda^+$ that $m \restriction \delta$ is a condition and $p \restriction \delta \leq m \restriction \delta$ for every $p \in F$.

For $\delta = 0$ this is trivial, and for a limit ordinal δ it follows from the fact that $|\mathrm{dom}(m)| < \lambda$. We are left with the successor steps, so fix $\delta = \alpha + 1 \in \lambda^+$. By the induction hypothesis, $m \restriction \alpha$ forces $\bar{A}_{p(\alpha)}$ to be a sequence of measure-one sets for every $p \in F$. Hence $m \restriction \delta$ is a condition. It remains to show that $p \restriction \delta \leq m \restriction \delta$ for every $p \in F$.

Fix $p \in F$. From the definition it follows that $s_{p(\alpha)} \trianglelefteq s_{m(\alpha)}$ and $\bar{A}_\xi^{m(\alpha)} \subseteq \bar{A}_\xi^{p(\alpha)}$ for every relevant ξ. Suppose that $\xi \in \mathrm{dom}(s_{m(\alpha)}) - \mathrm{dom}(s_{p(\alpha)})$. Since $s_{m(\alpha)}$ is the union of $s_{p(\alpha)}$s, there exists $r \in F$ for which $\xi \in \mathrm{dom}(s_{r(\alpha)})$ and $s_{m(\alpha)}(\xi) = s_{r(\alpha)}(\xi)$. Since F is directed, there exists $t \in F$ such that $r, p \leq t$. It follows that $s_{t(\alpha)}(\xi) = s_{r(\alpha)}(\xi)$, and hence $t \restriction \alpha \Vdash m(\alpha)(\xi) = s_{t(\alpha)}(\xi) \in \bar{A}_\xi^{p(\alpha)}$, so we are done.

$\square_{2.4}$

As mentioned before, we need the above lemma in order to prove a version of Laver's indestructibility. In the work of Laver, a supercompact cardinal λ is forced to be indestructible under any λ-directed-closed forcing notion. One cannot expect λ to be indestructible under every λ-strategically-closed forcing, since one can force a non-reflecting stationary set with such a forcing notion. However, if the set R is sufficiently sparse then one can prepare the universe so that the corresponding forcing notion $\mathbb{D}$ will preserve the supercompactness of λ. This is the content of our next lemma, and the main idea is that we define the Laver iteration only with respect to some strategically closed forcing notions.

Lemma 2.5. *Let λ be supercompact and assume* GCH. *There is a forcing notion $\mathbb{L}$ such that the following holds in the generic extension by $\mathbb{L}$. If $R \subseteq \lambda$ is a set of double-double successors of strongly inaccessible cardinals and $\mathscr{D}_\zeta$ is a ζ-complete filter over ζ^+ for every $\zeta \in R$ then the associated forcing $\mathbb{D}$ preserves the supercompactness of λ as well as* GCH.

Proof.
Let $\ell : \lambda \to V_\lambda$ be a Laver function. Let $\mathbb{L}$ be the Easton support iteration $(\mathbb{L}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \beta < \lambda, \alpha \leq \lambda)$ where $\underset{\sim}{\mathbb{Q}}_\beta$ is an $\mathbb{L}_\beta$-name of the trivial forcing unless β is strongly inaccessible, $\ell(\alpha) \in V_\beta$ for each $\alpha \in \beta$ and $\ell(\beta)$ is a pair of the form $(\gamma, \underset{\sim}{\tau})$ and $\underset{\sim}{\tau}$ is an $\mathbb{L}_\beta$-name of some $(< \beta)$-strategically-closed forcing notion that preserves GCH.

One can verify that $\mathbb{L}$ is λ-cc and if β is an inaccessible closure point of ℓ then β is preserved by $\mathbb{L}$. It follows from the choice of the $\underset{\sim}{\mathbb{Q}}_\beta$s and standard arguments that GCH holds in the generic extension by $\mathbb{L}$. Choose a V-generic set $G \subseteq \mathbb{L}$, and in $V[G]$ choose a $V[G]$-generic set $g \subseteq \mathbb{D}$. We claim that λ is supercompact in $V[G][g]$.

To see this, suppose that μ is an arbitrary ordinal and $\jmath : V \to M$ is a μ-supercompact elementary embedding, with $\jmath(\ell)(\lambda) = (\mu, \mathbb{D})$. By increasing μ if needed we may assume that $\mu = \mathrm{cf}(\mu) \geq \lambda^+$. We may also assume that $\jmath$ is an ultrapower embedding derived from some fine and normal measure over $\mathcal{P}_\lambda\mu$.

