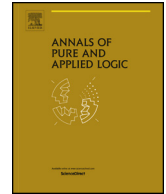




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Full Length Article

Borel sets without perfectly many overlapping translations, III [☆]Andrzej Rosłanowski ^{a,*}, Saharon Shelah ^{b,c}^a Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182-0243, USA^b Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel^c Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

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ABSTRACT

We expand the results of Rosłanowski and Shelah [11,10] to all perfect Abelian Polish groups $(\mathbb{H}, +)$. In particular, we show that if $\alpha < \omega_1$ and $4 \leq k < \omega$, then there is a ccc forcing notion adding a Σ_2^0 set $B \subseteq \mathbb{H}$ which has \aleph_α many pairwise k -overlapping translations but not a perfect set of such translations. The technicalities of the forcing construction led us to investigations of the question when, in an Abelian group, $X - X \subseteq Y - Y$ imply that a translation of X or $-X$ is included in Y .

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1. Introduction

For a Polish space X and a set $B \subseteq X \times X$ we say that B contains a μ -square (perfect square, respectively), if there is a set Z of cardinality μ (a perfect set Z , respectively) such that $Z \times Z \subseteq B$. The problem of Borel sets with large squares but no perfect squares was studied and resolved in Shelah [14].

Several questions can be phrased in a manner involving μ -squares and/or perfect squares *with some additional structure* on them. For instance, looking at a Polish group $(\mathbb{H}, +)$ we may ask for its Borel subsets with many, but not too many disjoint translations (or just translations with small overlaps). This leads to considering the *spectrum of translation k -disjointness of a set $A \subseteq \mathbb{H}$* ,

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* Corresponding author.

E-mail addresses: aroslanowski@unomaha.edu (A. Rosłanowski), shelah@math.huji.ac.il (S. Shelah).

URL: <http://shelah.logic.at> (S. Shelah).

$$\text{std}_k(A) = \{(x, y) \in \mathbb{H} \times \mathbb{H} : |(A + x) \cap (A + y)| \leq k\},$$

and asking if this set may contain a μ -square but not a perfect square. For $k = 0$ this is asking for μ many pairwise disjoint translations of A without a perfect set of such translations. This direction is related to works of Balcerzak, Rosłanowski and Shelah [1], Darji and Keleti [4], Elekes and Steprāns [6], Zakrzewski [15] and Elekes and Keleti [5].

It is still unresolved if we may repeat the results of [14] for the disjointness context, but there is some promising work in progress [13]. However a lot of progress has been made in the dual direction.

For a set $A \subseteq \mathbb{H}$ we consider its *spectrum of translation k -non-disjointness*,

$$\text{std}_k(A) = \{(x, y) \in \mathbb{H} \times \mathbb{H} : |(A + x) \cap (A + y)| \geq k\}.$$

Then a μ -square included in $\text{std}_\kappa(A)$ determines a family of μ many pairwise k -overlapping translations. These were studied extensively for the context of the Cantor space in Rosłanowski and Rykov [9], and Rosłanowski and Shelah [11,10,12]. Those works fully utilized the algebraic properties of $({}^\omega 2, +)$, leaving the general case of Polish groups unresolved.

In the current paper we aim at generalizing their results to perfect Abelian Polish groups. The main difficulty in this more general case lies in quite algebraic problem (\spadesuit) given below. Suppose $S \subseteq \mathbb{H}$ and $X \subseteq \mathbb{H}$ is a set of k -intersecting translations, i.e.,

$$(\diamond)_X^S \quad |(S + x) \cap (S + y)| \geq k \text{ for all } x, y \in X.$$

Then for all $c \in \mathbb{H}$ the property $(\diamond)_{X+c}^S$ also holds true. Thus the properties of objects added by our forcing should reflect some “translation invariance”. How can we know that a set Y is included in a translation of X ? Clearly, if $Y \subseteq X + c$ or $Y \subseteq c - X$, then $Y - Y \subseteq X - X$. It would be helpful in our forcing if we knew

(\spadesuit) when does $Y - Y \subseteq X - X$ imply that Y is included in a (small) neighborhood of a translation $X + c$ of X or of a translation $c - X$ of $-X$?

In the third section we introduce the main algebraic ingredient of our forcing notion: qifs and quasi independent sets. In forcing, we will use them in conjunction with differences of elements of the group, but a relative result for sums also seems interesting, so we present it in Section 4. The third and fourth section might be of interest independently from the rest of the paper, as they address the question (\spadesuit) giving interesting (though technical) properties of perfect Abelian Polish groups with few elements of rank 2.

Like in [14], the “no perfect set” property of the forcing extension results from the use of a “splitting rank” rk^{sp} . We remind its definition and basic properties in the second section. For the relevant proofs we refer the reader to [14,11].

In the fifth section we prove our main consistency result for groups with few elements of rank 2. The remaining case when \mathbb{H} has many elements of rank 2 is treated in Section 6. We close the paper with summary of our results and a list of open problems.

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [8] or Bartoszyński and Judah [2]). However, in forcing we keep the older convention that *a stronger condition is the larger one*.

(1) For a set u we let

$$u^{(2)} = \{(x, y) \in u \times u : x \neq y\}.$$

- (2) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. Finite ordinals (non-negative integers) will be denoted by letters $i, j, k, \ell, m, n, J, K, L, M, N$ and ι . The Greek letters λ and μ will stand for uncountable cardinals.
- (3) Finite sequences will be denoted σ, ς
- (4) For a forcing notion \mathbb{P} , all \mathbb{P} -names for objects in the extension via \mathbb{P} will be denoted with a tilde below (e.g., τ, \tilde{X}), and $\dot{G}_{\mathbb{P}}$ will stand for the canonical \mathbb{P} -name for the generic filter in \mathbb{P} .
- (5) $(\mathbb{H}, +, 0)$ is an Abelian group (in the main part of the paper it is a perfect Polish Abelian group). The elements of \mathbb{H} will be called a, b, c, d (with possible indices). For an integer ι and $a \in \mathbb{H}$, we use the notation ιa to denote the element of \mathbb{H} obtained by repeated addition of a (or $-a$) $|\iota|$ many times in the usual way.
- (6) For sets $A, B \subseteq \mathbb{H}$ we will write $-A = \{-a : a \in A\}$,

$$A + B = \{a + b : a \in A \wedge b \in B\} \quad \text{and} \quad A - B = \{a - b : a \in A \wedge b \in B\}.$$

2. Splitting rank rk^{sp}

Let us recall a rank used in previous papers which will be central for the results here too. We quote some definitions and theorems from [11, Section 2], however they were first given in [14, Section 1].

Let λ be a cardinal and \mathbb{M} be a model with the universe λ and a countable vocabulary τ .

Definition 2.1.

- (1) By induction on ordinals δ , for finite non-empty sets $w \subseteq \lambda$ we define when $\text{rk}(w, \mathbb{M}) \geq \delta$. Let $w = \{\alpha_0, \dots, \alpha_n\} \subseteq \lambda$, $|w| = n + 1$.
 - (a) $\text{rk}(w) \geq 0$ if and only if for every quantifier free formula $\varphi = \varphi(x_0, \dots, x_n) \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0, \dots, \alpha_k, \dots, \alpha_n]$ then the set

$$\{\alpha \in \lambda : \mathbb{M} \models \varphi[\alpha_0, \dots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \dots, \alpha_n]\}$$

is uncountable;

- (b) if δ is limit, then $\text{rk}(w, \mathbb{M}) \geq \delta$ if and only if $\text{rk}(w, \mathbb{M}) \geq \gamma$ for all $\gamma < \delta$;
- (c) $\text{rk}(w, \mathbb{M}) \geq \delta + 1$ if and only if for every quantifier free formula $\varphi = \varphi(x_0, \dots, x_n) \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0, \dots, \alpha_k, \dots, \alpha_n]$ then there is $\alpha^* \in \lambda \setminus w$ such that

$$\text{rk}(w \cup \{\alpha^*\}, \mathbb{M}) \geq \delta \quad \text{and} \quad \mathbb{M} \models \varphi[\alpha_0, \dots, \alpha_{k-1}, \alpha^*, \alpha_{k+1}, \dots, \alpha_n].$$

By a straightforward induction on δ one easily shows that if $\emptyset \neq v \subseteq w$ then

$$\text{rk}(w, \mathbb{M}) \geq \delta \geq \gamma \implies \text{rk}(v, \mathbb{M}) \geq \gamma.$$

Hence we may define the rank function on finite non-empty subsets of λ .

Definition 2.2. The rank $\text{rk}(w, \mathbb{M})$ of a finite non-empty set $w \subseteq \lambda$ is defined as:

- $\text{rk}(w, \mathbb{M}) = -1$ if $\neg(\text{rk}(w, \mathbb{M}) \geq 0)$,
- $\text{rk}(w, \mathbb{M}) = \infty$ if $\text{rk}(w, \mathbb{M}) \geq \delta$ for all ordinals δ ,
- for an ordinal δ : $\text{rk}(w, \mathbb{M}) = \delta$ if $\text{rk}(w, \mathbb{M}) \geq \delta$ but $\neg(\text{rk}(w, \mathbb{M}) \geq \delta + 1)$.

Definition 2.3. For an ordinal ε and a cardinal λ let $\text{NPr}^\varepsilon(\lambda)$ be the following statement:

“there is a model \mathbb{M}^* with the universe λ and a countable vocabulary τ^* such that $1 + \text{rk}(w, \mathbb{M}^*) \leq \varepsilon$ for all $w \in [\lambda]^{<\omega} \setminus \{\emptyset\}$.”

Let $\text{Pr}^\varepsilon(\lambda)$ be the negation of $\text{NPr}^\varepsilon(\lambda)$.

Note that NPr_ε of [11, Definition 2.4] differs from our NPr^ε : “ $\sup\{\text{rk}(w, \mathbb{M}^*) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \varepsilon$ ” there is replaced by “ $1 + \text{rk}(w, \mathbb{M}^*) \leq \varepsilon$ ” here. However, the proofs for [11, Propositions 2.6, 2.7] show the following results.

Proposition 2.4.

- (1) $\text{NPr}^1(\omega_1)$.
- (2) If $\text{NPr}^\varepsilon(\lambda)$, then $\text{NPr}^{\varepsilon+1}(\lambda^+)$.
- (3) If $\text{NPr}^\varepsilon(\mu)$ for $\mu < \lambda$ and $\text{cf}(\lambda) = \omega$, then $\text{NPr}^\varepsilon(\lambda)$.
- (4) If $\alpha < \omega_1$, then $\text{NPr}^\alpha(\aleph_\alpha)$ but $\text{Pr}^\alpha(\beth_{\omega_1})$ holds.

Definition 2.5. Let $\tau^\otimes = \{R_{n,j} : n, j < \omega\}$ be a fixed relational vocabulary where $R_{n,j}$ is an n -ary relational symbol (for $n, j < \omega$).

Definition 2.6. Assume that $\varepsilon < \omega_1$ and λ is an uncountable cardinal such that $\text{NPr}^\varepsilon(\lambda)$. By this assumption, we may fix a model $\mathbb{M}(\varepsilon, \lambda) = \mathbb{M} = (\lambda, \{R_{n,j}^\mathbb{M}\}_{n,j < \omega})$ in the vocabulary τ^\otimes with the universe λ such that:

- (*)_a for every n and a quantifier free formula $\varphi(x_0, \dots, x_{n-1}) \in \mathcal{L}(\tau^\otimes)$ there is $j < \omega$ such that for all $\alpha_0, \dots, \alpha_{n-1} \in \lambda$,

$$\mathbb{M} \models \varphi[\alpha_0, \dots, \alpha_{n-1}] \Leftrightarrow R_{n,j}[\alpha_0, \dots, \alpha_{n-1}],$$

- (*)_b the rank of every singleton is at least 0,
 (*)_c $1 + \text{rk}(v, \mathbb{M}) \leq \varepsilon$ for every $v \in [\lambda]^{<\omega} \setminus \{\emptyset\}$,
 (*)_d $\mathbb{M} \models R_{2,0}[\alpha_0, \alpha_1]$ if and only if $\alpha_0 < \alpha_1 < \lambda$.

For a nonempty finite set $v \subseteq \lambda$ let $\text{rk}^{\text{sp}}(v) = \text{rk}(v, \mathbb{M})$, and we fix witnesses $\mathbf{j}(v) < \omega$ and $\mathbf{k}(v) < |v|$ for the rank of v , so that the following demands (*)_e–(*)_g are satisfied. If $\{\alpha_0, \dots, \alpha_k, \dots, \alpha_{n-1}\}$ is the increasing enumeration of v and $k = \mathbf{k}(v)$ and $j = \mathbf{j}(v)$, then

- (*)_e if $\text{rk}^{\text{sp}}(v) \geq 0$, then $\mathbb{M} \models R_{n,j}[\alpha_0, \dots, \alpha_k, \dots, \alpha_{n-1}]$ but there is no $\alpha \in \lambda \setminus v$ such that

$$\text{rk}^{\text{sp}}(v \cup \{\alpha\}) \geq \text{rk}^{\text{sp}}(v) \quad \text{and} \quad \mathbb{M} \models R_{n,j}[\alpha_0, \dots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \dots, \alpha_{n-1}],$$

- (*)_f if $\text{rk}^{\text{sp}}(v) = -1$, then $\mathbb{M} \models R_{n,j}[\alpha_0, \dots, \alpha_k, \dots, \alpha_{n-1}]$ but the set

$$\{\alpha \in \lambda : \mathbb{M} \models R_{n,j}[\alpha_0, \dots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \dots, \alpha_{n-1}]\}$$

is countable,

- (*)_g for every $\beta_0, \dots, \beta_{n-1} < \lambda$, if $\mathbb{M} \models R_{n,j}[\beta_0, \dots, \beta_{n-1}]$ then $\beta_0 < \dots < \beta_{n-1}$.

The choices above define functions $\mathbf{j} : [\lambda]^{<\omega} \setminus \{\emptyset\} \rightarrow \omega$, $\mathbf{k} : [\lambda]^{<\omega} \setminus \{\emptyset\} \rightarrow \omega$, and $\text{rk}^{\text{sp}} : [\lambda]^{<\omega} \setminus \{\emptyset\} \rightarrow \{-1\} \cup (\varepsilon + 1)$.

3. QIFs and differences

Definition 3.1. Let $(\mathbb{H}, +, 0)$ be an Abelian group and $\mathbf{B} \subseteq \mathbb{H}$.

- (1) A $(2, n)$ -combination from \mathbf{B} is any sum of the form

$$\iota_0 b_0 + \iota_1 b_1 + \iota_2 b_2 + \dots + \iota_{n-1} b_{n-1}$$

where $b_0, b_1, \dots, b_{n-1} \in \mathbf{B}$ are pairwise distinct and $\iota_0, \iota_1, \iota_2, \dots, \iota_{n-1} \in \{-2, -1, 0, 1, 2\}$. The $(2, n)$ -combination is said to be *nontrivial* when not all $\iota_0, \dots, \iota_{n-1}$ are equal 0.

- (2) We say that the set \mathbf{B} is *quasi independent in \mathbb{H}* if $|\mathbf{B}| \geq 8$ and no nontrivial $(2, 8)$ -combination from \mathbf{B} equals to 0.
- (3) We say that a family \mathcal{V} of non-empty subsets of \mathbb{H} is an *n -good qif*¹ if $|\mathcal{V}| \geq n$, the sets in \mathcal{V} are pairwise disjoint and for distinct $V_0, \dots, V_{n-1} \in \mathcal{V}$, for each choice of $b_i, b'_i \in V_i$ (for $i < n$) and every $\iota_0, \iota'_0, \dots, \iota_{n-1}, \iota'_{n-1} \in \{-1, 0, 1\}$ such that $\sum_{i=0}^{n-1} (\iota_i + \iota'_i)^2 \neq 0$ we have

$$\iota_0 b_0 + \iota'_0 b'_0 + \iota_1 b_1 + \iota'_1 b'_1 + \dots + \iota_{n-1} b_{n-1} + \iota'_{n-1} b'_{n-1} \neq 0.$$

An expression as on the left hand side above will be called a *nontrivial $(2, \mathcal{V}, n)$ -combination* (or a *nontrivial $(2, n)$ -combination from \mathcal{V}*).

- (4) Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{P}(\mathbb{H}) \setminus \{\emptyset\}$. We will say that \mathcal{W} is *immersed in \mathcal{V}* if there is a bijection $\pi : \mathcal{W} \xrightarrow{1-1} \mathcal{V}$ such that
- $W \subseteq \pi(W)$ for all $W \in \mathcal{W}$, and
 - if $W_0, W_1 \in \mathcal{W}$, and $a, a' \in W_0$, $b \in W_1$, then $(a - a') + b \in \pi(W_1)$.

Observation 3.2.

- (1) If \mathbf{B} is quasi independent then all elements of \mathbf{B} have order at least 3 and $\{\{b\} : b \in \mathbf{B}\}$ is an 8-good qif.
- (2) If \mathcal{V} is an 8-good qif and $b_V \in V$ (for $V \in \mathcal{V}$) then $\{b_V : V \in \mathcal{V}\}$ is quasi independent.
- (3) Assume \mathbb{H} is an Abelian Polish group. Suppose also that, for $i < N < \omega$, $V_i \subseteq \mathbb{H}$ are disjoint open sets and $b_i \in V_i$. Then there are open sets W_i such that $b_i \in W_i \subseteq V_i$ for $i < N$, and $\{W_i : i < N\}$ is immersed in $\{V_i : i < N\}$.

Proposition 3.3. Assume that

- (i) $(\mathbb{H}, +, 0)$ is a perfect Abelian Polish group,
- (ii) the set of elements of \mathbb{H} of order larger than 2 is dense in \mathbb{H} ,
- (iii) U_0, \dots, U_{n-1} are nonempty open subsets of \mathbb{H} .

Then there are disjoint open sets $V_i \subseteq U_i$ (for $i < n$) such that $\{V_i : i < n\}$ is an n -good qif.

Proof. Let H_2 consist of all elements of \mathbb{H} of order ≤ 2 . Then H_2 is a closed subgroup of \mathbb{H} and, by the assumption (ii), it has empty interior. Consequently, for each $a \in \mathbb{H}$ and $i < n$ the set $(a + H_2) \cap U_i$ is meager. Therefore, for each $i < n$,

¹ short for quasi independent family.

$(\otimes)_i$ the set $\{a + H_2 : a \in \mathbb{H} \text{ and } (a + H_2) \cap U_i \neq \emptyset\}$ is infinite.

Let $m_0 = 10$ and $m_{i+1} = 10^{i+1} \cdot \prod_{j \leq i} m_j + 10$ (for $i < n$). For each $i < n$ choose a set $A_i \subseteq U_i \setminus H_2$ such that

$(\oplus)_0$ $|A_i| = m_i$ and

$(\oplus)_1$ if $a, b \in A_i$ and $a \neq b$, then $2a \neq 2b$.

(The choice is possible by $(\otimes)_i$ for each $i < n$.) For $0 < i < n$ let

$$X_i = \{\iota_0 a_0 + \dots + \iota_{i-1} a_{i-1} : a_0 \in A_0, \dots, a_{i-1} \in A_{i-1} \wedge \iota_0, \dots, \iota_{i-1} \in \{-2, -1, 0, 1, 2\}\}.$$

By the choice of m_j 's we know that $2 \cdot |X_i| < m_i = |A_i|$, so we may choose $b_i^* \in A_i$ such that $2b_i^*, b_i^* \notin X_i$. Let $b_0^* \in A_0$ be arbitrary. One easily verifies that every nontrivial $(2, n)$ -combination from $\{b_i^* : i < n\}$ is not zero, so for each $\iota_0, \iota'_0, \dots, \iota_{n-1}, \iota'_{n-1} \in \{-1, 0, 1\}$ such that $\sum_{i=0}^{n-1} (\iota_i + \iota'_i)^2 \neq 0$ we have

$$\iota_0 b_0^* + \iota'_0 b_0^* + \iota_1 b_1^* + \iota'_1 b_1^* + \dots + \iota_{n-1} b_{n-1}^* + \iota'_{n-1} b_{n-1}^* \neq 0.$$

For each such combination we may choose disjoint open sets $V_{\iota_0, \iota'_0, \dots, \iota_{n-1}, \iota'_{n-1}}^i$ such that $b_i^* \in V_{\iota_0, \iota'_0, \dots, \iota_{n-1}, \iota'_{n-1}}^i \subseteq U_i$ and for every $b_i, b'_i \in V_{\iota_0, \iota'_0, \dots, \iota_{n-1}, \iota'_{n-1}}^i$, $i < n$, we have

$$\iota_0 b_0 + \iota'_0 b'_0 + \iota_1 b_1 + \iota'_1 b'_1 + \dots + \iota_{n-1} b_{n-1} + \iota'_{n-1} b'_{n-1} \neq 0.$$

Now, for $i < n$ we set

$$V_i = \bigcap \{V_{\iota_0, \iota'_0, \dots, \iota_{n-1}, \iota'_{n-1}}^i : \iota_0, \iota'_0, \dots, \iota_{n-1}, \iota'_{n-1} \in \{-1, 0, 1\} \wedge (\iota_0 - \iota'_0)^2 + \dots + (\iota_{n-1} - \iota'_{n-1})^2 > 0\}.$$

It is clear that the sets V_i (for $i < n$) are as required. \square

Lemma 3.4. Suppose that $(\mathbb{H}, +, 0)$ is an Abelian group and ρ is a translation invariant metric on it. Let $\mathcal{W} \subseteq \mathcal{P}(\mathbb{H})$ be a finite 8-good qif. Assume that

- (a) \mathcal{W} is immersed in \mathcal{V} , $\mathcal{V} \subseteq \mathcal{P}(\mathbb{H})$,
- (b) $A' \subseteq A \subseteq \mathbb{H}$, $|A'| = 8$,
- (c) $A - A \subseteq \bigcup \{W - W' : W, W' \in \mathcal{W}\}$,
- (d) if $a, b \in A$, $a \neq b$, then $\rho(a, b) > \text{diam}_\rho(W)$ ($= \text{diam}_\rho(-W)$) for all $W \in \mathcal{W}$.

Then:

- (1) If $c \in \mathbb{H}$ is such that $A' + c \subseteq \bigcup \mathcal{W}$, then also $A + c \subseteq \bigcup \mathcal{V}$.
- (2) If $c \in \mathbb{H}$ is such that $c - A' \subseteq \bigcup \mathcal{W}$, then also $c - A \subseteq \bigcup \mathcal{V}$.

Proof. (1) Suppose that \mathcal{W}, \mathcal{V} , $A' \subseteq A \subseteq \mathbb{H}$ satisfy the assumptions of the Lemma and $c \in \mathbb{H}$ is such that $A' + c \subseteq \bigcup \mathcal{W}$.

Assume $a \in A \setminus A'$ and let us argue that $a + c \in \bigcup \mathcal{V}$.

Let $\langle a_i : i < 8 \rangle$ list the elements of A' . For $i < 8$ let $b_i = a_i + c \in W_i \in \mathcal{W}$ and note that all W_i 's are pairwise distinct (by assumption (d); remember ρ is translation invariant). It follows from assumption (c) that we may choose $b'_i \in W'_i \in \mathcal{W}$ and $b''_i \in W''_i \in \mathcal{W}$ such that $a - a_i = b'_i - b''_i$. Then, for each $i < 8$, we have

$$a + c = a + (b_i - a_i) = (b'_i - b''_i + a_i) + (b_i - a_i) = b'_i - b''_i + b_i.$$

Claim 3.4.1. *There are distinct $i^*, j^* < 8$ such that*

$$(\heartsuit)_{i^*, j^*} \quad W_{i^*} \notin \{W'_{j^*}, W''_{j^*}\} \text{ and } W_{j^*} \notin \{W'_{i^*}, W''_{i^*}\}.$$

Proof of the Claim. If for some $i_0 < 8$ we have $|\{j < 8 : W_{i_0} = W'_j \wedge j \neq i_0\}| \geq 3$, then choose $j_0 < j_1 < j_2 < 8$ distinct from i_0 and such that $W''_{j_0} = W''_{j_1} = W''_{j_2} = W_{i_0}$. Since all W_i 's are distinct, we may pick $i^* < 8$ such that $i^* \notin \{i_0, j_0, j_1, j_2\}$ and $W_{i^*} \notin \{W'_{j_0}, W'_{j_1}, W'_{j_2}\}$. Next let $j^* \in \{j_0, j_1, j_2\}$ be such that $W_{j^*} \notin \{W'_{i^*}, W''_{i^*}\}$. Then also $W_{i^*} \neq W_{i_0} = W_{j^*}$ and clearly $(\heartsuit)_{i^*, j^*}$ holds true.

If for some $i_0 < 8$ we have $|\{j < 8 : W_{i_0} = W'_j \wedge j \neq i_0\}| \geq 3$, then by the same argument (just interchanging W'_j 's and W''_j 's) we find i^*, j^* so that $(\heartsuit)_{i^*, j^*}$ holds true.

So now suppose that both $|\{j < 7 : W_7 = W'_j\}| \leq 2$ and $|\{j < 7 : W_7 = W''_j\}| \leq 2$. Then there are $j_0 < j_1 < j_2 < 7$ such that $W_7 \notin \{W'_{j_0}, W''_{j_0}, W'_{j_1}, W''_{j_1}, W'_{j_2}, W''_{j_2}\}$. Take $j^* \in \{j_0, j_1, j_2\}$ such that $W_{j^*} \notin \{W'_7, W''_7\}$ and note that then $(\heartsuit)_{7, j^*}$ holds true. \square

Let distinct $i^*, j^* < 8$ be such that $(\heartsuit)_{i^*, j^*}$ holds.

It follows from assumption (d) that $W_{i^*} \neq W''_{i^*}$ and $W'_{j^*} \neq W''_{j^*}$ (remember $a_{i^*} \neq a \neq a_{j^*}$). Now, if $W_{i^*} = W''_{i^*}$, then

$$a + c = b'_{i^*} + (b_{i^*} - b''_{i^*}) \in (W'_{i^*} + (W''_{i^*} - W'_{i^*})) \subseteq V'_{i^*}$$

where $W'_{i^*} \subseteq V'_{i^*} \in \mathcal{V}$ (so we are done). Similarly, if $W_{j^*} = W''_{j^*}$.

So suppose towards contradiction that both $W_{i^*} \neq W''_{i^*}$ and $W_{j^*} \neq W''_{j^*}$. Now,

$$b'_{i^*} - b''_{i^*} + b_{i^*} = a + c = b'_{j^*} - b''_{j^*} + b_{j^*},$$

so

$$(\otimes) \quad (b_{i^*} + b'_{i^*} + b''_{i^*}) - (b_{j^*} + b'_{j^*} + b''_{j^*}) = 0.$$

Considering known inequalities among $W_{i^*}, W'_{i^*}, W''_{i^*}, W_{j^*}, W'_{j^*}, W''_{j^*}$, we notice that no equality between them may involve more than two sets. Also $W_{i^*} \notin \{W_{j^*}, W'_{j^*}, W''_{j^*}\}$, so the expression on the left hand side of (\otimes) can be written as a nontrivial $(2, \mathcal{W}, 8)$ -combination, contradicting the assumption that \mathcal{W} is an 8-good qif.

(2) Follows from the first part applied to $-A$ and $-A'$. \square

Theorem 3.5. *Suppose that $(\mathbb{H}, +, 0)$ is an Abelian group and ρ is a translation invariant metric on it. Assume also that*

- (a) $\mathcal{W}, \mathcal{V}, \mathcal{Q} \subseteq \mathcal{P}(\mathbb{H})$ are finite 8-good qifs, and \mathcal{W} is immersed in \mathcal{V} and \mathcal{V} is immersed in \mathcal{Q} ,
- (b) $m \longrightarrow (10)_{2^{144}}^4$ (the Erdős–Rado arrow notation, see [7]),
- (c) $A \subseteq \mathbb{H}$, $|A| \geq m$ and
- (d) $A - A \subseteq \bigcup \{W - W' : W, W' \in \mathcal{W}\}$, and
- (e) if $a, b \in A$, $a \neq b$, then $\rho(a, b) > \text{diam}_\rho(Q)$ ($= \text{diam}_\rho(-Q)$) for all $Q \in \mathcal{Q}$.

Then exactly one of (A), (B) below holds true:

- (A) There is $a \in \mathbb{H}$ such that $A + a \subseteq \bigcup \mathcal{Q}$.
- (B) There is $a \in \mathbb{H}$ such that $a - A \subseteq \bigcup \mathcal{Q}$.

Proof. Let $\langle a_i : i < m \rangle$ be a sequence of pairwise distinct elements of A . Since $A - A \subseteq \bigcup \{W - W' : W, W' \in \mathcal{W}\}$, we may choose functions $\mathbf{b}_0, \mathbf{b}_1 : m \times m \rightarrow \bigcup \mathcal{W}$ and $\bar{W}_0, \bar{W}_1 : m \times m \rightarrow \mathcal{W}$ such that for all $i, j < m$

$$a_i - a_j = \mathbf{b}_0(i, j) - \mathbf{b}_1(i, j), \quad \mathbf{b}_0(i, j) \in \bar{W}_0(i, j), \quad \mathbf{b}_1(i, j) \in \bar{W}_1(i, j),$$

and $\mathbf{b}_0(i, j) = \mathbf{b}_1(j, i)$, and $\mathbf{b}_1(i, j) = \mathbf{b}_0(j, i)$. Let $\langle \varphi_\ell(i_0, i_1, i_2, i_3) : \ell < 144 \rangle$ list all formulas of the form

$$\bar{W}_j(i_x, i_y) = \bar{W}_{j'}(i_{x'}, i_{y'})$$

for $j, j' < 2$ and $x, y, x', y' < 4$, $x < y$, $x' < y'$.

Let $\mu : [m]^4 \rightarrow {}^{144}2$ be a coloring of quadruples from m such that if $i_0 < i_1 < i_2 < i_3 < m$, then

$$\mu(\{i_0, i_1, i_2, i_3\})(\ell) = 1 \quad \text{if and only if} \quad \varphi_\ell(i_0, i_1, i_2, i_3) \text{ holds true.}$$

Since $m \rightarrow (10)_{2^{144}}^4$, we may choose $u \in [m]^{10}$ homogeneous for μ . Without loss of generality, $u = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Claim 3.5.1. *Let $i, j, k < 10$ be pairwise distinct. Then*

- (1) $\bar{W}_0(i, j) \neq \bar{W}_1(i, j)$ and
- (2) $\mathbf{b}_0(i, k) - \mathbf{b}_1(i, k) = \mathbf{b}_0(i, j) - \mathbf{b}_1(i, j) + \mathbf{b}_0(j, k) - \mathbf{b}_1(j, k)$ and hence

$$\left(\bar{W}_0(i, k) - \bar{W}_1(i, k) \right) \cap \left(\left(\bar{W}_0(i, j) - \bar{W}_1(i, j) \right) + \left(\bar{W}_0(j, k) - \bar{W}_1(j, k) \right) \right) \neq \emptyset.$$

Proof of the Claim. (1) Follows from assumption (e) of the Theorem (remember every set from \mathcal{W} is a subset of a member of \mathcal{Q}).

(2) This follows by the equality $(a_i - a_j) + (a_j - a_k) = a_i - a_k$ and the choice of $\mathbf{b}_0(i, j), \bar{W}_0(i, j), \mathbf{b}_1(i, j), \bar{W}_1(i, j)$. \square

Claim 3.5.2. *If $\{\bar{W}_0(i, j) : i < j < 10\} \cap \{\bar{W}_1(i, j) : i < j < 10\} \neq \emptyset$, then either $\bar{W}_0(0, 1) = \bar{W}_1(1, 2)$, or $\bar{W}_1(0, 1) = \bar{W}_0(1, 2)$.*

Proof of the Claim. Suppose $i_0 < j_0 < 10$ and $i_1 < j_1 < 10$ are such that $\bar{W}_0(i_0, j_0) = \bar{W}_1(i_1, j_1)$. We shall consider all possible orders of i_0, j_0, i_1, j_1 and use the homogeneity to conclude one of the clauses in the assertion.

