CONSISTENCY OF SQUARE BRACKET PARTITION RELATION

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ABSTRACT. Characteristic earlier results were of the form $\mathrm{CON}(2^{\aleph_0} \to [\lambda]^2_{n,2})$, with $2^{\aleph_0}$ an ex-large cardinal, in the best case the first weakly Mahlo cardinal.

Characteristic new results are $\mathrm{CON}\big((2^{\aleph_0} = \aleph_m) + \aleph_\ell \to [\aleph_k]^2_{n,2}\big)$, where $k < \ell < m$. So we improve in three respects: the continuum may be small (e.g. not a Mahlo weakly inaccessible), we use no large cardinal, and the cardinals λ involved are $< 2^{\aleph_0}$ after the forcing.

§ 0. INTRODUCTION

In their seminal list of problems [EH71], Erdös and Hajnal posed the question (15(a)): does $2^{\aleph_0} \not\to [\aleph_2]^2_3$? Recently, Komjáth [Kom21] provided a comprehensive update on this topic.

We continue here works which start with the problem above:[She88, §2], [She92], [She89], [She95] [She96], [She00] and the work with Rabus [RS00].

The simplest case of our result is (recall 0.3 below):

Theorem 0.1. *Assume* GCH *for transparency.* *__Then__ for some ccc forcing notion of cardinality $\aleph_6$ in the universe $\mathbf{V}^{\mathbb{P}}$, we have $2^{\aleph_0} = \aleph_6$ and for any $n \geq 3$, $\aleph_5 \to [\aleph_1]^2_{n,2}$.*

Proof. Choose $(\mu, \theta, \partial, \lambda)$ as $(\aleph_6, \aleph_5, \aleph_1, \aleph_0)$ and apply Theorem 0.2 $\square_{0.1}$

The general case is:

Theorem 0.2. *Assume $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$ and $2^{\partial^{+\ell}} = \partial^{+\ell+1}$ for $\ell = 0, 1, 2, 3$ and $\partial^{+4} < \theta$. __Then__ for some λ^+-cc, $(< \lambda)$-complete forcing notion $\mathbb{P}$ (so the forcing does not collapse any cardinal and preserve cardinal arithmetic outside $[\lambda, \mu)$) of cardinality μ, in the universe $\mathbf{V}^{\mathbb{P}}$ we have, $2^\lambda = \mu$ and for every $\sigma < \lambda$, $\theta \to [\partial]^2_{\sigma,2}$.*

Proof. All this paper is dedicated to provingthis theorem. Pedantically Hypothesis 1.1 holds (see Fact 1.12) so we can apply Concusion 1.11. $\square_{1.11}$

We may replace $\theta \to [\partial]^2_{\sigma,2}$ by $(\forall \partial_1 < \partial)(\exists \theta_1 < \theta)[\theta_1 \to [\partial_1]^2_{\sigma,2}]$ and change the assumption on cardinal arithmetic accordingly.

Recall,

Definition 0.3. For possibly finite cardinals θ, ∂, σ and κ, let $\theta \to [\partial]^\kappa_{\sigma,\kappa}$ mean:

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- if $\mathbf{c}$ is a function from $[\theta]^2 := \{u \subseteq \theta \colon |u| = \kappa\}$ into σ, <u>then</u> there exists some subset $\mathscr{U}$ of θ of cardinality ∂ such that $\{\mathbf{c}(u) \colon u \in [\mathscr{U}]^2\}$ has at most κ-many members.

§ 0(A). **Preliminaries.**

Notation 0.4. $\mathrm{cof}(\delta)$ is the class of ordinals of cofinality $\mathrm{cf}(\delta)$.

Notation 0.5.
(1) $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ are forcing notions.
(2) p, q, r called *conditions* are members of a forcing notion.
(3) $\mathbf{q}$ is as in Definition 1.3, some kind of $(< \lambda)$-support iterated forcing with extra information.

Notation 0.6. We may write e.g. $N[\mathbf{q}, \beta, u]$ instead $N_{\mathbf{q},\beta,u}$ to help with sub-scripts (or super-script).

Definition 0.7. Let θ, ∂, κ and λ be infinite cardinals. We say that $\theta \rightarrow_{\mathrm{sq}} (\partial)^{\lambda,2}_{\kappa}$ <u>when</u> $\theta > \partial \geq \kappa \geq \lambda$ and:

$\boxplus$ If (a) then (b), where:
(a) $\mathscr{B}$ is an expansion of $(\mathscr{H}_{<\lambda}(\mu), \in, <_*)$, where $<_*$ is a well-ordering of $\mathscr{H}_{<\lambda}(\mu)$, $\mu^+ > \theta$, and its vocabulary $\tau_{\mathscr{B}}$ has cardinality $\leq \lambda$.
(b) There is a tuple $\mathbf{s} = (\mathscr{U}, \bar{N}, \bar{\pi})$ solving $\mathbf{p} = (\theta, \partial, \kappa, \lambda, \mathscr{B})$, which means: $\boxplus_{\mathbf{p},\mathbf{s}}$ for $u, v \in [\theta]^{\leq 2}$,
- $\bullet_1$ $\bar{N} = \langle N_u \colon u \in [\mathscr{U}]^{\leq 2} \rangle$,
- $\bullet_2$ $\mathscr{U} \subseteq \theta$ is such that $\mathrm{otp}(\mathscr{U}) = \partial$,
- $\bullet_3$ $N_u \prec \mathscr{B}$, $[N_u]^{<\lambda} \subseteq N_u$,
- $\bullet_4$ $\varepsilon[\mathbf{s}] := \min(\mathscr{U}_{\mathbf{s}})$,
- $\bullet_5$ $N_u \cap \mathscr{U} = u$,
- $\bullet_6$ $\|N_u\| = \kappa$ and $\kappa + 1 \subseteq N_u$,
- $\bullet_7$ $N_u \cap N_v \prec N_{u \cap v}$,
- $\bullet_8$ $\bar{\pi} = \langle \pi_{u,v} \colon u, v \in [\mathscr{U}]^{\leq 2}$ and $|u| = |v| \rangle$ such that if $|u| = |v|$, then $\pi_{u,v}$ is an isomorphism from N_v onto N_u mapping v onto u,
- $\bullet_9$ if $u_1 \subseteq u_2$ and $v_1 \subseteq v_2$ all from $[\mathscr{U}]^{\leq 2}$ and $|u_2| = |v_2|$ $\pi''_{v_2,u_2}(u_1) = v_1$ <u>then</u> $\pi_{v_1,u_1}, \pi_{v_2,u_2}$ are compatible functions.

§ 1. The forcing

Our aim here is to prove the consistency of the following configuration:

$$2 < \sigma < \lambda = \lambda^{<\lambda} \leq \partial = \partial^{<\lambda} \leq \mu = \mu^{\theta} = 2^{\lambda},$$

and having $\theta \rightarrow [\partial]^2_{\sigma,2}$.

A continuation is in preparation [S+], aiming to further develop the directions explored here, particularly for the case of superscript $\mathbf{n} > 2$, as dealt within [She92]. We also show there that we can weaken the demands on the cardinals.

Hypothesis 1.1. The parameter $\mathbf{p} = (\theta, \partial, \partial, \lambda, \mathscr{B})$ consists of the following:
(a) $\lambda = \lambda^{<\lambda} < \partial < \theta \leq \mu = \mu^{\theta}$,
(b) $\theta \rightarrow_{\mathrm{sq}} (\partial)^{\lambda,2}_{\partial}$ (see Definition 0.7, a variant of [She89, 2.1]).
(c) σ will vary on the cardinal numbers from $[2, \lambda)$ and the "nice" μ are such that $\gamma < \mu \Rightarrow |\gamma|^{\theta} < \mu$.
(d)
- $\mathscr{B}$ is a model expanding $(\mathscr{H}_{<\lambda}(\mu), \in, <_{\mathscr{B}}, \gamma, \mathbb{P}, \underset{\sim}{\mathbf{c}})$, where $<_{\mathscr{B}}$ is well-ordering of $\mathscr{H}_{<\lambda}(\mu)$,
- $\tau(\mathscr{B})$ is a vocabulary of cardinality $\leq \lambda$.