By elementarity, $\jmath(\mathbb{L}) = \mathbb{L} * \mathbb{D} * \mathbb{L}_{tail}$, where $\mathbb{L}_{tail}$ is μ^+-closed in $V[G][g]$. Let $F = \{\jmath(\underset{\sim}{q}) \mid q \in g\}$. By the closure of M and the fact that $\jmath \restriction \underset{\sim}{\mathbb{D}} \in M$ we see that $F \in M[G][g]$. Moreover, F satisfies the assumptions of Lemma 2.4. Indeed, by the choice of R we know that $\min(R - \lambda) > \lambda^+$ and $|F| = \lambda^+$. Hence Lemma 2.4 applies and there is a master condition m for F.

By GCH, which holds in $V[G][g]$, the number of dense subsets of $\mathbb{L}_{tail}$ is $|\jmath(2^{\lambda^+})|^V = |{}^{\mathcal{P}_\lambda\mu}(\lambda^{++})|^V = \mu^+$. Using the fact that $\mathbb{L}_{tail}$ is μ^+-closed, one can construct an $M[G][g]$-generic set h containing m. By the nature of m, $\jmath''g$ is contained in h. Hence Silver's criterion applies and $\jmath : V \to M$ lifts to $\jmath^+ : V[G][g] \to M[H][h]$. Thus λ is μ-supercompact in $V[G][g]$. But μ was arbitrary, hence one concludes that λ is supercompact in $V[G][g]$ and the proof is accomplished.

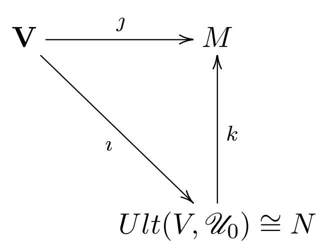
$\square_{2.5}$

The above setting is rendered in a universe satisfying GCH. Our goal, however, is to force the negative arrow relation when $2^\lambda > \lambda^+$. Furthermore, we wish to force this relation at $\lambda = \aleph_\omega$. We shall use the extender-based Prikry forcing in order to increase 2^λ, and the corresponding version with interleaved collapses in order to push down the result to $\aleph_\omega$. The presentation of these forcing notions is inspired from the work of Merimovich, see e.g. [Mer21].

Let W be the universe obtained by the preparatory forcing $\mathbb{L}$. Fix $R \subseteq \lambda$ such that every element of R is of the form ρ^{+4}, where ρ is a strongly inaccessible cardinal, and R is unbounded in λ. For every $\rho^{+4} \in R$ let

$c_\rho : [\rho^{+4}]^2 \to \{0,1\}$ be as guaranteed in Claim 2.1. Namely, c_ρ has no 1-monochromatic triangle and if $A \subseteq \rho^{+4}$ is 0-monochromatic under c_ρ then $A = \varnothing \mod \mathscr{D}_\rho$, where $\mathscr{D}_\rho$ is a ρ^{+3}-complete proper filter over ρ^{+4}. We denote the dual (proper) ideal by $\mathcal{I}_\rho$, so $\mathcal{I}_\rho$ is $< \rho^{+3}$-generated from the bounded subsets of ρ^{+4} and the 0-monochromatic sets of the coloring c_ρ.

Let $\mathbb{D}$ be the forcing notion associated with R, and let V be the extension of W by $\mathbb{D}$. Let $(f_\alpha \mid \alpha \in \lambda^+)$ be the generic scale. Fix a (λ, λ^{++})-extender E in V and let $\jmath : V \to M$ be the extender ultrapower map. Let $\mathscr{U}_0$ be the (unique) normal measure of E, and let $\imath : V \to Ult(V, \mathscr{U}_0) \cong N$ be the corresponding ultrapower embedding. Let K_0 be N-generic for the forcing $Col^N((\lambda^{+5})^N, < \imath(\lambda))$. Let $k : N \to M$ be the quotient map, and let K be the generic set generated by $k[K_0]$.



Claim 2.6. *K is M-generic for the forcing $Col^M((\lambda^{+5})^M, < \jmath(\lambda))$.*

Proof.
Let D be a dense subset of $Col^M((\lambda^{+5})^M, < \jmath(\lambda))$, and assume that $D \in M$. We must show that $D \cap K \neq \varnothing$. By the nature of $\jmath$, there are $a \in [\lambda^{++}]^n$ for some $n \in \omega$ and $f : [\lambda]^n \to V$ so that $\jmath f(a) = D$. Define $D' = \bigcap\{\imath f(z) \mid z \in [(\lambda^{++})^N]^n, \imath f(z)$ is a dense open subset of $Col^N((\lambda^{+5})^N, < \imath(\lambda))\}$.

