

ON AUTOMORPHISMS OF  $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ JAKOB KELLNER , SAHARON SHELAH , AND ANDA RAMONA TĂNĂSIE

**Abstract.** We investigate the statement “all automorphisms of  $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$  are trivial.” We show that MA implies the statement for regular uncountable  $\lambda < 2^{\aleph_0}$ , that the statement is false for measurable  $\lambda$  if  $2^\lambda = \lambda^+$ , and that for “densely trivial” it can be forced (together with  $2^\lambda = \lambda^{++}$ ) for inaccessible  $\lambda$ .

**§1. Introduction.** We investigate automorphisms of Boolean algebras of the form

$$P_\kappa^\lambda := \mathcal{P}(\lambda)/[\lambda]^{<\kappa}.$$

The instance  $P_\omega^\omega$ , i.e.,  $\mathcal{P}(\omega)/\text{FIN}$ , has been studied extensively for many years.<sup>1</sup> One can study variants for uncountable cardinals  $\lambda$ . Unsurprisingly, the behaviour here tends to be quite different to the countable case. One moderately popular<sup>2</sup> such generalisation is  $P_\omega^\lambda$ . Here, we study another obvious generalization of the countable case,  $P_\lambda^\lambda$ . Some results for general  $P_\kappa^\lambda$  can be found in [5].

The main result of the paper is:

*The following is equiconsistent with an inaccessible:  $\lambda$  is inaccessible,  $2^\lambda$  is  $\lambda^{++}$ , and all automorphisms of  $P_\lambda^\lambda$  are densely trivial.*

(Theorem A, Theorem 5.2)

Here,  $2^\lambda > \lambda^+$  is necessary, at least for measurables:

*If  $\lambda$  is measurable and  $2^\lambda = \lambda^+$ , then there is a nontrivial automorphism of  $P_\lambda^\lambda$ .*

(Theorem B, Theorem 4.1)

**REMARK 1.1.** From [12, Lemma 3.2] it would follow that Theorem B holds even when “measurable” is replaced by just “inaccessible.” However, the proof there turned out to be incorrect.<sup>3</sup>

For  $\lambda$  below the continuum we get the following result under Martin’s Axiom (MA). More explicitly,  $\text{MA}_{=\lambda}(\sigma\text{-centered})$  is sufficient, which is the statement that

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<sup>1</sup>Rudin [7, 8] showed in the 1950s that CH implies that there is a non-trivial automorphism; Shelah [9] showed that consistently all automorphisms are trivial. Further results can be found, e.g., in [1–4, 11, 12, 15].

<sup>2</sup>See, e.g., [13–15].

<sup>3</sup>A corrected version has been submitted (see <https://shelah.logic.at/papers/990a/>). This version again establishes the result only assuming inaccessibility.

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for any  $\sigma$ -centered poset  $P$  and  $\leq \lambda$  many open dense sets in  $P$  there is a filter  $G$  meeting all these open sets:

*For  $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$  and  $\kappa$  regular,  $MA_{=\lambda}(\sigma\text{-centered})$  implies that every automorphism of  $P_\kappa^\lambda$  is trivial.*

(Theorem C, Theorem 3.1)

Larson and McKenney [5] showed the same under  $MA_{\aleph_1}$  for the case  $\lambda = 2^{\aleph_0}$  and  $\kappa = \aleph_1$ .

Contrast this to the case  $\lambda = \kappa = \omega$ : Due to results of Veličković, Steprāns, and the second author, “Every automorphisms of  $\mathcal{P}(\omega)/[\omega]^{<\omega}$  is trivial” is implied by PFA [10], in fact even by  $MA+OCA$  [15], but not by  $MA$  alone [15] (not even for “somewhere trivial” [11]).

**Contents.** We start by introducing some notation and basic results in Section 2 (page 2).

The following sections are independent of each other:

In Section 3 (page 3) we show Theorem C, which we state as Theorem 3.1; in Section 4 (page 6), we show Theorem B, i.e., Theorem 4.1; and finally in the main part, Section 5 (page 7) we develop some forcing notions to prove Theorem A, i.e., Theorem 5.2.

**§2. Definitions.** We always assume that  $\lambda$  is a cardinal and  $\kappa \leq \lambda$  is regular.

- The case  $\kappa = \aleph_0$  or  $\lambda = \aleph_0$  is included only for completeness sake in the following definitions.
- In Section 3 we will assume that  $\aleph_1 \leq \kappa \leq \lambda < 2^{\aleph_0}$ .
- In Section 4 we assume that  $\lambda$  is measurable and  $\kappa = \lambda$ .
- In Section 5 we assume that  $\lambda$  is inaccessible and  $\kappa = \lambda$ .

Notation:

- We investigate the Boolean algebra (BA)  $P_\kappa^\lambda := \mathcal{P}(\lambda)/[\lambda]^{<\kappa}$ , i.e., the power set of  $\lambda$  factored by the ideal of sets of size  $< \kappa$ .
- For  $A \subseteq \lambda$ , we denote the equivalence class of  $A$  with  $[A]$ . We set  $\emptyset := [\emptyset]$ .
- $A \subseteq^* B$  means  $|B \setminus A| < \kappa$ , analogously for  $A =^* B$ ; and “for almost all  $\alpha \in A$ ” means for all but  $< \kappa$  many in  $A$ . In particular,  $A =^* \lambda$  means  $A \subseteq \lambda$  and  $|\lambda \setminus A| < \kappa$ .
- We denote the BA-operations in  $P_\kappa^\lambda$  with  $x \vee y$ ,  $x \wedge y$  and  $x^c$  (for the complement). So we have  $[A] \vee [B] = [A \cup B]$ ,  $[A] \wedge [B] = [A \cap B]$ , and  $[A]^c = [\lambda \setminus A]$ .
- A function  $\phi : P_\kappa^\lambda \rightarrow P_\kappa^\lambda$  is a BA-automorphism (which we will just call *automorphism*), if it is bijective, compatible with  $\wedge$  and the complement, and satisfies  $\phi(\emptyset) = \emptyset$ .
- Preimages of a function  $f$  are denoted by  $f^{-1}x$ , images by  $f''x$ .
- We sometimes identify  $\eta \in 2^\lambda$  with  $\eta^{-1}\{1\} \subseteq \lambda$  without explicitly mentioning it, by referring to  $\eta$  as element of  $2^\lambda$  or of  $P(\lambda)$ .

Let us note that  $P_\kappa^\lambda$  is  $<\kappa$ -complete<sup>4</sup> and  $\lambda^+$ -cc. Also, any automorphism  $\phi$  is closed under  $<\kappa$  unions:  $\phi(\bigvee_{i \in I} [A_i]) = \bigvee_{i \in I} \phi([A_i])$ .

An automorphism is trivial if it is induced by a function on  $\lambda$ . A standard definition to capture this concept is the following:

**DEFINITION 2.1.** An automorphism  $\phi : P_\kappa^\lambda \rightarrow P_\kappa^\lambda$  is trivial, if there is a  $g : \lambda \rightarrow \lambda$  such that  $\phi([A]) = [g^{-1}A]$  for all  $A \subseteq \lambda$ .

However, we prefer to use forward images instead of inverse images, which can easily be seen to be equivalent:

**DEFINITION 2.2.**

- For  $f : A_0 \rightarrow \lambda$  with  $A_0 =^* \lambda$ , define  $\pi_f : P_\kappa^\lambda \rightarrow P_\kappa^\lambda$  by  $\pi_f([B]) := [f''(B \cap A_0)]$  for all  $B \subseteq \lambda$ .
- $f$  is an almost permutation, if there are  $A_0 =^* \lambda$  and  $B_0 =^* \lambda$  with  $f : A_0 \rightarrow B_0$  bijective.

(Such a  $\pi_f$  is always a well-defined function.)

**LEMMA 2.3.** Let  $\phi : P_\kappa^\lambda \rightarrow P_\kappa^\lambda$  be a function. The following are equivalent:

- (1)  $\phi$  is a trivial automorphism.
- (2) There is an almost permutation  $f$  such that  $\phi = \pi_f$ .
- (3) (Assuming  $\kappa > \aleph_0$ .) There is a bijection  $f : \lambda \rightarrow \lambda$  such that  $\phi = \pi_f$ .

**PROOF.** (1) implies (2): Assume  $\phi$  is a trivial automorphism, witnessed by  $g$ .

Then  $X := g''\lambda =^* \lambda$  (as  $\phi([X]) = [g^{-1}X] = [\lambda]$ ), and  $Y := \{\alpha \in X : |g^{-1}\{\alpha\}| \neq 1\} =^* \emptyset$ . Otherwise, pick  $y_\alpha^0 \neq y_\alpha^1$  for each  $\alpha \in Y$  with  $g(y_\alpha^0) = g(y_\alpha^1) = \alpha$ . So  $y_\alpha^0 \in g^{-1}C$  iff  $y_\alpha^1 \in g^{-1}C$  for any  $C \subseteq \lambda$ . Set  $B^i := \{y_\alpha^i : \alpha \in Y\}$  for  $i = 0, 1$  and let  $[C] = \phi^{-1}([B^0])$ . So  $\phi([C]) = [g^{-1}C] = [B^0]$ , i.e., almost all  $y_\alpha^0$  are in  $g^{-1}C$ , but then almost all  $y_\alpha^1$  are in  $g^{-1}C$  as well, i.e.,  $[B^0] = \phi([C]) \geq [B^1]$ , a contradiction as  $B^0 \cap B^1 = \emptyset$ .

Set  $A_0 := X \setminus Y$ , and  $B_0 := g^{-1}A_0$ . Note that  $B_0 =^* \lambda$ , as  $\emptyset = \phi(\emptyset) = \phi([Y]) = [g^{-1}Y]$ . So  $g \upharpoonright B_0 \rightarrow A_0$  is bijective, and we can set  $f : A_0 \rightarrow B_0$  the inverse. Then  $f$  is an almost permutation, and  $\pi = \pi_f$ .

(2) implies (1): Let  $f : A_0 \rightarrow B_0$  be an almost permutation, and  $g : B_0 \rightarrow A_0$  the inverse (and let  $g$  be defined arbitrarily on  $\lambda \setminus B_0$ ). Then  $\pi_f([X]) = [f''(X \cap A_0)] = [g^{-1}(X)]$ . It remains to be shown that  $\pi_f$  is an automorphism:  $\pi_f(\{\emptyset\}) = [f''\emptyset] = [\emptyset]$ ;  $\pi_f([X \cap Y]) = [f''(X \cap Y \cap A_0)] = [f''(X \cap A_0) \cap f''(Y \cap A_0)]$ ; and  $\pi_f([\lambda \setminus X]) = [f''(A_0 \setminus X)] = [B_0 \setminus f''X]$ .

(2) implies (3) if  $\text{cf}(\kappa) > \aleph_0$ : This follows from the following lemma.  $\dashv$

**LEMMA 2.4** ( $\kappa > \aleph_0$ ). Let  $f$  be a  $\kappa$ -almost permutation. Then there is an  $S =^* \lambda$  such that  $f \upharpoonright S : S \rightarrow S$  is bijective.

**PROOF.** Set  $X_0 := A_0 = \text{dom}(f)$ , and  $X_{i+1} := X_i \cap f''X_i \cap f^{-1}X_i$ , and  $S := \bigcap_{i \in \omega} X_i$ .

The  $X_n$  are decreasing, and  $|\lambda \setminus X_n| < \kappa$  and thus  $|\lambda \setminus (f''X_n)| < \kappa$  for  $n < \omega$ . Accordingly,  $|\lambda \setminus S| < \kappa$ . We claim that  $g := f \upharpoonright S$  is a permutation of  $S$ . Clearly

<sup>4</sup>That is, if  $|I| < \kappa$  then  $\bigvee_{i \in I} [A_i] = [\bigcup_{i \in I} A_i]$ .

it is injective. If  $\alpha \in S$  then  $\alpha \in X_n$  for all  $n \in \omega$ , so  $f(\alpha) \in X_{n+1}$  for all  $n$ . So  $g : S \rightarrow S$ . If  $\alpha \in S$ , then  $\alpha \in X_{n+1}$  for all  $n$ , so  $f^{-1}(\alpha)$  exists and is in  $X_n$ .  $\dashv$

Remark: For  $\kappa = \lambda = \omega$ , there are trivial automorphisms that are not induced by “proper” bijections  $f : \omega \rightarrow \omega$ , e.g., the automorphism  $\phi$  induced by the almost permutation  $n \mapsto n + 1$ .<sup>5</sup>

We will investigate somewhere and densely trivial automorphisms. To simplify notation, we assume  $\kappa = \lambda > \aleph_0$ :

DEFINITION 2.5 ( $\lambda > \aleph_0$  regular). Let  $\phi : P_\lambda^\lambda \rightarrow P_\lambda^\lambda$  be an automorphism.

- $\phi$  is trivial on  $A \in [\lambda]^\lambda$ , if there is an  $f : A \rightarrow \lambda$  with  $\phi([B]) = [f''B]$  for all  $B \subseteq A$ .
- $\phi$  is somewhere trivial, if it is trivial on some  $A \in [\lambda]^\lambda$ .
- $\phi$  is densely trivial, if for all  $A \in [\lambda]^\lambda$  there is a  $B \subseteq A$  of size  $\lambda$  such that  $\phi$  is trivial on  $B$ .

Just as before it is easy to see that we can assume  $f$  to be a full permutation:

FACT 2.6 ( $\lambda > \aleph_0$  regular). An automorphism  $\phi : P_\lambda^\lambda \rightarrow P_\lambda^\lambda$  is trivial on  $A \in [\lambda]^\lambda$  iff there is a bijection  $f : \lambda \rightarrow \lambda$  such that  $\phi([B]) = [f''(B)]$  for all  $B \subseteq A$ .

LEMMA 2.7 ( $\lambda > \aleph_0$  regular). If every automorphism of  $P_\lambda^\lambda$  is somewhere trivial, then every automorphism of  $P_\lambda^\lambda$  is densely trivial.

PROOF. Assume  $\pi$  is an automorphism of  $P_\lambda^\lambda$ , and fix  $A \in [\lambda]^\lambda$ . If  $A =^* \lambda$  and if  $\pi$  is trivial on some  $B$ , then  $\pi$  is trivial on  $B \cap A \subseteq A$ , so we are done. So assume  $A \neq^* \lambda$ .

Pick some representative  $\pi^* : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda)$  of  $\pi$  such that  $\pi^*(A)$  and  $\pi^*(\lambda \setminus A)$  partition  $\lambda$ , and such that  $\pi^*(C) \subseteq \pi^*(A)$  for every  $C \subseteq A$ . Let  $i : \lambda \setminus A \rightarrow A$  and  $j : \pi^*(\lambda \setminus A) \rightarrow \pi^*(A)$  both be bijective. Let  $\pi'$  map  $[D]$  to  $[\pi^*(D \cap A) \cup j^{-1}\pi^*(i''(D \setminus A))]$ . This is an automorphism of  $P_\lambda^\lambda$ , so it is trivial on some  $D_0$ . If  $|D_0 \cap A| = \lambda$ , we are done, as  $\pi'$  restricted to  $D_0 \cap A$  is the same as  $\pi$  and trivial. So assume otherwise. Then  $\pi'$  is trivial on the large set  $D_0 \setminus A$ . Then  $\pi$  is trivial on  $i''(D_0 \setminus A) \subseteq A$ .  $\dashv$

### §3. Under MA, every automorphism is trivial for $\omega_1 \leq \lambda < 2^{\aleph_0}$ .

THEOREM 3.1. Assume  $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$ ,  $\kappa$  regular, and  $MA_{(=\lambda)}(\sigma\text{-centered})$  holds. Then every automorphism of  $P_\kappa^\lambda$  is trivial.

For the proof we will use that we can separate certain sets by closed sets.

A tree  $T$  is a subset of  $2^{<\omega}$  such that  $s \in T \cap 2^n$  and  $m \leq n$  implies  $s \restriction m \in T$ ; for such a  $T$  we set  $\lim(T) = \{\eta \in 2^\omega : (\forall n \in \omega) \eta \restriction n \in T\}$ . A subset of  $2^\omega$  is closed iff it is of the form  $\lim(T)$  for some tree  $T$ .

LEMMA 3.2. Assume  $\aleph_0 < \theta \leq \lambda < 2^{\aleph_0}$ ,  $\text{cf}(\theta) > \aleph_0$ , and  $MA_{(=\lambda)}(\sigma\text{-centered})$  holds. Assume  $A_0, A_1$  are disjoint subsets of  $2^\omega$  of size  $\leq \lambda$ ;  $|A_0| \geq \theta$ . Then there is a tree  $T_0$  in  $2^{<\omega}$  such that  $|A_0 \cap \lim(T_0)| \geq \theta$  and  $A_1 \cap \lim(T_0) = \emptyset$ .

If additionally  $|A_1| \geq \theta$ , we get an additional tree  $T_1$  such that  $|A_1 \cap \lim(T_1)| \geq \theta$ ,  $A_0 \cap \lim(T_1) = \emptyset$ , and  $T_0 \cap T_1 \subseteq 2^n$  for some  $n$ .

<sup>5</sup>A bijection  $f : \omega \rightarrow \omega$  has infinitely many  $n$  such that  $f(n) \neq n + 1$ , and therefore an infinite set  $A$  such that  $f''A$  is disjoint to  $\{n + 1 : n \in A\}$ .

**PROOF OF THE LEMMA.** In the following we identify an  $x \in 2^\omega$  with the according (infinite) branch  $b$  in the tree  $2^{<\omega}$ . So a branch  $b$  can be in  $A_0$  or in  $A_1$  (or in neither; but not both, as  $A_0$  and  $A_1$  are disjoint).

We define a poset  $\mathcal{Q}$  as follows: A condition  $q \in \mathcal{Q}$  is a triple  $(n_q, S_q, f_q)$ , where:

- $n_q \in \omega$ ,
- $S_q$  is a tree in  $2^{<\omega}$  of the following form:  $S_q$  is the union of  $2^{\leq n_q}$  and finitely many (infinite) branches  $\{b_j : j \in m\}$  for some  $m \in \omega$ , each  $b_j \in A_0 \cup A_1$ , and  $b_j \restriction n_q = b_k \restriction n_q$  implies  $(b_j \in A_i \text{ iff } b_k \in A_i)$ .

So every  $s \in S_q$  with  $|s| > n_q$  is either “in  $A_0$ -branches” (i.e., there is one or more  $b_j \in A_0$  with  $s \in b_j$ ), or “in  $A_1$ -branches” (but not in both).

Note that an  $s \in S_q$  of length  $n_q$  is either in  $A_0$ -branches, or in  $A_1$ -branches, or in neither (but not in both).

- $f_q : S_q \rightarrow 2$  such that, for  $i = 0, 1$ ,  $f_q(s) = i$  whenever  $s \in S_q$ ,  $|s| \geq n_q$  and  $s$  is in  $A_i$ -branches.

The order on  $\mathcal{Q}$  is the natural one:  $q \leq p$  if  $n_q \geq n_p$ ,  $S_q \supseteq S_p$ , and  $f_q$  extends  $f_p$ .

$\mathcal{Q}$  is  $\sigma$ -centered witnessed by  $(n_q, S_q, f_q) \mapsto (n_q, f_q \restriction 2^{\leq n_q})$ : If  $p, q$  are in  $\mathcal{Q}$  with  $n_p = n_q =: n$  and  $f_p \restriction 2^{\leq n} = f_q \restriction 2^{\leq n}$ , then  $(n, S_p \cup S_q, f_p \cup f_q)$  is a valid condition stronger than both  $p$  and  $q$ .

For  $x \in A_i$ , the set  $D_x$  of conditions containing  $x$  as branch is dense: Given  $p \in \mathcal{Q}$ , let  $n_q \geq n_p$  be such that all  $A_{1-i}$ -branches in  $p$  split off  $x$  below  $n_q$ ; set  $S_q := S_p \cup 2^{\leq n_q} \cup x$ ; and set  $f_q(s) = i$  for  $s \in S_q \setminus S_p$ .

Similarly, for all  $n \in \omega$ , the set  $D_n^*$  of conditions  $q$  with  $n_q \geq n$  is dense as well.

By  $\text{MA}_{(=\lambda)}$  ( $\sigma$ -centered) and  $|A_i| \leq \lambda$ , we can find a filter  $G$  which has nonempty intersection with each  $D_x$  for  $x \in A_0 \cup A_1$  as well as for each  $D_n^*$ . So  $F := \bigcup_{p \in G} f_p$  is a total function from  $2^{<\omega}$  to 2; and for all  $x \in A_i$  there is an  $n_x \in \omega$  such that  $m \geq n_x$  implies  $F(x \restriction m) = i$ .

As  $|A_0| \geq \theta$  and  $\text{cf}(\theta) > \aleph_0$  we can assume that there is an  $n_0^*$  such that  $n_x = n_0^*$  for  $\theta$  many  $x \in A_0$ . If additionally  $|A_1| \geq \theta$ , we analogously get an  $n_1^*$  and set  $n^* := \max(n_0^*, n_1^*)$ ; otherwise we set  $n^* := n_0^*$ . We set  $T_i^* := \{s \in 2^{<\omega} : |s| \geq n^*, (\forall n^* \leq k \leq |s|) (F(s \restriction k) = i)\}$  and generate a tree from it; i.e., we set  $T_i := T_i^* \cup \{s \restriction m : m < n^*, s \in T_i^*\}$ . As we have seen above,  $\lim(T_i) \cap A_i \geq \theta$  for  $i = 0$  (and, if  $|A_1| \geq \theta$ , for  $i = 1$  as well). Clearly  $T_0 \cap T_1 \subseteq 2^{n^*}$ ; and  $\lim(T_i) \cap A_{i-1}$  is empty, as for any  $x \in A_{i-1}$ , cofinally many  $n$  satisfy  $F(x \restriction n) = i - 1$ .  $\dashv$

**PROOF OF THE THEOREM.** Fix an automorphism  $\pi$  of  $P_\kappa^\lambda$  represented by some  $\pi^* : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda)$ , and let  $\pi^{-1*}$  represent  $\pi^{-1}$ . We have to show that  $\pi$  is trivial.

Fix an injective function  $\eta : \lambda \rightarrow 2^\omega$ . Set

$$C_n := \{x \in 2^\omega : x(n) = 0\} \text{ and } \Lambda_n := \eta^{-1}C_n = \{\alpha < \lambda : \eta(\alpha)(n) = 0\}.$$

Define  $v : \lambda \rightarrow 2^\omega$  by

$$v(\beta)(n) = 0 \text{ iff } \beta \in \pi^*(\Lambda_n).$$

In the following, “large” means “of cardinality  $\geq \kappa$ ”, and “small” means not large. We will show:

- ( $\ast_1$ )  $\pi^*(\eta^{-1}C) = {}^*v^{-1}C$  for  $C \subseteq 2^\omega$  closed.
- ( $\ast_2$ )  $Y \subseteq \lambda$  and  $|Y| \geq \kappa$  implies  $|v''Y| \geq \kappa$ .

- ( $\ast_3$ ) If  $A_0, A_1$  are disjoint subsets of  $2^\omega$ ,  $A_0 \subseteq v''\lambda$  large, then  $\pi^{-1\ast}(v^{-1}A_0) \setminus \eta^{-1}A_1$  is large.
- ( $\ast_4$ ) If  $A_0, A_1$  are disjoint subsets of  $2^\omega$ ,  $A_0 \subseteq \eta''\lambda$  large, then  $\pi^\ast(\eta^{-1}A_0) \setminus v^{-1}A_1$  is large.

