$\aleph_1$-FREE ABELIAN NON-ARCHIMEDEAN POLISH GROUPS

GIANLUCA PAOLINI AND SAHARON SHELAH

ABSTRACT. An uncountable $\aleph_1$-free group can not admit a Polish group topology but an uncountable $\aleph_1$-free abelian group can, as witnessed e.g. by the Baer-Specker group $\mathbb{Z}^\omega$, in fact, more strongly, $\mathbb{Z}^\omega$ is separable. In this paper we investigate $\aleph_1$-free abelian non-Archimedean Polish groups. We prove two main results. The first is that there are continuum many separable (and so torsionless, and so $\aleph_1$-free) abelian non-Archimedean Polish groups which are not topologically isomorphic to product groups and are pairwise not continuous homomorphic images of each other. The second is that the following four properties are complete co-analytic subsets of the space of closed abelian subgroups of S_∞: separability, torsionlessness, $\aleph_1$-freenees and $\mathbb{Z}$-homogeneity.

1. Introduction

Recall that a topological group G is said to be Polish if its group topology is Polish, i.e., separable and completely metrizable, and it is said to be non-Archimedean Polish if, in addition, it admits a countable neighborhood basis at the identity consisting of open subgroups. In a meeting in Durham in 1997, D. Evans asked if a non-Archimedean Polish group can be an uncountable free group. Around the same time H. Becker and A. Kechris asked the same question without the assumption of non-Archimedeanity (cf. [1]). In [14] the second author settled the first question, and in the subsequent paper [15] he settled also the second question. It was later discovered that some of the impossibility results from [15] followed already from an old important result of Dudley [5]. All these results were later vastly generalized in [12] by both authors of the present paper in the context of graph product of groups. In another direction, Khelif proved in [10] that no uncountable $\aleph_1$-free groups can be Polish, where we recall that a group G is said to be $\aleph_1$-free (resp. $\aleph_1$-free abelian) if every countable subgroups of G is free (resp. free abelian). Furthermore, a Polish group cannot be free abelian either, by [15]. In contrast, as well-known, the Baer-Specker group $\mathbb{Z}^\omega$ when topologized with the product topology is non-Archimedean Polish and $\aleph_1$-free (but not free). This interesting state of affairs motivated us to investigate the structure of the class of $\aleph_1$-free abelian non-Archimedean Polish groups, wondering in particular: how complicated are these topological groups?

Recently, there has been quite some interest in abelian (non-Archimedean) Polish groups. On one hand, with respect to the study of the Borel completexity of their actions, see e.g. [4], and on the other hand, in the context of homological algebra,

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see [2, 3]. Our study, which is algebraic in nature, complements these existing lines of inquiry by providing new structural algebraic insights into these groups.

We recall the following definitions as they will play a crucial role in our study:

Definition 1.1. Let A be a torsion-free abelian group.

(1) We say that A is separable if every finite subset of A is contained in a free direct summand of A.
(2) We say that A is torsionless if for every $0 \neq a \in A$ there is $f \in \mathrm{Hom}(A, \mathbb{Z})$ such that $f(a) \neq 0$.
(3) We A is $\mathbb{Z}$-homogenenous if every element has type $\mathbf{0}$ (cf. 3.7).

It is well-known, see e.g. [6, Corollary 2.9], that we have the following:

$$\text{separable} \;\Rightarrow\; \text{torsionless} \;\Rightarrow\; \aleph_1\text{-free} \;\Rightarrow \mathbb{Z}\text{-homogenenous.}$$

Now, it so happens that $\mathbb{Z}^\omega$ is not only $\aleph_1$-free but actually separable. Thus, the motivating question of the paper mentioned above (on the existence of $\aleph_1$-free abelian non-Archimedean Polish groups) can be strengthen to torsionless (resp. separable) abelian groups. We will deal this all these properties in our paper.

The first result that we present is on the number of groups and, with respect to the four properties mentioned above, it is of the strongest possible form.

Theorem 1.2. *There are continuum many separable (hence torsionless, hence $\aleph_1$-free) abelian non-Archimedean Polish groups which are not topologically isomorphic to product groups and are pairwise not continuous homomorphic images of each other. Furthermore, all these groups can be taken to be inverse limits of torsion-free completely decomposable groups (i.e., direct sums of* TFAB*'s of rank* 1*).*

The second result that we present is formulated in the framework of invariant descriptive set theory (see [8] for an excellent introduction to this subject). In [9], Kechris, Nies and Tent proposed the following program: as any non-Archimedean Polish group is topologically isomorphic to a closed subgroup of the topological group S_∞ of bijections of $\mathbb{N}$ onto $\mathbb{N}$, we can study classification problems on non-Archimedean Polish groups with respect to the Effros structure on the set of closed subgroups of S_∞, which is a standard Borel space (see [9] for details).

At the heart this program there is the following two-folded task:

- for natural classes of closed subgroups of S_∞, determine whether they are Borel;
- if a class $\mathcal{C}$ is Borel, study the Borel complexity of topol. isomorphism on $\mathcal{C}$.

In this respect, in our second theorem, we prove that the four properties mentioned above are not Borel and, more strongly, that they are complete co-analytic.

Theorem 1.3. *Determining if a non-Archimedean Polish group is in $\mathcal{C}$ is a complete co-analytic problem in the space of closed subgroups of S_∞ for the following classes $\mathcal{C}$ of abelian groups:*

(1) $\mathbb{Z}$-homogeneous;
(2) $\aleph_1$-free;
(3) torsionless;
(4) separable.

We conclude the paper with a result of independent interest:

Theorem 1.4. *There exists an inverse limit of free abelian groups (and so an abelian non-Archimedean Polish group) which is $\mathbb{Z}$-homogenenous but not $\aleph_1$-free.*

2. The descriptive set theory setting

From here on, as it is customary in set theory, we denote the set of natural numbers by ω. The aim of this section is to fix the coding, i.e., to specify which spaces we refer to in our results, together with introducing the necessary background.

Fact 2.1 ([4, Proposition 2.1])**.** The G be a topological group, then TFAE:

(1) G is a non-Archimedean Polish abelian group;
(2) G is isomorphic (as a topological group) to a closed subgroup of $\prod_{n<\omega} H_n$ (with the product topology), with each H_n abelian, countable and discrete;
(3) G is the inverse limit of an inverse system $(G_n, f_{(n,m)} : m \leqslant n < \omega)$, with each G_n abelian, countable, discrete, where on G we consider the product topology.

Notation 2.2. Let X be a Polish space. The Effros structure on X is the Borel space consisting of the family $\mathcal{F}(X)$ of closed subsets of X together with the σ-algebra generated by the following sets $\mathcal{C}_U$, where, for $U \subseteq X$ open, we let:

$$\mathcal{C}_U = \{D \in \mathcal{F}(X) : D \cap U \neq \emptyset\}.$$

Notation 2.3. We denote by S_∞ the topological group of bijections of ω onto ω, where the topology is the one induced by the following metric:

(1) if $x = y$, then $d(x, y) = 0$;
(2) if $x \neq y$, then $d(x, y) = 2^{-n}$, where n is least such that $x(n) \neq y(n)$.