Clearly $D' \in N$, and since $Col^N((\lambda^{+5})^N, < \imath(\lambda))$ is $(\lambda^{+5})^N$-distributive and D' is the intersection of less than $(\lambda^+)^N$ dense open sets, we see that D' is dense and open. Thus $D' \cap K \neq \varnothing$ and hence $k(D') \cap K \neq \varnothing$. Notice that $D = k(\imath f)(a)$ for $a \in k([(\lambda^{++})^N]^n)$, hence $D \supseteq k(D')$. In particular, $D \cap K \neq \varnothing$ as sought.

$\square_{2.6}$

Working in V, let $\mathbb{P}$ be the extender-based Prikry forcing with interleaved collapses, using the extender E and the set K as a guiding generic. Following [Mer21], for every $d \in [\lambda^{++}]^{\leq\lambda}$ we let $E(d) = \{x \in V_\lambda \mid (\jmath \restriction d)^{-1} \in \jmath(x)\}$. Thus $E(d)$ is a λ-complete ultrafilter over V_λ, and $(\jmath \restriction d)^{-1}$ is a partial increasing function from $\jmath(d)$ to d. Since $E(d)$ concentrates on such objects we may assume, without loss of generality, that if $A \in E(d)$ then A is a set of partial increasing functions with domain contained in d.

We define our forcing notion $\mathbb{P}$ as follows. A condition $p \in \mathbb{P}$ is a quadruple of the form $(\bar{c}^p, f^p, A^p, F^p)$, where:

- (a) f^p is a partial function from $\lambda^{++} - \lambda$ to $\lambda^{<\omega}$.
- (b) $d^p = \mathrm{dom}(f^p)$ satisfies $|d^p| \leq \lambda$.
- (c) $\lambda \in d^p$ for every p and we let $f^p(\lambda) = (\rho_i^p \mid i \in n)$.

(d) $\bar{c}^p \in \prod_{i\leq n} Col((\rho^p_{i-1})^{+5}, <\rho^p_i)$, where $\rho^p_{-1}=\omega$ and $\rho^p_n=\lambda$.
(e) $A^p \in E(d^p)$.
(f) If $\eta \in A^p$ then $\eta(\lambda) > \bigcup \mathrm{dom}(c^p_n)$, and $\eta(\lambda) > \rho^p_{n-1}$.
(g) F^p is a function from A^p to V_λ.
(h) For every $\eta \in A^p$, $F^p(\eta) \in Col(\eta(\lambda)^{+5}, <\lambda)$.
(i) $\jmath(F^p)((\jmath \restriction d^p)^{-1}) \in K$.

We have to define the forcing order. To this end, we define the direct order $\leq^*$ and one point extensions. Then we let the forcing order be a finite sequence of these two extensions.

Assume, therefore, that $p, q \in \mathbb{P}$. We shall say that q is a direct extension of p (denoted by $p \leq^* q$) iff the following hold:

(α) $f^p \subseteq f^q$. In particular, $\ell g(f^p(\lambda)) = \ell g(f^q(\lambda))$.
(β) $A^q \subseteq \{\eta \mid \eta \in d^p \in A^p\}$.
(γ) c^q_i extends c^p_i for every $i \leq n$.[3]
(δ) $F^q(\eta) \geq F^p(\eta)$ for every $\eta \in A^p$ (here $\geq$ is the order of the collapse).

Assume now that $p \in \mathbb{P}$ and $\eta \in A^p$. We define the one-point extension (or Prikry extension) $p^\frown\eta$ as a condition r satisfying the following requirements:

(α) $f^r(\alpha) = f^p(\alpha)$ if $\alpha \notin \mathrm{dom}(\eta)$, and $f^r(\alpha) = f^p(\alpha)^\frown\eta(\alpha)$ if $\alpha \in \mathrm{dom}(\eta)$.
(β) $\bar{c}^r = \bar{c}^{p\frown} F^p(\eta)$.
(γ) $A^r = A^p - \{\rho \in A^p \mid \rho(\lambda) \leq \max\{\eta(\lambda), \bigcup \mathrm{dom}(F^p(\eta))\}\}$.
(δ) $F^r = F^p \restriction A^r$.

Finally, q extends p iff q can be obtained from p by a finite sequence of direct extensions and one-point extensions. Clearly, if $p \leq_{\mathbb{P}} q$ then the process of deriving q from p can be introduced as one single direct extension followed by a finite sequence of one-point extensions.

