Archive for Mathematical Logic (2024) 63:623–654
https://doi.org/10.1007/s00153-023-00900-7

Mathematical Logic

Pcf without choice Sh835

Saharon Shelah[1,2]

Received: 23 November 2023 / Accepted: 19 December 2023 / Published online: 22 March 2024

Abstract

We mainly investigate models of set theory with restricted choice, e.g., ZF + DC + the family of countable subsets of λ is well ordered for every λ (really local version for a given λ). We think that in this frame much of pcf theory, (and combinatorial set theory in general) can be generalized. We prove here, in particular, that there is a proper class of regular cardinals, every large enough successor of singular is not measurable and we can prove cardinal inequalities. Solving some open problems, we prove that if $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$, then from a well ordering of $\mathscr{P}(\mathscr{P}(\kappa)) \cup {}^{\kappa>}\mu$ we can define a well ordering of ${}^{\kappa}\mu$.

Keywords Set theory · Weak axiom of choice · Pcf

Mathematics Subject Classification Primary 03E17 · Secondary: 03E05, 03E50

Annotated Content

This research was supported by the United States-Israel Binational Science Foundation, and several grants including grants with Maryanthe Malliaris number NFS 2051825, BSF 3013005232, and the ISF 2320/23, The Israel Science Foundation (ISF) (2023–2027). I would like to thank Alice Leonhardt for the beautiful typing. For later versions, the author would like to thank the typist for his work and is also grateful for the generous funding of typing services donated by a person who wishes to remain anonymous. References like [12, Def. 0.4 = Lz15] means the label of Def. 0.4 is z15. The reader should note that the version on my website is usually more updated than the one in the mathematical archive. Submitted to AML 1 July 2005. First Typed - 2004/Jan/20.

✉ Saharon Shelah
shelah@math.huji.ac.il
http://shelah.logic.at

1 Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, 91904 Jerusalem, Israel

2 Department of Mathematics, The State University of New Jersey, Hill Center - Busch Campus Rutgers, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

§1 Representing ${}^{\kappa}\lambda$

[We define $\mathrm{Fil}^{\ell}_{\kappa}$ and prove a representation theorem for ${}^{\kappa}\lambda$. Essentially under "reasonable choice" the set ${}^{\kappa}\lambda$ is the union of few well ordered sets, i.e., "their number depends on κ only". We end with a claim on $\Pi\mathfrak{a}$.]

§2 No decreasing sequence of subalgebras

[As suggested in the title we weaken the axioms. We deal with ${}^{\kappa}\lambda$ with λ^+ not measurable, existence of ladder $\bar{C}$ witnessing cofinality and prove that many λ^+ are regular (2.13).]

§3 Concluding remarks

[We prove that if $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$, then from a well-ordering of $\mathscr{P}(\mathscr{P}(\kappa)) \cup {}^{\kappa>}\mu$ we can define a well-ordering of ${}^{\kappa}\mu$, see 3.1. If e.g. μ is a strong limit singular of uncountable cofinality, using a well order of $\mathscr{H}(\mu)$ we can define a well ordering of $\mathscr{P}(\mu)$ hence of $\mathscr{H}(\mu^+)$, see 3.2. Lastly, we give sufficient conditions (in ZF+DC) for singular μ, that μ^+ is regular, see 3.3. Actually if $\mu = \mu^{\aleph_0} + 2^{2^{\kappa}}$, $\kappa = \kappa^{\aleph_0}$ and $X \subseteq \mu$ codes $\mathscr{P}(\mathscr{P}(\kappa))$ and ${}^{\omega}\mu$, then using X as a parameter we can define a well-ordering of ${}^{\kappa}\mu$, see 3.4.]

0 Introduction

0.1 Background, aims and results

The thesis of [9] was that pcf theory without full choice exists. Two theorems supporting this thesis were proved. The first ([9, 4.6,pg.117], we shall not mention ZF) is:

Theorem 0.1 [DC] *If $\mathscr{H}(\mu)$ is well ordered, μ strong limit singular of uncountable cofinality <u>then</u> μ^+ is regular not measurable (and 2^{μ} is an $\aleph$, i.e. $\mathscr{P}(\mu)$ can be well ordered and no $\lambda \in (\mu, 2^{\mu}]$ is measurable).*

Note that before this Apter and Magidor [1] had proved the consistency of "$\mathscr{H}(\mu)$ well ordered, $\mu = \beth_{\omega}$, $(\forall \kappa < \mu)\mathrm{DC}_{\kappa}$ and μ^+ is measurable" so 0.1 says that this consistency result cannot be fully lifted to uncountable cofinalities answering a question of them. Generally without full choice, a successor cardinal being not measurable is a piece of worthwhile information.
A second theorem ([9, §5]) is:

Theorem 0.2 *Assume*

(*a*) $\mathrm{DC} + \mathrm{AC}_{\kappa} + \kappa$ *regular uncountable.*
(*b*) $\langle \mu_i : i < \kappa \rangle$ *is increasing continuous with limit μ, $\mu > \kappa$, $\mathscr{H}(\mu)$ is well ordered, μ strong limit, (we need just a somewhat weaker version, the so-called $i < \kappa \Rightarrow Tw_{\mathscr{D}_{\kappa}}(\mu_i) < \mu$).*

<u>Then</u>, we cannot have two regular cardinals θ such that for some stationary $S \subseteq \kappa$, the sequence $\langle cf(\mu_i^+) : i \in S \rangle$ is constantly θ.

A dream was to prove that there is a class of regular cardinals from a restricted version of choice (see more in [9]).

Our original aim here is to improve those theorems. As for 0.1 we replace "$\mathscr{H}(\mu)$ well ordered" by "$[\mu]^{\aleph_0}$ is well ordered" and then by weaker statements.

We know (assuming full choice) that if, e.g., $\neg\exists 0^\#$ or there is no inner model with a measurable cardinal then though $\langle 2^\kappa : \kappa \text{ regular}\rangle$ is quite arbitrary, the size of $[\lambda]^\kappa, \lambda > \kappa$ is strictly controlled and equi-consistency results (by Easton forcing [2], and [8] and history there, and works of Gitik and history there respectively). It seemed that the situation here is parallel in some sense; under the restricted choice we assume, we cannot say much about the cardinality of $\mathscr{P}(\kappa)$ but can say something on the cardinality of $[\lambda]^\kappa$ for $\kappa \ll \lambda$.

In the proofs we fulfill a promise from [10, §5] about using $J[f, D]$ from Definition 0.13 instead of the nice filters used in [9] and, to some extent, in early versions of this work which require going through inner models to prove their existence. This work is continued in Larson-Shelah [5] and will be continued in [13]. On a different line with weak choice (say $\mathrm{DC}_{\aleph_0} + \mathrm{AC}_\mu, \mu$ fixed): see [6, 11, 12]. The present work fits the thesis of [8] which in particular says: it is better to look e.g. at $\langle \lambda^{\aleph_0} : \lambda \text{ a cardinal}\rangle$ then at $\langle 2^\lambda : \lambda \text{ a cardinal}\rangle$. Here instead well ordering $\mathscr{P}(\lambda)$ we well order $[\lambda]^{\aleph_0}$, this is enough for much.

A simply stated conclusion is (see 3.6):

Conclusion 0.3 [DC] *Assume $[\lambda]^{\aleph_0}$ is well ordered for every λ.*
(1) If 2^{2^κ} is well ordered <u>then</u> for every λ, $[\lambda]^\kappa$ is well ordered.
(2) For any set Y, there is a derived set Y_* so called $\mathrm{Fil}^4_{\aleph_1}(Y)$ of power near $\mathscr{P}(\mathscr{P}(Y))$ such that $\Vdash_{Levy(\aleph_0, Y)}$ "for every λ, ${}^Y\lambda$ is well ordered".

Thesis 0.4 (1) If $\mathbf{V} \models$ "ZF+ DC" and "every $[\lambda]^{\aleph_0}$ is well orderable" <u>then</u> $\mathbf{V}$ looks like the result of starting with a model of ZFC and using $\aleph_1$-complete forcing notions like Easton forcing, Levy collapses, and more generally, iterating of κ-complete forcing for $\kappa > \aleph_0$.
(2) This approach is dual to investigating $\mathbf{L}[\mathbb{R}]$ - <u>here</u> we assume ω-sequences are understood (or weaker versions) and we try to understand $\mathbf{V}$ (over this), <u>there</u> over the reals everything is understood.

Also though our original motivation was to look at the consequences of the so-called Ax_4, this was shadowed here by the try to use weaker relatives; see more in [13].

Explanation 0.5 How do we analyze $[\mu]^\kappa$ or equivalently ${}^\kappa\mu$ here? We use $\aleph_1$-complete filters on κ and a well-ordering of $[\alpha]^{\aleph_0}$ for appropriate α or less. We will consider $f : \kappa \to \mu$; now for every $\aleph_1$-complete filter D on κ, the ordinal $\alpha = \mathrm{rk}_D(f)$ gives us some information on f, but if $A, \kappa \backslash A \in D^+$ and $f \restriction A = 0_A$, then $\alpha = 0$ but we have no information on $f \restriction (\kappa \backslash A)$, then $\alpha = 0$ but we have no information on $f \restriction (\kappa \backslash A)$. Trying to correct this we consider the ideal $J[f, D] = \{A \subseteq \kappa : A = \emptyset \mod D \text{ or } A \in D^+ \text{ but } \mathrm{rk}_{D+A}(f) > \alpha\}$, this is an $\aleph_1$-complete ideal and so we may consider the pair $\bar{D} = (D_1, D_2) = (D, \mathrm{dual}(J[f, D]))$. Now α and the pair $\bar{D}$ gives more information on f; they determine f modulo D_2. This is not enough so we use an algebra $\mathscr{B}$ on μ with no infinite decreasing sequence of sub-algebras built using the assumption "$[\mu]^{\aleph_0}$ is well ordered". So there is $Z \in D_2$ such that $A = c\ell_{\mathscr{B}}(\mathrm{Rang}(f \restriction Z))$ is $\subseteq$-minimal.

Now the triple (D_1, D_2, Z) and the ordinal α almost determines f, we need one more piece of information with domain κ : $h(i) = \mathrm{otp}(\alpha \cap Z)$, hence an ordinal $<$ hrtg(Rang(f)). So we need a bound on it which depends on the choice of $\mathscr{B}$, usually, it is $\mathrm{hrtg}([\kappa]^{\aleph_0})$, natural by the construction of $\mathscr{B}$.

So $f \restriction Z$ is uniquely determined by the ordinal $\mathrm{rk}_D(f)$ and the quadruple (D_1, D_2, Z, h), which belongs to a set defined from κ, independently of μ.

Lastly, considering all such filters D (recalling we are assuming DC) we can find countably many quadruples (D_1^n, D_2^n, Z^n, h^n) which together are enough as $\bigcup\limits_n Z^n = \kappa$.

We thank for the attention and comments the audience in the advanced seminar in Rutgers 10/2004 (particularly Arthur Apter) and advanced course in logic in the Hebrew University 4,5/2005 and to Paul Larson and Shimoni Garti for many corrections as well as Tanmay Inamdar.

0.2 Preliminaries

Convention 0.6 We assume just $\mathbf{V} \models$ ZF if not said otherwise.

Notation 0.7 Let

(1) $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, i, j$ denote ordinals.
(2) $\kappa, \lambda, \mu, \chi$ denote cardinals, infinite if not said otherwise.
(3) n, m, k, ℓ denote natural numbers.
(4) D denotes a filter (on some set), I, J denote ideals on some set.

Definition 0.8 (1) $\mathrm{hrtg}(A) =$ Min$\{\alpha$: there is no function from A onto $\alpha\}$.
(2) $\mathrm{wlor}(A) =$ Min$\{\alpha$: there is no one-to-one function from α into A or $\alpha = 0 \wedge A = \emptyset\}$, so $\mathrm{wlor}(A) \leq \mathrm{hrtg}(A)$.

Remark 0.9 For many the meaning of "Hartogs number" is what is here called "wlor" (except that usually one would not make an exception for the empty set).

Definition 0.10 (1) For D an $\aleph_1$-complete filter on a set Y and $f \in {}^Y\mathrm{Ord}$ and $\alpha \in \mathrm{Ord} \cup \{\infty\}$ we define when $\mathrm{rk}_D(f) = \alpha$, by induction on α:

⊛ For $\alpha < \infty$, $\mathrm{rk}_D(f) = \alpha$ iff $\beta < \alpha \Rightarrow \mathrm{rk}_D(f) \neq \beta$ and for every $g \in {}^Y\mathrm{Ord}$ satisfying $g <_D f$ there is $\beta < \alpha$ such that $\mathrm{rk}_D(g) = \beta$.