- (a) If $i_0 < j_0 < i_1 < j_1$, then (by the homogeneity) $\bar{W}_0(0, 1) = \bar{W}_1(2, 3) = \bar{W}_1(4, 5) = \bar{W}_0(2, 3)$, so $\bar{W}_0(2, 3) = \bar{W}_1(2, 3)$, contradicting Claim 3.5.1(1).
- (b) If $i_0 < j_0 = i_1 < j_1$ then also $\bar{W}_0(0, 1) = \bar{W}_1(1, 2)$ (giving the conclusion of Claim 3.5.2).
- (c) If $i_0 < i_1 < j_0 < j_1$, then $\bar{W}_0(1, 4) = \bar{W}_1(2, 5) = \bar{W}_0(0, 3) = \bar{W}_1(1, 4)$, contradicting Claim 3.5.1(1).
- (d) If $i_0 < i_1 < j_0 = j_1$, then $\bar{W}_0(0, 3) = \bar{W}_1(1, 3) = \bar{W}_1(2, 3) = \bar{W}_0(1, 3)$, contradicting Claim 3.5.1(1).
- (e) If $i_0 < i_1 < j_1 < j_0$, then $\bar{W}_0(1, 4) = \bar{W}_1(2, 3) = \bar{W}_0(0, 5) = \bar{W}_1(1, 4)$, contradicting Claim 3.5.1(1).
- (f) If $i_0 = i_1 < j_0 < j_1$, then $\bar{W}_0(0, 1) = \bar{W}_1(0, 2) = \bar{W}_1(0, 3) = \bar{W}_0(0, 2)$, contradicting Claim 3.5.1(1).
- (g) The configuration $i_0 = i_1 < j_0 = j_1$ contradicts Claim 3.5.1(1).
- (h) If $i_0 = i_1 < j_1 < j_0$, then $\bar{W}_0(0, 2) = \bar{W}_1(0, 1) = \bar{W}_0(0, 3) = \bar{W}_1(0, 2)$, contradicting Claim 3.5.1(1).

- (i) The configuration $i_1 < i_0 < j_0 < j_1$ is not possible similarly to (e) (just interchange \bar{W}_0 and \bar{W}_1).
- (j) The configuration $i_1 < i_0 < j_0 = j_1$ is not possible similarly to (d).
- (k) The configuration $i_1 < i_0 < j_1 < j_0$ is not possible similarly to (c).
- (l) If $i_1 < i_0 = j_1 < j_0$, then $\bar{W}_1(0, 1) = \bar{W}_0(1, 2)$ (giving the conclusion of Claim 3.5.2).
- (m) The configuration $i_1 < j_1 < i_0 < j_0$, is not possible similarly to (a). \square

Now, we will consider three cases, showing that the first one is not possible. In the second case we will find $c \in \mathbb{H}$ such that $\{c - a_i : i < 8\} \subseteq \bigcup \mathcal{V}$. Then by Lemma 3.4 we will also have $c - A \subseteq \bigcup \mathcal{Q}$. Finally in the last case we will find $c \in \mathbb{H}$ such that $\{a_i + c : i < 8\} \subseteq \bigcup \mathcal{V}$, so by Lemma 3.4 we will also have $A + c \subseteq \bigcup \mathcal{Q}$.

For $\ell < 2$ and $i < j < 10$ let $\bar{V}_\ell(i, j) \in \mathcal{V}$ be the unique set such that $\bar{W}_\ell(i, j) \subseteq \bar{V}_\ell(i, j)$. Also, let $\mathcal{V}_\ell = \{\bar{V}_\ell(i, j) : i < j < 10\}$.

CASE 1: $\{\bar{W}_0(i, j) : i < j < 10\} \cap \{\bar{W}_1(i, j) : i < j < 10\} = \emptyset$.

By Claim 3.5.1(2) we have $\mathbf{b}_0(0, 2) - \mathbf{b}_1(0, 2) = \mathbf{b}_0(0, 1) - \mathbf{b}_1(0, 1) + \mathbf{b}_0(1, 2) - \mathbf{b}_1(1, 2)$ or

$$(\mathbf{b}_0(0, 1) - \mathbf{b}_0(0, 2) + \mathbf{b}_0(1, 2)) + (\mathbf{b}_1(0, 2) - \mathbf{b}_1(0, 1) - \mathbf{b}_1(1, 2)) = 0.$$

If $|\{\bar{W}_0(0, 1), \bar{W}_0(0, 2), \bar{W}_0(1, 2)\}| \leq 2$, then

either $\bar{W}_0(0, 1) = \bar{W}_0(1, 2)$ and by the homogeneity $\bar{W}_0(0, 1) = \bar{W}_0(i, j)$ for all $i < j < 9$, so $\mathbf{b}_0(0, 1) + \mathbf{b}_0(1, 2) - \mathbf{b}_0(0, 2) \in \bar{W}_0(0, 1) + (\bar{W}_0(0, 1) - \bar{W}_0(0, 1)) \subseteq \bar{V}_0(0, 1)$,
 or $\bar{W}_0(0, 2) = \bar{W}_0(0, 1)$ and then $\mathbf{b}_0(0, 1) - \mathbf{b}_0(0, 2) + \mathbf{b}_0(1, 2) \in (\bar{W}_0(0, 1) - \bar{W}_0(0, 1)) + \bar{W}_0(1, 2) \subseteq \bar{V}_0(1, 2)$,
 or $\bar{W}_0(0, 2) = \bar{W}_0(1, 2)$ and then $\mathbf{b}_0(1, 2) - \mathbf{b}_0(0, 2) + \mathbf{b}_0(0, 1) \in (\bar{W}_0(0, 2) - \bar{W}_0(0, 2)) + \bar{W}_0(0, 1) \subseteq \bar{V}_0(0, 1)$.

Therefore, if $|\{\bar{W}_0(0, 1), \bar{W}_0(0, 2), \bar{W}_0(1, 2)\}| \leq 2$ then $\mathbf{b}_0(0, 1) - \mathbf{b}_0(0, 2) + \mathbf{b}_0(1, 2) \in \bigcup \mathcal{V}_0$. If elements of $\{\bar{W}_0(0, 1), \bar{W}_0(0, 2), \bar{W}_0(1, 2)\}$ are all distinct, then they are respectively included in disjoint sets $\bar{V}_0(0, 1), \bar{V}_0(0, 2), \bar{V}_0(1, 2)$ (remember \mathcal{V} is an 8-good qif). Hence we may conclude that in any case $\mathbf{b}_0(0, 1) - \mathbf{b}_0(0, 2) + \mathbf{b}_0(1, 2)$ equals to a nontrivial $(2, \mathcal{V}_0, 3)$ -combination.

Similarly, if $|\{\bar{W}_1(0, 1), \bar{W}_1(0, 2), \bar{W}_1(1, 2)\}| \leq 2$, then

either $\bar{W}_1(0, 1) = \bar{W}_1(1, 2)$ and then $-(\mathbf{b}_1(0, 1) - \mathbf{b}_1(0, 2)) + \mathbf{b}_1(1, 2) \in -\bar{V}_1(1, 2)$,
 or $\bar{W}_1(0, 1) = \bar{W}_1(0, 2)$ and then $-(\mathbf{b}_1(0, 1) - \mathbf{b}_1(0, 2)) + \mathbf{b}_1(1, 2) \in -\bar{V}_1(1, 2)$,
 or $\bar{W}_1(0, 2) = \bar{W}_1(1, 2)$ and then $-(\mathbf{b}_1(1, 2) - \mathbf{b}_1(0, 2)) + \mathbf{b}_1(0, 1) \in -\bar{V}_1(0, 1)$.

Therefore easily in any case $\mathbf{b}_1(0, 2) - \mathbf{b}_1(0, 1) - \mathbf{b}_1(1, 2)$ equals to a nontrivial $(2, \mathcal{V}_1, 3)$ -combination.

Now, in the current case we have $\mathcal{V}_0 \cap \mathcal{V}_1 = \emptyset$, so we may conclude that $0 = (\mathbf{b}_0(0, 1) - \mathbf{b}_0(0, 2) + \mathbf{b}_0(1, 2)) + (\mathbf{b}_1(0, 2) - \mathbf{b}_1(0, 1) - \mathbf{b}_1(1, 2))$ is equal to a nontrivial $(2, \mathcal{V}, 8)$ -combination, contradicting the assumption that \mathcal{V} is an 8-good qif.

Thus Case 1 cannot happen and by Claim 3.5.2 either $\bar{W}_0(0, 1) = \bar{W}_1(1, 2)$, or $\bar{W}_1(0, 1) = \bar{W}_0(1, 2)$.

CASE 2: $\bar{W}_0(0, 1) = \bar{W}_1(1, 2)$.

By the homogeneity, $\bar{W}_0(j, 8) = \bar{W}_1(8, 9)$ for each $j < 8$. By Claim 3.5.1(2), for every $j < 8$, $a_j - a_9 = \mathbf{b}_0(j, 8) - \mathbf{b}_1(j, 8) + \mathbf{b}_0(8, 9) - \mathbf{b}_1(8, 9)$, so

$$(a_9 + \mathbf{b}_0(8, 9)) - a_j = (\mathbf{b}_1(8, 9) - \mathbf{b}_0(j, 8)) + \mathbf{b}_1(j, 8) \in (\bar{W}_0(j, 8) - \bar{W}_0(j, 8)) + \bar{W}_1(j, 8)$$

Since \mathcal{W} is immersed in \mathcal{V} , the set on the far right above is included in $\bar{V}_1(j, 8)$. Hence for $c = a_9 + \mathbf{b}_0(8, 9)$ and $A' = \{a_j : j < 8\}$ we have $c - A' \subseteq \bigcup \mathcal{V}$. Using Lemma 3.4(2) we may conclude that $c - A \subseteq \bigcup \mathcal{Q}$.

CASE 3: $\bar{W}_1(0, 1) = \bar{W}_0(1, 2)$.

By the homogeneity, $\bar{W}_1(j, 8) = \bar{W}_0(8, 9)$ for each $j < 8$. As before we use Claim 3.5.1(2) to get

$$(\mathbf{b}_1(8, 9) - a_9) + a_j = (\mathbf{b}_0(8, 9) - \mathbf{b}_1(j, 8)) + \mathbf{b}_0(j, 8) \in (\bar{W}_0(8, 9) - \bar{W}_0(8, 9)) + \bar{W}_0(j, 8)$$

Since \mathcal{W} is immersed in \mathcal{V} , the set on the far right above is included in $\bar{V}_0(j, 8)$. Thus for $c = \mathbf{b}_1(8, 9) - a_9$ and $A' = \{a_j : j < 8\}$ we have $A' + c \subseteq \bigcup \mathcal{V}$. By Lemma 3.4(1) we get $A + c \subseteq \bigcup \mathcal{Q}$.

Finally, to show that only one of (A) and (B) may take place, suppose $A + c \subseteq \bigcup \mathcal{Q}$ and $d - A \subseteq \bigcup \mathcal{Q}$ for some $c, d \in \mathbb{H}$. For $a \in A$ let $Q_a, Y_a \in \mathcal{Q}$ be such that $a + c \in Q_a$ and $d - a \in Y_a$.

Fix any $a \in A$ and choose $b \in A \setminus (\{a\} \cup (Y_a - c) \cup (d - Q_a))$ (it is possible as by the Assumption 3.5(e), $|A \cap (Y_a - c)| < 2$ and $|A \cap (d - Q_a)| < 2$). Now,

$$(a + c) + (d - a) = c + d = (b + c) + (d - b),$$

so $0 \in Q_a + Y_a - Q_b - Y_b$. By the choice of b we have $Q_b \neq Y_a$, $Q_a \neq Y_b$ and also (by 3.5(e)) $Q_a \neq Q_b$ and $Y_a \neq Y_b$. Therefore some nontrivial $(2, \mathcal{Q}, 4)$ -combination is equal to 0, contradicting \mathcal{Q} is a good qif. \square

4. Quasi independence and sums

In a special case when $\mathcal{Q}, \mathcal{V}, \mathcal{W}$ are all families consisting of singletons (and ρ is the discrete metric on \mathbb{H}), Theorem 3.5 gives the following result of its own interest.

Corollary 4.1. *Suppose that $(\mathbb{H}, +, 0)$ is an Abelian group and $\mathbf{B} \subseteq \mathbb{H}$ is quasi independent. Assume also that*

- (a) $m \rightarrow (10)_{2^{144}}^4$,
- (b) $A \subseteq \mathbb{H}$, $|A| \geq m$ and $A - A \subseteq \mathbf{B} - \mathbf{B}$.

Then exactly one of (A), (B) below holds true:

- (A) *There is a unique $c \in \mathbb{H}$ such that $A + c \subseteq \mathbf{B}$.*
- (B) *There is a unique $c \in \mathbb{H}$ such that $c - A \subseteq \mathbf{B}$.*

The above Corollary inspired our interest in its dual version when $A - A$ and $\mathbf{B} - \mathbf{B}$ are replaced by $A + A$ and $\mathbf{B} + \mathbf{B}$. This dual result (given in Theorem 4.4 below) is not used in the proof of our independence theorem, but we find it interesting.

Lemma 4.2. *Suppose that $(\mathbb{H}, +, 0)$ is an Abelian group and $\mathbf{B} \subseteq \mathbb{H}$ is quasi independent. Assume that $A' \subseteq A \subseteq \mathbb{H}$ and $c \in \mathbb{H}$ are such that*

- (a) $A + A \subseteq \mathbf{B} + \mathbf{B}$,
- (b) $A' + c \subseteq \mathbf{B}$ and $|A'| = 4$.

Then $A - c \subseteq \mathbf{B}$.

Proof. Suppose that $A' \subseteq A \subseteq \mathbb{H}$ satisfy the assumptions (a) and (b). Assume $a \in A$ and let us argue that $a - c \in \mathbf{B}$.

Let $\langle a_i : i < 4 \rangle$ list the elements of A' . For $i < 4$ let $b_i = a_i + c \in \mathbf{B}$ and note that all b_i 's are pairwise distinct. Since $a_i + a \in \mathbf{B} + \mathbf{B}$ we may also choose $b'_i, b''_i \in \mathbf{B}$ such that $a_i + a = b'_i + b''_i$. Then, for each $i < 4$, we have

$$a - c = a - (b_i - a_i) = b'_i + b''_i - a_i - (b_i - a_i) = b'_i + b''_i - b_i.$$

Thus for $i < j < 4$ we have

$$(*)_1 \quad 0 = (b'_i + b''_i + b_j) - (b'_j + b''_j + b_i).$$

If for some $i < j < 4$ both sets $\{b'_i, b''_i, b_j\}$ and $\{b'_j, b''_j, b_i\}$ had at least 2 elements, then the right hand side of $(*)_1$ would give a $(2, 8)$ -combination from \mathbf{B} with the value 0, so the combination would have to be a trivial one. Therefore

$$\begin{aligned} (*)_2 \quad & \text{for each } i < j < 4, \\ & \text{either (i) } b'_i = b''_i = b_j, \\ & \quad \text{or (ii) } b'_j = b''_j = b_i, \\ & \quad \text{or (iii) } \{b'_i, b''_i, b_j\} = \{b'_j, b''_j, b_i\}. \end{aligned}$$

Suppose that $i < j < 4$ are such that $(*)_2(\text{iii})$ holds true. Since $b_i \neq b_j$, we get $b_i \in \{b'_i, b''_i\}$ and hence $a - c = b'_i + b''_i - b_i \in \{b'_i, b''_i\} \subseteq \mathbf{B}$, and we are done.

Assume towards contradiction that

$$(*)_3 \quad \text{for each } i < j < 4, \text{ either } (*)_2(\text{i}) \text{ or } (*)_2(\text{ii}) \text{ holds true.}$$

Then for some $i_0 < 4$, $b'_j = b''_j$ whenever $j \neq i_0$. Necessarily,

$$(j_0 \neq j_1 \wedge i_0 \notin \{j_0, j_1\}) \Rightarrow b'_{j_0} \neq b'_{j_1}$$

(as $a + a_{j_0} \neq a + a_{j_1}$). Since there are no repetitions among b_j 's, we may now choose $j \neq i_0$ such that $b_j \neq b'_{i_0}$, $b'_j \neq b_{i_0}$ getting immediate contradiction with our assumption $(*)_3$. \square

Lemma 4.3. *Suppose that $(\mathbb{H}, +, 0)$ is an Abelian group and $\mathbf{B} \subseteq \mathbb{H}$ is quasi independent. Assume that $A' \subseteq A \subseteq \mathbb{H}$ are such that*

- (a) $A + A \subseteq \mathbf{B} + \mathbf{B}$,
- (b) $|A'| \geq 4$, and $A' + c \subseteq \mathbf{B}$ for some $c \in \mathbb{H}$.

Then $A + c \subseteq \mathbf{B}$ and the order of c is ≤ 2 .

Proof. Let $A' + c \subseteq \mathbf{B}$. It follows from Lemma 4.2 that $A - c \subseteq \mathbf{B}$. Applying that lemma again for A', A, \mathbf{B} and $-c$ we get $A + c \subseteq \mathbf{B}$.

Concerning the second part of the assertion, suppose towards contradiction that $c + c \neq 0$. Let a_0, a_1, a_2, a_3 be distinct elements of A . Then for distinct $i, j < 4$ we have

$$a_i + c \neq a_i - c, \quad a_i + c \neq a_j + c, \quad \text{and} \quad a_i - c \neq a_j - c,$$

and consequently we may find $i < 4$ such that $\{a_0 + c, a_0 - c\} \cap \{a_i + c, a_i - c\} = \emptyset$. Then, by the first paragraph of this proof, $a_0 + c, a_0 - c, a_i + c, a_i - c \in \mathbf{B}$ are all distinct and $(a_0 + c) - (a_0 - c) - (a_i + c) + (a_i - c) = 0$, contradicting the quasi independence of \mathbf{B} . \square

Theorem 4.4. *Suppose that $(\mathbb{H}, +, 0)$ is an Abelian group and $\mathbf{B} \subseteq \mathbb{H}$ is quasi independent. Assume also that*

- (a) $m \longrightarrow (6)_{2^{144}}^4$,
- (b) $A \subseteq \mathbb{H}$, $|A| \geq m$ and $A + A \subseteq \mathbf{B} + \mathbf{B}$.

Then there is a unique $c \in \mathbb{H}$ of order ≤ 2 such that $A + c \subseteq \mathbf{B}$.

Proof. Let $\langle a_i : i < m \rangle$ be a sequence of pairwise distinct elements of A . Since $A + A \subseteq \mathbf{B} + \mathbf{B}$, we may choose symmetric functions $\mathbf{b}_0, \mathbf{b}_1 : m \times m \longrightarrow \mathbf{B}$ such that

$$a_i + a_j = \mathbf{b}_0(i, j) + \mathbf{b}_1(i, j) \quad \text{for all } i, j < m.$$

Let $\langle \varphi_\ell(i_0, i_1, i_2, i_3) : \ell < 144 \rangle$ list all formulas of the form

$$\mathbf{b}_j(i_x, i_y) = \mathbf{b}_{j'}(i_{x'}, i_{y'})$$

for $j, j' < 2$ and $x < y < 4$, $x' < y' < 4$.

Let $\mu : [m]^4 \longrightarrow {}^{144}2$ be a coloring of quadruples from m such that if $i_0 < i_1 < i_2 < i_3 < m$, then

$$\mu(\{i_0, i_1, i_2, i_3\})(\ell) = 1 \quad \text{if and only if} \quad \varphi_\ell(i_0, i_1, i_2, i_3) \text{ holds true.}$$

Since $m \longrightarrow (6)_{2^{144}}^4$, we may choose $u \in [m]^6$ homogeneous for μ . Without loss of generality, $u = \{0, 1, 2, 3, 4, 5\}$.

Claim 4.4.1. *If $\{\mathbf{b}_0(i, j) : i < j < 6\} \cap \{\mathbf{b}_1(i, j) : i < j < 6\} \neq \emptyset$, then either $\mathbf{b}_0(0, 1) = \mathbf{b}_1(1, 2)$, or $\mathbf{b}_1(0, 1) = \mathbf{b}_0(1, 2)$, or $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$.*

Proof of the Claim. Suppose $i_0 < j_0 < 6$ and $i_1 < j_1 < 6$ are such that $\mathbf{b}_0(i_0, j_0) = \mathbf{b}_1(i_1, j_1)$. We shall consider all possible orders of i_0, j_0, i_1, j_1 and use the homogeneity to conclude one of the clauses in the assertion.

(a) If $i_0 < j_0 < i_1 < j_1$, then (by the homogeneity) $\mathbf{b}_0(0, 1) = \mathbf{b}_1(2, 3) = \mathbf{b}_1(4, 5) = \mathbf{b}_0(2, 3)$, so also $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$.

(b) If $i_0 < j_0 = i_1 < j_1$ then also $\mathbf{b}_0(0, 1) = \mathbf{b}_1(1, 2)$.

(c) If $i_0 < i_1 < j_0 < j_1$, then $\mathbf{b}_0(0, 3) = \mathbf{b}_1(2, 4) = \mathbf{b}_1(1, 4) = \mathbf{b}_0(0, 2)$ and also $\mathbf{b}_0(0, 1) = \mathbf{b}_1(1, 2)$.

(d) If $i_0 < i_1 < j_0 = j_1$, then $\mathbf{b}_0(0, 3) = \mathbf{b}_1(1, 3) = \mathbf{b}_1(2, 3) = \mathbf{b}_0(1, 3)$ and also $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$.

(e) If $i_0 < i_1 < j_1 < j_0$, then $\mathbf{b}_0(0, 5) = \mathbf{b}_1(3, 4) = \mathbf{b}_1(1, 2) = \mathbf{b}_0(0, 3)$ and also $\mathbf{b}_0(0, 1) = \mathbf{b}_1(1, 2)$.

(f) If $i_0 = i_1 < j_0 < j_1$, then $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 2) = \mathbf{b}_1(0, 3) = \mathbf{b}_0(0, 2)$, so also $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$.

(g) If $i_0 = i_1 < j_0 = j_1$ then $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$.

(h) If $i_0 = i_1 < j_1 < j_0$, then $\mathbf{b}_0(0, 2) = \mathbf{b}_1(0, 1) = \mathbf{b}_0(0, 3) = \mathbf{b}_1(0, 2)$, so also $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$.

(i) If $i_1 < i_0 < j_0 < j_1$, then $\mathbf{b}_1(0, 1) = \mathbf{b}_0(1, 2)$ similarly to (e), just interchange \mathbf{b}_0 and \mathbf{b}_1 .

(j) If $i_1 < i_0 < j_0 = j_1$, then $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$ similarly to (d).

- (k) If $i_1 < i_0 < j_1 < j_0$, then $\mathbf{b}_1(0, 1) = \mathbf{b}_0(1, 2)$ similarly to (c).
 (l) If $i_1 < i_0 = j_1 < j_0$, then $\mathbf{b}_1(0, 1) = \mathbf{b}_0(1, 2)$.
 (m) If $i_1 < j_1 < i_0 < j_0$, then $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$ similarly to (a). \square

Claim 4.4.2. If $\mathbf{b}_0(0, 3) = \mathbf{b}_0(1, 2)$, then $\mathbf{b}_0(0, 1) = \mathbf{b}_0(1, 2) = \mathbf{b}_0(2, 3) = \mathbf{b}_0(0, 3)$.

Similarly if \mathbf{b}_0 is replaced by \mathbf{b}_1 .

Proof of the Claim. Straightforward by the homogeneity of u . \square

Claim 4.4.3.

$$\mathbf{b}_0(0, 1) + \mathbf{b}_1(0, 1) - \mathbf{b}_0(1, 2) - \mathbf{b}_1(1, 2) + \mathbf{b}_0(2, 3) + \mathbf{b}_1(2, 3) = \mathbf{b}_0(0, 3) + \mathbf{b}_1(0, 3).$$

Proof of the Claim. Follows by the choice of $\mathbf{b}_0(i, j)$, $\mathbf{b}_1(i, j)$ and

$$(a_0 + a_1) - (a_1 + a_2) + (a_2 + a_3) = a_0 + a_3. \quad \square$$

Now, we will consider six cases, showing that the first four of them are not possible. In the remaining two cases we will find $c \in \mathbb{H}$ such that $\{a_i + c : i < 4\} \subseteq \mathbf{B}$. Then by Lemma 4.3 we will also have $A + c \subseteq \mathbf{B}$.

CASE 1: $\{\mathbf{b}_0(i, j) : i < j < 6\} \cap \{\mathbf{b}_1(i, j) : i < j < 6\} = \emptyset$ and $\mathbf{b}_1(0, 3) \notin \{\mathbf{b}_1(0, 1), \mathbf{b}_1(1, 2), \mathbf{b}_1(2, 3)\}$.

Then $\mathbf{b}_1(0, 3) \notin \{\mathbf{b}_0(0, 1), \mathbf{b}_1(0, 1), \mathbf{b}_0(1, 2), \mathbf{b}_1(1, 2), \mathbf{b}_0(2, 3), \mathbf{b}_1(2, 3), \mathbf{b}_0(0, 3)\}$ and by Claim 4.4.3

$$\mathbf{b}_1(0, 3) = \mathbf{b}_0(0, 1) + \mathbf{b}_1(0, 1) - \mathbf{b}_0(1, 2) - \mathbf{b}_1(1, 2) + \mathbf{b}_0(2, 3) + \mathbf{b}_1(2, 3) - \mathbf{b}_0(0, 3),$$

contradicting quasi independence of \mathbf{B} .

CASE 2: $\{\mathbf{b}_0(i, j) : i < j < 6\} \cap \{\mathbf{b}_1(i, j) : i < j < 6\} = \emptyset$ and $\mathbf{b}_0(0, 3) \notin \{\mathbf{b}_0(0, 1), \mathbf{b}_0(1, 2), \mathbf{b}_0(2, 3)\}$.

By an argument similar to Case 1, one shows that this case is not possible as well.

CASE 3: $\{\mathbf{b}_0(i, j) : i < j < 6\} \cap \{\mathbf{b}_1(i, j) : i < j < 6\} = \emptyset$ and $\mathbf{b}_0(0, 3) \in \{\mathbf{b}_0(0, 1), \mathbf{b}_0(1, 2), \mathbf{b}_0(2, 3)\}$ and $\mathbf{b}_1(0, 3) \in \{\mathbf{b}_1(0, 1), \mathbf{b}_1(1, 2), \mathbf{b}_1(2, 3)\}$.

SUBCASE 3A: $\mathbf{b}_0(0, 3) = \mathbf{b}_0(1, 2)$.

Then by Claim 4.4.2, $\mathbf{b}_0(0, 3) = \mathbf{b}_0(0, 1) = \mathbf{b}_0(1, 2) = \mathbf{b}_0(2, 3)$.

If $\mathbf{b}_1(0, 3) = \mathbf{b}_1(0, 1)$, then $a_0 + a_3 = a_0 + a_1$ and $a_3 = a_1$, a contradiction.

If $\mathbf{b}_1(0, 3) = \mathbf{b}_1(2, 3)$, then $a_0 + a_3 = a_2 + a_3$ and $a_0 = a_2$, a contradiction.

If $\mathbf{b}_1(0, 3) = \mathbf{b}_1(1, 2)$, then Claim 4.4.2 implies $\mathbf{b}_1(0, 3) = \mathbf{b}_1(0, 1)$ and we already know that this leads to a contradiction.

Consequently Subcase 3A is not possible.

SUBCASE 3B: $\mathbf{b}_1(0, 3) = \mathbf{b}_1(1, 2)$.

Similarly as in Subcase 3A one argues that this is not possible.

SUBCASE 3C: $\mathbf{b}_0(0, 3) = \mathbf{b}_0(0, 1)$ and $\mathbf{b}_1(0, 3) = \mathbf{b}_1(0, 1)$.

Then $a_0 + a_1 = a_0 + a_3$ and $a_1 = a_3$ giving a contradiction.

SUBCASE 3D: $\mathbf{b}_0(0, 3) = \mathbf{b}_0(2, 3)$ and $\mathbf{b}_1(0, 3) = \mathbf{b}_1(2, 3)$.

Like Subcase 3C, this is not possible.

SUBCASE 3E: $\mathbf{b}_0(0, 3) = \mathbf{b}_0(0, 1)$ and $\mathbf{b}_1(0, 3) = \mathbf{b}_1(2, 3)$.

If we had $\mathbf{b}_1(0, 1) = \mathbf{b}_1(1, 2)$, then also (by the homogeneity) $\mathbf{b}_1(1, 2) = \mathbf{b}_1(2, 3)$ so $\mathbf{b}_1(0, 1) = \mathbf{b}_1(0, 3)$ and we get a contradiction like in Subcase 3C.

If we had $\mathbf{b}_0(1, 2) = \mathbf{b}_0(2, 3)$ then also $\mathbf{b}_0(2, 3) = \mathbf{b}_0(0, 1)$ so $\mathbf{b}_0(2, 3) = \mathbf{b}_0(0, 3)$ and we get a contradiction like in Subcase 3D.

Consequently, there must be no repetitions in $\{\mathbf{b}_0(1, 2), \mathbf{b}_0(2, 3), \mathbf{b}_1(0, 1), \mathbf{b}_1(1, 2)\}$. By Claim 4.4.3 and the assumption of the current subcase we have

$$\mathbf{b}_1(0, 1) + \mathbf{b}_0(2, 3) - \mathbf{b}_0(1, 2) - \mathbf{b}_1(1, 2) = 0,$$

a contradiction with the quasi independence of \mathbf{B} .

SUBCASE 3F: $\mathbf{b}_0(0, 3) = \mathbf{b}_0(2, 3)$ and $\mathbf{b}_1(0, 3) = \mathbf{b}_1(0, 1)$.

Like Subcase 3E, this is not possible.

The next three cases cover the possibility when $\{\mathbf{b}_0(i, j) : i < j < 6\} \cap \{\mathbf{b}_1(i, j) : i < j < 6\} \neq \emptyset$. By Claim 4.4.1, this implies that either $\mathbf{b}_0(0, 1) = \mathbf{b}_1(1, 2)$, or $\mathbf{b}_1(0, 1) = \mathbf{b}_0(1, 2)$, or $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$.

CASE 4: $\mathbf{b}_0(0, 1) = \mathbf{b}_1(0, 1)$

Then for all $i < j < 6$ we have $\mathbf{b}_0(i, j) = \mathbf{b}_1(i, j)$.

If for some $i_0 < j_0 \leq i_1 < j_1$ we had $\mathbf{b}_0(i_0, j_0) = \mathbf{b}_0(i_1, j_1)$, then by the homogeneity we would have had $\mathbf{b}_0(0, 1) = \mathbf{b}_0(i, j) = \mathbf{b}_1(i, j)$ for all $i < j < 5$ and

$$4b_0(0, 1) = (a_0 + a_1) + (a_1 + a_2) = 2a_1 + 2b_0(0, 1).$$

Hence $2b_0(0, 1) + (a_0 + a_1) = 4b_0(0, 1) = 2a_1 + 2b_0(0, 1)$ and $a_0 = a_1$, a contradiction.

Therefore, $\mathbf{b}_0(i_0, j_0) \neq \mathbf{b}_0(i_1, j_1)$ whenever $i_0 < j_0 \leq i_1 < j_1 \leq 3$. Now, by Claim 4.4.3,

$$\begin{aligned} 2\mathbf{b}_0(0, 3) &= \mathbf{b}_0(0, 3) + \mathbf{b}_1(0, 3) = \\ &= \mathbf{b}_0(0, 1) + \mathbf{b}_1(0, 1) - \mathbf{b}_0(1, 2) - \mathbf{b}_1(1, 2) + \mathbf{b}_0(2, 3) + \mathbf{b}_1(2, 3) = \\ &= 2\mathbf{b}_0(0, 1) - 2\mathbf{b}_0(1, 2) + 2\mathbf{b}_0(2, 3). \end{aligned}$$

If we had $\mathbf{b}_0(0, 3) = \mathbf{b}_0(1, 2)$, then by the homogeneity $\mathbf{b}_0(1, 2) = \mathbf{b}_0(0, 5) = \mathbf{b}_0(2, 3)$, contradicting what we said above. Therefore, $\mathbf{b}_0(0, 3) \neq \mathbf{b}_0(1, 2)$ and $\mathbf{b}_0(0, 1), \mathbf{b}_0(1, 2), \mathbf{b}_0(2, 3)$ are pairwise distinct. Hence

$$2\mathbf{b}_0(0, 1) - 2\mathbf{b}_0(1, 2) + 2\mathbf{b}_0(2, 3) - 2\mathbf{b}_0(0, 3)$$

is a nontrivial $(2, 8)$ -combination with value 0, a contradiction with the quasi independence of \mathbf{B} .