Definition 2.4. By a tree on ω we mean a non-empty subset of $\omega^{<\omega}$ closed under initial segments (so in particular the empty sequence $\emptyset$ is the root of the tree).

Fact 2.5. Let $\mathcal{T}(S_\infty)$ denote the set of trees on ω with no leafs topologized defining the distance between two trees $T_1 \neq T_2$ as $\frac{1}{2^n}$, where n is least n such that $T_1 \cap \omega^n \neq T_1 \cap \omega^n$. For $C \in \mathcal{F}(S_\infty)$, define:

$$\mathbf{B} : C \mapsto T_C = \{g \restriction n : g \in C \text{ and } n < \omega\} \cup \{g^{-1} \restriction n : g \in C \text{ and } n < \omega\}.$$

Then the map $\mathbf{B}$ is a Borel isomorphism from $\mathcal{F}(S_\infty)$ onto $\mathcal{T}(S_\infty)$.

In virtue of Fact 2.5, we can work with $\mathcal{T}(S_\infty)$ instead of $\mathcal{F}(S_\infty)$.

Fact 2.6. The closed subgroups of S_∞ form a Borel subset of $\mathcal{F}(S_\infty)$ (see [9]), which we denote by $\mathrm{Sgp}(S_\infty)$, together with the Borel structure inherited from $\mathcal{F}(S_\infty)$.

Proviso 2.7. *From now on we only look at inverse systems with binding onto maps.*

Notation 2.8. We can look at an inverse system $A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ as a first-order structure in a language $L_{\mathrm{inv}} = \{P_n, \cdot_n, f_{(n,m)} : m \leqslant n < \omega\}$, where:

(1) the P_n's are disjoint predicates denoting the groups A_n's;
(2) $\cdot_n$ is a binary function symbol, interpreted as the group operation on A_n;
(3) the $f_{(n,m)}$'s are function symbols interpreted as morphisms from A_n to A_m.

Notation 2.9. Of course with respect to the language considered in 2.8 saying that a structure of the form $A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ is an inverse system is axiomatizable in first-order logic. Thus, the usual setting of invariant descriptive set theory applies and we can consider the Borel space of countable Pro-Groups with domain (a subset of) ω. For concreteness we also assume that our inverse systems

$A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ are such that e.g. $\mathrm{dom}(A_n) \subseteq \{p_n^m : 0 < m < \omega\}$, where p_n is the n-th prime number, so that the different sorts A_n's are disjoint, and thus in particular $n \neq m$ implies that $e_{A_n} \neq e_{A_m}$ (where e_{A_n} is the neutral element of A_n). Finally, naturally when the groups A_n are abelian, then we move to additive notation and so we write $+_n$ instead of $\cdot_n$ and 0_{A_n} instead of e_{A_n}.

Definition 2.10. Let $A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ be an inverse system as in 2.9, so that in particular $\mathrm{dom}(A_n) \subseteq \{p_n^m : 0 < m < \omega\}$, where p_n is the n-th prime number. For every $\bar{a} = (a_n : n < \omega) \in \varprojlim(A)$ we define a permutation $\pi_{\bar{a}}$ of ω as:

$$\pi_{\bar{a}}(k) = \begin{cases} k +_{A_n} a_n, & \text{if there is } n < \omega \text{ s.t. } k \in A_n, \\ k, & \text{if } k \notin \bigcup_{n<\omega} A_n. \end{cases}$$

Observation 2.11. The map $\bar{a} \mapsto \pi_{\bar{a}}$ from 2.10 is an embedding of $\varprojlim(A)$ into S_∞.

We omit the details of the next proposition as its proof is standard (use e.g. 2.5).

Proposition 2.12. *The map $A \mapsto \varprojlim(A)$ from* Pro-Groups *(recalling 2.9) into* $\mathrm{Sgp}(S_\infty)$ *(recalling 2.6 and the embedding described in 2.10) is Borel.*

3. A few crucial claims on TFAB's

3.1. The engine

The main source of construction of Polish non-Archimedean TFAB's present in this paper is described in the following definition, which we refer to as *the engine*.

Notation 3.1. $\mathbb{P}$ denotes the set of prime numbers.

Definition 3.2. Let $T \subseteq \omega^{<\omega}$ be a tree with no leafs and $L : T \to \mathcal{P}(\mathbb{P})$ be s.t.:
(a) if $\eta \trianglelefteq \nu$, then $L(\nu) \subseteq L(\eta)$;
(b) if $\eta \in T$, then $L(\eta) = \bigcup\{L(\nu) : \nu \in T \text{ and } \nu \text{ is a successor of } \eta\}$.
We define an inverse system $\mathrm{inv}(T, L) = (G_n, f_{(m,n)} : m \leqslant n < \omega)$ as follows:
(i) $Z_n := \sum\{x_\eta : \eta \in T \cap \omega^n\} \leqslant G_n \leqslant \sum\{\mathbb{Q}x_\eta : \eta \in T \cap \omega^n\} := Q_n$;
(ii) $G_n = \langle \frac{1}{p} x_\eta : \eta \in T \cap \omega^n \text{ and } p \in L(\eta)\rangle_{Q_n}$;
(iii) for $m \leqslant n < \omega$, $f_{(n,m)}$ is the unique homomorphism from G_n into G_m s.t.:

$$\eta \in T \cap \omega^n \;\Rightarrow\; x_\eta \mapsto x_{\eta \restriction m}.$$

Notice that the inverse system $\mathrm{inv}(T, L) = (G_n, f_{(n,m)} : m \leqslant n < \omega)$ is well-defined because of conditions (a) and (b) above. Being it well-defined, we do the following:

$$(T, L) \mapsto \mathrm{inv}(T, L) \mapsto \varprojlim(\mathrm{inv}(T, L)) := G(T, L).$$

So the group $G(T, L)$ is a well-defined Polish non-Archimedean abelian group, when equipped with the topology inherited from the product topology on $\prod_{n<\omega} G_n$.

Remark 3.3. Notice that the inverse systems $\mathrm{inv}(T, L) = (G_n, f_{(n,m)} : m \leqslant n < \omega)$ defined in 3.2 have bounding maps $f_{(n,m)}$ which are onto, this is because we ask that the tree T has no leafs and because we ask that condition (b) from there holds.

Remark 3.4. Notice also that for $\mathrm{inv}(T, L) = (G_n, f_{(n,m)} : m \leqslant n < \omega)$ as in 3.2 we have that the groups G_n's are completely decomposable, i.e., they are direct sums of groups of rank 1, in fact it follows immediately from the definitions that:

$$G_n = \sum_{\eta \in \omega^n \cap T} \langle x_\eta \rangle^*_{G_n},$$

where $\langle X\rangle^*_{G_n}$ denotes pure closure (cf. [7, pg. 151]).