(Note that ( $\ast_2$ ) is the only place where we use that  $\kappa$  is regular.)

Proof:

- ( $\ast_1$ )  $\pi^\ast(\eta^{-1}C_n) = v^{-1}C_n$  holds by definition of  $v$ . As  $\pi$  honors  $<\kappa$ -unions and complements, and as the  $C_n$  generate the open sets, this equation (with  $=^\ast$ ) holds whenever  $C$  is generated by  $<\kappa$ -unions and complements from the open sets, in particular, if  $C$  is closed.
- ( $\ast_2$ ) Fix  $x \in 2^\omega$ . Then  $\eta^{-1}\{x\}$  has at most one element (as  $\eta$  is injective), and  $\eta^{-1}\{x\} =^\ast \pi^{-1\ast}v^{-1}\{x\}$  by ( $\ast_1$ ). That is,  $v^{-1}\{x\}$  is small. And  $Y \subseteq \bigcup_{x \in v''Y} v^{-1}\{x\}$ , so as  $\kappa$  is regular we get  $|v''Y| \geq \kappa$ .
- ( $\ast_3$ ) Using the previous lemma (with  $\kappa$  as  $\theta$ ) we get a tree  $T_0$  separating  $A_0$  and  $A_1$ . That is,  $\lim(T_0) \cap A_1 = \emptyset$  and  $X := \lim(T_0) \cap A_0$  is large. As  $X \subseteq A_0 \subseteq v''\lambda$ , we get that  $v^{-1}X$  is large. And  $v^{-1}X = v^{-1}\lim(T_0) \cap v^{-1}A_0 =^\ast \pi^\ast(\eta^{-1}\lim(T_0)) \cap v^{-1}A_0$ , the last equation by ( $\ast_1$ ). This implies  $\eta^{-1}\lim(T_0) \cap \pi^{-1\ast}(v^{-1}A_0)$  is large, and so  $\pi^{-1\ast}(v^{-1}A_0) \setminus \eta^{-1}A_1$  is large.
- ( $\ast_4$ ) We get an analogous result when interchanging  $v$  and  $\eta$  and using  $\pi^\ast$  instead of  $\pi^{-1\ast}$ .

We claim that the following sets  $N_i$  are all small:

- (1)  $N_1 := \{\alpha \in \lambda : (\neg \exists \beta \in \lambda) \eta(\alpha) = v(\beta)\}$ .
- (2)  $N_2 := \{\alpha \in \lambda : (\exists^{\geq 2} \beta \in \lambda) \eta(\alpha) = v(\beta)\}$ .
- (3)  $N_3 := \{\beta \in \lambda : (\neg \exists \alpha \in \lambda) \eta(\alpha) = v(\beta)\}$ .

Proof.

- (3) Assume  $N_3$  is large. Set  $A_0 := v''N_3$ , which is large by ( $\ast_2$ ); and  $A_1 := \eta''\lambda$ . So  $A_0$  and  $A_1$  are disjoint, and by ( $\ast_3$ )  $\pi^{-1\ast}v^{-1}A_0 \setminus \eta^{-1}A_1$  is large, but  $\eta^{-1}A_1 = \lambda$ .
- (1) Assume  $N_1$  is large. Set  $A_0 = \eta''N_1$  (large, as  $\eta$  is injective) and  $A_1 := v''\lambda$ . So  $A_0$  and  $A_1$  are disjoint, and by ( $\ast_4$ )  $\pi^\ast(\eta^{-1}A_0) \setminus v^{-1}A_1$  is large, but  $v^{-1}A_1 = \lambda$ .
- (2) Assume that  $N_2$  is large. For every  $\alpha \in N_2$ , let  $\beta_\alpha^0 \neq \beta_\alpha^1$  in  $\lambda$  be such that  $\eta(\alpha) = v(\beta_\alpha^0) = v(\beta_\alpha^1)$ . For  $i \in \{0, 1\}$ , set  $Y_i := \{\beta_\alpha^i : \alpha \in N_2\}$  and  $X_i := \pi^{-1\ast}(Y_i)$  (without loss of generality disjoint), and  $A_i := \eta''X_i$ . So the  $A_i$  are large and disjoint, and we can find a tree  $T_0$  such that  $A_0 \cap \lim(T_0)$  is large, and  $A_1 \cap \lim(T_0)$  is empty.

As  $A_0 \subseteq \eta''\lambda$ , this implies that the inverse  $\eta$ -image of  $A_0 \cap \lim(T_0)$  is also large. That is,

$\eta^{-1}(A_0 \cap \lim(T_0)) = \eta^{-1}A_0 \cap \eta^{-1}\lim(T_0) =^\ast X_0 \cap \pi^{-1\ast}v^{-1}\lim(T_0)$  is large (for the last equation we use ( $\ast_1$ )). Therefore also  $Y_0 \cap v^{-1}\lim(T_0)$  is large, and so, by ( $\ast_2$ ),  $v''(Y_0 \cap v^{-1}\lim(T_0)) = \lim(T_0) \cap v''Y_0$  is large as well.

On the other hand,  $\lim(T_0) \cap A_1$  is empty, so  $0 =^\ast \pi^\ast\eta^{-1}(\lim(T_0) \cap A_1) =^\ast \pi^\ast\eta^{-1}\lim(T_0) \cap \pi^\ast\eta^{-1}A_1$ . Using ( $\ast_1$ ) for  $\lim(T_0)$ , and noting that  $\pi^\ast\eta^{-1}A_1 = Y_1$ , this set is (almost) equal to  $Y_1 \cap v^{-1}\lim(T_0)$  which therefore is also small, and so  $\lim(T_0) \cap v''Y_1$  is small.

So we know that  $\lim(T_0) \cap v''Y_0$  is large and  $\lim(T_0) \cap v''Y_1$  is small, but  $v''Y_0 = v''Y_1$ , a contradiction.

Note that this implies:

- ( $\ast_5$ )  $X \cap Y$  small implies  $v''X \cap v''Y$  small, for  $X, Y \subseteq \lambda$ .
- ( $\ast_6$ )  $v^{-1}v''X =^* X$  for  $X \subseteq \lambda$ .

Proof:

- ( $\ast_5$ ) Assume otherwise. Without loss of generality we can assume that  $X$  and  $Y$  are disjoint, and by (3) that  $v''X$  and  $v''Y$  both are subsets of  $\eta''\lambda$ . Then  $v''X \cap v''Y \subseteq \eta''N_2$  is small.
- ( $\ast_6$ ) Set  $Y := v^{-1}v''X \setminus X$ . Then  $v''Y \subseteq N_2 \cup N_3$  is small, and by ( $\ast_2$ )  $Y$  is small.

Set  $D := \lambda \setminus (N_1 \cup N_2)$  and define  $e : D \rightarrow \lambda$  such that  $e(\alpha)$  is the (unique)  $\beta \in \lambda$  with  $\eta(\alpha) = v(\beta)$ . Clearly  $e$  is injective. We claim that  $e$  generates  $\pi$ , i.e., that the following are small (where we can assume  $X \subseteq D$ ):

- (4)  $N_4 := \pi^*(X) \setminus e''X$ .
- (5)  $N_5 := e''X \setminus \pi^*(X)$ .

Proof.

- (4) Assume that  $N_4$  is large. Set  $Y = \pi^{-1*}(N_4)$ , without loss of generality  $Y \subseteq X$  and  $\pi^*(Y) = N_4$ . So  $\pi^*(Y)$  is disjoint from  $e''Y$  (as it is even disjoint from  $e''X$ ). We set  $A_0 := v''\pi^*(Y)$  and  $A_1 := v''e''Y$ , by ( $\ast_5$ ) we can assume they are disjoint, and by ( $\ast_2$ ) both are large ( $e$  is injective).

By ( $\ast_3$ ),  $\pi^{-1*}(v^{-1}A_0) \setminus \eta^{-1}A_1$  is large.

$\eta^{-1}(A_1) = Y$ , as  $v(e(\alpha)) = \eta(\alpha)$  for all  $\alpha \in D$ . And  $\pi^{-1*}(v^{-1}A_0) =^* Y$  by definition and ( $\ast_6$ ), a contradiction.

- (5) The same proof works: This time we set  $Y = e^{-1}N_5$ ; see that  $\pi^*(Y)$  and  $e''Y$  are disjoint and large; set  $A_0 := v''\pi^*(Y)$  and  $A_1 := v''e''Y$ ; use ( $\ast_3$ ) to see that  $Y \setminus \eta^{-1}v''e''Y = Y \setminus Y$  is large, a contradiction.  $\dashv$

#### §4. For measurables, GCH implies a nontrivial automorphism.

**THEOREM 4.1.** *If  $\lambda$  is measurable and  $2^\lambda = \lambda^+$ , then there is a nontrivial automorphism of  $P_\lambda^\lambda$ .*

**PROOF.** Let  $\mathcal{D}$  be a normal ultrafilter on  $\lambda$  and denote by  $\mathcal{I} := [\lambda]^\lambda \setminus \mathcal{D}$  its dual ideal restricted to sets of size  $\lambda$ .

Since  $2^\lambda = \lambda^+$ , we can list all permutations of  $\lambda$  as  $\{e_\alpha : \alpha < \lambda^+\}$ ; and analogously all elements of  $\mathcal{I}$  as  $\{X_\alpha : \alpha < \lambda^+\}$ .

We will construct, by induction on  $\alpha < \lambda^+$  a set  $A_\alpha \in \mathcal{I}$  and a permutation  $f_\alpha$  of  $A_\alpha$ , such that for  $\alpha < \beta$ :

- (1)  $A_\alpha \subseteq^* A_\beta$ .
- (2)  $X_\alpha \subseteq A_{\alpha+1}$ .
- (3)  $f_\alpha(x) = f_\beta(x)$  for almost all  $x \in A_\alpha \cap A_\beta$ .
- (4) There is some  $X \subseteq A_{\alpha+1}$  of size  $\lambda$  such that  $e''_\alpha X$  and  $f''_{\alpha+1} X$  are disjoint.

(Note that by  $x \subseteq^* y$  we mean  $|y \setminus x| = \lambda$ , not  $y \setminus x \in \mathcal{I}$ ; and the same for ‘almost all’.)

The construction:

- Successor stages  $\alpha + 1$ : Fix any  $B \in \mathcal{I}$  disjoint to  $A_\alpha$  such that  $A_\alpha \cup B \supseteq X_\alpha$ . Set  $C := e''_\alpha B \cap A_\alpha$ .

First assume that  $|C| = \lambda$ . Then set  $A_{\alpha+1} = A_\alpha \cup B$  and let  $f_{\alpha+1}$  extend  $f_\alpha$  by the identity on  $B$ . Then (4) is witnessed by  $X := e_\alpha^{-1}C$ . So we assume  $|C| < \lambda$ . Partition  $B$  into large sets  $B_0, B_1, B_2$  such that  $e''_\alpha B_i$  is disjoint to  $A_\alpha$  for  $i = 0, 1$ . Set  $A_{\alpha+1} := A_\alpha \cup B \cup e''_\alpha B$ , and define  $f_{\alpha+1}$  on  $B$  such that the restriction to  $B_i$  is a bijection on  $e''_\alpha B_{1-i}$  for  $i = 0, 1$ , and the restriction to  $B_2$  a bijection to  $e''_\alpha B_2 \setminus A$ . Then (4) is witnessed by  $X := B_0$ .

- Limit stages  $\delta$  of cofinality  $<\lambda$ : Let  $\xi := \text{cf}(\delta)$  and choose  $\langle \alpha_i : i < \xi \rangle$  a cofinal increasing sequence converging to  $\delta$ . The union  $\bigcup_{i < \xi} A_{\alpha_i}$  is, by  $<\lambda$  completeness, in  $\mathcal{I}$ . Remove  $<\lambda$  many points to get a subset  $A_\delta$  such that:
  - For all  $i < j < \xi$ ,  $f_i$  and  $f_j$  agree on  $A_{\alpha_i} \cap A_\delta$ .
  - For all  $i < \xi$ ,  $f_i \upharpoonright (A_{\alpha_i} \cap A_\delta)$  is a full permutation (we can do this as in Lemma 2.4).

Then  $f_\delta$ , defined as the union of the  $f_{\alpha_i}$ , is a permutation of  $A_\delta$  and almost extends each  $f_{\alpha_i}$ .

- Limit stages  $\delta$  of cofinality  $\lambda$ : We choose an increasing cofinal sequence  $\langle \alpha_i : i < \lambda \rangle$  converging to  $\delta$ . By induction on  $i \in \lambda$  we construct  $A'_i =^* A_{\alpha_i}$ , such that:
  - $A'_i \cap i = \emptyset$ .
  - The  $f_{\alpha_i}$ 's fully extend each other on the  $A'_i$ 's, i.e., if  $x \in A'_i \cap A'_j$  then  $f_{\alpha_i}(x) = f_{\alpha_j}(x)$ .
  - $f_{\alpha_i} : A'_i \rightarrow A'_i$  is a “full” permutation.

We can do this by removing from  $A_{\alpha_i}$ : the points less than  $i$ , the points where  $f_{\alpha_i}$  disagrees with some previous  $f_{\alpha_j}$  for any  $j < i$ ; and by removing  $<\lambda$  many points to get a full permutation.

Now we can set  $A_\delta$  and  $f_\delta$  to be the unions of  $A'_i$  and  $f_{\alpha_i}$ , respectively, for  $i < \delta$ . Note that  $A_\delta$  is in  $\mathcal{I}$  (as it is a subset of the diagonal union); and  $f_\delta$  is a permutation of  $A_\delta$  satisfying (3).

Note that for all  $X \subseteq \lambda$ , either  $X \in \mathcal{I}$  or  $\lambda \setminus X \in \mathcal{I}$  (but not both), i.e., either  $X$  or  $\lambda \setminus X$  is  $\subseteq^* A_\alpha$  for coboundedly many  $\alpha < \lambda$ .

This allows us to define the automorphism  $\pi$  as follows: For  $X \in [\lambda]^\lambda$ ,

$$\pi([X]) := \begin{cases} [f''_\alpha X], & \text{if } X \in \mathcal{I}, X \subseteq^* A_\alpha \text{ for some } \alpha < \lambda^+ \text{ (Case 1),} \\ [\lambda \setminus f''_\alpha(\lambda \setminus X)], & \text{if } X \notin \mathcal{I}, \lambda \setminus X \subseteq^* A_\alpha \text{ for some } \alpha < \lambda^+ \text{ (Case 2).} \end{cases}$$

Note that in Case 2,  $\pi([X]) = [(\lambda \setminus A_\alpha) \cup (A_\alpha \setminus f''_\alpha(A_\alpha \setminus X))] = [(\lambda \setminus A_\alpha) \cup f''_\alpha(X \cap A_\alpha)]$ , as  $f''_\alpha A_\alpha =^* A_\alpha$ .

$\pi$  is well defined on  $[\lambda]^\lambda$ , as exactly one of  $X$  or  $\lambda \setminus X$  will eventually be  $\subseteq^* A_\alpha$ .

$\pi$  is an automorphism:  $\pi([\emptyset]) = \emptyset$ .  $\pi$  honors complements: If  $X$  is Case 1, then  $\pi([\lambda \setminus X])$  is by definition (Case 2)  $[\lambda \setminus f''_\alpha(X)]$ .  $\pi$  honors intersections  $X \cap Y$ : This is clear if both sets are the same Case. Assume that  $X$  is Case 1 and  $Y$  Case 2. Then  $X \cap Y \subseteq X$  is Case 1, and for any  $\alpha$  suitable for both  $X$  and  $Y$  we have

$$\begin{aligned} \pi([X]) \wedge \pi([Y]) &= [f''_\alpha X \cap ((\lambda \setminus A_\alpha) \cup f''_\alpha(Y \cap A_\alpha))] \\ &= [f''_\alpha X \cap f''_\alpha(Y \cap A_\alpha)] = [f''_\alpha(X \cap Y)]. \end{aligned}$$



$\pi$  is not trivial: Every automorphism  $e$  is an  $e_\alpha$  for some  $\alpha \in \lambda^+$ ; and according to (4) there is some  $X_\alpha \subseteq A_{\alpha+1}$  (and therefore in  $\mathcal{I}$ ) of size  $\lambda$  such that  $e''_\alpha X_\alpha$  is disjoint to  $f''_{\alpha+1} X_\alpha$ , a representative of  $\pi([X_\alpha])$ .  $\dashv$

**§5. For inaccessible  $\lambda$ , all automorphisms can be densely trivial.** In this section, we always assume the following (in the ground model):

ASSUMPTION 5.1.  $\lambda$  is inaccessible and  $2^\lambda = \lambda^+$ . We set  $\mu := \lambda^{++}$ .

In the rest of the paper, we will show the following:

THEOREM 5.2. ( $\lambda$  is inaccessible and  $2^\lambda = \lambda^+$ .) *There is a  $\lambda$ -proper,  $<\lambda$ -closed,  $\lambda^{++}$ -cc poset  $P$  (in particular, preserving all cofinalities) that forces:  $2^\lambda = \lambda^{++}$ , and every automorphism of  $P^\lambda_\lambda$  is densely trivial.*

By Lemma 2.7, it is enough to show that every automorphism is somewhere trivial.

### 5.1. The single forcing $\mathcal{Q}$ .

DEFINITION 5.3. We fix a strictly increasing sequence  $(\theta^*_\zeta)_{\zeta < \lambda}$  with  $\theta^*_\zeta < \lambda$  regular and  $\theta^*_\zeta > 2^{|\zeta|}$ .

- Let  $(I^*_\zeta)_{\zeta \in \lambda}$  be an increasing interval partition of  $\lambda$  such that  $I^*_\zeta$  has size  $2^{\theta^*_\zeta}$ ; and fix a bijection of  $I^*_\zeta$  and  $2^{\theta^*_\zeta}$ . Using this (unnamed) bijection, we set  $[s] := \{\ell \in I^*_\zeta : \ell > s\}$  for  $s \in 2^{<\theta^*_\zeta}$ .

So the  $[s]$  are cones, i.e., the set of all branches in  $I^*_\zeta$  extending  $s$ .

For  $\zeta < \lambda$ , we set  $I^*(<\zeta) := \bigcup_{\ell < \zeta} I^*_\ell$ , and analogously  $I^*(\leq \zeta) := I^*(<\zeta + 1)$ ,  $I^*(\geq \zeta) := \lambda \setminus I^*(<\zeta)$ , and  $I^*(\geq \zeta, < \xi) := I^*(\geq \zeta) \cap I^*(< \xi)$ .

- A condition  $q$  of the forcing notion  $\mathcal{Q}$  is a function with domain  $\lambda$  such that, for all  $\zeta \in \lambda$ ,  $q(\zeta)$  is a partial function from  $I^*_\zeta$  to 2, and such that for a club-set  $C^q \subseteq \lambda$ :
  - if  $\zeta \notin C^q$ , then  $q(\zeta)$  is total,
  - otherwise, the domain of  $q(\zeta)$  is  $I^*_\zeta \setminus [s^q_\zeta]$  for some  $s^q_\zeta \in 2^{<\theta^*_\zeta}$ .

$C^q$  and  $s^q_\zeta$  are uniquely determined by  $q$ ; and  $q$  is uniquely determined by the partial function  $\eta^q : \lambda \rightarrow 2$  defined as  $\bigcup_{\zeta \in \lambda} q(\zeta)$ .

- $q$  is stronger than  $p$  if  $\eta^q$  extends  $\eta^p$ .  
(This implies that  $C^q \subseteq C^p$ , and that  $s^q_\zeta$  extends  $s^p_\zeta$  for all  $\zeta \in C^q$ .)

The following is straightforward:

LEMMA 5.4.  $\mathcal{Q}$  has size  $2^\lambda$ , is  $<\lambda$ -closed, and adds a generic real  $\eta := \bigcup_{q \in G} \eta^q$  in  $2^\lambda$ .

PROOF.  $<\lambda$ -closure is obvious, but for later reference we would like to point out the “problematic cases”:

Let  $(p_i)_{i < \delta}$  be decreasing for a limit ordinal  $\delta < \lambda$ .

As a first approximation, set  $\eta^* := \bigcup_{i < \delta} \eta^{p_i}$  (a partial function) and  $C^* := \bigcap_{i < \delta} C^{p_i}$  (a club set) and  $s^*_\zeta := \bigcup_{i < \delta} s^{p_i}_\zeta \in 2^{\leq \theta^*_\zeta}$  for  $s \in C^*$ . For  $\zeta \notin C^*$ ,  $\eta^*$  is indeed total on  $I^*_\zeta$ , and for  $\zeta \in C^*$  the domain in  $I^*_\zeta$  is  $I^*_\zeta \setminus [s^*_\zeta]$ .

The *problematic case* is when  $s^*_\zeta$  is unbounded in  $\theta^*_\zeta$ . (This can only happen if  $\text{cf}(\delta) = \theta^*_\zeta$ , in particular for at most one  $\zeta$ .) In this case we can just pick any

extension  $\eta^q$  of  $\eta^*$  by filling all values in  $I_{\leq \zeta}^*$ . This gives the desired  $q$ , with  $C^{q\delta} = C^* \setminus \zeta + 1$ .  $\dashv$

#### REMARKS.

- The limits of  $<\lambda$ -sequences of conditions are not “canonical” if there are problematic  $\zeta$ ’s, as we have to fill in arbitrary values.
- $\eta$  determines the generic filter, by  $G = \{p \in Q : \eta^p \subseteq \eta\}$ . This follows from the following facts:
  - $p$  and  $q$  are compatible (as conditions in  $Q$ ) iff  $\eta^p$  and  $\eta^q$  are compatible as partial functions and  $X_{p,q} := \{\zeta \in C^p : s_\zeta^p \text{ and } s_\zeta^q \text{ are incomparable}\}$  is non-stationary.
  - If  $p, q$  are such that  $X_{p,q}$  is stationary, then the set of conditions  $r$  such that  $\eta^r$  and  $\eta^q$  are incompatible (as partial functions) is dense below  $p$ .

### 5.2. Properness of $Q$ : Fusion and pure decision.

**DEFINITION 5.5.** We say  $q \leq_\xi p$ , if  $q \leq p$ ,  $\xi \in C^q$ , and  $q \restriction \xi = p \restriction \xi$ .  $q \leq_\xi^+ p$  means  $q \leq_\xi p$  and  $q(\xi) = p(\xi)$ .

(Note the difference between  $q \leq_\xi^+ p$  and  $q \leq_{\xi+1} p$ : The former does not require  $\xi + 1 \in C^q$ .)

**LEMMA 5.6.** Let  $\delta \leq \lambda$  be a limit ordinal,  $\xi \in \lambda$ , and  $(q_i)_{i < \delta}$  a sequence in  $Q$ .

- (1) If  $\delta < \lambda$  and  $q_j <_\xi^+ q_i$  for all  $i < j < \delta$ , then there is a  $q_\infty$  such that  $q_\infty <_\xi^+ q_i$  for all  $i$ .
- (2) If  $q_j <_{\xi_i} q_i$  for  $i < j < \delta$ , where  $(\xi_i)_{i < \delta}$  is a strictly increasing<sup>6</sup> sequence in  $\lambda$ , then there is a (canonical) limit  $q_\infty$  such that  $q_\infty <_{\xi_i} q_i$  for all  $i$ .