(2) We can replace D by the dual ideal. If $f \in {}^Z\mathrm{Ord}$ and $Z \in D$ then we let $\mathrm{rk}_D(f) = \mathrm{rk}_{D+Z}(f \cup 0_{Y \setminus Z})$.

Galvin-Hajnal [3] use the rank for the club filter on ω_1. This was continued in [7] where varying D was extensively used.

Claim 0.11 [DC] *In Definition 0.10,* $\mathrm{rk}_D(f)$ *is always an ordinal and if* $\alpha \leq \mathrm{rk}_D(f)$ *<u>then</u> for some* $g \in \prod\limits_{y \in Y} (f(y) + 1)$ *we have* $\alpha = \mathrm{rk}_D(g)$, *(if* $\alpha < \mathrm{rk}_D(f)$ *we can add*

$g <_D f$; if $\mathrm{rk}_D(f) < \infty$ then DC *is not necessary; if* $\mathrm{rk}_D(f) = \alpha$ *this is trivial, as we can choose $g = f$).*

Claim 0.12 *(1)* [DC] *If D is an $\aleph_1$-complete filter on Y and $f \in {}^Y\mathrm{Ord}$ and $Y = \cup\{Y_n : n < \omega\}$ then* $\mathrm{rk}_D(f) =$ *$Min\{ rk_{D+Y_n}(f) : n < \omega$ and $Y_n \in D^+\}$, ([7]).*

(2) [DC + AC_{α^*}] If D is a κ-complete filter on Y, κ a cardinal $> \aleph_0$ and $f \in {}^Y\mathrm{Ord}$ and $Y = \cup\{Y_\alpha : \alpha < \alpha^*\}, \alpha^* < \kappa$ then $\mathrm{rk}_D(f) =$ *$Min\{ rk_{D+Y_\alpha}(f) : \alpha < \alpha^*$ and $Y_\alpha \in D^+\}$.*

Proof (1) By [7], in fact, $\mathrm{AC}_{\aleph_0}$ suffice.
(2) By [7], in fact, DC is not necessary.

Definition 0.13 For Y, D, f as in 0.10 let $J[f, D] =: \{Z \subseteq Y : Y\backslash Z \in D$ or $Y\backslash Z \in D^+$ and $\mathrm{rk}(f)_{D+(Y\backslash Z)} > \mathrm{rk}_D(f)\}$.

Claim 0.14 *[DC+$\mathrm{AC}_{<\kappa}$]* Assume D is a κ-complete filter on $Y, \kappa > \aleph_0$.

(1) If $f \in {}^Y$ Ord then $J[f, D]$ is a κ-complete ideal on Y.
(2) If $f_1, f_2 \in {}^Y\mathrm{Ord}$ and $J = J[f_1, D] = J[f_2, D]$ then $\mathrm{rk}_D(f_1) < \mathrm{rk}_D(f_2) \Rightarrow f_1 < f_2 \bmod J$ and $\mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2) \Rightarrow f_1 = f_2 \bmod J$.

Proof Straightforward or see [10, §5] and the reference there to [9] (and [7]).

Definition 0.15 (1) Here $Y \leq_{\mathrm{qu}} Z$ or $|Y| \leq_{\mathrm{qu}} |Z|$ or $|Y| \leq_{\mathrm{qu}} Z$ or $Y \leq_{\mathrm{qu}} |Z|$ means that $Y = \emptyset$ or there is a function from Z (equivalently from a subset of Z) onto Y.
(2) $\mathrm{reg}(\alpha) = \mathrm{Min}\{\partial : \partial \geq \alpha$ is a regular cardinal$\}$.

Definition 0.16 For a set Y, cardinal κ and ordinal γ we define $\mathscr{H}_{<\kappa,\gamma}(Y)$ by induction on γ: if $\gamma = 0$, $\mathscr{H}_{<\kappa,\gamma}(Y) = Y$, if $\gamma = \beta + 1$ then $\mathscr{H}_{<\kappa,\gamma}(Y) = \mathscr{H}_{<\kappa,\beta}(Y) \cup \{u : u \subseteq \mathscr{H}_{<\kappa,\beta}(Y)$ and $|u| < \kappa\}$ and if γ is a limit ordinal then $\mathscr{H}_{<\kappa,\gamma}(Y) = \cup\{\mathscr{H}_{<\kappa,\beta}(Y) : \beta < \gamma\}$.

Observation 0.17 *(1) If λ is the disjoint union of $\langle W_z : z \in Z\rangle$ and $z \in Z \Rightarrow |W_z| < \lambda$ and* $\mathrm{wlor}(Z) \leq \lambda$ *then* $\lambda = \sup\{$ *$otp(W_z) : z \in Z\}$ hence* $\mathrm{cf}(\lambda) < \mathrm{hrtg}(Z)$.
(2) If $\lambda = \cup\{W_z : z \in Z\}$ and $\mathrm{wlor}(\mathscr{P}(Z)) \leq \lambda$ *then* $\sup\{$ *$otp(W_z) : z \in Z\} = \lambda$.*
(3) If $\lambda = \cup\{W_z : z \in Z\}$ and $|Z| < \lambda$ then $\lambda = sup\{ otp(W_z) : z \in Z\}$.
(4) If $Z \subseteq$ Ord, $\bar{W} = \langle W_\alpha : \alpha \in Z\rangle$, $W_\alpha \subseteq$ Ord and $\lambda \geq \aleph_0, |Z|, |W_\alpha|$ for $\alpha \in Z$ then $\cup\{W_\alpha : \alpha \in Z\}$ has cardinality $\leq \lambda$.

Proof (1) Let $Z_1 = \{z \in Z : W_z \neq \emptyset\}$, so the mapping $z \mapsto \mathrm{Min}(W_z)$ exemplifies that Z_1 is well ordered hence by the definition of $\mathrm{wlor}(Z_1)$ the power $|Z_1|$ is an aleph $< \mathrm{wlor}(Z_1) \leq \mathrm{wlor}(Z)$ and by assumption $\mathrm{wlor}(Z) \leq \lambda$. Now if the desirable conclusion fails then $\gamma^* = \sup(\{\mathrm{otp}(W_z) : z \in Z_1\} \cup \{|Z_1|\})$ is an ordinal $< \lambda$, so we can find a sequence $\langle u_\gamma : \gamma < \gamma^*\rangle$ such that $\mathrm{otp}(u_\gamma) \leq \gamma^*, u_\gamma \subseteq \lambda$ and $\lambda = \cup\{u_\gamma : \gamma < \gamma^*\}$, so $\gamma^* < \lambda \leq |\gamma^* \times \gamma^*|$, easy contradiction.
(2) For $x \subseteq Z$ let $W_x^* = \{\alpha < \lambda : (\forall z \in Z)(\alpha \in W_z \equiv z \in x)\}$ hence λ is the disjoint union of $\{W_x^* : x \in \mathscr{P}(Z)\backslash\{\emptyset\}\}$. So the result follows by part (1).
(3) So let $<_*$ be a well-ordering of Z and let $W'_z = \{\alpha \in W_z$: if $y <_* z$ then $\alpha \notin W_y\}$, so $\langle W'_z : z \in Z\rangle$ is a well-defined sequence of pairwise disjoint sets with union equal to $\cup\{W_z : z \in Z\} = \lambda$ and $\mathrm{otp}(W'_z) \leq \mathrm{otp}(W_z)$. Hence if $|W_z| = \lambda$ for some $z \in Z$ the desirable conclusion is obvious, otherwise the result follows by part (1).
(4) Should be clear.

Definition 0.18 (1) We say that $c\ell$ is a very weak closure operation on λ of character (μ, κ) when:

(a) $c\ell$ is a function from $\mathscr{P}(\lambda)$ to $\mathscr{P}(\lambda)$
(b) $u \in [\lambda]^{\leq \kappa} \Rightarrow |c\ell(u)| \leq \mu$
(c) $u \subseteq \lambda \Rightarrow u \cup \{0\} \subseteq c\ell(u)$, the 0 for technical reasons.

1A) We say that $c\ell$ is a weak closure[1] operation on λ of character (μ, κ) when (a),(b),(c) above and:

(d) $u \subseteq v \subseteq \lambda \Rightarrow u \subseteq c\ell(u) \subseteq c\ell(v)$
(e) $c\ell(u) = \cup\{c\ell(v) : v \subseteq u, |v| \leq \kappa\}$.

So we may identify $c\ell$ with $c\ell \restriction [\lambda]^{\leq \kappa}$.

(1B) Let "... character $(< \mu, \kappa)$ or $(\mu, < \kappa)$, or $(< \mu, < \kappa)$" have the obvious meaning but if μ is an ordinal not a cardinal, then "$< \mu$" means of order type $< \mu$; similarly for "$< \kappa$". Let "... character (μ, Y)" means "character $(< \mu^+, < \mathrm{hrtg}(Y))$"

(1C) We omit the weak when in addition:

(f) $c\ell(u) = c\ell(c\ell(u))$ for $u \subseteq \lambda$.

(2) We say λ is f-inaccessible when $\delta \in \lambda \cap \mathrm{Dom}(f) \Rightarrow f(\delta) < \lambda$.
(3) We say $c\ell : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$ is well founded when for no sequence $\langle \mathscr{U}_n : n < \omega \rangle$ of subsets of λ do we have $c\ell(\mathscr{U}_{n+1}) \subset \mathscr{U}_n$ for $n < \omega$.
(4) For $c\ell$ a partial function from $\mathscr{P}(\alpha)$ to $\mathscr{P}(\alpha)$ (for simplicity assume $\alpha = \cup\{u : u \in \mathrm{Dom}(c\ell)\}$) let $c\ell^1_{\varepsilon, <\kappa}$ be the function from $\mathscr{P}(\alpha)$ to $\mathscr{P}(\alpha)$ defined by induction on the ordinal ε as follows:

(a) $c\ell^1_{0, <\kappa}(u) = u$
(b) $c\ell^1_{\varepsilon+1, <\kappa}(u) = \{0\} \cup c\ell^1_{\varepsilon, <\kappa}(u) \cup \bigcup\{c\ell(v) : v \subseteq c\ell^1_{\varepsilon, <\kappa}(u)$ and $v \in \mathrm{Dom}(c\ell), |v| < \kappa\}$
(c) for limit ε let $c\ell^1_{\varepsilon, <\kappa}(u) = \cup\{c\ell^1_{\zeta, <\kappa}(u) : \zeta < \varepsilon\}$.

(4A) Instead "$< \kappa$" we may use "$\leq \kappa$".
(5) For any function $F : [\lambda]^{\aleph_0} \to \lambda$ and countable $u \subseteq \lambda$ we define $c\ell^2_\varepsilon(u, F)$ by induction on $\varepsilon \leq \omega_1$

(a) $c\ell^2_0(u, F) = u \cup \{0\}$
(b) $c\ell^2_{\varepsilon+1}(u, F) = c\ell^2_\varepsilon(u, F) \cup \{F(c\ell^2_\varepsilon(u, F))\}$
(c) $c\ell^2_\varepsilon(u, F) = \cup\{c\ell^2_\zeta(u, F) : \zeta < \varepsilon\}$ when $\varepsilon \leq \omega_1$ is a limit ordinal.

(6) For countable u and F as in part (5) let $c\ell^3_F(u) = c\ell^3(u, F) := c\ell^2_{\omega_1}(u, F)$ and for any $u \subseteq \lambda$ let $c\ell^4_F(u) := u \cup \bigcup\{c\ell^3_F(v) : v \in [u]^{\aleph_0}\}$.
(7) For a cardinal ∂ we say that $c\ell : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$ is ∂-well founded when for no $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial \rangle$ of subsets of λ do we have $\varepsilon < \zeta < \partial \Rightarrow c\ell(\mathscr{U}_\zeta) \not\supseteq \mathscr{U}_\varepsilon$.
(8) If $F : [\lambda]^{\leq \kappa} \to \lambda$ and $u \subseteq \lambda$ then we let $c\ell_F(u) = c\ell^1_F(u)$ be the minimal subset v of λ such that $w \in [v]^{\leq \kappa} \Rightarrow F(w) \in v$ and $u \subseteq v$ (exists).

[1] so by actually only $c\ell \restriction [\lambda]^{\leq \kappa}$ count.

Observation 0.19 *For $F : [\lambda]^{\aleph_0} \to \lambda$, the operation $u \mapsto c\ell^3_F(u)$ is a very weak closure operation of character* $(\aleph_1, \aleph_0)$.

Remark 0.20 So for any very weak closure operation, $\aleph_0$-well founded is a stronger property than well founded, but if $u \subseteq \lambda \Rightarrow c\ell(c\ell(u)) = c\ell(u)$ which is reasonable, they are equivalent.