Consequently, Case 4 is also impossible.

CASE 5: $\mathbf{b}_0(0, 1) = \mathbf{b}_1(1, 2)$.

By the homogeneity, for each $j < 4$ we have then $\mathbf{b}_0(j, 4) = \mathbf{b}_1(4, 5)$. Hence for every $j < 4$ we have

$$a_j + a_4 = \mathbf{b}_0(j, 4) + \mathbf{b}_1(j, 4) = \mathbf{b}_1(4, 5) + \mathbf{b}_1(j, 4),$$

and consequently

$$a_j + (a_4 - \mathbf{b}_1(4, 5)) = \mathbf{b}_1(j, 4) \in \mathbf{B}.$$

Thus letting $c = a_4 - \mathbf{b}_1(4, 5)$ we will have $\{a_i + c : i < 4\} \subseteq \mathbf{B}$. By Lemma 4.3 we also have $A + c \subseteq \mathbf{B}$.

CASE 6: $\mathbf{b}_1(0, 1) = \mathbf{b}_0(1, 2)$.

Similarly to Case 5, for each $j < 4$ we have $\mathbf{b}_1(j, 4) = \mathbf{b}_0(4, 5)$ and

$$a_j + a_4 = \mathbf{b}_0(j, 4) + \mathbf{b}_1(j, 4) = \mathbf{b}_0(j, 4) + \mathbf{b}_0(4, 5).$$

Hence $a_j + (a_4 - \mathbf{b}_0(4, 5)) = \mathbf{b}_0(j, 4) \in \mathbf{B}$ and the rest is clear.

Concerning the uniqueness of c , suppose towards contradiction that $c \neq d$ are such that $A + c \subseteq \mathbf{B}$ and $A + d \subseteq \mathbf{B}$. Let a_0, a_1, a_2, a_3 be distinct elements of A . Then for distinct $i, j < 4$ we have

$$a_i + c \neq a_i + d, \quad a_i + c \neq a_j + c, \quad \text{and} \quad a_i + d \neq a_j + d,$$

and we may find $i < 4$ such that $\{a_0 + c, a_0 + d\} \cap \{a_i + c, a_i + d\} = \emptyset$. Then the elements $a_0 + c, a_0 + d, a_i + c, a_i + d$ belong to \mathbf{B} , they are all distinct and $(a_0 + c) - (a_0 + d) - (a_i + c) + (a_i + d) = 0$, contradicting the quasi-independence of \mathbf{B} .

Finally, Lemma 4.3 gives that c must be of order at most 2. \square

5. Forcing for Abelian groups with few elements of order two

In this and the next section, we will keep the following notation/assumptions concerning our group \mathbb{H} .

Assumption 5.1.

- (1) $(\mathbb{H}, +, 0)$ is an Abelian perfect Polish group with the topology generated by a complete metric ρ . We also require that the metric ρ is translation invariant (possible by the Birkhoff–Kakutani theorem and [3, Corollary 1.2.2]).
- (2) $\mathbf{D} \subseteq \mathbb{H}$ is a countable dense set.
- (3) The open ball in the metric ρ with radius 2^{-n} and center at 0 is denoted \mathbf{B}_n and we let $\mathcal{U} = \{d + \mathbf{B}_n : d \in \mathbf{D} \wedge n < \omega\}$. By the invariance of the metric ρ , the family \mathcal{U} is a countable base of the topology of \mathbb{H} .

Note that if $P \subseteq B \subseteq \mathbb{H}$ then $x + y \in (B + x) \cap (B + y)$ for each $x, y \in P$. Consequently, if $P \subseteq B$ is a perfect set, then it witnesses that B has a perfect set of pairwise non-disjoint translations. But for $k \geq 2$ we may and will introduce a forcing notion adding a Borel set $B \subseteq \mathbb{H}$ which has many pairwise k -overlapping translations but no perfect set of such translations.

The technical details force us to break up the construction into two cases. First, we will assume that the group \mathbb{H} has only a few elements of rank 2. So, in addition to the assumptions and notation specified in 5.1, in this section we assume the following:

Assumption 5.2.

- (1) The set of elements of \mathbb{H} of order larger than 2 is dense in \mathbb{H} .
- (2) $1 < k < \omega$.
- (3) ε is a countable ordinal and λ is an uncountable cardinal such that $\text{NPr}^\varepsilon(\lambda)$ holds true. The model $\mathbb{M}(\varepsilon, \lambda)$ and functions $\text{rk}^{\text{sp}}, \mathbf{j}$ and \mathbf{k} on $[\lambda]^{<\omega} \setminus \{\emptyset\}$ are as fixed in Definition 2.6.

We will define a forcing notion \mathbb{P} adding λ many (distinct) elements $\langle \eta_\alpha : \alpha < \lambda \rangle$ of the group \mathbb{H} as well as a sequence $\langle F_m : m < \omega \rangle$ of closed subsets of \mathbb{H} . The Σ_2^0 subset $S = \bigcup_{m < \omega} F_m$ of \mathbb{H} will have the property that (in the forcing extension)

(\heartsuit)₁ there is *no* perfect set $P \subseteq \mathbb{H}$ satisfying

$$(\forall x, y \in P) (|(x + S) \cap (y + S)| \geq k).$$

At the same we will make sure that

$(\heartsuit)_2 \quad |(-\eta_\alpha + S) \cap (-\eta_\beta + S)| \geq k$ for all $\alpha, \beta < \lambda$.

To ensure $(\heartsuit)_2$ holds, the forcing will also add witnesses for it: group elements $\nu_{i,\alpha,\beta} = \nu_{i,\beta,\alpha} \in \mathbb{H}$ and integers $h_{\alpha,\beta} < \omega$ such that $\eta_\alpha + \nu_{i,\alpha,\beta} \in F_{h_{\alpha,\beta}}$ (for $i < k$, $\alpha, \beta < \lambda$).

A condition $p \in \mathbb{P}$ will give a “finite information” on objects mentioned above. Thus for some finite $w^p \subseteq \lambda$, for all distinct $\alpha, \beta \in w^p$, the condition p provides a basic open neighborhood $U_\alpha^p(n^p)$ of η_α , basic open neighborhood $W_{i,\alpha,\beta}^p$ of $\nu_{i,\alpha,\beta}$ and the values of $h_{\alpha,\beta} = h^p(\alpha, \beta)$. An approximation to the closed set $F_m \subseteq \mathbb{H}$ will be given by its open neighborhood

$$F(p, m) = \bigcup \{U_\alpha^p(n^p) + W_{i,\alpha,\beta}^p : (\alpha, \beta) \in (w^p)^{(2)} \wedge i < k \wedge h^p(\alpha, \beta) = m\}.$$

Clause $(\heartsuit)_1$ as well as the ccc of the forcing \mathbb{P} will result from the involvement of the rank rk^{sp} and additional technical pieces of information carried by conditions $p \in \mathbb{P}$: basic open sets $Q_{i,\alpha,\beta}^p, V_{i,\alpha,\beta}^p$ and integers r_m^p .

Definition 5.3. (A) Let \mathbb{P} be the collection of all tuples

$$p = (w^p, M^p, \bar{r}^p, n^p, \tilde{\Upsilon}^p, \bar{V}^p, h^p) = (w, M, \bar{r}, n, \tilde{\Upsilon}, \bar{V}, h)$$

such that the following demands $(\boxtimes)_1 - (\boxtimes)_8$ are satisfied.

- $(\boxtimes)_1 \quad w \in [\lambda]^{<\omega}$, $|w| \geq 4$, $0 < M < \omega$, $3 \leq n < \omega$ and $\bar{r} = \langle r_m : m < M \rangle \subseteq \omega$ with $r_m \leq n - 2$ for $m < M$.
- $(\boxtimes)_2 \quad \tilde{\Upsilon} = \langle \bar{U}_\alpha : \alpha \in w \rangle$ where each $\bar{U}_\alpha = \langle U_\alpha(\ell) : \ell \leq n \rangle$ is a \subseteq -decreasing sequence of elements of the basis \mathcal{U} .
- $(\boxtimes)_3 \quad \bar{V} = \langle Q_{i,\alpha,\beta}, V_{i,\alpha,\beta}, W_{i,\alpha,\beta} : i < k, (\alpha, \beta) \in w^{(2)} \rangle \subseteq \mathcal{U}$ and $Q_{i,\alpha,\beta} = Q_{i,\beta,\alpha} \supseteq V_{i,\alpha,\beta} = V_{i,\beta,\alpha} \supseteq W_{i,\alpha,\beta} = W_{i,\beta,\alpha}$ for all $i < k$ and $(\alpha, \beta) \in w^{(2)}$.
- $(\boxtimes)_4$ (a) The indexed family $\langle U_\alpha(n-2) : \alpha \in w \rangle \smallfrown \langle Q_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is an 8-good qif (so in particular the sets in this system are pairwise disjoint), and
 (b) $\langle U_\alpha(n) : \alpha \in w \rangle \smallfrown \langle W_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is immersed in $\langle U_\alpha(n-1) : \alpha \in w \rangle \smallfrown \langle V_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ and $\langle U_\alpha(n-1) : \alpha \in w \rangle \smallfrown \langle V_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is immersed in $\langle U_\alpha(n-2) : \alpha \in w \rangle \smallfrown \langle Q_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$; see Definition 3.1(4) (so all these families are 8-good qifs).
- $(\boxtimes)_5$ (a) If $\alpha, \beta \in w$, $\ell \leq n$ and $U_\alpha(\ell) \cap U_\beta(\ell) \neq \emptyset$, then $U_\alpha(\ell) = U_\beta(\ell)$, and
 (b) if $\alpha, \beta, \gamma \in w$, $\ell \leq n$, $U_\alpha(\ell) \neq U_\beta(\ell)$ and $a \in U_\alpha(\ell)$, $b \in U_\beta(\ell)$, then $\rho(a, b) > \text{diam}_\rho(U_\gamma(\ell))$ ($= \text{diam}_\rho(-U_\gamma(\ell))$).
- $(\boxtimes)_6 \quad h : w^{(2)} \xrightarrow{\text{onto}} M$ is such that $h(\alpha, \beta) = h(\beta, \alpha)$ for $(\alpha, \beta) \in w^{(2)}$.
- $(\boxtimes)_7$ Assume that $u, u' \subseteq w$, π and $\ell \leq n$ are such that
 - $4 \leq |u| = |u'|$ and $\pi : u \rightarrow u'$ is a bijection,
 - $r_{h(\alpha,\beta)} \leq \ell$ for all $(\alpha, \beta) \in u^{(2)}$,
 - $U_\alpha(\ell) \cap U_\beta(\ell) = \emptyset$ and $h(\alpha, \beta) = h(\pi(\alpha), \pi(\beta))$ for all distinct $\alpha, \beta \in u$,
 - for some $c \in \mathbb{H}$,

either for all $\alpha \in u$, we have $(U_\alpha(\ell) + c) \cap U_{\pi(\alpha)}(\ell) \neq \emptyset$
 or for all $\alpha \in u$, we have $(c - U_\alpha(\ell)) \cap U_{\pi(\alpha)}(\ell) \neq \emptyset$.

Then $\text{rk}^{\text{sp}}(u) = \text{rk}^{\text{sp}}(u')$, $\mathbf{j}(u) = \mathbf{j}(u')$, $\mathbf{k}(u) = \mathbf{k}(u')$ and for $\alpha \in u$

$$|\alpha \cap u| = \mathbf{k}(u) \quad \Leftrightarrow \quad |\pi(\alpha) \cap u'| = \mathbf{k}(u).$$

$(\boxtimes)_8$ Assume that

- $\emptyset \neq u \subseteq w$, $\text{rk}^{\text{sp}}(u) = -1$, $\ell \leq n$ and

- $\alpha \in u$ is such that $|\alpha \cap u| = \mathbf{k}(u)$, and
 - $r_{h(\beta, \beta')} \leq \ell$ and $U_\beta(\ell) \cap U_{\beta'}(\ell) = \emptyset$ for all $(\beta, \beta') \in u^{(2)}$.
- Then there is **no** $\alpha' \in w \setminus u$ such that $U_\alpha(\ell) = U_{\alpha'}(\ell)$ and $h(\alpha, \beta) = h(\alpha', \beta)$ for all $\beta \in u \setminus \{\alpha\}$.

(B) For $p \in \mathbb{P}$ and $m < M^p$ we define

$$F(p, m) = \bigcup \{U_\alpha^p(n^p) + W_{i, \alpha, \beta}^p : (\alpha, \beta) \in (w^p)^{(2)} \wedge i < k \wedge h^p(\alpha, \beta) = m\}.$$

(C) For $p, q \in \mathbb{P}$ we declare that $p \leq q$ if and only if

- $w^p \subseteq w^q$, $M^p \leq M^q$, $\bar{r}^q \upharpoonright M^p = \bar{r}^p$, $n^p \leq n^q$, $h^q \upharpoonright (w^p)^{(2)} = h^p$, and
- if $\alpha \in w^p$ and $\ell \leq n^p$ then $U_\alpha^q(\ell) = U_\alpha^p(\ell)$, and
- if $(\alpha, \beta) \in (w^p)^{(2)}$, $i < k$, then $Q_{i, \alpha, \beta}^q \subseteq Q_{i, \alpha, \beta}^p$, $V_{i, \alpha, \beta}^q \subseteq V_{i, \alpha, \beta}^p$, and $W_{i, \alpha, \beta}^q \subseteq W_{i, \alpha, \beta}^p$, and
- if $m < M^p$, then $F(q, m) \subseteq F(p, m)$.

Lemma 5.4.

- (1) (\mathbb{P}, \leq) is a partial order of size λ .
- (2) The following sets are dense in \mathbb{P} :
 - (i) $D_{\gamma, M, n}^0 = \{p \in \mathbb{P} : \gamma \in w^p \wedge M^p > M \wedge n^p > n\}$ for $\gamma < \lambda$ and $M, n < \omega$.
 - (ii) $D_N^1 = \{p \in \mathbb{P} : \text{for all } i, j < k \text{ and } (\alpha, \beta), (\gamma, \delta) \in (w^p)^{(2)} \text{ it holds that } \text{diam}_\rho(U_\alpha^p(n^p - 2)) < 2^{-N} \text{ and } \text{diam}_\rho(Q_{i, \alpha, \beta}^p) < 2^{-N} \text{ and } \text{diam}_\rho(U_\alpha^p(n^p - 2) + Q_{i, \alpha, \beta}^p) < 2^{-N} \text{ and if } (i, \alpha^*, \alpha, \beta) \neq (j, \gamma^*, \gamma, \delta) \text{ then } (U_{\alpha^*}^p(n^p) + W_{i, \alpha, \beta}^p) \cap (U_{\gamma^*}^p(n^p) + W_{i, \gamma, \delta}^p) = \emptyset\}$ for $N < \omega$.
- (3) Assume $p \in \mathbb{P}$. Then there is $q \geq p$ such that $n^q \geq n^p + 3$, $w^q = w^p$ and
 - for all $\alpha \in w^p$, $\text{cl}(U_\alpha^q(n^q - 2)) \subseteq U_\alpha^p(n^p)$, and
 - for all $i < k$ and $(\alpha, \beta) \in (w^p)^{(2)}$,

$$\text{cl}(U_\alpha^q(n^q - 2) + Q_{i, \alpha, \beta}^q) \subseteq U_\alpha^p(n^p) + W_{i, \alpha, \beta}^p \quad \text{and} \quad \text{cl}(Q_{i, \alpha, \beta}^q) \subseteq W_{i, \alpha, \beta}^p.$$

Proof. (2)(i) Suppose $p \in \mathbb{P}$ and $\gamma \in \lambda \setminus w^p$. Let $\alpha^* = \min(w^p)$ and let $w = w^p \cup \{\gamma\}$ and $n = n^p + 3$. Using Proposition 3.3 we may choose $U_\alpha(n - 2) \in \mathcal{U}$ (for $\alpha \in w$) and $Q_{i, \alpha, \beta} \in \mathcal{U}$ (for $i < k$, $\alpha < \beta$, $\alpha, \beta \in w$) such that

- $U_\alpha(n - 2) \subseteq U_\alpha^p(n^p)$ and $Q_{i, \alpha, \beta} \subseteq W_{i, \alpha, \beta}^p$ when $\alpha, \beta \in w^p$,
- $U_\gamma(n - 2) \subseteq U_{\alpha^*}^p(n^p)$,
- $\langle U_\alpha(n - 2) : \alpha \in w \rangle \smallfrown \langle Q_{i, \alpha, \beta} : i < k, \alpha < \beta, \alpha, \beta \in w \rangle$ is an 8-good qif,
- $\text{diam}_\rho(U_\delta(n - 2)) = \text{diam}_\rho(-U_\delta(n - 2)) < \rho(a, b)$ for all $\delta \in w$, $(\alpha, \beta) \in w^{(2)}$, $a \in U_\alpha(n - 2)$ and $b \in U_\beta(n - 2)$.

Then by Observation 3.2(3) we may choose $U_\alpha(n - 1), U_\alpha(n), V_{i, \alpha, \beta}, W_{i, \alpha, \beta} \in \mathcal{U}$ (for $\alpha < \beta$ from w and $i < k$) such that $U_\alpha(n) \subseteq U_\alpha(n - 1) \subseteq U_\alpha(n - 2)$, $W_{i, \alpha, \beta} \subseteq V_{i, \alpha, \beta} \subseteq Q_{i, \alpha, \beta}$ and

- $\langle U_\alpha(n - 1) : \alpha \in w \rangle \smallfrown \langle V_{i, \alpha, \beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is immersed in $\langle U_\alpha(n - 2) : \alpha \in w \rangle \smallfrown \langle Q_{i, \alpha, \beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$, and
- $\langle U_\alpha(n) : \alpha \in w \rangle \smallfrown \langle W_{i, \alpha, \beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is immersed in $\langle U_\alpha(n - 1) : \alpha \in w \rangle \smallfrown \langle V_{i, \alpha, \beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$.

Put $\bar{\Upsilon} = \langle \bar{U}_\alpha : \alpha \in w \rangle$, where $\bar{U}_\alpha = \bar{U}_\alpha^p \smallfrown \langle U_\alpha(n-2), U_\alpha(n-1), U_\alpha(n) \rangle$ if $\alpha \in w^p$ and $\bar{U}_\gamma = \bar{U}_\gamma^p \smallfrown \langle U_\gamma(n-2), U_\gamma(n-1), U_\gamma(n) \rangle$. Let $Q_{i,\beta,\alpha} = Q_{i,\alpha,\beta}$, $V_{i,\beta,\alpha} = V_{i,\alpha,\beta}$ and $W_{i,\beta,\alpha} = W_{i,\alpha,\beta}$ (for $i < k$, $\alpha < \beta$ from w), and let $\bar{V} = \langle Q_{i,\alpha,\beta}, V_{i,\alpha,\beta}, W_{i,\alpha,\beta} : i < k, (\alpha, \beta) \in w^{(2)} \rangle$. Let $M = M^p + |w^p|$ and let $h : w^{(2)} \rightarrow M$ be such that

- $h(\alpha, \beta) = h^p(\alpha, \beta)$ when $(\alpha, \beta) \in (w^p)^{(2)}$,
- $h(\alpha, \gamma) = h(\gamma, \alpha) = M^p + j$ when $\alpha \in w^p$ and $j = |w^p \cap \alpha|$.

We also define $\bar{r} : M \rightarrow (n-1)$ so that $\bar{r} \restriction M^p = \bar{r}^p$ and $r_m = n-2$ for $m \in [M^p, M)$.

Put $q = (w, M, r, n, \bar{\Upsilon}, \bar{V}, h)$. Let us argue that $q \in \mathbb{P}$. To this end we have to verify conditions $(\boxtimes)_1 - (\boxtimes)_8$ of Definition 5.3. Of these the first six demands follow immediately by our choices. To show $(\boxtimes)_7$, suppose $u, u' \subseteq w$ and $\pi : u \rightarrow u'$ and $\ell \leq n$ and $c \in \mathbb{H}$ satisfy the assumptions there. If $\alpha \in w^p$ then $h(\gamma, \alpha) \geq M^p$ and therefore $\gamma \in u$ if and only if $\gamma \in u'$. If $\gamma \notin u$, then $u \cup u' \subseteq w^p$ and clause $(\boxtimes)_7$ for p (applied to $\min(n^p, \ell)$ instead of ℓ) gives the needed conclusion. If $\gamma \in u$, then $\gamma \in u'$ too and we look at $h(\beta, \gamma)$ for $\beta \in u \cap w^p$. Each of these values is taken by h exactly one time, so $h(\pi(\beta), \pi(\gamma)) = h(\beta, \gamma)$ for all $\beta \in w^p$ implies that $\pi(\gamma) = \gamma$ and $\pi(\beta) = \beta$ for $\beta \in u \cap w^p$. Hence $u = u'$ and π is the identity, so the desired conclusion follows.

Now suppose $\ell \leq n$, $\alpha \in u \subseteq w$ are as in the assumptions of $(\boxtimes)_8$ (so by 2.6(*)_b also $|u| \geq 2$). If $\gamma \notin u$, then applying $(\boxtimes)_8$ for p to α, u and $\ell' = \min(\ell, n^p)$ we see that there is no $\alpha' \in w^p \setminus u$ with $U_\alpha(\ell) = U_{\alpha'}(\ell)$, and $h(\alpha, \beta) = h(\alpha', \beta)$ for all $\beta \in u \setminus \{\alpha\}$. The values of $h(\beta, \gamma)$ (for $\beta \in u$) are above M^p , so they cannot be equal to $h(\beta, \alpha)$ either. Consequently, the conclusion of $(\boxtimes)_8$ holds in this case. So assume now that $\gamma \in u \setminus \{\alpha\}$. The value of $h(\gamma, \alpha)$ is taken exactly once, so no $\alpha' \in w \setminus \{\gamma, \alpha\}$ satisfies $h(\gamma, \alpha) = h(\gamma, \alpha')$ and the desired conclusion should be clear now. Finally, assume $\gamma = \alpha$. As we said, $|u| \geq 2$ so we may take $\beta \in u \setminus \{\gamma\}$ and look at $h(\gamma, \beta)$. There is no $\alpha' \in w \setminus \{\gamma\}$ with $h(\alpha', \beta) = h(\gamma, \beta)$, so the desired conclusion follows, finishing the proof of $(\boxtimes)_8$.

Now one easily deduces (2)(i).

(ii) Assume $p \in \mathbb{P}$ and $N < \omega$.

First note that if $\langle U_\ell, V_\ell : \ell < m \rangle \subseteq \mathcal{U}$, $m < \omega$, then there are $U'_\ell \subseteq U_\ell$, $V'_\ell \subseteq V_\ell$ from \mathcal{U} (for $\ell < m$) such that $(U'_{\ell_0} + V'_{\ell_0}) \cap (U'_{\ell_1} + V'_{\ell_1}) = \emptyset$ whenever $\ell_0 < \ell_1 < m$ and $\text{diam}_\rho(U'_\ell + V'_\ell) < 2^{-N}$ for $\ell < m$. Therefore, for $(\alpha, \beta) \in (w^p)^{(2)}$ and $i < k$ we may choose $U_\alpha(n^p + 1), Q_{i,\alpha,\beta} \in \mathcal{U}$ such that $U_\alpha(n^p + 1) \subseteq U_\alpha^p(n^p)$, $Q_{i,\alpha,\beta} = Q_{i,\beta,\alpha} \subseteq W_{i,\alpha,\beta}^p$ and ρ -diameters of $U_\alpha(n^p + 1), Q_{i,\alpha,\beta}$ and $U_\alpha(n^p + 1) + Q_{i,\alpha,\beta}$ are all smaller than 2^{-N} and if $(i, \alpha^*, \alpha, \beta) \neq (j, \gamma^*, \gamma, \delta)$ then $(U_{\alpha^*}^p(n^p) + W_{i,\alpha,\beta}^p) \cap (U_{\gamma^*}^p(n^p) + W_{i,\gamma,\delta}^p) = \emptyset$.

Note that $\langle U_\alpha(n^p + 1) : \alpha \in w^p \rangle \smallfrown \langle Q_{i,\alpha,\beta} : i < k, \alpha < \beta, \alpha, \beta \in w^p \rangle$ is an 8-good qif. Next, use Observation 3.2 to choose $U_\alpha(n^p + 2), U_\alpha(n^p + 3), V_{i,\alpha,\beta}, W_{i,\alpha,\beta} \in \mathcal{U}$ such that $U_\alpha(n^p + 3) \subseteq U_\alpha(n^p + 2) \subseteq U_\alpha(n^p + 1)$, and $W_{i,\alpha,\beta} = W_{i,\beta,\alpha} \subseteq V_{i,\alpha,\beta} = V_{i,\beta,\alpha} \subseteq Q_{i,\alpha,\beta}$ (for $(\alpha, \beta) \in (w^p)^{(2)}$ and $i < k$), and

- $\langle U_\alpha(n^p + 3) : \alpha \in w^p \rangle \smallfrown \langle W_{i,\alpha,\beta} : i < k, \alpha < \beta, \alpha, \beta \in w^p \rangle$ is immersed in $\langle U_\alpha(n^p + 2) : \alpha \in w^p \rangle \smallfrown \langle V_{i,\alpha,\beta} : i < k, \alpha < \beta, \alpha, \beta \in w^p \rangle$, and
- $\langle U_\alpha(n^p + 2) : \alpha \in w^p \rangle \smallfrown \langle V_{i,\alpha,\beta} : i < k, \alpha < \beta, \alpha, \beta \in w^p \rangle$ is immersed in $\langle U_\alpha(n^p + 1) : \alpha \in w^p \rangle \smallfrown \langle Q_{i,\alpha,\beta} : i < k, \alpha < \beta, \alpha, \beta \in w^p \rangle$.

Now, for $\alpha \in w^p$ let $\bar{U}_\alpha = \bar{U}_\alpha^p \smallfrown \langle U_\alpha(n^p + 1), U_\alpha(n^p + 2), U_\alpha(n^p + 3) \rangle$ and then let $\bar{\Upsilon} = \langle \bar{U}_\alpha : \alpha \in w^p \rangle$ and $\bar{V} = \langle Q_{i,\alpha,\beta}, V_{i,\alpha,\beta}, W_{i,\alpha,\beta} : i < k, (\alpha, \beta) \in (w^p)^{(2)} \rangle$. These choices clearly determine a condition $q = (w^p, M^p, \bar{r}^p, n^p + 3, \bar{\Upsilon}, \bar{V}, h^p) \in D_N^1$ stronger than p .

(3) Analogous. \square

Lemma 5.5. Suppose that $p \in \mathbb{P}$ and $\alpha, \beta, \gamma, \delta \in w^p$ are such that $\alpha \neq \beta$. If

$$\left(U_\alpha^p(n^p - 2) - U_\beta^p(n^p - 2) \right) \cap \left(U_\gamma^p(n^p - 2) - U_\delta^p(n^p - 2) \right) \neq \emptyset,$$

then $\alpha = \gamma$ and $\beta = \delta$.

Proof. Let $n = n^p$. Suppose that $a \in U_\alpha^p(n - 2)$, $b \in U_\beta^p(n - 2)$, $c \in U_\gamma^p(n - 2)$ and $d \in U_\delta^p(n - 2)$ are such that $a - b = c - d$. Then $a + (c - a) = c$ and $b + (c - a) = d$, so as ρ is invariant we have $\rho(a, b) = \rho(c, d)$. Demand 5.3(A)(\boxtimes)₅(b) implies that $\rho(a, b) > \text{diam}_\rho(U_\gamma^p(n - 2))$ and hence $\gamma \neq \delta$. Now look at $a + d - b - c$: since $\alpha \neq \beta$ and $\gamma \neq \delta$ it is a (2,4)-combination from an 8-good qif $\langle U_\zeta^p(n - 2) : \zeta \in w \rangle$. Since the value of the combination is 0, it has to be trivial. Hence immediately $\alpha = \gamma$ and $\beta = \delta$. \square

Lemma 5.6. The forcing notion \mathbb{P} has the Knaster property.

Proof. Suppose $\langle p_\varepsilon : \varepsilon < \omega_1 \rangle$ is a sequence of pairwise distinct conditions from \mathbb{P} . Applying standard Δ -lemma based cleaning procedure we may find $w_0 \subseteq \lambda$ and $A \in [\omega_1]^{\omega_1}$ such that for distinct $\xi, \zeta \in A$ the following demands $(*)_1 + (*)_2$ are satisfied.

$$(*)_1 \quad |w^{p_\xi}| = |w^{p_\zeta}|, \quad w_0 = w^{p_\xi} \cap w^{p_\zeta}, \quad M^{p_\xi} = M^{p_\zeta}, \quad n^{p_\xi} = n^{p_\zeta}, \quad \bar{r}^{p_\xi} = \bar{r}^{p_\zeta}.$$

(*)₂ If $\pi^* : w^{p_\zeta} \longrightarrow w^{p_\xi}$ is the order isomorphism, then

- $\pi^* \upharpoonright w_0$ is the identity,
- $\bar{U}_\alpha^{p_\zeta}(\ell) = \bar{U}_{\pi^*(\alpha)}^{p_\xi}(\ell)$ whenever $\alpha \in w^{p_\zeta}$, $\ell \leq n^{p_\zeta}$,
- if $(\alpha, \beta) \in (w^{p_\zeta})^{(2)}$, $i < k$, then $h^{p_\zeta}(\alpha, \beta) = h^{p_\xi}(\pi^*(\alpha), \pi^*(\beta))$, and

$$Q_{i,\alpha,\beta}^{p_\zeta} = Q_{i,\pi^*(\alpha),\pi^*(\beta)}^{p_\xi}, \quad V_{i,\alpha,\beta}^{p_\zeta} = V_{i,\pi^*(\alpha),\pi^*(\beta)}^{p_\xi} \quad \text{and} \quad W_{i,\alpha,\beta}^{p_\zeta} = W_{i,\pi^*(\alpha),\pi^*(\beta)}^{p_\xi},$$

- if $\emptyset \neq u \subseteq w^{p_\zeta}$, then $\text{rk}^{\text{sp}}(u) = \text{rk}^{\text{sp}}(\pi^*[u])$, $\mathbf{j}(u) = \mathbf{j}(\pi^*[u])$ and $\mathbf{k}(u) = \mathbf{k}(\pi^*[u])$.

Note that then for all $\xi \in A$ we have

$$(*)_3 \quad \text{if } u \subseteq w_0, \alpha \in w^{p_\xi} \setminus w_0 \text{ and } \text{rk}^{\text{sp}}(u \cup \{\alpha\}) = -1, \text{ then } \mathbf{k}(u \cup \{\alpha\}) \neq |u \cap \alpha|.$$

[Why? Suppose towards contradiction that $\mathbf{k}(u \cup \{\alpha\}) = |u \cap \alpha|$. For $\zeta \in A$ let $\alpha_\zeta \in w^{p_\zeta}$ be such that $|\alpha_\zeta \cap w^{p_\zeta}| = |\alpha \cap w^{p_\xi}|$. By $(*)_2$ we have

$$j \stackrel{\text{def}}{=} \mathbf{j}(u \cup \{\alpha\}) = \mathbf{j}(u \cup \{\alpha_\zeta\}) \quad \text{and} \quad \mathbf{k}(u \cup \{\alpha_\zeta\}) = \mathbf{k}(u \cup \{\alpha\}) = |u \cap \alpha| = |u \cap \alpha_\zeta| \stackrel{\text{def}}{=} k.$$

Therefore, letting $u \cup \{\alpha\} = \{\alpha_0, \dots, \alpha_{\ell-1}\}$ be the increasing enumeration, we have $\alpha_k = \alpha$ and

$$\mathbb{M} \models R_{\ell,j}[\alpha_0, \dots, \alpha_{k-1}, \alpha_\zeta, \alpha_{k+1}, \dots, \alpha_{\ell-1}] \quad \text{for all } \zeta \in A.$$

However, this contradicts the choice of \mathbf{j}, \mathbf{k} in Definition 2.6 and the assumption $\text{rk}^{\text{sp}}(u \cup \{\alpha\}) = -1$.