We intend to use $(<\lambda)$-support iterated forcing of quite a special kind but first, we define the iterand.

Definition 1.2.
(1) Let $\mathbf{A}$ be the set of objects $\mathbf{a}$ consisting of (so $\gamma = \gamma_{\mathbf{a}}$, $N_u = N_{\mathbf{a},u}$):

(a)
- $\gamma \le \mu$ and $\sigma \in (2, \lambda)$,
- $\mathbb{P}$ is a forcing notion such that:
$$p \in \mathbb{P} \Rightarrow \mathrm{dom}(p) \in [\gamma]^{<\lambda} \wedge (\forall \alpha \in \mathrm{dom}(p))[p(\alpha) \in [\omega \cup \gamma]^{<\lambda}],$$
- $\mathbb{P}$ is λ^+-cc and $(<\lambda)$-complete,
- the order $\le_{\mathbb{P}}$ is: $p \le_{\mathbb{P}} q$ iff:
$$\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \wedge (\forall \alpha \in \mathrm{dom}(p))[p(\alpha) \subseteq q(\alpha)],$$

(b)
- $\underset{\sim}{\mathbf{c}}$ is a $\mathbb{P}$-name of a function from $[\theta]^2$ to σ (we may write $\underset{\sim}{\mathbf{c}}(\alpha, \beta)$ instead $\underset{\sim}{\mathbf{c}}\{\alpha, \beta\}$ for $\alpha \ne \beta < \theta$).

(c) There is a triple $(\mathscr{U}, \bar{N}, \pi)$ solving $\mathbf{p} = (\theta, \partial, \partial, \lambda, \mathscr{B})$ (see Definition 0.7⊞(b), 1.1) such that $\underset{\sim}{\mathbf{c}} \in N_u$ for every $u \in [\mathscr{U}]^{\le \partial}$.

(1A) In the context of Definition 1.2(1), $\mathbf{a} = (\gamma, \mathscr{B}, \mathbb{P}, \underset{\sim}{\mathbf{c}}, \mathscr{U}, \bar{N}, \bar{\pi}) = (\gamma_{\mathbf{a}}, \mathscr{B}_{\mathbf{a}}, ...)$, so e.g. $N_{\mathbf{a},u} = N_u$.

(2) We say that the pair $(p, \bar{\iota})$ is a *solution* of $\mathbf{a} \in \mathbf{A}$, denoted by $(\mathbf{a}, p, \bar{\iota}) \in \mathbf{A}^+$, <u>when</u>,

(a) $\bar{\iota} = (\iota_1, \iota_2) \in \sigma \times \sigma$,
(b) $p \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$,
(c) if $p \le q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ and $\zeta_1 < \zeta_2$ are from $\mathscr{U}$ <u>then</u> there are q_1, q_2, r_1, r_2 such that for $\ell = 1, 2$, we have:
  - $\bullet_0$ $q \le_{\mathbb{P}_{\mathbf{a}}} q_\ell$,
  - $\bullet_1$ $q_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ and $q_1 \restriction (N_{\mathbf{a},\emptyset} \cap \lg(\mathbf{q})) = q_2 \restriction (N_{\mathbf{a},\emptyset} \cap \lg(\mathbf{q}))$,
  - $\bullet_2$ $r_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\zeta_1,\zeta_2\}}$,
  - $\bullet_3$ $r_\ell \Vdash$ "$\underset{\sim}{\mathbf{c}}(\zeta_1, \zeta_2) = \iota_\ell$",
  - $\bullet_4$ $r_\ell \restriction N_{\mathbf{a},\{\zeta_1\}}$ is $\le_{\mathbb{P}_{\mathbf{a}}}$-below $\pi^{\mathbf{a}}_{\{\zeta_1\},\{\varepsilon[\mathbf{a}]\}}(q_\ell)$,
  - $\bullet_5$ $r_\ell \restriction N_{\mathbf{a},\{\zeta_2\}}$ is $\le_{\mathbb{P}_{\mathbf{a}}}$-below $\pi^{\mathbf{a}}_{\{\zeta_2\},\{\varepsilon[\mathbf{a}]\}}(q_{\zeta_{3-\ell}})$.

(3) If $\mathbf{b} = (\mathbf{a}, p, \bar{\iota}) \in \mathbf{A}^+$ then let $\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}$ be the $\mathbb{P}$-name of the following forcing notion:

$(*)$ For $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$,
  (a) the set of elements of $\mathbb{Q}_{\mathbf{b}} = \underset{\sim}{\mathbb{Q}}_{\mathbf{b}}[\mathbf{G}]$ is:

$$\Big\{u \in [\mathscr{U}]^{<\lambda}\colon \text{ if } \zeta_1 < \zeta_2 \text{ in } \mathscr{U}, \text{ then } \underset{\sim}{\mathbf{c}}\{\zeta_1, \zeta_2\}[\mathbf{G}] \in \{\iota_1, \iota_2\}, \text{ moreover}$$
$$\text{for some } q_1, q_2, r_1, r_2 \text{ as in Definition 1.2(1)(c)}(\bullet_1)\text{-}(\bullet_5), \text{ we have } r_1 \in \mathbf{G} \text{ or } r_2 \in \mathbf{G}\Big\},$$

  (b) the order of $\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}[\mathbf{G}]$ is the inclusion,
  (c) the generic is $\underset{\sim}{\mathcal{V}}_{\mathbf{b}} = \bigcup \mathbf{G}_{\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}}$.

Definition 1.3.
(1) Let $\mathbf{Q} := \mathbf{Q}_{\mathbf{p}}$ be the class of $\mathbf{q}$ which consist of (below, $\alpha \le \lg(\mathbf{q})$ and $\beta < \lg(\mathbf{q})$ and e.g. $\mathbb{P}_\alpha = \mathbb{P}_{\mathbf{q},\alpha}$):

(a) $\lg(\mathbf{q})$ is an ordinal $\le \mu$,
(b) $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \colon \alpha \le \lg(\mathbf{q}), \beta < \lg(\mathbf{q})\rangle$ is a $(<\lambda)$-support iteration,
(c) $\mathbb{P}_\beta$ satisfies the λ^+-cc,
(d) $\underset{\sim}{\mathbb{Q}}_\beta$ is $\underset{\sim}{\mathbb{Q}}_{\mathbf{b}_\beta}$, where:
  - $\bullet_1$ $\mathbf{b}_\beta := (\mathbf{a}_\beta, p^*_\beta, \bar{\iota}^*_\beta) \in \mathbf{A}^+$, and
  - $\bullet_2$ $\mathbf{a}_\beta := (\gamma_\beta, \mathscr{B}_\beta, \mathbb{P}^\bullet_\beta, \underset{\sim}{\mathbf{c}}_\beta, \mathscr{U}_\beta, \bar{N}_\beta, \bar{\pi}_\beta) \in \mathbf{A}$,
  - $\bullet_3$ $\mathbb{P}^\bullet_\beta$ is equal to $\mathbb{P}'_{\xi(\beta)}$ for some $\xi(\beta) = \xi_{\mathbf{q}}(\beta) \le \beta$ (on $\mathbb{P}'_\beta$, see below).