**PROOF.** (1): We perform the same construction as in the proof of Lemma 5.4. If there is a problematic case  $\zeta$ , then  $\zeta > \xi$  (as for  $\zeta' \leq \xi$  the conditions  $q_i(\zeta')$  are constant). We can then make  $\eta^*$  total on  $I^*(\geq \xi, \leq \zeta)$ . (It may not be enough to make it total on  $I_\zeta^*$ , as  $C^* \setminus \{\zeta\}$  might not be club.)

(2): Define  $q_\infty(\zeta) := \bigcup_{i \in \delta} q_i(\zeta)$  for  $\zeta \in \lambda$ .

This is a non-total function (on  $I_\zeta^*$ ) iff  $\zeta \in C^{q_\infty} := \bigcap_{i < \delta} C^{q_i}$ , which is closed as intersection of closed sets, and also unbounded: If  $\delta < \lambda$  because we have a small intersections of clubs, if  $\delta = \lambda$  as it contains each  $\xi_i$ .

There are no problematic cases: If  $\zeta$  is below some  $\xi_i$ , then  $q_j(\zeta)$  is eventually constant. If  $\zeta$  is above all  $\xi_i$ , which can only happen if  $\delta < \lambda$ , then  $\text{cf}(\delta) \leq \delta \leq \sup(\xi_i) \leq \zeta < \theta_\zeta^*$ .  $\dashv$

So  $Q$  satisfies fusion; and we will now show that it also satisfies “pure decision”; standard arguments then imply that  $Q$  is  $\lambda$ -proper and  $\lambda^\lambda$ -bounding.

**DEFINITION 5.7.** Let  $\xi \in \lambda$ ,  $q \in Q$ .

- $\text{POSS}^Q(\xi) := 2^{I^*(\leq \xi)}$ . So in the extension  $V[G]$ , for each  $\xi$  there will be exactly one  $x \in \text{POSS}^Q(\xi)$  compatible with (or equivalently: an initial segment of) the generic real  $\eta$ . We write “ $x \subseteq \eta$ ” or “ $G$  chooses  $x$ ” for this  $x$ .

<sup>6</sup>For  $\delta = \lambda$ , it is enough that the  $\xi_i$  converge to  $\lambda$ . For  $\delta < \lambda$ , we use that the  $\xi_i$  are increasing and that  $\sup(\xi_i) \geq \text{cf}(\delta)$ .

- $\text{poss}(q, \xi)$  is the set of  $x \in \text{POSS}^Q(\xi)$  compatible with  $\eta^q$  (as partial functions), or equivalently:  $x \in \text{poss}(q, \xi)$  iff  $\neg q \Vdash x \not\subseteq \eta$ . So  $q$  forces that exactly one  $x \in \text{poss}(q, \xi)$  is chosen by  $G$ .
- Let  $\tau$  be a name for an ordinal. We say that  $q$   $\xi$ -decides  $\tau$ , if there is for all  $x \in \text{poss}(q, \xi)$  an ordinal  $\tau^x$  such that  $q$  forces  $x \subseteq \eta \rightarrow \tau = \tau^x$ .

Note that for  $p \in Q$  and  $\zeta \in C^p$ ,  $q \leq_\zeta^+ p$  is equivalent to  $\text{poss}(q, \zeta + 1) = \text{poss}(p, \zeta + 1)$ , while  $q \leq_\zeta p$  is equivalent to  $\zeta \in C^q$  and  $\text{poss}(q, \zeta) = \text{poss}(p, \zeta)$ .

LEMMA 5.8. Assume  $p \in Q$ ,  $\zeta \in C^p$ ,  $x \in \text{poss}(p, \zeta + 1)$ , and  $r \leq p$  extends<sup>7</sup>  $x$ . Then there is a  $q \leq_\zeta^+ p$  forcing:  $x \subseteq \eta \rightarrow r \in G$ . This condition is denoted by  $r \vee (p \restriction \zeta + 1)$ .

PROOF. We set  $q(\ell)$  to be  $p(\ell)$  for  $\ell \leq \zeta$ , and  $r(\ell)$  otherwise. If  $q' \leq q$  forces  $x \subseteq \eta$  then  $q'$  extends  $x$  and thus  $q' \leq r$ .  $\dashv$

COROLLARY 5.9. (“Pure decision”) Let  $\tau$  be a name for an ordinal,  $p \in Q$ , and  $\zeta \in C^p$ . Then there is a  $q \leq_\zeta^+ p$  which  $(\zeta + 1)$ -decides  $\tau$ .

PROOF. Let  $(x_i)_{i \in \delta}$  enumerate  $\text{poss}(p, \zeta + 1)$ , for some  $\delta < \lambda$ . Set  $p_0 = p$ , and define a  $\leq_\zeta^+$ -decreasing sequence  $p_j$  by induction on  $j \leq \delta$ : For limits use Lemma 5.6(1), and for successors choose some  $r \leq p_i$  deciding  $\tau$  with a stem extending  $x_i$  and set  $p_{i+1}$  to  $r \vee p_i \restriction (\zeta + 1)$ .  $\dashv$

From fusion and pure decision we get bounding and  $\lambda$ -proper, via “continuous reading of names.” This is a standard argument, and we will not give it here; we will anyway prove a more “general” variant (for an iteration of  $Q$ ’s), in Lemmas 5.25 and 5.27.

FACT 5.10.

- $Q$  has continuous reading of names: If  $\bar{\sigma}$  is a  $Q$ -name for a  $\lambda$ -sequence of ordinals, and  $p \in Q$ , then there is a  $q \leq p$  and there are  $\xi_i \in \lambda$  such that  $q \xi_i$ -decides  $\bar{\sigma}(i)$  for all  $i \in \lambda$ .
- $Q$  is  $\lambda^\lambda$ -bounding. That is, for every name  $\bar{\sigma} \in \lambda^\lambda$  and  $p \in Q$  there is an  $f \in \lambda^\lambda$  and  $q \leq p$  such that  $q$  forces  $f(i) > \bar{\sigma}(i)$  for all  $i \in \lambda$ .
- $Q$  is  $\lambda$ -proper. This means: If  $N$  is a  $< \lambda$ -closed elementary submodel of  $H(\chi)$  of size  $\lambda$  containing  $Q$ , with  $\chi$  sufficiently large and regular, and if  $p \in Q \cap N$ , then there is a  $q \leq p$   $N$ -generic (i.e., forcing that each name of an ordinal which is in  $N$  is evaluated to an ordinal in  $N$ ).

For completeness, we also mention the following well-known fact (the proof is straightforward):

FACT 5.11. Assume  $\kappa$  is regular, and that the forcing notion  $R$  is  $\kappa^\kappa$ -bounding. Then  $R$  preserves the regularity of  $\kappa$ , and every club-subset of  $\kappa$  in the extension contains a ground model club-set.

<sup>7</sup>By which we mean  $x \subseteq \eta^r$ .

### 5.3. The iteration $P$ .

Let us first recall some well-known facts:

**FACTS 5.12.** *A  $<\lambda$ -closed forcing preserves cofinalities  $\leq \lambda$  and also the inaccessibility of  $\lambda$ . The  $\leq \lambda$ -support iteration of  $<\lambda$ -closed forcings is  $<\lambda$ -closed.*

We will iterate the forcings  $Q$  from the previous section in a  $<\lambda$ -closed  $\leq \lambda$ -support iteration of length  $\mu := \lambda^{++}$ :

**DEFINITION 5.13.** Let  $(P_\alpha, Q_\alpha)_{\alpha < \mu}$  be the  $\leq \lambda$ -support iteration such that each  $Q_\alpha$  is the forcing  $Q$  (evaluated in the  $P_\alpha$ -extension). We will write  $P$  to denote the limit.

**REMARK.** One way to see that  $P$  is proper is to use the framework of [6]. However, we will need an explicit form of continuous reading for  $P$  anyway, which in turn gives properness for free.

**DEFINITION 5.14.** Assume that  $w \in [\mu]^{<\lambda}$  and  $\xi \in \lambda$ .

- $\tilde{\eta} = (\eta_\alpha)_{\alpha \in \mu}$  is the sequence of  $Q_\alpha$ -generic reals added by  $P$ .
- $\text{POSS}(w, \xi) := 2^{w \times I^*(\xi)}$ . Exactly one  $x \in \text{POSS}(w, \xi)$  is extended by  $\tilde{\eta}$ , we write “ $x$  is selected by  $G$ ,” or “ $x \triangleleft G$ .”
- $\text{poss}(p, w, \xi) := \{x \in \text{POSS}(w, \xi) : \neg p \Vdash \neg x \triangleleft G\}$ .
- Let  $\tau$  be a name of an ordinal.  $\tau$  is  $(w, \xi)$ -decided by  $q$ , if there are  $(\tau^x)_{x \in \text{poss}(q, w, \xi)}$  such that  $q$  forces  $x \triangleleft G \rightarrow \tau = \tau^x$ .

Clearly, if  $\tau$  is  $(w, \xi)$ -decided by  $q$ , and if  $q' \leq q$ ,  $w' \supseteq w$  and  $\xi' \geq \xi$ , then  $\tau$  is  $(w', \xi')$ -decided by  $q'$ .

**REMARK.** If  $q \in P(w, \xi)$ -decides some  $P_\alpha$ -name  $\tau$ , then the *same*  $q$  will generally *not*  $(w \cap \alpha, \xi)$ -decide  $\tau$  for any  $\xi$ .<sup>8</sup>

In the following, whenever we say that  $q(w, \xi)$ -decides something, we implicitly assume that  $w \in [\mu]^{<\lambda}$  and  $\xi \in \lambda$ .

**DEFINITION 5.15.** Let  $\sigma$  be a  $P$ -name for a  $\lambda$ -sequence of ordinals.

- $q$  continuously reads  $\sigma$ , if there are  $(w_i, \xi_i)_{i \in \lambda}$  such that  $q(w_i, \xi_i)$ -decides  $\sigma(i)$  for each  $i \in \lambda$ .
- $P$  has continuous reading, if for each such  $\sigma$  and  $p \in P$  there is some  $q \leq p$  continuously reading  $\sigma$ .

The following is a straightforward standard argument:

**FACT 5.16.** *If  $P$  has continuous reading, then it is  $\lambda^\lambda$ -bounding.*

As a first step towards pure decision, let us generalize the  $\leq_\xi$ -notation we defined for  $Q$ :

<sup>8</sup>For example: For a  $p$ -condition  $Q$ , let  $\text{ODD}^p$  be the set of odd elements of  $C^p$  (or any other unbounded subset  $X$  of  $C^p$  such that  $C^p \setminus X$  is still club), and set  $\text{ODD}_\eta^p := \bigcup_{\zeta \in \text{ODD}^p} I_\zeta^* \setminus \text{dom}(\eta^p)$ . Note that for any  $x : \text{ODD}_\eta^p \rightarrow 2$ ,  $\eta^p \cup x$  defines a condition in  $Q$  (stronger than  $p$ ). So if we fix any  $p(0) \in P_1$ , and define the  $P_1$ -name  $\tau \in \{0, 1\}$  to be 0 iff  $\eta_\alpha \upharpoonright \text{ODD}_\eta^{p(0)}$  is eventually constant to 0, then  $\tau$  cannot be  $(\{0\}, \zeta)$ -decided by  $p(0)$  for any  $\zeta$ . And if  $p(1)$  is any condition with  $p(0) \Vdash \eta^{p(1)}(0) = \tau$ , then  $\tau$  is  $(\{1\}, 1)$ -decided by  $q := (p(0), p(1))$ .

DEFINITION 5.17. Let  $p \in P$ ,  $w \in [\mu]^{<\lambda}$ , and  $\xi \in \lambda$ .

- $p$  fits  $(w, \xi)$ , if  $w \subseteq \text{dom}(p)$  and  $p \restriction \alpha \Vdash \xi \in C^{p(\alpha)}$  for all  $\alpha \in w$ .
- $q \leq_{w, \xi} p$  means:  $q \leq p$ , and for all  $\alpha \in w$ ,  $q \restriction \alpha$  forces  $q(\alpha) <_{\xi} p(\alpha)$ .
- $q \leq_{w, \xi}^+ p$  is defined analogously using  $<_{\xi}^+$  instead of  $<_{\xi}$ .

Obviously  $q \leq_{w, \xi}^+ p$  implies  $q \leq_{w, \xi} p$ ; and  $q \leq_{w, \xi} p$  implies that both  $p$  and  $q$  fit  $(w, \xi)$ .

REMARK. In contrast to the single forcing (or a product of such forcings),  $q \leq_{w, \xi} p$  (or  $q \leq_{w, \xi}^+ p$ ) does *not* imply  $\text{poss}(q, w, \xi) = \text{poss}(p, w, \xi)$ .<sup>9</sup> More explicitly, setting  $w = \{0, 1\}$ , it is possible that  $x \in \text{poss}(p, w, \xi)$  but  $p$  does not force that  $x(0) \subseteq \eta_0$  implies  $x(1) \in \text{poss}(p(1), \xi)$ . (But see Section 5.5.)

#### 5.4. Continuous reading and properness of $P$ .

LEMMA 5.18. If  $q_i$  is a  $\leq_{w, \xi}^+$ -decreasing sequence of length  $\delta < \lambda$ , then there is an  $r \leq_{w, \xi}^+ q_i$  for all  $i < \delta$ .

PROOF. Set  $\text{dom}(r) := \bigcup_{i < \delta} \text{dom}(q_i)$ , without loss of generality closed under limits. By induction on  $\alpha \in \text{dom}(r)$  we know that  $r \restriction \alpha \leq q_i \restriction \alpha$  for all  $i$ , and define  $r(\alpha)$  as follows: If  $\alpha \in w$ , we know that the  $q_i(\alpha)$  are  $\leq_{\xi}^+$ -increasing. Using Lemma 5.6(1), we pick some  $r(\alpha)$  such that  $r(\alpha) \leq_{\xi}^+ q_i(\alpha)$  for all  $i$ . If  $\alpha \notin w$ , we just pick any  $r(\alpha) \leq q_i(\alpha)$  for all  $i$ .  $\dashv$

It is easy to see that  $P$  satisfies a version of fusion:

LEMMA 5.19. Assume  $(p_i)_{i < \delta}$  is a sequence of length  $\delta \leq \lambda$ , such that  $p_j \leq_{w_i, \xi_i} p_i$  for  $i \leq j < \delta$ ,  $w_i \in [\mu]^{<\lambda}$  increasing,  $\xi_i \in \lambda$  strictly increasing. Set  $w_{\infty} := \bigcup_{i < \delta} w_i$ ,  $\text{dom}_{\infty} := \bigcup_{i < \delta} \text{dom}(p_i)$ , and  $\xi_{\infty} := \sup_{i < \delta} \xi_i$ . If  $\delta = \lambda$ , we additionally assume  $w_{\infty} = \text{dom}_{\infty}$ .

Then there is a limit  $q_{\infty}$  with  $\text{dom}(q_{\infty}) = \text{dom}_{\infty}$  such that  $q_{\infty} \leq_{w_i, \xi_i} p_i$  for all  $i < \delta$ .

If  $\delta < \lambda$ , then  $q_{\infty}$  fits  $(w_{\infty}, \xi_{\infty})$ .

(If  $w_{\infty} = \text{dom}_{\infty}$ , then the limit  $q_{\infty}$  is “canonical”).

PROOF. We define  $q_{\infty}(\alpha)$  by induction on  $\text{dom}_{\infty}$ . We assume that we already have  $q' := q_{\infty} \restriction \alpha$  which satisfies  $q' \leq_{w_i \cap \alpha, \xi_i} p_i$  for all  $i < \delta$ .

Case 1:  $\alpha \notin w_{\infty}$  (this can only happen if  $\delta < \lambda$ ): We know that  $q'$  forces that  $(p_i(\alpha))_{i < \delta}$  is a decreasing sequence, and we just pick some  $q_{\infty}(\alpha)$  stronger than all of them.

Case 2:  $\alpha \in w_{\infty}$ : Let  $i^*$  be minimal such that  $\alpha \in w_{i^*}$ . We know that  $q'$  forces for all  $i^* \leq i < j < \delta$  that  $p_j(\alpha) <_{\xi_i} p_i(\alpha)$ , so according to Lemma 5.6(2) there is a limit  $q_{\infty}(\alpha) <_{\xi_i} p_i(\alpha)$  (so in particular  $q' \Vdash \xi_i \in C^{q_{\infty}(\alpha)}$  for all  $i \geq i^*$ ).

<sup>9</sup>An example:  $\text{dom}(p) = \text{dom}(q) = w = \{0, 1\}$ ,  $\min(C^{p(0)}) = \min(C^{q(0)}) = \xi$ , and both  $p(0)$  and  $q(0)$  have trunk  $a \in \text{POSS}^Q(\xi)$ .  $p(0)$  forces that  $p(1) = q(1)$ , that  $\min(C^{p(1)}) = \xi$  and that the trunk of  $p(1)$  is either  $b$  or  $c$  (elements of  $\text{POSS}^Q(\xi)$ ); both are possible with  $p(0)$ . Now  $q(0) \leq_{\xi}^+ p(0)$  decides that the trunk of  $p(1)$  is  $b$ . Then  $q \leq_{w, \xi}^+ p$ , and  $(a, c)$  is in  $\text{poss}(p, w, \xi) \setminus \text{poss}(q, w, \xi)$ . In particular  $(a, c) \in \text{poss}(p, w, \xi)$  but  $p$  does not force that  $a \subseteq \eta_0$  implies  $c \in \text{poss}(p(1), \xi)$ .

Now assume  $\delta < \lambda$ . If  $\alpha \in w_\infty$ , then it is in  $w_i$  for coboundedly many  $i < \delta$ . In other words,  $p_j \restriction \alpha \Vdash \zeta_i \in C^{p_j(\alpha)}$  for coboundedly many  $i \in \delta$  and all  $j > i$ , which implies  $q_\infty \restriction \alpha \Vdash \xi_\infty \in C^{q_\infty(\alpha)}$ .  $\dashv$

**PRELIMINARY LEMMA 5.20.** *Let  $p$  fit  $(w, \zeta)$ ,  $x \in \text{poss}(p, w, \zeta + 1)$ , and let  $r \leq p$  extend  $x$ , i.e.,  $r \restriction x \triangleleft G$ . Then there is a  $q \leq_{w, \zeta}^+ p$  forcing that  $x \triangleleft G$  implies  $r \in G$ .*

**PROOF.** Set  $\text{dom}(q) := \text{dom}(r)$ . We define  $q(\alpha)$  by induction on  $\alpha \in \text{dom}(q)$  and show inductively:

- $q \restriction \alpha \leq_{w \cap \alpha, \zeta}^+ p \restriction \alpha$ .
- $q \restriction \alpha \Vdash (x \restriction \alpha \triangleleft G_\alpha \rightarrow r \restriction \alpha \in G_\alpha)$ .

For notational convenience, we assume  $\text{dom}(p) = \text{dom}(r)$  (by setting  $p(\alpha) = 1_Q$  for any  $\alpha$  outside the original domain of  $p$ ).

Assume we already have constructed  $q_0 = q \restriction \alpha$ . Work in the  $P_\alpha$ -extension  $V[G_\alpha]$  with  $q_0 \in G$ .

Case 1:  $r \restriction \alpha \notin G_\alpha$ . Set  $q(\alpha) := p(\alpha)$ .

Case 2:  $r \restriction \alpha \in G_\alpha$ . Then  $r(\alpha) \leq p(\alpha)$ . If  $\alpha \notin w$ , we set  $q(\alpha) := r(\alpha)$ ; otherwise we set  $q(\alpha)$  to be  $r(\alpha) \vee (p(\alpha) \restriction \zeta + 1)$  as in Lemma 5.8.

If  $\alpha \in w$ , then in both cases we get  $q \restriction \alpha \Vdash q(\alpha) \leq_\zeta^+ p(\alpha)$ . Also, if  $G_{\alpha+1}$  selects  $x \restriction (\alpha + 1)$ , then at stage  $\alpha$  we used, by induction, Case 2; so then  $r(\alpha) \in G(\alpha)$  as  $x(\alpha) \subseteq \eta_\alpha$ .  $\dashv$

We can iterate the construction for all elements of  $\text{poss}(w, \zeta + 1)$ , which gives us:

**LEMMA 5.21.** *If  $p$  fits  $(w, \zeta)$  and  $\tau$  is a name for an ordinal, then there is a  $q \leq_{w, \zeta}^+ p$  which  $(w, \zeta + 1)$ -decides  $\tau$ .*

**PROOF.** We enumerate  $\text{poss}(p, w, \zeta + 1)$  as  $(x_i)_{i \in \delta}$ . We start with  $p_0 := p$ . Inductively we construct  $p_\ell$ : If at step  $\ell$ , if  $x_\ell$  is not in  $\text{poss}(p_\ell, w, \zeta + 1)$  any more, then we set  $p_{\ell+1} := p_\ell$ . Otherwise, pick an  $r \leq p_\ell$  that decides  $\tau$  to be some  $\tau^{x_\ell}$  and extends  $x_\ell$ . Then apply 5.20 to get  $p_{\ell+1} \leq_{w, \zeta}^+ p_\ell$  which forces that  $x_\ell \triangleleft G$  implies  $\tau = \tau^{x_\ell}$ . At limits use Lemma 5.18.  $\dashv$

For the proof of Lemma 5.23 we will need a variant where the “height”  $\zeta$  is not the same for all elements of  $w$ , more specifically:

**PRELIMINARY LEMMA 5.22.** *Assume that  $p$  fits  $(w, \zeta)$  and  $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$ , and that  $\tau$  is a name for an ordinal. Then there is a  $q \leq_{w, \zeta}^+ p$  such that  $q \restriction \alpha^* \Vdash q(\alpha^*) \leq_{\zeta^*}^+ p(\alpha^*)$  and there is a (ground model) set  $A$  of size  $< \lambda$  such that  $q \restriction \tau \in A$ .*

**PROOF.** This is just a notational variation of the previous proof. For notational simplicity we assume  $\alpha^* \notin w$ .

First we have to modify Preliminary Lemma 5.20: A candidate is a pair  $(x, a)$  where  $x \in \text{POSS}(w, \zeta)$  and  $a^* \in \text{POSS}^Q(\zeta^*)$ . Assume that  $(x, a)$  is a candidate, that  $p \in P$  fits  $(w, \zeta)$ , and that  $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$ , and assume that  $r \leq p$  extends  $(x, a)$ , i.e.,  $r \restriction (x \triangleleft G \ \& \ a^* \subseteq \eta_{\alpha^*})$ . Then there is a  $q$  such that

$$q \leq_{w, \zeta}^+ p, \quad q \restriction \alpha^* \Vdash q(\alpha^*) \leq_{\zeta^*}^+ p(\alpha^*), \quad \text{and} \quad q \restriction ((x \triangleleft G \ \& \ a^* \subseteq \eta_{\alpha^*}) \rightarrow r \in G). \tag{*}$$

The same proof works, with the obvious modifications:

When defining  $q(\alpha)$ , we inductively show:

- $q \upharpoonright \alpha \leq_{w \cap \alpha, \zeta}^+ p \upharpoonright \alpha$  and if  $\alpha > \alpha^*$  then  $q \upharpoonright \alpha^* \Vdash q(\alpha^*) \leq_{\zeta^*}^+ p(\alpha^*)$ ,
- $q \upharpoonright \alpha \Vdash ((x \upharpoonright \alpha \triangleleft G_\alpha \ \& \ a^* \subseteq \eta_{\alpha^*}) \rightarrow r \upharpoonright \alpha \in G_\alpha)$ , unless  $\alpha \leq \alpha^*$  in which case we omit the clause about  $\alpha^*$ .