Notice that $(\mathbb{P}, \leq^*)$ is σ-closed since $E(d)$ is λ-complete and the collapses are $\aleph_1$-complete. It is known that $\mathbb{P}$ satisfies the strong Prikry property, preserves cardinals above λ (including λ itself), forces $\lambda = \aleph_\omega$, preserves GCH below λ and forces $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Claim 2.7. *Let $\underset{\sim}{\bar{\rho}} = (\rho_n \mid n \in \omega)$ be a name for the normal Prikry sequence. Suppose that $p \in \mathbb{P}$ and $\underset{\sim}{\tau}$ is a $\mathbb{P}$-name of an ordinal in ρ_n^{+5}. There exists a condition $q \in \mathbb{P}$ so that $p \leq^* q$ and a function g with $\mathrm{dom}(g) = \lambda$, such that $g(\alpha) \in [\alpha^{+5}]^{\leq \alpha^{++}}$ for every $\alpha \in \lambda$ and $q \Vdash \underset{\sim}{\tau} \in g(\rho_n)$.*

Proof.
Let D be a dense open sets deciding the value of $\underset{\sim}{\tau}$. By the strong Prikry property there are a natural number n and a condition q such that $p \leq^* q$ and every extension of q by n-many one-point extensions belongs to D. Recall that d^p is the domain of f^p. Let $\eta_0, \ldots, \eta_{n-1}$ be a sequence of n-many one-point extensions of q such that the value of $\underset{\sim}{\tau}$ is decided by q and these ηs.

[3]We include the extension of the collapses in the definition of the direct order.

Since $d^p \in [\lambda^{++}]^{\leq\lambda}$ and $((\jmath \restriction d^p)^{-1}) \in \jmath(A^q)$ we may assume (by shrinking A^q if needed) that for every $\eta \in A^q$ and every $\alpha \in \mathrm{dom}(\eta) \subseteq d^p$ it is true that $\eta(\alpha) \leq \eta(\lambda)^{++}$. Moreover, by fixing a bijection $h : \lambda \to d^p$ and shrinking A^q further we may assume that $\mathrm{dom}(\eta) = h''\eta(\lambda)$. Consequently, $|\{\eta \in A^q \mid \alpha = \eta(\lambda)\}| \leq |[\alpha^{++}]^{\alpha}| = \alpha^{++}$ for every $\alpha \in \lambda$. Hence we can define $g(\alpha)$ as the set of all possible values for $\underset{\sim}{\tau}$ using $\eta_0, \ldots, \eta_{n-1}$ with $\eta_{n-1}(\lambda) = \alpha$, and get $q \Vdash \underset{\sim}{\tau} \in g(\rho_n)$ as required.

$\square_{2.7}$

For every $n \in \omega$ let $\mathscr{D}_n$ be the ρ_n^{+3}-complete filter over ρ_n^{+4} derived from Claim 2.1. The forcing $\mathbb{P}$ certainly introduces new bounded subsets of λ, so we must show that the relevant assumptions of Theorem 2.2 remain true in the generic extension by $\mathbb{P}$. We commence with the following:

Claim 2.8. *Let $c : [\rho_n^{+4}]^2 \to \{0,1\}$ be the coloring described in Claim 2.1. Let $\underset{\sim}{I}$ be a name of a 0-monochromatic set under c. Then $\underset{\sim}{I}$ is (forced to be) contained in the union of less than ρ_n^{+3}-many 0-monochromatic sets from the ground model.*

Proof.
Fix an enumeration of $\mathcal{P}^V(\rho_n^{+4})$, for every $n \in \omega$, in the ground model. We are assuming GCH in the ground model, hence the size of the enumeration is ρ_n^{+5}. Let p be an arbitrary condition that forces $\underset{\sim}{I}$ to be 0-monochromatic. By extending $f^p(\lambda)$ if needed we may assume that $\ell g(f^p(\lambda)) > n$. In other words, ρ_n is decided by p.

Let $\mathbb{C} = \prod_{i \leq n} Col(\rho_{i-1}^{+4}, < \rho_i)$, that is, $\mathbb{C}$ is the product of the first $n+1$ collapses mentioned in the condition p. Note that $|\mathbb{C}| = \rho_n$ and the next collapses are at least ρ_n^{+5}-closed. Let $((s_\alpha, \xi_\alpha) \mid \alpha \in \rho_n^{+4})$ enumerate $\mathbb{C} \times \rho_n^{+4}$. By induction on $\alpha \in \rho_n^{+4}$ we define a condition p_α such that:

- (א) $(p_\alpha \mid \alpha \in \rho_n^{+4})$ is $\leq^*$-increasing.
- (ב) $\bar{c}^{p_\alpha} \restriction n+1$ is constant.
- (ג) If there is a direct extension r of p_α which is the same as p_α except that $\bar{c}^{p_\alpha} = s_\alpha{}^\frown c_{n+1}^{p_\alpha}$ and r decides the truth value of the statement $\check{\xi}_\alpha \in \underset{\sim}{I}$ then p_α already decides this statement.

Let q be an upper bound of $(p_\alpha \mid \alpha \in \rho_n^{+4})$. Such an upper bound exists since all the relevant components are sufficiently closed. Let $q[s]$ denote the condition obtained from q upon replacing $\bar{c}^q$ by $s^\frown(\bar{c}^q \restriction [n, \ell g(\bar{c}^q)))$.