We will argue now that for $\xi, \zeta \in A$ the conditions p_ξ, p_ζ are compatible. So let $\xi < \zeta$ be from A and let $\pi^* : w^{p_\zeta} \longrightarrow w^{p_\xi}$ be the order isomorphism. Set $w = w^{p_\xi} \cup w^{p_\zeta}$, $M = M^{p_\xi} + |w^{p_\xi} \setminus w^{p_\zeta}|^2$, $n = n^{p_\xi} + 3$ and let $\bar{r} = \langle r_m : m < M \rangle$ be such that $r_m = r_m^{p_\xi}$ if $m < M^{p_\xi}$, and $r_m = n - 2$ if $M^{p_\xi} \leq m < M$.

Use Proposition 3.3 and Observation 3.2(iii) to choose $U_\alpha(n - 2)$, $U_\alpha(n - 1)$, $U_\alpha(n)$, $Q_{i,\alpha,\beta}$, $V_{i,\alpha,\beta}$ and $W_{i,\alpha,\beta}$ from \mathcal{U} for $i < k$ and $(\alpha, \beta) \in w^{(2)}$ so that

(*)₄ (a) demands 5.3(\boxtimes)₃-(\boxtimes)₅ are satisfied and

- (b) if $(\alpha, \beta) \in (w^{p_\xi})^{(2)}$, $i < k$, then $U_\alpha(n-2) \subseteq U_\alpha^{p_\xi}(n^{p_\xi})$ and $Q_{i,\alpha,\beta} \subseteq W_{i,\alpha,\beta}^{p_\xi}$, and
 (c) if $(\alpha, \beta) \in (w^{p_\zeta})^{(2)}$, $i < k$, then $U_\alpha(n-2) \subseteq U_\alpha^{p_\zeta}(n^{p_\zeta})$ and $Q_{i,\alpha,\beta} \subseteq W_{i,\alpha,\beta}^{p_\zeta}$.

Let $\bar{U}_\alpha = \bar{U}_\alpha^{p_\xi} \smallfrown \langle U_\alpha(n-2), U_\alpha(n-1), U_\alpha(n) \rangle$ if $\alpha \in w^{p_\xi}$ and $\bar{U}_\alpha = \bar{U}_\alpha^{p_\zeta} \smallfrown \langle U_\alpha(n-2), U_\alpha(n-1), U_\alpha(n) \rangle$ if $\alpha \in w^{p_\zeta}$, and let \bar{Y}, \bar{V} be defined naturally. Choose $h : w^{(2)} \rightarrow M$ extending both h^{p_ξ} and h^{p_ζ} in such a manner that $h(\alpha, \beta) = h(\beta, \alpha)$ for $(\alpha, \beta) \in w^{(2)}$ and the mapping

$$(w^{p_\xi} \setminus w_0) \times (w^{p_\zeta} \setminus w_0) \ni (\alpha, \beta) \mapsto h(\alpha, \beta)$$

is a bijection onto $[M^{p_\xi}, M]$. Finally we set $q = (w, M, \bar{r}, n, \bar{Y}, \bar{V}, h)$.

Let us argue that $q \in \mathbb{P}$ (once we are done with that, it should be clear that q is stronger than both p_ξ and p_ζ). The only potentially unclear demands to verify are $(\boxtimes)_7$ and $(\boxtimes)_8$ of 5.3.

First, to demonstrate $(\boxtimes)_7$, suppose that $u, u' \subseteq w$ and $\pi : u \rightarrow u'$ and $\ell \leq n$ and $c \in \mathbb{H}$ are as in the assumptions there. Let us consider the following three cases.

CASE 1: $u \subseteq w^{p_\xi}$.

Then for each $(\alpha, \beta) \in u^{(2)}$ we have $h(\alpha, \beta) < M^{p_\xi}$, so this also holds for all $(\gamma, \delta) \in (u')^{(2)}$. Consequently, either $u' \subseteq w^{p_\xi}$ or $u' \subseteq w^{p_\zeta}$.

If $u' \subseteq w^{p_\xi}$, then let $\ell' = \min(\ell, n^{p_\xi})$ and consider u, u', π, ℓ' . Using clause $(\boxtimes)_7$ for p_ξ we immediately obtain the desired conclusion.

If $u' \subseteq w^{p_\zeta}$, then we let $\ell' = \min(\ell, n^{p_\xi})$ and we consider $u, \pi^*[u'], \ell'$ and $\pi^* \circ \pi$ (where, remember, $\pi^* : w^{p_\zeta} \rightarrow w^{p_\xi}$ is the order isomorphism). By $(*)_1 + (*)_2$, clause $(\boxtimes)_7$ for p_ξ applies to them and we get

- $\text{rk}^{\text{sp}}(u) = \text{rk}^{\text{sp}}(\pi^*[u']), \mathbf{j}(u) = \mathbf{j}(\pi^*[u']), \mathbf{k}(u) = \mathbf{k}(\pi^*[u'])$ and
- for $\alpha \in u$, $|\alpha \cap u| = \mathbf{k}(u) \Leftrightarrow |(\pi^* \circ \pi)(\alpha) \cap \pi^*[u']| = \mathbf{k}(u)$.

Now $(*)_1, (*)_2$ immediately imply the desired conclusion.

CASE 2: $u \subseteq w^{p_\zeta}$.

Same as the previous case, just interchanging ξ and ζ .

CASE 3: $u \setminus w^{p_\xi} \neq \emptyset \neq u \setminus w^{p_\zeta}$.

Choose $\alpha \in u \setminus w^{p_\xi}$ and $\beta \in u \setminus w^{p_\zeta}$. Then $h(\alpha, \beta) \geq M^{p_\xi}$ and therefore $n-2 = r_{h(\alpha,\beta)} \leq \ell$.

We will argue that π is the identity on u and $u = u'$ (so the needed assertion is immediate). Suppose towards contradiction that we got a $\gamma \in u$ such that $\pi(\gamma) \neq \gamma$. Since $|u| \geq 4$ we may also pick $\gamma' \in u$ such that $\{\gamma, \pi(\gamma)\} \cap \{\gamma', \pi(\gamma')\} = \emptyset$. Now we consider two subcases determined by the property of $c \in \mathbb{H}$.

Suppose $(U_\delta(\ell) + c) \cap U_{\pi(\delta)}(\ell) \neq \emptyset$ for all $\delta \in u$. Then for some $b \in U_\gamma(\ell)$, $b' \in U_{\pi(\gamma)}(\ell)$, $b'' \in U_{\gamma'}(\ell)$ and $b''' \in U_{\pi(\gamma')}(\ell)$ we have $b' - b = c = b''' - b''$. However, this (and the choice of γ and γ') gives immediate contradiction with $\langle U_\delta(\ell) : \delta \in w \rangle$ being a good qif (remember $\ell \geq n-2$).

Assume now that $(c - U_\delta(\ell)) \cap U_{\pi(\delta)}(\ell) \neq \emptyset$ for all $\delta \in u$. Then for some $b \in U_\gamma(\ell)$, $b' \in U_{\pi(\gamma)}(\ell)$, $b'' \in U_{\gamma'}(\ell)$ and $b''' \in U_{\pi(\gamma')}(\ell)$ we have $b' + b = c = b''' + b''$, getting immediate contradiction with $\langle U_\delta(\ell) : \delta \in w \rangle$ being a good qif.

Now, concerning $(\boxtimes)_8$, suppose that $u \subseteq w$, $\ell \leq n$ and $\alpha \in u$ are such that

- $|\alpha \cap u| = \mathbf{k}(u)$ and $\text{rk}^{\text{sp}}(u) = -1$ and
- $r_{h(\beta,\beta')} \leq \ell$ and $U_\beta(\ell) \cap U_{\beta'}(\ell) = \emptyset$ for all $(\beta, \beta') \in u^{(2)}$.

We want to argue that there is no $\alpha' \in w$ such that

$$(\boxtimes)^{\alpha'} \quad \alpha' \notin u, h(\alpha, \beta) = h(\alpha', \beta) \text{ for all } \beta \in u \setminus \{\alpha\}, \text{ and } U_\alpha(\ell) = U_{\alpha'}(\ell).$$

This is immediate if $\ell \geq n - 2$, so let us assume $\ell \leq n^{p_\zeta}$. Then we must also have $r_{h(\beta, \beta')} \leq n^{p_\zeta}$ for all $(\beta, \beta') \in u^{(2)}$, so either $u \subseteq w^{p_\zeta}$ or $u \subseteq w^{p_\zeta}$. By the symmetry, we may assume that $u \subseteq w^{p_\zeta}$.

If $u \subseteq w^{p_\zeta} \cap w^{p_\xi}$ then we may first use $(\boxtimes)_8$ for p_ξ to assert that there is no $\alpha' \in w^{p_\xi}$ satisfying $(\boxtimes)^{\alpha'}$ and then in the same manner argue that no $\alpha' \in w^{p_\zeta}$ satisfies $(\boxtimes)^{\alpha'}$.

If $u \subseteq w^{p_\xi}$ but $u \setminus w^{p_\zeta} \neq \emptyset$ and $\alpha \in w^{p_\xi} \cap w^{p_\zeta}$, then $(\boxtimes)_8$ for p_ξ implies there is no $\alpha' \in w^{p_\xi}$ satisfying $(\boxtimes)^{\alpha'}$. Also if $\alpha' \in w^{p_\zeta} \setminus w^{p_\xi}$ then for $\beta \in u \setminus w^{p_\zeta}$ we have $h(\alpha, \beta) < M^{p_\xi} \leq h(\alpha', \beta)$, so $(\boxtimes)^{\alpha'}$ fails then too.

Thus we are left only with the possibility that $\alpha \in w^{p_\xi} \setminus w^{p_\zeta}$. Like before, $(\boxtimes)_8$ for p_ξ implies there is no $\alpha' \in w^{p_\xi}$ satisfying $(\boxtimes)^{\alpha'}$. So suppose now $\alpha' \in w^{p_\zeta} \setminus w^{p_\xi}$. By $(*)_3$ we know that $(u \setminus \{\alpha\}) \setminus w^{p_\zeta} \neq \emptyset$, so let $\beta \in u \setminus w^{p_\zeta}$, $\beta \neq \alpha$. Then we have $h(\alpha, \beta) < M^{p_\xi} \leq h(\alpha', \beta)$, so $(\boxtimes)^{\alpha'}$ fails. The proof of $(\boxtimes)_8$ is complete now. \square

Lemma 5.7. For each $(\alpha, \beta) \in \lambda^{(2)}$ and $i < k$,

$$\Vdash_{\mathbb{P}} \text{“the sets} \\ \bigcap \{U_\alpha^p(n^p) : p \in G_{\mathbb{P}} \wedge \alpha \in w^p\} \quad \text{and} \quad \bigcap \{W_{i, \alpha, \beta}^p : p \in G_{\mathbb{P}} \wedge \alpha, \beta \in w^p\} \\ \text{have exactly one element each.”}$$

Proof. Follows from Lemma 5.4(2)(ii), (3). \square

Definition 5.8.

(1) For $(\alpha, \beta) \in \lambda^{(2)}$ and $i < k$ let η_α , $\nu_{i, \alpha, \beta}$ and $h_{\alpha, \beta}$ be \mathbb{P} -names such that

$$\begin{aligned} \Vdash_{\mathbb{P}} \text{“}\{\eta_\alpha\} &= \bigcap \{U_\alpha^p(n^p) : p \in G_{\mathbb{P}} \wedge \alpha \in w^p\}, \\ \{\nu_{i, \alpha, \beta}\} &= \bigcap \{W_{i, \alpha, \beta}^p : p \in G_{\mathbb{P}} \wedge \alpha, \beta \in w^p\} \\ h_{\alpha, \beta} &= h^p(\alpha, \beta) \text{ for some (all) } p \in G_{\mathbb{P}} \text{ such that } \alpha, \beta \in w^p.” \end{aligned}$$

(2) For $m < \omega$ let \mathbf{F}_m be a \mathbb{P} -name such that

$$\Vdash_{\mathbb{P}} \text{“}\mathbf{F}_m = \bigcap \{F(p, m) : p \in G_{\mathbb{P}} \wedge m < M^p\}.”$$

(Remember $F(p, m)$ was defined in Definition 5.3(B).)

Lemma 5.9.

(1) For each $m < \omega$, $\Vdash_{\mathbb{P}} \text{“}\mathbf{F}_m \text{ is a closed subset of } \mathbb{H}.”$

(2) For $i < k$ and $(\alpha, \beta) \in \lambda^{(2)}$ we have

$$\Vdash_{\mathbb{P}} \text{“}\eta_\alpha, \nu_{i, \alpha, \beta} \in \mathbb{H}, \quad h_{\alpha, \beta} < \omega, \quad \nu_{i, \alpha, \beta} = \nu_{i, \beta, \alpha} \quad \text{and} \quad \eta_\alpha + \nu_{i, \alpha, \beta} \in \mathbf{F}_{h_{\alpha, \beta}}.”$$

(3) $\Vdash_{\mathbb{P}} \text{“}\langle \eta_\alpha, \nu_{i, \alpha, \beta} : i < k, \alpha < \beta < \lambda \rangle \text{ is quasi independent (so they are also distinct).”}$

(4) $\Vdash_{\mathbb{P}} \text{“}\left| \left(-\eta_\alpha + \bigcup_{m < \omega} \mathbf{F}_m \right) \cap \left(-\eta_\beta + \bigcup_{m < \omega} \mathbf{F}_m \right) \right| \geq k.”$

Proof. Should be clear (remember Lemma 5.4). \square

Lemma 5.10. Let $p = (w, M, \bar{r}, n, \bar{\Upsilon}, \bar{V}, h) \in D_1^1 \subseteq \mathbb{P}$ (cf. 5.4(2)(ii)) and $a_\ell, b_\ell \in \mathbb{H}$ and $U_\ell, W_\ell \in \mathcal{U}$ (for $\ell < 4$) be such that the following conditions are satisfied.

- (\otimes)₁ $U_\ell \in \{U_\alpha(n) : \alpha \in w\}$, $W_\ell \in \{W_{i,\alpha,\beta} : i < k, (\alpha, \beta) \in w^{(2)}\}$ (for $\ell < 4$).
- (\otimes)₂ • $(U_0 + W_0) \cap (U_1 + W_1) = \emptyset$,
 • $(U_1 + W_1) \cap (U_3 + W_3) = \emptyset$,
 • $(U_2 + W_2) \cap (U_3 + W_3) = \emptyset$,
 • $(U_0 + W_0) \cap (U_2 + W_2) = \emptyset$.
- (\otimes)₃ $a_\ell \in U_\ell$ and $b_\ell \in W_\ell$ and $a_\ell + b_\ell \in \bigcup_{m < M} F(p, m)$ for $\ell < 4$.
- (\otimes)₄ $(a_0 + b_0) - (a_1 + b_1) = (a_2 + b_2) - (a_3 + b_3)$.

Then for some $(\alpha, \beta) \in w^{(2)}$ and distinct $i, j < k$ we have

either $U_0 = U_2 = U_\alpha(n)$, $U_1 = U_3 = U_\beta(n)$, $W_0 = W_1 = W_{i,\alpha,\beta}$, and $W_2 = W_3 = W_{j,\alpha,\beta}$,
 or $U_0 = U_1 = U_\alpha(n)$, $U_2 = U_3 = U_\beta(n)$, $W_0 = W_2 = W_{i,\alpha,\beta}$, and $W_1 = W_3 = W_{j,\alpha,\beta}$.

Proof. For $\ell < 4$ let U_ℓ^- and V_ℓ be such that

- if $U_\ell = U_\alpha(n)$ then $U_\ell^- = U_\alpha(n-1)$,
- if $W_\ell = W_{i,\alpha,\beta}$ then $V_\ell = V_{i,\alpha,\beta}$.

Also, let

$$\text{LHS}_a = a_0 - a_1 - a_2 + a_3, \quad \text{LHS}_b = b_0 - b_1 - b_2 + b_3, \quad \text{and} \quad \text{LHS} = \text{LHS}_a + \text{LHS}_b = 0.$$

Put $\mathcal{U}^* = \{U_0, U_1, U_2, U_3\}$, $\mathcal{W}^* = \{W_0, W_1, W_2, W_3\}$, $\mathcal{U}_-^* = \{U_0^-, U_1^-, U_2^-, U_3^-\}$, and $\mathcal{V}^* = \{V_0, V_1, V_2, V_3\}$.

(a) $|\mathcal{U}^*| > 1$.

Why? If not, then $U_0 = U_1 = U_2 = U_3$ and by the assumption (\otimes)₂ of the Lemma we have $\{W_0, W_3\} \cap \{W_1, W_2\} = \emptyset$. By 5.3(A)(\boxtimes)₄, the latter also means that $\{V_0, V_3\} \cap \{V_1, V_2\} = \emptyset$. Now,

$$\text{LHS} = ((a_0 - a_1) + b_0) + ((a_3 - a_2) + b_3) - b_1 - b_2,$$

and using 5.3(A)(\boxtimes)₄(b) we have $(a_0 - a_1) + b_0 \in V_0$, $(a_3 - a_2) + b_3 \in V_3$, $b_1 \in V_1$ and $b_2 \in V_2$. Since $\{V_0, V_3\} \cap \{V_1, V_2\} = \emptyset$ we see that LHS is a nontrivial (2, 4)-combination from \mathcal{V}^* , so it cannot be 0, contradicting assumption (\otimes)₄ of the Lemma.

(b) $|\mathcal{W}^*| > 1$.

Why? Fully parallel to (a).

(c) If for some W we have $|\{\ell < 4 : W_\ell = W\}| = 3$, then LHS_b is a nontrivial (2, 4)-combination from \mathcal{V}^* .

Why? Suppose $W_0 = W_1 = W_2 \neq W_3$. Then, by 5.3(A)(\boxtimes)₄(b), we have $(b_1 - b_0) + b_2 \in V_2$ and $b_3 \in V_3$. Hence $\text{LHS}_b = -((b_1 - b_0) + b_2) + b_3 \in -V_2 + V_3$ and $V_2 \neq V_3$.

Suppose $W_0 = W_1 = W_3 \neq W_2$. Then, by 5.3(A)(\boxtimes)₄(b), we have $(b_0 - b_1) + b_3 \in V_3$ and $b_2 \in V_2$, so $\text{LHS}_b = (b_0 - b_1) + b_3 - b_2 \in V_3 - V_2$ and $V_2 \neq V_3$.

The other cases are fully parallel.

(d) If for some U we have $|\{\ell < 4 : U_\ell = U\}| = 3$, then LHS_a is a nontrivial (2, 4)-combination from \mathcal{U}_-^* .

Why? Same argument as for (c), just using U_ℓ instead of W_ℓ .

(e) For every W we have $|\{\ell < 4 : W_\ell = W\}| < 3$.

Why? We already know that $|\{\ell < 4 : W_\ell = W\}| < 4$ (by (b)), so suppose $\{\ell < 4 : W_\ell = W\}$ has exactly 3 elements. It follows from (c) that then LHS_b is a nontrivial $(2, 4)$ -combination from \mathcal{V}^* . By (a) we know that $|\mathcal{U}^*| > 1$. If for some U we have $|\{\ell < 4 : U_\ell = U\}| = 3$, then we may use (d) to claim that LHS_a is a (nontrivial) $(2, 4)$ -combination from \mathcal{U}^* and then LHS is a nontrivial $(2, 8)$ -combination from $\mathcal{U}^* \cup \mathcal{V}^*$, contradicting $(\otimes)_4$ (remember 5.3(A)(\boxtimes) $_4$).

So suppose that for each U we have $|\{\ell < 4 : U_\ell = U\}| \leq 2$. Then LHS_a is a (possibly trivial) $(2, 4)$ -combination from \mathcal{U}^* and consequently LHS is a nontrivial $(2, 8)$ -combination from $\mathcal{U}^* \cup \mathcal{V}^*$, so also from $\mathcal{U}^* \cup \mathcal{V}^*$, again contradicting $(\otimes)_4$.

(f) For each U , $|\{\ell < 4 : U_\ell = U\}| < 3$.

Why? Same argument as for (e), just using (a) and (c) instead of (b) and (d).

Since $p \in D_1^1$, it follows from our assumption $(\otimes)_3$ that for each $\ell < 4$, for some $\alpha = \alpha(\ell), \beta = \beta(\ell)$, and $i = i(\ell)$ we have $U_\ell = U_\alpha(n)$ and $W_\ell = W_{i,\alpha,\beta}$. It follows from (e)+(f) that LHS is a $(2, 8)$ -combination from $\mathcal{U}^* \cup \mathcal{W}^*$. Necessarily it is a trivial combination (as $\text{LHS} = 0$ by $(\otimes)_4$). Consequently,

- $(\odot)_1$ either $U_0 = U_1 \neq U_2 = U_3$, or $U_0 = U_2 \neq U_1 = U_3$, and
- $(\odot)_2$ either $W_0 = W_1 \neq W_2 = W_3$, or $W_0 = W_2 \neq W_1 = W_3$.

Suppose $U_0 = U_1 \neq U_2 = U_3$. Then by $(\otimes)_2$ we must have $W_0 \neq W_1, W_2 \neq W_3$ and by $(\odot)_2$ we get $W_0 = W_2$ and $W_1 = W_3$. Thus for some $(\alpha, \beta) \in w^{(2)}$ and $i, j < k, i \neq j$, we have

$$U_0 = U_1 = U_\alpha(n), \quad U_2 = U_3 = U_\beta(n), \quad W_0 = W_2 = W_{i,\alpha,\beta}, \quad W_1 = W_3 = W_{j,\alpha,\beta}.$$

Suppose now that $U_0 = U_2$ and $U_1 = U_3$. By $(\otimes)_2$ we must have then $W_0 \neq W_2$ and $W_1 \neq W_3$. Therefore, by $(\odot)_2$, we may conclude that $W_0 = W_1$ and $W_2 = W_3$. Consequently, for some $(\alpha, \beta) \in w^{(2)}$ and distinct $i, j < k$ we have

$$U_0 = U_2 = U_\alpha(n), \quad U_1 = U_3 = U_\beta(n), \quad W_0 = W_1 = W_{i,\alpha,\beta}, \quad W_2 = W_3 = W_{j,\alpha,\beta}. \quad \square$$

Lemma 5.11. Let $p = (w, M, \bar{r}, n, \bar{\Upsilon}, \bar{V}, h) \in D_1^1$ and $\mathbf{X} \subseteq \mathbb{H}$, $|\mathbf{X}| \geq 5$. Suppose that $a_i(x, y), b_i(x, y), U_i(x, y)$ and $W_i(x, y)$ for $x, y \in \mathbf{X}$, $x \neq y$ and $i < k$ satisfy the following demands (i)–(iv) (for all $x \neq y, i \neq i'$).

- (i) $U_i(x, y) \in \{U_\alpha(n) : \alpha \in w\}$, $W_i(x, y) \in \{W_{j,\alpha,\beta} : j < k, (\alpha, \beta) \in w^{(2)}\}$.
- (ii) • $(U_i(x, y) + W_i(x, y)) \cap (U_i(y, x) + W_i(y, x)) = \emptyset$,
• $(U_i(x, y) + W_i(x, y)) \cap (U_{i'}(x, y) + W_{i'}(x, y)) = \emptyset$.
- (iii) $a_i(x, y) \in U_i(x, y)$ and $b_i(x, y) \in W_i(x, y)$, and
 $a_i(x, y) + b_i(x, y) \in \bigcup_{m < M} F(p, m)$.
- (iv) $x - y = (a_i(x, y) + b_i(x, y)) - (a_i(y, x) + b_i(y, x))$.

Then

$$(1) \quad \mathbf{X} - \mathbf{X} \subseteq \bigcup \{U_\alpha(n - 2) - U_\beta(n - 2) : \alpha, \beta \in w\}.$$

(2) If $(x, y) \in \mathbf{X}^{(2)}$ and $x - y \in U_\alpha(n-2) - U_\beta(n-2)$, $\alpha, \beta \in w$, then $\alpha \neq \beta$ and for each $i < k$ we have $a_i(x, y) + b_i(x, y), a_i(y, x) + b_i(y, x) \in F(p, h(\alpha, \beta))$.

Proof. (1) Fix $x, y \in \mathbf{X}$, $x \neq y$, for a moment.

Let $i \neq i'$, $i, i' < k$. We may apply Lemma 5.10 for $U_i(x, y), W_i(x, y), U_i(y, x), W_i(y, x), a_i(x, y), b_i(x, y), a_i(y, x), b_i(y, x)$ here as $U_0, W_0, U_1, W_1, a_0, b_0, a_1, b_1$ there and for similar objects with i' in place of i as $U_2, W_2, U_3, W_3, a_2, b_2, a_3, b_3$ there. This will produce distinct $\alpha = \alpha(x, y, i, i'), \beta = \beta(x, y, i, i') \in w$ and distinct $j = j(x, y, i, i'), j' = j'(x, y, i, i') < k$ such that

either (A) $U_i(x, y) = U_{i'}(x, y) = U_\alpha(n), U_i(y, x) = U_{i'}(y, x) = U_\beta(n),$
 $W_i(x, y) = W_i(y, x) = W_{j, \alpha, \beta}, W_{i'}(x, y) = W_{i'}(y, x) = W_{j', \alpha, \beta},$
 or (B) $U_i(x, y) = U_i(y, x) = U_\alpha(n), U_{i'}(x, y) = U_{i'}(y, x) = U_\beta(n),$
 $W_i(x, y) = W_{i'}(x, y) = W_{j, \alpha, \beta}, W_i(y, x) = W_{i'}(y, x) = W_{j', \alpha, \beta}.$

Note that if for some $i \neq i'$, $i, i' < k$, the possibility (A) above holds, then it holds for all $i, i' < k$ and

$$x - y = (a_i(x, y) + b_i(x, y)) - (a_i(y, x) + b_i(y, x)) = \left(a_i(x, y) + (b_i(x, y) - b_i(y, x)) \right) - a_i(y, x)$$

and $a_i(x, y) + (b_i(x, y) - b_i(y, x)) \in U_\alpha(n) + (W_{j, \alpha, \beta} - W_{j', \alpha, \beta}) \subseteq U_\alpha(n-1) \subseteq U_\alpha(n-2)$. Hence $x - y \in U_\alpha(n-2) - U_\beta(n-2)$.

Now unfix x, y . By what we have said, the first assertion of the Lemma will follow once we show that

(\heartsuit) for all $x, y \in \mathbf{X}$, $x \neq y$, there are $i \neq i'$ such that possibility (A) above holds for them.

Here the argument breaks into two cases: $k \geq 3$ and $k = 2$, with the former being somewhat simpler.

CASE $k \geq 3$.

Let $x, y \in \mathbf{X}$, $x \neq y$. Suppose towards contradiction that in the previous considerations both for $x, y, 0, 1$ and for $x, y, 1, 2$ the second (i.e., (B)) possibility takes place. This gives us α, β, j, j' such that $\alpha \neq \beta, j \neq j'$ and

- (*)₁ $U_0(x, y) = U_0(y, x) = U_\alpha(n),$
- (*)₂ $U_1(x, y) = U_1(y, x) = U_\beta(n),$
- (*)₃ $W_0(x, y) = W_1(x, y) = W_{j, \alpha, \beta},$
- (*)₄ $W_0(y, x) = W_1(y, x) = W_{j', \alpha, \beta},$

and we also get $\gamma, \delta, \ell, \ell'$ such that $\gamma \neq \delta$ and $\ell \neq \ell'$ and

- (*)₅ $U_1(x, y) = U_1(y, x) = U_\gamma(n),$
- (*)₆ $U_2(x, y) = U_2(y, x) = U_\delta(n),$
- (*)₇ $W_1(x, y) = W_2(x, y) = W_{\ell, \gamma, \delta},$
- (*)₈ $W_1(y, x) = W_2(y, x) = W_{\ell', \gamma, \delta}.$

It follows from (*)₂ + (*)₅ that $\gamma = \beta$ and from (*)₃ + (*)₇ we have $\ell = j$ and $\delta = \alpha$. Finally, (*)₄ + (*)₈ imply $\ell' = j'$. Consequently,

$$U_0(x, y) = U_2(x, y), U_0(y, x) = U_2(y, x), W_0(x, y) = W_2(x, y), W_0(y, x) = W_2(y, x),$$

contradicting assumption (ii).

CASE $k = 2$.

We will argue that (\heartsuit) holds true in this case as well. First, however, we have to establish some auxiliary facts.

For each $x, y \in \mathbf{X}$, $x \neq y$, we may choose $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ and $j = j(x, y)$ such that $\alpha \neq \beta$ and

$$\begin{aligned} \text{either } (A)_{x,y}^{\alpha,\beta,j} \quad & U_0(x, y) = U_1(x, y) = U_\alpha(n), \quad U_0(y, x) = U_1(y, x) = U_\beta(n), \\ & W_0(x, y) = W_0(y, x) = W_{j,\alpha,\beta}, \quad W_1(x, y) = W_1(y, x) = W_{1-j,\alpha,\beta}, \\ \text{or } (B)_{x,y}^{\alpha,\beta,j} \quad & U_0(x, y) = U_0(y, x) = U_\alpha(n), \quad U_1(x, y) = U_1(y, x) = U_\beta(n), \\ & W_0(x, y) = W_1(x, y) = W_{j,\alpha,\beta}, \quad W_0(y, x) = W_1(y, x) = W_{1-j,\alpha,\beta}, \end{aligned}$$

Note that for each $(x, y) \in w^{(2)}$, either there are α, β, j such that $(A)_{x,y}^{\alpha,\beta,j}$ holds true or there are α, β, j such that $(B)_{x,y}^{\alpha,\beta,j}$ is true, but not both. Also, remembering 5.3(A)(\boxtimes)₄(b),

$$(\triangle)_1 \text{ if } (A)_{x,y}^{\alpha,\beta,j} \text{ holds, then } x - y \in U_\alpha(n-1) - U_\beta(n-1) \text{ and if } (B)_{x,y}^{\alpha,\beta,j} \text{ is satisfied, then } x - y \in V_{j,\alpha,\beta} - V_{1-j,\alpha,\beta}.$$

Define functions $\chi : \mathbf{X}^{(2)} \rightarrow 2$ and $\Theta : \mathbf{X}^{(2)} \rightarrow [w]^2 \times 2$ as follows. Assuming $(x, y) \in \mathbf{X}^{(2)}$,

- if for some α, β, j the demand $(A)_{x,y}^{\alpha,\beta,j}$ holds, then $\chi(x, y) = 1$ and $\Theta(x, y) = (\{\alpha, \beta\}, j)$,
- if for some α, β, j the demand $(B)_{x,y}^{\alpha,\beta,j}$ is satisfied, then $\chi(x, y) = 0$ and $\Theta(x, y) = (\{\alpha, \beta\}, j)$.