Proof. This is immediate. ∎

Notation 3.5. Given a tree $T \subseteq \omega^{<\omega}$ we denote by $[T]$ the set of $\eta \in \omega^\omega$ such that $\eta \restriction n \in T$ for all $n < \omega$, i.e., the set of infinite branches of T. Recall that in 3.2 we only consider trees with no leafs and so, for T's as in 3.2, $[T]$ is always non-empty.

For background on the notions that we now introduce see e.g. [7, Chapter 13].

Definition 3.6. Let $A \in$ TFAB. Let $(p_i : i < \omega)$ be the list of the prime numbers in increasing order. For $a \in A$, we define the characteristic of a, denoted as $\chi(a)$, as follows: $\chi(a) = (h_{p_i}(a) : i < \omega)$, where $h_{p_i}(a)$ is the supremum of the $k < \omega$ such that $p_i^k \mid a$, where the supremum is taken in $\omega \cup \{\infty\}$ (so the value ∞ is allowed).

Definition 3.7. (1) Two characteristics $(k_i : i < \omega)$ and $(\ell_i : i < \omega)$ are said to be equivalent if $k_i = \ell_i$ for for almost all $i < \omega$ and both k_i and ℓ_i are finite whenever $k_i \neq \ell_i$. The equivalence classes of characteristics are called *types*.
(2) Given $A \in$ TFAB we denote by $\mathbf{t}_A(a)$ the type of a in A; we may simply write $\mathbf{t}(a)$ when A is clear from the context.
(3) We denote by $\mathbf{0}$ the type of the characteristic $(0, 0, ...)$.
(4) For $A \in$ TFAB and $a, b \in A$ we say that $\mathbf{t}(a) \leqslant \mathbf{t}(b)$ if there are characteristics $\chi = (k_i : i < \omega) \in \mathbf{t}(a)$ and $\nu = (\ell_i : i < \omega) \in \mathbf{t}(b)$ such that $(k_i : i < \omega) \leqslant (\ell_i : i < \omega)$, where the order is the pointwise order, i.e., $k_i \leqslant \ell_i$, for all $i < \omega$.

We will need the following well-known facts.

Fact 3.8 ([7, (C) on pg. 411])**.** Let G and H be torsion-free abelian and suppose that there is an homomorphism $f : G \to H$, then, for every $g \in G$, $\mathbf{t}(g) \leqslant \mathbf{t}(f(g))$.

Fact 3.9. [6, Theorem 2.3] Let $A \in$ TFAB. Then A is $\aleph_1$-free if and only if, for every finite $S \subseteq A$, the pure closure of S in A (denoted $\langle S\rangle^*_A$) is free abelian.

3.2. Nice pairs

Definition 3.10. We call (T, L) a nice pair when for some S and $\bar p$ we have that:
(a) S is a subtree of T and T is a subtree of $\omega^{<\omega}$ with no leafs;
(b) $S \subseteq T$ and $\eta \in S$ implies that $0 \notin \mathrm{ran}(\eta)$;
(c) $\bar p = (p_\eta : \eta \in S)$ is a sequence of primes without repetitions;
(d) $S^+ = \{\eta^\frown\nu : \eta \in S,\ \nu \in \{0\}^{<\omega}\} \subseteq T$;
(e) $p_{\eta^\frown\nu} = p_\eta$, when $\eta \in S$ and $\nu \in \{0\}^{<\omega}$;
(f) L is the function with domain T defined as follows:

$$L(\eta) = \{p_\nu : (\nu \triangleleft \eta \text{ or } \eta \trianglelefteq \nu) \text{ and } \nu \in S^+\}.$$

Observation 3.11. If (T, L) a nice pair, then (T, L) is as in 3.2.

Convention 3.12. *For the rest of this subsection we fix (T, L), S, S^+ and $\bar p$ as in 3.10 and we let $G = G(T, L)$. For $\eta \in [T]$, we let $L(\eta) = \bigcap\{L(\eta \restriction n) : n < \omega\}$. Also, for $n < \omega$, we let $f_{(\omega,n)}$ be the obvious projection of G onto G_n. For $\nu \in [T]$, we let $x_\nu = (x_{\nu\restriction n} : n < \omega) \in G$.*

Claim 3.13. (Recalling 3.12.) If S has an ω-branch (i.e., there is $\eta \in [T]$ such that, for all $n < \omega$, $\eta \restriction n \in S$), then G is not $\mathbb{Z}$-homogeneous, and so G is not $\aleph_1$-free.

Proof. In this case $L(\eta) = \bigcap\{L(\eta \restriction n) : n < \omega\} = \{p_{\eta\restriction n} : n < \omega\}$ is infinite, and so in G we have that x_η is divisible by p for every $p \in L(\eta)$. ∎

Definition 3.14. For $\eta \in T \cup [T]$, let K_η be the additive subgroup of $\mathbb{Q}$ generated by $K^-_\eta = \{\frac{1}{p} : p \in L(\eta)\} \subseteq \mathbb{Q}$, explicitly, K_η is the subgroup of $\mathbb{Q}$ consisting of:

$$\{\frac{b}{\prod_{\ell<k} p_\ell} : b \in \mathbb{Z},\ k < \omega,\ p_\ell \in L(\eta),\ (p_\ell : \ell < k) \text{ with no repetitions}\}.$$

Main Claim 3.15. If S has no ω-branch, then G is separable (and so G torsionless, and so G is $\aleph_1$-free).

Proof. We first show that G is torsionless. For any $y \in G \setminus \{0\}$ we want to find $f \in \mathrm{Hom}(G, \mathbb{Z})$ such that $f(y) \neq 0$. To this extent, fix $y \in G \setminus \{0\}$, so $y = (y_n : n < \omega)$.

$(\star_1)$ For $n < \omega$, let $y_n = \sum\{a_\nu x_\nu : \nu \in Y_n\}$, where:
- $(\cdot_1)$ $Y_n \subseteq \omega^n \cap T$ is finite;
- $(\cdot_2)$ $a_\nu \in K_\nu \setminus \{0\}$ (recall 3.14).

$(\star_2)$ We can find $n_* < \omega$ such that, for every $n_* \leqslant n < \omega$, $y_n = f_{(\omega,n)}(y) \neq 0$.