Let $r_0 = \bar{c}^q \restriction n+1$. For every $s \in \mathbb{C}$ let $I_s = \{\alpha \in \rho_n^{+4} \mid q[s] \Vdash \check{\alpha} \in \underset{\sim}{I}\}$, so $I_s \in V$ and $\Vdash_{\mathbb{P}} \underset{\sim}{I} \subseteq \bigcup\{I_s \mid r_0 \leq_{\mathbb{C}} s\}$. Furthermore, I_s is 0-monochromatic under c for every s such that $r_0 \leq_{\mathbb{C}} s$, hence $I_s \in \mathcal{I}_n$. We conclude, therefore, that $\underset{\sim}{I}$ is (forced to be) covered by a ground model set in $\mathcal{I}_n$, as required.

$\square_{2.8}$

Let $R = \{\rho_n^{+4} \mid n \in \omega\}$. We claim that $(f_\alpha \restriction R \mid \alpha \in \lambda^+)$ is a scale satisfying the required assumption of Theorem 2.2. Specifically, we must show that for every sequence of sets $(A_n \mid n \in \omega)$ where $A_n \in \mathscr{D}_n$ for each $n \in \omega$, there exists $\gamma \in \lambda^+$ such that if $\gamma \leq \beta \in \lambda^+$ then for some $n_\beta \in \omega$ one has $f_\beta(n) \in A_n$ whenever $n_\beta \leq n \in \omega$.

By Claim 2.8 we may assume that $A_n \in V$ for every $n \in \omega$. Notice, however, that the whole sequence $(A_n \mid n \in \omega)$ need not be in V. Let p be an arbitrary condition in $\mathbb{P}$. Applying Claim 2.7 repeatedly and using the enumeration of the ground model sets, we construct a $\leq^*$-increasing sequence $(q_n \mid n \in \omega)$ such that $q_0 \geq p$ and $q_n \Vdash A_n \in \mathcal{A}_n(\rho_n)$, where $|\mathcal{A}_n(\gamma)| \leq \gamma^{++}$ for every $\gamma \in \lambda$.

Since $(\mathbb{P}, \leq^*)$ is σ-closed, there exists a single condition q such that $q_n \leq q$ for each $n \in \omega$. For every $\zeta \in R$ let $B_\zeta = \bigcap\{A_n \mid n \in \omega, A_n \in \mathcal{A}_n(\zeta)\}$. By the closure of $\mathscr{D}_\zeta$ we see that $B_\zeta \in \mathscr{D}_\zeta$. By the construction, $(B_\zeta \mid \zeta \in R)$ belongs to the ground model and hence there is an ordinal $\gamma \in \lambda^+$ such that for every $\gamma \leq \beta \in \lambda^+$ there exists $\zeta_\beta \in \lambda$ so that $f_\beta(\zeta) \in B_\zeta$ whenever $\zeta_\beta \leq \zeta \in R$. This completes the proof, as $B_{\rho_n^{+3}} \subseteq A_n$ for every $n \in \omega$.

In order to force the failure of SCH at λ, as done in our results, one has to assume the existence of a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ in the ground model. This fundamental result was proved by Gitik in [Git89] and in [Git91]. In our constructions we started from a supercompact cardinal in the ground model. The gap between these large cardinals invites the following:

Question 2.9. Let λ be a strong limit singular cardinal.

- (ℵ) What is the consistency strength of the negative arrow relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ with $2^\lambda > \lambda^+$?
- (ב) What is the consistency strength of the same negative relation with $2^\lambda > \lambda^+$ where $\lambda = \aleph_\omega$?
- (ג) What is the consistency strength of the negative relation at every strong limit singular cardinal λ, in a universe in which $2^\lambda > \lambda^+$ at every such λ?

3. A negative relation from stick

In the previous sections we used pcf assumptions in order to cope with the problem of Erdős and Hajnal. We move now to the second approach in which prediction principles play an important role. The prediction principle that we need for the combinatorial proof is called *stick*. The idea of stick as a prediction principle is well-articulated in [BBCE, Chapter 4(12)]: "It consults its stick, its rod directs it". Here we need the mathematical incarnation of this idea. We commence with the following definition.

Definition 3.1. Suppose that $\kappa \leq \lambda$.

(ℵ) $\overset{\bullet}{|}(\kappa, \lambda) = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq [\lambda]^\kappa \wedge \forall y \in [\lambda]^\lambda \exists x \in \mathcal{F}, x \subseteq y\}$.

(ℶ) Denote $\overset{\bullet}{|}(\lambda, \lambda^+)$ by $\overset{\bullet}{|}(\lambda)$.