Our goal is to show that the function χ never takes value 0 (as this will imply that the assertion (\heartsuit) holds true). Note that

$$(\triangle)_2 \text{ if } \chi(x, y) = 0 \text{ and } \Theta(x, y) = (\{\alpha, \beta\}, j), \text{ then } \chi(y, x) = 0 \text{ and } \Theta(y, x) = (\{\alpha, \beta\}, 1-j), \text{ so also } \Theta(x, y) \neq \Theta(y, x).$$

Also,

$$(\triangle)_3 \text{ if } x, y, z \in \mathbf{X} \text{ are pairwise distinct and } \chi(x, y) = \chi(y, z) = 1, \text{ then } \chi(x, z) = 1.$$

Why? Assume $\chi(x, z) = 0$. Then, by $(\triangle)_1$, for some j, ξ, ζ we have $x - z \in V_{j,\xi,\zeta} - V_{1-j,\xi,\zeta}$. However, $x - y \in U_\alpha(n-1) - U_\beta(n-1)$ and $y - z \in U_\gamma(n-1) - U_\delta(n-1)$ (for some $\alpha \neq \beta$ and $\gamma \neq \delta$), so

$$x - z \in U_\alpha(n-1) - U_\beta(n-1) + U_\gamma(n-1) - U_\delta(n-1).$$

Thus for some $a \in U_\alpha(n-1)$, $b \in U_\beta(n-1)$, $c \in U_\gamma(n-1)$, $d \in U_\delta(n-1)$, $e \in V_{j,\xi,\zeta}$, and $f \in V_{1-j,\xi,\zeta}$ we have $a - b + c - d + f - e = 0$. The left hand side of this equation represents a nontrivial $(2,8)$ -combination from $\langle U_\zeta(n-1) : \zeta \in w \rangle \smallfrown \langle V_{0,\zeta,\zeta'}, V_{1,\zeta,\zeta'} : (\zeta, \zeta') \in w^{(2)} \rangle$ (remember $\alpha \neq \beta$, $\gamma \neq \delta$, $\xi \neq \zeta$), a contradiction.

$$(\triangle)_4 \text{ If } x, y, z \in \mathbf{X} \text{ are pairwise distinct and } \chi(x, y) = \chi(y, z) = 0, \text{ then } \Theta(x, y) = \Theta(y, z) = \Theta(z, x) \text{ and } \chi(x, z) = 0.$$

Why? Let $\Theta(x, y) = (\{\alpha, \beta\}, i)$, $\Theta(y, z) = (\{\gamma, \delta\}, j)$, and $\Theta(x, z) = (\{\xi, \zeta\}, \ell)$. If $\{\alpha, \beta\} \neq \{\gamma, \delta\}$, then

$$x - z = (x - y) + (y - z) \in V_{i,\alpha,\beta} - V_{1-i,\alpha,\beta} + V_{j,\gamma,\delta} - V_{1-j,\gamma,\delta}$$

and $\{V_{0,\alpha,\beta}, V_{1,\alpha,\beta}\} \cap \{V_{0,\gamma,\delta}, V_{1,\gamma,\delta}\} = \emptyset$. Since either $x - z \in V_{\ell,\xi,\zeta} - V_{1-\ell,\xi,\zeta}$ or $x - z \in (U_\xi(n-1) - U_\zeta(n-1)) \cup (U_\zeta(n-1) - U_\xi(n-1))$, we easily get that some nontrivial (2,8)-combination from $\langle U_\zeta(n-1) : \zeta \in w \rangle \smallfrown \langle V_{0,\zeta,\zeta'}, V_{1,\zeta,\zeta'} : (\zeta, \zeta') \in w^{(2)} \rangle$ equals 0, a contradiction. Consequently, $\{\alpha, \beta\} = \{\gamma, \delta\}$, i.e., $\Theta(x, y) = (\{\alpha, \beta\}, i)$ and $\Theta(y, z) = (\{\alpha, \beta\}, j)$ for some α, β, i, j .

If $i \neq j$ then $x - z \in V_{i,\alpha,\beta} - V_{1-i,\alpha,\beta} + V_{1-i,\alpha,\beta} - V_{i,\alpha,\beta}$. But also either $x - z \in U_\xi(n-1) - U_\zeta(n-1)$, or $x - z \in U_\zeta(n-1) - U_\xi(n-1)$, or $x - z \in V_{\ell,\xi,\zeta} - V_{1-\ell,\xi,\zeta}$. In the first case we get

$$0 \in \left((V_{i,\alpha,\beta} - V_{i,\alpha,\beta}) + U_\xi(n-1) \right) - \left((V_{1-i,\alpha,\beta} - V_{1-i,\alpha,\beta}) + U_\zeta(n-1) \right) \subseteq U_\xi(n-2) - U_\zeta(n-2),$$

and symmetrically in the second case. In the last case we have

$$0 \in \left((V_{i,\alpha,\beta} - V_{i,\alpha,\beta}) + V_{\ell,\xi,\zeta} \right) - \left((V_{1-i,\alpha,\beta} - V_{1-i,\alpha,\beta}) + V_{1-\ell,\xi,\zeta} \right) \subseteq Q_{\ell,\xi,\zeta} - Q_{1-\ell,\xi,\zeta}.$$

In any case this gives a contradiction with 5.3(A)(\boxtimes)₄. Consequently $i = j$ and $\Theta(x, y) = \Theta(y, z) = (\{\alpha, \beta\}, i)$.

By considerations as above we see that necessarily $\chi(x, z) = 0$ and $\Theta(x, z) = (\{\alpha, \beta\}, \ell)$. If $\ell = i$, then

$$x - z \in V_{i,\alpha,\beta} - V_{1-i,\alpha,\beta} \quad \text{and} \quad x - z \in V_{i,\alpha,\beta} - V_{1-i,\alpha,\beta} + V_{i,\alpha,\beta} - V_{1-i,\alpha,\beta}.$$

Hence

$$0 \in \left((V_{i,\alpha,\beta} - V_{i,\alpha,\beta}) + V_{i,\alpha,\beta} \right) - \left((V_{1-i,\alpha,\beta} - V_{1-i,\alpha,\beta}) + V_{1-i,\alpha,\beta} \right) \subseteq Q_{i,\alpha,\beta} - Q_{1-i,\alpha,\beta},$$

a contradiction.

Consequently, $\ell = 1 - i$ and $\Theta(z, x) = (\{\alpha, \beta\}, i) = \Theta(x, y)$ (and $\chi(x, z) = 0$).

Now, suppose towards contradiction that (\heartsuit) is not true and $x, y \in \mathbf{X}$ are such that $x \neq y$ and $\chi(x, y) = 0$. Let $z \in \mathbf{X} \setminus \{x, y\}$. We cannot have $\chi(x, z) = \chi(y, z) = 1$ (as then (Δ)₃ would give a contradiction with $\chi(x, y) = 0$). So one of them is 0, and then (Δ)₄ implies that the other is 0 as well and

$$\chi(x, y) = \chi(y, z) = \chi(x, z) = 0 \quad \text{and} \quad \Theta(x, y) = \Theta(y, z) = \Theta(z, x).$$

Taking $t \in \mathbf{X} \setminus \{x, y, z\}$ by similar considerations we obtain

$$\chi(x, t) = \chi(y, t) = 0 \quad \text{and} \quad \Theta(x, y) = \Theta(y, t) = \Theta(t, x).$$

Now consider x, z, t : since $\chi(x, z) = \chi(x, t) = 0$ we may use (Δ)₄ to conclude that

$$\chi(z, t) = 0 \quad \text{and} \quad \Theta(x, z) = \Theta(z, t) = \Theta(t, x).$$

But we have established already that $\Theta(t, x) = \Theta(x, y) = \Theta(z, x)$, a contradiction (remember (Δ)₂). The proof of Lemma 5.11(1) is complete now.

(2) Suppose $(x, y) \in \mathbf{X}^{(2)}$. In the previous part we showed that for all $i < i' < k$ possibility (A) holds true. More precisely, there are distinct $\alpha, \beta \in w$ such that for all $i < k$ for some $j < k$ we have $a_i(x, y) \in U_\alpha(n)$ and $a_i(y, x) \in U_\beta(n)$, and $b_i(x, y), b_i(y, x) \in W_{j,\alpha,\beta}$. Then also

- $a_i(x, y) + b_i(x, y) \in U_\alpha(n) + W_{j,\alpha,\beta} \subseteq F(p, h(\alpha, \beta)),$
- $a_i(y, x) + b_i(y, x) \in U_\beta(n) + W_{j,\alpha,\beta} \subseteq F(p, h(\alpha, \beta)).$

We also know that for these α, β we have $x - y \in U_\alpha(n-2) - U_\beta(n-2)$. To complete the proof we note that, by Lemma 5.5, for any α', β'

$$(U_\alpha(n-2) - U_\beta(n-2)) \cap (U_{\alpha'}(n-2) - U_{\beta'}(n-2)) \neq \emptyset \text{ implies } \alpha = \alpha' \text{ and } \beta = \beta'. \quad \square$$

Lemma 5.12.

$$\Vdash_{\mathbb{P}} \text{“there is no perfect set } P \subseteq \mathbb{H} \text{ such that } (\forall x, y \in P) (|(x + \bigcup_{m < \omega} \mathbf{F}_m) \cap (y + \bigcup_{m < \omega} \mathbf{F}_m)| \geq k).”$$

Proof. Suppose towards contradiction that $G \subseteq \mathbb{P}$ is generic over \mathbf{V} and in $\mathbf{V}[G]$ the following assertion holds true:

for some perfect set $P \subseteq \mathbb{H}$ we have

$$\left| (x + \bigcup_{m < \omega} \mathbf{F}_m^G) \cap (y + \bigcup_{m < \omega} \mathbf{F}_m^G) \right| \geq k$$

for all $x, y \in P$.

Then for any distinct $x, y \in P$ there are $b_0, c_0, \dots, b_{k-1}, c_{k-1} \in \bigcup_{m < \omega} \mathbf{F}_m^G$ such that $b_i \neq b_j$ whenever $i \neq j$ and $x - y = b_i - c_i$ (for all $i < k$).

For $\bar{\ell} = \langle \ell_i : i < k \rangle \subseteq \omega$, $\bar{m} = \langle m_i : i < k \rangle \subseteq \omega$ and $N < \omega$ let

$$Z_{\bar{\ell}, \bar{m}}^N = \{(x, y) \in P^2 : \text{there are } b_i \in \mathbf{F}_{\ell_i}^G, c_i \in \mathbf{F}_{m_i}^G \text{ (for } i < k) \text{ such that } \\ x - y = b_i - c_i \text{ and } 2^{-N} < \min(\rho(b_i, b_j), \rho(c_i, c_j)) \text{ for all distinct } i, j < k\}.$$

By our assumption on P we know that

$$(\square)_0 \text{ for all } x, y \in P, x \neq y, \text{ there are } \bar{\ell}, \bar{m} \text{ and } N \text{ such that } (x, y) \in Z_{\bar{\ell}, \bar{m}}^N.$$

Remember our Assumption 5.1: \mathbf{D} is a fixed countable dense subset of \mathbb{H} and \mathbf{B}_n is the ball of radius 2^{-n} centered at 0. Now, the sets $Z_{\bar{\ell}, \bar{m}}^N \subseteq P^2$ are Σ_1^1 , so they have the Baire property (in P^2). Therefore, for every open set $U \subseteq \mathbb{H} \times \mathbb{H}$ with $U \cap P^2 \neq \emptyset$ there are $d_0, d_1 \in \mathbf{D}$ and the corresponding basic open set $(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1}) \subseteq U$ such that $[(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1})] \cap P^2 \neq \emptyset$ and

- either $Z_{\bar{\ell}, \bar{m}}^N \cap [(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1})]$ is a meager subset of P^2 ,
- or $[(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1})] \cap P^2 \setminus Z_{\bar{\ell}, \bar{m}}^N$ is a meager subset of P^2 .

Now we may choose closed nowhere dense subsets F_j of P^2 (for $j < \omega$) such that for each $d_0, d_1 \in \mathbf{D}$ and $n_0, n_1 < \omega$ and $N, \bar{\ell}, \bar{m}$ as before we have

$$(\square)_1^a \text{ if } Z_{\bar{\ell}, \bar{m}}^N \cap [(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1})] \text{ is meager in } P^2, \text{ then}$$

$$Z_{\bar{\ell}, \bar{m}}^N \cap [(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1})] \subseteq \bigcup_{j < \omega} F_j,$$

$(\square)_1^b$ if $[(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1})] \cap P^2 \setminus Z_{\bar{\ell}, \bar{m}}^N$ is meager in P^2 , then

$$[(d_0 + \mathbf{B}_{n_0}) \times (d_1 + \mathbf{B}_{n_1})] \cap P^2 \setminus Z_{\bar{\ell}, \bar{m}}^N \subseteq \bigcup_{j < \omega} F_j.$$

Then $\bigcup_{j < \omega} F_j$ is a meager subset of P^2 . Let $B^* = P^2 \setminus \bigcup_{j < \omega} F_j$.

We are going to choose now a sequence $0 = n_0^* = n_0 < n_1^* < n_1 < n_2^* < n_2 < n_3^* < n_3 < \dots$ and a system $\langle d_\sigma : \sigma \in {}^{\iota}2, \iota < \omega \rangle \subseteq \mathbf{D}$ such that the following demands $(\square)_2^a - (\square)_2^e$ are satisfied.

$(\square)_2^a$ If $\iota < \omega$, $\sigma, \sigma' \in {}^{\iota}2$, $\sigma \neq \sigma'$, then $(d_\sigma + \mathbf{B}_{n_\iota}) \cap P \neq \emptyset$ and $\rho(d_\sigma, d_{\sigma'}) > 2^{2-n_\iota}$.

$(\square)_2^b$ If $\iota < \omega$, $\sigma \in {}^{\iota}2$, then $\text{cl}(d_{\sigma \restriction \langle 0 \rangle} + \mathbf{B}_{n_{\iota+1}}) \cup \text{cl}(d_{\sigma \restriction \langle 1 \rangle} + \mathbf{B}_{n_{\iota+1}}) \subseteq (d_\sigma + \mathbf{B}_{n_\iota})$.

$(\square)_2^c$ If $\iota < \omega$ and $\sigma, \sigma' \in {}^{\iota}2$, $\sigma \neq \sigma'$, and

$$(x, y), (x', y') \in B^* \cap [(d_\sigma + \mathbf{B}_{n_\iota}) \times (d_{\sigma'} + \mathbf{B}_{n_\iota})],$$

then for all $\bar{\ell} \subseteq n_\iota^*$, $\bar{m} \subseteq n_\iota^*$ and $N < n_\iota^*$ we have

$$(x, y) \in Z_{\bar{\ell}, \bar{m}}^N \Leftrightarrow (x', y') \in Z_{\bar{\ell}, \bar{m}}^N.$$

$(\square)_2^d$ If $\iota < \omega$ and $\sigma, \sigma' \in {}^{\iota}2$, $\sigma \neq \sigma'$, and $(x, y) \in B^* \cap [(d_\sigma + \mathbf{B}_{n_\iota}) \times (d_{\sigma'} + \mathbf{B}_{n_\iota})]$, then there are $\bar{\ell} \subseteq n_\iota^*$, $\bar{m} \subseteq n_\iota^*$ and $N < n_\iota^*$ such that $(x, y) \in Z_{\bar{\ell}, \bar{m}}^N$.

$(\square)_2^e$ If $\iota < \omega$ and $\sigma, \sigma' \in {}^{\iota}2$, $\sigma \neq \sigma'$, then $[(d_\sigma + \mathbf{B}_{n_\iota}) \times (d_{\sigma'} + \mathbf{B}_{n_\iota})] \cap \bigcup_{j < \iota} F_j = \emptyset$.

The construction is by induction on $\iota < \omega$. We start with choosing any $d_{\langle \rangle} \in \mathbf{D}$ such that $(d_{\langle \rangle} + \mathbf{B}_0) \cap P \neq \emptyset$. We also set $n_0 = n_0^* = 0$. Let us describe in more detail choices for $\iota = 1$ as they have all the ingredients used later. So, first find open sets $V^\dagger, V^{\dagger\dagger}$ such that $V^\dagger \cap P \neq \emptyset \neq V^{\dagger\dagger} \cap P$ and $\text{cl}(V^\dagger) \cup \text{cl}(V^{\dagger\dagger}) \subseteq (d_{\langle \rangle} + \mathbf{B}_0)$, $\text{cl}(V^\dagger) \cap \text{cl}(V^{\dagger\dagger}) = \emptyset$. Let $N_0, \bar{\ell}_0, \bar{m}_0$ be such that the set $Z_{\bar{\ell}_0, \bar{m}_0}^{N_0} \cap [V^\dagger \times V^{\dagger\dagger}]$ is not meager in P^2 and let n_1^* be such that $N_0 < n_1^*$, $\bar{\ell}_0 \subseteq n_1^*$ and $\bar{m}_0 \subseteq n_1^*$. Now we repeatedly use the Baire property of the sets $Z_{\bar{\ell}, \bar{m}}^N$ to find open sets $V' \subseteq V^\dagger$ and $V'' \subseteq V^{\dagger\dagger}$ such that $V' \cap P \neq \emptyset \neq V'' \cap P$ and

(A) $[(V' \times V'') \cap P^2] \setminus Z_{\bar{\ell}_0, \bar{m}_0}^{N_0}$ is meager in P^2 (where $N_0, \bar{\ell}_0, \bar{m}_0$ are the ones fixed above), and

(B) for every $\bar{\ell} \subseteq n_1^*$, $\bar{m} \subseteq n_1^*$ and $N < n_1^*$, either $[(V' \times V'') \cap P^2] \setminus Z_{\bar{\ell}, \bar{m}}^N$ is meager in P^2 , or $(V' \times V'') \cap Z_{\bar{\ell}, \bar{m}}^N$ is meager in P^2 .

Since F_0 is a nowhere dense subset of P^2 , we may find open sets $V^* \subseteq V'$ and $V^{**} \subseteq V''$ such that $V^* \cap P \neq \emptyset \neq V^{**} \cap P$ and $(V^* \times V^{**}) \cap F_0 = \emptyset$. Now, after fixing some $x \in V^* \cap P$ and $y \in V^{**} \cap P$ we choose $n > n_1^*$ so large that $\rho(x, y) > 2^{3-n}$ and $x + \mathbf{B}_n \subseteq V^*$ and $y + \mathbf{B}_n \subseteq V^{**}$. Then we set $n_1 = n + 2$ and choose $d_{\langle 0 \rangle} \in (x + \mathbf{B}_{n_1}) \cap \mathbf{D}$ and $d_{\langle 1 \rangle} \in (y + \mathbf{B}_{n_1}) \cap \mathbf{D}$. Note that $x \in d_{\langle 0 \rangle} + \mathbf{B}_{n_1} \subseteq x + \mathbf{B}_n$ and $y \in d_{\langle 1 \rangle} + \mathbf{B}_{n_1} \subseteq y + \mathbf{B}_n$.

Assuming $n_\iota^* < n_\iota < \omega$ and $\langle d_\sigma : \sigma \in {}^{\iota}2 \rangle \subseteq \mathbf{D}$ have been selected, we first pick open sets $\langle V_\varsigma^\dagger : \varsigma \in {}^{\iota+1}2 \rangle$ such that for all $\sigma \in {}^{\iota}2$ we have $V_{\sigma \restriction \langle 0 \rangle}^\dagger \cap P \neq \emptyset \neq V_{\sigma \restriction \langle 1 \rangle}^\dagger \cap P$, $\text{cl}(V_{\sigma \restriction \langle 0 \rangle}^\dagger) \cup \text{cl}(V_{\sigma \restriction \langle 1 \rangle}^\dagger) \subseteq (d_\sigma + \mathbf{B}_{n_\iota})$, $\text{cl}(V_{\sigma \restriction \langle 0 \rangle}^\dagger) \cap \text{cl}(V_{\sigma \restriction \langle 1 \rangle}^\dagger) = \emptyset$. Next, letting $\langle \langle \varsigma_j', \varsigma_j'' \rangle : j < j^* \rangle$ be an enumeration of $({}^{\iota+1}2)^{\langle 2 \rangle}$, we choose inductively open sets $V_\varsigma^\dagger = V_\varsigma^0 \supseteq V_\varsigma^1 \supseteq \dots \supseteq V_\varsigma^{j^*}$ and integers $n_\iota = N_\varsigma^0 \leq N_\varsigma^1 \leq \dots \leq N_\varsigma^{j^*}$ (for $\varsigma \in {}^{\iota+1}2$), as well as $N_j, \bar{\ell}_j, \bar{m}_j$, in such a manner that the following demands (a)–(d) are satisfied for all $j < j^*$.

(a) If $\varsigma \in {}^{\iota+1}2 \setminus \{\varsigma_j', \varsigma_j''\}$, then $V_\varsigma^{j+1} = V_\varsigma^j$ and $N_\varsigma^{j+1} = N_\varsigma^j$.

- (b) $N_j, \bar{\ell}_j, \bar{m}_j$ are such that the set $Z_{\bar{\ell}_j, \bar{m}_j}^{N_j} \cap [V_{\zeta'_j}^j \times V_{\zeta''_j}^j]$ is not meager in P^2 and $N_{\zeta'_j}^{j+1} = N_{\zeta''_j}^{j+1}$ is such that $N_j + N_{\zeta'_j}^j + N_{\zeta''_j}^j < N_{\zeta'_j}^{j+1}$, $\bar{\ell}_j \subseteq N_{\zeta'_j}^{j+1}$ and $\bar{m}_j \subseteq N_{\zeta'_j}^{j+1}$.
- (c) Open sets $V_{\zeta'_j}^{j+1} \subseteq V_{\zeta'_j}^j$ and $V_{\zeta''_j}^{j+1} \subseteq V_{\zeta''_j}^j$ are such that $V_{\zeta'_j}^{j+1} \cap P \neq \emptyset \neq V_{\zeta''_j}^{j+1} \cap P$ and
- (A) $[(V_{\zeta'_j}^{j+1} \times V_{\zeta''_j}^{j+1}) \cap P^2] \setminus Z_{\bar{\ell}_j, \bar{m}_j}^{N_j}$ is meager in P^2 (where $N_j, \bar{\ell}_j, \bar{m}_j$ are the ones fixed in (b) above).
- (d) $(V_{\zeta'_j}^{j+1} \times V_{\zeta''_j}^{j+1}) \cap \bigcup_{i \leq \iota} F_i = \emptyset$.

Then we set $n_{\iota+1}^* = \max\{N_{\zeta}^{j^*} : \zeta \in {}^{\iota+1}2\}$ and we choose inductively open sets

$$V_{\zeta}^{j^*} \supseteq V_{\zeta}^{j^*+1} \supseteq V_{\zeta}^{j^*+2} \supseteq \dots V_{\zeta}^{j^*+j^*}$$

(for $\zeta \in {}^{\iota+1}2$) so that the following conditions (e)–(f) are satisfied.

- (e) If $\zeta \in {}^{\iota+1}2 \setminus \{\zeta'_j, \zeta''_j\}$, then $V_{\zeta}^{j^*+j+1} = V_{\zeta}^{j^*+j}$.
- (f) Open sets $V_{\zeta'_j}^{j^*+j+1} \subseteq V_{\zeta'_j}^{j^*+j}$ and $V_{\zeta''_j}^{j^*+j+1} \subseteq V_{\zeta''_j}^{j^*+j}$ are such that $V_{\zeta'_j}^{j^*+j+1} \cap P \neq \emptyset \neq V_{\zeta''_j}^{j^*+j+1} \cap P$ and
- (B) for every $\bar{\ell} \subseteq n_{\iota+1}^*$, $\bar{m} \subseteq n_{\iota+1}^*$ and $N < n_{\iota+1}^*$, either $[(V_{\zeta'_j}^{j^*+j+1} \times V_{\zeta''_j}^{j^*+j+1}) \cap P^2] \setminus Z_{\bar{\ell}, \bar{m}}^N$ is meager in P^2 , or $(V_{\zeta'_j}^{j^*+j+1} \times V_{\zeta''_j}^{j^*+j+1}) \cap Z_{\bar{\ell}, \bar{m}}^N$ is meager in P^2 .

Next, we fix $x_{\zeta} \in V_{\zeta}^{2j^*} \cap P$ for $\zeta \in {}^{\iota+1}2$. Choose $n > n_{\iota+1}^*$ so large that

- $\rho(x_{\zeta}, x_{\zeta'}) > 2^{3-n}$ for distinct $\zeta, \zeta' \in {}^{\iota+1}2$, and
- $x_{\zeta} + \mathbf{B}_n \subseteq V_{\zeta}^{j^*}$ for all $\zeta \in {}^{\iota+1}2$.

Then we set $n_{\iota+1} = n + 2$ and choose $d_{\zeta} \in (x_{\zeta} + \mathbf{B}_{n_{\iota+1}}) \cap \mathbf{D}$.

This completes the description of the inductive construction.

It follows from $(\square)_2^a + (\square)_2^b$ that for each $\eta \in {}^{\omega}2$ the set $\bigcap_{\ell < \omega} d_{\eta \upharpoonright \ell} + \mathbf{B}_{n_{\ell}}$ is a singleton included in P . By

$(\square)_2^c$ we know that for $\eta \neq \eta'$

$$\bigcap_{\ell < \omega} (d_{\eta \upharpoonright \ell} + \mathbf{B}_{n_{\ell}}) \times \bigcap_{\ell < \omega} (d_{\eta' \upharpoonright \ell} + \mathbf{B}_{n_{\ell}}) \subseteq B^*.$$

For $\sigma \in {}^{\iota}2$ and $\ell < \omega$ let $\sigma *_{\ell} 0 = \sigma \smallfrown \underbrace{\langle 0, \dots, 0 \rangle}_{\ell}$ and let $x_{\sigma}^* \in \mathbb{H}$ be such that

$$(\square)_3 \quad \{x_{\sigma}^*\} = \bigcap_{\ell < \omega} (d_{\sigma *_{\ell} 0} + \mathbf{B}_{n_{\iota+\ell}}); \text{ so } x_{\sigma}^* \in P \text{ and if } \sigma \neq \sigma' \text{ are from } {}^{\iota}2 \text{ then } (x_{\sigma}^*, x_{\sigma'}^*) \in B^*.$$

Let $\underline{P}, \underline{F}_j, \underline{n}_{\iota}^*, \underline{n}_{\iota}, \underline{d}_{\sigma}, \underline{x}_{\sigma}^*$ be \mathbb{P} -names for the objects appearing in $(\square)_2 - (\square)_3$. Still working in $\mathbf{V}[G]$, we may choose a sequence $\langle p_{\iota}, q_{\iota} : \iota < \omega \rangle \subseteq G$ such that:

- $(\square)_4^a$ $p_0 \Vdash_{\mathbb{P}}$ “ \underline{P} is a perfect subset of \mathbb{H} , \underline{F}_j are closed nowhere dense subsets of P^2 , and $\underline{n}_{\iota}^*, \underline{n}_{\iota}, \underline{d}_{\sigma}, \underline{x}_{\sigma}^*$ have the properties stated in $(\square)_1^a - (\square)_1^b$, $(\square)_2^a - (\square)_2^c$, $(\square)_3$ ”, and
- $(\square)_4^b$ p_{ι} decides the values of $\underline{n}_{\iota}^*, \underline{n}_{\iota}$ and \underline{d}_{σ} for $\sigma \in {}^{\iota}2$, $\iota > 0$,
- $(\square)_4^c$ $p_{\iota} \leq q_{\iota} \leq p_{\iota+1}$ and $p_{\iota}, q_{\iota} \in D_{n_{\iota}}^2 \cap D_{0, n_{\iota}, n_{\iota}}^0 \cap G$ (see 5.4(2)) and $n^{p_{\iota}} + 10 < n^{q_{\iota}}$ and $w^{p_{\iota}} = w^{q_{\iota}}$.

The properties of conditions from \mathbb{P} stated in 5.3(A) are absolute, so they hold in $\mathbf{V}[G]$ as well (with \mathbf{B}_{ℓ} being \mathbf{B}_{ℓ}^G etc). Now, still working in $\mathbf{V}[G]$, for $0 < \iota < \omega$ let $\mathbf{X}_{\iota} = \{x_{\sigma}^* : \sigma \in {}^{\iota}2\}$. Note that $x_{\sigma}^* \neq x_{\sigma'}^*$, and

$(x_\sigma^*, x_{\sigma'}^*) \in B^*$ when $\sigma, \sigma' \in {}^\iota 2$ are distinct, and $\mathbf{X}_\iota \subseteq \mathbf{X}_{\iota'}$ when $\iota \leq \iota' < \omega$. It follows from $(\square)_2^d + (\square)_3$ that for $x, y \in \mathbf{X}_\iota$, $x \neq y$, we have $(x, y) \in Z_{\bar{\ell}, \bar{m}}^N$ for some $N = N(x, y) < n_\iota^*$, $\bar{\ell} = \bar{\ell}(x, y)$, $\bar{m} = \bar{m}(x, y) \subseteq n_\iota^*$. By clause $(\square)_2^c$, these $N(x, y), \bar{\ell}(x, y), \bar{m}(x, y)$ may be chosen in such a manner that

$(\square)_5$ if $\sigma, \sigma' \in {}^\iota 2$, $\iota^* < \iota$, $\sigma \upharpoonright \iota^* = \sigma' \upharpoonright \iota^*$ but $\sigma(\iota^*) \neq \sigma'(\iota^*)$, and $\varsigma = \sigma \upharpoonright (\iota^* + 1)$, $\varsigma' = \sigma' \upharpoonright (\iota^* + 1)$, then $\bar{\ell}(x_\sigma, x_{\sigma'}) = \bar{\ell}(x_\varsigma, x_{\varsigma'})$, $\bar{m}(x_\sigma, x_{\sigma'}) = \bar{m}(x_\varsigma, x_{\varsigma'})$, and $N(x_\sigma, x_{\sigma'}) = N(x_\varsigma, x_{\varsigma'})$.

Let $J < \omega$ be such that the arrow property $J \longrightarrow (10)_{2^{144}}^4$ holds true and fix a $\iota \geq J$ for a while.

Fix $x, y \in \mathbf{X}_\iota$, $x \neq y$, and let $N = N(x, y) < n_\iota^*$, $\bar{\ell} = \bar{\ell}(x, y)$, $\bar{m} = \bar{m}(x, y) \subseteq n_\iota^*$. Then there are $c_i \in \mathbf{F}_{\bar{\ell}_i}^G$ and $d_i \in \mathbf{F}_{\bar{m}_i}^G$ (for $i < k$) such that for $i \neq i'$ we have

$$x - y = c_i - d_i \quad \text{and} \quad 2^{-n_\iota} < 2^{-N} < \rho(c_i, c_{i'}), \quad \text{and} \quad 2^{-n_\iota} < 2^{-N} < \rho(d_i, d_{i'}).$$

The reasons for the use of q_ι rather than p_ι in what follows will become clear at the end. Since $n_\iota < M^{q_\iota}$ and $\mathbf{F}_m^G \subseteq F(q_\iota, m)$ for all $m < M^{q_\iota}$, we get $c_i \in U_\alpha^{q_\iota}(n^{q_\iota}) + W_{j, \alpha, \beta}^{q_\iota}$ for some $j < k$ and $(\alpha, \beta) \in (w^{q_\iota})^{(2)} = (w^{p_\iota})^{(2)}$ and similarly for d_i . Therefore, for each $i < k$ we may pick

- $U_i(x, y), U_i(y, x) \in \{U_\alpha^{q_\iota}(n^{q_\iota}) : \alpha \in w^{q_\iota}\}$, and
- $W_i(x, y), W_i(y, x) \in \{W_{j, \alpha, \beta}^{q_\iota} : (\alpha, \beta) \in (w^{q_\iota})^{(2)}\}$, and
- $a_i(x, y) \in U_i(x, y)$, $a_i(y, x) \in U_i(y, x)$ and $b_i(x, y) \in W_i(x, y)$, $b_i(y, x) \in W_i(y, x)$

such that $x - y = (a_i(x, y) + b_i(x, y)) - (a_i(y, x) + b_i(y, x))$ and for $i \neq i'$

$$\begin{aligned} 2^{-n_\iota} &< \rho(a_i(x, y) + b_i(x, y), a_{i'}(x, y) + b_{i'}(x, y)) \\ 2^{-n_\iota} &< \rho(a_i(y, x) + b_i(y, x), a_{i'}(y, x) + b_{i'}(y, x)). \end{aligned}$$

Since the metric ρ is invariant (and by $(\square)_2^a$), we also have

$$2^{-n_\iota} < \rho(x, y) = \rho(a_i(x, y) + b_i(x, y), a_i(y, x) + b_i(y, x)).$$

Since $q_\iota \in D_{n_\iota}^2$ we know that for all relevant j, α, β ,

$$\text{diam}_\rho(U_\alpha^{q_\iota}(n^{q_\iota}) + W_{j, \alpha, \beta}^{q_\iota}) < 2^{-n_\iota},$$

and consequently each of the sets $U_\alpha^{q_\iota}(n^{q_\iota}) + W_{j, \alpha, \beta}^{q_\iota}$ contains at most one element from each of the sets $\{a_i(x, y) + b_i(x, y), a_i(y, x) + b_i(y, x)\}$, $\{a_i(x, y) + b_i(x, y), a_{i'}(x, y) + b_{i'}(x, y)\}$ and $\{a_i(y, x) + b_i(y, x), a_{i'}(y, x) + b_{i'}(y, x)\}$. Since $q_\iota \in D_{n_\iota}^2$, different sets of the form $U_\alpha^{q_\iota}(n^{q_\iota}) + W_{j, \alpha, \beta}^{q_\iota}$ are disjoint, and thus we see that the assumptions (i)–(iv) of Lemma 5.11 are satisfied.