Observation 0.21 $[\alpha]^\partial$ *is well ordered iff* ${}^\partial\alpha$ *is well ordered when* $\alpha \geq \partial$.

Proof Use a pairing function on α for showing $|{}^\partial\alpha| \leq [\alpha]^\partial$, so $\Rightarrow$ holds. If ${}^\partial\alpha$ is well ordered by $<_*$ map $u \in [\alpha]^\partial$ to the $<_*$-first $f \in {}^\partial\alpha$ satisfying $\mathrm{Rang}(f) = u$.

1 Representing ${}^\kappa\lambda$

Here we give a simple case to illustrate what we do (see later on improvements in the hypothesis and the conclusion). Specifically, if Y is uncountable and $[\lambda]^{\aleph_0}$ is well ordered, then the set ${}^Y\lambda$ can be analyzed modulo countable union over few (i.e., their number depends on Y but not on λ) well ordered sets.

Definition 1.1 (1)

- (a) $\mathrm{Fil}_{\aleph_1}(Y) = \mathrm{Fil}^1_{\aleph_1}(Y) = \{D : D$ is an $\aleph_1$-complete filter on $Y\}$, so Y is defined from D as $\cup\{X : X \in D\}$
- (b) $\mathrm{Fil}^2_{\aleph_1}(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are $\aleph_1$-complete filters on Y, $(\emptyset \notin D_2$, of course$)\}$; in this context $Z \in \bar{D}$ means $Z \in D_2$
- (c) $\mathrm{Fil}^3_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h) : (D_1, D_2) \in \mathrm{Fil}^2_{\aleph_1}(Y)$ and $h : Y \to \alpha$ for some $\alpha < \mu\}$, if we omit μ we mean $\mu = \mathrm{hrtg}(Y) \cup \omega$
- (d) $\mathrm{Fil}^4_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h, Z) : (D_1, D_2, h) \in \mathrm{Fil}^3_{\aleph_1}(Y, \mu), Z \in D_2\}$; omitting μ means as above.

(2) For $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y, \mu)$ let $Y = Y^{[\mathfrak{y}]} = Y[\mathfrak{y}]$ and $\mathfrak{y} = (D^{\mathfrak{y}}_1, D^{\mathfrak{y}}_2, h^{\mathfrak{y}}, Z^{\mathfrak{y}}) = (D_1[\mathfrak{y}], D_2[\mathfrak{y}], h[\mathfrak{y}], Z[\mathfrak{y}])$; similarly for the others and let $D^{\mathfrak{y}} = D[\mathfrak{y}]$ be $D^{\mathfrak{y}}_1 + Z^{\mathfrak{y}}$.
(3) We can replace $\aleph_1$ by any $\kappa > \aleph_1$ (the results can be generalized easily assuming DC + $\mathrm{AC}_{<\kappa}$, used in §2).

Theorem 1.2 *[DC] Assume* $[\lambda]^{\aleph_0}$ *is well ordered.*
Then we can find a sequence $\langle \mathscr{F}_{\mathfrak{y}} : \mathfrak{y} \in Fil^4_{\aleph_1}(Y)\rangle$ *satisfying*

- (α) $\mathscr{F}_{\mathfrak{y}} \subseteq {}^{Z[\mathfrak{y}]}\lambda$
- (β) $\mathscr{F}_{\mathfrak{y}}$ *is a well ordered set by* $f_1 <_{\mathfrak{y}} f_2 \Leftrightarrow rk_{D[\mathfrak{y}]}(f_1) < rk_{D[\mathfrak{y}]}(f_2)$ *so* $f \mapsto rk_{D[\mathfrak{y}]}(f)$ *is a one-to-one mapping from* $\mathscr{F}_{\mathfrak{y}}$ *into the ordinals*
- (γ) *if* $f \in {}^Y\lambda$ *then we can find a sequence* $\langle \mathfrak{y}_n : n < \omega\rangle$ *with* $\mathfrak{y}_n \in Fil^4_{\aleph_1}(Y)$ *such that* $n < \omega \Rightarrow f \restriction Z^{\mathfrak{y}_n} \in \mathscr{F}_{\mathfrak{y}_n}$ *and* $\cup\{Z^{\mathfrak{y}_n} : n < \omega\} = Y$.

An immediate consequence of 1.2 is

Conclusion 1.3 *(1) [DC +* ${}^\omega\alpha$ *is well-orderable for every ordinal* α*].*
For any set Y and cardinal λ *there is a sequence* $\langle \mathscr{F}_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(Fil^4_{\aleph_1}(Y))\rangle$ *such that*

(a) ${}^Y\lambda = \cup\{\mathscr{F}_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathit{Fil}^4_{\aleph_1}(Y))\}$

(b) $\mathscr{F}_{\bar{\mathfrak{x}}}$ *is well orderable for each* $\bar{\mathfrak{x}} \in {}^\omega(\mathit{Fil}^4_{\aleph_1}(Y))$

$(b)^+$ *moreover, uniformly, i.e., there is a sequence* $\langle <_{\bar{\mathfrak{x}}}: \bar{\mathfrak{x}} \in {}^\omega(\mathit{Fil}^4_{\aleph_1}(Y)\rangle$ *such that* $<_{\bar{\mathfrak{x}}}$ *is a well order of* $\mathscr{F}_{\bar{\mathfrak{x}}}$

(c) *there is a function* F *with domain* $\mathscr{P}({}^Y\lambda)\backslash\{\emptyset\}$ *such that: if* $S \subseteq {}^Y\lambda$ *is non-empty then* $F(S)$ *is a non-empty subset of* S *of power* $\leq_{\text{qu}} {}^\omega(\mathit{Fil}^4_{\aleph_1}(Y)))$ *recalling Definition 0.15. In fact, some ordinal* $\alpha(*)$ *and* $\bar{u}$ *we have:*

(α) $\bar{u} = \langle \mathscr{U}_\alpha : \alpha < \alpha(*)\rangle$ *is a partition of* ${}^Y\lambda$

(β) *if* $S \subseteq {}^Y\lambda$ *then* $F(S) = \mathscr{U}_{f(S)} \cap S$ *where* $f(S) = \mathit{Min}\{\alpha : \mathscr{U}_\alpha \cap S \neq \emptyset\}$

(γ) *if* $\alpha < \alpha(*)$ *then* $|\mathscr{U}_\alpha| < \text{hrtg}({}^\omega(\mathit{Fil}^4_{\aleph_1}(Y)))$.

(2) [DC] For any Y, λ *above, if* $[\alpha(*)]^{\aleph_0}$ *is well ordered where* $\alpha(*) = \cup\{\mathit{rk}_D(f)+1 : f \in {}^Y\lambda$ *and* $D \in \mathit{Fil}^1_{\aleph_1}(Y)\}$ *<u>then</u>* ${}^Y\lambda$ *satisfies the conclusion of part (1).*

Remark 1.4 So clause (c) of 1.3(1) is a weak form of choice.

Proof <u>Proof of 1.3</u> (1) Let $\langle \mathscr{F}_{\mathfrak{y}} : \mathfrak{y} \in \text{Fil}^4_{\aleph_1}(Y)\rangle$ be as in 1.2.

For each $\bar{\mathfrak{x}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))$ (so $\bar{\mathfrak{x}} = \langle \mathfrak{x}_n : n < \omega\rangle$) let

$$\begin{aligned}\mathscr{F}'_{\bar{\mathfrak{x}}} = \{f : \ &f \text{ is a function from } Y \text{ to } \lambda \text{ such that} \\ &n < \omega \Rightarrow f \restriction Z^{\mathfrak{x}_n} \in \mathscr{F}_{\mathfrak{x}_n} \text{ and } Y = \cup\{Z^{\mathfrak{x}_n} : n < \omega\}\}.\end{aligned}$$

Now

$(*)_1$ ${}^Y\lambda = \cup\{\mathscr{F}'_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))\}$.

[Why? By clause (γ) of 1.2.]

Let $\alpha(*) = \cup\{\text{rk}_D(f) + 1 : f \in {}^Y\lambda$ and $D \in \text{Fil}^1_{\aleph_1}(Y)\}$. For $\bar{\mathfrak{x}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))$ we define the function $G_{\bar{\mathfrak{x}}} : \mathscr{F}'_{\bar{\mathfrak{x}}} \to {}^\omega\alpha(*)$ by $G_{\bar{\mathfrak{x}}}(f) = \langle \text{rk}_{D_1[\mathfrak{x}_n]}(f) : n < \omega\rangle$.

Next

$(*)_2$
- (α) $\bar{G} = \langle G_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))\rangle$ exists
- (β) $G_{\bar{\mathfrak{x}}}$ is a function from $\mathscr{F}'_{\bar{\mathfrak{x}}}$ to ${}^\omega\alpha(*)$
- (γ) $G_{\bar{\mathfrak{x}}}$ is one to one.

[Should be clear, e.g. for $(*)_2(\gamma)$ read the definition of $\mathscr{F}'_{\mathfrak{x}'}$ and clause (β) of Theorem 1.2.]

Let $<_*$ be a well ordering of ${}^\omega\alpha(*)$ and for $\bar{\mathfrak{x}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))$ let $<_{\bar{\mathfrak{x}}}$ be the following two place relation on $\mathscr{F}'_{\bar{\mathfrak{x}}}$:

$(*)_3$ $f_1 <_{\bar{\mathfrak{x}}} f_2$ <u>iff</u> $G_{\bar{\mathfrak{x}}}(f_1) <_* G_{\bar{\mathfrak{x}}}(f_2)$.

Obviously

$(*)_4$
- (α) $\langle <_{\bar{\mathfrak{x}}}: \bar{\mathfrak{x}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))\rangle$ exists
- (β) $<_{\bar{\mathfrak{x}}}$ is a well ordering of $\mathscr{F}'_{\bar{\mathfrak{x}}}$.

By $(*)_1 + (*)_4$ we have proved clauses (a),(b),(b)$^+$ of the conclusion. Now clause (c) follows: for non-empty $S \subseteq {}^Y\lambda$, let $f(S)$ be $\min\{\text{otp}(\{g : g <_{\bar{\mathfrak{y}}} f\}, <_{\bar{\mathfrak{y}}}) : \bar{\mathfrak{y}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))$ and $f \in \mathscr{F}'_{\bar{\mathfrak{y}}} \cap S\}$. Also for any ordinal γ let $\mathscr{U}^1_\gamma := \{f$: for some $\bar{\mathfrak{y}} \in {}^\omega(\text{Fil}^4_{\aleph_1}(Y))$ we have $\gamma = \text{otp}(\{g : g <_{\bar{\mathfrak{y}}} f\}, <_{\bar{\mathfrak{y}}})\}$ and $\mathscr{U}_\gamma = \mathscr{U}^1_\gamma \setminus \cup \bigcup\{\mathscr{U}^1_\beta : \beta < \gamma\}$.

Lastly, we let $F(S) = \mathscr{U}_{f(S)} \cap S$. Now check.

(2) Similarly.

Proof Proof of Theorem 1.2 First

⊛$_1$ there are a cardinal μ and a sequence $\bar{u} = \langle u_\alpha : \alpha < \mu\rangle$ listing $[\lambda]^{\aleph_0}$.

[Why? By the assumption.]

Second, we can deduce

⊛$_2$ there are $\mu_1 \leq \mu$ and a sequence $\bar{u} = \langle u_\alpha : \alpha < \mu_1\rangle$ such that:

- (a) $u_\alpha \in [\lambda]^{\aleph_0}$
- (b) if $u \in [\lambda]^{\leq\aleph_0}$ then for some finite $w \subseteq \mu_1, u \subseteq \cup\{u_\beta : \beta \in w\}$
- (c) u_α is not included in $u_{\alpha_0} \cup \ldots \cup u_{\alpha_{n-1}}$ when $n < \omega, \alpha_0, \ldots, \alpha_{n-1} < \alpha$.

[Why? Let $\bar{u}^0$ be of the form $\langle u^0_\alpha : \alpha < \alpha^*\rangle$ such that $(a) + (b)$ holds and $\ell g(\bar{u}^0)$ is minimal; it is well defined and $\ell g(\bar{u}^0) \leq \mu$ by ⊛$_1$. Let $W = \{\alpha < \ell g(\bar{u}^0) : u^0_\alpha \not\subseteq \cup\{u^0_\beta : \beta \in w\}$ when $w \subseteq \alpha$ is finite$\}$. Let $\mu_1 = |W|$ and let $f : \mu_1 \to W$ be one-to-one onto, let $u_\alpha = u^0_{f(\alpha)}$ so $\langle u_\alpha : \alpha < \mu_1\rangle$ satisfies $(a)+(b)$ and $\mu_1 = |W| \leq \ell g(\bar{u}^0)$. So by the choice of $\bar{u}^0$ we have $\ell g(\bar{u}^0) = \mu_1$. So we can choose f such that it is increasing hence $\bar{u}$ is as required.]