The stick principle is closely related to the club principle, but no stationary sets are involved in the prediction. This fact makes $\overset{\bullet}{|}(\lambda)$ very useful when λ is a singular cardinal. Let us recall the definition of the club principle (or *tiltan*), which appeared for the first time in [Ost76]. If $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $S \subseteq \kappa$ is stationary, then a tiltan sequence $(T_\delta \mid \delta \in S)$ is a sequence of sets, where T_δ is a cofinal subset of δ for each $\delta \in S$,[4] and if $A \in [\kappa^+]^{\kappa^+}$ then $S_A = \{\delta \in S \mid T_\delta \subseteq A\}$ is stationary. One says that $\clubsuit_S$ holds if there exists such a sequence.

We proceed to the combinatorial result. Our goal is to prove the relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ from the stick principle $\overset{\bullet}{|}(\lambda)$. Negative partition relations at successors of a regular cardinal κ follow from $\overset{\bullet}{|}(\kappa)$ as shown in [CGW20]. Here we apply a similar idea to successors of singular cardinals. We need the following lemma about free sets from [HM75]. The lemma and its proof also appear in [EHMR84, Lemma 20.3].

Lemma 3.2. *Let κ be a regular cardinal. Suppose that $E = \bigcup_{\alpha \in \kappa} E_\alpha$, and $|E_\alpha| > \kappa$ for every $\alpha \in \kappa$. Assume further that $f : E \to \mathcal{P}(E)$ is a set mapping, and $|f(x) \cap E_\alpha| < \kappa$ for every $x \in E, \alpha \in \kappa$. Then there exists a free set X for f so that $X \cap E_\alpha \neq \varnothing$ for every $\alpha \in \kappa$.*

We can state now the following:

Theorem 3.3. *suppose that $\theta = \mathrm{cf}(\lambda) < \lambda$ and assume that $\overset{\bullet}{|}(\lambda)$ holds. Then $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$.*

Proof.
Let $(\kappa_i \mid 1 \leq i \in \theta)$ be an increasing sequence of infinite cardinals such that $\lambda = \bigcup_{i \in \theta} \kappa_i$. Notice that the enumeration of these cardinals begins with κ_1 since we wish to save the first color to the full-sized independent subsets of the graph. We shall define a partition $\mathcal{P} = (\mathcal{P}_i \mid i \in \theta)$ of $[\lambda^+]^2$.[5]

[4] We assume, tacitly, that S consists of limit ordinals. There is no loss of generality here since S is stationary.

[5] The elements of this partition are not required to be disjoint, so we use here the term *partition* in an unusual way.

Then, essentially, for $\alpha < \beta < \lambda^+$ we will set $c(\alpha, \beta) = i$ iff $\{\alpha, \beta\} \in \mathcal{P}_i$.[6] The partition $\mathcal{P}$ will be based on a sequence of set-mappings in the following way. For every $i \in (0, \theta)$ we shall define $f_i : \lambda^+ \to [\lambda^+]^{\leq \kappa_i}$ such that $f_i(\alpha) \subseteq \alpha$ for every $i \in \theta, \alpha \in \lambda^+$. We let $\mathcal{P}_i = \{\{\alpha, \beta\} \mid \alpha < \beta < \lambda^+, \alpha \in f_i(\beta)\}$. This procedure defines $\mathcal{P}_i$ for $i > 0$, and we let $\mathcal{P}_0 = [\lambda^+]^2 - \bigcup\{\mathcal{P}_i \mid 1 \leq i < \theta\}$.

The construction of each f_i is by induction on $\alpha \in \lambda^+$, where at the αth stage, $f_i(\alpha)$ is defined simultaneously for each $i \in (0, \theta)$. Fix $\alpha \in \lambda^+$ and suppose that $f_i(\gamma)$ is already defined for every $\gamma \in \alpha$ and every $i \in \theta$. Let $(T_\eta \mid \eta \in \lambda^+)$ be a $\tiltan(\lambda)$ sequence, so $T_\eta \in [\lambda^+]^\lambda$ for every $\eta \in \lambda^+$. Let $\mathcal{S}_\alpha = \{T_\eta \mid \eta \in \alpha, T_\eta \subseteq \alpha\}$. Notice that $|\mathcal{S}_\alpha| \leq |\alpha| \leq \lambda$ and hence there exists a decomposition of the form $\mathcal{S}_\alpha = \bigcup\{\mathcal{S}_i^\alpha \mid 1 \leq i \in \theta\}$, where $i < j \Rightarrow \mathcal{S}_i^\alpha \cap \mathcal{S}_j^\alpha = \varnothing$ and $|\mathcal{S}_i^\alpha| \leq \kappa_i$ for every $i \in (0, \theta)$.