Again, in the  $P_\alpha$ -extension we have:

Case 1:  $r \upharpoonright \alpha \notin G_\alpha$ . Set  $q(\alpha) := p(\alpha)$ .

Case 2:  $r \upharpoonright \alpha \in G_\alpha$ . Then  $r(\alpha) \leq p(\alpha)$ . If  $\alpha \notin w \cup \{\alpha^*\}$ , we set  $q(\alpha) := r(\alpha)$ ; otherwise we set  $q(\alpha)$  to be  $r(\alpha) \vee (p(\alpha) \upharpoonright \zeta + 1)$  as in Lemma 5.8.

Then we can show  $(*)$  as before.

We then enumerate all candidates (there are  $< \lambda$  many) as  $(x_\ell, a_\ell)$ , and at step  $\ell$ , if  $(x_\ell, a_\ell)$  is compatible with  $p_\ell$ , use  $(*)$  to decide  $\tau$  to be some  $\tau^\ell$ .  $\dashv$

We will now show that  $P$  is  $\lambda^\lambda$ -bounding and proper. We first give two preliminary lemmas that assume this is already the case for all  $P_{\beta'}$  with  $\beta' < \beta$ .

**PRELIMINARY LEMMA 5.23.** *Let  $\beta \leq \mu$ , and assume that  $P_{\beta'}$  is  $\lambda^\lambda$ -bounding for all  $\beta' < \beta$ .*

*Assume  $p \in P_\beta$  fits  $(w, \zeta)$ ,  $\tilde{C} \subseteq \lambda$  is club, and  $\alpha^* < \beta$ .*

*Then there is a  $q \leq_{w, \zeta}^+ p$  and a  $\zeta \in \tilde{C}$  such that  $q$  fits  $(w \cup \{\alpha^*\}, \zeta)$ .*

*If additionally  $\alpha^* \in \text{dom}(p)$  and  $p \upharpoonright \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$  for some  $\zeta^* \in \lambda$ , then we can additionally get  $q \upharpoonright \alpha^* \Vdash q(\alpha^*) \leq_{\zeta^*}^+ p(\alpha^*)$ .*

**PROOF.** For notational simplicity assume  $\alpha^* \notin w$  and  $\min(\tilde{C}) > \max(\zeta, \zeta^*)$ . By induction on  $\alpha \leq \beta$  we show that the result holds for all  $w, \alpha^*$  with  $w \cup \{\alpha^*\} \subseteq \alpha$ .

**Successor case  $\alpha + 1$ :** Set  $w_0 := w \cap \alpha$ .

By our assumption  $P_\alpha$  is  $\lambda^\lambda$ -bounding, so every club-set in the  $P_\alpha$ -extension contains a ground-model club (see Fact 5.11). In particular,  $C^{p(\alpha)}$  contains some ground-model  $C^*$ . By Lemma 5.21 (or Preliminary Lemma 5.22, if  $\alpha^* < \alpha$ ) there is a  $p' \leq_{w_0, \zeta}^+ p \upharpoonright \alpha$  (also dealing with  $\alpha^*$ , if  $\alpha^* < \alpha$ ) leaving only  $< \lambda$  many possibilities for  $C^*$ . So we can intersect them all, resulting in  $C'$ . Set  $C'' := C' \cap \tilde{C}$ . Apply the induction hypothesis in  $P_\alpha$  to get  $q' \leq_{w_0, \zeta}^+ p'$  and  $\xi$  in  $C''$  such that  $q'$  fits  $(w_0, \xi)$  (and also  $(\{\alpha^*\}, \xi)$ , if  $\alpha^* < \alpha$ ). Set  $q := q' \cup \{(\alpha, p(\alpha))\}$ , so trivially  $q \leq_{w, \zeta}^+ p$  (and, if  $\alpha = \alpha^*$ , then  $q \upharpoonright \alpha \Vdash q(\alpha) \leq_{\zeta^*}^+ p(\alpha)$ ), and  $q$  fits  $(w \cup \{\alpha^*\}, \xi)$ .

**Limit case:** If  $w$  is bounded in  $\alpha$  there is nothing to do. So assume  $w$  is cofinal.

Set  $\alpha_0 := \min(w \setminus \alpha^*)$  and  $w_0 := (w \cap \alpha_0) \cup \{\alpha^*\}$ . Use the induction hypothesis in  $P_{\alpha_0}$  using  $(p \upharpoonright \alpha_0, w_0, \zeta, \alpha^*, \zeta^*)$  as  $(p, w, \zeta, \alpha^*, \zeta^*)$ . This gives us some  $p'_0 \leq_{w \cap \alpha_0, \zeta}^+ p \upharpoonright \alpha_0$  fitting  $(w_0, \zeta_0)$  and dealing with  $\alpha^*$ , for some  $\zeta_0 \in \tilde{C}$ . Set  $p_0 := p' \wedge p$ .

Enumerate  $w \setminus w_0$  increasingly as  $(\alpha_i)_{i < \delta}$ , and set  $w_j := w_0 \cup \{\alpha_i : i < j\}$  for  $j \leq \delta$ .

We will construct  $p'_i$  in  $P_{\alpha_i}$  and  $(\zeta_i)_{i < \delta}$  a strictly increasing sequence in  $\tilde{C}$ , and we set  $p_j := p'_j \wedge p$  and will get:  $p_\ell$  fits  $(w_\ell, \zeta_\ell)$ , and  $p_\ell \leq_{w_i, \zeta_i}^+ p_i$  for all  $i < \ell \leq j$ .

For successors  $\ell = i + 1$ , we use the induction hypothesis in  $P_{\alpha_{i+1}}$ , using  $(p_i \upharpoonright \alpha_{i+1}, w_i, \zeta_i, \alpha_i, \zeta)$  as  $(p, w, \zeta, \alpha^*, \zeta^*)$ . This gives us  $p'_{i+1} \leq_{w_i, \zeta_i}^+ p_i \upharpoonright \alpha_{i+1}$  and some  $\zeta_{i+1} > \zeta_i$  in  $\tilde{C}$  such that  $p_{i+1}$  fits  $(w_{i+1}, \zeta_{i+1})$  and  $p_{i+1} \upharpoonright \alpha_i \Vdash p_{i+1}(\alpha_i) \leq_{\zeta}^+ p_i(\alpha_i)$ .



For  $j$  limit, we set  $\zeta_j := \sup_{i < j} \zeta_i$  (which is in  $\tilde{C}$ ), and let  $p_j$  be a limit of the  $(p_i)_{i < j}$ . That is,  $\text{dom}(p_j) = \bigcup_{i < j} \text{dom}(p_i)$ , and for  $\beta \in \text{dom}(p_j)$  let  $p_j(\beta)$  be as follows: If  $\beta \notin w$ , fix some condition  $p_j(\beta)$  stronger than all  $p_i(\beta)$ . Otherwise, there is a minimal  $i_0 < j$  such that  $\beta \in w_{i_0}$ , and  $p_\ell(\beta) <_{\zeta_i}^+ p_i(\beta)$  for all  $i_0 \leq i < \ell < j$ . In that case let  $p_j(\beta)$  be the (canonical) limit of the  $(p_i(\beta))_{i_0 \leq i < j}$ , and note that  $\zeta_j \in C^{p_j(\beta)}$ .  $\dashv$

**PRELIMINARY LEMMA 5.24.** *Let  $\beta \leq \mu$ , and assume that  $P_{\beta'}$  is  $\lambda^\lambda$ -bounding for all  $\beta' < \beta$ .*

*Assume that  $p \in P_\beta$  fits  $(w, \zeta)$ , and  $\sigma$  is a  $P_\beta$ -name for a  $\lambda$ -sequence of ordinals. Then there is a  $q \leq_{w, \zeta}^+ p$  continuously reading  $\sigma$ .*

**PROOF.** Set  $p_0 := p$ ,  $\zeta_0 := \zeta$ ,  $w_0 := w$ . We construct by induction on  $i < \lambda p'_i$ ,  $p_i$ ,  $\zeta_i$ ,  $\alpha_i$ , and  $w_i$  as follows:

- Given  $p_j$ ,  $w_j$ , and  $\zeta_j$ , pick  $\alpha_j \in \text{dom}(p_j) \setminus w_j$  by bookkeeping (so that in the end the domains of all conditions will be covered).
- Successor  $j = i + 1$ : Set  $w_{i+1} := w_i \cup \{\alpha_i\}$ . Find  $p'_{i+1} \leq_{w_i, \zeta_i}^+ p_i$  and  $\zeta_{i+1} > \zeta_i$  such that  $p'_{i+1}$  fits  $(w_{i+1}, \zeta_{i+1})$  (using the previous preliminary lemma).
- Limit  $j$ : Let  $p'_j$  be the canonical limit of the  $(p_i)_{i < j}$ ,  $\zeta_j := \sup_{i < j} (\zeta_i)$ , and  $w_j := \bigcup_{i < j} w_i$ . Note that  $p'_j$  fits  $(w_j, \zeta_j)$ .
- In any case, given  $p'_j$  we pick some  $p_j \leq_{w_j, \zeta_j}^+ p'_j$  which  $(w_j, \zeta_j + 1)$ -decides  $\sigma(\zeta_j)$ .

Then the limit  $q$  of the  $p_i$  continuously reads  $\sigma$ .  $\dashv$

**LEMMA 5.25.**  *$P$  has continuous reading (and in particular is  $\lambda^\lambda$ -bounding).*

**PROOF.** Assume by induction that  $P_{\beta'}$  is  $\lambda^\lambda$ -bounding for all  $\beta < \beta'$ . Then the previous lemma gives us that  $P_\beta$  has continuous reading of names, and thus is  $\lambda^\lambda$ -bounding.  $\dashv$

The same construction shows  $\lambda$ -properness:

**DEFINITION 5.26.** Let  $\chi \gg \mu$  be sufficiently large and regular. An “elementary model” is an  $M \preceq H(\chi)$  of size  $\lambda$  which is  $<\lambda$ -closed and contains  $\lambda$  and  $\mu$  (and thus  $P$ ).

**LEMMA 5.27.** *If  $M$  is an elementary model containing  $p \in P$ , then there is a  $q \leq p$  which is strongly  $M$ -generic in the following sense: For each  $P$ -name  $\tau$  in  $M$  for an ordinal,  $q(w, \zeta)$ -decides  $\tau$  via a decision function in  $M$  (so in particular  $q \Vdash \tau \in M$ ).*

(The decision function being in  $M$  is equivalent to  $w \subseteq M$ , as  $M$  is  $<\lambda$  closed.)

**PROOF.** Let  $\sigma$  be a sequence of all  $P$ -names for ordinals that are in  $M$ . Starting with  $p_0 \in M$ , perform the successor step of the previous construction within  $M$ ; as  $M$  is closed the limits at steps  $<\lambda$  are in  $M$  as well. Then the  $\lambda$ -limit is  $M$ -generic.  $\dashv$

**5.5. Canonical conditions.** We will use conditions that “continuously read themselves.”



DEFINITION 5.28.  $p \in P$  is  $(w, \zeta)$ -canonical if  $p$  fits  $(w, \zeta)$  and  $p(\alpha) \restriction (\zeta + 1)$  is  $(w \cap \alpha, \zeta + 1)$ -decided by  $p \restriction \alpha$  for all  $\alpha \in w$ .

FACTS 5.29. Let  $p$  be canonical for  $(w, \zeta)$ .

- (1) If  $q \leq_{w, \zeta}^+ p$ , then  $q$  is canonical for  $(w, \zeta)$  and  $\text{poss}(p, w, \zeta + 1) = \text{poss}(q, w, \zeta + 1)$ .
- (2) Let  $x \in \text{poss}(p, w, \zeta + 1)$ . There is a naturally defined  $p \wedge x \leq p$  such that  $p \Vdash (p \wedge x \in G \leftrightarrow x \triangleleft G)$ .  $\{p \wedge x : x \in \text{poss}(p, w, \zeta + 1)\}$  is a maximal antichain below  $p$ .
- (3) Let  $x \in \text{poss}(p, w, \zeta + 1)$ . In an intermediate  $P_\alpha$ -extension  $V[G_\alpha]$  with  $x \restriction \alpha \triangleleft G_\alpha$  the rest of  $x$ , i.e.,  $x \restriction [\alpha, \mu]$ , is compatible with  $p/G_\alpha$  in the quotient forcing.

Or equivalently: If  $r_0 \leq p \restriction \alpha$  in  $P_\alpha$  extends  $x \restriction \alpha$ , then there is an  $r \leq r_0$  extending  $x$ .

DEFINITION 5.30. Assume  $p \in P$ , and  $\sigma$  is a  $P$ -name for a  $\lambda$ -sequence of ordinals. Let  $E \subseteq \lambda$  be a club-set and  $\bar{w} = (w_\zeta)_{\zeta \in E}$  an increasing sequence in  $[\mu]^{<\lambda}$ .  $p$  canonically reads  $\sigma$  as witnessed by  $\bar{w}$  if the following holds:

- $\text{dom}(p) = \bigcup_{\zeta \in E} w_\zeta$ .
- $p$  is  $(w_\zeta, \zeta)$ -canonical for all  $\zeta \in E$ .
- $p \restriction \alpha \Vdash C^{p(\alpha)} = E \setminus (\zeta'_\alpha)$  for some (ground model)  $\zeta'_\alpha$ .
- $\sigma \restriction I^*(\leq \zeta + 1)$  is  $(w_\zeta, \zeta + 1)$ -decided by  $p$  for all  $\zeta \in E$ .

If  $\sigma$  is the constant 0 sequence (or any sequence in  $V$ ), we just say “ $p$  is canonical” (as witnessed by  $\bar{w}$ ).

LEMMA 5.31. For  $p, \sigma$  as above, there is a  $q \leq p$  canonically reading  $\sigma$ .

If  $p \in P_\alpha$  and  $\sigma$  is a  $P_\alpha$ -name for some  $\alpha < \mu$ , then  $q \in P_\alpha$ .

PROOF. We just have to slightly modify the proof of Lemma 5.24.

We will construct  $p_j, \xi_j$ , and  $\alpha_j$  by induction on  $j \in \lambda$ , setting  $w_j := \{\alpha_i : i < j\}$ , such that for  $0 < j < k$  the following holds:

- $p_k \leq_{w_j, \xi_j}^+ p_j$ .
- $p_j$  is  $(w_j, \xi_j)$ -canonical.
- $p_j(w_j, \xi_j + 1)$ -decides  $\sigma \restriction I^*(\leq \xi_j + 1)$ .
- In  $p_k$ , for  $\alpha_j \in w_k$ ,  $\{\xi_i : j < i < k\}$  is (forced to be) an initial segment of  $C^{p_k(\alpha_j)}$ .
- The  $\alpha_j$  are chosen (by some book-keeping) so that  $\{\alpha_i : i \in \lambda\} = \bigcup_{i \in \lambda} \text{dom}(p_i)$ .

Then the limit of the  $p_j$  is as required, with  $E = \{\xi_i : i \in \lambda\}$  and, for  $\zeta = \xi_j$  in  $E$ , we use  $w_j$  as  $w_\zeta$ .

Set  $p_0 \leq p$  such that  $|\text{dom}(p_0)| = \lambda$ , and set  $\xi_0 := 0$ . Assume we already have  $p_i, \alpha_i$  for  $i < j$  (so we also have  $w_j$ ).

- For  $j$  limit, let  $s$  be a limit of  $(p_i)_{i < j}$ , and set  $\xi_j := \sup_{i < j} \xi_i$ . Note that  $s$  fits  $(w_j, \xi)$ .
- Successor case  $j = i + 1$ : Find  $s_0 \leq_{w_i, \xi_i}^+ p_i$  and  $\xi_j > \xi_i$  such that  $s$  fits  $(w_j, \xi_j)$ . (As in Lemma 5.23. Recall that  $w_j = w_i \cup \{\alpha_i\}$ .) Strengthen  $s_0$  to  $s \leq_{w_i, \xi_i}^+$  so that:

- $s$  still fits  $(w_j, \xi_j)$ ,
- the trunk at  $\alpha_i$  has length  $\xi_j$ , i.e.,  $s \restriction \alpha_i \Vdash \min(C^{s(\alpha_i)}) = \xi_j$ ,
- for  $\alpha_{i'}, i' < i$ , there are no elements in  $C^{s(\alpha_{i'})}$  between  $\xi_i$  and  $\xi_j$ .
- Construct  $s^* \restriction \alpha$  by recursion on  $\alpha \in w_j$ , such that  $s^* \restriction \alpha \leq_{w_j \cap \alpha, \xi_j}^+ s \restriction \alpha$  and  $s^* \restriction \alpha(w_j \cap \alpha, \xi_j + 1)$ -decides  $s(\alpha) \restriction (\xi_j + 1)$  (which is the same as  $s^*(\alpha) \restriction (\xi_j + 1)$ ). This gives  $s^* \leq_{w_j, \xi_j}^+ s$ .
- Find  $p_j \leq_{w_j, \xi_j}^+ s^*$  which  $(w_j, \xi_j + 1)$  decides  $\mathcal{G} \restriction I^*(\leq \xi + 1)$ .
- Choose  $\alpha_j \in \text{dom}(p_j) \setminus w_j$  by bookkeeping. ⊢

FACTS 5.32.

- (1) If a  $P_\beta$ -name  $\tilde{x} \subseteq \lambda$  is continuously read (by some  $P_\beta$ -condition  $p$ ), and  $\text{cf}(\beta) > \lambda$ , then there is an  $\alpha < \beta$  such that:  $p \in P_\alpha$ , and  $\tilde{x}$  is already a  $P_\alpha$ -name (formally: there is a  $P_\alpha$ -name  $\tilde{y}$  such that  $p \Vdash \tilde{x} = \tilde{y}$ ).
- (2) There are at most  $|\alpha|^\lambda \leq \lambda^+$  many pairs<sup>10</sup>  $(p, \tilde{x})$  such that  $p$  canonically reads  $\tilde{x}$  in  $P_\alpha$ .

**5.6.  $\Delta$  systems.** In this section we define  $\Delta$ -systems and show that such systems exist, which we will in the indirect proofs of Lemmas 5.39 and 5.54.

In Section 5.10 we will then fix a specific  $\Delta$ -system for the rest of the paper.

From now on, we assume that  $p_*$  forces

$$\pi : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda) \text{ represents the automorphism } \tilde{\phi} : P_\lambda^\lambda \rightarrow P_\lambda^\lambda, \quad (5.33)$$

and we set, for  $\beta \in \mu$ ,

$$\tilde{a}_\beta := \pi(\tilde{\eta}_\beta),$$

where, as usual, we identify  $\tilde{\eta}_\beta \in 2^\lambda$  with  $\tilde{\eta}_\beta^{-1}\{1\} \subseteq \lambda$ .

Note that, other than  $\tilde{\eta}_\beta$ ,  $\tilde{a}_\beta$  is a priori not a  $P_{\beta+1}$ -name (but see Section 5.9).

We also fix a  $P$ -name for a representation of the inverse automorphism  $\tilde{\phi}^{-1}$ .

Abusing notation, we call it  $\tilde{\pi}^{-1}$ .

With  $S_{\lambda^+}^\mu$  we denote the stationary subset of  $\mu$  consisting of ordinals with cofinality  $\lambda^+$ .

**DEFINITION 5.34.** Let  $S \subseteq S_{\lambda^+}^\mu$  be stationary,  $\chi \gg \mu$  sufficiently large, and regular, and  $z \in H(\chi)$ . “An elementary  $S$ -system” (using parameter  $z$ ) is a sequence  $(M_\beta, p_\beta)_{\beta \in S}$  such that, for each  $\beta \in S$ ,  $M_\beta$  is an elementary model (as in Definition 5.26) and contains  $z$ ,  $\beta$ ,  $p_*$ ,  $\tilde{\phi}$ ,  $\tilde{\pi}$  and  $\tilde{\pi}^{-1}$ , and  $p_\beta \in P \cap M_\beta$  canonically reads  $\tilde{a}_\beta$  witnessed by some  $(w_\zeta^{p_\beta})_{\zeta \in E^{p_\beta}}$ , which  $E^{p_\beta} \subseteq \lambda$  club (cf. Definition 5.30).

By a simple  $\Delta$ -system argument we can make an  $S$ -system homogeneous:

<sup>10</sup>Depending on the formal definition, we could/should add “modulo equivalence,” i.e., there is a  $\leq |\alpha|^\lambda$ -sized set  $Z$  of such pairs such that whenever  $p$  canonically reads  $\tilde{y}$  in  $P_\alpha$  then there is a  $\tilde{x}$  such that  $(p, \tilde{x}) \in Z$  and  $p \Vdash \tilde{x} = \tilde{y}$ .

DEFINITION 5.35.  $(M_\beta, p_\beta)_{\beta \in S}$  forms a “ $\Delta$ -system,” if  $\bar{M}, \bar{p}$  is an elementary  $S$ -system with parameter  $z$ , and is homogeneous in the following sense: For  $\beta$  and  $\beta_1 < \beta_2$  in  $S$ , we get:

- (1)  $M_{\beta_1} \cap M_{\beta_2} \cap \mu$  is constant. We call this set the “heart” and, abusing notation, denote it with  $\Delta$ . Obviously  $\Delta \supseteq \lambda$ ,  $\Delta \supseteq \text{dom}(p_*)$ ,  $\lambda^+ \in \Delta$ , etc.
- (2)  $M_\beta \cap \beta = \Delta$ . So in particular  $\beta$  is the minimal element of  $M_\beta$  above  $\Delta$ . All the non-heart elements of  $M_{\beta_2}$  are above all elements of  $M_{\beta_1}$ . That is,  $\sup(M_{\beta_1} \cap \mu) < \beta_2$ .
- (3) There is an  $\in$ -isomorphism  $h_{\beta_1, \beta_2}^* : M_{\beta_1} \rightarrow M_{\beta_2}$ , mapping  $\beta_1$  to  $\beta_2$ ,  $p_{\beta_1}$  to  $p_{\beta_2}$ ,  $\bar{q}_{\beta_1}$  to  $\bar{q}_{\beta_2}$  and fixing  $\lambda, \mu, \phi, \pi$  as well as each  $\alpha$  in  $\Delta$ .

Note that this implies that the continuous reading of  $q_\beta$  works the same way for all  $\beta$ . In particular the  $E^{p_\beta}$  are that same  $E$  for all  $\beta$ ; and if  $F_\zeta^\beta$  is the function mapping  $\text{POSS}(w_\zeta^{p_\beta}, \zeta + 1)$  to the value of  $\bar{q}_\beta \upharpoonright I^*(\leq \zeta + 1)$  (for  $\zeta \in E$ ), then  $h_{\beta_1, \beta_2}^*(F_\zeta^{\beta_1}) = F_\zeta^{\beta_2}$  and in particular  $h_{\beta_1, \beta_2}^*(w_\zeta^{p_{\beta_1}}) = w_\zeta^{p_{\beta_2}}$ ; i.e., they are the same apart from shifting coordinates above  $\Delta$ .

LEMMA 5.36. Assume  $S \subseteq S_{\lambda^+}^\mu$  is stationary.