$(\star_3)$ By induction on $n \geqslant n_*$ we choose ν_n and b_ν such that:
- (a) $\nu_n \in Y_n$;
- (b) $n = m + 1 > n_*$ implies $\nu_m \triangleleft \nu_n$;

Why can we carry the induction? For $n = n_*$ let $\nu_n \in Y_n$. This is possible by $(\star_2)$. For $n = m+1$, note that $f_{(n+1,k_n)} \circ f_{(\omega,n+1)} = f_{(\omega,n)}$ and so $a_{\nu_m} = \sum\{a_\rho : \nu_m \triangleleft \rho \in Y_{n+1}\}$. As $\nu_m \in Y_m$ we have $a_{\nu_n} \neq 0$, hence there is $\rho \in Y_{n+1}$ such that $\nu_m \triangleleft \rho$, so letting $\nu_n = \rho$ we can satisfy clauses (a) and (b).

$(\star_4)$ Let $\nu = \bigcup\{\nu_n : n \geqslant n_*\} \in [T]$, for $(\nu_n : n \geqslant n_*)$ as in $(\star_3)$.

Now there are two cases.
Case 1. For all $n < \omega$, $\nu \restriction n \in S$.
This is impossible, as we are assuming that S has no ω-branch.
Case 2. For some $n < \omega$, $\nu \restriction n \notin S$.
If this is the case, then w.l.o.g. $n > n_*$ and easily we have that $L(\nu \restriction (n + 1))$ is finite. Let $\eta = \nu_{n+1}$.

$(\star_5)$ Let $g \in \mathrm{Hom}(G, K_\eta x_\eta)$ be defined for $z \in G$ by $g(z) = \pi_{(n,\eta)}(f_{(\omega,n)})(z)$, where $\pi_{(n,\eta)}$ if the projection from $G_{\mathrm{lg}(\eta)}$ onto $\langle x_\nu\rangle^*_{G_{\mathrm{lg}(\eta)}}$ (recall that $G_{\mathrm{lg}(\eta)}$ is completely decomposable, cf. 3.4).

Clearly we have that:

$(\star_6)$ (a) $g(y) = a_\eta x_\nu \neq 0$;
(b) $g \in \mathrm{Hom}(G, K_\eta x_\eta)$.

[Why? See $(\star_1)(\cdot_2)$.]
But then, as $L(\eta)$ is finite, clearly $K_\eta x_\eta \cong \mathbb{Z}x_\eta$, and so we are done proving that G is torsionless.

We now show that G is actually separable. To this extent, let $X \subseteq G$ be finite.

$(*_1)$ It suffices to prove that there are $n < \omega$ and finite $Y \subseteq \omega^n \cap T$ such that letting $\pi_{(n,Y)}$ be the natural projection of G_n onto $G_{(n,Y)} := \langle Y\rangle^*_{G_n}$ we have:
- (a) for every $\nu \in Y$ we have that $L(\nu)$ is finite;
- (b) $g = \pi_{(n,Y)} \circ f_{(\omega,n)}$;
- (c) $g \in \mathrm{Hom}(G, G_{(n,Y)})$;
- (d) $g_0 := g \restriction \langle X\rangle^*_G$ is an isomorphism from $\langle X\rangle^*_G$ onto $G_{(n,Y)} = \langle Y\rangle^*_{G_n}$.

We show that $(*_1)$ holds. To this extent, first of all, notice that $G_{(n,Y)}$ is a free direct summand of G, because $G_n = \bigoplus\{K_\nu x_\nu : \nu \in T \cap \omega\}$ and, for every $\nu \in Y$,

$L(\nu)$ is finite and so $K_\nu \cong \mathbb{Z}$. Secondly, observe that then also $\langle X\rangle^*_G$ is free, since by (c), g_0 is an isomorphism. Observe now that, by clause (b), we have that $g_0^{-1}\circ g$ is an homomorphism from G into $\langle X\rangle^*_G$. Also, by clause (c), clearly g_0^{-1} is an isomorphism from $G_{(n,Y)}$ onto $\langle X\rangle^*_G$. Notice now that for $a \in \langle X\rangle^*_G$ we have that:

$$g_0^{-1}\circ g(a) = g_0^{-1}(g(a)) = g_0^{-1}(g_0^{-1}(a)) = a.$$

Hence, we have that $g_0^{-1}\circ g$ is a retraction of G onto the free subgroup $\langle X\rangle^*_G$ and so X is contained in a free direct summand of G, as desired.

($*_2$) For every $n<\omega$, $f_{(n+1,n)}$ maps $f_{(\omega,n+1)}(\langle X\rangle^*_G)$ onto $f_{(\omega,n)}(\langle X\rangle^*_G)$, hence:
 ($\cdot_1$) $\mathrm{rk}(f_{(\omega,n+1)}(\langle X\rangle^*_G)) \geqslant \mathrm{rk}(f_{(\omega,n)}(\langle X\rangle^*_G))$;
 ($\cdot_2$) those ranks are $\leqslant |X|$ which is finite.

($*_3$) For some $n_* < \omega$, $(\mathrm{rk}(f_{(\omega,m)}(\langle X\rangle^*_G)) : m \geqslant n_*)$ is constant, call it k_*, and for every $n_2 > n_1 \geqslant n_*$, $f_{(n_2,n_1)} \restriction f_{(\omega,n_2)}(\langle X\rangle^*_G)$ is an isomorphism from $f_{(\omega,n_2)}(\langle X\rangle^*_G)$ onto $f_{(\omega,n_1)}(\langle X\rangle^*_G)$.

[Why? As $n_2 > n_1 \geqslant n_*$ and so $\mathrm{rk}(f_{(\omega,n_2)}(\langle X\rangle^*_G)) = \mathrm{rk}(f_{(\omega,n_1)(\langle X\rangle^*_G)})$, recalling that $f_{(n_2,n)}$ maps $f_{(\omega,n_2)}(\langle X\rangle^*_G)$ onto $f_{(\omega,n_1)}(\langle X\rangle^*_G)$, so the map $f_{(n_2,n_1)} \restriction f_{(\omega,n_2)}(\langle X\rangle^*_G)$ has trivial kernel and so it is 1-to-1, and trivially it is also onto.]

($*_4$) If $n \geqslant n_*$, then $f_{(\omega,n)} \restriction \langle X\rangle^*_G$ is an isomorphism from $\langle X\rangle^*_G$ onto $f_{(\omega,n)}(\langle X\rangle^*_G)$.

Why ($*_4$) holds? Let $n_* \leqslant n < \omega$ and suppose the negation of the thesis. Then there is $y \in \langle X\rangle^*_G \setminus \{0\}$ such that $f_{(\omega,n)}(y) = 0$. Let $y = (y_m : m<\omega)$. Then we can find $k \geqslant n$ such that $f_{(\omega,k)}(y) = y_k \neq 0$. But then $y_k \in G_k \setminus \{0\}$ and $f_{(k,n)}(y_k) = 0$, and so $f_{(k,n)}$ has non-trivial kernel, contradicting the choice of n_* (recall that by hypothesis $n \geqslant n_*$ and that $f_{(k,n)} : G_k \to G_n$ is onto G_n). So ($*_4$) holds indeed.