(e) $\mathbb{P}'_\alpha$ is a dense subset of $\mathbb{P}_\alpha$, where,

- $\mathbb{P}'_\alpha := \mathbb{P}_\alpha \restriction \{p \in \mathbb{P}_\alpha\colon$ if $\beta \in \mathrm{dom}(p)$ then $p(\beta)$ is a member of $\mathbf{V}$ (not just a $\mathbb{P}_\alpha$-name) and if $\zeta_1 < \zeta_2$ are in $p(\beta) \cap \mathscr{U}_\beta$, then there are q_1, q_2, r_1, r_2 as in Definition 1.2(2)(c)($\bullet_1$)-($\bullet_5$) with $\mathbf{a}_\beta, \mathbf{b}_\beta$ here standing for $\mathbf{a}, \mathbf{b}$ there and

$$\bigvee_{\ell=1}^{2} (\forall \gamma \in \mathrm{dom}(r_\ell))[\gamma \in \mathrm{dom}(p) \wedge r_\ell(\gamma) \subseteq p(\gamma)]\}.$$

(f) $\gamma_{\mathbf{q}} := \gamma(\mathbf{q}) := \sup\{\gamma_{\mathbf{q},\beta}\colon \beta < \lg(\mathbf{q})\}$, so $\mathbb{P}'_{\gamma(\mathbf{q})} \subseteq \mathscr{H}_{<\lambda}(\gamma_{\mathbf{q}})$; let $\mathbb{P}_{\mathbf{q}} := \mathbb{P}_{\lg(\mathbf{q})}$ and $\mathbb{P}'_{\mathbf{q}} := \mathbb{P}'_{\lg(\mathbf{q})}$.

(1A) We may write either $\mathbb{P}_{\mathbf{q},\alpha}$ or $\mathbb{P}_\alpha$ whenever $\mathbf{q}$ is clear and $(\iota_{\mathbf{q},\beta,1}, \iota_{\mathbf{q},\beta,2})$ is $\bar{\iota}_{\mathbf{b}_\beta}$,

(2) Let $\leq_{\mathbf{p}}$ be the following two-place relation on $\mathbf{Q}_{\mathbf{p}}$:

$$\mathbf{q}_1 \leq_{\mathbf{p}} \mathbf{q}_2 \text{ \underline{iff} } \mathbf{q}_1 = \mathbf{q}_2 \restriction \lg(\mathbf{q}_1), \text{ see below.}$$

(3) For $\mathbf{q}_2 \in \mathbf{Q}_{\mathbf{p}}$ and $\alpha_* \leq \lg(\mathbf{q}_2)$, we define $\mathbf{q}_1 := \mathbf{q}_2 \restriction \alpha_*$ by:

(a) $\lg(\mathbf{q}_1) = \alpha_*$,
(b) $(\mathbb{P}_{\mathbf{q}_1,\alpha}, \mathbb{P}'_{\mathbf{q}_1,\alpha}) = (\mathbb{P}_{\mathbf{q}_2,\alpha}, \mathbb{P}'_{\mathbf{q}_2,\alpha})$ for $\alpha \leq \alpha_*$,
(c) $(\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_1,\beta}, \mathbf{b}_{\mathbf{q}_1,\beta}) = (\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_2,\beta}, \mathbf{b}_{\mathbf{q}_2,\beta})$ for $\beta < \alpha_*$.

(4) We say that two conditions $p, q \in \mathbb{P}'_\alpha$ are *isomorphic*, <u>when</u>:

(a) $\mathrm{otp}(\mathrm{dom}(p)) = \mathrm{otp}(\mathrm{dom}(q))$, and
(b) if $\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ then:
   - $\bullet_1$ $\mathrm{otp}(p(\beta)) = \mathrm{otp}(q(\beta))$,
   - $\bullet 2$ if $\varepsilon \in p(\beta) \cap q(\beta)$ then $\mathrm{otp}(\varepsilon \cap p(\beta)) = \mathrm{otp}(\varepsilon \cap q(\beta))$,
   - $\bullet_3$ if $\varepsilon \in p(\beta), \zeta \in q(\beta)$ and $\mathrm{otp}(\varepsilon \cap p(\varepsilon)) = \mathrm{otp}(\zeta \cap q(\beta))$ then

$$\pi_{\beta,\{\zeta\},\{\varepsilon\}}(p \restriction N_{\beta,\{\varepsilon\}}) = p \restriction N_{\beta,\{\zeta\}}.$$

Remark 1.4. If we prefer in clause (d) ($\bullet_3$) of Definition 1.3 (1) to have $\xi(\beta) = \beta$, i.e., $\mathbb{P}^\bullet_\beta = \mathbb{P}'_\beta$, we need to add, e.g. "$\mu$ is regular and force with $(\{\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}\colon \lg(\mathbf{q}) < \mu\}, \triangleleft)$".

Claim 1.5.

(0) For $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, we have: $\mathbb{P}'_{\mathbf{q}} \models$ "$p \leq q$" iff $\{p, q\} \subseteq \mathbb{P}'_{\mathbf{q}}$, $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and $\beta \in \mathrm{dom}(p) \Rightarrow p(\beta) \subseteq q(\beta)$.

(1) For $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, any increasing sequence of members of length $< \lambda$ of $\mathbb{P}'_{\mathbf{q}}$ has a lub, in fact, if $\delta < \lambda$, $\bar{p} = \langle p_i\colon i < \delta\rangle \in {}^{\delta}(\mathbb{P}'_{\mathbf{q}})$ is increasing, <u>then</u> the following $p \in \mathbb{P}'_{\mathbf{q}}$ is a lub of $\bar{p}$; defined by: $\mathrm{dom}(p) = \bigcup\{\mathrm{dom}(p_i)\colon i < \delta\}$, and if $\beta \in \mathrm{dom}(p)$ then

$$p(\beta) = \bigcup \{p_i(\beta)\colon i < \delta \text{ and } \beta \in \mathrm{dom}(p_i)\}.$$

We denote this p by $\lim(\bar{p})$.

(2) For $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, we have:

- *$p \in \mathbb{P}'_{\mathbf{q}}$ <u>iff</u>:*
  (a) p is a function with domain $\in [\lg(\mathbf{q})]^{<\lambda}$,
  (b) if $\beta \in \mathrm{dom}(p)$ then $p(\beta)$ belongs to $[\mathscr{U}_\beta]^{<\lambda}$.
  (c) If $\beta \in \mathrm{dom}(p)$ and $(\iota_1, \iota_2) = (\iota_{\mathbf{q},\beta,1}, \iota_{\mathbf{q},\beta,2})$ then for every $\zeta_1 < \zeta_2$ from $p(\beta)$, $(p \restriction \beta) \restriction N_{\mathbf{q},\beta,\{\zeta_1,\zeta_2\}} \Vdash_{\mathbb{P}_{\mathbf{q},\beta}}$ "$\underset{\sim}{\mathbf{c}}\{\zeta_1, \zeta_2\} \in \{\iota_1, \iota_2\}$". Moreover, there are q_1, q_2, r_1, r_2 as in Definition 1.2(2)(c)($\bullet_1$)-($\bullet_5$) and

$$\bigvee_{\ell=1}^{2} (\forall \gamma \in \mathrm{dom}(r_\ell))[\gamma \in \mathrm{dom}(p) \cap \beta \wedge r_\ell(\gamma) \subseteq p(\gamma)].$$

(3) If $\mathbf{q} \in \mathbf{Q_p}$ and $\alpha \le \lg(\mathbf{q})$ then $\mathbf{q} \restriction \alpha \in \mathbf{Q_p}$.
(4) $\le_{\mathbf{p}}$ is a partial order on $\mathbf{Q_p}$.
(5) If $\bar{\mathbf{q}} = \langle \mathbf{q}_j : j < \delta \rangle$ is $\le_{\mathbf{p}}$-increasing <u>then</u> it has a $\le_{\mathbf{p}}$-lub, $\lim(\bar{\mathbf{q}})$.
(6) If $\beta < \lg(\mathbf{q})$, $\mathbf{a} = \mathbf{a}_{\mathbf{q},\beta}$, $u \in [\mathscr{U}_{\mathbf{a},\beta}]^{\le \partial}$ and $N_u = N_{\mathbf{a},u}$, <u>then</u>:

- $\bullet_1$ *if $p \in \mathbb{P}'_{\mathbf{q}}$ and $\gamma \in \mathrm{dom}(p) \cap N_u$, then $p(\gamma) \in N_u$, and*
- $\bullet_2$ *if $p \in \mathbb{P}'_{\mathbf{q}}$ then $p \restriction (\beta \cap N_u) \in N_u$.*

(7) If (A) then (B), where:

(A) (a) $i_* < \lambda$,
(b) $p_i \in \mathbb{P}'_{\mathbf{q}}$ for $i < i_*$,
(c) if $i < j < i_*$, <u>then</u> p_i and q_i are essentially compatible, i.e.:
- *if $\beta \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j)$ <u>then</u> $p_i(\beta) \subseteq p_j(\beta)$ or $p_j(\beta) \subseteq p_i(\beta)$.*

(B) (a) $\{p_i : i < i_*\}$ have a common upper bound in $\mathbb{P}'_{\mathbf{q}}$,
(b) moreover, p is the least common upper bound when:
- *$\mathrm{dom}(p) = \bigcup\{\mathrm{dom}(p_i) : i < i_*\}$,*
- *if $\beta \in \mathrm{dom}(p)$, then*

$$p(\beta) = \bigcup\{p_i(\beta) : i < i_* \text{ satisfying } \beta \in \mathrm{dom}(p_i)\}.$$

Proof. Part (2) is crucial but easy to verify. Parts (0), (1), (3), and (4) are also easy.

(5) For this, define $\mathbf{q} \coloneqq \lim(\bar{\mathbf{q}})$ naturally, but we elaborate.

$(*)$ (a) $\lg(\mathbf{q}) = \bigcup\{\lg(\mathbf{a}_i) : i < \delta\}$,
(b) if $i < \delta$ and $\alpha \le \lg(\mathbf{q}_i)$, then $(\mathbb{P}_{\mathbf{q},\alpha}, \mathbb{P}'_{\mathbf{q},\alpha}) = (\mathbb{P}_{\mathbf{q}_i,\alpha}, \mathbb{P}'_{\mathbf{q},\alpha})$,
(c) if $i < \delta$ and $\beta < \lg(\mathbf{q}_i)$, then $(\mathbb{Q}_{\mathbf{q},\beta}, \mathbf{a}_{\mathbf{q},\beta}, \mathbf{b}_{\mathbf{q},\beta}) = (\mathbb{Q}_{\mathbf{q}_i,\beta}, \mathbf{a}_{\mathbf{q}_i,\beta}, \mathbf{b}_{\mathbf{q}_i,\beta})$,
(d) $(\mathbb{P}_{\mathbf{q},\lg(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})})$ is $(\bigcup\{\mathbb{P}_{\mathbf{q}_i} : i < \partial\}, \bigcup\{\mathbb{P}'_{\mathbf{q}_i} : i < \partial\})$ when $\mathrm{cf}(\delta) \ge \lambda$,
(e) if $\mathrm{cf}(\delta) < \lambda$, then $(\mathbb{P}_{\mathbf{q},\lg(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})})$ are defined as inverse limit. Then,
- $\mathbb{P}'_{\mathbf{q}} \coloneqq \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})}$ is dense in $\mathbb{P}_{\mathbf{q}}$ because by Definition 1.2(3), for each $\beta < \lg(\mathbf{q}_j)$ with $j < \delta$, $\mathbb{Q}_{\mathbf{b}[\beta,\mathbf{q}_j]}$ is closed under increasing unions of length $< \lambda$.

Recalling that in Definition 1.3(1)(c), we use β and not α, "$\mathbb{P}_{\mathbf{q}}$ satisfies the λ^+-cc" is not required for proving 1.5 (5), only "if $\beta < \lg(\mathbf{q})$ then $\mathbb{P}_{\mathbf{q},\beta}$ satisfies the λ^+-cc", which is clear.

(6) For $\bullet_1$, as $\gamma \in N_u$ and $\mathbf{q} \restriction \xi_{\mathbf{q},\beta}(\gamma) \in N_u$ necessarily, $\mathscr{U}_{\mathbf{a}_\gamma} \in N_u$ so recalling that $[N_u]^{<\lambda} \subseteq N_u \cap \partial + 1 \subseteq N_u \wedge |\mathscr{U}_{\mathbf{a}_\gamma}| = \partial$, we have that $\mathscr{U}_{\mathbf{a}_\gamma} \subseteq N_u$, $[\mathscr{U}_{\mathbf{a}_\gamma}]^{<\lambda} \subseteq N_u$ and $p(\gamma) \in [\mathscr{U}_{\mathbf{a}_\gamma}]^{<\lambda}$, hence $p(\gamma) \in N_u$.

For $\bullet_2$, use $\bullet_1$ and "$[N_u]^{<\lambda} \subseteq N_u$".

(7) Follow by (6) and our definitions. $\square_{1.5}$

Still,

Crucial Claim 1.6. *If $\mathbf{q} \in \mathbf{Q_p}$ <u>then</u> $\mathbb{P}_{\mathbf{q}}$ satisfies λ^+-cc.*

Proof. It suffices to prove that $\mathbb{P}'_{\mathbf{q}} = \mathbb{P}'_{\mathbf{q}}(\lg(\mathbf{q}))$ satisfies the λ^+-cc, so assume:

$(*)_1$ (a) Let $\bar{p} = \langle p_\xi : \xi < \lambda^+ \rangle$, where $p_\xi \in \mathbb{P}'_{\mathbf{q}}$,
(b) it suffice to prove that for some $\zeta < \xi < \lambda^+$, p_ζ and p_ξ are compatible.

[Why? By the definitions.]

$(*)_2$ For some stationary set $S \subseteq \mathrm{cof}(\lambda) \cap \lambda^+$, we have:
- $\bullet_1$ $\langle \mathrm{dom}(p_\xi) : \xi \in S \rangle$ is a Δ-system with heart $w_* \in [\lg(\mathbf{q})]^{<\lambda}$, and
- $\bullet_2$ if $\beta \in w_*$ then $\langle p_\xi(\beta) : \xi \in S \rangle$ is a Δ-system.

[Why? By the Delta system lemma, the proof using Fodor's lemma recalling $\lambda = \lambda^{<\lambda}$.]

$(*)_3$ Without loss of generality, $\langle p_\xi : \xi \in S\rangle$ are pairwise isomorphic (see Definition 1.3(4)).

[Why? Easy.]

$(*)_4$ We fix $\xi(1) \neq \xi(2)$ from S and we shall prove that $p_{\xi(1)}$ and $p_{\xi(2)}$ have a common upper bound; this suffices for proving Crucial Claim 1.6.

Let $\bar\beta = \langle \beta_i : i \leq i_*\rangle$ list the closure of $\{\alpha, \alpha+1 : \alpha \in w_*\} \cup \{0, \lg(\mathbf{q})\}$ in increasing order so necessarily $i_* < \lambda$ and clearly it suffice:

$(*)_5$ To choose $q_i \in \mathbb{P}'_{\mathbf{q},\beta_i}$ a common upper bound of $\{p_{\xi(1)} \restriction \beta_i, p_{\xi(2)} \restriction \beta_i\}$ increasing with $i \leq i_*$ by induction on $i \leq i_*$.

Let us carry the induction.

Case 1: $i = 0$. Clearly, this case is trivial.

Case 2: i is a limit ordinal.