- For every  $z \in H(\chi)$  and  $(p'_\beta)_{\beta \in S}$  there are  $M_\beta$  and  $p_\beta \leq p'_\beta$  such that  $\bar{M}, \bar{p}$  is an  $S$ -system with parameter  $z$ .
- If  $\bar{M}, \bar{p}$  is an  $S$ -system then there is an  $S' \subseteq S$  stationary such that  $(M_\beta, p_\beta)_{\beta \in S'}$  is a  $\Delta$ -system on  $S'$ .

PROOF. The first item is trivial, using the fact that everything can be read canonically.

Using  $2^\lambda = \lambda^+$ , a standard  $\Delta$ -system argument (or: Fodor’s Lemma argument) lets us thin out  $S$  to some  $S^2$  so that  $(M_\beta \cap \mu)_{\beta \in S^2}$  satisfies (1–3). For  $\beta \in S^2$  let  $\iota_\beta : M_\beta \cup \{M_\beta\} \rightarrow H(\lambda^+)$  be the transitive collapse, and assign to  $\beta$  the tuple of the  $\iota_\beta$ -images of the following objects:

- $M_\beta, p_\beta, \bar{q}_\beta, \mu, \phi, \pi$ , and  $E^{p_\beta}$ .
- For  $\zeta \in E^{p_\beta}$ , the object  $w_\zeta^{p_\beta}$ .
- For  $\zeta \in E^{p_\beta}$  and  $\gamma \in w_\zeta^{p_\beta}$ , the object  $F_\gamma^{p_\beta}$ .

Again, there are  $|H(\lambda^+)|^\lambda < \mu$  many possibilities, so the objects are constant on a stationary  $S' \subseteq S^2$ .

For  $\alpha < \beta$  in  $S'$ , we define  $h_{\beta_1, \beta_2}^* := \iota_{\beta_2}^{-1} \circ \iota_{\beta_1}$ . (Note that  $\iota_{\beta_1}(\alpha) = \iota_{\beta_2}(\alpha)$  for  $\alpha \in \Delta$ .) ⊢

So in particular if we have a  $\Delta$ -system on  $S$ , then  $p_\beta \upharpoonright \sup(\Delta) = p_\beta \upharpoonright \beta \in M_\beta$  is the same for all  $\beta \in S$ , and outside of  $\Delta$  the domains of the  $p_\beta$  are disjoint for  $\beta \in S$ . In particular we get:

FACT 5.37. For a  $\Delta$ -system with domain  $S$ , and  $A \subseteq S$  of size  $\leq \lambda$ , the union of the  $(p_\beta)_{\beta \in A}$  is a condition in  $P$  (and stronger than each  $p_\beta$ ).

Whenever  $r \in P_\beta \cap M_\beta$  (as is the case for  $r = p_\beta \upharpoonright \beta$ ), we know that  $r \in P_\alpha$  for  $\alpha \in \Delta$  (as  $M_\beta$  knows that  $\beta$  has cofinality  $\lambda^+$ ).

Instead of “ $r \in P_\alpha$  for some  $\alpha \in \Delta$ ” we will sometimes just state the weaker but shorter  $r \in P_{\sup(\Delta)}$ .

REMARK. This is an important effect also for some names. Generally, a  $P_\beta$ -name in  $M_\beta$  is of course not a  $P_\alpha$ -name for any  $\alpha < \beta$  (just take the  $P_\beta$ -generic filter  $G_\beta$ ). However, as we will explicitly state in Lemma 5.42, such names for subsets of  $\lambda$  are, modulo some condition,  $P_\alpha$ -names for some  $\alpha \in \Delta$  and independent of  $\beta$ . In the specific case of the  $P_\beta$ -name  $p_\beta(\beta)$  we do not have to increase the condition:

DEFINITION AND LEMMA 5.38.  $\tilde{p} := p_\beta(\beta)$  is a  $P_{\sup(\Delta)}$ -name independent of  $\beta \in S$ .

PROOF.  $p_\beta(\beta) \restriction \zeta + 1$  is  $(w_\zeta^{P_\beta}, \zeta + 1)$ -determined for cofinally many  $\zeta \in E$ , where  $w_\zeta^{P_\beta} \in [\beta]^{<\lambda}$  is a subset of  $M_\beta$ . So  $w_\zeta^{P_\beta} \subseteq \Delta$ , and the isomorphisms between the  $M_\beta$  guarantee that each  $w_\zeta^{P_\beta}$  is the same, and that  $p_\beta(\beta) \restriction \zeta + 1$  is decided the same way. So  $\tilde{p}$  is a  $P_\gamma$ -name for  $\gamma = \sup(w_\zeta^{P_\beta})_{\zeta \in E}$ . This  $\gamma$  is independent of  $\beta \in S$ , and is in  $\Delta$ . So  $\tilde{p}$  is actually a  $P_\alpha$ -name for some  $\alpha \in \Delta$ ; and certainly a  $P_{\sup(\Delta)}$ -name.  $\dashv$

For later reference we note:

LEMMA 5.39. For all but non-stationary many  $\beta$ ,  $p_*$  forces  $\underline{q}_\beta \notin V_\beta$ .

(Here,  $V_\beta$  denotes the  $P_\beta$ -extension of the ground model.)

PROOF. Assume that  $p_\beta \leq p_*$  forces that  $\underline{q}_\beta = \underline{x}_\beta$  for a  $P_\beta$ -name  $\underline{x}_\beta$  for all  $\beta \in S^*$  stationary. We can also assume that  $p_\beta$  canonically reads  $\underline{q}_\alpha$ . Pick  $M_\beta$  containing  $p_\beta$  and  $S \subseteq S^*$  such that  $(M_\beta, p_\beta)_{\beta \in S}$  is a  $\Delta$ -system, where we can assume (or get from homogeneity) that  $h_{\beta_0, \beta_1}^*(\underline{x}_{\beta_0}) = \underline{x}_{\beta_1}$ . So the  $\underline{x}_\beta$  are  $P_\beta$ -names in  $M_\beta$  and therefore  $P_{\sup(\Delta)}$ -names, and are the same for all  $\beta$ . Choose  $\beta_1 > \beta_0$  in  $S$ . So  $p_{\beta_0} \wedge p_{\beta_1}$  force that  $\underline{q}_{\beta_0} = \underline{x} = \underline{q}_{\beta_1}$ , which contradicts the injectivity of  $\phi$  and the fact that  $\eta_{\beta_0} \neq \eta_{\beta_1}$ .  $\dashv$

## 5.7. Preservation of cofinalities, catching canonical names.

COROLLARY 5.40.  $P$  is  $\lambda^{++}$ -cc and preserves all cofinalities.

PROOF. Cofinalities  $\leq \lambda$  are preserved as  $P$  is  $<\lambda$ -closed.

Cofinality  $\lambda^+$  is preserved by properness: Assume that it is forced by  $p$  that  $\kappa$  has a cofinal  $\lambda$ -sequence  $\tilde{\alpha} := (\alpha_i)_{i \in \lambda}$ . Then there is an elementary model  $M$  containing  $p$  and  $\tilde{\alpha}$ . If  $q \leq p$  is  $M$ -generic, and  $G$  a  $P$ -generic filter containing  $q$ , then  $\alpha_i[G] \in M$  for all  $i < \lambda$ , so  $M \cap \kappa$  is a cofinal subset of  $\kappa$  of size  $\lambda$  in the ground model.

Cofinality  $\geq \lambda^{++}$  is preserved as  $P$  has the  $\lambda^{++}$ -cc, which we have shown in a very roundabout way with the fact about  $\Delta$ -systems: If  $(p'_\alpha)_{\alpha \in \mu}$  are arbitrary conditions, then  $(M_\beta, p_\beta)$  form a  $\Delta$ -system from some  $p_\beta < p'_\beta$  and stationary  $S$ , and any two (in fact,  $\leq \lambda$  many)  $p_\beta$  are compatible for  $\beta \in S$ .  $\dashv$

REMARK 5.41. This shows that  $P$  is  $(\mu, \lambda)$ -Knaster, i.e., for every  $A \in [P]^\mu$  there is a  $B \in [A]^\mu$  which is  $\lambda$ -linked.

The  $\lambda^{++}$ -cc also implies: For every name  $\underline{x}$  for a subset of  $\lambda$  (or of  $\lambda^+$ ) there is a  $\beta < \mu$  and a  $P_\beta$ -name  $\underline{y}$  such that the empty condition forces that  $\underline{x} = \underline{y}$ .

Given  $\alpha < \mu$ , there are  $< \mu$  many pairs  $(p, \underline{x})$  where  $p$  canonically reads  $\underline{x} \subseteq \lambda$  in  $P_\alpha$  (see Fact 5.32(2)). So there is a  $g(\alpha) < \mu$  such that for each such  $p, \underline{x}$ , both  $\pi(\underline{x})$

and  $\pi^{-1}(\tilde{x})$  are equivalent (modulo the empty condition) to some  $P_{g(\alpha)}$ -name. Let  $C^* \subseteq \mu$  be the club set with  $(\zeta \in C^* \ \& \ \alpha < \zeta) \rightarrow g(\alpha) < \zeta$ .

Given a  $\Delta$ -system on  $S$  we can restrict it to a  $\Delta$ -system on  $S \cap C^*$ ; so we will assume from now on that each  $\Delta$ -system we consider satisfies  $S \subseteq C^*$ .

To summarize:

LEMMA 5.42.

- (1) If  $\beta \in S$ ,  $p \in P_\beta$ , and  $\tilde{x}$  a  $P_\beta$ -name for a subset of  $\lambda$ , then there is an  $\alpha < \beta$  and a  $q \leq p$  canonically reading  $\tilde{x}$ ,  $\tau(\tilde{x})$ ,  $\tau^{-1}(\tilde{x})$  as  $P_\alpha$ -names.

More explicitly: There is a  $P_\alpha$ -name  $\tilde{y}$  which is canonically read by  $q$  such that  $q \Vdash \tilde{y} = \tilde{x}$ . (And analogously for  $\tau(\tilde{x})$  and  $\tau^{-1}(\tilde{x})$  instead of  $\tilde{x}$ .)

- (2) If additionally  $p \leq p_\beta \restriction \beta$  in  $P_\beta$  and  $(p, \tilde{x}) \in M_\beta$ , then we can additionally get:  $\tilde{x}$ ,  $\tau(\tilde{x})$  and  $\tau^{-1}(\tilde{x})$  are  $P_\alpha$ -names in  $M_\beta$  independent of  $\beta \in S$ .

More explicitly: Let  $\tilde{y}$  be as above (for  $\tilde{x}$ ). Then  $\alpha \in \Delta$ ,  $q$  and  $\tilde{y}$  are in  $M_\beta$ , and if  $\beta' \in S$  and  $h := h_{\beta, \beta'}^*$ , then  $h$  acts as identity on  $\alpha$ ,  $q$ , and  $\tilde{y}$ , and  $(M_{\beta'}, \text{ knows that } q \Vdash \tilde{y} = h(\tilde{x}))$ . (And analogously for  $\tau(\tilde{x})$  and  $\tau^{-1}(\tilde{x})$  instead of  $\tilde{x}$ .)

PROOF. (1): Use Lemma 5.31 to get a  $q_1 \in P_\beta$  canonically reading  $\tilde{x}$ . And if  $\beta \in S$  then  $\text{cf}(\beta) = \lambda^+$ , so  $\text{dom}(p)$  is bounded by some  $\alpha' < \beta$  and, by Fact 5.32(1),  $q_1 \in P_{\alpha'}$  for some  $\alpha' \leq \alpha_1 < \beta$ . As  $\beta \in C^*$ ,  $\tau(\tilde{x})$  and  $\tau^{-1}(\tilde{x})$  are  $P_\beta$ -names. So repeat the same argument to get  $q \leq q_1$  in  $P_\alpha$  canonically reading all three subsets of  $\lambda$ .

(2): Apply (1) inside  $M_\beta$ . As  $\alpha \in \beta \cap M_\beta$ , we get  $\alpha \in \Delta$ . As  $q$  canonically reads itself as well as  $\tilde{y}$ , we know that  $h$  does not change  $q$  and  $\tilde{y}$ . As  $h$  is an isomorphism, we know that  $h(q) = q$  forces that  $h(\tilde{x}) = h(\tilde{y}) = \tilde{y}$ .  $\dashv$

**5.8. Majority decisions.** For any  $(a_1, a_2, a_3)$  with  $a_i \in \{0, 1\}$  there is a  $b \in \{0, 1\}$  such that  $b = a_i$  for at least two  $i \in \{1, 2, 3\}$ . We write  $b = \text{major}_{i=1,2,3}(a_i)$ .

Similarly, if  $f_1, f_2, f_3$  are functions  $A \rightarrow 2$  we write  $\text{major}_{i=1,2,3}(f_i)$  for the function  $A \rightarrow 2$  that maps  $\ell$  to  $\text{major}_{i=1,2,3}(f_i(\ell))$ .

The following is a central point of the whole construction:

LEMMA 5.43. Let  $(M_\alpha, p_\alpha)_{\alpha \in S}$  be a  $\Delta$ -system. Pick  $\beta_0 < \beta_1 < \beta_2 < \beta_3$  in  $S$ .

- (1)  $p_*$  forces: If  $\eta_{\beta_0} = {}^* \text{major}_{i=1,2,3}(\eta_{\beta_i})$ , then  $\eta_{\beta_0} = {}^* \text{major}_{i=1,2,3}(\eta_{\beta_i})$ .
- (2) Let  $s = \bigwedge_{i < 4} P_{\beta_i}$ . Recall that  $s(\beta_i)$  is the same  $P_{\sup(\Delta)}$ -name called  $\tilde{p}$  for all  $i$ . We can strengthen  $s$  by strengthening, for  $i = 1, 2, 3$ , the condition  $s(\beta_i) = \tilde{p}$  to some  $P_{\beta_0+1}$ -names  $r_i \leq \tilde{p}$  (without changing  $C^{\tilde{p}}$ ) such that the resulting condition forces  $\eta_{\beta_0} = \text{major}_{i=1,2,3}(\eta_{\beta_i})$ .

(We do not have to strengthen  $s(\beta_0)$  for this, i.e., we can use  $r_0 := \tilde{p}$ .)

We describe this by “ $(r_i)_{i < 4}$  honors majority.”

Recall that  $v_1 = {}^* v_2$  denotes that  $v_1(\ell) = v_2(\ell)$  for all but  $< \lambda$  many  $\ell \in \lambda$ .

PROOF. (1) Identifying  $2^\lambda$  with  $P(\lambda)$ , we have  $\text{major}_{i=1,2,3} f_i = (f_1 \cap f_2) \cup (f_2 \cap f_3) \cup (f_1 \cap f_3)$  for any tuple  $(f_i)_{i=1,2,3}$ . As  $\pi$  represents an automorphism, we get  $\pi(\text{major}_{i=1,2,3}(f_i)) = {}^* \text{major}_{i=1,2,3}(\pi(f_i))$ . Apply this to  $f_i := \eta_{\beta_i}$ .

(2) Work in the  $P_{\beta_0+1}$ -extension. Recall  $\tilde{p} := p_{\beta_0}(\beta_0)$ . So both  $\tilde{p}$  and  $\eta_{\tilde{\beta}_0}$  are already determined, and  $\eta_{\tilde{\beta}_0}$  extends  $\eta^{\tilde{p}}$ . Set  $r_0 := \tilde{p}$ .

Set  $s_1 := (0, 0)$ ,  $s_2 := (0, 1)$ ,  $s_3 := (1, 0)$ . For  $\zeta \in C^{\tilde{p}}$  and  $i = 1, 2, 3$ , we define  $r_i(\zeta) \supseteq \tilde{p}(\zeta)$  as follows:

$$\text{Extend } s_\zeta^{\tilde{p}} \text{ by } s_i, \text{ i.e., } s_\zeta^{r_i} := (s_\zeta^{\tilde{p}}) \smallfrown s_i; \text{ and set } r_i(\zeta)(\ell) := \eta_{\tilde{\beta}}(\ell) \text{ for } \ell \in [s_\zeta^{\tilde{p}}] \setminus [s_\zeta^{r_i}]. \quad (5.44)$$

So  $\eta^{r_i}$  agrees on its domain with  $\eta_{\tilde{\beta}_0}$ , and each  $\ell \in \lambda$  is in  $\text{dom}(\eta^{r_i})$  for at least two  $i \in \{1, 2, 3\}$ . Accordingly, an extension by a generic filter  $G$  with  $r_i \in G(\beta_i)$  for all  $i < 4$  will satisfy  $\eta_{\tilde{\beta}_0} = \text{major}_{i=1,2,3}(\eta_{\tilde{\beta}_i})$ . (We do not even have to assume that any  $p_\beta \in G$ .)  $\dashv$

**REMARK 5.45.** Let  $p'_{\beta_1}$  be the condition where we strengthen  $p_{\beta_1}(\beta_1)$  to  $r_1$ . Note that  $p'_{\beta_1}$  is not in  $M_{\beta_1}$ , as  $\beta_0 \notin M_{\beta_1}$  and  $r_1$  is defined using  $\eta_{\tilde{\beta}_0}$ . Similarly (basically the same):  $r_1[G_{\beta_1}] \notin M_{\beta_1}[G_{\beta_1}]$ , even if we assume that  $G_{\beta_1}$  is  $M_{\beta_1}$ -generic. But generally we will not be interested in  $M_\beta$ -generic conditions or extensions (we needed generic conditions only in Lemma 5.27, which in turn is needed for Corollary 5.40). And while usually most conditions we consider can be constructed within (and therefore will be elements of) some  $M_\beta$ , this is generally not required (an example are the  $s_i$ 's in the following lemma).

The same proof works if we do not start with the  $p_\beta$  but with any stronger conditions, as long as they still “cohere” in the way that the  $p_{\beta_i}$  cohere:

**LEMMA 5.46.** *Let  $(M_\alpha, p_\alpha)_{\alpha \in S}$  be a  $\Delta$ -system,  $\beta_0 < \beta_1 < \beta_2 < \beta_3$  in  $S$ , and  $s_i \leq p_{\beta_i}$  for  $i = 0, 1, 2, 3$  such that:*

- $\text{dom}(s_i) \subseteq M_{\beta_i}$ .
- $s^* := s_i \restriction \beta_i$  is the same for all  $i$ .
- $s^*$  forces that the  $s_i(\beta_i)$  are the same for all  $i$ .

*(In the usual sense: The  $s_i(\beta_i)$  are continuously read from generics below  $\beta_0$  in the same way for each  $i < 4$ .)*

*Then there is condition stronger than all  $s_i$  forcing that  $\eta_{\tilde{\beta}_0} = \text{major}_{i=1,2,3}(\eta_{\tilde{\beta}_i})$  and thus  $q_{\beta_0} =^* \text{major}_{i=1,2,3}(q_{\beta_i})$ .*

**5.9.  $q_\beta$  is in the  $\beta + 1$  extension.** We now show that  $q_\beta$  can be assumed to be a  $P_\beta$ -name.

The following definitions, in particular everything concerning the notion of coherence, is used only in this section. In the rest of the paper, we will use from this section only Lemma 5.54, i.e., the fact that  $q_\beta \in V_{\beta+1}$ .

**REMARK.** Why do we introduce this (rather annoying) notion of coherence? Well, we would like to simultaneously construct something like  $s_i \leq p_{\beta_i}$  where each  $s_i$  ends up in  $M_{\beta_i}$ . We cannot directly do this in  $M_{\beta_0}$ , as  $M_{\beta_0}$  does not know about, e.g.,  $\beta_1$ . So instead, we construct four different  $s'_i \leq p_{\beta_0}$  in  $M_{\beta_0}$  in such a way (a “coherent” way) and use  $s_i := h_{\beta_0, \beta_i}^*(s'_i)$ .

Let us for now (until Lemma 5.54) fix an arbitrary  $\Delta$ -system  $(M_\beta, p_\beta)_{\beta \in S}$  as well as  $\beta_0 < \beta_1 < \beta_2 < \beta_3$  in  $S$ . For notational convenience, set

$$\beta := \beta_0.$$

DEFINITION 5.47.

- $\bar{q} = (q_i)_{i < 4}$  in  $M_\beta$  is called coherent, if each  $q_i$  is stronger than  $p_\beta$  and  $q_i \restriction (\beta + 1)$  is the same for all  $i < 4$ .
- If  $\bar{q}$  is coherent, then  $\bigwedge_{i < 4} h_{\beta, \beta_i}^*(q_i)$  is a valid condition in  $P$ , and we call it  $q^*$ .  
I.e.,  $q^*$  is the union of the copies of  $q_i$  in  $M_{\beta_i}$ ; and the copy for  $q_0$  is just  $q_0$ .  
 $r \in P$  is called coherent, if  $r = q^*$  for some coherent  $\bar{q} \in M_\beta$ .

FACTS.

- The  $p_{\beta_i}$  are coherent, more correctly:  
The condition  $\bigwedge_{i \in 4} p_{\beta_i}$  is coherent; equivalently: The tuple  $(h_{\beta, \beta_i}^{*-1}(p_{\beta_i}))_{i < 4}$  is coherent.
- Any coherent  $r$  is stronger than  $\bigwedge_{i < 4} p_{\beta_i}$ .
- If  $\bar{q}$  is coherent,  $r_i \leq q_i$  in  $M_\beta$  for  $i < 4$ , and  $r_i \restriction \beta_i$  is the same for all  $i < 4$ , then  $\bigwedge_{i < 4} h_{\beta, \beta_i}^*(r_i)$  is (a valid condition and) compatible with  $q^*$ .
- $r \in P$  is coherent iff:  $\text{dom}(r) \subseteq \bigcup_{i < 4} M_{\beta_i}$ ,  $r \restriction (\mu \cap M_{\beta_i}) \in M_{\beta_i}$  is stronger than  $p_{\beta_i}$ , and each  $r(\beta_i)$  is forced to be the same condition.  
In that case,  $r = q^*$  for  $q_i := h_{\beta, \beta_i}^{*-1}(r_i)$  and  $r_i := r \restriction (\mu \cap M_{\beta_i})$ .

LEMMA 5.48. If  $r$  is coherent, then it can be strengthened<sup>11</sup> to force<sup>12</sup>  $g_{\beta_0} = \text{major}_{i=1,2,3} g_{\beta_i}$ .

PROOF. This follows from Lemma 5.46, using  $s_i := r \restriction (\mu \cap M_{\beta_i})$ .  $\dashv$

DEFINITION 5.49.

- $\bar{w} = (w_i)_{i < 4}$  is coherent, if  $w_i \in [\mu]^{<\lambda}$  is in  $M_\beta$  and  $w_i \cap (\beta + 1)$  is independent of  $i$ .

In the following we always assume that  $\bar{q}$  and  $\bar{w}$  are coherent.

- $\bar{q}$  fits  $(\bar{w}, \zeta)$ , if each  $q_i$  fits  $(w_i, \zeta)$ .
- $\bar{q}$  is  $(\bar{w}, \zeta)$ -canonical, if each  $q_i$  is  $(w_i, \zeta)$ -canonical.
- $\bar{r} \leq_{\bar{w}, \zeta}^+ \bar{q}$  means:  $\bar{r}$  is coherent, and  $r_i \leq_{w_i, \zeta}^+ q_i$  for all  $i < 4$ .
- $\bar{x} = (x_i)_{i < 4}$  is defined to be in  $\text{poss}(\bar{q}, \bar{w}, \zeta)$  if  $x_i \in \text{poss}(q_i, w_i, \zeta)$  and  $x_i \restriction \beta$  is independent of  $i$ . Such a  $\bar{x}$  will be called coherent possibility.