($*_5$) W.l.o.g. $|X| = k_*$, equivalently, X is independent.

($*_6$) For $n<\omega$.
 (a) Let $Y_n \subseteq T \cap \omega^n$ be finite and such that we have:

$$f_{(\omega,n)}(\langle X\rangle^*_G) \subseteq G_{(n,Y_n)} = \bigoplus\{K_\nu x_\nu : \nu \in Y_n\}.$$

 (b) For $y \in X$, let $f_{(\omega,n)}(y) = \sum\{a_{(y,n,\nu)}x_\nu : \nu \in Y_n\}$, where $a_{(y,n,\nu)} \in K_\nu$.
 (c) W.l.o.g. for every $n \geqslant n_*$ and $\nu \in Y_n$, there is $y \in X$ such that $a_{(y,n,\nu)} \neq 0$.
 (d) For $\nu \in Y_n$, let $\bar{a}_\nu = (a_{(y,n,\nu)} : y \in Y_n)$.

[Why? Obvious.]

($*_7$) Let $Y^\star_n = \{\nu \in Y_n : L(\nu) \text{ is finite}\}$.

($*_8$) $(\mathrm{rk}(\pi_{(n,Y^\star_n)} \circ f_{(\omega,n)}(\langle X\rangle^*_G)) : n \geqslant n_*)$ is eventually constant, say for $n \geqslant n_\star$, and let $k_\star$ be this constant value.

($*_9$) Assume $n_* \leqslant n_1 \leqslant n_2 < \omega$, then:
 ($\cdot_1$) if $\nu \in Y_{n_1}$, then there is $\rho \in Y_{n_2}$ such that $\nu \triangleleft \rho$;
 ($\cdot_2$) if $\nu \in Y^\star_n$ and $\nu \triangleleft \rho \in Y_{n_2}$, then $\rho \in Y^\star_{n_2}$.

($*_{10}$) W.l.o.g. $n_\star = n_*$ (we shall use this freely).

($*_{11}$) Let $(y_\ell : \ell < k)$ listing $|X|$ (recall ($*_5$)).

($*_{12}$) For every $n<\omega$ and $\ell < k_*$, let:

$$y_{(\ell,n)} = \pi_{(n,Y^\star_n)} \circ f_{(\omega,n)}(y_\ell),$$

where $\pi_{(n,Y^\star_n)} : G_n \to \langle Y^\star_n\rangle^*_{G_n}$ is the obvious projection onto $\langle Y^\star_n\rangle^*_{G_n}$.

$(*_{13})$ Hence, we have:

$$y_{(\ell,n)} = \sum\{a_{(y_\ell,n,\nu)}x_\nu : \nu \in Y^\star_n\}.$$

$(*_{14})$ For $n \geqslant m \geqslant n_*$, let:
(a) $Y^\star_{(n,m)} = \{\nu \in Y^\star_n : \nu \restriction m \in Y^\star_m\}$;
(b) $y_{(\ell,n,m)} = \sum\{a_{(y_\ell,n,\nu)}x_\nu : \nu \in Y^\star_{(n,m)}\}$.

$(*_{15})$ For $n \geqslant m \geqslant n_*$, $y_{(\ell,n,m)} = \pi_{(n,Y^\star_{(n,m)})}(y_\ell)$.

$(*_{16})$ $k_\star = k_*(= |X|)$ (recall $(*_5)$ and $(*_8)$).

We prove $(*_{16})$. Toward contradiction, assume that $k_\star < k_*$. For $n \geqslant n_*$, the sequence $(y_{(\ell,n)} : \ell \leqslant k_\star)$ cannot be independent, so for some $\ell(n) \leqslant k_\star$ the element $y_{(\ell(n),n)}$ depends on $\langle y_{(\ell,n)} : \ell \neq \ell(n),\ \ell \leqslant k_\star\rangle_{G_n}$. But if $n_2 \geqslant n_1 \geqslant n_*$, then using $f_{(n_2,n_1)}$ and $y_{(\ell,n_2)}$, $y_{(\ell,n_2,n_1)}$ we get that $\ell(n_2)$ can serve as $\ell(n_1)$, and so w.l.o.g. $\ell(n) = \ell(n_*)$ for all $n \geqslant n_*$, so renaming w.l.o.g. we have that $\ell(n) = k_\star$.

$(*_{16.1})$ Let $\sum\{b^n_\ell y_{(\ell,n)} : \ell \leqslant k_\star\} = 0$, where, for $\ell \leqslant k_\star$, $b^n_\ell \in \mathbb{Z}$ and $b^n_{k_\star} \neq 0$.

$(*_{16.2})$ W.l.o.g. for every $n_2 \geqslant n_1 \geqslant n_*$ we have that, for every $\ell \leqslant k_\star$, $b^{n_2}_\ell = b^{n_1}_\ell$, call this number b_ℓ, so that in particular $b_{k_\star} \neq 0$.

$(*_{16.3})$ However, in G we have that $z := \sum\{b_\ell y_\ell : \ell \leqslant k_\star\} \neq 0$ and $z \notin \langle y_\ell : \ell < k_*\rangle^*_G$, so there is $n_+ > n_*$ such that $f_{(\omega,n_+)}(z) \neq 0$, so we can find $\nu \in Y_{n_+}$ such that $a_{(z,n_+,\nu)} \neq 0$.

$(*_{16.4})$ $\nu \notin Y^\star_{n_+}$.

[Why? Because $\pi_{(n_+,Y^\star_{n_+})} \circ f_{(\omega,n_+)}(z) = \sum\{b^n_\ell y_{(\ell,n)} : \ell \leqslant k_\star\} = 0$ by $(*_{16.1})$.]

$(*_{16.5})$ There are $n_\dagger$ and ρ such that:
(a) $n_\dagger \geqslant n_+ \geqslant n_*$;
(b) $\rho \in Y_{n_\dagger}$ and $\nu \triangleleft \rho$ (where ν is as in $(*_{16.3})$, $(*_{16.4})$);
(c) $\rho \in Y^\star_{n_\dagger}$;
(d) $a_{(z,n_\dagger,\rho)} \neq 0$.

Why does $(*_{16.5})$ hold? We choose ν_n by induction on $n \geqslant n_+$ such that:
(a) $\nu_{n_+} = \nu \in Y_{n_+}$;
(b) $\nu_n \in Y_n$;
(c) if $n = m+1 > n_+$, then $\nu_m \triangleleft \nu_n$;
(d) $a_{(z,n,\nu_n)} \neq 0$.

As in the beginning of the proof we can carry the induction. Now, as we are assuming that S has no ω-branch, clearly for some $n_\dagger > n_+$ we have that $\nu_{n_\dagger} \notin S$, but then necessarily $L(\nu_{n_\dagger})$ has be finite, i.e., $\nu_n \in Y^\star_n$, since by assumption S is a subtree of T and we defined, for $\eta \in T$, $L(\eta) = \{p_\nu : (\nu \triangleleft \eta \text{ or } \eta \trianglelefteq \nu) \text{ and } \nu \in S^+\}$.