In this case, let $q_i := \lim\langle q_j : j < i\rangle$, so by Claim 1.5(1), q_i is well-defined and is as required by the definition of the order.

Case 3: $i = j + 1$ and $\beta_j \notin w_*$.

In this case, $\mathrm{dom}(p_{\xi(1)}) \cap \mathrm{dom}(p_{\xi(2)}) \cap \beta_i \subseteq \beta_j$, hence the condition

$$q_i := q_j \cup \big(p_{\xi(1)} \restriction [\beta_j, \beta_i] \cup (p_{\xi(2)} \restriction [\beta_j, \beta_i])\big)$$

is as promised.

Case 4: $i = j + 1$ and $\beta_j \in w_*$.

By the choice of $\bar\beta$, clearly $\beta_i = \beta_j + 1$.

In this case, for $\ell \in \{1, 2\}$, consider the sequence $\langle \alpha_{\xi(\ell),\varepsilon} : \varepsilon < \varepsilon_*\rangle$ listing the set $p_{\xi(\ell)}(\beta_j)$ in increasing order (the two sequences have the same length because $p_{\xi(1)}, p_{\xi(2)}$ are isomorphic, see Definition 1.3(4) $(\bullet_1)$). Let $\mathscr{S} := \{\varepsilon < \varepsilon_* : \alpha_{\xi(1),\varepsilon} \neq \alpha_{\xi(2),\varepsilon}\}$, so by Definition 1.3 (4) $\bullet_2$ the sets $\{\alpha_{\xi(1),\varepsilon} : \varepsilon \in \mathscr{S}\}$, $\{\alpha_{\xi(2),\varepsilon} : \varepsilon \in \mathscr{S}\}$ are disjoint and disjoint to $\{\alpha_{\xi(1),\varepsilon} : \varepsilon \in \varepsilon_* \setminus \mathscr{S}\} = \{\alpha_{\xi(2),\varepsilon} : \varepsilon \in \varepsilon_* \setminus \mathscr{S}\}$.

Recalling 1.3(1)(d) and 0.7(b)$(\bullet_8)$, we have:

$(*)_6$ $\mathbf{a}_{\beta_j} = \mathbf{a}_{\mathbf{q},\beta_j}$ determine:
- (a) $\bar\pi_{\beta_j} = \langle \pi_{u,v} : u, v \in [\mathscr{U}_{\beta_j}]^{\leq 2}$ and $|u| = |v|\rangle$,
- (b) $\bar N = \langle N_u : u \in [\mathscr{U}_{\beta_j}]^{\leq 2}\rangle$,
- (c) for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$, let:
  - $v[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1),\varepsilon(1)}, \alpha_{\xi(1),\varepsilon(1)}\}$, and
  - $u[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1),\varepsilon(1)}, \alpha_{\xi(2),\varepsilon(2)}\}$.
- (d) for $\varepsilon \in \mathscr{S}$, let $v[\varepsilon] = \{\alpha_{\xi(1),\varepsilon}\}$, $u[\varepsilon] = \{\alpha_{\xi(2),\varepsilon}\}$,
- (e) $\bar\iota = \bar\iota^*_{\beta_j}$, see 1.3 (1) (d) $(\bullet_1)$,
- (f) $\gamma_j = \xi_{\mathbf{q}}(\beta_j)$; see 1.3(1)(d) $(\bullet_3)$.

We shall now define $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$ such that:

$(*)_7$
- (a) $p_{\varepsilon(1),\varepsilon(2)} \in \mathbb{P}_{\gamma_j} \cap N_{u[\varepsilon(1),\varepsilon(2)],v[\varepsilon(1),\varepsilon(2)]}$, hence $\mathrm{dom}(p_{\varepsilon(1),\varepsilon(2)}) \subseteq \gamma_j \cap N_{u[\varepsilon(1),\varepsilon(2)],v[\varepsilon(1),\varepsilon(2)]}$,
- (b) $p_{\varepsilon(1),\varepsilon(2)} \restriction (\gamma_j \cap N_{v[\varepsilon(1)]})$, $p_{\xi(1)} \restriction N_{v[\varepsilon(1)]}$ are essentially compatible; see 1.5(7)(A)(c),
- (c) $p_{\varepsilon(1),\varepsilon(2)} \restriction (\gamma_j \cap N_{v[\varepsilon(2)]})$, $p_{\xi(2)} \restriction N_{v[\varepsilon(2)]}$ are essentially compatible,
- (d) $p_{\varepsilon(1),\varepsilon(2)}$ satisfies 1.3(1)(e)$(\bullet)$ with $(\gamma_j, \varepsilon(1), \varepsilon(2))$ here standing for $(\beta, \zeta_1, \zeta_2)$ there,
- (e) $\{q_j \restriction N_\emptyset\} \cup \{p_{\varepsilon(1),\varepsilon(2)} \restriction N_\emptyset : \varepsilon(1), \varepsilon(2) \in \mathscr{S}\}$ are pairwise essentially compatible,
- (f) if $\varepsilon(1) \neq \varepsilon(2)$ then $p_{\varepsilon(1),\varepsilon(2)} \restriction N_{\{\varepsilon(\ell)\}} \leq p_{\xi(\ell)} \restriction N_{\{\varepsilon(\ell)\}}$ for $\ell = 1, 2$.

We have to show two things: $\boxplus_1$ and $\boxplus_2$. The first saying we can choose them (the $p_{\varepsilon(1),\varepsilon(2)}$-s), the second that this is enough.

$\boxplus_1$ we can choose $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1),\varepsilon(2)\in\mathscr{S}$ as required in $(*)_7$.

We consider two possible cases:

Case 4.1: $\varepsilon(1)\neq\varepsilon(2)$.

Let $p_{\varepsilon(1),\varepsilon(2)}=\pi(p_{\xi(1)}\restriction N_{\{\varepsilon(1),\varepsilon(2)\}})$, where $\pi=\pi_{u[\varepsilon(1),\varepsilon(2)],v[\varepsilon(1),\varepsilon(2)]}$.

Case 4.2: $\varepsilon(1)=\varepsilon(2)$.

In this case, we deal with all those pairs together; so we pick some sequence $\langle p_{\varepsilon,\varepsilon}\colon\varepsilon\in\mathscr{S}\rangle$ by choosing $p_{\varepsilon,\varepsilon}$ by induction on $\varepsilon\in\mathscr{S}$. Now, $p_{\varepsilon,\varepsilon}\in\mathbb{P}'_\beta\cap N_{u[\varepsilon(1),\varepsilon(2)]}$ is such that:

$(*)$ (a) $p_{\varepsilon,\varepsilon}$ is $\leq_{\mathbb{P}'_{\mathbf{q},\beta}}$-above $p_{\xi(1)}\restriction N_{v[\varepsilon]}$ and above the restriction $p_{\xi(2)}\restriction N_{u[\varepsilon]}$,

(b) $\langle p_{\zeta,\zeta}\restriction N_\emptyset\colon\zeta\in(\varepsilon+1)\cap\mathscr{S}\rangle$ is $\leq_{\mathbb{P}_{\beta[j]}}$-increasing, and

(c) there are q_1,q_2,r_1,r_2 as in Definition 1.3(2)(c) $(\bullet_1)$-$(\bullet_5)$ with $\mathbf{b}_{\mathbf{q},\beta}$ standing here for $(\mathbf{a},p,\bar\iota)$ there such that:

$$\bigvee_{\ell=1}^{2}(\forall\gamma\in\mathrm{dom}(r_\ell))\left[\gamma\in\mathrm{dom}(p_{\varepsilon,\varepsilon})\wedge r_\ell(\gamma)\subseteq p_{\varepsilon,\varepsilon}(\gamma)\right].$$

We can choose $p_{\varepsilon,\varepsilon}$ by the properties of $\mathbf{b}_{\beta_j}$.