Unfixing x, y , we may use Lemma 5.11(1) to conclude that

$$(\square)_6 \quad \mathbf{X}_\iota - \mathbf{X}_\iota \subseteq \bigcup \{U_\alpha^{q_\iota}(n^{q_\iota} - 2) - U_\beta^{q_\iota}(n^{q_\iota} - 2) : \alpha, \beta \in w^{q_\iota}\}$$

and hence also $\mathbf{X}_\iota - \mathbf{X}_\iota \subseteq \bigcup \{U_\alpha^{p_\iota}(n^{p_\iota}) - U_\beta^{p_\iota}(n^{p_\iota}) : \alpha, \beta \in w^{p_\iota}\}$. Moreover, by 5.11(2), we also conclude that

$$(\square)_7 \quad \text{if } x, y \in \mathbf{X}_\iota \text{ and } 0 \neq x - y \in U_\alpha^{q_\iota}(n^{q_\iota} - 2) - U_\beta^{q_\iota}(n^{q_\iota} - 2), \text{ then } \alpha \neq \beta \text{ and } \bar{m}(x, y)(i) = \bar{\ell}(x, y)(i) = h^{q_\iota}(\alpha, \beta) = h^{p_\iota}(\alpha, \beta) \text{ for all } i < k.$$

Since $\{U_\alpha^{p_\iota}(n^{p_\iota}) : \alpha \in w^{p_\iota}\}$, $\{U_\alpha^{p_\iota}(n^{p_\iota} - 1) : \alpha \in w^{p_\iota}\}$, $\{U_\alpha^{p_\iota}(n^{p_\iota} - 2) : \alpha \in w^{p_\iota}\}$ and \mathbf{X}_ι satisfy the assumptions of Theorem 3.5, we get that exactly one of $(A)_\iota$, $(B)_\iota$ below holds true.

$(A)_\iota$ There is a $c_\iota \in \mathbb{H}$ such that $\mathbf{X}_\iota + c_\iota \subseteq \bigcup \{U_\alpha^{p_\iota}(n^{p_\iota} - 2) : \alpha \in w^{p_\iota}\}$.

$(B)_\iota$ There is a $c_\iota \in \mathbb{H}$ such that $c_\iota - \mathbf{X}_\iota \subseteq \bigcup \{U_\alpha^{p_\iota}(n^{p_\iota} - 2) : \alpha \in w^{p_\iota}\}$.

Unfixing $\iota < \omega$ we let

$$A = \{\iota < \omega : J \leq \iota \text{ and case } (A)_\iota \text{ holds true}\}$$

$$B = \{\iota < \omega : J \leq \iota \text{ and case } (B)_\iota \text{ holds true}\}.$$

One of the sets A, B is infinite and this leads us to two very similar cases.

CASE: The set A is infinite.

For $\iota \in A$ let $\mathbf{X}_\iota, c_\iota$ be as before. Let $w_\iota = \{\alpha \in w^{p_\iota} : U_\alpha^{p_\iota}(n^{p_\iota} - 2) \cap (\mathbf{X}_\iota + c_\iota) \neq \emptyset\}$. Since $\text{diam}_\rho(U_\alpha^{p_\iota}(n^{p_\iota} - 2)) < 2^{-n_\iota} < \rho(x, y)$ for $\alpha \in w_\iota$ and distinct $x, y \in \mathbf{X}_\iota$, we get $|U_\alpha^{p_\iota}(n^{p_\iota} - 2) \cap (\mathbf{X}_\iota + c_\iota)| = 1$ for $\alpha \in w_\iota$. Consequently, we have a natural bijection $\varphi_\iota : \mathbf{X}_\iota \rightarrow w_\iota$ such that $x + c_\iota \in U_{\varphi_\iota(x)}^{p_\iota}(n^{p_\iota} - 2)$.

For $\iota < \iota'$ from A we have $\mathbf{X}_\iota \subseteq \mathbf{X}_{\iota'}$ and the mapping $\pi_{\iota, \iota'} = \varphi_{\iota'} \circ \varphi_\iota^{-1} : w_\iota \rightarrow w_{\iota'}$ is an injection. Clearly, if $x \in \mathbf{X}_\iota$, $\alpha = \varphi_\iota(x) \in w_\iota$ then

$$(\square)_8 \quad x + c_{\iota'} \in \left(U_{\alpha}^{p_{\iota'}}(n^{p_{\iota'}} - 2) + (c_{\iota'} - c_\iota) \right) \cap U_{\pi_{\iota, \iota'}(\alpha)}^{p_{\iota'}}(n^{p_{\iota'}} - 2) \neq \emptyset.$$

Suppose now that $x, y \in \mathbf{X}_\iota$, $x \neq y$. By $(\square)_6$, there are $\alpha, \beta \in w^{q_\iota}$ such that $x - y \in U_\alpha^{q_\iota}(n^{q_\iota} - 2) - U_\beta^{q_\iota}(n^{q_\iota} - 2)$ (and, by $(\square)_7$, $\alpha \neq \beta$). Then also

$$x - y \in \left(U_\alpha^{p_\iota}(n^{p_\iota} - 2) - U_\beta^{p_\iota}(n^{p_\iota} - 2) \right) \cap \left(U_{\varphi_\iota(x)}^{p_\iota}(n^{p_\iota} - 2) - U_{\varphi_\iota(y)}^{p_\iota}(n^{p_\iota} - 2) \right).$$

By Lemma 5.5 we conclude that $\alpha = \varphi_\iota(x)$ and $\beta = \varphi_\iota(y)$. Together with $(\square)_7$ this gives us that

$$(\square)_9^i \quad \text{if } (x, y) \in (\mathbf{X}_\iota)^{\langle 2 \rangle}, \text{ then } \bar{m}(x, y)(i) = \bar{\ell}(x, y)(i) = h^{p_\iota}(\varphi_\iota(x), \varphi_\iota(y)) \text{ for all } i < k.$$

Putting together $(\square)_9^i$ and $(\square)_9^{\iota'}$ we see that

$$(\square)_{10} \quad \text{if } \iota < \iota' \text{ are from } A \text{ and } (x, y) \in (\mathbf{X}_\iota)^{\langle 2 \rangle}, \text{ then}$$

$$h^{p_\iota}(\varphi_\iota(x), \varphi_\iota(y)) = h^{p_{\iota'}}(\varphi_{\iota'}(x), \varphi_{\iota'}(y)).$$

In other words, if $(\alpha, \beta) \in w_\iota$ then

$$h^{p_\iota}(\alpha, \beta) = h^{p_{\iota'}}(\alpha, \beta) = h^{p_{\iota'}}(\pi_{\iota, \iota'}(\alpha), \pi_{\iota, \iota'}(\beta)).$$

It follows from $(\square)_8 + (\square)_{10}$ and 5.3(A)(\boxtimes) $_7$ that for $\iota < \iota'$ from A we have

$$(\square)_{11} \quad \text{rk}^{\text{sp}}(w_\iota) = \text{rk}^{\text{sp}}(\pi_{\iota, \iota'}[w_\iota]), \mathbf{j}(w_\iota) = \mathbf{j}(\pi_{\iota, \iota'}[w_\iota]), \mathbf{k}(w_\iota) = \mathbf{k}(\pi_{\iota, \iota'}[w_\iota]) \text{ and}$$

$$|\alpha \cap w_\iota| = \mathbf{k}(w_\iota) \quad \Leftrightarrow \quad |\pi_{\iota, \iota'}(\alpha) \cap \pi_{\iota, \iota'}[w_\iota]| = \mathbf{k}(w_\iota) \quad \text{for all } \alpha \in w_\iota.$$

(Note that $r_m^{p_{\iota'}} \leq n^{p_\iota} - 2$ when $m = h^{p_{\iota'}}(\alpha, \beta)$, $\alpha, \beta \in w_\iota \subseteq w^{p_\iota}$, $\alpha \neq \beta$.)

Choose a strictly increasing sequence $\langle \iota(\ell) : \ell < \omega \rangle \subseteq A$ such that

$(\square)_{12}$ for each $\ell < \omega$,

$$2^{2-n_{\iota(\ell+1)}-1} < \text{diam}_\rho \left(U_0^{p_{\iota(\ell)}}(n^{p_{\iota(\ell)}} - 2) \right) = \text{diam}_\rho \left(U_0^{p_{\iota(\ell+1)}}(n^{p_{\iota(\ell)}} - 2) \right)$$

(remember n_ι 's were chosen in $(\square)_2$ and $p_{\iota(\ell)} \in D_{0, n_{\iota(\ell)}, n_{\iota(\ell)}}^0$ so also $0 \in w^{p_{\iota(\ell)}}$).

Fix $\ell < \omega$ for a moment and suppose $\varsigma \in {}^{\iota(\ell)}2$ is such that

$$|\varphi_{\iota(\ell)}(x_\varsigma^*) \cap w_{\iota(\ell)}| = \mathbf{k}(w_{\iota(\ell)}).$$

Let $\varsigma^* \in {}^{\iota(\ell+1)}2$ be such that $\varsigma \triangleleft \varsigma^*$, $\varsigma^*(n) = 0$ for $n \in [\iota(\ell), \iota(\ell+1))$, and let $\sigma \in {}^{\iota(\ell+1)}2$ be such that $\varsigma^* \upharpoonright (\iota(\ell+1) - 1) \triangleleft \sigma$ and $\sigma(\iota(\ell+1) - 1) = 1$. Then $x_{\varsigma^*}^* = x_\varsigma^*$ and $\rho(x_{\varsigma^*}^*, x_\sigma^*) < 2^{1-n_{\iota(\ell+1)}-1}$. By $(\square)_{12}$ we have then $\rho(x_\varsigma^* + c_{\iota(\ell+1)}, x_\sigma^* + c_{\iota(\ell+1)}) = \rho(x_{\varsigma^*}^*, x_\sigma^*) < \text{diam}_\rho(U_0^{p_{\iota(\ell+1)}}(n^{p_{\iota(\ell)}} - 2))$. Consequently,

$$(\square)_{13} \quad U_{\varphi_{\iota(\ell+1)}(x_\varsigma^*)}^{p_{\iota(\ell+1)}}(n^{p_{\iota(\ell)}} - 2) = U_{\varphi_{\iota(\ell+1)}(x_\sigma^*)}^{p_{\iota(\ell+1)}}(n^{p_{\iota(\ell)}} - 2)$$

(remember 5.3(A)(\boxtimes)₅(b)). It follows from $(\square)_2^{c,d} + (\square)_5$ that for each $x \in X_{\iota(\ell)} \setminus \{x_\varsigma^*\}$ we have $\bar{\ell}(x, x_\varsigma^*) = \bar{\ell}(x, x_\sigma^*)$ and $\bar{m}(x, x_\varsigma^*) = \bar{m}(x, x_\sigma^*)$, so by $(\square)_9^{\iota(\ell+1)}$ we also have

$$(\square)_{14} \quad h^{p_{\iota(\ell+1)}}(\varphi_{\iota(\ell+1)}(x), \varphi_{\iota(\ell+1)}(x_\varsigma^*)) = h^{p_{\iota(\ell+1)}}(\varphi_{\iota(\ell+1)}(x), \varphi_{\iota(\ell+1)}(x_\sigma^*)).$$

Condition 5.3(A)(\boxtimes)₇ for $p_{\iota(\ell+1)}$ together with $(\square)_{11}$ imply now that, letting $w^{\iota(\ell), \sigma} = (\pi_{\iota(\ell), \iota(\ell+1)}[w_{\iota(\ell)}] \setminus \{\varphi_{\iota(\ell+1)}(x_\varsigma^*)\}) \cup \{\varphi_{\iota(\ell+1)}(x_\sigma^*)\}$, we have

$$\begin{aligned} (\square)_{15} \quad \text{rk}^{\text{sp}}(w^{\iota(\ell), \sigma}) &= \text{rk}^{\text{sp}}(\pi_{\iota(\ell), \iota(\ell+1)}[w_{\iota(\ell)}]) = \text{rk}^{\text{sp}}(w_{\iota(\ell)}), \\ \mathbf{j}(w^{\iota(\ell), \sigma}) &= \mathbf{j}(\pi_{\iota(\ell), \iota(\ell+1)}[w_{\iota(\ell)}]) = \mathbf{j}(w_{\iota(\ell)}), \text{ and} \\ \mathbf{k}(w^{\iota(\ell), \sigma}) &= \mathbf{k}(\pi_{\iota(\ell), \iota(\ell+1)}[w_{\iota(\ell)}]) = \mathbf{k}(w_{\iota(\ell)}) = |\varphi_{\iota(\ell+1)}(x_\sigma^*) \cap w^{\iota(\ell), \sigma}|. \end{aligned}$$

(Remember, $r_m^{p_{\iota(\ell+1)}} \leq n^{p_{\iota(\ell)}} - 2$ when $m = h^{p_{\iota(\ell+1)}}(\alpha, \beta)$, $\alpha, \beta \in \pi_{\iota(\ell), \iota(\ell+1)}[w_{\iota(\ell)}]$ are distinct.) Consequently, if $\text{rk}^{\text{sp}}(w_{\iota(\ell)}) \geq 0$ then $\text{rk}^{\text{sp}}(w_{\iota(\ell+1)}) \leq \text{rk}^{\text{sp}}(\pi_{\iota(\ell), \iota(\ell+1)}[w_{\iota(\ell)}] \cup \{\varphi_{\iota(\ell+1)}(x_\sigma^*)\}) < \text{rk}^{\text{sp}}(w_{\iota(\ell)})$ (remember Definition 2.6(\otimes)_e).

Unfixing $\ell < \omega$, we see that for some ℓ^* we have $\text{rk}^{\text{sp}}(w_{\iota(\ell^*)}) = -1$. However, applying to ℓ^* the procedure described above we get $\sigma \in {}^{\iota(\ell^*+1)}2$ such that $\varphi_{\iota(\ell^*+1)}(x_\sigma^*)$ contradicts clause 5.3(A)(\boxtimes)₈ for $p_{\iota(\ell^*+1)}$ (remember $(\square)_{13} + (\square)_{15}$).

CASE: The set B is infinite.

Almost identical to the previous case. Defining φ_ι we use the condition $c_\iota - x \in U_{\varphi_\iota(x)}^{p_\iota}(n^{p_\iota} - 2)$, but then not much other changes is needed. Even in $(\square)_8$ we have

$$c_{\iota'} - x = (c_\iota - x) + (c_{\iota'} - c_\iota) \in \left(U_{\alpha}^{p_{\iota'}}(n^{p_\iota} - 2) + (c_{\iota'} - c_\iota) \right) \cap U_{\pi_{\iota, \iota'}(\alpha)}^{p_{\iota'}}(n^{p_\iota} - 2) \neq \emptyset$$

(where $\alpha = \varphi_\iota(x) \in w_\iota$). \square

The following theorem is the consequence of results presented in this section.

Theorem 5.13. *Assume that*

(1) $(\mathbb{H}, +, 0)$ *is an Abelian perfect Polish group,*

- (2) the set of elements of \mathbb{H} of order larger than 2 is dense in \mathbb{H} ,
- (3) $2 \leq k < \omega$ and
- (4) $\varepsilon < \omega_1$ and λ is an uncountable cardinal such that $\text{NPr}^\varepsilon(\lambda)$ holds true.

Then there is a ccc forcing notion \mathbb{P} of cardinality λ such that

$$\begin{aligned} \Vdash_{\mathbb{P}} \text{ "for some } \Sigma_2^0 \text{ subset } B \text{ of } \mathbb{H} \text{ we have:} \\ \text{there is a set } X \subseteq \mathbb{H} \text{ of cardinality } \lambda \text{ such that} \\ (\forall x, y \in X) (|(x+B) \cap (y+B)| \geq k) \\ \text{but there is no perfect set } P \subseteq \mathbb{H} \text{ such that} \\ (\forall x, y \in P) (|(x+B) \cap (y+B)| \geq k) \text{".} \end{aligned}$$

6. Forcing for groups with all elements of order ≤ 2

Let us consider the situation when the main (algebraic) assumption of the previous section fails: the set of elements of \mathbb{H} of order larger than 2 is NOT dense in \mathbb{H} . Let $H_2 = \{a \in \mathbb{H} : a + a = 0\}$, so H_2 is a closed subgroup of \mathbb{H} and its complement $\mathbb{H} \setminus H_2$ is not dense in \mathbb{H} . Consequently, the interior of H_2 is not empty and thus also H_2 is an open subset of \mathbb{H} . If \mathbb{H} is a perfect Polish group, so is H_2 . Each coset of H_2 is clopen and consequently \mathbb{H}/H_2 is countable.

Suppose that $\mathbf{T} \subseteq H_2$ is a Borel set with λ many k -overlapping translations but without a perfect set of such translations. Then \mathbf{T} is also a Borel subset of \mathbb{H} and it still has λ many k -overlapping translations. If $P \subseteq \mathbb{H}$ is a perfect set, then (as $|\mathbb{H}/H_2| \leq \omega$) for some $a \in \mathbb{H}$ the intersection $P \cap (H_2 + a)$ is uncountable. Consider $Q = (P \cap (H_2 + a)) - a \subseteq H_2$ — it is a closed uncountable subset of H_2 (so contains a perfect set) and by the assumptions on \mathbf{T} there are $c, d \in Q$ such that $|(\mathbf{T} + c) \cap (\mathbf{T} + d)| < k$. Then $c + a, d + a \in P$ and $|(\mathbf{T} + (c + a)) \cap (\mathbf{T} + (d + a))| = |((\mathbf{T} + c) \cap (\mathbf{T} + d)) + a| < k$.

Consequently, to completely answer the problem of Borel sets with non-disjoint translations it is enough to deal with the case of all elements of \mathbb{H} being of order ≤ 2 . The arguments in this case are similar to those from Section 5, but they are simpler. However, there is one substantial difference. If \mathbb{H} is a Polish group with all elements of order ≤ 2 and $B \subseteq \mathbb{H}$ is an uncountable Borel set, then B has a perfect set of pairwise 2-overlapping translations. Namely, choosing a perfect set $P \subseteq B$ we will have $x + y, 0 \in (B + x) \cap (B + y)$ for each $x, y \in P$. Moreover, if $x + b_0 = y + b_1$, then also $x + b_1 = y + b_0$. Therefore, if $x \neq y$ and $(B + x) \cap (B + y)$ is finite, then $|(B + x) \cap (B + y)|$ must be even. For that reason the meaning of k in our forcing here will be slightly different: the translations of the new Borel set will have at least $2k$ elements.

Assumption 6.1. In the rest of the section we assume the following:

- (1) $(\mathbb{H}, +, 0)$, \mathbf{D} , ρ and \mathcal{U} are as in Assumption 5.1.
- (2) All elements of \mathbb{H} have orders at most 2.
- (3) $1 < k < \omega$.
- (4) ε is a countable ordinal and λ is an uncountable cardinal such that $\text{NPr}^\varepsilon(\lambda)$ holds true. The model $\mathbb{M}(\varepsilon, \lambda)$ and functions $\text{rk}^{\text{sp}}, \mathbf{j}$ and \mathbf{k} on $[\lambda]^{<\omega} \setminus \{\emptyset\}$ are as fixed in Definition 2.6.

In groups with all elements of order two we should use a weaker notion of independence.

Definition 6.2. Let $(\mathbb{H}, +, 0)$ be an Abelian group

- (1) A set $\mathbf{B} \subseteq \mathbb{H}$ is *quasi⁻ independent* in \mathbb{H} if $|\mathbf{B}| \geq 8$ and if for all distinct $b_0, b_1, b_2, \dots, b_7 \in \mathbf{B}$ and any $e_0, e_1, e_2, \dots, e_7 \in \{0, 1\}$ not all equal 0, we have

$$e_0b_0 + e_1b_1 + e_2b_2 + e_3b_3 + e_4b_4 + e_5b_5 + e_6b_6 + e_7b_7 \neq 0.$$

- (2) A family $\{V_i : i \leq n\}$ of disjoint subsets of \mathbb{H} is a qif^- if for each choice of $b_i \in V_i$, $i \leq n$, the set $\{b_i : i \leq n\}$ is quasi $^-$ independent.

Proposition 6.3. *Assume that*

- (i) $(\mathbb{H}, +, 0)$ is a perfect Abelian Polish group,
- (ii) U_0, \dots, U_n are nonempty open subsets of \mathbb{H} , $n \geq 7$.

Then there are non-empty open sets $V_i \subseteq U_i$ (for $i \leq n$) such that $\{V_i : i \leq n\}$ is a qif^- .

Proof. Similar to Proposition 3.3. \square

The forcing notion used in the case of groups with all elements of order ≤ 2 is almost the same as the one introduced in Definition 5.3. The only difference is that instead of 8-good qifs we use the weaker concept of qifs^- . (There are no 8-good qifs in the current case.) Since in the current case, $a - b = a + b$ for $a, b \in \mathbb{H}$, we still can repeat all needed ingredients of Section 5. To stress the importance of this property we will consistently use the addition $+$ rather than subtraction $-$.

Definition 6.4. (A) Let \mathbb{Q} be the collection of all tuples

$$p = (w^p, M^p, \bar{r}^p, n^p, \bar{\Upsilon}^p, \bar{V}^p, h^p) = (w, M, \bar{r}, n, \bar{\Upsilon}, \bar{V}, h)$$

such that the following demands $(\otimes)_1 - (\otimes)_8$ are satisfied.

- $(\otimes)_1$ $w \in [\lambda]^{<\omega}$, $|w| \geq 4$, $0 < M < \omega$, $3 \leq n < \omega$ and $\bar{r} = \langle r_m : m < M \rangle$ where $r_m \leq n - 2$ for $m < M$.
- $(\otimes)_2$ $\bar{\Upsilon} = \langle \bar{U}_\alpha : \alpha \in w \rangle$ where each $\bar{U}_\alpha = \langle U_\alpha(\ell) : \ell \leq n \rangle$ is a \subseteq -decreasing sequence of elements of \mathcal{U} .
- $(\otimes)_3$ $\bar{V} = \langle Q_{i,\alpha,\beta}, V_{i,\alpha,\beta}, W_{i,\alpha,\beta} : i < k, (\alpha, \beta) \in w^{(2)} \rangle \subseteq \mathcal{U}$ and $Q_{i,\alpha,\beta} = Q_{i,\beta,\alpha} \supseteq V_{i,\alpha,\beta} = V_{i,\beta,\alpha} \supseteq W_{i,\alpha,\beta} = W_{i,\beta,\alpha}$ for all $i < k$ and $(\alpha, \beta) \in w^{(2)}$.
- $(\otimes)_4$ (a) The indexed family $\langle U_\alpha(n-2) : \alpha \in w \rangle \smallfrown \langle Q_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is a qif^- (so in particular the sets in this system are pairwise disjoint), and
(b) $\langle U_\alpha(n) : \alpha \in w \rangle \smallfrown \langle W_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is immersed in $\langle U_\alpha(n-1) : \alpha \in w \rangle \smallfrown \langle V_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ and $\langle U_\alpha(n-1) : \alpha \in w \rangle \smallfrown \langle V_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$ is immersed in $\langle U_\alpha(n-2) : \alpha \in w \rangle \smallfrown \langle Q_{i,\alpha,\beta} : i < k, \alpha, \beta \in w, \alpha < \beta \rangle$.
- $(\otimes)_5$ (a) If $\alpha, \beta \in w$, $\ell \leq n$ and $U_\alpha(\ell) \cap U_\beta(\ell) \neq \emptyset$, then $U_\alpha(\ell) = U_\beta(\ell)$, and
(b) if $\alpha, \beta, \gamma \in w$, $\ell \leq n$, $U_\alpha(\ell) \neq U_\beta(\ell)$ and $a \in U_\alpha(\ell)$, $b \in U_\beta(\ell)$, then $\rho(a, b) > \text{diam}_\rho(U_\gamma(\ell))$.
- $(\otimes)_6$ $h : w^{(2)} \xrightarrow{\text{onto}} M$ is such that $h(\alpha, \beta) = h(\beta, \alpha)$ for $(\alpha, \beta) \in w^{(2)}$.
- $(\otimes)_7$ Assume that $u, u' \subseteq w$, π and $\ell \leq n$ are such that
 - $4 \leq |u| = |u'|$ and $\pi : u \rightarrow u'$ is a bijection,
 - $r_{h(\alpha,\beta)} \leq \ell$ for all $(\alpha, \beta) \in u^{(2)}$,
 - $U_\alpha(\ell) \cap U_\beta(\ell) = \emptyset$ and $h(\alpha, \beta) = h(\pi(\alpha), \pi(\beta))$ for all distinct $\alpha, \beta \in u$,
 - for some $c \in \mathbb{H}$, for all $\alpha \in u$, we have $(U_\alpha(\ell) + c) \cap U_{\pi(\alpha)}(\ell) \neq \emptyset$.
 Then $\text{rk}^{\text{sp}}(u) = \text{rk}^{\text{sp}}(u')$, $\mathbf{j}(u) = \mathbf{j}(u')$, $\mathbf{k}(u) = \mathbf{k}(u')$ and for $\alpha \in u$

$$|\alpha \cap u| = \mathbf{k}(u) \quad \Leftrightarrow \quad |\pi(\alpha) \cap u'| = \mathbf{k}(u).$$

- $(\otimes)_8$ Assume that
 - $\emptyset \neq u \subseteq w$, $\text{rk}^{\text{sp}}(u) = -1$, $\ell \leq n$ and

- $\alpha \in u$ is such that $|\alpha \cap u| = \mathbf{k}(u)$, and
 - $r_{h(\beta, \beta')} \leq \ell$ and $U_\beta(\ell) \cap U_{\beta'}(\ell) = \emptyset$ for all $(\beta, \beta') \in u^{(2)}$.
- Then there is **no** $\alpha' \in w \setminus u$ such that $U_\alpha(\ell) = U_{\alpha'}(\ell)$ and $h(\alpha, \beta) = h(\alpha', \beta)$ for all $\beta \in u \setminus \{\alpha\}$.

(B) For $p \in \mathbb{Q}$ and $m < M^p$ we define

$$F(p, m) = \bigcup \{U_\alpha^p(n^p) + W_{i, \alpha, \beta}^p : (\alpha, \beta) \in (w^p)^{(2)} \wedge i < k \wedge h^p(\alpha, \beta) = m\}.$$

(C) For $p, q \in \mathbb{Q}$ we declare that $p \leq q$ if and only if

- $w^p \subseteq w^q$, $M^p \leq M^q$, $\bar{r}^q \upharpoonright M^p = \bar{r}^p$, $n^p \leq n^q$, $h^q \upharpoonright (w^p)^{(2)} = h^p$, and
- if $\alpha \in w^p$ and $\ell \leq n^p$ then $U_\alpha^q(\ell) = U_\alpha^p(\ell)$, and
- if $(\alpha, \beta) \in (w^p)^{(2)}$, $i < k$, then $Q_{i, \alpha, \beta}^q \subseteq Q_{i, \alpha, \beta}^p$, $V_{i, \alpha, \beta}^q \subseteq V_{i, \alpha, \beta}^p$, and $W_{i, \alpha, \beta}^q \subseteq W_{i, \alpha, \beta}^p$, and
- if $m < M^p$, then $F(q, m) \subseteq F(p, m)$.

Lemma 6.5.

- (1) (\mathbb{Q}, \leq) is a partial order of size λ .
- (2) The following sets are dense in \mathbb{Q} :
 - (i) $D_{\gamma, M, n}^0 = \{p \in \mathbb{Q} : \gamma \in w^p \wedge M^p > M \wedge n^p > n\}$ for $\gamma < \lambda$ and $M, n < \omega$.
 - (ii) $D_N^1 = \{p \in \mathbb{Q} : \text{for all } i, j < k \text{ and } (\alpha, \beta), (\gamma, \delta) \in (w^p)^{(2)} \text{ it holds that } \text{diam}_\rho(U_\alpha^p(n^p - 2)) < 2^{-N} \text{ and } \text{diam}_\rho(Q_{i, \alpha, \beta}^p) < 2^{-N} \text{ and } \text{diam}_\rho(U_\alpha^p(n^p - 2) + Q_{i, \alpha, \beta}^p) < 2^{-N} \text{ and if } (i, \alpha^*, \alpha, \beta) \neq (j, \gamma^*, \gamma, \delta) \text{ then } (U_{\alpha^*}^p(n^p) + W_{i, \alpha, \beta}^p) \cap (U_{\gamma^*}^p(n^p) + W_{i, \gamma, \delta}^p) = \emptyset\} \text{ for } N < \omega.$
 - (iii) $D_N^2 = \{p \in D_N^1 : \text{for some } \langle Q_{i, \alpha, \beta}^* : i < k, \alpha, \beta \in w^p, \alpha < \beta \rangle \subseteq \mathcal{U} \text{ the system } \langle U_\alpha^p(n - 3) : \alpha \in w^p \rangle \frown \langle Q_{i, \alpha, \beta}^* : i < k, \alpha, \beta \in w^p, \alpha < \beta \rangle \text{ is a qif}^- \text{ and } \langle U_\alpha^p(n - 2) : \alpha \in w^p \rangle \frown \langle Q_{i, \alpha, \beta} : i < k, \alpha, \beta \in w^p, \alpha < \beta \rangle \text{ is immersed in it} \}$.
- (3) Assume $p \in \mathbb{Q}$. Then there is $q \geq p$ such that $n^q \geq n^p + 3$, $w^q = w^p$ and
 - for all $\alpha \in w^p$, $\text{cl}(U_\alpha^q(n^q - 2)) \subseteq U_\alpha^p(n^p)$, and
 - for all $i < k$ and $(\alpha, \beta) \in (w^p)^{(2)}$,

$$\text{cl}(U_\alpha^q(n^q - 2) + Q_{i, \alpha, \beta}^q) \subseteq U_\alpha^p(n^p) + W_{i, \alpha, \beta}^p \quad \text{and} \quad \text{cl}(Q_{i, \alpha, \beta}^q) \subseteq W_{i, \alpha, \beta}^p.$$

Proof. Same as for 5.4 (just using Proposition 6.3). \square

Lemma 6.6. Suppose that $p \in D_1^2$ and $\alpha, \beta, \gamma, \delta \in w^p$ are such that $\alpha \neq \beta$. If

$$\left(U_\alpha^p(n^p - 2) + U_\beta^p(n^p - 2) \right) \cap \left(U_\gamma^p(n^p - 2) + U_\delta^p(n^p - 2) \right) \neq \emptyset,$$

then $\{\alpha, \beta\} = \{\gamma, \delta\}$.

Proof. Similar to 5.5, remembering $\langle U_\alpha^p(n - 2) : \alpha \in w^p \rangle$ is immersed in a $\text{qif}^- \langle U_\alpha^p(n - 3) : \alpha \in w^p \rangle$; see 6.5(2)(iv). \square

Lemma 6.7. The forcing notion \mathbb{Q} has the Knaster property.

Proof. Same as Lemma 5.6, but when defining a bound q to p_ξ, p_ζ modify the demands to have $n^q = n^{p_\zeta} + 4$ and $q \in D_1^2$. \square

Lemma 6.8. For each $(\alpha, \beta) \in \lambda^{(2)}$ and $i < k$,

$$\Vdash_{\mathbb{Q}} \text{“the sets} \\ \bigcap \{U_{\alpha}^p(n^p) : p \in G_{\mathbb{Q}} \wedge \alpha \in w^p\} \quad \text{and} \quad \bigcap \{W_{i,\alpha,\beta}^p : p \in G_{\mathbb{Q}} \wedge \alpha, \beta \in w^p\} \\ \text{have exactly one element each.”}$$

Proof. Follows from Lemma 6.5. \square

Definition 6.9.

(1) For $(\alpha, \beta) \in \lambda^{(2)}$ and $i < k$ let η_{α} , $\nu_{i,\alpha,\beta}$ and $h_{\alpha,\beta}$ be \mathbb{Q} -names such that

$$\begin{aligned} \Vdash_{\mathbb{Q}} \text{“}\{\eta_{\alpha}\} &= \bigcap \{U_{\alpha}^p(n^p) : p \in G_{\mathbb{Q}} \wedge \alpha \in w^p\}, \\ \{\nu_{i,\alpha,\beta}\} &= \bigcap \{W_{i,\alpha,\beta}^p : p \in G_{\mathbb{Q}} \wedge \alpha, \beta \in w^p\} \\ h_{\alpha,\beta} &= h^p(\alpha, \beta) \text{ for some (all) } p \in G_{\mathbb{Q}} \text{ such that } \alpha, \beta \in w^p\text{.”} \end{aligned}$$

(2) For $m < \omega$ let \mathbf{F}_m be a \mathbb{Q} -name such that

$$\Vdash_{\mathbb{Q}} \text{“}\mathbf{F}_m = \bigcap \{F(p, m) : m < M^p \wedge p \in G_{\mathbb{Q}}\}\text{.”}$$

Lemma 6.10.