(Note that the  $x_i(\beta)$  in a coherent possibility can be different for different  $i < 4$ . Also note that such an  $\bar{x}$  is automatically in  $M_\beta$ , which is  $<\lambda$ -closed.)

Note that if  $\bar{r} \leq_{\bar{w}, \zeta}^+ \bar{q}$  and  $\bar{q}$  is  $(\bar{w}, \zeta)$ -canonical, then  $\bar{r}$  and  $\bar{q}$  have the same coherent  $(\bar{w}, \zeta + 1)$ -possibilities (see Fact 5.29(1)).

Several of the previous constructions result in coherent 4-tuples when applied to coherent 4-tuples. In particular:

<sup>11</sup>To a condition that will generally not be coherent.

<sup>12</sup>Here we write  $\beta_0$  instead of  $\beta$  to stress the interaction with  $\beta_1, \dots, \beta_3$ , but recall that  $\beta := \beta_0$ .



LEMMA 5.50.

- (1) Assume  $(\bar{q}^j)_{j \in \delta}$  is a sequence of coherent 4-tuples such that, for each  $i < 4$ , the  $i$ -part  $(q_i^j)_{j \in \delta}$  satisfies the assumptions of Lemma 5.18.

Then for each  $i$ , the lemma (in  $M_\beta$ ) gives us a limit  $r_i$ , which we call  $q_i^\delta$ .

We can choose the  $q_i^\delta$  so that they form a coherent 4-tuple.

- (2) The same applies to Lemma 5.19. That is, we can get a coherent fusion limit from a  $\lambda$ -sequence of coherent tuples.
- (3) Assume  $\bar{p}$  fits  $(\bar{w}, \zeta)$ , and  $\alpha_i \in \mu$  such that  $w'_i := w_i \cup \{\alpha_i\}$  is coherent. Then there is a  $\xi \geq \zeta$  and a  $\bar{q} \leq_{\bar{w}, \xi}^+ \bar{p}$  which fits  $(\bar{w}', \xi)$  and is  $(\bar{w}', \xi)$ -canonical.
- (4) Assume  $\bar{q}$  is coherent and (for simplicity)  $(\bar{w}, \zeta)$ -canonical with  $\beta \in w_i$  (which is independent of  $i < 4$ ), and  $\tau_i$  are names of ordinals. Then there is an  $\bar{r} \leq_{\bar{w}, \zeta}^+ \bar{q}$  such that  $\bar{r}$  is  $(\bar{w}, \zeta + 1)$ -decided by  $\bar{r}$ .

By this we mean that  $\tau_i$  is  $(w_i, \zeta + 1)$ -decided by  $r_i$  for all  $i < 4$ .

PROOF. For the first items, we just have to look at the proofs of the according lemmas (For (3) this is Preliminary Lemmas 5.23 and 5.24) and note that coherent input gives us coherent output. In the following we will prove (4). We work in  $M_\beta$ .

Enumerate all coherent possibilities as  $(\bar{x}_k)_{k \in K}$ . Set  $\bar{r}^0 := \bar{q}$ . We now construct  $\bar{r}^{k+1}$  from  $\bar{r} := \bar{r}^k$  where we assume  $\bar{r}^k \leq_{\bar{w}, \zeta}^+ \bar{q}$ .

- Find  $s_0$  stronger than  $r_0$  and extending  $x_0$ , deciding  $\tau_0$ .
- $s^* := (s_0 \upharpoonright \beta) \wedge r_1$  is stronger than  $r_1$ , as  $\bar{r}$  is coherent. Strengthen  $s^*(\beta) = r_1(\beta) = r_0(\beta)$  to  $s_0(\beta)$ , but replace the trunk with  $x_1(\beta)$ . Then  $s^* \upharpoonright \beta$  forces that  $s^*(\beta) \leq r_1(\beta)$ , as  $x_1 \upharpoonright \beta = x_0 \upharpoonright \beta$  and as  $x_1(\beta)$  is guaranteed to be possible, because  $r_1$  is canonical. Further strengthen  $s^*$  (above  $\beta$ ) to extend (the rest of)  $x_1$ ; and then strengthen the whole condition once more to decide  $\tau_1$ . Call the result  $s_1$ .
- Do the same for  $i = 2$ , starting with  $s_1$ , resulting in  $s_2$ , and then for  $i = 3$ , starting with  $s_2$ , resulting in some  $s_3$ . So  $s_i \leq r_i$  extends  $x_i$  and decides  $\tau_i$ , and  $s_3 \upharpoonright \beta \leq s_i \upharpoonright \beta$  and  $s_3(\beta)$  is stronger than  $s_i(\beta)$  “above  $\zeta + 1$ .”
- We define  $r'_i \leq r_i$  as follows:  $\text{dom}(r'_i) = (\text{dom}(s_3) \cap \beta) \cup \text{dom}(s_i)$ . We define  $r'_i(\alpha)$  inductively such that  $r'_i \upharpoonright \alpha \leq_{w_i \cap \alpha, \zeta}^+ r_i$  forces that  $x_i \upharpoonright \alpha \triangleleft G$  implies  $s_i \upharpoonright \alpha \in G$ .

– For  $\alpha \leq \beta$ :

If  $s_3 \upharpoonright \alpha \notin G_\alpha$ , set  $r'_i(\alpha) = r_i(\alpha)$ . Assume otherwise. So  $s_3(\alpha)$  is defined and stronger than  $r_i(\alpha) = r_3(\alpha)$ . If  $\alpha \notin w_i$  (which implies  $\alpha < \beta$ ), set  $r'_i(\alpha) = s_3(\alpha)$ . Otherwise, use  $s_3(\alpha) \vee (r_3(\alpha) \upharpoonright \zeta + 1)$ , as in Lemma 5.8.

– For  $\alpha > \beta$ , we do the same but we use  $s_i$  instead of  $s_3$ . In more detail:

If  $s_i \upharpoonright \alpha \notin G_\alpha$ , set  $r'_i(\alpha) = r_i(\alpha)$ . Assume otherwise. If  $\alpha \notin w_i$ , set  $r'_i(\alpha) = s_i(\alpha)$ . Otherwise, use  $s_i(\alpha) \vee (r_i(\alpha) \upharpoonright \zeta + 1)$ .

We can use this  $\bar{r}'$  as  $\bar{r}^{k+1}$ : It is coherent,  $\bar{r}' \leq_{\bar{w}, \zeta}^+ \bar{r}^k$ , and  $r'_i$  decides  $\tau_i$  assuming  $x_i \triangleleft G$ .

Coherent tuples  $\bar{q}$  naturally define a  $P$ -condition  $q^*$ . However, we have to assume that  $\bar{q}$  is canonical to guarantee that coherent  $\bar{q}$  possibilities correspond to  $q^*$ -possibilities:

LEMMA 5.51. Assume  $\bar{q}$  and  $\bar{w}$  coherent. We set  $w^* := \bigcup_{i < 4} h_{\beta, \tau_i}^*(w_i)$ . Let  $\bar{x}$  be in  $\text{poss}(\bar{q}, \bar{w}, \zeta + 1)$ .



- (1)  $\bar{q}$  fits  $(\bar{w}, \zeta)$  iff  $q^*$  fits  $(w^*, \zeta)$ .
- (2)  $\bar{r} \leq_{\bar{w}, \zeta}^+ \bar{q}$  iff  $r^* \leq_{w^*, \zeta}^+ q^*$ .
- (3) Assume  $\bar{q}$  fits  $(\bar{w}, \zeta)$ . Then  $\bar{q}$  is  $(\bar{w}, \zeta)$ -canonical iff  $q^*$  is  $(w^*, \zeta)$ -canonical.
- (4) Assume that  $\bar{q}$  is  $(\bar{w}, \zeta)$ -canonical. Let  $x^*$  be the union of the  $h_{\beta, \beta_i}^*(x_i)$ . Then  $x^* \in \text{poss}(q^*, w^*, \zeta + 1)$ ; and every element of  $\text{poss}(q^*, w^*, \zeta + 1)$  is such an  $x^*$  for some  $\bar{x} \in \text{poss}(\bar{q}, \bar{w}, \zeta + 1)$ .
- (5) Assume that  $\bar{q}$  is  $(\bar{w}, \zeta)$ -canonical. Then  $\bar{q}(\bar{w}, \zeta + 1)$ -decides  $(\tau_i)_{i < 4}$  iff  $q^*(w^*, \zeta + 1)$ -decides all  $h_{\beta, \beta_i}^*(\tau_i)$ .

PROOF. Assume  $\alpha \in w_i$ . Set  $\alpha' := h_{\beta, \beta_i}^*(\alpha) \in w^*$  and  $q' := h_{\beta, \beta_i}^*(q_i)$ .

(1) Assume  $q_i, \alpha$  satisfy  $q_i \restriction \alpha \Vdash \zeta \in C^{q_i(\alpha)}$ . By absoluteness they satisfy it in  $M_\beta$ , so the  $h_{\beta, \beta_i}^*$ -images  $q', \alpha'$  satisfy it in  $M_{\beta_i}$ , which again is absolute; and  $q^* \restriction \alpha' \leq q' \restriction \alpha'$  forces that  $q^*(\alpha') = q'(\alpha')$ . For the other direction, assume (in  $M_\beta$ ) some  $s \leq q_i \restriction \alpha$  forces  $\zeta \notin C^{q_i(\alpha)}$ . Then  $h_{\beta, \beta_i}^*(s)$  is compatible with  $q^*$  and forces  $\zeta \notin C^{q_i'(\alpha')} = C^{q^*(\alpha')}$ .

In the same way we can show (2), as well as (5) and the trivial directions of (3) and (4). For example, if  $\bar{q}$  is  $(\bar{w}, \zeta)$ -canonical, then  $q^*$  is  $(w^*, \zeta)$ -canonical. For this, use the fact that every element  $y^* \in \text{poss}(q^*, w^*, \zeta + 1)$  “induces” a coherent possibility  $\bar{y}$  (which is true whether  $\bar{q}$  is canonical or not). And if additionally  $\bar{x} \in \text{poss}(\bar{q}, \bar{w}, \zeta + 1)$ , then  $x^* \in \text{poss}(q^*, w^*, \zeta + 1)$ ; and if each  $q_i$  forces that  $x_i \triangleleft G$  implies  $\tau_i = x^i$ , then  $q^*$  forces that  $x^* \triangleleft G$  implies  $h_{\beta, \beta_1}^*(\tau_i) = h_{\beta, \beta_1}^*(x^i)$ .

We omit the (also straightforward) proofs of the other directions of (3) and (4) (which we do not need in this paper).  $\dashv$

In the following, whenever we mention  $q^*$  or  $w^*$ , we assume  $\bar{w}, \bar{q}$  to be coherent and in  $M_\beta$ . We will (and can) use  $x^*$  only if  $\bar{q}$  additionally is canonical (otherwise  $x^*$  will generally not be a possibility for  $q^*$ ). In this case, every  $P$ -generic filter containing  $q^*$  will select an  $x^*$  for some coherent possibility  $\bar{x}$ .

LEMMA 5.52. Assume  $\bar{q}$  is coherent,  $\sigma_i$  are  $P$ -names in  $M_\beta$  for elements of  $2^\lambda$ , and<sup>13</sup>  $q_0 \Vdash \bar{q}_0 \notin V_{\beta+1}$ . Then there is a coherent  $\bar{r} \leq \bar{q}$ , and sequences  $(\zeta^j)_{j \in \lambda}$  and  $(\bar{w}^j)_{j \in \lambda}$  such that  $\bar{r}$  is  $(\bar{w}^j, \zeta^j)$ -canonical for all  $j$ , and for all  $\bar{x} \in \text{poss}(\bar{r}, \bar{w}^j, \zeta^j + 1)$  there is some  $\ell \in I^*(\zeta^j, < \zeta^{j+1})$  and  $\bar{b} = (b_i)_{i < 4}$ , with  $b_i \in 2$ , violating majority<sup>14</sup> such that for all  $i < 4$

$$r_i \Vdash x_i \triangleleft G \rightarrow \sigma_i(\ell) = b_i.$$

As the  $p_{\beta_i}$  are coherent, we can apply the lemma to  $\sigma_i := q_\beta$  (for all  $i$ ) and get:

COROLLARY 5.53. If  $p_\beta \Vdash q_\beta \notin V_{\beta+1}$ , then there is a coherent  $r^* \leq \bigwedge_{i < 4} p_{\beta_i}$  forcing that

$$\neg (q_{\beta_0} =^* \text{major}_{i=1,2,3}(q_{\beta_i})).$$

PROOF OF THE LEMMA. We will construct (in  $M_\beta$ ), by induction on  $j \in \lambda$ ,  $\zeta^j, \bar{w}^j$  and  $\bar{r}^j$  with  $r_i^0 = q_i$ , such that the following holds:

<sup>13</sup>As usual,  $V_{\beta+1}$  denoted the  $P_{\beta+1}$ -extension.

<sup>14</sup>That is,  $b_0 = 1 - \text{major}_{i=1,2,3}(b_i)$ .

- (1)  $\bar{r}^j$  is coherent.
- (2)  $\bar{w}^j$  is coherent, for each  $i < 4$  the  $w_i^j$  are increasing with  $j$ , and their union covers  $\bigcup_{j \in \lambda} \text{dom}(r_i^j)$ .
- (3)  $\bar{r}^j$  is  $(\bar{w}^j, \zeta^j)$ -canonical.
- (4)  $\bar{r}^k \leq_{\bar{w}^j, \zeta^j}^+ \bar{r}^j$  for  $j < k$ .
- (5) If  $\bar{x} \in \text{poss}(\bar{r}^j, \bar{w}^j, \zeta^j + 1)$ , then there is an  $\ell \in I^*(\zeta^j, \zeta^{j+1})$  and a  $b \in 2$  such that for at least two  $i_1, i_2$  in  $\{1, 2, 3\}$ ,  $r_i^{j+1}$  forces that  $x_i \triangleleft G$  implies

$$\mathfrak{g}_0(\ell) = 1 - b, \quad \mathfrak{g}_{i_1}(\ell) = b, \quad \mathfrak{g}_{i_2}(\ell) = b. \tag{*}$$

Then we take the usual fusion limits, as in Lemma 5.50(2), and are done.

For limits  $j$ , let  $\bar{r}'$  be a (coherent) limit of  $(\bar{r}^{j'})_{j' < j}$ , and set  $\zeta^* := \sup_{j' < j} (\zeta^{j'})$  and  $w_i^* := \bigcup_{j' < j} w_i^{j'}$  for each  $i < 4$ . Note that  $\bar{r}'$  fits  $(\bar{w}^*, \zeta^*)$ . Then we can find coherent  $\bar{r}^* \leq_{\bar{w}^*, \zeta^*}^+ \bar{r}'$  which is  $(\bar{w}^*, \zeta^*)$ -canonical, as in Lemma 5.50(3).

In successor cases  $j = j' + 1$  set  $(\bar{r}^*, \bar{w}^*, \zeta^*) := (\bar{r}^{j'}, \bar{w}^{j'}, \zeta^{j'})$ .

In any case we want to construct  $\bar{r}^j$ ,  $\bar{w}^j$ , and  $\zeta^j$ .

Enumerate  $\text{poss}(\bar{r}^*, \bar{w}^*, \zeta^* + 1)$  as  $(\bar{x}^k)_{k \in K}$ .

We define  $\bar{s}^k$  for  $k \leq K$ , with  $\bar{s}^0 := \bar{r}^*$  and, as usual, taking (coherent) limits at limits, such that:

- $\bar{s}^k$  is coherent.
- $\bar{s}^\ell \leq_{\bar{w}^*, \zeta^*}^+ \bar{s}^k$  for  $k < \ell < K$ . (This implies that  $\bar{s}^k$  is  $(\bar{w}^*, \zeta^*)$ -canonical.)
- There is a  $\zeta^k$  and an  $\ell \in I^*(\zeta^*, \zeta^k)$  and a  $b \in 2$  such that

$$s_0^{k+1} \Vdash x_0^k \triangleleft G \rightarrow \mathfrak{z}_0(\ell) = 1 - b \quad \text{and} \quad (\exists \geq^2 i \in \{1, 2, 3\}) s_i^{k+1} \Vdash x_i^k \triangleleft G \rightarrow \mathfrak{z}_i(\ell) = b. \tag{**}$$

Assume we can construct these  $\bar{s}^k, \zeta^k$  for all  $k \in K$ , then let  $\bar{s}^K$  be again a (coherent) limit. We set  $w_i^j := w_i^* \cup \{\alpha_j\}$  such that  $\bar{w}^j$  is coherent (and such that, by bookkeeping, all elements of  $\text{dom}(p_i^j)$  will be eventually covered), and find some  $\zeta^j > \sup_{k \in K} (\zeta^k)$  and  $\bar{r}^j \leq_{\bar{w}^*, \zeta^*}^+ \bar{r}^*$  which is  $(\bar{w}^j, \zeta^j)$ -canonical, again as in Lemma 5.50(3). Then  $\bar{r}^j, \bar{w}^j$  and  $\zeta^j$  are as required.

So it remains to construct, for  $k \in K$ ,  $\bar{s}^{k+1}$  and  $\zeta^k$ , which we will do in the rest of the proof. Set  $\bar{s} := \bar{s}^k, \bar{x} := \bar{x}^k, \bar{w} := \bar{w}^*,$  and  $\zeta := \zeta^*$ . Recall that  $\bar{s}$  is  $(\bar{w}, \zeta)$ -canonical,  $\bar{x} \in \text{poss}(\bar{s}, \bar{w}, \zeta)$ , and we are looking for  $\bar{s}^{k+1} \leq_{\bar{w}, \zeta}^+ \bar{s}$  which satisfies (\*\*) for  $\bar{x}$ .

Set  $s_i' := s_i \wedge x_i$ . It is enough to construct  $t_i \leq s_i'$  such that:

- Both  $t_i \restriction \beta$  and  $t_i(\beta) \restriction (\lambda \setminus \zeta + 1)$  are independent of  $i$ .
- $t_0 \Vdash \mathfrak{z}_0(\ell) = 1 - b$  and  $(\exists \geq^2 i \in \{1, 2, 3\}) t_i \Vdash \mathfrak{z}_i(\ell) = b$ .

Then we can define  $\bar{s}^{k+1}$  in the usual way:  $\text{dom}(s_i^{k+1}) = \text{dom}(t_i)$  (and we can assume  $\text{dom}(s_i) = \text{dom}(t_i)$ , by using trivial conditions). For  $\alpha \in \text{dom}(t_i)$ , if  $t_i \restriction \alpha \notin G_\alpha$  then set  $s_i^{k+1}(\alpha)$  to be  $s_i(\alpha)$ , otherwise  $t_i(\alpha) \vee (s_i(\alpha) \restriction \zeta + 1)$  if  $\alpha \in w_i$  and  $t_i(\alpha)$  otherwise. The resulting  $\bar{s}^{k+1} \leq_{\bar{w}, \zeta}^+ \bar{s}$  is coherent and  $s_i^{k+1}$  forces that  $x_i \triangleleft G$  implies  $t_i \in G$ .

We have to introduce more notation: Fix  $j \neq i$ , and  $a \leq s_j'$  and  $b \leq s_i' \restriction \beta + 1$  (in  $P_{\beta+1}$ ) such that  $b \restriction \beta \leq a$  and  $b \restriction \beta$  forces that  $b(\beta)$  is stronger than  $a(\beta)$  above  $\zeta$

(i.e.,  $b \restriction \beta \Vdash (\forall \xi > \zeta) b(\beta)(\xi) \supseteq a(\beta)(\xi)$ ). Then we define  $b^{[j]} \wedge a$  by

$$(b^{[j]} \wedge a)(\alpha)(\xi) = \begin{cases} b(\alpha)(\xi), & \text{if } \alpha < \beta, \\ x_j(\beta)(\xi), & \text{if } \alpha = \beta \text{ and } \xi \leq \zeta, \\ b(\beta)(\xi), & \text{if } \alpha = \beta \text{ and } \xi > \zeta, \\ a(\alpha)(\xi), & \text{otherwise.} \end{cases}$$

Note that  $b^{[j]} \wedge a$  is stronger than  $a$ , but generally not stronger than  $b$ .

By our assumption,  $q_0$  and therefore  $s'_0$  forces  $\mathcal{G}_0 \notin V_{\beta+1}$ . So in an intermediate model  $V[G_{\beta+1}]$ , there is some  $\ell \in I^*(>\zeta)$  such that  $s'_0/G_{\beta+1}$  does not decide  $\mathcal{G}_0(\ell)$ . Back in  $V$ , fix some  $b_0 \leq s'_0 \restriction (\beta+1)$  in  $P_{\beta+1}$  which determines this  $\ell$ .

Find  $r'_1 \leq b_0^{[1]} \wedge s'_1$  which determines  $\mathcal{G}_1(\ell)$  to be  $j_1$  for some  $j_1 \in 2$ . Find  $r'_2 \leq (r'_1 \restriction \beta+1)^{[2]} \wedge s'_2$  which determines  $\mathcal{G}_2(\ell)$  to be  $j_2$ ; analogously find  $r'_3 \leq (r'_2 \restriction \beta+1)^{[3]} \wedge s'_3$  which determines  $\mathcal{G}_3(\ell)$  to be some  $j_3$ . Let  $j \in 2$  be equal to at least two of  $j_1, j_2, j_3$ .

Set  $p := (r'_3 \restriction \beta+1)^{[0]} \wedge s'_0$ . In any  $P_{\beta+1}$ -extension honoring  $p \restriction \beta+1$ ,  $\mathcal{G}_0(\ell)$  is not determined by  $p/G_{\beta+1}$ , i.e., there is a  $t_0 \leq p$  forcing that  $\mathcal{G}_0(\ell) = 1 - j$ .

We now set and  $t_i := (t_0 \restriction \beta+1)^{[i]} \wedge r'_i$  for  $i = 1, 2, 3$ . Note that  $t_i \leq r'_i \leq s'_i$  extends  $x_i$  and forces  $\mathcal{G}_i(\ell)$  to be  $1 - j$  if  $i = 0$  and to be  $j$  for at least two  $i$  in  $\{1, 2, 3\}$ .  $\dashv$

We can now easily show:

LEMMA 5.54. *For all but non-stationary many  $\beta \in S_{\lambda^+}^\mu$*

$$p_* \Vdash \mathcal{G}_\beta \in V_{\beta+1}.$$

PROOF. We started in this section with an arbitrary  $\Delta$ -system and showed that Corollary 5.53 and Lemma 5.48 hold for this system.

We now use a specific  $\Delta$ -system:

Assume towards a contradiction that on a non-stationary set  $S'$  there are  $p_\beta \leq p_*$  forcing  $\mathcal{G}_\beta \notin V_{\beta+1}$ . By strengthening we can assume that  $p_\beta$  canonically reads  $\mathcal{G}_\beta$ . Let  $M_\beta$  contain  $p_\beta$  and let  $S \subseteq S'$  be such that  $(M_\beta, p_\beta)_{\beta \in S}$  is a  $\Delta$ -system. Fix  $\beta_0 < \beta_1 < \beta_2 < \beta_3$  in  $S$ . By Corollary 5.53 we get a coherent  $\bar{r}$  stronger than  $\bar{p}$  such that  $r^* \Vdash \neg(\mathcal{G}_{\beta_0} =^* \text{major}_{i=1,2,3}(\mathcal{G}_{\beta_i}))$ . This contradicts Lemma 5.48.  $\dashv$

**5.10. Fixing the  $\Delta$ -system.** We now know that there is a stationary set  $S^0 \subseteq S_{\lambda^+}^\mu$  such that for all  $\beta \in S^0$ ,  $\mathcal{G}_\beta$  is forced (by  $p_*$ ) to be in  $V_{\beta+1}$  but not in  $V_\beta$  (see Lemmas 5.39 and 5.54).