Having defined all the $p_{\varepsilon(1),\varepsilon(2)}$-s we can proceed.

$\boxplus_2$ The following set of members of $\mathbb{P}_{\beta_i}$ has a common upper bound q_*:
- $p_{\xi(1)},p_{\xi(2)}$, and
- $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1),\varepsilon(2)\in\mathscr{S}$.

[Why? Recall Claim 1.5(2) and 1.2(1)(c)($\bullet_1$) by 1.5(7), clause (A) there holds, in particular sub-clause (A)(c). The main point is that:

$(*)$ $\langle N_{\{\alpha_{\varepsilon(1)},\alpha_{\varepsilon(2)}\}}\cap\gamma_j\setminus(N_{\{\alpha_{\xi(1)},\varepsilon(1)\}}\cup N_{\{\alpha_{\xi(2)},\varepsilon(1)\}})\colon\varepsilon(1),\varepsilon(2)\in\mathscr{S}\rangle$ is a sequence of pairwise disjoint sets.

Why? As "$N_u\cap N_v\subseteq N_{u\cap v}$ for $u,v\in[\mathscr{U}_{\beta_j}]^{<\partial}$ by 0.7($\bullet_7$)].]

So q_* from $\boxplus_2$ is a common upper bound of $p_{\xi(1)},p_{\xi(2)}$, as promised. $\square_{1.6}$

Remark 1.7. No need so far, but we may add in $(*)_4$ of the proof of Crucial Claim 1.6 the following item:

(d) if $\beta\in w_*$ and $\langle\alpha_{\zeta,\beta,i}\colon i<\iota_{\zeta,\beta}\rangle$ list in increasing order the members of $p_\zeta(\beta)$ for $\zeta\in S$, then:
- $\langle\iota_{\zeta,\beta}\colon\zeta\in S\rangle$ is constant called i_β,
- for $i<i_\beta$, the sequence $\langle\alpha_{\zeta,\beta,i}\colon\zeta\in S\rangle$ is constant or increasing,
- if $i,j<i_\beta$ the sequence of truth values
$$\langle\text{Truth value}(\alpha_{\zeta,\beta,i}<\alpha_{\xi,\beta,j})\colon\zeta<\xi\text{ are from }S\rangle$$
is constant, and
- if $i,j<i_\beta$, $\zeta\neq\xi$ are from S and $\alpha_{\zeta,\beta,i}=\alpha_{\xi,\beta,j}$ then $i=j$.

Claim 1.8. *If (A) then (B), where:*

(A) (a) $\mathbf{q}\in\mathbf{G}_{\mathbf{p}}$,

(b) $\sigma<\lambda$,

(c) $\underset{\sim}{\mathbf{c}}$ *is a* $\mathbb{P}_{\mathbf{q}}$*-name of a function from* $[\theta]^2$ *into* σ.

(B) There is some $\mathbf{b}\in\mathbf{A}^+$ *such that* $\mathbb{P}_{\mathbf{b}}=\mathbb{P}'_{\mathbf{q}}$ *and* $\underset{\sim}{\mathbf{c}}_{\mathbf{b}}=\underset{\sim}{\mathbf{c}}$.

Proof. Recalling Hypothesis 1.1(b), on the one hand it is clear how to choose $\mathbf{a}\in\mathbf{A}$ such that $\mathbb{P}_{\mathbf{a}}=\mathbb{P}'_{\mathbf{q}}$ and $\underset{\sim}{\mathbf{c}}_{\mathbf{a}}=\underset{\sim}{\mathbf{c}}$. On the other hand, the choice of $p_{\mathbf{b}}$ and $\bar\iota_{\mathbf{b}}$ is similar to the proof of [She88, 2.1], but we elaborate.

First, we can find $\mathbf{a}$ such that:

$(*)^1_{\mathbf{a}}$ (a) $\mathbf{a}\in\mathbf{A}$,

(b) $\mathbb{P}_{\mathbf{a}}=\mathbb{P}_{\mathbf{q}}$,

(c) $\gamma = \lg(\mathbf{q})$,
(d) $\underset{\sim}{\mathfrak{c}}_{\mathbf{a}} = \underset{\sim}{\mathfrak{c}}$.

[Why? Because we have chosen $\mathbb{P}_{\mathbf{a}}$ as in $(*)^1_{\mathbf{a}}$(b), it is λ^+-cc by Claim 1.6; also $\gamma, \underset{\sim}{\mathfrak{c}}_{\mathbf{a}}$ are as is required in Definition 1.2. Next, it is easy to choose $\mathscr{B}_{\mathbf{a}}$ as required and lastly we can choose $(\mathscr{U}_{\mathbf{a}}, \bar{N})$ as is required because $\theta \to (\partial)^{\lambda,2}_{\partial}$ holds by Hypothesis 1.1 clause (b). We are left with choosing some appropriate $(p, \bar{\iota})$ and then $\mathbf{b} = (\mathbf{a}, p, \bar{\iota})$.]

Let

$$Y := \{(q_1, q_2) \colon q_1, q_2 \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}} \text{ and } q_1 \restriction (N_{\mathbf{a},\emptyset} \cap \lg(\mathbf{q})) = q_2 \restriction (N_{\mathbf{a},\emptyset} \cap \lg(\mathbf{q}))\},$$

and let $\leq_Y$ be the following two place relation on Y:

$(*)_2$ $(p_1, p_2) \leq_Y (q_1, q_2)$ iff:
  (a) $(p_1, p_2) \in Y$ and $(q_1, q_2) \in Y$,
  (b) $p_1 \leq_{\mathbb{P}_{\mathbf{q}}} q_1$ and $p_2 \leq_{\mathbb{P}_{\mathbf{q}}} q_2$.

Clearly,

$(*)_3$ $(Y, \leq_Y)$ is a $(< \lambda)$-complete partial order.

[Why? Recalling 1.5(1).]

$(*)_4$ For $(p_1, p_2) \in Y$, let
  (a) $\mathrm{solv}(p_1, p_2)$ be the set of pairs (ι_0, ι_1) such that for any $\zeta_1 < \zeta_2$ from $\mathscr{U}$, there are r_1, r_2 such that for $\ell = 1, 2$ clauses $\bullet_2$-$\bullet_5$ of Definition 1.2(2)(e) holds.
  (b) $\mathrm{solv}^+(p_1, p_2) := \bigcap\{\mathrm{solv}(q_1, q_2) \colon (p_1, p_2) \leq_Y (q_1, q_2) \in Y\}$.

$(*)_5$ (a) if $(p_1, p_2) \leq_Y (q_1, q_2)$ then:

$$\mathrm{solv}(p_1, p_2) \supseteq \mathrm{solv}(q_1, q_2) \supseteq \mathrm{solv}^+(q_1, q_2) \supseteq \mathrm{solv}^+(p_1, p_2),$$

  (b) if $(p_1, p_2) \in Y$ then $\mathrm{solv}(p_1, p_2) \neq \emptyset$.

[Why? The first inclusion in Clause (a) holds because $\leq_{\mathbb{P}_{\mathbf{q}}}$ is transitive. The other inclusions are clear, and Clause (b) is easy too.]

$(*)_6$ If $(p_1, p_2) \in Y$ then for some (q_1, q_2) and $\bar{\iota}$, we have:
  (a) $(p_1, p_2) \leq_Y (q_1, q_2) \in Y$,
  (b) if $(q_1, q_2) \leq_Y (q_1', q_2')$ then $\bar{\iota} \in \mathrm{solv}(q_1', q_2')$, moreover, $\mathrm{solv}(q_1, q_2) = \mathrm{solv}(q_2', q_2') = \mathrm{solv}^+(q_1', q_2') = \mathrm{solv}^+(q_1, q_2)$.