$(*_{16.6})$ $\pi_{(n_\dagger,Y^\star_{n_\dagger})} \circ f_{(\omega,n_\dagger)}(z) \neq 0$.

[Why? By $(*_{16.5})$(d).]

$(*_{16.7})$ $\pi_{(n_\dagger,Y^\star_{n_\dagger})} \circ f_{(\omega,n_\dagger)}(z) = 0$.

[Why? Notice that:

$$\begin{array}{rcl}\pi_{(n_\dagger,Y^\star_{n_\dagger})} \circ f_{(\omega,n_\dagger)}(z) & = & \pi_{(n_\dagger,Y^\star_{n_\dagger})} \circ f_{(\omega,n_\dagger)}(\sum\{b_\ell y_\ell : \ell \leqslant k_\star\}) \\ & = & \sum\{b^{n_\dagger}_\ell y_{(n_\dagger,\ell)} : \ell \leqslant k_\star\} \\ & = & \sum\{b_\ell y_{(n_\dagger,\ell)} : \ell \leqslant k_\star\} \\ & = & 0\end{array},$$

where the first equality is by the definition of z in $(*_{16.3})$; the second equality is by $(*_{12})$, the third equality is by $(*_{16.2})$ and the fourth equality is by $(*_{16.1})$.]

But $(*_{16.7})$ contradicts $(*_{16.6})$. Hence, $(*_{16})$ holds indeed.

$(*_{17})$ (a) Let $f^\star_{(\omega,n)} = \pi_{(n,Y^\star_n)} \circ f_{(\omega,n)}$;

(b) $f^\star_{(\omega,n)}$ is an homomorphism from G into $\langle Y^\star_n\rangle^*_{G_n} \leqslant G_n$.

$(*_{18})$ If $n \geqslant n_*$, then $g_0 = f^\star_{(\omega,n)} \restriction \langle X\rangle^*_G$ is 1-to-1 from $\langle X\rangle^*_G$ onto $\langle Y^\star_n\rangle^*_{G_n}$.

[Why? By $(*_{16})$ and $(*_{17})$, as in the proof of $(*_4)$.]

$(*_{19})$ $(*_1)$ holds and so the proof is complete.

[Why? Let $n \geqslant n_*$ and let Y from $(*_1)$ be $Y^\star_n$, then by $(*_{18})$ we are done.] ■

4. Proofs of main theorems

In this section we show how to use our machinery to prove Theorems 1.2, 1.3.

4.1. Proof of Theorem 1.2

Definition 4.1. Let $\mathbb{P}_*$ be an infinite set of primes. Let S be a well-founded (i.e., no ω-branches) subtree of $\omega^{<\omega}$ such that for every $\eta \in S$ we have that $\{\nu \in S : \eta \triangleleft \nu\}$ is infinite and $0 \notin \mathrm{ran}(\eta)$. We define a pair $(\omega^{<\omega}, L_{\mathbb{P}_*})$ as in 3.10 by letting $T = \omega^{<\omega}$, $S \subseteq T$ the tree we just fixed and $(p_\eta : \eta \in S)$ be a list of $\mathbb{P}_*$ without repetitions.

Observation 4.2. By 3.15, for $(\omega^{<\omega}, L_{\mathbb{P}})$ as in 4.1, $G(\omega^{<\omega}, L_{\mathbb{P}})$ (cf. 3.2) is $\aleph_1$-free.

We need the following definition from [11].

Definition 4.3. For a category $\mathcal{C}$ by Pro-$\mathcal{C}$ we denote the category of ω-inverse systems where the morphisms are given as sequences of onto maps $f_n : A_{\varphi(n)} \to B_n$ where $\varphi : \omega \to \omega$ is increasing, and all is commutative, that is, letting $A = (A_n, \pi^A_{(n,m)} : m \leqslant n < \omega)$ and $B = (B_n, \pi^B_{(n,m)} : m \leqslant n < \omega)$ in Pro-$\mathcal{C}$, for every $n \leqslant m < \omega$, we have that $f_m \circ \pi^A_{(\varphi(n),\varphi(m))} = \pi^B_{(n,m)} \circ f_n$.

Lemma 4.4. *Let $A = (A_n, \pi^A_{(n,m)} : m \leqslant n < \omega)$ and $B = (B_n, \pi^B_{(n,m)} : m \leqslant n < \omega)$ be inverse systems of groups and let $G = \varprojlim(A)$ and $H = \varprojlim(B)$. Then if there is a continuous surjection from G onto H, then there is a morphism of Pro-Groups from A to B.*

Proof. Suppose that $\gamma : G \to H$ is a continous surjection (epimorphism). Recall that procountable group means that there is a nbhd basis of e_G of normal open subgroups. So if G is procountable, D is discrete and $f : G \to D$ is a continuous epimorphism, then the kernel of f being an open subgroup contains a subgroup from this nbhd basis. This yields the desired morphism of Pro-Groups from A to B. ■

Lemma 4.5. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be infinite sets of primes such that $\mathbb{P}_1 \cap \mathbb{P}_2$ is finite, and let $(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ be as in 4.1 and $G_\ell = G(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ (recall 3.2). Then there is no continuous surjection from G_1 onto G_2.*

Proof. Let $G_1 = \varprojlim(G_{(1,n)})$ and $G_2 = \varprojlim(G_{(2,n)})$, where the $G_{(\ell,n)}$'s as from the inverse system $\mathrm{inv}(\omega^{<\omega}, L_{\mathbb{P}_\ell})$, of course. Suppose that there is a continuous surjection from G_1 onto G_2, then by 4.4 there is $n < \omega$ and a surjection from $G_{(1,n)}$ onto $G_{(2,0)}$, but by 3.8 we reach a contradiction as we can find $a \in G_{(1,n)}$ such that $\mathbf{t}(a) \nleqslant \mathbf{t}(b)$ for all $b \in G_{(2,0)}$ recalling that by assumption $\mathbb{P}_1 \cap \mathbb{P}_2$ is finite, and so there are infinitely many primes in $\mathbb{P}_1 \setminus \mathbb{P}_2$ (as by assumption $\mathbb{P}_1$ is infinite). ■

We recall that by *product group* we mean a group of the form $\prod_{n<\omega} H_n$.