⊛$_3$ we can define $\mathbf{n} : [\lambda]^{\leq\aleph_0} \to \omega$ and partial functions $F_\ell : [\lambda]^{\leq\aleph_0} \to \mu_1$ for $\ell < \omega$ (so $\langle F_\ell : \ell < \omega\rangle$ exists) as follows:

- (a) u infinite $\Rightarrow F_0(u) = \text{Min}\{\alpha$: for some finite $w \subseteq \alpha, u \subseteq u_\alpha \cup \bigcup\{u_\beta : \beta \in w\}$ mod finite$\}$
- (b) u finite $\Rightarrow F_0(u)$ undefined
- (c) $F_{\ell+1}(u) := F_0(u \setminus (u_{F_0(u)} \cup \ldots \cup u_{F_\ell(u)}))$ for $\ell < \omega$ when $F_\ell(u)$ is defined
- (d) $\mathbf{n}(u) := \text{Min}\{\ell : F_\ell(u)$ undefined$\}$.

Then

⊛$_4$ (a) $F_{\ell+1}(u) < F_\ell(u) < \mu_1$ when they are well defined
- (b) $\mathbf{n}(u)$ is a well defined natural number and $u \setminus \cup\{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\}$ is finite and $k < \mathbf{n}(u) \Rightarrow (u \setminus \cup\{u_{F_\ell(u)} : \ell < k\}) \cap u_{F_k(u)}$ is infinite
- (c) if $u_1, u_2 \in [\lambda]^{\aleph_0}, u_1 \subseteq u_2$ and $u_2 \setminus u_1$ is finite then $F_\ell(u_1) = F_\ell(u_2)$ for $\ell < \mathbf{n}(u_1)$ and $\mathbf{n}(u_1) = \mathbf{n}(u_2)$

⊛$_5$ define $F_* : [\lambda]^{\aleph_0} \to \lambda$ by $F_*(u) = \text{Min}(\cup\{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\} \cup \{0\} \setminus u)$ if well defined, zero otherwise

[Note: the reader may wonder: as you add $\{0\}$ then $\text{Min}(-) = 0$ in all cases. However, if $0 \in u$ then by "$\setminus u$", zero does not belong to the set from which we choose a minimal ordinal.]

⊛$_6$ if $u \in [\lambda]^{\aleph_0}$ then (recalling 0.18(4), (5), (6)):

(α) $c\ell^3(u, F_*) = c\ell^3_{F_*}(u)$ is $F'(u) := u \cup \bigcup\{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\} \cup \{0\}$
(β) $c\ell^3_{F_*}(u) = c\ell^2_{\varepsilon(u)}(F)$ for some $\varepsilon(u) < \omega_1$
(γ) there is $\bar{F} = \langle F'_\varepsilon : \varepsilon < \omega_1\rangle$ such that: for every $u \in [\lambda]^{\aleph_0}$, $c\ell^3_{F_*}(u) = \{F'_\varepsilon(u) : \varepsilon < \varepsilon(u)\}$ and $F'_\varepsilon(u) = 0$ if $\varepsilon \in [\varepsilon(u), \omega_1)$
(δ) in fact $F'_\varepsilon(u)$ is the ε-th member of $c\ell^3_{F_*}(u)$ if $\varepsilon < \varepsilon(u)$.

[Why? Define w^ε_u by induction on ε by $w^0_u = u$, $w^{\varepsilon+1}_u = w^\varepsilon_u \cup \{F_*(w^\varepsilon_u)\}$ and for limit ordinal ε we let $w^\varepsilon_u = \cup\{w^\zeta_u : \zeta < \varepsilon\}$. We can prove by induction on ε that $w^\varepsilon_u \subseteq F'(u)$ which is countable. The partial function g with domain $F'(u)\backslash u$ to Ord, $g(\alpha) = \text{Min}\{\varepsilon : \alpha \in w^{\varepsilon+1}_u\}$ is one to one onto an ordinal call it $\varepsilon(*)$, so $w^{\varepsilon(*)}_u \subseteq F'(u)$ and if they are not equal that $F_*(w^{\varepsilon(*)}_u) \in F'(u)\backslash w^{\varepsilon(*)}_u$ hence $w^{\varepsilon(*)}_u \subsetneqq w^{\varepsilon(*)+1}_u$ contradicting the choice of $\varepsilon(*)$. So clause (α) holds. In fact, $c\ell^3(u, F_*) = w^{\varepsilon(*)}_u$ and clause (β) holds. CLauses (γ), (δ) should be clear.]

$\circledast_7$ there is no sequence $\langle \mathscr{U}_n : n < \omega\rangle$ such that:

(a) $\mathscr{U}_{n+1} \subseteq \mathscr{U}_n \subset \lambda$
(b) $\mathscr{U}_n$ is closed under F_*, i.e. $u \in [\mathscr{U}_n]^{\aleph_0} \Rightarrow F_*(u) \in \mathscr{U}_n$
(c) $\mathscr{U}_{n+1} \neq \mathscr{U}_n$.

[Why? Assume toward contradiction that $\langle \mathscr{U}_n : n < \omega\rangle$ satisfies clauses (a),(b),(c). Let $\alpha_n = \text{Min}(\mathscr{U}_n\backslash\mathscr{U}_{n+1})$ for $n < \omega$ hence the sequence $\bar{\alpha} = \langle \alpha_n : n < \omega\rangle$ is well defined with no repetitions and let $\beta_{m,\ell} := F_\ell(\{\alpha_n : n \geq m\})$ for $m < \omega$ and $\ell < \mathbf{n}_m := \mathbf{n}(\{\alpha_n : n \in [m, \omega)\})$. As $\bar{\alpha}$ is with no repetition, $\mathbf{n}_m > 0$ and by $\circledast_4(c)$ clearly $\mathbf{n}_m = \mathbf{n}_0$ for $m < \omega$ and $\beta_{m,\ell} = \beta_{0,\ell}$ for $m < \omega$, $\ell < \mathbf{n}_0$. So letting $v_m = \cup\{u_{F_\ell(\{\alpha_n : n \in [m,\omega)\})} : \ell < \mathbf{n}_m\}$, it does not depend on m so $v_m = v_0$, and by the choice of F_*, as $\{\alpha_n : n \in [m, \omega)\} \subseteq \mathscr{U}_m$ and $\mathscr{U}_m$ is closed under F_* clearly $v_m \subseteq \mathscr{U}_m$. Together $v_0 = v_m \subseteq \mathscr{U}_m$ so $v_0 \subseteq \cap\{\mathscr{U}_m : m < \omega\}$. Also, by the definition of the F_ℓ's, $\{\alpha_n : n < \omega\}\backslash v_0$ is finite so for some $k < \omega$, $\{\alpha_m : n \in [k, \omega)\} \subseteq v_0$ but $v_0 \subseteq \mathscr{U}_{k+1}$ contradicting the choice of α_k.]

Moreover, recalling Definition 0.18(6):

$\circledast'_7$ there is no sequence $\langle \mathscr{U}_n : n < \omega\rangle$ such that

(a) $\mathscr{U}_{n+1} \subseteq \mathscr{U}_n \subseteq \lambda$
(b) $\mathscr{U}_n\backslash c\ell^4_{F_*}(\mathscr{U}_{n+1}) \neq \emptyset$.

[Why? As above but letting $\alpha_n = \text{Min}(\mathscr{U}_n\backslash c\ell^3_{F_*}(\mathscr{U}_{n+1}))$.]

Now we define for $(D_1, D_2, h, Z) \in \text{Fil}^4_{\aleph_1}(Y)$ and ordinal α the following, recalling Definition 0.18(6) for clauses (e),(f):

$\circledast_8$ $\mathscr{F}_{(D_1,D_2,h,Z),\alpha} =: \{f :$ (a) f is a function from Z to λ
(b) $\text{rk}_{D_1+Z}(f \cup 0_{(Y\backslash Z)}) = \alpha$
(c) $D_2 = \{Y\backslash X : X \subseteq Y$ satisfies $X = \emptyset \mod D_1$ or $X \in D_1^+$ and $\text{rk}_{D_1+X}(f \cup 0_{(Y\backslash Z)}) > \alpha$ that is $\text{rk}_{D_1+X}(f) > \alpha\}$
(d) $Z \in D_2$, really follows
(e) if $Z' \subseteq Z \wedge Z' \in D_2$ then

$$c\ell^3_{F_*}(\mathrm{Rang}(f \restriction Z')) = c\ell^3_{F_*}(\mathrm{Rang}(f))$$
$$(f) \quad y \in Z \Rightarrow f(y) = \text{the } h(y)\text{-th member of } c\ell^3_{F_*}(\mathrm{Rang}(f))\}.$$

So we have:

$\circledast_9$ $\mathscr{F}_{(D_1,D_2,h,Z),\alpha}$ has at most one member; call it $f_{(D_1,D_2,h,Z),\alpha}$ (when defined; pedantically we should write $f_{(D_1,D_2,h,Z),c\ell,\alpha}$)

$\circledast_{10}$ $\mathscr{F}_{(D_1,D_2,h,Z)} =: \cup\{\mathscr{F}_{(D_1,D_2,h,Z),\alpha} : \alpha$ an ordinal$\}$ is a well ordered set.

[Why? Define $<_{(D_1,D_2,h,Z)}$ by the α's, i.e. $f^1 < f^2$ iff there are $\alpha_1 < \alpha_2$ such that $f^\ell = f_{(D_1,D_2,h,2),\alpha_\ell}$ for $\ell = 1, 2$.]

$\circledast_{11}$ if $f : Y \to \lambda$ and $Z \subseteq Y$ then the set $\mathrm{Rang}(f \restriction Z)$ has cardinality $< \mathrm{hrtg}(Z)$.

[Why? By the definition of hrtg$(-)$ this should be clear.]

$\circledast_{12}$ if $f : Z \to \lambda$ and $Z \subseteq Y$ then $c\ell^4_{F_*}(\mathrm{Rang}(f)) \subseteq \lambda$ has cardinality $< \mathrm{hrtg}([Z]^{\aleph_0})$ or is finite.

Why? This will take some time. If $\mathrm{Rang}(f)$ is countable more holds by 0.19. Otherwise, by $\circledast_6(\beta)$ recallng Definition 0.18(6) we have $c\ell^4_{F_*}(\mathrm{Rang}(f)) = \mathrm{Rang}(f) \cup \{F'_\varepsilon(u) : u \in [\mathrm{Rang}(f)]^{\aleph_0}$ and $\varepsilon < \omega_1\}$.

Let $\alpha(*)$ be minimal such that $\mathrm{Rang}(f) \cap \alpha(*)$ has order type ω_1. Let $h_1, h_2 : \omega_1 \to \omega_1$ be such that $h_\ell(\varepsilon) < \max\{\varepsilon, 1\}$ and for every $\varepsilon_1, \varepsilon_2 < \omega_1$ there is $\zeta \in [\varepsilon_1 + \varepsilon_2 + 1, \omega_1)$ such that $h_\ell(\zeta) = \varepsilon_\ell$ for $\ell = 1, 2$. Define $F : [Z]^{\aleph_0} \to \lambda$ as follows: if $u \in [\mathrm{Rang}(f)]^{\aleph_0}$, let $\varepsilon_\ell(u) = h_\ell(\mathrm{otp}(u \cap \alpha(*))$ for $\ell = 1, 2$ and $F(u) = F'_{\varepsilon_2(u)}(\{\alpha \in u$: if $\alpha < \alpha(*)$ then $\mathrm{otp}(u \cap \alpha) < \varepsilon_1(u)\})$.

Now

$\bullet_1$ if $u \in [\mathrm{Rang}(f)]^{\aleph_0}$ then $F(u)$ is $F_\varepsilon(v)$ for some $v \in [Z]^{\aleph_0}$ and $\varepsilon < \omega_1$.

[Why? As $F(u) \in \mathrm{Rang}(F'_{\varepsilon_2(u)} \restriction [\mathrm{Rang}(f)]^{\aleph_0})$]

$\bullet_2$ $\{F(u) : u \in [\mathrm{Rang}(f)]^{\aleph_0}\} \subseteq c\ell^4_{F_*}(\mathrm{Rang}(f))$.

[Why? By $\bullet_1$ recalling $\circledast_6$.]

$\bullet_3$ if $u \in [\mathrm{Rang}(f)]^{\aleph_0}$ and $\varepsilon < \omega_1$ then $F'_\varepsilon(u)$ is $F(u)$ for some $v \in [\mathrm{Rang}(f)]^{\aleph_0}$.