[Why? Recalling $\sigma < \lambda$, hence $|\sigma \times \sigma| < \lambda$ and $(Y, \leq_Y)$ is λ-complete by $(*)_3$.]

$(*)_7$ For $p \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$, let $\mathrm{solv}(p)$ be the set of $\iota \in \sigma \times \sigma$ such that if $q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ is $\leq_{\mathbb{P}_{\mathbf{a}}}$-above then there is $(q_1, q_2) \in Y$:
  $\bullet_1$ $q \leq_{\mathbb{P}_{\mathbf{q}}} q_1$, $q \leq_{\mathbb{P}_{\mathbf{q}}} q_2$ and
  $\bullet_2$ $(q_1, q_2) \in Y$,
  $\bullet_3$ $\bar{\iota} \in \mathrm{solv}^+(q_1, q_2)$,
  $\bullet_4$ $\mathrm{solv}(q_1, q_2) = \mathrm{solv}^+(q_1, q_2)$.

$(*)_8$ (a) if $p \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ then $\mathrm{solv}(p) \neq \emptyset$,
  (b) if $p \leq_{\mathbb{P}_{\mathbf{a}}} q$ are from $\mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ then $\mathrm{solv}(p) \supseteq \mathrm{solv}(q)$,
  (c) if $p \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ then for some q and $\bar{\iota}$, for every q', we have $q \leq \mathbb{P}_{\mathbf{q}} \wedge q' \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}} \Rightarrow \bar{\iota} \in \mathrm{solv}(q')$.

[Why? Clause (a) follows by $(*)_6$, Clause (b) by the definitions, and Clause (c) holds as $\mathbb{P}_{\mathbf{a}}$ and even $\mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ is λ-complete and $|\sigma \times \sigma| < \lambda$.]

Now, $(*)_8$(c) finish the proof of 1.8. $\square_{1.8}$

Claim 1.9. *If (A) then (B), where:*

(A) *(a)* $\mathbf{q} \in \mathbb{Q}_{\mathbf{p}}$ *and* $\mathbf{q}_0 <_{\mathbf{p}} \mathbf{q}$,
  (b) $\gamma(\mathbf{q}) < \mu$, *so* $\lg(\mathbf{q}) < \mu$,
  (c) $\mathbf{b} \in \mathbf{A}_{\mathbf{p}}$ *and* $\mathbb{P}_{\mathbf{b}} = \mathbb{P}_{\mathbf{q}_0}$.

(B) There exists some $\mathbf{q}_1$ *such that:*
  (a) $\mathbf{q} \leq_{\mathbf{p}} \mathbf{q}_1$,
  (b) $\lg(\mathbf{q}_1) = \lg(\mathbf{q}) + 1$,
  (c) $\mathbf{b}_{\lg(\mathbf{q})}[\mathbf{q}_1] = \mathbf{b}$.

Proof. Easy. $\square_{1.9}$

Lastly, before arriving at the main conclusion, we have to prove the following.

Claim 1.10.
(1) Assume $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, $\alpha < \lg(\mathbf{q})$ *and* $\mathbf{b} = \mathbf{b}_{\mathbf{q},\alpha} = (\mathbf{a}_\alpha, p_\alpha, \bar{\iota}_\alpha) = (\mathbf{a}, p, \bar{\iota})$*, then:*

- $\Vdash_{\mathbb{P}_{\mathbf{q},\alpha+1}}$ "$\underset{\sim}{\mathscr{V}}_{\mathbb{Q}_{\mathbf{b}}} \in [\mathscr{U}_{\mathbf{a}_\alpha}]^\partial$ *and for every* $\alpha \neq \beta \in \underset{\sim}{\mathcal{V}}_{\mathbb{Q}_{\mathbf{b}}}$, $\mathbf{c}_{\mathbf{a}_\alpha}\{\alpha, \beta\} \in \{\iota_1, \iota_2\}$".

(2) If $\mathbf{b} = (\mathbf{a}, p, \bar{\iota}) \in \mathbf{A}^+$, $\mathrm{cf}(\partial) > \lambda$*, and in* $\mathcal{V}^{\mathbb{P}_{\mathbf{a}}}$, $\mathbb{Q}_{\mathbf{b}}$ *satisfies the* λ^+*-cc,* <u>*then*</u> *for some* $p \in \mathbb{Q}_{\mathbf{b}} \cap \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ *we have* $p \Vdash_{\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}}$ "$\underset{\sim}{\mathscr{V}}_{\mathbb{Q}_{\mathbf{b}}} \in [\mathscr{U}_{\mathbf{a}}]^\partial$ *and for every* $\alpha \neq \beta \in \underset{\sim}{\mathcal{V}}_{\mathbb{Q}_{\mathbf{b}}}$, $\mathbf{c}_{\mathbf{a}}\{\alpha, \beta\} \in \{\iota_1, \iota_2\}$".

Proof. (1) The second phrase in both conclusion holds by the definitions of $\mathbb{Q}_{\mathbf{b}}$.

By the proof of "$\mathbb{P}_{\mathbf{q}}$ satisfies the λ^+-cc", we can show for $\varepsilon < \partial$, the density of the set

$$\mathscr{I}_\varepsilon := \{p \in \mathbb{P}'_{\mathbf{q}} : \alpha \in \mathrm{dom}(p) \text{ and there is } \beta \in p(\alpha) \text{ such that } \varepsilon < \mathrm{otp}(\mathscr{U}_{\mathbf{a}_\alpha} \cap \beta)\}.$$

(2) Easily, for every $\beta \in \mathscr{U}_{\mathbf{a}}$ we can choose $p^0_\beta = \{\beta\}$, $q_\beta = \{(\beta, p^0_\beta)\}$. Clearly, $q_\beta \in \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_{\mathbf{b}}$ for $\beta \in \mathscr{U}_{\mathbf{a}}$. So by the λ^+-cc for some $\beta \in \mathscr{U}_{\mathbf{a}}$, $q_\beta \Vdash$"$\{\varepsilon \in \mathscr{U}_{\mathbf{a}} : q_\varepsilon \in \underset{\sim}{\mathbb{Q}}_{\mathbf{b}}\} \in [\mathscr{U}_{\mathbf{a}}]^\partial$ well assuming $\mathrm{cf}(\theta) > \lambda$. $\square_{1.10}$

Conclusion 1.11. *There exists a forcing notion* $\mathbb{P}$ *satisfying the following conditions:*
  (a) $\mathbb{P}$ *is* λ^+*-cc of cardinality* μ.
  (b) $\mathbb{P}$ *is* $(<\lambda)$*-complete; hence, it collapses no cardinals, changes no cofinalities, and preserves cardinal arithmetic outside the interval* $[\lambda, \mu)$.
  (c) $\Vdash_{\mathbb{P}}$ "$2^\lambda = \mu$".
  (d) $\Vdash_{\mathbb{P}}$ "$\theta \to [\partial]^2_{\sigma,2}$" *for every* $\sigma \in (2, \lambda)$.

Proof. Choose a $\leq_{\mathbf{p}}$-increasing continuous sequence $\langle \mathbf{q}_\alpha : \alpha < \mu \rangle \in {}^\mu(\mathbf{Q}_{\mathbf{p}})$ such that $\lg(\mathbf{q}_\alpha) = \alpha$, $\mathbb{P}_{\mathbf{q}_\alpha}$ has cardinality $\leq (|\alpha| + \lambda)^{<\lambda}$ and,

- if $\alpha < \mu$ and $\Vdash_{\mathbb{P}_{\mathbf{q}_\alpha}}$ "$\underset{\sim}{\mathbf{c}} : [\theta]^2 \to \sigma$", then for some $\beta \in [\alpha, \mu)$, $\underset{\sim}{\mathbf{c}}_{\mathbf{q}_{\beta+1,\beta}} = \underset{\sim}{\mathbf{c}}$.