For each  $\beta \in S^0$  there is a  $p'_\beta \leq p_*$  in  $P$  forcing that  $\mathcal{G}_\beta$  is equal to some  $P_{\beta+1}$ -name, call it  $\mathcal{G}_\beta^*$ , and we choose  $p_\beta \leq p'_\beta$  (we only have to strengthen the part below  $\beta+1$ ) which canonically reads  $\mathcal{G}_\beta^*$ .<sup>15</sup>

We now fix, as usual, for each  $\beta \in S^0$ , some elementary model  $M_\beta$  containing  $p_\beta$ , and fix  $S \subseteq S^0$  such that  $(M_\beta, p_\beta)_{\beta \in S}$  is a  $\Delta$ -system.

<sup>15</sup>So  $p_\beta \restriction \beta+1$  reads  $\mathcal{G}_\beta^*$ , but generally the whole  $p_\beta$  may be required to force  $\mathcal{G}_\beta = \mathcal{G}_\beta^*$ .

So  $p_{**} := p_\beta \restriction \beta \leq p_*$  is independent of  $\beta \in S$  (it is a  $P_\alpha$ -condition for some  $\alpha \in \Delta$ , independent of  $\beta \in S$ ); and  $q_\beta^*$  is read continuously by  $p_\beta \restriction \beta + 1$  via  $(w'_\zeta)_{\zeta \in E'}$  for some  $E' \subseteq \lambda$  club, with  $w'_\zeta \subseteq \beta + 1$ . As usual, due to homogeneity  $E'$  is independent of  $\beta \in S$ , and the  $w'_\zeta$  are independent of  $\beta$  apart from the shifting of the final coordinate  $\beta$  via the mapping  $h_{\beta_0, \beta_1}^*$ ; the same holds for the decision functions that map  $\text{poss}(p_\zeta, w'_\zeta, \zeta + 1)$  to  $q_\beta \restriction I^*(<\zeta + 1)$ .

Let  $E$  be the limit points of  $E'$ , and set  $w'_\zeta := \bigcup_{\nu < \zeta} w'_\nu$ . Then  $q_\beta \restriction I^*(<\xi)$  is  $(w'_\xi, \xi)$ -determined by  $p_\beta$  for all  $\xi \in E$ .

In the  $P_\beta$ -extension, only  $\eta_\beta$  remains undetermined, i.e., there are  $f_\xi$  for  $\xi \in E$  such that  $p_\beta/G_\beta$  forces  $q_\beta \restriction I^*(<\xi) = f_\xi(\eta_\beta \restriction I^*(<\xi))$ . The  $f_\xi$  are canonically read from  $p_\beta \restriction \beta$  in a way independent of  $\beta$  (due to homogeneity).

Recall that  $x \in \text{poss}(\tilde{p}, \xi)$  is equivalent to:  $x \in 2^{I^*(<\xi)}$  and  $x$  extends  $\eta_{\tilde{p}} \restriction I^*(<\xi)$ . So the domain of  $f_\xi$  is  $\text{poss}(\tilde{p}, \xi)$ .

To summarize:

FACT 5.55.  $(M_\beta, p_\beta)_{\beta \in S}$  satisfies:

- $p_\beta \restriction \beta =: p_{**} \leq p_*$  is a  $P_{\sup(\Delta)}$ -condition independent of  $\beta \in S$ .
- $p_\beta(\beta) =: \tilde{p}$  is a  $P_{\sup(\Delta)}$ -name independent of  $\beta \in S$ .
- There is a club-set  $E \subseteq \lambda$  and, for  $\xi \in E$ ,  $P_{\sup(\Delta)}$ -names  $f_\xi : \text{poss}(\tilde{p}, \xi) \rightarrow 2^{I^*(<\xi)}$  such that for all  $\beta \in S$  and  $\xi \in E$

$$p_\beta \Vdash q_\beta \restriction I^*(<\xi) = f_\xi(\eta_\beta \restriction I^*(<\xi)).$$

- If  $\beta \in S$ ,  $\tilde{x} \subseteq \lambda$  is a  $P_\beta$ -name,  $q \leq p_{**}$  in  $P_\beta$ , and  $q, \tilde{x}$  are in  $M_\beta$ , then we can find  $\alpha \in \Delta$  and  $p'_{**} \leq q$  in  $P_\alpha$  which continuously reads  $\tilde{x}$ ,  $\tau(\tilde{x})$ , and  $\tau^{-1}(\tilde{x})$  independently<sup>16</sup> of  $\beta$ .

The last item follows from Lemma 5.42; and we will use it several times: Before Corollary 5.59 we find  $p_{**}^2 \leq p_{**}$  to get names for  $U, F_\xi$  etc. that are independent of  $\beta$ ; before Lemma 5.63 we get  $p_{**}^3 \leq p_{**}^2$  to get independent names for some unions, intersections, and  $\tau$ -images; and finally after Corollary 5.70 we choose  $q \leq p_{**}^3$  to get an independent name for the generator  $f_{\text{gen}}$ .

**5.11. Local reading.** So we know that we can determine initial segments of  $q_\beta$  from initial segments of  $\eta_\beta$ , more specifically, we can determine  $\eta_\beta \restriction I$  from  $q_\beta \restriction I$  for  $I := I^*(<\xi)$ .

In this section we show that on unboundedly many disjoint intervals of the form  $A := I^*(\geq \xi, < \nu)$ , we can read  $q_\beta \restriction A$  from just  $\eta_\beta \restriction A$  (without having to use the  $\eta_\beta$ -values below  $A$ ).

The following definition (the notion of candidate) is only used in this section. In the rest of the paper we only need Corollary 5.59.

<sup>16</sup>This means:  $p'_{**} \in M_\gamma$  for all  $\gamma \in S$ , and there is a way (independent of  $\gamma \in S$ ) to continuously read  $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$  modulo  $p'_{**}$  from the generics below  $\alpha$ , and for all  $\gamma \in S$  we have that  $p'_{**} \wedge p_\gamma$  forces  $\tilde{y}_1 = \tilde{x}', \tilde{y}_2 = \tau(\tilde{x}')$  and  $\tilde{y}_3 = \tau^{-1}(\tilde{x}')$ , where  $\tilde{x}' := h_{\beta, \gamma}^*(\tilde{x})$ .

In the following, we work in  $V_\beta$ , the  $P_\beta$ -extension  $V[G_\beta]$  where we assume  $\beta \in S$  and  $p_{**} \in G_\beta$ .

DEFINITION 5.56. (In  $V_\beta$ )

- For  $A \subseteq \lambda$  and  $\bar{x} = (x_i)_{i < 4}$ ,  $x_i : A \rightarrow 2$ , we say  $\bar{x}$  honors majority above  $\zeta$ , if

$$x_0(\ell) = \text{major}_{i=1,2,3} x_i(\ell) \text{ for all } \ell \in A \cap I^*(\geq \zeta).$$

We say  $\bar{x}$  honors  $\tilde{p}$ , if each  $x_i$  is compatible with  $\eta^{\tilde{p}}$  (as partial functions).

- $\bar{x} = (x_i)_{i < 4}$  is a  $(\zeta_0, \zeta_1)$ -candidate, (for  $\zeta_0 \leq \zeta_1$  both in  $E$ ) if the  $x_i \in \text{poss}(\tilde{p}, \zeta_1)$  honor majority above  $\zeta_0$ .

(As elements of  $\text{poss}(\tilde{p}, \zeta_1)$  they automatically honor  $\tilde{p}$ .)

- If  $\bar{x}$  is a  $(\zeta_0, \zeta_1)$ -candidate, we say “ $\bar{y}$  extends  $\bar{x}$ ” if  $\bar{y}$  is a  $(\zeta_1, \zeta_2)$ -candidate<sup>17</sup> for some  $\zeta_2 \geq \zeta_1$  and each  $y_i$  extends  $x_i$ .

Equivalently,  $\bar{y} = \bar{x} \hat{\smallfrown} \bar{b}$  for some  $\bar{b}$ , with  $b_i : I^*(\geq \zeta_1, < \zeta_2) \rightarrow 2$ , which honors both majority and  $\tilde{p}$ .

- A  $(\zeta_0, \zeta_1)$ -candidate  $\bar{y}$  is “good,” if for every candidate  $\bar{z}$  of height  $\xi > \zeta_1$  that extends  $\bar{y}$  we have:

$$f_\xi(z_0)(\ell) = \text{major}_{i=1,2,3} f_\xi(z_i)(\ell) \text{ for all } \ell \in I^*(\geq \zeta_1, < \xi). \quad (*_1)$$

PRELIMINARY LEMMA 5.57. (In  $V_\beta$ .) Every candidate can be extended to a good candidate.

PROOF. Assume otherwise, i.e., there is a  $(\zeta', \zeta_0)$ -candidate  $\bar{x}$  which is a counterexample, which means:

Whenever  $\bar{y}$  is a  $(\zeta_0, \zeta_1)$ -candidate extending  $\bar{x}$  then there is a

$\xi > \zeta_1$  and a  $(\zeta_1, \xi)$ -candidate  $\bar{z}$  extending  $\bar{y}$  which violates  $(*_1)$ .

We now construct  $r_0 \leq \tilde{p}$  and, for  $i = 1, 2, 3$ ,  $Q_\beta$ -names  $r_i \leq \tilde{p}$ . All these conditions live on the same  $C^* \subseteq E$  with  $\min(C^*) = \zeta_0$ . The trunk of  $r_i$  is  $x_i$ .

We now construct inductively  $C^* \restriction \zeta$  and  $r_i \restriction \zeta$ .

Assume we have determined that  $\zeta \in C^*$  and we have constructed each  $r_i$  below  $\zeta$ . Set  $r_0(\zeta) := \tilde{p}(\zeta)$  and pick  $r_i(\zeta)$  as in (5.44), i.e., they have majority  $\eta_{\tilde{p}}$  and leave enough freedom to form a valid condition.

We will now construct the  $C^*$ -successor  $\xi$  of  $\zeta$ , together with  $r_i$  on  $I^*(> \zeta, < \xi)$ .

Enumerate all  $(\zeta_0, \zeta + 1)$ -candidates extending  $\bar{x}$  as  $(\bar{y}^k)_{k \in K}$ .

Let  $\bar{a}^0$  be the empty 4-tuple and set  $\xi_0 := \zeta + 1$ . We will construct, for  $k \in K$ ,  $\xi_k$  and some  $\bar{a}^k$  that honors majority and  $\tilde{p}$ , where  $\bar{a}_i^k$  has domain  $I^*(\geq \zeta + 1, < \xi_k)$  and extends  $\bar{a}_i^j$  if  $j < k$ .

If  $k$  is a limit, let  $\bar{a}^x$  be the (pointwise) union of  $\bar{a}^j$  with  $j < k$ , and set  $\xi_k := \sup_{j < k} (\xi_j)$ .

Assume we already have  $\bar{a}^j$ . Extend  $\bar{y}^j \hat{\smallfrown} \bar{a}^j$  to some candidate  $\bar{y}^j \hat{\smallfrown} \bar{a}^{j+1}$  of some height  $\xi_{j+1}$  in  $E$  such that

$$\bar{y}^j \hat{\smallfrown} \bar{a}^{j+1} \text{ violates } (*_1) \text{ for some } \ell \in I^*(\geq \xi_j, < \xi_{j+1}). \quad (*_3)$$

We can do this due to  $(*_2)$ .

<sup>17</sup>Or equivalently, a  $(\zeta_0, \zeta_2)$ -candidate.

So in the end we get some  $\xi > \zeta$  in  $E$  and  $\bar{b}^\xi$  with domain  $I^*(>\zeta, <\xi)$  honoring majority and  $\tilde{p}$  such that

for every  $(\zeta_0, \zeta + 1)$  – candidate  $\bar{y}$  extending  $\bar{x}$ ,  $\bar{y} \frown \bar{b}^\xi$  is a  $(\zeta_0, \xi)$  – candidate violating  $(*)_1$  for some  $\ell \in I^*(>\zeta, <\xi)$ . ( $*$ <sub>4</sub>)

We then define  $C^*$  below  $\xi + 1$  by adding only  $\xi$ , i.e.,  $\xi$  is the  $C^*$ -successor of  $\zeta$ . We extend the conditions  $r_i$  by  $b_i^\xi$  for  $i < 4$ . That is, we have  $\eta^{r_i}(\ell) = b_i^\xi(\ell)$ . This ends the construction of  $r_i \leq \tilde{p}$ .

Back in  $V$ , assume that  $(*)_2$  is forced by some  $q' \leq p_\beta \restriction \beta$ . Pick an increasing sequence  $\beta_i$  ( $i < 4$ ) in  $S$ . We take the union of  $q'$  and the  $p_{\beta_i}$ , call it  $s$ , and strengthen  $s(\beta_i) = \tilde{p}$  to  $r_i$ . The resulting condition  $s'$  forces the following:

- $\mathcal{a}_{\beta_i} \restriction I^*(<\xi) = f_\xi(\eta_{\beta_i} \restriction I^*(<\xi))$  for all  $\xi \in C^*$ . This is because  $s' \leq p_{\beta_i}$  (cf. Fact 5.55).
- The  $\eta_{\beta_i}$  honor majority above  $\zeta_0$ . This is because for all  $\zeta \in C^*$ , the  $r_i(\zeta)$  are chosen as in (5.44) and therefore honor majority; and for  $\zeta \in \lambda \setminus (C^* \cup \zeta_0)$  we use values  $\bar{b}$  which honor majority.
- Accordingly, the  $\mathcal{a}_{\beta_i}$  honor majority above some  $\gamma < \lambda$  (cf. Lemma 5.43(1)). Pick  $\zeta_1$  such that  $\sup(I^*(<\zeta_1)) > \gamma$ .
- So for all  $\xi > \zeta_1$  the  $f_\xi(\eta_{\beta_i} \restriction I^*(<\xi))$  honor majority above  $\zeta_1$ .
- Pick some  $\zeta > \zeta_0, \zeta_1$  in  $C^*$  with  $C^*$ -successor  $\xi$ . By construction of the  $r_i, \eta_{\beta_i} \restriction I^*(\geq \zeta + 1, <\xi)$  is  $b_i^\xi$ . As  $r_i$  extends  $x_i$ ,  $\bar{y} := \eta_{\beta_i} \restriction I^*(<\zeta + 1)$  is a  $(\zeta_0, \zeta + 1)$ -candidate extending  $\bar{x}$ . So by  $(*)_4$ , the  $\eta_{\beta_i} \restriction I^*(<\xi)$  violate  $(*)_1$  at some  $\ell \in I^*(>\zeta, <\xi)$ , a contradiction. ⊥

Let  $U \subseteq \lambda$  be club. Set  $U^{\text{odd}}$  to be the odd elements<sup>18</sup> of  $U$ . For  $\xi \in U^{\text{odd}}$  with  $U$ -successor  $v$ , set

$$A_\xi^U := I^*(\geq \xi, <v)$$

LEMMA 5.58. (In  $V_\beta$ .) There is an  $r_0 \leq \tilde{p}$ , a club  $U \subseteq C^{r_0} \subseteq E$ , and, for  $\xi \in U^{\text{odd}}$ , an  $F_\xi : 2^{A_\xi^U} \rightarrow 2^{A_\xi^U}$  such that:

- $r_0 \wedge p_\beta / G_\beta$  forces that  $F_\xi(\eta_\beta \restriction A_\xi^U) = \mathcal{a}_\beta \restriction A_\xi^U$ .
- $F_\xi$  is not constant: There are, for  $k = 0, 1$ ,  $z_\xi^k$  in  $\text{poss}(r_0, I^*(<v))$  and  $\ell_\xi \in A_\xi^U$  such that  $F_\xi(z_\xi^k \restriction A_\xi^U)(\ell_\xi) = k$ . (Again,  $v$  is the  $U$ -successor of  $\xi$ .)

(Note: Only those elements of  $2^{A_\xi^U}$  that are compatible with  $r_0$  are relevant as arguments for  $F_\xi$ .)

PROOF. We construct  $r_i$  for  $i < 4$  and  $U$  iteratively;  $C^{r_i}$  will be independent of  $i$ , call it  $C$ .

<sup>18</sup>That is, if  $(u_\alpha)_{\alpha < \lambda}$  is the canonical enumeration of  $U$ , then  $\zeta \in U$  is in  $U^{\text{odd}}$  if  $\zeta = u_{\delta+2n+1}$  for  $\delta$  a limit (or 0) and  $n \in \omega$ .

All  $r_i$  have the same trunk as  $\tilde{p}$ ; i.e.,  $\min(C) = \min(C^{\tilde{p}}) =: \zeta_0$  and  $r_i \upharpoonright \zeta_0 := \tilde{p} \upharpoonright \zeta_0$ . We also set  $\min(U) = \zeta_0$ .

For all  $\zeta \in C$ , we choose some  $r_i^*(\zeta)$  as in (5.44), i.e.,  $r_0^*(\zeta) = \tilde{p}(\zeta)$ , and the  $r_i^*(\zeta)$  for  $i = 1, 2, 3$  are such that the majority of their generics would be the  $r_0^*(\zeta)$ -generic.

Assume that we already know that some  $\zeta$  is in  $U$  (which is a subset of  $C$ ), and that we know  $r_i \upharpoonright \zeta$  for  $i < 4$ .

We now construct the  $U$ -successor  $\xi$  of  $\zeta$ ,  $C \upharpoonright [\zeta, \xi]$ , and  $r_i(v)$  for  $i < 4$  and  $v \in [\zeta, \xi]$ .

- Even case: If  $\zeta$  is an even element of  $U$ , we start with  $r_i(\zeta) := r_i^*(\zeta)$ , but then add a “shield,” or “isolator” above  $\zeta$ : As in the previous proof, we iterate over all  $\zeta + 1$ -candidates  $\bar{y}^j$ , but but in  $(*_3)$ , instead of violating  $(*_1)$  for some  $\ell$ , we demand that  $\bar{y}^j \frown \bar{z}^{j+1}$  is good. (We already know that every candidate can be extended to a good one.) Accordingly, we get some  $\xi > \zeta$  and  $\bar{b}^\xi$  with domain  $I^*(>\zeta, <\xi)$  (and honoring majority and  $\tilde{p}$ ) such that  $\bar{y} \frown \bar{b}^\xi$  is good for every candidate  $\bar{y}$  of height  $\zeta + 1$ ; i.e.:

If  $\bar{z}$  is a  $(\zeta + 1, v)$ -candidate whose restriction to  $I^*(>\zeta, <\xi)$  is  $\bar{b}^\xi$ , then the  $f_v(z_i)$  honor majority above  $\xi$ . (\*'4)

We now let this  $\xi$  be the successor of  $\zeta$  in both  $C$  and  $U$  (and extend each  $p_i(\zeta)$  by  $b_i$ ).

- Odd case: Now assume  $\zeta$  is odd in  $U$ . Then we choose some  $\xi > \zeta$  in  $C^{\tilde{p}}$  large enough such that there are, for  $k = 0, 1$ ,  $z_\xi^k$  in  $\text{poss}(\tilde{p}, \xi)$  compatible with all the  $r_0$  constructed so far, such that the  $f_\xi(z_\xi^k)(\ell) = k$  for some  $\ell > I^*(<\zeta)$ . (Such  $\xi$  and  $\ell$  have to exist as  $q_\beta$  is not in  $V_\beta$ .)

We let  $C$  restricted to  $[\zeta, \xi]$  be the same as  $C^{\tilde{p}}$ , and set  $r_i(v) := r_i^*(v)$  for  $v \in C \cap [\zeta, \xi]$ . (For  $\zeta \in [\zeta, \xi] \setminus C$  there is no freedom left, i.e.,  $\tilde{p}(\zeta)$  is already completely determined, so the only choice for any  $r \leq \tilde{p}$  is  $r(\zeta) = \tilde{p}(\zeta)$ .)

This ends the construction of  $U$  and of  $r_i$  (for  $i < 4$ ).

Pick  $\xi \in U^{\text{odd}}$ , let  $\zeta$  be the  $U$ -predecessor and  $v$  the  $U$ -successor. We have to show that we can determine (modulo  $p_\beta$ )  $q_\beta \upharpoonright I^*(\geq \xi, < v)$  from  $\eta_\beta \upharpoonright I^*(\geq \xi, < v)$  alone. (We already know that we can determine it from  $\eta_\beta \upharpoonright I^*(< v)$ .)

Fix any  $z_\xi^\zeta \in \text{poss}(r_0, \zeta + 1)$ . Let  $x_0 \in \text{poss}(r_0, v)$ . In particular  $x_0$  extends  $b_0^\zeta$ . For  $i = 1, 2, 3$ , let  $x_i$  be the copy of  $x_0$  with the initial segment  $x_0 \upharpoonright \xi$  replaced by  $z_\xi^\zeta \frown b_i^\zeta$ . Note that  $\bar{x}$  is a candidate extending  $\bar{b}^\xi$ . Accordingly the  $f_v(x_i)$  honor majority above  $\xi$ . So we can define

$$F_\xi(x_0 \upharpoonright A_\xi^U) := \text{major}_{i=1,2,3} f_v(x_i) \upharpoonright A_\xi^U = f_v(x_0) \upharpoonright A_\xi^U.$$

This is well-defined,<sup>19</sup> and  $r_0 \wedge p_\beta / G_\beta$  forces that  $F_\xi(x_0 \upharpoonright A_\xi^U) = q_\beta \upharpoonright A_\xi^U$ . ⊢

We now summarize this lemma, which was shown in  $V_\beta$  for some  $\beta \in S$ , from the point of view of the ground model. The lemma only uses the parameters  $\eta_\beta$  and  $q_\beta$

<sup>19</sup> Assume  $y$  and  $x$  in  $\text{poss}(r_0, v)$  are the identical restricted to  $A_\xi^U$ . Then  $y$  defines the same  $(x_i)_{i=1,2,3}$  and thus the same  $F_\xi$ .

(and  $\tilde{p}$ , which is just  $\eta_{\beta}(\beta)$ ), so by absoluteness  $M_{\beta}$  knows that the Lemma is forced by  $p_{**}$ . Accordingly, we can find  $P_{\beta}$ -names for  $U, F_{\xi}$ , etc. in  $M_{\beta}$ . Using the last item of Fact 5.55, we can strengthen  $p_{**}$  to  $p_{**}^2$  to canonically read these names:

**COROLLARY 5.59.** *There is an  $\alpha \in \Delta$ , a  $p_{**}^2 \leq p_{**}$  in  $P_{\alpha}$  and  $P_{\alpha}$ -names for: A condition  $r_0 \leq \tilde{p}$ , a set  $U$ , and a sequence  $(F_{\xi}, z_{\xi}^0, z_{\xi}^1, \ell_{\xi}^0, \ell_{\xi}^1)_{\xi \in U}$ , such that the following holds for all  $\beta \in S$ , where we set*

$p_{\beta}^+$  to be the condition  $p_{**}^2 \wedge p_{\beta}$  where we strengthen  $p_{\beta}(\beta)$  to  $r_0$ .