Proof of 1.2. Relying on 4.5, it suffices to take a collection $\{\mathbb{P}_\alpha : \alpha < 2^{\aleph_0}\}$ of almost disjoint subsets of the set of prime numbers, where A, B are almost disjoint if $A \cap B$ is finite. We are only left to show that each $G(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ is not topologically isomorphic to a product group. Clearly a $\aleph_1$-free abelian *product* group has to have the form $H = \prod_{n<\omega} H_n$ with each H_n countable and free abelian. Suppose that there is surjective continuous homomorphism from $G = G(\omega^{<\omega}, L_{\mathbb{P}_\alpha}) = \varprojlim(G_n)$ onto H for some $\alpha < 2^{\aleph_0}$, then by 4.4 there is $n < \omega$ and a surjective homomorphism from G_n onto a free abelian group, but, by 3.8, this leads to a contradiction, as there is $g \in G_n$ such that $\mathbf{t}(g) \neq \mathbf{0}$, while every $h \in H$ is such that $\mathbf{t}(h) = \mathbf{0}$. ■

4.2. Proof of Theorem 1.3

Definition 4.6. Let $A \subseteq \omega^{<\omega}$ be an infinite tree (in this case, crucially, possibly with leafs). For $\eta \in A$, let $\eta^{+1} = (\eta(n) + 1 : n < \omega)$ and let $A^{+1} = S = \{\eta^{+1} : \eta \in A\}$. Let $\bar{p} = \{p_\eta : \eta \in S\}$ and $T = \omega^{<\omega}$. And let $G(\omega^{<\omega}, L_A)$ be defined as in 3.10 for this choice of S and $\bar{p}$.

Lemma 4.7. *For $(\omega^{<\omega}, L_A)$ as in 4.6, letting $G_A = G(\omega^{<\omega}, L_A)$ we have:*
(1) if A is well-founded, then G_A is separable, and so torsionless, and so $\aleph_1$-free, and so $\mathbb{Z}$-homogenenous;
(2) if A is not well-founded, then G_A is not $\mathbb{Z}$-homogenenous, and so not $\aleph_1$-free, and so not torsionless, and so not separable.

Proof. This is by 3.13 and 3.15. ■

Proof of 1.3. This is by 2.12 and 4.7, together with the observation that the map $A \mapsto \mathrm{inv}(\omega^{<\omega}, L_A)$ is Borel, when considering models whose domain is $\subseteq \omega$. ■

5. Other results

5.1. Separating $\mathbb{Z}$-homogeneous and $\aleph_1$-free

In this section we prove Theorem 1.4, this result is of independent interest.

Fact 5.1. For every $2 \leqslant \mathbf{n} < \omega$ and $\mathbb{P}_*$ an infinite set of primes, there are $K_{\mathbf{n}}$, $\bar{x}_{\mathbf{n}} = (x_i : i < \mathbf{n})$ and $\bar{y}_{\mathbf{n}} = (y_p : p \in \mathbb{P}_*)$ such that:
(1) $Z_{\mathbf{n}} := \mathbb{Z}^{\mathbf{n}} \leqslant K_n \leqslant \mathbb{Q}^{\mathbf{n}} := H_{\mathbf{n}}$;
(2) for every $p \in \mathbb{P}_*$, there is $y_p \in K_{\mathbf{n}} \setminus Z_{\mathbf{n}}$ such that $py_p \in H_{\mathbf{n}}$;
(3) for every $X \subseteq K_{\mathbf{n}}$ of size $< \mathbf{n}$ we have that $\langle X \rangle^*_{K_{\mathbf{n}}}$ is free;
(4) $\langle \bar{x} \rangle^*_{K_{\mathbf{n}}}$ is not free, in fact, more strongly, if $\mathbb{P}_0 \subseteq \mathbb{P}_*$, then the group

$$K_{(\mathbf{n}, \mathbb{P}_0)} = \langle x_\ell, y_p : \ell < \mathbf{n} \text{ and } p \in \mathbb{P}_0 \rangle_{K_{\mathbf{n}}} \leqslant \langle \bar{x} \rangle^*_{K_{\mathbf{n}}}$$

is not free and disjoint from $\{y_p : p \in \mathbb{P}_* \setminus \mathbb{P}_0\}$.

Claim 5.2. Let $G = \prod_{n<\omega} K_n$. Then G is $\mathbb{Z}$-homogenenous but not $\aleph_1$-free.

Proof. Easy. ■

Crucially the group G from 5.2 is not an inverse limit of completely decomposable groups, we want to strengthen 5.2 so as to have in addition this.

Convention 5.3. *(1)* $\mathbf{n} \in [2, \omega)$*;*
(2) $K_{\mathbf{n}}, H_{\mathbf{n}}, Z_{\mathbf{n}}$ *are as in 5.1;*

(3) $(p_n : n < \omega)$ *lists* $\mathbb{P}_\star$ *in increasing order.*

Definition 5.4. For $n < \omega$, let $G^*_n = \langle x_\ell, y_{p_i} : \ell < \mathbf{n} \text{ and } i < n\rangle_{K_\mathbf{n}} \leqslant K_\mathbf{n} \leqslant \langle \bar{x}\rangle^*_{H_\mathbf{n}}$.

Claim 5.5. (1) For every $n < \omega$, G^*_n is free and $G^*_n \leqslant H_\mathbf{n}$;
(2) For every $n < \omega$, G^*_n is completely decomposable;
(3) $(G^*_n : n < \omega)$ is $\leqslant$-decreasing with intersection $K_\mathbf{n}$.

Proof. It suffices to consider the following group:

$$G^\star_n = \langle \frac{1}{\prod_{i<n} p_\ell} x_0, ..., \frac{1}{\prod_{i<n} p_\ell} x_{\mathbf{n}-1}\rangle_{H_\mathbf{n}}$$

and to observe that $G^*_n \leqslant G^\star_n$ and that $G^\star_n$ is free. ∎

Claim 5.6. There is an inverse system $(G_n, f_{(n,m)} : m \leqslant n < \omega)$ such that:

(a) for every $n < \omega$, $G_n = \bigoplus\{G_{(n,j)} : j < \omega\}$;
(b) $G_{(n,0)} = G^*_n$ and, for $j < \omega$, $G_{(n,j+1)} = \mathbb{Z}x_{(n,j+1)}$;
(c) for each $n < \omega$, there is $g_n : \omega \to \omega$ onto such that for $j < \omega$:
$$h_{(n,j)} := f_{(n+1,n} \restriction G_{(n+1,j)} \in \mathrm{Hom}(G_{(n+1,j)}, G_{(n,g_n(j))});$$
(d) for each $n < \omega$, $h_{(n,0)}$ is the inclusion map from G^*_{n+1} into G^*_{n+1}, recalling that $G^*_{n+1} \leqslant G^*_n$;
(e) $g_n(2j + 1) = 0$ and $(h_{(n,2j+1)}(x_{(n+1,2j+1)}) : j < \omega)$ lists the elements of $G_{(n,0)}$;
(f) $g_n \restriction \{2j+2 : j < \omega\}$ is 1-to-1 onto $\omega\backslash\{0\}$ and $h_{(n,2j+2)}(x_{(n+1,2j+2)}) = x_{g_n(2j+2)}$.