In order to define $f_i(\alpha)$ for each $i \in (0, \theta)$, fix an ordinal i and apply Lemma 3.2, where κ_i^+ here stands for κ there, and $f_i \restriction \alpha$ here stands for f there. Notice that $|f_i(\gamma)| \leq \kappa_i$ for each $\gamma \in \alpha$ by the induction hypothesis, so the assumptions of the lemma hold. By the conclusion of the lemma, there exists a free set $X = X_{\alpha i}$ for $f_i \restriction \alpha$ which satisfies $X \cap T \neq \varnothing$ for every $T \in \mathcal{S}_i^\alpha$. By removing elements from X if needed, we may assume that $|X| \leq |\mathcal{S}_i^\alpha| \leq \kappa_i$, so we can define $f_i(\alpha) = X = X_{\alpha i}$. This completes the definition of our set mappings, and consequently the definition of $\mathcal{P}$, the partition of $[\lambda^+]^2$.

We define, at this stage, the coloring $c : [\lambda^+]^2 \to \theta$ by letting $c(\alpha, \beta) = i$ iff $i \in \theta$ is the first ordinal so that $\{\alpha, \beta\} \in \mathcal{P}_i$. We claim that c witnesses the negative relation to be proved. To see this, let us show firstly that there are no $\alpha < \beta < \delta < \lambda^+$ and $i \in \theta$ such that $c(\alpha, \beta) = c(\alpha, \delta) = c(\beta, \delta) = i$ where $i > 0$. Indeed, if $\alpha < \beta < \delta < \lambda^+$ and $c(\alpha, \delta) = c(\beta, \delta) = i$ then $\{\alpha, \delta\}, \{\beta, \delta\} \in \mathcal{P}_i$. This means that $\alpha, \beta \in f_i(\delta) = X$. But X is a free set with respect to $f_i \restriction \delta$, and $\beta \in X$, hence $f_i(\beta) \cap X = \varnothing$. Since $\alpha \in X$ one concludes that $\alpha \notin f_i(\beta)$. Therefore, $c(\alpha, \beta) \neq i$.

Secondly, we argue that there is no 0-monochromatic subset of λ^+ of size λ^+. To see this, fix $A \in [\lambda^+]^{\lambda^+}$. Choose an ordinal $\eta \in \lambda^+$ such that $T_\eta \subseteq A$. If $\xi > \eta$ and $\xi > \sup(T_\eta)$ then, by definition, $T_\eta \in \mathcal{S}_\xi$. Since A is unbounded in λ^+, one can choose $\xi > \eta, \sup(T_\eta)$ such that $\xi \in A$. Recall that we had a partition $\mathcal{S}_\xi = \bigcup\{\mathcal{S}_i^\xi \mid 1 \leq i \in \theta\}$, hence $T_\eta \in \mathcal{S}_i^\xi$ for some $i \in (0, \theta)$.

By the choice of $f_i(\xi)$ we know that $T_\eta \cap f_i(\xi) \neq \varnothing$, so one can choose $\alpha \in T_\eta \cap f_i(\xi)$. The fact that $\alpha \in f_i(\xi)$ implies that $\{\alpha, \xi\} \in \mathcal{P}_i$. Hence $c(\alpha, \xi) \neq 0$. Since $\alpha \in T_\eta \subseteq A$ and $\xi \in A$, one concludes that $c''[A]^2 \neq \{0\}$, so we are done.

$\square_{3.3}$

We would like to emphasize an important aspect of the proof. The size of each T_η is λ, as our guessing principle is $\tiltan(\lambda) = \tiltan(\lambda, \lambda^+)$. Life would be much

[6] We say *essentially* since the elements of the partition here are not necessarily disjoint, so the formal definition of the coloring will take the first i for which $\{\alpha, \beta\} \in \mathcal{P}_i$.

simpler if we could replace $⫯(\lambda)$ by $⫯(\kappa, \lambda^+)$ or by $\clubsuit_{S^{\lambda^+}_{\kappa}}$ for some $\kappa < \lambda$. One should ask, therefore, whether $⫯(\lambda)$ is essential for the combinatorial proof above. To wit, one should ask why do we insist on guessing sets of size λ. The point lies in Lemma 3.2. In order to find a free set X that meets every E_α, one must verify that E_α is sufficiently large. In the context of our proof, one needs $|T_\eta| > \kappa_i$ for some relevant $\kappa_i < \lambda$, where T_η plays the role of E_α within the proof of the theorem. However, we do not know in advance the identity of κ_i, and it might appear as any κ_i in the sequence $(\kappa_i \mid i \in \theta)$. It seems that the only way to cope with this problem is by taking the T_ηs to be of cardinality λ.