The existence of $\mathbf{b}_\beta[\mathbf{q}_{\beta+1}]$ with $\underset{\sim}{\mathbf{c}}[\mathbf{b}_\beta[\mathbf{q}_{\beta+1}]] = \underset{\sim}{\mathbf{c}}$ as required hold by Claim 1.8 and Claim 1.9.

Clearly $\bigcup\{\mathbb{P}_{\mathbf{q}_\beta} : \beta < \mu\}$ is a forcing notion as is required. $\square_{1.11}$

Conclusion 1.11 is meaningful because:

Fact 1.12. Assume that $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$, $\alpha < \mu \Rightarrow |\alpha|^\lambda < \mu$, $\theta > \beth_4(\partial)$ and $\partial = \partial^{<\lambda}$. <u>Then</u> the demands in Hypothesis 1.1 hold.

Remark 1.13. To justify the assumption, notice that:
  (A) Omitting $\partial = \partial^{<\lambda}$ does not help.
  (B) $\theta \to_{\mathrm{sq}} (\partial)^{\lambda,\leq 2}_\partial$ implies $\theta \to (\partial)^2_{2^\partial}$, hence $\theta > 2^{2^\partial}$.

With stronger lower bound on θ, see [She89] and anyhow just $\theta < \partial^{+\omega}$ and GCH in $[\partial, \partial^{+\omega}]$ would suffice for me.

The main point is proving $\theta \to_{\mathrm{sq}} (\partial)^{\leq\lambda,2}_{\partial}$. For this, see [She89], $\theta = \beth_m(\partial)$ for some small m suffice, we intend to return for better bound, see [S+].

Proof. The point is to prove $\theta \to (\partial)^{\lambda,2}_{\partial}$. Let $\mathscr{B}$ be as in 0.7(a).

For $u \subseteq \mathscr{B}$, let N^*_u be the minimal $N \prec \mathscr{B}$ such that $u \subseteq N$, $\partial + 1 \subseteq N$ and $[N]^{<\lambda} \subseteq N$, which exists as $<^{\mathscr{B}}_*$ is a well-ordering.

We define $\mathbf{c}\colon [\theta]^3 \to 2^\partial$ such that from $\mathbf{c}(u)$ we can compute the isomorphism type of $(N_u, \alpha)_{\alpha \in u}$. By Erdös-Rado theorem, $\beth_4(\partial)^+ \to ((2^\theta)^+)^3_{2^\partial}$ there is $\mathscr{U}_1 \subseteq \theta$ of order type $(2^\partial)^+$ such that $\mathbf{c} \restriction [\mathscr{U}]^3$ is constant and $\mathrm{otp}(\mathscr{U}_1) = (2^\partial)^+$.

Clearly, if $\alpha < \beta < \gamma$ are from $\mathscr{U}_1$, then $x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}$ implies $\beta < \gamma(1) \in \mathscr{U} \Rightarrow x \in N_{\{\alpha,\beta\}} \cap N_{\{\alpha,\gamma(1)\}}$, and

$$\alpha < \beta(1) < \gamma(1) \wedge \{\beta(1), \gamma(1)\} \subseteq \mathscr{U} \Rightarrow x \in N^*_{\{\alpha,\beta(1)\}} \cap N^*_{\{\alpha,\gamma(1)\}}.$$

So,

- $X_0 \coloneqq \{x\colon$ for some β, γ from $\mathscr{U}_1$, $\alpha < \beta < \gamma$ and $x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}\}$ have cardinality ∂ and it includes $N^*_{\{\alpha\}}$.

Similarly, X_1, X_2 has cardinality ∂, where:

- $X_1 \coloneqq \{x\colon$ for some $\beta, \gamma \in \mathscr{U}$, we have $\beta < \alpha < \gamma$ and $x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}\}$, and
- $X_2 \coloneqq \{x\colon$ for some $\beta, \gamma \in \mathscr{U}$, we have $\beta < \gamma < \alpha$ and $x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}\}$.

Let $X \coloneqq \bigcup_{\ell=0}^2 X_\ell$. For $\alpha \in \mathscr{U}_1$, let $N_{\{\alpha\}} \coloneqq N^*_X$, so N^*_X has cardinality ∂. Now,

$(*)_1$ The sets $\langle N^*_{\{\alpha,\beta\}} \setminus (N_{\{\alpha\}} \cup N_{\{\beta\}})\colon \alpha < \beta$ are from $\mathscr{U}_1\rangle$ is a sequence of pairwise disjoint sets.

$(*)_2$ If $\gamma \in \mathscr{U}_1$ then $\Lambda_\gamma \coloneqq \{\{\alpha, \beta\}\colon \alpha < \beta$ are from $\mathscr{U}_1$ and $(N^*_{\{\alpha,\beta\}} \setminus (N_{\{\alpha\}} \cup N_{\{\beta\}})) \cap N_{\{\gamma\}} \neq \emptyset\}$ has cardinality $\leq \partial$.

So for some $\mathscr{U}_2 \subseteq \mathscr{U}_1$ of cardinality $(2^\partial)^+$, we have:

$(*)_3$ (a) $\langle N_{\{\gamma\}}\colon \gamma \in \mathscr{U}_2\rangle$ is a Δ-system with heart called $N_\emptyset$,
(b) The $N_{\{\gamma\}}$ for $\gamma \in \mathscr{U}_2$ are pairwise isomorphic over $N_\emptyset$.

Lastly choose $N_{\{\alpha,\beta\}}$ for $\alpha \neq \beta \in \mathscr{U}_2$ as $N^*_{\{\alpha,\beta\}}$. Replacing $\mathscr{U}_2$ by $\mathscr{U} \subseteq \mathscr{U}_2$ of order type ∂, we are done. $\square_{1.12}$

References

[EH71] Paul Erdős and Andras Hajnal, *Unsolved problems in set theory*, Axiomatic Set Theory (Providence, R.I.), Proc. of Symp. in Pure Math., vol. XIII Part I, AMS, 1971, pp. 17–48.

[Kom21] Péter Komjáth, *Notes on some Erdős-Hajnal problems*, Journal of Symbolic Logic **86** (2021), no. 3, 1116–1123.

[RS00] Mariusz Rabus and Saharon Shelah, *Covering a function on the plane by two continuous functions on an uncountable square—the consistency*, Ann. Pure Appl. Logic **103** (2000), no. 1-3, 229–240, arXiv: math/9706223. MR 1756147

[S+] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F2407.

[She88] Saharon Shelah, *Was Sierpiński right? I*, Israel J. Math. **62** (1988), no. 3, 355–380. MR 955139

[She89] ______, *Consistency of positive partition theorems for graphs and models*, Set theory and its applications (Toronto, ON, 1987), Lecture Notes in Math., vol. 1401, Springer, Berlin, 1989, pp. 167–193. MR 1031773

[She92] ______, *Strong partition relations below the power set: consistency; was Sierpiński right? II*, Sets, graphs and numbers (Budapest, 1991), Colloq. Math. Soc. János Bolyai, vol. 60, North-Holland, Amsterdam, 1992, arXiv: math/9201244, pp. 637–668. MR 1218224

[She95] ______, *Possibly every real function is continuous on a non-meagre set*, Publ. Inst. Math. (Beograd) (N.S.) **57(71)** (1995), 47–60, arXiv: math/9511220. MR 1387353

[She96] ______, *Was Sierpiński right? III. Can continuum-c.c. times c.c.c. be continuum-c.c.?*, Ann. Pure Appl. Logic **78** (1996), no. 1-3, 259–269, arXiv: math/9509226. MR 1395402

[She00] ______, *Was Sierpiński right? IV*, J. Symbolic Logic **65** (2000), no. 3, 1031–1054, arXiv: math/9712282. MR 1791363

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