(1) For each $m < \omega$, $\Vdash_{\mathbb{Q}} \text{“}\mathbf{F}_m \text{ is a closed subset of } \mathbb{H}\text{.”}$

(2) For $i < k$ and $(\alpha, \beta) \in \lambda^{(2)}$ we have

$$\Vdash_{\mathbb{Q}} \text{“}\eta_{\alpha}, \nu_{i,\alpha,\beta} \in \mathbb{H}, \quad h_{\alpha,\beta} < \omega, \quad \nu_{i,\alpha,\beta} = \nu_{i,\beta,\alpha} \quad \text{and} \quad \eta_{\alpha} + \nu_{i,\alpha,\beta} \in \mathbf{F}_{h_{\alpha,\beta}}\text{.”}$$

(3) $\Vdash_{\mathbb{Q}} \text{“}\langle \eta_{\alpha}, \nu_{i,\alpha,\beta} : i < k, \alpha < \beta < \lambda \rangle \text{ is quasi- independent.”}$

(4) $\Vdash_{\mathbb{Q}} \text{“}\nu_{0,\alpha,\beta}, \dots, \nu_{k-1,\alpha,\beta}, (\eta_{\alpha} + \nu_{0,\alpha,\beta}), \dots, (\eta_{\alpha} + \nu_{k-1,\alpha,\beta}) \text{ are distinct elements of } (\eta_{\alpha} + \bigcup_{m < \omega} \mathbf{F}_m) \cap (\eta_{\beta} + \bigcup_{m < \omega} \mathbf{F}_m)\text{.”}$

Proof. Should be clear. \square

Lemma 6.11. Let $p = (w, M, \bar{r}, n, \bar{\Upsilon}, \bar{V}, h) \in D_1^1 \subseteq \mathbb{Q}$ (cf. 6.5(ii)) and $a_{\ell}, b_{\ell} \in \mathbb{H}$ and $U_{\ell}, W_{\ell} \in \mathcal{U}$ (for $\ell < 4$) be such that the following conditions are satisfied.

- (*)₁ $U_{\ell} \in \{U_{\alpha}(n) : \alpha \in w\}$, $W_{\ell} \in \{W_{i,\alpha,\beta} : i < k, (\alpha, \beta) \in w^{(2)}\}$ (for $\ell < 4$).
- (*)₂ $(U_{\ell} + W_{\ell}) \cap (U_{\ell'} + W_{\ell'}) = \emptyset$ for $\ell < \ell' < 4$.
- (*)₃ $a_{\ell} \in U_{\ell}$ and $b_{\ell} \in W_{\ell}$ and $a_{\ell} + b_{\ell} \in \bigcup_{m < M} F(p, m)$ for $\ell < 4$.
- (*)₄ $(a_0 + b_0) + (a_1 + b_1) = (a_2 + b_2) + (a_3 + b_3)$.

Then for some $(\alpha, \beta) \in w^{(2)}$ and distinct $i, j < k$ one of the following three conditions holds.

- (A) $\{\{U_0 + W_0, U_1 + W_1\}, \{U_2 + W_2, U_3 + W_3\}\} = \{\{U_{\alpha}(n) + W_{i,\alpha,\beta}, U_{\beta}(n) + W_{i,\alpha,\beta}\}, \{U_{\alpha}(n) + W_{j,\alpha,\beta}, U_{\beta}(n) + W_{j,\alpha,\beta}\}\}.$
- (B) $\{\{U_0 + W_0, U_1 + W_1\}, \{U_2 + W_2, U_3 + W_3\}\} = \{\{U_{\alpha}(n) + W_{i,\alpha,\beta}, U_{\alpha}(n) + W_{j,\alpha,\beta}\}, \{U_{\beta}(n) + W_{i,\alpha,\beta}, U_{\beta}(n) + W_{j,\alpha,\beta}\}\}.$

$$(C) \left\{ \{U_0 + W_0, U_1 + W_1\}, \{U_2 + W_2, U_3 + W_3\} \right\} = \left\{ \{U_\alpha(n) + W_{i,\alpha,\beta}, U_\beta(n) + W_{j,\alpha,\beta}\}, \{U_\alpha(n) + W_{j,\alpha,\beta}, U_\beta(n) + W_{i,\alpha,\beta}\} \right\}.$$

Proof. The arguments here are very similar to those in Lemma 5.10. Note that the assumption $(\otimes)_2$ here is slightly stronger than there (to compensate for weaker qifs). Also our conclusion here is arguably weaker, but this is a necessity caused by the fact that $a - b = a + b$ in \mathbb{H} .

For $\ell < 4$ let U_ℓ^-, U_ℓ^{--} and V_ℓ, Q_ℓ be such that

- if $U_\ell = U_\alpha(n)$ then $U_\ell^- = U_\alpha(n-1)$, $U_\ell^{--} = U_\alpha(n-2)$,
- if $W_\ell = W_{i,\alpha,\beta}$ then $V_\ell = V_{i,\alpha,\beta}$, $Q_\ell = Q_{i,\alpha,\beta}$.

Using steps as in 5.10, one can show that

(*) For every W, U we have

$$|\{\ell < 4 : W_\ell = W\}| < 3 \quad \text{and} \quad |\{\ell < 4 : U_\ell = U\}| < 3.$$

Now, since $p \in D_1^2$, it follows from our assumption $(\otimes)_3$ that

(**) for each $\ell < 4$, for some $\alpha = \alpha(\ell)$, $\beta = \beta(\ell)$, and $i = i(\ell)$ we have $U_\ell = U_\alpha(n)$ and $W_\ell = W_{i,\alpha,\beta}$.

By assumption $(\otimes)_4$ we know that

$$0 \in U_0 + U_1 + U_2 + U_3 + W_0 + W_1 + W_2 + W_3.$$

If all of U_i 's are distinct, then $0 \in U_0^- + U_1^- + U_2^- + U_3^- + X$, where $X = \{0\}$ or $X = W_i + W_j$ for some $i < j < 4$ with $W_i \neq W_j$ or $X = W_0 + W_1 + W_2 + W_3$ with all W_i 's distinct (remember (*)). This contradicts 6.4(A)(\otimes)₄. Similarly if all W_i 's are distinct.

So suppose $|\{U_0, U_1, U_2, U_3\}| = 3$. Then for some $\ell < \ell' < 4$, $U_\ell \neq U_{\ell'}$ and

$$0 \in U_\ell^- + U_{\ell'}^- + W_0 + W_1 + W_2 + W_3 \subseteq U_\ell^{--} + U_{\ell'}^{--} + X,$$

where $X = \{0\}$ or $X = W_i + W_j$ for some $i < j < 4$ with $W_i \neq W_j$ or $X = W_0 + W_1 + W_2 + W_3$ with all W_i 's distinct (remember (*)). This again contradicts 6.4(A)(\otimes)₄. Similarly if $|\{W_0, W_1, W_2, W_3\}| = 3$.

Consequently, $|\{U_0, U_1, U_2, U_3\}| = 2 = |\{W_0, W_1, W_2, W_3\}|$. Moreover for some distinct $\alpha, \beta \in w$ we have

$$|\{\ell < 4 : U_\ell = U_\alpha(n)\}| = |\{\ell < 4 : U_\ell = U_\beta(n)\}| = 2$$

and for some $(i, \gamma, \delta) \neq (j, \varepsilon, \zeta)$ we have

$$|\{\ell < 4 : W_\ell = W_{i,\gamma,\delta}\}| = |\{\ell < 4 : W_\ell = W_{j,\varepsilon,\zeta}\}| = 2.$$

Now we consider all possible configurations.

CASE 1 $U_0 = U_1$, $U_2 = U_3$, say they are respectively $U_\alpha(n)$ and $U_\beta(n)$.

Necessarily $W_0 \neq W_1$ and $W_2 \neq W_3$.

If $W_0 = W_2$, $W_1 = W_3$ then recalling (**) above, we also get $\{\gamma, \delta\} = \{\varepsilon, \zeta\} = \{\alpha, \beta\}$ and (possibly after interchanging i and j) $W_0 = W_{i,\alpha,\beta}$, $W_1 = W_{j,\alpha,\beta}$. This gives conclusion (B).

If $W_0 = W_3$, $W_1 = W_2$ then again by (**) above, we get $\{\gamma, \delta\} = \{\varepsilon, \zeta\} = \{\alpha, \beta\}$ and (possibly after interchanging i and j) $W_0 = W_{i,\alpha,\beta}$, $W_1 = W_{j,\alpha,\beta}$. This also gives conclusion (B).

CASE 2 $U_0 = U_2$, $U_1 = U_3$, say they are respectively $U_\alpha(n)$ and $U_\beta(n)$.

Necessarily $W_0 \neq W_2$ and $W_1 \neq W_3$.

If $W_0 = W_1$, $W_2 = W_3$ then recalling (**) above, we also get $\{\gamma, \delta\} = \{\varepsilon, \zeta\} = \{\alpha, \beta\}$. After possibly interchanging i and j , $W_0 = W_{i,\alpha,\beta}$, $W_2 = W_{j,\alpha,\beta}$ and we get conclusion (A).

If $W_0 = W_3$, $W_1 = W_2$ then again by (**), we have $\{\gamma, \delta\} = \{\varepsilon, \zeta\} = \{\alpha, \beta\}$. After possibly interchanging i and j , $W_0 = W_{i,\alpha,\beta}$, $W_1 = W_{j,\alpha,\beta}$. This leads to conclusion (C).

CASE 3 $U_0 = U_3$, $U_1 = U_2$, say they are respectively $U_\alpha(n)$ and $U_\beta(n)$.

Necessarily $W_0 \neq W_3$ and $W_1 \neq W_2$.

If $W_0 = W_1$, $W_2 = W_3$ then like above we get $\{\gamma, \delta\} = \{\varepsilon, \zeta\} = \{\alpha, \beta\}$. After possibly interchanging i and j , $W_0 = W_{i,\alpha,\beta}$, $W_2 = W_{j,\alpha,\beta}$ and we get conclusion (A).

If $W_0 = W_2$, $W_1 = W_3$ then we also have $\{\gamma, \delta\} = \{\varepsilon, \zeta\} = \{\alpha, \beta\}$ and after possibly interchanging i and j , $W_0 = W_{i,\alpha,\beta}$, $W_1 = W_{j,\alpha,\beta}$. This leads to conclusion (C). \square

Lemma 6.12. Let $p = (w, M, \bar{r}, n, \bar{\Upsilon}, \bar{V}, h) \in D_1^2$ and $\mathbf{X} \subseteq \mathbb{H}$, $|\mathbf{X}| \geq 5$. Suppose that $a_i(x, y), b_i(x, y), U_i(x, y)$ and $W_i(x, y)$ for $x, y \in \mathbf{X}$, $x \neq y$ and $i < k$ satisfy the following demands (i)–(iv) (for all $x \neq y$, $i \neq i'$).

- (i) $U_i(x, y) \in \{U_\alpha(n) : \alpha \in w\}$, $W_i(x, y) \in \{W_{j,\alpha,\beta} : j < k, (\alpha, \beta) \in w^{(2)}\}$.
- (ii) • $(U_i(x, y) + W_i(x, y)) \cap (U_i(y, x) + W_i(y, x)) = \emptyset$,
• $(U_i(x, y) + W_i(x, y)) \cap (U_{i'}(x, y) + W_{i'}(x, y)) = \emptyset$,
• $(U_i(x, y) + W_i(x, y)) \cap (U_{i'}(y, x) + W_{i'}(y, x)) = \emptyset$.
- (iii) $a_i(x, y) \in U_i(x, y)$ and $b_i(x, y) \in W_i(x, y)$, and
 $a_i(x, y) + b_i(x, y) \in \bigcup_{m < M} F(p, m)$.
- (iv) $x + y = (a_i(x, y) + b_i(x, y)) + (a_i(y, x) + b_i(y, x))$.

Then

- (1) $\mathbf{X} + \mathbf{X} \subseteq \bigcup \{U_\alpha(n-2) + U_\beta(n-2) : \alpha, \beta \in w\}$.
- (2) If $(x, y) \in \mathbf{X}^{(2)}$ and $x + y \in U_\alpha(n-2) + U_\beta(n-2)$, $\alpha, \beta \in w$, then $\alpha \neq \beta$ and for each $i < k$ we have
 $a_i(x, y) + b_i(x, y), a_i(y, x) + b_i(y, x) \in F(p, h(\alpha, \beta))$.

Proof. (1) Fix $x, y \in \mathbf{X}$, $x \neq y$, for a moment.

Let $i \neq i'$, $i, i' < k$. We may apply Lemma 6.11 for $U_i(x, y)$, $W_i(x, y)$, $U_i(y, x)$, $W_i(y, x)$, $a_i(x, y)$, $b_i(x, y)$, $a_i(y, x)$, $b_i(y, x)$ here as $U_0, W_0, U_1, W_1, a_0, b_0, a_1, b_1$ there and for similar objects with i' in place of i as $U_2, W_2, U_3, W_3, a_2, b_2, a_3, b_3$ there. This will produce distinct $\alpha = \alpha(x, y, i, i'), \beta = \beta(x, y, i, i') \in w$ and distinct $j = j(x, y, i, i'), j' = j'(x, y, i, i') < k$ such that

$$\begin{aligned} & \text{either } (A)_{x,y,i,i'}^{\alpha,\beta,j,j'} : \\ & \{ \{U_i(x, y) + W_i(x, y), U_i(y, x) + W_i(y, x)\}, \{U_{i'}(x, y) + W_{i'}(x, y), U_{i'}(y, x) + W_{i'}(y, x)\} \} = \\ & \{ \{U_\alpha(n) + W_{j,\alpha,\beta}, U_\beta(n) + W_{j,\alpha,\beta}\}, \{U_\alpha(n) + W_{j',\alpha,\beta}, U_\beta(n) + W_{j',\alpha,\beta}\} \}, \\ & \text{or } (B)_{x,y,i,i'}^{\alpha,\beta,j,j'} : \\ & \{ \{U_i(x, y) + W_i(x, y), U_i(y, x) + W_i(y, x)\}, \{U_{i'}(x, y) + W_{i'}(x, y), U_{i'}(y, x) + W_{i'}(y, x)\} \} = \\ & \{ \{U_\alpha(n) + W_{j,\alpha,\beta}, U_\alpha(n) + W_{j',\alpha,\beta}\}, \{U_\beta(n) + W_{j,\alpha,\beta}, U_\beta(n) + W_{j',\alpha,\beta}\} \}, \\ & \text{or } (C)_{x,y,i,i'}^{\alpha,\beta,j,j'} : \\ & \{ \{U_i(x, y) + W_i(x, y), U_i(y, x) + W_i(y, x)\}, \{U_{i'}(x, y) + W_{i'}(x, y), U_{i'}(y, x) + W_{i'}(y, x)\} \} = \\ & \{ \{U_\alpha(n) + W_{j,\alpha,\beta}, U_\beta(n) + W_{j',\alpha,\beta}\}, \{U_\alpha(n) + W_{j',\alpha,\beta}, U_\beta(n) + W_{j,\alpha,\beta}\} \}. \end{aligned}$$

Plainly,

$(\odot)_1^{x,y}$ if for some $i \neq i'$ and α, β, j, j' the clause $(A)_{x,y,i,i'}^{\alpha,\beta,j,j'}$ holds true,
then $x + y \in U_\alpha(n-2) + U_\beta(n-2)$.

It should be also clear that for each $x \neq y$ and distinct i, i', i'' ,

$(\odot)_2$ if $(B)_{x,y,i,i'}^{\alpha,\beta,j,j'}$ holds true, then also $(B)_{x,y,i,i''}^{\alpha,\beta,j,j'}$ holds true,

and

$(\odot)_3$ if $(C)_{x,y,i,i'}^{\alpha,\beta,j,j'}$ holds true, then also $(C)_{x,y,i,i''}^{\alpha,\beta,j,j'}$ holds true,

Consequently, if $k \geq 3$ then by argument similar to 5.11 (case $k \geq 3$) for any $x \neq y$ from \mathbf{X} neither of possibilities $(B)_{x,y,i,i'}^{\alpha,\beta,j,j'}$ nor $(C)_{x,y,i,i'}^{\alpha,\beta,j,j'}$ can hold. Therefore we may easily finish the proof of Lemma 6.12 (when $k \geq 3$).

So assume $k = 2$. For each $x \neq y$ from \mathbf{X} we fix $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ such that either $(A)_{x,y,0,1}^{\alpha,\beta,0,1}$ or $(B)_{x,y,0,1}^{\alpha,\beta,0,1}$ or $(C)_{x,y,0,1}^{\alpha,\beta,0,1}$. Let $\chi(x, y) = \chi(y, x) \in \{A, B, C\}$ and $\theta(x, y) = \theta(y, x) \in [w]^2$ be such that $(\chi(x, y))^{\theta(x,y),0,1}_{x,y,0,1}$ holds true.

Claim 6.12.1. *If $x, y, z \in \mathbf{X}$ are distinct and $\chi(x, y) = \chi(y, z) = A$, then $\chi(x, z) = A$.*

Proof of the Claim. Let $\chi(x, y) = A = \chi(y, z)$ and $\theta(x, y) = \{\alpha, \beta\}$, $\theta(y, z) = \{\gamma, \delta\}$. Assume towards contradiction that $\chi(x, z) \in \{B, C\}$ and let $\theta(x, z) = \{\xi, \zeta\}$. Then for some $\xi', \zeta' \in \{\xi, \zeta\}$ we have

$$x + z \in U_{\xi'}(n) + U_{\zeta'}(n) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta}.$$

• If $|\{\alpha, \beta\} \cap \{\gamma, \delta\}| = 1$, say $\alpha = \gamma$, $\beta \neq \delta$, and $\{\xi, \zeta\} = \{\xi', \zeta'\} = \{\beta, \delta\}$, then

$$x + z \in U_\alpha(n) + U_\alpha(n) + U_\beta(n) + U_\delta(n) + W_{i,\alpha,\beta} + W_{i,\alpha,\beta} + W_{j,\alpha,\delta} + W_{j,\alpha,\delta}$$

but also $x + z \in U_\beta(n) + U_\delta(n) + W_{0,\beta,\delta} + W_{1,\beta,\delta}$. Consequently,

$$\begin{aligned} 0 \in & ((U_\alpha(n) + U_\alpha(n)) + U_\beta(n)) + (U_\beta(n) + (U_\delta(n) + U_\delta(n))) + \\ & ((W_{i,\alpha,\beta} + W_{i,\alpha,\beta}) + W_{0,\beta,\delta}) + ((W_{j,\alpha,\delta} + W_{j,\alpha,\delta}) + W_{1,\beta,\delta}) \subseteq \\ & U_\beta(n-1) + U_\beta(n-1) + V_{0,\beta,\delta} + V_{1,\beta,\delta} \subseteq Q_{0,\beta,\delta} + Q_{1,\beta,\delta}. \end{aligned}$$

This immediately contradicts 6.4(A)(\otimes)₄.

• If $|\{\alpha, \beta\} \cap \{\gamma, \delta\}| = 1$, say $\alpha = \gamma$, $\beta \neq \delta$, and $\xi' \neq \zeta'$, $|\{\xi', \zeta'\} \cap \{\beta, \delta\}| = 1$, say $\xi' = \beta$, then by similar considerations we arrive to

$$\begin{aligned} 0 \in & ((U_\alpha(n) + U_\alpha(n)) + U_\delta(n)) + ((U_\beta(n) + U_\beta(n)) + U_{\zeta'}(n)) + \\ & ((W_{i,\alpha,\beta} + W_{i,\alpha,\beta}) + W_{0,\beta,\zeta'}) + ((W_{j,\alpha,\delta} + W_{j,\alpha,\delta}) + W_{1,\beta,\zeta'}) \subseteq \\ & U_\delta(n-1) + U_{\zeta'}(n-1) + V_{0,\beta,\zeta'} + V_{1,\beta,\zeta'}. \end{aligned}$$

In our case necessarily $\delta \neq \zeta'$ so we easily get contradiction with 6.4(A)(\otimes)₄.

• If $|\{\alpha, \beta\} \cap \{\gamma, \delta\}| = 1$, say $\alpha = \gamma$, $\beta \neq \delta$, and $\xi' \neq \zeta'$ and $\{\xi', \zeta'\} \cap \{\beta, \delta\} = \emptyset$, then

$$\begin{aligned} 0 \in & ((U_\alpha(n) + U_\alpha(n)) + U_\beta(n)) + U_\delta(n) + U_\xi(n) + U_{\zeta'}(n) + \\ & ((W_{i,\alpha,\beta} + W_{i,\alpha,\beta}) + W_{0,\xi,\zeta}) + ((W_{j,\alpha,\delta} + W_{j,\alpha,\delta}) + W_{1,\xi,\zeta}) \subseteq \\ & U_\beta(n-1) + U_\delta(n) + U_\xi(n) + U_{\zeta'}(n) + V_{0,\xi,\zeta} + V_{1,\xi,\zeta}, \end{aligned}$$

and $\beta, \delta, \xi, \zeta$ are all pairwise distinct. This again contradicts 6.4(A)(\otimes)₄.

- If $|\{\alpha, \beta\} \cap \{\gamma, \delta\}| = 1$, say $\alpha = \gamma$, $\beta \neq \delta$, and $\xi' = \zeta'$, then

$$0 \in ((U_\alpha(n) + U_\alpha(n)) + U_\beta(n)) + ((U_{\xi'}(n) + U_{\xi'}(n)) + U_\delta(n)) + \\ ((W_{i,\alpha,\beta} + W_{i,\alpha,\beta}) + W_{0,\xi,\zeta}) + ((W_{j,\alpha,\delta} + W_{j,\alpha,\delta}) + W_{1,\xi,\zeta}) \subseteq \\ U_\beta(n-1) + U_\delta(n-1) + V_{0,\xi,\zeta} + V_{1,\xi,\zeta},$$

and $\beta \neq \delta$. Again contradiction with 6.4(A)(\otimes)₄.

- If $\{\alpha, \beta\} = \{\gamma, \delta\}$, then we arrive to

$$0 \in ((U_\alpha(n) + U_\alpha(n)) + U_{\xi'}(n)) + ((U_\beta(n) + U_\beta(n)) + U_{\zeta'}(n)) + \\ ((W_{i,\alpha,\beta} + W_{i,\alpha,\beta}) + W_{0,\xi,\zeta}) + ((W_{j,\alpha,\beta} + W_{j,\alpha,\beta}) + W_{1,\xi,\zeta}) \subseteq \\ U_{\xi'}(n-1) + U_{\zeta'}(n-1) + V_{0,\xi,\zeta} + V_{1,\xi,\zeta}.$$

Considering cases $\xi' = \zeta'$ and $\xi' \neq \zeta'$ separately we easily get a contradiction with 6.4(A)(\otimes)₄.

- If $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$, then

$$0 \in ((W_{i,\alpha,\beta} + W_{i,\alpha,\beta}) + U_\alpha(n)) + ((W_{j,\gamma,\delta} + W_{j,\gamma,\delta}) + U_\beta(n)) + \\ U_\gamma(n) + U_\delta(n) + U_{\xi'}(n) + U_{\zeta'}(n) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta} \subseteq \\ U_\alpha(n-1) + U_\beta(n-1) + U_\gamma(n) + U_\delta(n) + U_{\xi'}(n) + U_{\zeta'}(n) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta}.$$

If $\xi' = \zeta'$ then this gives

$$0 \in U_\alpha(n-1) + U_\beta(n-1) + U_\gamma(n) + U_\delta(n-1) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta},$$

a contradiction. So $\xi' \neq \zeta'$ and we ask what is the intersection $\{\xi', \zeta'\} \cap \{\alpha, \beta, \gamma, \delta\}$. In each possible case we also get a contradiction. \square

Claim 6.12.2. *If $\chi(x, y) = A$ and $z \in \mathbf{X} \setminus \{x, y\}$, then either $\chi(x, z) \neq A$ or $\theta(x, z) \neq \theta(x, y)$.*

Proof of the Claim. Suppose $\chi(x, y) = \chi(x, z) = A$ and $\theta(x, y) = \theta(x, z) = \{\alpha, \beta\}$. By 6.12.1 we know that $\chi(y, z) = A$. Hence for some $\xi \neq \zeta$ and $i < 2$ we have

$$y + z \in U_\xi(n) + U_\zeta(n) + W_{i,\xi,\zeta} + W_{i,\xi,\zeta}.$$

Also,

$$y + z = y + x + x + z \in U_\alpha(n) + U_\beta(n) + W_{0,\alpha,\beta} + W_{0,\alpha,\beta} + U_\alpha(n) + U_\beta(n) + W_{0,\alpha,\beta} + W_{0,\alpha,\beta}.$$

Hence $0 \in U_\xi(n-1) + U_\zeta(n-1) + V_{i,\xi,\zeta} + V_{i,\xi,\zeta}$, and we get contradiction as usual. \square

Claim 6.12.3. $\chi(x, y) \neq B$ for any distinct $x, y \in \mathbf{X}$.

Proof of the Claim. Suppose $\chi(x, y) = B$, $\theta(x, y) = \{\alpha, \beta\}$. By 6.12.2 we may choose $z \in \mathbf{X} \setminus \{x, y\}$ such that

$$(\star)_z \quad (\chi(x, z), \theta(x, z)) \neq (A, \{\alpha, \beta\}) \neq (\chi(y, z), \theta(y, z)).$$

[Why is it possible? First take $t \notin \{x, y\}$ and ask if it has the property described in $(\star)_t$. If not, then $\theta(x, t) = \theta(y, t) = \{\alpha, \beta\}$ and either $\chi(x, t) = A$ or $\chi(y, t) = A$. Say the former holds true. Pick $u \in \mathbf{X} \setminus \{x, y, t\}$ and ask if this element has the property $(\star)_u$. By Claim 6.12.2 we have

$$(\chi(x, u), \theta(x, u)) \neq (\chi(x, t), \theta(x, t)) = (A, \{\alpha, \beta\}),$$

so if $(\star)_u$ fails this can be only because $(\chi(y, u), \theta(y, u)) = (A, \{\alpha, \beta\})$. Taking $z \in \mathbf{X} \setminus \{x, y, t, u\}$ and applying Claim 6.12.2 twice (with $\{x, t\}$ and $\{y, u\}$) we immediately see that $(\star)_z$ holds true.]

By Claim 6.12.1 we know that either $\chi(x, z) \neq A$ or $\chi(y, z) \neq A$; by the symmetry we may assume $\chi(x, z) \neq A$. Now we consider the other possibilities for the value of $\chi(x, z)$.

(i) If $\chi(x, z) = B$ and $\theta(x, z) = \{\alpha, \beta\}$, then

$$x + y, x + z \in U_\alpha(n) + U_\alpha(n) + W_{0,\alpha,\beta} + W_{1,\alpha,\beta}.$$

Hence $y + z \in U_\alpha(n-1) + U_\alpha(n-1) + U_\alpha(n) + U_\alpha(n)$. Also, for some $\xi', \zeta' \in \theta(y, z) = \{\xi, \zeta\}$ and $i, j < 2$ we have

$$y + z \in U_{\xi'}(n) + U_{\zeta'}(n) + W_{i,\xi,\zeta} + W_{j,\xi,\zeta},$$

where either $\xi' \neq \zeta'$ or $i \neq j$. Thus

$$0 \in U_\alpha(n-1) + U_\alpha(n-1) + U_\alpha(n) + U_\alpha(n) + U_{\xi'}(n) + U_{\zeta'}(n) + W_{i,\xi,\zeta} + W_{j,\xi,\zeta} \stackrel{\text{def}}{=} Y.$$

If $\xi' = \zeta'$ then $i \neq j$ and

$$Y \subseteq U_\alpha(n-1) + U_\alpha(n-1) + U_\alpha(n-1) + U_\alpha(n) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta} \subseteq Q_{0,\xi,\zeta} + Q_{1,\xi,\zeta},$$

and we get a contradiction with 6.4(A)(\otimes)₄. If $\xi' \neq \zeta'$ then

$$Y \subseteq U_{\xi'}(n-2) + U_{\zeta'}(n-1) + W_{i,\xi,\zeta} + W_{j,\xi,\zeta}$$

and regardless of i being equal to j or not, we may get a contradiction too.

(ii) If $\chi(x, z) = B$ and $\theta(x, z) = \{\gamma, \delta\} \neq \{\alpha, \beta\}$, then

$$\begin{aligned} x + z &\in U_\gamma(n) + U_\gamma(n) + W_{0,\gamma,\delta} + W_{1,\gamma,\delta} \quad \text{and} \\ x + y &\in U_\alpha(n) + U_\alpha(n) + W_{0,\alpha,\beta} + W_{1,\alpha,\beta}. \end{aligned}$$

Hence $y + z \in V_{0,\gamma,\delta} + W_{1,\gamma,\delta} + V_{0,\alpha,\beta} + W_{1,\alpha,\beta}$. Like before, for some $\xi', \zeta' \in \theta(y, z) = \{\xi, \zeta\}$ and $i, j < 2$ we have

$$y + z \in U_{\xi'}(n) + U_{\zeta'}(n) + W_{i,\xi,\zeta} + W_{j,\xi,\zeta},$$

where either $\xi' \neq \zeta'$ or $i \neq j$. Since $\{V_{0,\gamma,\delta}, V_{1,\gamma,\delta}\} \cap \{V_{0,\alpha,\beta}, V_{1,\alpha,\beta}\} = \emptyset$, like before we get a contradiction with 6.4(A)(\otimes)₄.

(iii) If $\chi(x, z) = C$ and $\theta(x, z) = \{\alpha, \beta\}$, then

$$\begin{aligned} x + y &\in U_\alpha(n) + U_\alpha(n) + W_{0,\alpha,\beta} + W_{1,\alpha,\beta} \\ x + z &\in U_\alpha(n) + U_\beta(n) + W_{0,\alpha,\beta} + W_{1,\alpha,\beta}. \end{aligned}$$

Also, $y + z \in U_{\xi'}(n) + U_{\zeta'}(n) + W_{i,\xi,\zeta} + W_{j,\xi,\zeta}$, where $\xi', \zeta' \in \theta(y, z) = \{\xi, \zeta\}$, $i, j < 2$ and either $\xi' \neq \zeta'$ or $i \neq j$. We consider 2 subcases now.

If $i = j$ then ($\xi' \neq \zeta'$ and) $\chi(y, z) = A$ so by the choice of z at the beginning we know that $\theta(y, z) \neq \{\alpha, \beta\}$. So we arrive to

$$0 \in ((U_\alpha(n) + U_\alpha(n)) + U_\alpha(n)) + ((W_{i,\xi,\zeta} + W_{i,\xi,\zeta}) + U_\beta(n)) + \\ ((W_{0,\alpha,\beta} + W_{0,\alpha,\beta}) + U_\xi(n)) + ((W_{1,\alpha,\beta} + W_{1,\alpha,\beta}) + U_\zeta(n)) \subseteq \\ U_\alpha(n-1) + U_\beta(n-1) + U_\xi(n-1) + U_\zeta(n-1)$$

and since $\{\xi, \zeta\} \neq \{\alpha, \beta\}$ a contradiction follows.

If $i \neq j$ then we get

$$0 \in ((U_\alpha(n) + U_\alpha(n)) + U_\alpha(n)) + ((W_{0,\alpha,\beta} + W_{0,\alpha,\beta}) + U_\beta(n)) + \\ ((W_{1,\alpha,\beta} + W_{1,\alpha,\beta}) + U_{\xi'}(n)) + U_{\zeta'}(n) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta} \subseteq \\ U_\alpha(n-1) + U_\beta(n-1) + U_{\xi'}(n-1) + U_{\zeta'}(n) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta},$$

and again a contradiction.