[Why? Let $\varepsilon_1 = \mathrm{otp}(u \cap \alpha(*))$, $\varepsilon_2 = \varepsilon$; now let $\zeta < \omega_1$ be such that $h_\ell(\zeta) = \varepsilon_\ell$ for $\ell = 1, 2$. Let $v = u \cup \{\alpha : \alpha \in \mathrm{Rang}(f) \cap \alpha(*)$ and $\alpha \geq \sup(u \cap \alpha(*)) + 1$ and $\mathrm{otp}(\mathrm{Rang}(f) \cap \alpha \backslash (\sup(u \cap \alpha(*) + 1)) < (\zeta - \varepsilon_1))\}$.]

So $F(u) = F'_\varepsilon(u)$. By $\bullet_2 + \bullet_3$ we can conclude:

$\bullet_4$ in $\bullet_2$ we have equality.

Together $c\ell^4_{F_*}(\mathrm{Rang}(f)) = \{F(u) : u \in [\mathrm{Rang}(f)]^{\aleph_0}\} \cup \mathrm{Rang}(f)$ so it is the union of two sets; by the definition of hrtg$(-)$ the first is of cardinality $< \mathrm{hrtg}([Z]^{\aleph_0})$ and the second is of cardinality $< \mathrm{hrtg}[Z]$, so we are easily done proving $\circledast_{12}$

$\circledast_{13}$ if $f : Y \to \lambda$ then for some sequence $\langle(\mathfrak{y}_n, \alpha_n) : n < \omega\rangle$ we have $\mathfrak{y}_n \in \mathrm{Fil}^4_{\aleph_1}(Y)$ and $\alpha_n \in \mathrm{Ord}$ for $n < \omega$ and $f = \cup\{f_{\mathfrak{y}_n,\alpha_n} : n < \omega\}$.

[Why? Let

$$\mathscr{I}^0_f = \{Z \subseteq Y : \text{ for some } \mathfrak{y} \in \text{Fil}^4_{\aleph_1}(Y) \text{ satisfying } Z^{\mathfrak{y}} = Z \text{ and ordinal } \alpha, f_{\mathfrak{y},\alpha} \text{ is well defined and equal to } f \restriction Z\}$$

$$\mathscr{I}_f = \{Z \subseteq Y : Z \text{ is included in a countable union of members of } \mathscr{I}^0_f\}.$$

So recalling we are assuming DC it is enough to show that $Y \in \mathscr{I}_f$.

Toward contradiction assume not. Let $D_1 = \{Y \backslash Z : Z \in \mathscr{I}_f\}$, clearly it belongs to $\text{Fil}_{\aleph_1}(Y)$, noting that $Y \notin \mathscr{I}_f$. So $\alpha(*) := \text{rk}_{D_1}(f)$ is well defined (by 0.11) recalling that only DC = $\text{DC}_{\aleph_0}$ is needed.

Let

$$D_2 = \{X \subseteq Y : X \in D_1 \text{ or } \text{rk}_{D_1+(Y\backslash X)}(f) > \alpha(*)\}.$$

By 0.13 + 0.14 clearly D_2 is an $\aleph_1$-complete filter on Y extending D_1.

Now we try to choose $Z_n \in D_2$ for $n < \omega$ such that $Z_{n+1} \subseteq Z_n$ and $c\ell^4_{F_*}(\text{Rang}(f \restriction Z_{n+1}))$ does not include $\text{Rang}(f \restriction Z_n)$.

For $n = 0$, $Z_0 = Y$ is O.K.

By $\circledast'_7$ we cannot have such ω-sequence $\langle Z_n : n < \omega\rangle$; so by DC for some (unique) $n = n(*)$, Z_n is chosen but not Z_{n+1}.

Let $h : Z_n \to \text{hrtg}([Y]^{\aleph_0}) \cup \omega_1$ be:

$$h(y) = \text{ otp}(f(y) \cap c\ell^4_{F_*}(\text{Rang}(f \restriction Z_n))).$$

Now h is well defined by $\circledast_{12}$. Easily

$$f \restriction Z_n \in \mathscr{F}_{(D_1+Z_n, D_2, h, Z_n), \alpha(*)}$$

hence $Z_n \in \mathscr{I}^0_f \subseteq \mathscr{I}_f$, contradiction to $Z_n \in D_2$, $D_1 \subseteq D_2$.

So we are done proving $\circledast_{13}$.]

Now clause (β) of the conclusion holds by the definition of $\mathscr{F}_{\mathfrak{y}}$, clause (α) holds by $\circledast_{10}$ recalling $\circledast_8$, $\circledast_9$ and clause (γ) holds by $\circledast_{12}$.

Remark 1.5 We can improve 1.2 in some way by weakening the demands on $\bar{u}$.

We may replace the assumption "$[\lambda]^{\aleph_0}$ is well ordered" by:

$(*)$ there is $\langle u_\alpha : \alpha < \alpha^*\rangle$, a sequence of members of $[\lambda]^{\aleph_0}$ such that $(\forall u \in [\lambda]^{\aleph_0})(\exists \alpha)(u \cap u_\alpha$ infinite).

[Why? We define $F_\varepsilon : [\lambda]^{\aleph_0} \to \alpha^*$ by induction on $\varepsilon < \omega_1$ by $F_\varepsilon(v) := \text{Min}\{\alpha < \alpha^* : (v \backslash \cup \{u_{F_\zeta(v)} : \zeta < \varepsilon\}) \cap u_\alpha \text{ infinite}\}$ if well defined and let $F : [\lambda]^{\aleph_0} \to [\lambda]^{\aleph_0}$ be defined by $F(v) = \cup\{F_\varepsilon(v) : \varepsilon < \omega_1, F_\varepsilon(v) \text{ well defined}\}$.

Lastly, let $F_*(u) = \min(F(u)\backslash u)$.]

Observation 1.6 *(1) The power of* $\mathrm{Fil}^4_{\aleph_1}(Y,\mu)$ *is smaller or equal to the power of the set* $(\mathscr{P}(\mathscr{P}(Y)))^2 \times \mathscr{P}(Y) \times \mu^{|Y|}$*; if* $\aleph_0 \le |Y|$ *this is equal to the power of* $\mathscr{P}(\mathscr{P}(Y)) \times {}^Y\mu$.
(2) The power of $\mathrm{Fil}^4_{\aleph_1}(Y)$ *is smaller or equal to the power of the set* $(\mathscr{P}(\mathscr{P}(Y)))^2 \times \mathscr{P}(Y) \times \cup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}$.
(3) In part (2), if $\aleph_0 \le |Y|$ *this is equal to* $|\mathscr{P}(\mathscr{P}(Y))| \times \cup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}$*; also* $\alpha < \mathrm{hrtg}([Y]^{\aleph_0}) \Rightarrow |\mathscr{P}(\mathscr{P}(Y)) \times {}^Y\alpha| = |\mathscr{P}(\mathscr{P}(Y))|$ *and* $|\mathrm{Fil}^4_{\aleph_1}(Y)| \le_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y \times Y))$.

Remark 1.7 (1) As we are assuming DC, the case $\aleph_0 \not\le |Y|$ means that Y is finite, so degenerated. Now, if $|Y| < \aleph_0$, then $\mathrm{Fil}^1_{\aleph_1}(Y) = \{\{Z \subseteq Y : Z \supseteq X\} : X \subseteq Y\}$ hence $|\mathrm{Fil}^1_{\aleph_1}(Y)| = |\mathscr{P}(Y)|$ hence $\mathrm{FIL}^4_{\aleph_1}(Y,\mu)$ has the same power as ${}^3\mathscr{P}(Y) \times {}^\omega\mu$ this is a dull case.

Proof 1.6 (1) Reading the definition of $\mathrm{Fil}^4_{\aleph_1}(Y,\mu)$ clearly its power is $\le$ the power of $\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y) \times \mu^{|Y|}$. If $\aleph_0 \le |Y|$ then $|\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y)| \le |\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y))| = 2^{|\mathscr{P}(Y))} \times 2^{|\mathscr{P}(Y)|} \le 2^{|\mathscr{P}(Y)|+|\mathscr{P}(Y)|} = 2^{|\mathscr{P}(Y)|} = |\mathscr{P}(\mathscr{P}(Y))| \le |\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y) \times \mu^{|Y|}|$ as $\mathscr{P}(Y) + \mathscr{P}(Y) = 2^{|Y|} \times 2 = 2^{|Y|+1} = 2^{|Y|}$; so the second conclusion follows.
(2) Read the definitions.
(3) If $\alpha < \mathrm{hrtg}([Y]^{\aleph_0})$ then let f be a function from $[Y]^{\aleph_0}$ onto α and for $\beta < \alpha$ let $A_{f,\beta} = \{u \in [Y]^{\aleph_0} : f(u) < \beta\}$. So $\beta \mapsto A_{f,\beta}$ is a one-to-one function from α onto $\{A_{f,\gamma} : \gamma < \alpha\} \subseteq \mathscr{P}(\mathscr{P}(Y))$ so $|{}^Y\alpha| \le \mathscr{P}(\mathscr{P}(Y))$ and $\mathscr{P}(\mathscr{P}(Y)) \times |{}^Y\alpha| \le \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y)) \le 2^{|\mathscr{P}(Y)|+|\mathscr{P})Y)|} = 2^{|\mathscr{P}(Y)|}$. Better, for f a function from $[Y]^{\aleph_0}$ onto $\alpha < \mathscr{P}(Y)$ let $A_f = \{(y_1, y_2) : f(y_1) < f(y_2)\} \subseteq Y \times Y$. Define $F : \mathscr{P}(Y \times Y) \to \mathrm{hrtg}(Y)$ by $F(A) = \alpha$ if $A = A_f$ and f, α are as above, and $F(A) = 0$ otherwise.

So $|\mathscr{P}(\mathscr{P}(Y)) \cup \bigcup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}| \le_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y \times Y))) = |\mathscr{P}(\mathscr{P}(Y \times Y))|$. By the proof above we easily get $|\mathrm{Fil}^4_{\aleph_1}(Y)| \le_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y \times Y))$.

Claim 1.8 *[DC] Assume*

(a) $\mathfrak{a}$ *is a countable set of limit ordinals*
(b) $<_*$ *is a well ordering of* $\prod\mathfrak{a}$
(c) $\theta \in \mathfrak{a} \Rightarrow cf(\theta) \ge \kappa$ *where* $\kappa = \mathrm{hrtg}(\mathscr{P}(\omega))$ *or just* $\prod\mathfrak{a}/[\mathfrak{a}]^{<\aleph_0}$ *is* $<\kappa$*-directed.*

Then we can define $(\bar{J}, \bar{\mathfrak{b}}, \bar{\mathbf{f}})$ *such that*
(α)

(i) $\bar{J} = \langle J_i : i \le i(*)\rangle$ *where* $i(*) < \mathrm{hrtg}(\mathscr{P}(\omega))$
(ii) J_i *is an ideal on* $\mathfrak{a}$ *(though not necessarily a proper ideal)*
(iii) J_i *is increasing continuous with* i, $J_0 = \{\emptyset\}$, $J_{i(*)} = \mathscr{P}(\mathfrak{a})$
(iv) $\bar{\mathfrak{b}} = \langle \mathfrak{b}_i : i < i(*)\rangle$, $\mathfrak{b}_i \subseteq \mathfrak{a}$ *and* $J_{i+1} = J_i + \mathfrak{b}_i \ne J_i$,
(v) *so* J_i *is the ideal on* $\mathfrak{a}$ *generated by* $\{\mathfrak{b}_j : j < i\}$

(β)

(i) $\bar{\mathbf{f}} = \langle \bar{f}^i : i < i(*)\rangle$

(ii) $\bar{f}^i = \langle f^i_\alpha : \alpha < \alpha_i \rangle$
(iii) $f^i_\alpha \in \prod \mathfrak{a}$ *is* $<_{J_i}$*-increasing with* $\alpha < \alpha_i$
(iv) $\{f^i_\alpha : \alpha < \alpha_i\}$ *is cofinal in* $(\prod \mathfrak{a}, <_{J_i+(\mathfrak{a}\backslash \mathfrak{b}_i)})$

(γ)

(i) $\mathrm{cf}(\prod \mathfrak{a}) \leq \sum_{i<i(*)} \alpha_i$
(ii) *for every* $f \in \Pi\mathfrak{a}$ *for some* n *and finite set* $\{(i_\ell, \gamma_\ell) : \ell < n\}$ *such that* $i_\ell < i(*), \gamma_\ell < \alpha_{i_\ell}$ *we have* $f < \max_{\ell<n} f^{i_\ell}_{\gamma_\ell}$*, i.e.,* $(\forall \theta \in \mathfrak{a})(\exists \ell < n)[f(\theta) < f^{i_\ell}_{\gamma_\ell}(\theta)]$.