Claim 5.7. For $\mathfrak{s} = (G_n, f_{(n,m)} : m \leqslant n < \omega)$ and $G = \varprojlim(\mathfrak{s})$ as in 5.6 we have:

(1) each G_n is completely decomposable (even free);
(2) G is $\mathbb{Z}$-homogeneous;
(3) G is not $\aleph_1$-free.

Proof. Item (1) is clear by 5.5 and 5.6. Concerning (3), let $G_{(\omega,0)}$ be the subgroup of G consisting of the elements $(g_n : n < \omega) \in G$ such that for every $n < \omega$, $g_n \in G_{(n,0)}$. Then $G_{(\omega,0)} \cong K_\mathbf{n}$ (recall 5.5(c)) and so obviously G is not $\aleph_1$-free. Finally, we show (2), i.e., that G is $\mathbb{Z}$-homogenenous. Let $g = (g_n : n < \omega) \in G\backslash\{0\}$. We distinguish two cases.
<u>Case 1</u>. For some $n < \omega$, $g_n \notin G_{(n,0)}$.
If this is the case, then for some $n_* < \omega$ we have that, for every $m \geqslant n_*$, $f_{(\omega,m)}(g) \neq 0$. But then, as in earlier proofs, we can find $h_* \in \mathrm{Hom}(G, G_{(n_*,j_*)})$, for some $j_* > 0$, such that $h_*(g) = x_{(n_*,j_*)}$ and so easily $\langle g\rangle^*_G$ is free.
<u>Case 2</u>. For every $n < \omega$, $g_n \in G_{(n,0)}$.
If this is the case, then $g \in G_{(\omega,0)}$ and so as $G_{(\omega,0)}$ is pure in G it suffices to observe that $G_{(\omega,0)} \cong K_\mathbf{n}$ is $\mathbb{Z}$-homogeneous. ∎

5.2. For every $\eta \in [T]$, $\bigcap\{L(\eta \restriction n)\}$ is finite is not sufficient

In a previous version of this paper we were claiming that if (T, L) is as in 3.2 and in addition, for every $\eta \in [T]$, $\bigcap\{L(\eta \restriction n)\}$ is finite, then $G(T, L)$ is $\aleph_1$-free. The next example shows that this is *not* the case. This motivated the introduction of nice pairs in Section 3.2. We include the example for completeness of exposition.

Claim 5.8. Let $(p_n : n \in \omega)$ be distinct primes. Then, for every $n < \omega$, there are $(\bar{p}_n, \bar{a}_n, \bar{Q}_n)$ such that the following conditions hold:

(a) $\bar{p}_n = \{p_\eta : \eta \in 2^n\}$ where $p_\eta \in \{p_\ell : \ell \leqslant n\}$;

(b) $\bar{a}_n = \{a_\eta : \eta \in 2^n\}$, where $a_\eta \in \mathbb{Z} \setminus \{0\}$;
(c) $\bar{Q}_n = \{Q_\eta : \eta \in 2^n\}$, $Q_\eta \subseteq \{p_\nu : \nu \in 2^n\} \setminus \{p_\eta\}$;
(d) $(\prod Q_\eta)$ divides a_η in $\mathbb{Z}$ (when η is the root of the tree, we let $\prod Q_\eta = \{1\}$);
(e) p_η does not divide a_η;
(f) If $n = m+1$ and $\eta \in 2^m$ then:
  (i) $p_{\eta^\frown(0)} = p_\eta$;
  (ii) $p_{\eta^\frown(1)} = p_{n+1}$
  (iii) $\bar{Q}_{\eta^\frown(0)} = \bar{Q}_\eta \cup \{p_{n+1}\}$;
  (iv) $\bar{Q}_{\eta^\frown(1)} = \bar{Q}_\eta \cup \{p_\eta\}$;
  (v) $a_\eta = a_{\eta^\frown(0)} + a_{\eta^\frown(1)}$.

Proof. Case 1. $n = 0$
Let $a_{()} = 1$, $Q_{()} = \emptyset$, $P_{()} = p_0$.
Case 2. $n = m+1$ Now $(p_\eta, Q_\eta : \eta \in 2^m)$ are determined by the other conditions, but we have to find for $\eta \in 2^m$, the numbers $a_{\eta^\frown(0)}, a_{\eta^\frown(1)} \in \mathbb{Z} \setminus \{0\}$ satisfying (d)-(f). To this extent, choose $a'_0, a'_1 \in \mathbb{Z}$ such that:

- $p_\eta \nmid a'_0$ and $p_{n+1} \nmid a'_1$;
- $p_\eta a'_0 + p_{n+1} a'_1 = 1$.

Let then:

- $a_{\eta^\frown(0)} = a_\eta p_\eta a'_0$;
- $a_{\eta^\frown(1)} = a_\eta p_{n+1} a'_1$.

■

Claim 5.9. Let $T = 2^{<\omega}$ and, for $\eta \in T$, let $L(\eta) = \{p_\nu : \eta \trianglelefteq \nu \in T\}$. Then (T, L) is as in 3.2 and we have that there is $y \in G$ such that:
(i) $f_{(\omega,n)}(y) = \sum\{a_\eta x_\eta : \eta \in 2^n\}$;
(ii) in G we have that $\prod_{\ell<n} p_\ell$ divides y, for every $n < \omega$;
(iii) G is not $\mathbb{Z}$-homogenenous.

Proof. Let $y_n = \sum\{a_\eta x_\eta : \eta \in 2^n\}$. Now, clearly $y_n \in G_n$. Furthermore, if $n < m < \omega$, then $f_{(m,n)}(y_m) = y_n$. To see this it suffices to prove it for $n = m+1$. To this extent notice:

$$\begin{aligned} f_{(n+1,n)}(y_n) &= f_{(n+1,n)}(\textstyle\sum\{a_\eta x_\eta : \eta \in 2^{n+1}\}) \\ &= \textstyle\sum\{a_\eta f_{(n+1,n)}(x_\eta) : \eta \in 2^{n+1}\} \\ &= \textstyle\sum\{a_\eta x_{\eta\restriction n} : \eta \in 2^{n+1}\} \\ &= \textstyle\sum\{\sum\{a_\eta : \nu \triangleleft \eta \in 2^{n+1}\} : \nu \in 2^n\} \\ &= \textstyle\sum\{(a_{\nu^\frown(1)} + a_{\nu^\frown(0)}) x_\nu : \nu \in 2^n\}, \end{aligned}$$

but, by 5.8(f)(v), $a_{\nu^\frown(1)} + a_{\nu^\frown(0)} = a_\nu$ and so we are done. ■

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Department of Mathematics "Giuseppe Peano", University of Torino, Via Carlo Alberto 10, 10123, Italy.
E-mail address: gianluca.paolini@unito.it

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel and Department of Mathematics, Rutgers University, U.S.A.
E-mail address: shelah@math.huji.ac.il