- (1)  $\alpha$ , the condition  $p_{**}^2$  and all the names are in  $M_{\beta}$ .
- (2)  $p_{**}^2 \Vdash U \subseteq C^{r_0} \subseteq \lambda$  club.
- (3) For  $k = 0, 1$ :  $p_{**}^2 \Vdash \forall \xi \in U^{\text{ODD}} \left( z_{\xi}^k \in \text{poss}(r_0, I^*(<v)) \ \& \ \ell_{\xi} \in A_{\xi}^U \ \& \ F_{\xi}(z_{\xi}^k \restriction A_{\xi}^U)(\ell_{\xi}) = k \right)$ .
- (4)  $p_{\beta}^+ \Vdash (\forall \xi \in U^{\text{ODD}}) F_{\xi}(\eta_{\beta} \restriction A_{\xi}^U) = \underline{a}_{\beta} \restriction A_{\xi}$ , where we define

$A_{\xi}$  to be  $I^*(\geq \xi, <v)$  with  $v$  the  $U$ -successor of  $\xi$ .

**5.12. Finding the generator.** In this section we use these  $p_{**}^2, r_0, (F_{\xi}, z_{\xi}^0, z_{\xi}^1, \ell_{\xi}^0, \ell_{\xi}^1)_{\xi \in U}$ .

We start working in  $V_{\beta} = V[G_{\beta}]$ , where we assume  $p_{**}^2 \in G_{\beta}$ .

Let  $\xi \in U^{\text{ODD}}$  and  $v$  its  $U$ -successor. Set

$$\begin{aligned} A_{\xi} &:= I^*(\geq \xi, <v), & A_{\xi}^? &:= A_{\xi} \setminus \text{dom}(\eta^{r_0}), \\ \text{ODD} &:= \bigcup_{\xi \in U^{\text{ODD}}} A_{\xi}, & \text{ODD}^? &:= \bigcup_{\xi \in U^{\text{ODD}}} A_{\xi}^? = \text{ODD} \setminus \text{dom}(\eta^{r_0}). \end{aligned} \tag{5.60}$$

For  $F_{\xi}$  it is enough to use  $\eta_{\beta} \restriction A_{\xi}^?$  as input (the part in  $A_{\xi} \setminus A_{\xi}^?$  is determined anyway by  $r_0$ ), and every element of  $2^{A_{\xi}^?}$  is compatible with  $r_0$  (and thus a possible input for  $F_{\xi}$ ). Identifying  $2^B$  and  $\mathcal{P}(B)$  as usual, we get

$$F_{\xi} : \mathcal{P}(A_{\xi}^?) \rightarrow \mathcal{P}(A_{\xi})$$

is such that  $p_{\beta}^+/G_{\beta}$  forces

$$F_{\xi}(\eta_{\beta} \cap A_{\xi}^?) = \underline{a}_{\beta} \cap A_{\xi}.$$

We now define

$$F : \mathcal{P}(\text{ODD}^?) \rightarrow \mathcal{P}(\text{ODD}) \quad \text{by} \quad x \mapsto \bigcup_{\xi \in U^{\text{ODD}}} F_{\xi}(x \cap A_{\xi}^?).$$

So in particular  $p_{\beta}^+/G_{\beta}$  forces that

$$F(\eta_{\beta} \cap \text{ODD}^?) = \underline{a}_{\beta} \cap \text{ODD}. \tag{5.61}$$



Note that for every  $z \subseteq \text{ODD}^?$  (in  $V_\beta$  that is) there is an  $r' \leq r_0$  forcing that  $\eta_\beta \cap \text{ODD}^? = z$ . ( $C' := U \setminus U^{\text{ODD}}$  is club, so it is enough to leave freedom at  $C'$  and we may assign arbitrary values everywhere else.)

Back in the ground model  $V$ , using the last item of Fact 5.55 again, we can strengthen  $p_{**}^2$  to  $p_{**}^3$  so that<sup>20</sup>

$p_{**}^3$  canonically reads each of the following (countably many) sets: (5.62)

- $(A_\xi)_{\xi \in U^{\text{ODD}}, \text{ODD}}, r_0, (A_\xi^?)_{\xi \in U^{\text{ODD}}, \text{ODD}^?}$  (actually, these are already read by  $r_{**}^2$ ).
- The closure of these sets under  $\pi, \pi^{-1}$ , finite unions, and finite intersections.

In particular, the (names for) all these sets are independent of  $\beta \in S$ , modulo  $p_{**}^3$ .<sup>21</sup>

LEMMA 5.63. (In  $V$ )  $p_{**}^3 \Vdash |\pi(\text{ODD}^?) \cap \text{ODD}| = \lambda$ .

PROOF. Let  $q \leq p_{**}^3$  in  $P_\beta$  be arbitrary. We have to show that  $q$  does not force (in  $P_\beta$ )  $|\pi(\text{ODD}^?) \cap \text{ODD}| < \lambda$ .

For  $\xi \in U^{\text{ODD}}$  and  $k = 0, 1$ , use  $r_0, p_\beta^+, z_\xi^k$  and  $\ell_\xi$  as in Corollary 5.59 and set  $b_\xi^k := z_\xi^k \cap A_\xi^?$ .

For  $k = 0, 1$ , set  $B^k := \bigcup_{\xi \in U^{\text{ODD}}} (b_\xi^k)$ . Note that  $F(B^1) \setminus F(B^0)$  contains  $\{\ell_\xi : \xi \in U^{\text{ODD}}\}$ , a set of size  $\lambda$ .

Pick increasing  $(\beta_i)_{i < 4}$  in  $S$  with  $\beta_0 = \beta$ . Set  $s := q \wedge \bigwedge_{i < 4} p_{\beta_i}^+ \in P$ .

Now for each  $i < 4$ , strengthen  $s(\beta_i)$  (i.e.,  $r_0$ ) as follows: At the even intervals in some way that together they honor majority; and at the odd intervals (where we do not have to leave freedom) to the value  $B^{\text{sgn}(i)}$  (where  $\text{sgn}(k) = 0$  for  $k = 0$  and 1 for  $k = 1, 2, 3$ ).

Accordingly, we have

$$\pi(\eta_{\beta_i}) \cap \text{ODD} = F(\eta_{\beta_i} \cap \text{ODD}^?) = F(B^{\text{sgn}(i)}),$$

or, when we split  $\pi(\eta_{\beta_i})$  into the parts in and out of  $\pi(\text{ODD}^?)$ :

$$\left( (\pi(\eta_{\beta_i}) \setminus \pi(\text{ODD}^?)) \cap \text{ODD} \right) \cup \left( \pi(\eta_{\beta_i}) \cap \pi(\text{ODD}^?) \cap \text{ODD} \right) =^* F(B^{\text{sgn}(i)}).$$

Now assume towards a contradiction that  $\pi(\text{ODD}^?) \cap \text{ODD} =^* \emptyset$ . Then we get

$$(\pi(\eta_{\beta_i}) \setminus \pi(\text{ODD}^?)) \cap \text{ODD} =^* F(B^{\text{sgn}(i)}). \tag{5.64}$$

But on the other hand we have

$$\begin{aligned} \eta_{\beta_0} \setminus \text{ODD}^? &= \text{major}_{i=1,2,3}(\eta_{\beta_i} \setminus \text{ODD}^?), \text{ so} \\ \pi(\eta_{\beta_0}) \setminus \pi(\text{ODD}^?) &=^* \pi(\eta_{\beta_0} \setminus \text{ODD}^?) = \pi(\text{major}_{i=1,2,3}(\eta_{\beta_i} \setminus \text{ODD}^?)) =^* \end{aligned}$$

<sup>20</sup>We can do this for  $\lambda$  many sets, of course; but we cannot assume, e.g., that  $\pi(z) \in V_\beta$  for all  $z \in V_\beta$ , let alone that each such  $\pi(z)$  is canonically read by  $p_{**}^3$ .

<sup>21</sup>But we need  $p_\beta^+$  to force that these names have anything to do with  $a_\beta$ .

$$=^* \text{major}_{i=1,2,3} (\pi(\eta_{\beta_i} \setminus \text{ODD}^?) ) =^* \text{major}_{i=1,2,3} (\pi(\eta_{\beta_i}) \setminus \pi(\text{ODD}^?)), \text{ and}$$

$$(\pi(\eta_{\beta_0}) \setminus \pi(\text{ODD}^?)) \cap \text{ODD} =^* \text{major}_{i=1,2,3} \left( (\pi(\eta_{\beta_i}) \setminus \pi(\text{ODD}^?)) \cap \text{ODD} \right).$$

Applying (5.64) to both sides of the last line, we get  $F(B^0) =^* \text{major}_{i=1,2,3} F(B^{\text{sgn}(i)}) = F(B^1)$ , a contradiction.  $\dashv$

Set

$$\tilde{X} := \text{ODD}^? \cap \pi^{-1}(\text{ODD}). \quad (5.65)$$

By choice of  $p_{**}^3$ ,  $\tilde{X}$  and  $\pi(\tilde{X})$  are canonically read by  $p_{**}^3$  (and independent of  $\beta$ ).

We now show that  $F(z) \cap \pi(\tilde{X}) = \pi(z)$  for  $z \subseteq \tilde{X}$ . Again, here we are talking about  $z \in V_\beta$ . To make that more explicit, let us formulate in the ground model  $V$ :

LEMMA 5.66. *For  $\beta \in S$ ,  $p_{**}^3 \Vdash_{P_\beta} \left( |\tilde{X}| = \lambda, \text{ and for all } z \subseteq \tilde{X}, p_\beta^+/G_\beta \Vdash \pi(z) =^* F(z) \cap \pi(\tilde{X}) \right)$ .*

(Note that, other than  $F(z)$ ,  $\pi(z)$  will generally not be in  $V_\beta$ , and we have to force with  $p_\beta^+/G_\beta$ .)

PROOF. Work in  $V_\beta$ .  $|\tilde{X}| = \lambda$  follows from Lemma 5.63, as  $\pi(\tilde{X}) =^* \pi(\text{ODD}^?) \cap \text{ODD}$ .

Set  $y := \eta_\beta \cap \text{ODD}^?$ . So by (5.61),  $p_\beta^+/G_\beta \leq r_0$  forces:  $F(y) = \pi(\eta_\beta) \cap \text{ODD}$ . As  $\pi(\tilde{X}) \subseteq^* \text{ODD}$ , we get  $F(y) \cap \pi(\tilde{X}) =^* \pi(\eta_\beta) \cap \pi(\tilde{X})$ . Then  $y \subseteq^* \pi^{-1}(\text{ODD})$  (or equivalently,  $y \subseteq^* \tilde{X}$ ) implies  $y =^* y \cap \pi^{-1}(\text{ODD}) = \eta_\beta \cap \tilde{X}$  and thus  $\pi(y) =^* \pi(\eta_\beta) \cap \pi(\tilde{X})$ . To summarize:

$$p_\beta^+/G_\beta \Vdash \left( y \subseteq^* \tilde{X} \rightarrow \pi(y) =^* F(y) \cap \pi(\tilde{X}), \text{ for } y := \eta_\beta \cap \text{ODD}^? \right). \quad (*)$$

Now back in  $V$  assume towards a contradiction that some  $q \leq p_\beta^+$  forces that the lemma fails, i.e., that  $\tilde{z} \subseteq \tilde{X}$  in  $V_\beta$  is a counterexample (in the final extension). By absoluteness, we can assume that  $q$  and  $\tilde{z}$  are in  $M_\beta$ , in particular  $\tilde{z}$  is a  $P_\beta$ -name in  $M_\beta$ . Strengthen  $q \upharpoonright \beta$  to canonically read  $\tilde{z}$ . So for every  $\beta' \in S$ ,  $h_{\beta,\beta'}^*(\tilde{z})$  will be evaluated in  $V_{\beta'}$  to the same  $z \subseteq \lambda$  as  $\tilde{z}$  in  $V_\beta$ .

Chose a  $\beta'$  above  $\text{supp}(q)$ . Then we can strengthen  $q \wedge p_{\beta'}$  at index  $\beta'$ , i.e.,  $r_0$ , to some  $r_1$  that forces  $\eta_{\beta'} \cap \text{ODD}^? = h_{\beta,\beta'}^*(\tilde{z})$ . (Recall that we can fix the values in the odd intervals, as the even intervals still form a club). Let  $G$  be  $P$ -generic containing  $q \wedge p_{\beta'}^+ \wedge r_1$ . Then we have:

- The evaluation of  $h_{\beta,\beta'}^*(\tilde{z})$  in  $V_{\beta'}$ , is the same as the evaluation of  $\tilde{z}$  in  $V_\beta$ , call it  $z$ .
- Also the evaluation of  $\tilde{X}$  and  $F$  are the same  $\beta$  and  $\beta'$  (cf. (5.62)).
- $z \subseteq \tilde{X}$  is a counterexample (as this is forced by  $q$ ).

In particular,  $z \subseteq \tilde{X}$  and  $\pi(z) \neq^* F(z) \cap \pi(\tilde{X})$  in the final extension.

- $p_{\beta'} \wedge r_1$  forces in  $V_{\beta'+1}$  that  $\eta_{\beta} \cap \text{ODD}^? = z$ ; also we have just seen that  $z \subseteq \tilde{X}$ ; and so  $\pi(z) = {}^* F(z) \cap \pi(X)$  by  $(*)$ , a contradiction.  $\dashv$

For  $\zeta \in U^{\text{ODD}}$ , we define the following  $P_\beta$ -names (independent of  $\beta$ ):<sup>22</sup>

$$\begin{aligned} \tilde{x}_\zeta &:= A_\zeta^? \cap \tilde{X} & \tilde{y}_\zeta &:= A_\zeta \cap \pi(\tilde{X}), \\ \text{so } \bigcup_{\zeta \in U^{\text{ODD}}} \tilde{x}_\zeta &= \tilde{X} & \bigcup_{\zeta \in U^{\text{ODD}}} \tilde{y}_\zeta &= \text{ODD} \cap \pi(\tilde{X}) = {}^* \pi(\tilde{X}), \end{aligned}$$

as well as

$$F'_\zeta : \mathcal{P}(\tilde{x}_\zeta) \rightarrow \mathcal{P}(\tilde{y}_\zeta) \quad \text{by} \quad a \mapsto F_\zeta(a) \cap \pi(\tilde{X}),$$

$$\text{and } F' : P(\tilde{X}) \rightarrow P(\pi(\tilde{X})) \quad \text{by} \quad z \mapsto \bigcup_{\zeta \in U^{\text{ODD}}} F'_\zeta(z \restriction \tilde{x}_\zeta) = F(z) \cap \pi(\tilde{X}).$$

So the  $p_{**}^3$  forces that for all  $z \in V_\beta$  the following is forced by  $p_\beta^+/G_\beta$ :

$$\begin{aligned} z \subseteq \tilde{X} \rightarrow F'(z) = {}^* \pi(z), \quad & \text{in particular } F'(\tilde{X}) = {}^* \pi(\tilde{X}), \\ & \text{also } F'(z) \subseteq \pi(\tilde{X}) \text{ for all } z. \end{aligned} \tag{5.67}$$

**LEMMA 5.68.**  $p_{**}^3$  forces: For almost all  $\zeta \in U^{\text{ODD}}$ ,  $F'_\zeta$  is a Boolean algebra isomorphism from  $P(\tilde{x}_\zeta)$  to  $P(\tilde{y}_\zeta)$ .

**PROOF.** *All and nothing:* We claim that for almost all  $\zeta$ ,  $F'_\zeta(\tilde{x}_\zeta) = \tilde{y}_\zeta$ . Assume that  $\ell \in \tilde{y}_\zeta \setminus F'_\zeta(\tilde{x}_\zeta) \subseteq \pi(\tilde{X})$ . Then  $\ell \in \pi(\tilde{X})$ , and  $\ell$  is not in  $F'(\tilde{X}) = {}^* \pi(\tilde{X})$ , so there cannot be many such  $\ell$ . Similarly  $F'_\zeta(\emptyset) = \emptyset$  for almost all  $\zeta$ .

*Unions:* We claim that for almost all  $\zeta$ ,  $F'_\zeta(a) \cup F'_\zeta(b) = F'_\zeta(a \cup b)$  for all subsets  $a, b$  of  $\tilde{x}_\zeta$ . Let  $A \subseteq \lambda$  be the set of counterexamples, i.e., for  $\zeta \in A$  there are  $\ell_\zeta \in \tilde{y}_\zeta$ , and  $a_\zeta, b_\zeta$  subsets of  $\tilde{x}_\zeta$  such that  $\ell_\zeta \in (F'_\zeta(a_\zeta) \cup F'_\zeta(b_\zeta)) \Delta F'_\zeta(a_\zeta \cup b_\zeta)$ . Set  $x := \bigcup_{\zeta \in A} a_\zeta$  and  $y := \bigcup_{\zeta \in A} b_\zeta$ . Then  $\ell_\zeta$  is in  $(F'(x) \cup F'(y)) \Delta F'(x \cup y) = {}^* \emptyset$ , so  $A$  cannot be large.

*Complements:* We claim that for almost all  $\zeta$ ,  $F'_\zeta(a) \cap F'_\zeta(\tilde{x}_\zeta \setminus a) = \emptyset$ . Let  $A$  be the set of counterexamples, i.e., for  $\zeta \in A$  there is an  $a_\zeta \subseteq \tilde{x}_\zeta$  and  $\ell \in \tilde{y}_\zeta$  such that  $\ell_\zeta \in F'_\zeta(a_\zeta) \cap F'_\zeta(\tilde{x}_\zeta \setminus a_\zeta)$ . Then  $\ell_\zeta$  is in  $F'(\bigcup_{\zeta \in A} a_\zeta) \cap F'(\bigcup_{\zeta \in A} \tilde{x}_\zeta \setminus a_\zeta) = {}^* \emptyset$ , so  $A$  cannot be large.

*Injectivity:* We already know that union and complements (and thus disjointness) are preserved, so it is enough to show that a nonempty set is mapped to a nonempty set.

Assume this fails often, then we get an  $x \subseteq \tilde{X}$  of size  $\lambda$  such that  $\emptyset = F'(x) = {}^* \pi(x)$ , a contradiction.

*Surjectivity:* Assume surjectivity fails often; i.e., there are many  $b_\zeta \subseteq \pi(\tilde{X}) \cap \text{ODD}$  not in the range of  $F'_\zeta$ . Let  $y$  be the union of those  $b_\zeta$ . Pick  $x \subseteq \lambda$  such that  $\pi(x) = {}^* y \subseteq \pi(\tilde{X})$ . So we can assume  $x \subseteq \tilde{X}$  and so  $F'(x) = {}^* y$ , which implies that  $F'_\zeta(x \cap \tilde{x}_\zeta) = y \cap A_\zeta = b_\zeta$  for almost all  $\zeta$ , a contradiction.  $\dashv$

<sup>22</sup>More concretely, canonically read by  $p_{***}^3$ , see (5.62).

LEMMA 5.69. *For each  $\beta \in S$ :  $p_{**}^3$  forces (in  $P_\beta$ ): There is a  $f_{\text{gen}} : \mathcal{X} \rightarrow \pi(\mathcal{X})$  bijective such that for all  $z \subseteq \mathcal{X}$  (in  $V_\beta$ ),  $p_\beta^+/G_\beta$  forces  $\pi(z) =^* f''_{\text{gen}} z$ .*

PROOF. Every Boolean algebra isomorphism from  $P(A)$  to  $P(B)$  is generated by a bijection from  $A$  to  $B$  (the restriction to the atoms). So there is a  $U' \subseteq U^{\text{ODD}}$  with  $|U^{\text{ODD}} \setminus U'| < \lambda$  such that  $\zeta \in U'$  implies that  $F'_\zeta$  is generated by some bijection  $g_\zeta : \mathcal{X}_\zeta \rightarrow \mathcal{Y}_\zeta$ . So  $F'$  is generated by  $g := \bigcup_{\zeta \in U'} g_\zeta$ ; and we can change  $g$  into a bijection from  $\mathcal{X}$  to  $\pi(\mathcal{X})$  by changing less than  $\lambda$  many values.  $\dashv$

We now strengthen  $p_{**}^3$  to some  $q$  to continuously read  $f_{\text{gen}}$  (independently of  $\beta$ ), again using Fact 5.55.

So to summarize, we have the following (where we start with the  $\Delta$ -system  $(M_\beta, p_\beta)_{\beta \in S}$  of Section 5.10):

COROLLARY 5.70. *There is  $\alpha \in \Delta$ ,  $q \in P_\alpha$  stronger than all  $p_\beta \restriction \beta$  and canonically reading  $r_0 \leq \tilde{p}$ ,  $\mathcal{X}$ ,  $f_{\text{gen}}$ , and  $\pi(\mathcal{X})$ , such that the following holds for all  $\beta \in S$ :*

- $q \wedge p_\beta$  with the condition<sup>23</sup> at index  $\beta$  strengthened to  $r_0$  is a valid condition, called  $p_\beta^{++}$ .
- $\alpha$ ,  $p_\beta^{++}$  and the names are in  $M_\beta$ .
- $q$  forces in  $P_\beta$ :  $|\mathcal{X}| = \lambda$ ,  $f_{\text{gen}} : \mathcal{X} \rightarrow \pi(\mathcal{X})$  is a bijection, and if  $z \subseteq \mathcal{X}$  is in  $V_\beta$ , then  $p_\beta^{++}/G_\beta \Vdash \pi(z) =^* f''_{\text{gen}} z$ .

**5.13. Putting everything together.**

COROLLARY 5.71. *(Assuming  $\lambda$  is inaccessible and  $2^\lambda = \lambda^+$ .)  $P$  forces that every automorphism of  $P^\lambda_\lambda$  is somewhere trivial.*

PROOF. Assume towards a contradiction that some  $p_*$  forces that  $\phi$  is a nowhere trivial automorphism represented by  $\pi$ .

As described in Section 5.10 we find a  $\Delta$ -system  $(M_\beta, p_\beta)_{\beta \in S}$  with  $p_\beta \restriction \beta \leq p_*$  for all  $\beta \in S$ , and we find  $q$ ,  $\mathcal{X}$ ,  $f_{\text{gen}}$  as in Corollary 5.70, so in particular:  $q \leq p_\beta \restriction \beta$  for all  $S$ ; and  $q$  forces that  $|\mathcal{X}| = \lambda$  and that  $f_{\text{gen}} : \mathcal{X} \rightarrow \pi(\mathcal{X})$  is a bijection.

As  $\pi$  is nowhere trivial,  $f_{\text{gen}}$  cannot be a generator, i.e., there is some  $z \subseteq \mathcal{X}$  with  $\pi(z) \neq^* f''_{\text{gen}} z$ . Fix a name for this  $z$  and let  $q^* \leq q$  canonically read  $z$ .

Pick  $\beta \in S$  above  $\text{dom}(q^*)$ . So  $q^* \wedge p_\beta^{++}$  is a valid condition, which forces that in the final extension  $V[G]$  the following holds:

- $z \subseteq \mathcal{X}$  with  $\pi(z) \neq^* f''_{\text{gen}} z$ , as this is forced by  $q^*$ .
- $z \in V_\beta$ , as  $q^*$  canonically reads  $z$ .
- By Corollary 5.70 and as  $p_\beta^{++} \in G$ , we get  $\pi(z) =^* f''_{\text{gen}} z$ , a contradiction.  $\dashv$

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<sup>23</sup>which is  $p_\beta(\beta) = \tilde{p}$

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