Remark 1.9 Note that there is no harm in having more than one occurence of $\theta \in \mathfrak{a}$. See more in [13], e.g. on uncountable $\mathfrak{a}$.

Proof 1.8 Note that:

$\circledast_1$ clause (γ) follows from $(\alpha) + (\beta)$.

[Why? Easily $(\gamma)(ii) \Rightarrow (\gamma)(i)$. Now let $g \in \Pi\mathfrak{a}$ and let $I_g = \{\mathfrak{b} \subseteq \mathfrak{a}$: we can find $n < \omega$ and $i_\ell < i(*)$ and $\beta_\ell < \alpha_{i_\ell}$ for $\ell < n$ such that $\theta \in \mathfrak{b} \Rightarrow (\exists \ell < n)(g(\theta) < f^{i_\ell}_{\beta_\ell}(\theta))\}$.

Easily I_g is an ideal on $\mathfrak{a}$ though not necessarily a proper ideal. Note that if $\mathfrak{a} \in I_g$ we are done. So assume $\mathfrak{a} \notin I_g$. Note that $I_g \subseteq J_{i(*)}$ hence $j_g = \min\{i \leq i(*)$: some $\mathfrak{c} \in \mathscr{P}(\mathfrak{a})\backslash I_g$ belongs to $J_i\}$ is well defined (as $\mathfrak{a} \in \mathscr{P}(\mathfrak{a})\backslash I_g \wedge \mathfrak{a} \in J_{i(*)}$). As $J_0 = \{\emptyset\}$ and clearly as $\emptyset \in I_g$, so $\mathfrak{c} = \mathfrak{a}$ witness $j_g > 0$. As $\langle J_i : i \leq i(*)\rangle$ is $\subseteq$-increasing continuous, necessarily j_g is a successor ordinal say $j_g = i_g + 1$ and let $i(g) = i_g$ and choose $\mathfrak{c} \in J_{j_g}\backslash I_g$, clearly $J_{i(g)} \subseteq I_g$ so $\mathfrak{c}$ belongs to $J_{j_g}\backslash J_{i_g}$. By clause $(\beta)(iv)$ there is $\alpha < \alpha_{i(g)}$ such that $g < f^i_\alpha \mod (J_{i(g)} + (\mathfrak{a}\backslash \mathfrak{b}_{i(g)}))$.

Now let $\mathfrak{d} = \{\theta \in \mathfrak{a} : g(\theta) < f^i_\alpha(\theta)\}$ so by the choice of α we have $\mathfrak{d} = \mathfrak{a} \mod (J_{i(g)} + (\mathfrak{a}\backslash \mathfrak{b}_{(g)}))$, which means that $\mathfrak{b}_{i(g)} \subseteq \mathfrak{d} \mod J_{i(g)}$ so as $J_{i(g)+1} = J_{i(g)} + \mathfrak{b}_{i,g}$ and $\mathfrak{c} \in J_{i(g)+1}\backslash J_{i(g)}$ clearly $\mathfrak{c} \subseteq \mathfrak{b}_{i(g)} \mod J_{i(g)}$.

But by the definition of the ideal $J_{i(g)}$ and of $\mathfrak{d}$ necessarily $\mathfrak{d} \in J_{i(g)}$ and recall $J_{i(g)} \subseteq J_{i(g)}$, contradicting the conclusion of the last sentence.]

Since (γ) follows from $(\alpha) + (\beta)$, it suffices to prove these parts. By induction on $i < \kappa$ we try to choose $(\bar{J}^i, \bar{\mathfrak{b}}^i, \bar{\mathbf{f}}^i)$ where $\bar{J}^i = \langle J_j : j \leq i\rangle$, $\bar{\mathfrak{b}}^i = \langle \mathfrak{b}^i_j : j < i\rangle$, $\bar{\mathbf{f}}^i = \langle \bar{f}^j : j < i\rangle$ which satisfies the relevant parts of the conclusion and do it uniformly from $(\mathfrak{a}, <_*)$. Once we arrive at i such that $J_i = \mathscr{P}(\mathfrak{a})$ we are done.

For $i = 0$ recalling $J_0 = \{\emptyset\}$ there is no problem.

For i limit recalling that $J_i = \cup\{J_j : j < i\}$ there is no problem and note that if $j < i \Rightarrow \mathfrak{a} \notin J_j$ then $\mathfrak{a} \notin J_i$.

So assume that $(\bar{J}^i, \mathfrak{b}^i, \bar{\mathbf{f}}^i)$ is well defined and $\mathfrak{a} \notin J_i$ and we shall define for $i + 1$.

We try to choose $\bar{g}^{i,\varepsilon} = \langle g^{i,\varepsilon}_\alpha : \alpha < \delta_{i,\varepsilon}\rangle$ and $\mathfrak{b}_{i,\varepsilon}$ by induction on $\varepsilon < \omega_1$ and for each ε we try to choose $g^{i,\varepsilon}_\alpha \in \Pi\mathfrak{a}$ by induction on α (in fact $\alpha < \mathrm{hrtg}(\Pi\mathfrak{a})$ suffice, we shall get stuck earlier) such that:

$\circledast^2_{i,\varepsilon}$

(a) if $\beta < \alpha$ then $g^{i,\varepsilon}_\beta <_{J_i} g^{i,\varepsilon}_\alpha$,

(b) if $\zeta < \varepsilon$ then $\delta_{i,\zeta} \geq \delta_{i,\varepsilon}$ and $\alpha < \delta_{i,\varepsilon}$ implies $g^{i,\zeta}_\alpha \leq g^{i,\varepsilon}_\alpha$,

(c) if $\mathrm{cf}(\alpha) = \aleph_1$ then $g^{i,\varepsilon}_\alpha$ is defined by

$$\theta \in \mathfrak{a} \Rightarrow g^{i,\varepsilon}_\alpha(\theta) = \ \mathrm{Min}\{\bigcup_{\beta \in C} g^{i,\varepsilon}_\beta(\theta) : C \text{ is a club of } \alpha\},$$

(d) if α is a limit ordinal and $\mathrm{cf}(\alpha) \neq \aleph_1, \alpha \neq 0$ then $g^{i,\varepsilon}_\alpha$ is the $<_*$-first $g \in \Pi\mathfrak{a}$ satisfying clauses (a) + (b),
(e) if we have $\langle g^{i,\varepsilon}_\beta : \beta < \alpha\rangle$, $\mathrm{cf}(\alpha) > \aleph_1$, moreover $\mathrm{cf}(\alpha) \geq \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$ and there is no g as required in clause (d) then $\delta_{i,\varepsilon} = \alpha$,
(f) if $\alpha = 0$ or α is a successor, then $g^{i,\varepsilon}_\alpha$ is the $<_*$-first $g \in \Pi\mathfrak{a}$ such that:

$\bullet_1$ $\zeta < \varepsilon \wedge \alpha < \delta_{i,\zeta} \Rightarrow g^{i,\zeta}_\alpha \leq g$,
$\bullet_2$ $\beta < \alpha \Rightarrow g^{i,\varepsilon}_\beta < g^{i,\varepsilon}_\alpha \mod J_i$,
$\bullet_3$ $\varepsilon = \zeta + 1 \Rightarrow (\forall \beta < \delta_{i,\zeta})[\neg(g \leq_{J_i} g^{i,\zeta}_\beta)]$, follows if $\alpha > 0$.

(g) J_i is the ideal on $\mathscr{P}(\mathfrak{a})$ generated by $\{\mathfrak{b}_j : j < i\}$,
(h) $\mathfrak{b}_{i,\varepsilon} \in (J_i)^+$ so $\mathfrak{b}_{i,\varepsilon} \subseteq \mathfrak{a}$,
(i) $\bar{g}^{i,\varepsilon}$ is increasing and cofinal in $(\Pi(\mathfrak{a}), <_{J_i+(\mathfrak{a}\backslash\mathfrak{b}_{i,\varepsilon})})$,
(j) $\mathfrak{b}_{i,\varepsilon}$ is such that under clauses $(h)+(i)$ the set $\{\mathrm{otp}(\mathfrak{a}\cap\theta) : \theta \in \mathfrak{b}_{i,\varepsilon}\}$ is $<_*$-minimal recalling the claim assumptions,
(k) if $\zeta < \varepsilon$ then $\mathfrak{b}_{i,\zeta} \subseteq \mathfrak{b}_{i,\varepsilon} \mod J_i$ (follows by "if $\zeta < \varepsilon$ then $g^{i,\varepsilon}_0$ is a $<_{J_i+\mathfrak{b}_{i,\zeta}}$-upper bound of $\bar{g}^{i,\zeta}$".

Clearly in stage ε we first choose $g^{i,\varepsilon}_\alpha$ by induction on α. As $\beta < \alpha \Rightarrow g^{i,\varepsilon}_\beta \neq g^{i,\varepsilon}_\alpha$ we are stuck in some $\delta_{i,\varepsilon}$ and then choose $\mathfrak{b}_{i,\varepsilon}$.

We now give details on some points:

$(*)_0$ if $\alpha = 0$ then we can choose $g^{2,\varepsilon}_0$.

[Why? Trivial.]

$(*)_1$ Clause (c) is O.K., that is: if we arrive to (ε, α), $\mathrm{cf}(\alpha) = \aleph_1$ then we can define $g^{i,\varepsilon}_\alpha$.

[Why? We already have $\langle g^{i,\varepsilon}_\alpha : \alpha < \delta\rangle$ and $\langle g^{i,\zeta}_\alpha : \alpha < \delta_{i,\zeta}, \zeta < \varepsilon\rangle$, and we define $g^{i,\varepsilon}_\delta$ as there. Now $g^{i,\varepsilon}_\delta(\theta)$ is well defined as the "Min" is taken on a non-empty set of ordinals as we are assuming $\mathrm{cf}(\delta) = \aleph_1$ and by DC, $\aleph_1$ is regular. The value is $< \theta$ because for some club C of δ, $\mathrm{otp}(C) = \omega_1$, so $g^{i,\varepsilon}_\delta(\theta) \leq \cup\{g^{i,\varepsilon}_\beta(\theta) : \beta \in C\}$ but this set is $\subseteq \theta$ while $\mathrm{cf}(\theta) > \aleph_1$ by clause (c) of the assumption. By $AC_{\aleph_0}$ we can find a sequence $\langle C_\theta : \theta \in \mathfrak{a}\rangle$ such that: C_θ is a club of δ of order type ω_1 satisfying $g^{i,\varepsilon}_\delta(\theta) = \cup\{g^{i,\varepsilon}_\alpha(\theta) : \alpha \in C_\theta\}$ hence for every club C of δ included in C_θ we have $g^{i,\varepsilon}_\delta(\theta) = \cup\{g^{i,\varepsilon}_\alpha(\theta) : \alpha \in C_\theta\}$. Now $\theta \in \mathfrak{a} \Rightarrow g^{i,\varepsilon}_\delta(\theta) = \bigcup_{\alpha \in C} g^{i,\varepsilon}_\alpha(\theta)$ when $C := \cap\{C_\sigma : \sigma \in \mathfrak{a}\}$, because C too is a club of δ recalling $\mathfrak{a}$ is countable. So if $\alpha < \delta$ then for some β we have $\alpha < \beta \in C$ hence the set $\mathfrak{c} := \{\theta \in \mathfrak{a} : g^{i,\varepsilon}_\alpha(\theta) \geq g^{i,\varepsilon}_\beta(\theta)\}$ belongs to J_i and $\theta \in \mathfrak{a}\backslash\mathfrak{c} \Rightarrow g^{i,\varepsilon}_\alpha(\theta) < g^{i,\varepsilon}_\beta(\theta) \leq g^{i,\varepsilon}_\delta(\theta)$, so indeed $g^{i,\varepsilon}_\alpha <_{J_i} g^{i,\varepsilon}_\delta$.

Lastly, why $\zeta < \varepsilon \Rightarrow g^{i,\zeta}_\delta \leq g^{i,\varepsilon}_\delta$? As we can find a club C of δ which is as above for both $g^{i,\zeta}_\delta$ and $g^{i,\varepsilon}_\delta$ and recall that clause (b) of $\circledast_{i,\varepsilon}$ holds for every $\beta \in C$. Together $g^{i,\varepsilon}_\delta$ is as required.]

$(*)_2$ $\mathrm{cf}(\delta_{i,\varepsilon}) > \aleph_1$ and even $\mathrm{cf}(\delta_{i,\varepsilon}) \geq \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$.