(iv) If $\chi(x, z) = C$ and $\theta(x, z) = \{\gamma, \delta\} \neq \{\alpha, \beta\}$, then

$$x + z \in U_\gamma(n) + U_\delta(n) + W_{0,\gamma,\delta} + W_{1,\gamma,\delta} \quad \text{and} \\ x + y \in U_\alpha(n) + U_\alpha(n) + W_{0,\alpha,\beta} + W_{1,\alpha,\beta}, \quad \text{and} \\ y + z \in U_{\xi'}(n) + U_{\zeta'}(n) + W_{i,\xi,\zeta} + W_{j,\xi,\zeta},$$

where $\xi', \zeta' \in \theta(y, z) = \{\xi, \zeta\}$, $i, j < 2$ and either $\xi' \neq \zeta'$ or $i \neq j$. Thus

$$0 \in (U_\gamma(n) + (U_\alpha(n) + U_\alpha(n))) + U_\delta(n) + W_{0,\gamma,\delta} + W_{1,\gamma,\delta} + \\ W_{0,\alpha,\beta} + W_{1,\alpha,\beta} + U_{\xi'}(n) + U_{\zeta'}(n) + W_{i,\xi,\zeta} + W_{j,\xi,\zeta} \subseteq \\ U_\gamma(n-1) + U_\delta(n) + U_{\xi'}(n) + U_{\zeta'}(n) + \\ W_{0,\gamma,\delta} + W_{1,\gamma,\delta} + W_{0,\alpha,\beta} + W_{1,\alpha,\beta} + W_{i,\xi,\zeta} + W_{j,\xi,\zeta}.$$

Since $W_{0,\gamma,\delta}, W_{1,\gamma,\delta}, W_{0,\alpha,\beta}$ and $W_{1,\alpha,\beta}$ are all distinct we get a contradiction in the usual manner. \square

Claim 6.12.4. $\chi(x, y) \neq C$ for any distinct $x, y \in \mathbf{X}$.

Proof of the Claim. Suppose towards contradiction $\chi(x, y) = C$ and let $\theta(x, y) = \{\alpha, \beta\}$. Let $z \in \mathbf{X} \setminus \{x, y\}$. By 6.12.1 we know that either $\chi(x, z) \neq A$ or $\chi(y, z) \neq A$; by the symmetry we may assume $\chi(x, z) \neq A$. By 6.12.3 we know that $\chi(x, z) \neq B$, so $\chi(x, z) = C$.

If $\theta(x, y) = \theta(x, z) = \{\alpha, \beta\}$, then $y + z \in U_\alpha(n-1) + U_\alpha(n) + U_\beta(n-1) + U_\beta(n)$. We know that $\chi(y, z) \in \{A, C\}$ and in both cases $y + z \in U_\gamma(n) + U_\delta(n) + W_{i,\gamma,\delta} + W_{j,\gamma,\delta}$, where $\theta(y, z) = \{\gamma, \delta\}$ and $i, j < 2$. Now we may conclude

$$0 \in U_\alpha(n-1) + U_\alpha(n) + U_\beta(n-1) + U_\beta(n) + U_\gamma(n) + U_\delta(n) + W_{i,\gamma,\delta} + W_{j,\gamma,\delta} \stackrel{\text{def}}{=} S.$$

If $i \neq j$ then $S \subseteq U_\gamma(n-2) + U_\delta(n-2) + W_{0,\gamma,\delta} + W_{1,\gamma,\delta}$ and an immediate contradiction with 6.4(A)(\otimes)₄ follows. If $i = j$ then $S \subseteq U_\gamma(n-2) + U_\delta(n-2) + W_{i,\gamma,\delta} + W_{i,\gamma,\delta} \subseteq U_\gamma(n-3) + U_\delta(n-3)$ and we get a contradiction with $p \in D_1^3$.

If $\theta(x, z) = \{\xi, \zeta\} \neq \theta(x, y) = \{\alpha, \beta\}$, then $\{W_{0,\xi,\zeta}, W_{1,\xi,\zeta}\} \cap \{W_{0,\alpha,\beta}, W_{1,\alpha,\beta}\} = \emptyset$ and

$$y + z \in U_\alpha(n) + U_\beta(n) + W_{0,\alpha,\beta} + W_{1,\alpha,\beta} + U_\xi(n) + U_\zeta(n) + W_{0,\xi,\zeta} + W_{1,\xi,\zeta}.$$

Now, by considerations as before, we get a contradiction with 6.4(A)(\otimes)₄. \square

Therefore,

$$(\odot) \quad \chi(x, y) = A \text{ for all distinct } x, y \in \mathbf{X}.$$

Hence, if $x \neq y$ are from \mathbf{X} and $\theta(x, y) = \{\alpha, \beta\}$, then $x + y \in U_\alpha(n-2) + U_\beta(n-2)$.

(2) Like Lemma 5.11, using Lemma 6.6. \square

Lemma 6.13. Let $p = (w, M, \bar{r}, n, \bar{\Upsilon}, \bar{V}, h) \in D_1^2$ and $\mathbf{X} \subseteq \mathbb{H}$, $|\mathbf{X}| \geq 5$. Suppose that

- (a) $\mathbf{X} + \mathbf{X} \subseteq \bigcup \{U_\alpha(n) + U_\beta(n) : \alpha, \beta \in w\}$, and
- (b) $\text{diam}_\rho(U_\alpha(n)) < \rho(x, y)$ for all $\alpha \in w$, $(x, y) \in \mathbf{X}^{(2)}$.

Then there is a $c \in \mathbb{H}$ such that

$$\mathbf{X} + c \subseteq \bigcup \{U_\alpha(n-1) : \alpha \in w\}.$$

Proof. By assumption (b), if $x, y \in \mathbf{X}$ are distinct and $x + y \in U_\alpha(n) + U_\beta(n)$, $\alpha, \beta \in w$, then $\alpha \neq \beta$. Also, if $(x, y) \in \mathbf{X}^{(2)}$ and $x + y \in (U_\alpha(n) + U_\beta(n)) \cap (U_\gamma(n) + U_\delta(n))$, then $\{\alpha, \beta\} = \{\gamma, \delta\}$ (by 6.4(A)(\otimes)₄). Consequently, for each $(x, y) \in \mathbf{X}^{(2)}$ we may let $\theta(x, y)$ to be the unique $\{\alpha, \beta\} \in [w]^2$ such that $x + y \in U_\alpha(n) + U_\beta(n)$.

Claim 6.13.1.

$$|\theta(x, y) \cap \theta(x, z)| = 1$$

whenever $x, y, z \in \mathbf{X}$ are distinct.

Proof of the Claim. Let $\theta(x, y) = \{\alpha, \beta\}$, $\theta(x, z) = \{\gamma, \delta\}$ and $\theta(y, z) = \{\xi, \zeta\}$. Then

$$y + z \in (U_\alpha(n) + U_\beta(n) + U_\gamma(n) + U_\delta(n)) \cap (U_\xi(n) + U_\zeta(n)).$$

Hence $0 \in U_\alpha(n) + U_\beta(n) + U_\gamma(n) + U_\delta(n) + U_\xi(n) + U_\zeta(n)$. Since $\alpha \neq \beta$, $\gamma \neq \delta$ and $\xi \neq \zeta$ we conclude that $\{\alpha, \beta\} \cap \{\gamma, \delta\} \neq \emptyset$ (remember 6.4(A)(\otimes)₄). If we had $\{\alpha, \beta\} = \{\gamma, \delta\}$, then $0 \in U_\xi(n-1) + U_\zeta(n-1)$, a contradiction as well. Consequently $|\{\alpha, \beta\} \cap \{\gamma, \delta\}| = 1$. \square

Fix distinct $x_0, y_0, z_0 \in \mathbf{X}$. Let $\theta(x_0, y_0) = \{\alpha_0, \beta_0\}$, $\theta(x_0, z_0) = \{\gamma_0, \alpha_0\}$ and let $a', a'' \in U_{\alpha_0}(n)$, $b_0 \in U_{\beta_0}(n)$, $c_0 \in U_{\gamma_0}(n)$ be such that $x_0 + y_0 = a' + b_0$ and $x_0 + z_0 = a'' + c_0$.

Let $c = a' + x_0$. We will show that $x + c \in \bigcup \{U_\alpha(n-1) : \alpha \in w\}$ for all $x \in \mathbf{X}$. To this end, first note that

- $x_0 + c = x_0 + a' + x_0 = a' \in U_{\alpha_0}(n)$,
- $y_0 + c = y_0 + a' + x_0 = a' + b_0 + a' = b_0 \in U_{\beta_0}(n)$,
- $z_0 + c = z_0 + a' + x_0 = a'' + c_0 + a' \in U_{\gamma_0}(n) + (U_{\alpha_0}(n) + U_{\alpha_0}(n)) \subseteq U_{\gamma_0}(n-1)$.

Now suppose $x \in \mathbf{X} \setminus \{x_0, y_0, z_0\}$. Let $\theta(x, x_0) = \{\delta, \zeta\}$, $x + x_0 = d + e$, $d \in U_\delta(n)$, $e \in U_\zeta(n)$.

(*) $\alpha_0 \in \{\delta, \zeta\}$.

Why? By Claim 6.13.1 we have $|\theta(x_0, x) \cap \theta(x_0, y_0)| = |\theta(x_0, x) \cap \theta(x_0, z_0)| = 1$. Hence if $\alpha_0 \notin \{\delta, \zeta\}$, then $\theta(x, x_0) = \{\beta_0, \gamma_0\}$. Take $x' \in \mathbf{X} \setminus \{x_0, y_0, z_0, x\}$ and note that (again by Claim 6.13.1)

$$|\theta(x_0, x') \cap \{\alpha_0, \beta_0\}| = |\theta(x_0, x') \cap \{\alpha_0, \gamma_0\}| = |\theta(x_0, x') \cap \{\gamma_0, \beta_0\}| = 1,$$

and this is clearly impossible.

By symmetry we may assume $\alpha_0 = \delta$. But now

$$x + c = x + x_0 + a' = (d + a') + e \in U_\zeta(n-1),$$

so we are done. \square

Lemma 6.14.

$\Vdash_{\mathbb{Q}}$ “there is no perfect set $P \subseteq \mathbb{H}$ such that $(\forall x, y \in P)(|(x + \bigcup_{m < \omega} \mathbf{F}_m) \cap (y + \bigcup_{m < \omega} \mathbf{F}_m)| \geq 2k)$.”

Proof. Suppose towards contradiction that $G \subseteq \mathbb{Q}$ is generic over \mathbf{V} and in $\mathbf{V}[G]$ the following assertion holds true:

for some perfect set $P \subseteq \mathbb{H}$ we have

$$\left| \left(x + \bigcup_{m < \omega} \mathbf{F}_m^G \right) \cap \left(y + \bigcup_{m < \omega} \mathbf{F}_m^G \right) \right| \geq 2k$$

for all $x, y \in P$.

Then for any distinct $x, y \in P$ there are $b_0, c_0, \dots, b_{k-1}, c_{k-1} \in \bigcup_{m < \omega} \mathbf{F}_m^G$ such that $x + y = b_i + c_i$ (for all $i < k$) and $\{b_i, c_i\} \cap \{b_{i'}, c_{i'}\} = \emptyset$ (for $i < i' < k$); remember $x + y = b_i + c_i$ implies that $x + b_i, x + c_i$ are distinct elements of $(x + \bigcup_{m < \omega} \mathbf{F}_m^G) \cap (y + \bigcup_{m < \omega} \mathbf{F}_m^G)$. For $\bar{\ell} = \langle \ell_i : i < k \rangle \subseteq \omega$, $\bar{m} = \langle m_i : i < k \rangle \subseteq \omega$ and $N < \omega$ let

$$Z_{\bar{\ell}, \bar{m}}^N = \left\{ (x, y) \in P^2 : \text{there are } c_i \in \mathbf{F}_{\ell_i}^G, d_i \in \mathbf{F}_{m_i}^G \text{ (for } i < k) \text{ such that} \right. \\ \left. x + y = b_i + c_i \text{ and } 2^{-N} < \min(\rho(b_i, b_j), \rho(c_i, b_j), \rho(b_i, c_j)) \right. \\ \left. \text{for all distinct } i, j < k \right\}.$$

Now we continue as in 5.12, but instead of 3.5 we use 6.13. In $(\square)_4^c$ as there we demand $p_\iota, q_\iota \in D_{n_\iota}^2$. Also under current assumptions on \mathbb{H} , $\mathbf{X}_\iota + c_\iota = c_\iota - \mathbf{X}_\iota$, so we have only one case. Otherwise the same proof works. \square

The following theorem is a consequence of results presented in this section.

Theorem 6.15. Assume that

- (1) $(\mathbb{H}, +, 0)$ is an Abelian perfect Polish group,
- (2) all elements of \mathbb{H} have order at most 2,
- (3) $2 \leq k < \omega$ and
- (4) $\varepsilon < \omega_1$ and λ is an uncountable cardinal such that $\text{NPr}^\varepsilon(\lambda)$ holds true.

Then there is a ccc forcing notion \mathbb{Q} of cardinality λ such that

$\Vdash_{\mathbb{Q}}$ “for some Σ_2^0 subset B of \mathbb{H} we have:
there is a set $X \subseteq \mathbb{H}$ of cardinality λ such that
 $(\forall x, y \in X)(|(x + B) \cap (y + B)| \geq 2k)$
but there is no perfect set $P \subseteq \mathbb{H}$ such that
 $(\forall x, y \in P)(|(x + B) \cap (y + B)| \geq 2k)$.”

7. Conclusions and questions

Let us recall from the Introduction, that *the spectrum of translation k -non-disjointness of a set $A \subseteq \mathbb{H}$* is

$$\text{stnd}_k(A) = \text{stnd}_k(A, \mathbb{H}) = \{(x, y) \in \mathbb{H} \times \mathbb{H} : |(A + x) \cap (A + y)| \geq k\}.$$

By the definition, $X \times X \subseteq \text{stnd}_k(A)$ if and only if

$$(\forall x, y \in X) (|(x + A) \cap (y + A)| \geq k).$$

In particular, there is a perfect square $P \times P$ included in $\text{stnd}_k(A)$ if and only if A has a perfect set P of k -overlapping translations.

Conclusion 7.1. Assume that

- (a) $\mathbb{H} = (\mathbb{H}, 0, +)$ is a perfect Abelian Polish group,
- (b) $1 < \iota < \omega$ and
 - $k = \iota$ if $\{c \in \mathbb{H} : c + c \neq 0\}$ is dense in \mathbb{H} , and
 - $k = 2\iota$ otherwise,
- (c) λ is an uncountable cardinal such that $\text{NPr}^\varepsilon(\lambda)$ holds true for some countable ordinal ε , and
- (d) $\lambda = \lambda^{\aleph_0} \leq \mu = \mu^{\aleph_0}$.

Then there is a ccc forcing notion \mathbb{P}^* and a \mathbb{P}^* -name \underline{B} for a Σ_2^0 subset of \mathbb{H} such that

- (1) $\Vdash_{\mathbb{P}^*} "2^{\aleph_0} = \mu"$,
- (2) $\Vdash_{\mathbb{P}^*} "$ there is a set $X \subseteq \mathbb{H}$ of cardinality λ such that $X \times X \subseteq \text{stnd}_k(\underline{B})"$, but
- (3) $\Vdash_{\mathbb{P}^*} "$ there is **no** set $X \subseteq \mathbb{H}$ of cardinality λ^+ such that $X \times X \subseteq \text{stnd}_k(\underline{B})"$, and
- (4) $\Vdash_{\mathbb{P}^*} "$ there is **no** perfect set $P \subseteq \mathbb{H}$ such that $P \times P \subseteq \text{stnd}_k(\underline{B})"$.

Proof. Let us consider the case when (in assumption (b) of the Corollary) the set $\{c \in \mathbb{H} : c + c \neq 0\}$ is dense in \mathbb{H} . The other case is fully parallel. So we assume

- $(\mathbb{H}, +, 0)$, \mathbf{D} , ρ and \mathcal{U} are as in Assumption 5.1 and Assumption 5.2,
- $k, \varepsilon, \lambda, \text{rk}^{\text{sp}}, \mathbf{j}, \mathbf{k}$ and μ satisfy Assumption 5.2 and assumption (d) of the Corollary.

Let \mathbb{P} be the forcing notion discussed in Section 5 (cf Theorem 5.13) and let \mathbb{C}_μ be the forcing notion adding μ Cohen reals, where conditions are finite functions with domains included in μ and values 0, 1.

Let $\mathbb{P}^* = \mathbb{P} \times \mathbb{C}_\mu$.

By standard arguments, \mathbb{P}^* is a ccc forcing notion and $\Vdash_{\mathbb{P}^*} 2^{\aleph_0} = \mu$. Let \underline{B} be a \mathbb{P} -name for the Σ_2^0 subset of \mathbb{H} added by $\mathbb{P} \triangleleft \mathbb{P}^*$.

Claim 7.1.1.

- (2) $\Vdash_{\mathbb{P}^*} "$ there is a set $X \subseteq \mathbb{H}$ of cardinality λ such that

$$(\forall x, y \in X) (|(x + \underline{B}) \cap (y + \underline{B})| \geq k) ",$$

but

(4) $\Vdash_{\mathbb{P}^*}$ “there is **no** perfect set $P \subseteq \mathbb{H}$ such that

$$(\forall x, y \in P)(|(x + \mathcal{B}) \cap (y + \mathcal{B})| \geq k) ”.$$

Proof of the Claim. If $H \subseteq \mathbb{C}_\mu$ is generic over \mathbf{V} , then in $\mathbf{V}[H]$ we may look at the definition of the forcing notion \mathbb{P} as all the ingredients still have the required properties. Identifying $\mathbf{B}_n^{\mathbf{V}}$ with $\mathbf{B}_n^{\mathbf{V}[H]}$ we easily see that $\mathbb{P}^{\mathbf{V}} = \mathbb{P}^{\mathbf{V}[H]}$. Hence \mathbb{P}^* is equivalent to the iteration $\mathbb{C}_\mu * \mathbb{P}$ and consequently the results of Section 5 give the desired conclusion. \square

Claim 7.1.2.

(3) $\Vdash_{\mathbb{P}^*}$ “there is **no** set $X \subseteq \mathbb{H}$ of cardinality λ^+ such that

$$(\forall x, y \in X)(|(x + \mathcal{B}) \cap (y + \mathcal{B})| \geq k) ”.$$

Proof of the Claim. Assume $\lambda < \mu$ (otherwise clear). Suppose towards contradiction that $G = G_0 \times G_1 \subseteq \mathbb{P} \times \mathbb{C}_\mu$ is generic over \mathbf{V} and in $\mathbf{V}[G_0][G_1]$ there are distinct $x_\alpha \in \mathbb{H}$ (for $\alpha < \lambda^+$) such that

$$|(x_\alpha + \mathcal{B}^G) \cap (x_\beta + \mathcal{B}^G)| \geq k \quad \text{for } \alpha, \beta < \lambda^+.$$

Then in $\mathbf{V}[G_0]$ we may find a condition $q \in G_1$ and \mathbb{C}_μ -names \dot{x}_α , $\alpha < \lambda^+$, for elements of the group \mathbb{H} such that

$$q \Vdash_{\mathbb{C}_\mu} “\dot{x}_\alpha \neq \dot{x}_\beta \text{ and } |(\dot{x}_\alpha + \mathcal{B}) \cap (\dot{x}_\beta + \mathcal{B})| \geq k”$$

for all $\alpha < \beta < \lambda^+$. Each of the names \dot{x}_α is actually a \mathbb{C}_{A_α} -name for some countable set $A_\alpha \subseteq \mu$. Since $\mathbf{V}[G_0] \models 2^{\aleph_0} = \lambda$, we may choose a set $I \in [\lambda^+]^{\lambda^+}$ and a set $u \subseteq \mu$ such that the following two demands are satisfied (in $\mathbf{V}[G_0]$).

(♣)₁ $\text{otp}(A_\alpha) = \text{otp}(A_\beta)$ for $\alpha, \beta \in I$.

(♣)₂ For each $\alpha < \beta$ from I , letting $\pi_{\alpha, \beta} : A_\alpha \rightarrow A_\beta$ be the order isomorphism, we have

$$u = A_\alpha \cap A_\beta, \quad \pi_{\alpha, \beta} \upharpoonright u = \text{id}_u \quad \text{and} \quad A_\alpha \setminus u \text{ is infinite.}$$

Let $u^* = u \cup \text{dom}(q) \subseteq \mu$. Dismissing finitely many elements of I we may assume that $A_\alpha \setminus u = A_\alpha \setminus u^*$ for all $\alpha \in I$.

Let $G_1^* = G_1 \cap \mathbb{C}_{u^*}$ and let us work in $\mathbf{V}[G_0][G_1^*]$ for a moment. Each name \dot{x}_α (for $\alpha \in I$) can be thought of as a $\mathbb{C}_{A_\alpha \setminus u^*}$ -name now. Let $\xi = \text{otp}(A_\alpha \setminus u^*)$ for some (equivalently, all) $\alpha \in I$. Since $\mathbf{V}[G_0][G_1^*] \models 2^{\aleph_0} = \lambda$, we may find $I^* \in [I]^{\lambda^+}$ and a Borel function $\tau : {}^\xi 2 \rightarrow \mathbb{H}$ such that

(♣)₃ $\Vdash \dot{x}_\alpha = \tau(\mathcal{C}_\alpha \circ \pi^\alpha)$, where $\pi^\alpha : {}^\xi 2 \rightarrow A_\alpha \setminus u^*$ is the order isomorphism and \mathcal{C}_α is (a name for) the Cohen real added by $\mathbb{C}_{A_\alpha \setminus u^*}$.

Consequently, if $\alpha \neq \beta$ are from I^* , then

(♣)₄ $\Vdash_{\mathbb{C}_{A_\alpha \setminus u^*} \times \mathbb{C}_{A_\beta \setminus u^*}} “|(\tau(\mathcal{C}_\alpha \circ \pi^\alpha) + \mathcal{B}^{G_0}) \cap (\tau(\mathcal{C}_\beta \circ \pi^\beta) + \mathcal{B}^{G_0})| \geq k \text{ and } \tau(\mathcal{C}_\alpha \circ \pi^\alpha) \neq \tau(\mathcal{C}_\beta \circ \pi^\beta)”$

Therefore,

(♣)₅ if $d_0, d_1 \in {}^\xi 2$ are (mutually) Cohen reals over $\mathbf{V}[G_0][G_1^*]$, then

$$\mathbf{V}[G_0][G_1^*][d_0, d_1] \models |(\tau(d_0) + \mathcal{B}^{G_0}) \cap (\tau(d_1) + \mathcal{B}^{G_0})| \geq k \text{ and } \tau(d_0) \neq \tau(d_1).$$

Adding one Cohen real c over \mathbf{V} results in adding a perfect set P such that (in $\mathbf{V}[c]$) if $x, y \in P$ are distinct then x is a Cohen real over $\mathbf{V}[y]$. Thus taking $\alpha \in I$ we will have that in $\mathbf{V}^* = \mathbf{V}[G_0][G_1^*][G_1 \cap \mathbb{C}_{A_\alpha \setminus u^*}]$ there is a perfect set $P \subseteq {}^\xi 2$ of mutually Cohen reals over $\mathbf{V}[G_0][G_1^*]$. By (♣)₅ we know

$$\mathbf{V}^* \models \tau \upharpoonright P \text{ is one-to-one and } |(\tau(x) + \mathcal{B}^{G_0}) \cap (\tau(y) + \mathcal{B}^{G_0})| \geq k \text{ for } x, y \in P.$$

By upward absoluteness of Σ_3^1 sentences we may assert now that

$$\mathbf{V}[G_0 \times G_1] \models \text{there is a perfect set } P^* \subseteq \mathbb{H} \text{ such that} \\ (\forall x, y \in P^*) (|(x + \mathcal{B}^{G_0}) \cap (y + \mathcal{B}^{G_0})| \geq k).$$

This, however, contradicts Claim 7.1.1. $\square \quad \square$

Conclusion 7.2 (See [11, Proposition 3.3(5)]). Assume that

- (1) \mathbb{H} is a perfect Polish group and $B \subseteq \mathbb{H}$ is a Borel set,
- (2) a cardinal λ is such that $\text{Pr}^\varepsilon(\lambda)$ holds true for every $\varepsilon < \omega_1$, and
- (3) $1 < k < \omega$, and
- (4) there is a set $X \subseteq \mathbb{H}$ of cardinality λ such that $X \times X \subseteq \text{stnd}_k(B)$.

Then there is a perfect set $P \subseteq \mathbb{H}$ such that $P \times P \subseteq \text{stnd}_k(B)$.

Proof. Under our assumptions on λ , if an analytic set $B \subseteq {}^\omega 2 \times {}^\omega 2$ includes a λ -square, it includes a perfect square (see [14, Claim 1.12(1)]).

The space \mathbb{H} is Borel isomorphic with ${}^\omega 2$; let $f : \mathbb{H} \rightarrow {}^\omega 2$ be a Borel isomorphism and let $f_2 : \mathbb{H} \times \mathbb{H} \rightarrow {}^\omega 2 \times {}^\omega 2 : (x, y) \mapsto (f(x), f(y))$. Then the set $f_2[\text{stnd}_k(B)]$ is analytic and $f[X] \times f[X] \subseteq f_2[\text{stnd}_k(B)]$. Consequently there is a perfect set $P^* \subseteq {}^\omega 2$ such that $P^* \times P^* \subseteq f_2[\text{stnd}_k(B)]$. We may choose a perfect set $P \subseteq f^{-1}[P^*] \subseteq \mathbb{H}$ – it will also satisfy $P \times P \subseteq \text{stnd}_k(B)$. \square

Now, in Claim 7.1.2 we used the upward absoluteness to show $\Vdash_{\mathbf{P}^*}$ “(3)”. If the group \mathbb{H} is compact and $B \subseteq \mathbb{H}$ is Σ_2^0 , then the set $\text{stnd}_k(B)$ is Σ_2^0 and hence the assertion in (4) of 7.1 is Π_2^1 , so also absolute. However, in the case of general \mathbb{H} the corresponding assertion appears to be Π_3^1 so not so obviously absolute. The absoluteness would be helpful for natural consequences under MA. In [10] a similar issue in the Cantor space ${}^\omega 2$ was dealt with by consideration of “finite approximations” to a perfect set of non-disjoint translations and a rank function on those approximations.

Problem 7.3. Develop the rank and the results parallel to ndrk_k and cute \mathcal{YZR} -systems presented in [10] for the case of general perfect Abelian Polish groups.

The forcing notions presented in this article for various Abelian Polish groups look similar, but the particular group structures may have different impacts.

Problem 7.4. Is it consistent that for some perfect Abelian Polish groups \mathbb{H}_0 , \mathbb{H}_1 and $2 < k < \omega$ and an uncountable cardinal λ we have:

- (1) for some Borel set $B_0 \subseteq \mathbb{H}_0$,

- (a) there is a set $X \subseteq \mathbb{H}_0$ of cardinality λ such that $X \times X \subseteq \text{stnd}_k(B_0, \mathbb{H}_0)$ (i.e., $\text{stnd}_k(B_0, \mathbb{H}_0)$ includes a λ -square), but
 - (b) there is **no** perfect set $P \subseteq \mathbb{H}_0$ such that $P \times P \subseteq \text{stnd}_k(B_0, \mathbb{H}_0)$ (i.e., $\text{stnd}_k(B_0, \mathbb{H}_0)$ does not include any perfect square)
- and
- (2) for every Borel set $B \subseteq \mathbb{H}_1$, if $\text{stnd}_k(B, \mathbb{H}_1)$ includes a λ -square, then it includes a perfect square?

Considering differences caused by various choices of parameters, it is natural to ask about the impact of k .

Problem 7.5. Is it consistent that for some perfect Abelian Polish group \mathbb{H} and $2 < k < \ell < \omega$ and an uncountable cardinal λ the following two statements are true.

- (1) For some Borel set $B_0 \subseteq \mathbb{H}$,
 - (a) there is a set $X \subseteq \mathbb{H}$ of cardinality λ such that $X \times X \subseteq \text{stnd}_\ell(B_0, \mathbb{H})$, but
 - (b) there is **no** perfect set $P \subseteq \mathbb{H}_0$ such that $P \times P \subseteq \text{stnd}_\ell(B_0, \mathbb{H}_0)$.
- (2) For every Borel set $B \subseteq \mathbb{H}$, if $\text{stnd}_k(B, \mathbb{H})$ includes a λ -square, then it includes a perfect square.

Of course, the next steps could be to investigate stnd_ω and stnd_{ω_1} :

Problem 7.6. Let \mathbb{H} be a perfect Abelian Polish group. Is it consistent that for some Borel set $B \subseteq \mathbb{H}$:

- there is an uncountable set $X \subseteq \mathbb{H}$ such that $(B+x) \cap (B+y)$ is uncountable for every $x, y \in X$, but
- for every perfect set $P \subseteq \mathbb{H}$ there are $x, y \in P$ with $(B+x) \cap (B+y)$ countable?

Similarly if “uncountable / countable” are replaced with “infinite / finite”, respectively.

Let us also remind two other questions related to our results. The first one calls for a “dual” results.

Problem 7.7. Is it consistent to have a Borel set $B \subseteq \mathbb{H}$ such that

- B has uncountably many pairwise disjoint translations, but
- there is no perfect of pairwise disjoint translations of B ?

Assumptions of Conclusion 7.1 and Conclusion 7.2 bring the question what is the value of the first cardinal $\lambda = \lambda_{\omega_1}$ such that $\text{Pr}^\varepsilon(\lambda)$ holds true every $\varepsilon < \omega_1$.

Problem 7.8. Is $\lambda_{\omega_1} = \aleph_{\omega_1}$? Does $\text{Pr}^\varepsilon(\aleph_{\omega_1})$ hold true for all $\varepsilon < \omega_1$?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

- [1] Marek Balcerzak, Andrzej Rosłanowski, Saharon Shelah, Ideals without ccc, *J. Symb. Log.* 63 (1998) 128–147, arXiv:math/9610219.
- [2] Tomek Bartoszyński, Haim Judah, *Set Theory: On the Structure of the Real Line*, A K Peters, Wellesley, Massachusetts, 1995.
- [3] Howard Becker, Alexander S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Mathematical Society Lecture Notes Series, vol. 232, Cambridge University Press, Cambridge, 1996.
- [4] Udayan B. Darji, Tamás Keleti, Covering \mathbb{R} with translates of a compact set, *Proc. Am. Math. Soc.* 131 (2003) 2593–2596.
- [5] Márton Elekes, Tamás Keleti, Decomposing the real line into Borel sets closed under addition, *Math. Log. Q.* 61 (2015) 466–473.
- [6] Márton Elekes, Juris Steprāns, Less than 2^ω many translates of a compact nullset may cover the real line, *Fundam. Math.* 181 (2004) 89–96.
- [7] Paul Erdős, Richard Rado, A partition calculus in set theory, *Bull. Am. Math. Soc.* 62 (1956) 427–489.
- [8] Thomas Jech, *Set Theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.
- [9] Andrzej Rosłanowski, Vyacheslav V. Rykov, Not so many non-disjoint translations, *Proc. Am. Math. Soc. Ser. B* 5 (2018) 73–84, arXiv:1711.04058.
- [10] Andrzej Rosłanowski, Saharon Shelah, Borel sets without perfectly many overlapping translations II, in: Melvin Fitting, Massoud Pourmahdian, Adrian Rezus, Ali Sadegh Daghigh (Eds.), *Research Trends in Contemporary Logic*, submitted for publication, arXiv:1909.00937.
- [11] Andrzej Rosłanowski, Saharon Shelah, Borel sets without perfectly many overlapping translations, *Rep. Math. Log.* 54 (2019) 3–43, arXiv:1806.06283.
- [12] Andrzej Rosłanowski, Saharon Shelah, Borel sets without perfectly many overlapping translations IV, *Colloq. Math.* (2025), Online First, doiA-cm9104-12-2024, arXiv:2302.12964.
- [13] Saharon Shelah, Disjoint translations.
- [14] Saharon Shelah, Borel sets with large squares, *Fundam. Math.* 159 (1999) 1–50, arXiv:math/9802134.
- [15] Piotr Zakrzewski, On Borel sets belonging to every invariant ccc σ -ideal on $2^{\mathbb{N}}$, *Proc. Am. Math. Soc.* 141 (2013) 1055–1065.