Though we do not know how to force $⫯(\lambda)$ with the failure of SCH at λ, we can describe a possible strategy towards this goal. Let λ be a supercompact cardinal and let $\mathscr{U}$ be a normal ultrafilter over λ. One says that $\mathrm{Gal}(\mathscr{U}, \lambda^+, \lambda^+)$ holds every family $\{A_\alpha \mid \alpha \in \lambda\} \subseteq \mathscr{U}$ contains a subfamily $\{A_{\alpha_i} \mid i \in \lambda^+\}$ so that $\bigcap_{i\in\lambda^+} A_{\alpha_i} \in \mathscr{U}$. One can force such an ultrafilter over a supercompact cardinal λ, see e.g. [BGP23]. If one forces Prikry through such an ultrafilter then every new set of size λ^+ in the Prikry generic extension contains an old set of size λ^+. Thus if stick or tiltan hold at λ^+ in the ground model, this will be preserved in the generic extension. Moreover, this is true for a variety of Prikry-type forcing notions, including Prikry forcing with interleaved collapses. There are also several ways to force stick or tiltan at a supercompact cardinal λ while increasing 2^λ above λ^+. We do not know, however, to force these two things together:

Question 3.4. Is it consistent that λ is supercompact, $\mathscr{U}$ is a normal ultrafilter over λ satisfying $\mathrm{Gal}(\mathscr{U}, \lambda^+, \lambda^+)$, $⫯(\lambda)$ holds and $2^\lambda > \lambda^+$?

References

[AM10] Uri Abraham and Menachem Magidor. Cardinal arithmetic. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1149–1227. Springer, Dordrecht, 2010.

[BBCE] Hosea Ben Be'eri. The book of Hosea. In *Prophets*. 727 B.C.E.

[BGP23] Tom Benhamou, Shimon Garti, and Alejandro Poveda. Negating the Galvin property. *J. Lond. Math. Soc. (2)*, 108(1):190–237, 2023.

[BM90] Maxim R. Burke and Menachem Magidor. Shelah's pcf theory and its applications. *Ann. Pure Appl. Logic*, 50(3):207–254, 1990.

[CGW20] William Chen, Shimon Garti, and Thilo Weinert. Cardinal characteristics of the continuum and partitions. *Israel J. Math.*, 235(1):13–38, 2020.

[EH71] P. Erdős and A. Hajnal. Unsolved problems in set theory. In *Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967)*, pages 17–48. Amer. Math. Soc., Providence, R.I., 1971.

[EHMR84] Paul Erdős, András Hajnal, Attila Máté, and Richard Rado. *Combinatorial set theory: partition relations for cardinals*, volume 106 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1984.

[EHR65] P. Erdős, A. Hajnal, and R. Rado. Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hungar.*, 16:93–196, 1965.

[Git89] Moti Gitik. The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Ann. Pure Appl. Logic*, 43(3):209–234, 1989.

[Git91] Moti Gitik. The strength of the failure of the singular cardinal hypothesis. *Ann. Pure Appl. Logic*, 51(3):215–240, 1991.

[Git10] Moti Gitik. Prikry-type forcings. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1351–1447. Springer, Dordrecht, 2010.

[GS12] Shimon Garti and Saharon Shelah. A strong polarized relation. *J. Symbolic Logic*, 77(3):766–776, 2012.

[Hay23] Yair Hayut. Prikry type forcings and the Bukovský-Dehornoy phemomena. *preprint*, 2023.

[HM75] András Hajnal and Attila Máté. Set mappings, partitions, and chromatic numbers. In *Logic Colloquium '73 (Bristol, 1973)*, volume Vol. 80 of *Stud. Logic Found. Math.*, pages 347–379. North-Holland, Amsterdam-Oxford, 1975.

[Kom25] Péter Komjáth. The Erdos-Hajnal problem list. *Bull. Symolic Logic*, page To appear, 2025.

[Lav78] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel J. Math.*, 29(4):385–388, 1978.

[Mag77a] Menachem Magidor. On the singular cardinals problem. I. *Israel J. Math.*, 28(1-2):1–31, 1977.

[Mag77b] Menachem Magidor. On the singular cardinals problem. II. *Ann. of Math. (2)*, 106(3):517–547, 1977.

[Mer21] Carmi Merimovich. Mathias like criterion for the extender based Prikry forcing. *Ann. Pure Appl. Logic*, 172(9):Paper No. 102994, 6, 2021.

[Ost76] A. J. Ostaszewski. On countably compact, perfectly normal spaces. *J. London Math. Soc. (2)*, 14(3):505–516, 1976.

[She94] Saharon Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

Email address: shimon.garty@mail.huji.ac.il

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

Email address: yair.hayut@mail.huji.ac.il

Institute of Mathematics The Hebrew University of Jerusalem, Jerusalem 91904, Israel and Department of Mathematics Rutgers University New Brunswick, NJ 08854, USA

Email address: shelah@math.huji.ac.il

URL: http://www.math.rutgers.edu/~shelah