[Why? We have to prove that arriving to $\alpha > 0$, if $\mathrm{cf}(\alpha) < \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$ then we can choose $g^{i,\varepsilon}_\alpha$ as required. The cases $\mathrm{cf}(\alpha) = \aleph_1, \alpha = 0$ are covered by $(*)_1$, $(*)_0$ respectively, otherwise let $u \subseteq \alpha$ be unbounded of order type $\mathrm{cf}(\alpha)$, and define a function g from $\mathfrak{a}$ to the ordinals by $g(\theta) = \sup(\{g^{i,\varepsilon}_\beta(\theta) : \beta \in u\} \cup \{g^{i,\zeta}_\alpha(\theta) : \zeta < \varepsilon\})$. This is a subset of θ of cardinality $< |\mathfrak{a}| + \mathrm{cf}(\alpha)$ which is $< \theta = \mathrm{cf}(\theta)$ hence $g \in \Pi\mathfrak{a}$, easily is as required, i.e. satisfies clauses (a) + (b) and the $<_*$-first such g is $g^{i,\varepsilon}_\alpha$.]

Note that clause (e) of $\circledast_{i,\varepsilon}$ follows.

$(*)_3$ if $\zeta < \varepsilon$ then $\delta_{i,\varepsilon} \leq \delta_{i,\zeta}$.

[Why? Otherwise $g^{i,\varepsilon}_{\delta_{i,\zeta}}$ contradict clause (e) of $\circledast^2_{i,\zeta}$.]

$(*)_4$ if $\bar{g}^{i,\varepsilon} = \langle g^{i,\varepsilon}_\alpha : \alpha < \delta_{i,\varepsilon}\rangle$ is well defined and $\mathrm{cf}(\delta_{i,\varepsilon}) \geq \kappa$ then $\mathfrak{b}_{i,\varepsilon}$ is well defined.

[Why? Clearly, it suffices to prove that there is $\mathfrak{b}$ as required on $\mathfrak{b}_{i,\varepsilon}$ (in clauses (b),(i)). So toward contradiction assume that for every $\mathfrak{b} \in J_i^+$, $\bar{g}^{i,\varepsilon}$ is not $<_{J_i + (\mathfrak{a}\setminus\mathfrak{b})}$-cofinal in $\Pi\mathfrak{a}$ hence there is $h \in \Pi\mathfrak{a}$ such that $\alpha < \delta_{i,\varepsilon} \Rightarrow h \nleq_{J_i} g^{i,\varepsilon}_\alpha$ and let $h_\mathfrak{b}$ be the $<_*$-minimal such h. Let h_* be the function with domain $\mathfrak{a}$ such that $h(\theta) = \cup\{h_\mathfrak{b}(\theta) + 1 : \mathfrak{b} \in J_i^+\}$.

As $\mathrm{hrtg}(J_i^+) \leq \mathrm{hrtg}(\mathscr{P}(\mathfrak{a})) < \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$, clearly $h_* \in \Pi\mathfrak{a}$. Now for $\alpha < \delta_{i,\varepsilon}$ let $\mathfrak{d}_{i,\varepsilon,\alpha} = \{\theta \in \mathfrak{a} : g^{i,\varepsilon}_\alpha(\theta) \leq h_*(\theta)\}$. So $\langle \mathfrak{d}_{i,\varepsilon,\alpha}/J_i : \alpha < \delta_{i,\varepsilon}\rangle$ is $\leq$-increasing in the Boolean Algebra $\mathscr{P}(\mathfrak{a})/J_i$, so for some $\beta_{i,\varepsilon} < \delta_{i,\varepsilon}$ we have $\alpha \in (\beta_{i,\varepsilon}, \delta_{i,\varepsilon}) \Rightarrow \mathfrak{d}_{i,\varepsilon,\alpha} = \mathfrak{d}_{i,\varepsilon,\beta_{i,\varepsilon}} \mod J_i$. This implies $\mathfrak{d}_{i,\varepsilon}$ can serve as $\mathfrak{b}_{i,\varepsilon}$.]

To finish consider the following two cases.

<u>Case 1</u>: We succeed to carry the induction, i.e. choose $\bar{g}^{i,\varepsilon}$ for every $\varepsilon < \kappa$.

So $\langle \mathfrak{b}_{i,\varepsilon} : \varepsilon < \kappa\rangle$ is a sequence of subsets of $\mathfrak{a}$, pairwise distinct (by $\circledast^2_{\kappa,0}$ clauses (g) + (b)), but $\kappa \geq \mathrm{hrtg}(\mathscr{P}(\omega))$ and $\mathfrak{a}$ is countable; contradiction.

<u>Case 2</u>: We are stuck in $\varepsilon < \kappa$.

For $\varepsilon = 0$ there is no problem to define $g^{i,\varepsilon}_\alpha$ by induction on α till we are stuck, say in α, necessarily α is of large enough cofinality $\geq \kappa$ by $(*)_2$, and so $\bar{g}^{i,\varepsilon}$ is well defined. We then prove $\mathfrak{b}_{i,\varepsilon}$ exists by $(*)_4$ again using $<_*$.

For ε limit we can also choose $\bar{g}^\varepsilon$.

For $\varepsilon = \zeta + 1$, if $\mathfrak{a} \in J_\varepsilon$ then we are done; otherwise $g^{i,\varepsilon}_0$ as required can be chosen by $(*)_0$, and then we can prove that $\bar{g}^{i,\varepsilon}$, $\mathfrak{b}_{i,\varepsilon}$ exists as above.

Remark 1.10 From 1.8 we can deduce bounds on $\mathrm{hrtg}({}^Y(\aleph_\delta))$ when $\delta < \aleph_1$ and more like the one on $\aleph_\omega^{\aleph_0}$ (even better, the bound on $\mathrm{pp}(\aleph_\omega)$).

2 No decreasing sequence of subalgebras

In this section we concentrate on weaker axioms. We consider Theorem 1.2 under weaker assumptions than "$[\lambda]^{\aleph_0}$ is well orderable". We are also interested in replacing ω by ∂ in "no decreasing ω-sequence of $c\ell$-closed sets", but the reader may consider

$\partial = \aleph_0$ only. Note that for the full version, Ax^4_α, i.e., $[\alpha]^\partial$ is well orderable, the case of $\partial = \aleph_0$ is implied by the $\partial > \aleph_0$ version and suffices for the results. But for other versions, the axioms for different ∂'s seem incomparable.

Note that if we add many Cohens (not well ordering them) then Ax^4_λ fails below even for $\partial = \aleph_0$, whereas the other axioms are not affected. But forcing by $\aleph_1$-complete forcing notions preserve Ax_4.

Hypothesis 2.1 DC_∂ and let $\partial(*) = \partial + \aleph_1$. Actually we use only DC in 2.5(1) and DC_∂ in 2.5(3) and the later claims. We fix a regular cardinal ∂.

Definition 2.2 Below, pedantically we should, e.g. write $\mathrm{Ax}^\ell_\partial$ instead of Ax^ℓ and assume $\alpha > \mu > \kappa \geq \partial$. If $\kappa = \partial$ we may omit it.

(1) $\mathrm{Ax}^0_{\alpha,\mu,\kappa}$ means that there is a weak closure operation on λ of character (μ, κ), see Definition 0.18(1A), such that there is no $\subseteq$-decreasing ∂-sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial \rangle$ of subsets of α with $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$. We may here and below replace κ by $< \kappa$; similarly writing $\leq \mu$ has the obvious meaning; let $< |Y|^+$ means $|Y|$.

(2) Let $\mathrm{Ax}^1_{\alpha,<\mu,\kappa}$ mean there is $c\ell$, a weak closure operation on λ of character $(< \mu, \kappa)$, so may think $c\ell : [\alpha]^{\leq \kappa} \to [\alpha]^{<\mu}$ such that there is no $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial \rangle$ of members of $[\alpha]^{\leq\kappa}$ such that $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$.

(2A) Writing Y instead of κ means $c\ell : [\alpha]^{<\mathrm{hrtg}(Y)} \to [\alpha]^{<\mu}$. Let $c\ell_{[\varepsilon]} : \mathscr{P}(\alpha) \to \mathscr{P}(\alpha)$ be $c\ell^1_{\varepsilon,<\mathrm{reg}(\kappa^+)}$ as defined in 0.18(4) recalling $\mathrm{reg}(\gamma) = \mathrm{Min}\{\chi : \chi$ a regular cardinal $\geq \gamma\}$.

(2B) In parts (1) and (2) omitting μ mean $\mu = \mathrm{hrtg}(\mathscr{P}(\kappa)$ and omitting μ and κ mean $\kappa = \partial(*)$

(3) Ax^2_α means that there is $\mathscr{A} \subseteq [\alpha]^\partial$ which is well orderable and for every $u \in [\alpha]^\partial$ for some $v \in \mathscr{A}$, $u \cap v$ has power $= \partial$.

(4) Ax^3_α means that $\mathrm{cf}([\alpha]^{\leq\partial}, \subseteq)$ is below some cardinal, i.e., some cofinal $\mathscr{A} \subseteq [\alpha]^\partial$ (under $\subseteq$) is well orderable.

(5) Ax^4_α means that $[\alpha]^{\leq\partial}$ is well orderable.

(6) Above omitting α (or writing ∞) means "for every α", omitting μ we mean "$< \mathrm{hrtg}(\mathscr{P}(\partial))$".

(7) Lastly, let $\mathrm{Ax}_\ell = \mathrm{Ax}^\ell$ for $\ell = 1, 2, 3$.

So easily (or we have shown in the proof of 1.2):

Claim 2.3 *(1)* Ax^4_α *implies* Ax^3_α, Ax^3_α *implies* Ax^2_α, Ax^2_α *implies* Ax^1_α *and* Ax^1_α *implies* Ax^0_α. *Similarly for* $\mathrm{Ax}^\ell_{\alpha,<\mu,\kappa}$.

(2) In Definition 2.2(2), the last demand only $c\ell \restriction [\alpha]^{\leq\partial}$ *is relevant, in fact, an equivalent demand is that if* $\langle \beta_\varepsilon : \varepsilon < \partial \rangle \in {}^\partial\alpha$ *then for some* ε, $\beta_\varepsilon \in c\ell\{\beta_\zeta : \zeta \in (\varepsilon, \partial)\}$.

(3) If $\mathrm{Ax}^0_{\alpha,<\mu_1,<\theta}$ *and* $\theta \leq \mathrm{hrtg}(Y)$ *and*[2] $\mu_2 = \sup\{\mathrm{hrtg}(\mu_1 \times [\beta]^\theta) : \beta < \mathrm{hrtg}(Y)\}$ *<u>then</u>* $Ax^0_{\alpha,<\mu_2,<\mathrm{hrtg}(Y)}$.

Proof (1) Clearly $\mathrm{Ax}^2_{\alpha,<\mu,\kappa} \Rightarrow \mathrm{Ax}^1_{\alpha,<\mu,\kappa}$ holds similarly to the proof of 1.5; the other implications hold by inspection.

[2] Can do somewhat better; we can replace $[\alpha]^{<\mu_1}$ by $\{v \subseteq \alpha : \mathrm{otp}(v) \subseteq \mu_1\}$

(2) First assume that we have a $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial \rangle$ such that $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$. Let $\beta_\varepsilon = \min(\mathscr{U}_\varepsilon \backslash c\ell(\mathscr{U}_{\varepsilon+1}))$ for $\varepsilon < \partial$ so clearly $\bar{\beta} = \langle \beta_\varepsilon : \varepsilon < \partial \rangle$ exists; so by monotonicity $c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1, \partial)\} \subseteq c\ell(\mathscr{U}_{\varepsilon+1})$ hence $\beta_\varepsilon \notin c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1, \partial)\}$.

Second, assume that $\bar{\beta} = \langle \beta_\varepsilon : \varepsilon < \partial \rangle \in {}^\partial\alpha$ satisfies $\beta_\varepsilon \notin c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1, \partial)\}$ for $\varepsilon < \partial$. Now letting $\mathscr{U}'_\varepsilon = \{\beta_\zeta : \zeta < \partial$ satisfies $\varepsilon \leq \zeta\}$ for $\varepsilon < \partial$ clearly $\langle \mathscr{U}'_\varepsilon : \varepsilon < \partial \rangle$ exists, is $\subseteq$-decreasing and $\varepsilon < \partial \Rightarrow \beta_\varepsilon \notin c\ell(\mathscr{U}'_{\varepsilon+1}) \wedge \beta_\varepsilon \in \mathscr{U}'_\varepsilon$. So we have shown the equivalence.

(3) Let $c\ell(-)$ witness $\mathrm{Ax}^0_{\alpha,<\mu_1,<\theta}$. We define the function $c\ell'$ with domain $[\alpha]^{<\mathrm{hrtg}(Y)}$ by $c\ell'(u) = \cup\{c\ell(v) : v \subseteq u$ has cardinality $< \theta\}$.

Now

$(*)_0$ $c\ell'$ is a function from $[\alpha]^{<\mathrm{hrtg}(Y)}$ into $[\alpha]^{<\mu_2}$.

For this, it is enough to note:

$(*)_1$ if $u \in [\alpha]^{<\mathrm{hrtg}(Y)}$ then $c\ell'(u)$ has cardinality $< \mu_2 := \sup\{\mathrm{hrtg}(\mu_1 \times [\beta]^\theta : \beta < \mathrm{hrtg}(Y)\}$.

[Why? Let $C_u = \{(v, \varepsilon) : v \subseteq u$ has cardinality $< \theta$ and $\varepsilon < \mathrm{otp}(c\ell(v))$ which is $< \mu_1\}$. Clearly $|c\ell'(u)| < \mathrm{hrtg}(C_u)$ and $|C_u| = |\mu_1 \times [\mathrm{otp}(u)]^{<\theta}|$, so $(*)_1$ holds. Note that if $\alpha_* < \mu_1^+$ we can replace the demand $v \in [u]^{<\theta} \Rightarrow |c\ell(v)| < \mu_1$ by $v \in [u]^{<\theta} \Rightarrow \mathrm{otp}(c\ell(v)) < \alpha_*$.]

$(*)_2$ If $\langle u_\varepsilon : \varepsilon < \partial \rangle$ is $\subseteq$-decreasing where $u_\varepsilon \subseteq \alpha$ then $u_\varepsilon \subseteq c\ell'(u_{\varepsilon+1})$ for some $\varepsilon < \partial$.

[Why? If not we can choose a sequence $\langle \beta_\varepsilon : \varepsilon < \partial \rangle$ by letting $\varepsilon < \partial \Rightarrow \beta_\varepsilon = \min(u_\varepsilon \backslash c\ell'(u_{\varepsilon+1}))$. Let $u'_\varepsilon = \{\beta_\zeta : \zeta \in [\varepsilon, \partial)\}$. As $\langle u'_\varepsilon : \varepsilon < \partial \rangle$ is $\subseteq$-decreasing by the choice of $c\ell(-)$ for some $\varepsilon, \beta_\varepsilon \in c\ell\{\beta_\zeta : \zeta \in (\varepsilon+1, \partial)\}$, but this set is $\subseteq c\ell'(u_{\varepsilon+1})$ by the definition of $c\ell'(-)$, so we are done.]

Claim 2.4 *Assume $c\ell$ witness $\mathrm{Ax}^0_{\alpha,<\mu,\kappa}$ so $\partial \leq \kappa < \mu$ and so $c\ell : [\alpha]^{\leq\kappa} \to [\alpha]^{<\mu}$ and recall $c\ell^1_{\varepsilon,\leq\kappa} : \mathscr{P}(\alpha) \to \mathscr{P}(\alpha)$ is from 2.2(2A), 0.18(4).*

(1) $c\ell^1_{1,\leq\kappa}$ is a weak closure operation, it has character (μ_κ, κ) whenever $\partial \leq \kappa \leq \alpha$ and $\mu_\kappa = \mathrm{hrtg}(\mu \times \mathscr{P}(\kappa))$, see Definition 0.18.

(2) $c\ell^1_{reg(\kappa^+),\leq\kappa}$ is a closure operation and it has character $(< \mu'_\kappa, \kappa)$ when $\partial \leq \kappa \leq \alpha$ and $\mu'_\kappa = \mathrm{hrtg}(\mathscr{H}_{<\partial^+}(\mu \times \kappa))$.

Proof (1) By its definition $c\ell^1_{1,\leq\kappa}$ is a weak closure operation.

Assume $u \subseteq \alpha$, $|u| \leq \kappa$; non-empty for simplicity. Clearly $\mu \times [|u|]^{<\partial}$ has the same power as $\mu \times [u]^{<\partial}$. Define[3] the function G with domain $\mu \times [u]^{<\partial}$ as follows: if $\alpha < \mu$ and $v \in [u]^{\leq\partial}$ then $G((\alpha, v))$ is the α-th member of $c\ell(v)$ if $\alpha < \mathrm{otp}(c\ell(v))$ and $G((\alpha, v)) = \min(u)$ otherwise.

So G is a function from $\mu \times [u]^{\leq\partial}$ onto $c\ell^1_{1,\leq\kappa}(u)$. This proves that $c\ell^1_{1,\leq\kappa}$ has character $(< \mu_\kappa, \kappa)$ as $\mu_\kappa = \mathrm{hrtg}(\mu \times \mathscr{P}(\kappa))$.

(2) If $\langle u_\varepsilon : \varepsilon \leq \mathrm{reg}(\kappa^+) \rangle$ is an increasing continuous sequence of sets <u>then</u> $[u_{\partial^+}]^{\leq\partial} = \cup\{[u_\varepsilon]^{\leq\partial} : \varepsilon < \mathrm{reg}(\kappa^+)\}$ as $\mathrm{reg}(\kappa^+)$ is regular (even of cofinality $> \partial$ suffice) by its definition, note $\mathrm{reg}(\partial^+) = \partial^+$ when AC_∂ holds when DC_∂ holds.

[3] clearly we can replace $< \mu$ by $< \gamma$ for $\gamma \in (\mu, \mu^+)$

Second, let $u \subseteq \alpha$, $|u| \leq \kappa$ and let $u_\varepsilon = c\ell^1_{\varepsilon,\kappa}(u)$ for $\varepsilon \leq \partial^+$; it is enough to show that $|u_{\partial^+}| < \mu'_\kappa$. The proof is similar to earlier one.

Definition/Claim 2.5 Let $c\ell$ exemplify $\mathrm{Ax}^0_{\lambda,<\mu,Y}$ and Y be an uncountable set such that $\partial(*) \leq_{\rm qu} Y$.
(1) Let $\mathscr{F}_\mathfrak{y}, \mathscr{F}_{\mathfrak{y},\alpha}$ be as in the proof of Theorem 1.2 for $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu)$ and ordinal α (they depend on λ and $c\ell$ but note that $c\ell$ determines λ; so if we derive $c\ell$ by Ax^4_λ then they depend indirectly on the well ordering of $[\lambda]^\partial$) so we may write $\mathscr{F}_{\mathfrak{y},\alpha} = \mathscr{F}_\mathfrak{y}(\alpha, c\ell)$, etc.

That is, fully

$(*)_1$ for $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu)$ and ordinal α let $\mathscr{F}_{\mathfrak{y},\alpha}$ be the set of f such that:

- (a) f is a function from $Z^\mathfrak{y}$ to λ,
- (b) $\mathrm{rk}_{D[\mathfrak{y}]}(f) = \alpha$ recalling that this means $\mathrm{rk}_{D^\mathfrak{y}_1+Z^\mathfrak{y}}(f \cup 0_{Y\setminus Z^\mathfrak{y}}) = \alpha$ by Definition 0.10(2),
- (c) $D^\mathfrak{y}_2 = D^\mathfrak{y}_1 \cup \{Y\setminus A : A \in J[f, D^\mathfrak{y}_1 + Z^\mathfrak{y}]\}$, see Definition 0.13,
- (d) $Z^\mathfrak{y} \in D^\mathfrak{y}_2$,
- (e) if $Z \in D^\mathfrak{y}_2$ and $Z \subseteq Z^\mathfrak{y}$ then $c\ell(\{f(y) : y \in Z\}) \supseteq \{f(y) : y \in Z^\mathfrak{y}\}$,
- (f) $h^\mathfrak{y}$ is a function with domain $Z^\mathfrak{y}$ such that $y \in Z^\mathfrak{d} \Rightarrow h^\mathfrak{h}(y) = \mathrm{otp}(f(y) \cap \{c\ell(\{f(z) : z \in Z^\mathfrak{y}\})$.

$(*)_2$ $\mathscr{F}_\mathfrak{y} = \cup\{\mathscr{F}_{\mathfrak{y},\alpha} : \alpha$ an ordinal$\}$.

(2) Notice that $\mathscr{F}_{\mathfrak{y},\alpha}$ is a singleton or the empty set. Let $\Xi_\mathfrak{y} = \Xi_\mathfrak{y}(c\ell) = \Xi_\mathfrak{y}(\lambda, c\ell) = \{\alpha : \mathscr{F}_{\mathfrak{y},\alpha} \neq \emptyset\}$ and $f_{\mathfrak{y},\alpha}$ is the function $f \in \mathscr{F}_{\mathfrak{y},\alpha}$ when $\alpha \in \Xi_\mathfrak{y}$; it is well defined.
(3) If $D \in \mathrm{Fil}_{\partial(*)}(Y)$, $\mathrm{rk}_D(f) = \alpha$ and $f \in {}^Y\lambda$ <u>then</u> $\alpha \in \Xi_D(\lambda, c\ell)$ and $f \restriction Z^\mathfrak{y} = f_{\mathfrak{y},\alpha}$ for some $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$; moreover, $(D^\mathfrak{y}_1, D^\mathfrak{y}_2) = (D, \mathrm{dual}(J(J[f,D]))$ where $\Xi_D(\lambda, c\ell) := \cup\{\Xi_\mathfrak{y} : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y)$ and $D^\mathfrak{y}_1 = D\}$.
(4) If $D \in \mathrm{Fil}_{\partial(*)}(Y)$, $f \in {}^Y\lambda$, $Z \in D^+$ and $\mathrm{rk}_{D+Z}(f) \geq \alpha$ <u>then</u> for some $g \in \prod_{y\in Y}(f(y)+1) \subseteq {}^Y(\lambda+1)$ we have $\mathrm{rk}_D(g) = \alpha$ hence $\alpha \in \Xi_D(\lambda, c\ell)$.
(5) So we should write $\mathscr{F}_\mathfrak{y}[c\ell]$, $\Xi_\mathfrak{y}[\lambda, c\ell]$, $f_{\mathfrak{y},\alpha}[c\ell]$.

Proof As in the proof of 1.2 recalling "$c\ell$ exemplifies $\mathrm{Ax}^0_{\lambda,<\mu,\mathrm{hrtg}(Y)}$" holds, this replaces the use of F_* there; and see the proof of 2.11 below in part (3), for this we need:
⊞ if $D \in \mathrm{Fil}^1_\partial(Y)$ and $f \in {}^\kappa\partial$, <u>then</u> for some $Z \in D$ we have:

- if $Y \subseteq Z$ belongs to D then $c\ell(\mathrm{Rang}(f\restriction Y) = c\ell(\mathrm{Rang}(f\restriction Z))$.

[Why ⊞ holds? By Definition 2.2(2) using the axiom DC_∂.]

Claim 2.6 *We have ξ_2 is an ordinal and $\mathrm{Ax}^0_{\xi_2,<\mu_2,Y}$ holds <u>when</u>,(note that μ_2 is not much larger than μ_1):*

- (a) $Ax^0_{\xi_1,<\mu_1,Y}$ *so* $\partial < \mathrm{hrtg}(Y)$,
- (b) *$c\ell$ witnesses clause (a),*
- (c) $D \in Fil_{\partial(*)}(Y)$,
- (d) $\xi_2 = \{\alpha : f_{\mathfrak{y},\alpha}[c\ell]$ *is well defined for some* $\mathfrak{y} \in Fil^4_{\partial(*)}(Y,\mu_1)$ *which satisfies* $D^\mathfrak{y}_1 = D$ *and necessarily* $\mathrm{Rang}(f_{\mathfrak{y},\alpha}[c\ell]) \subseteq \xi_1\}$,

(*e*) *μ_2 is defined as $\mu_{2,3}$ where:*

(α) *let* $\mu_{2,0} = \text{hrtg}(Y)$,
(β) $\mu_{2,1} = \sup_{\beta<\mu_{2,0}} \text{hrtg}(\beta \times Fil^4_{\partial(*)}(Y, \mu_1))$,
(γ) $\mu_{2,2} = \sup_{\alpha<\mu_{2,1}} \text{hrtg}(\mu_1 \times [\alpha]^{\leq\partial})$,
(δ) $\mu_{2,3} = \sup\{\text{hrtg}({}^Y\beta \times \text{Fil}_{\partial(*)}(Y)) : \beta < \mu_{2,2}\}$
(this is an overkill).

Proof $\oplus_1$ ξ_2 is an ordinal.
[Why? To prove that ξ_2 is an ordinal we have to assume $\alpha < \beta \in \xi_2$ and prove $\alpha \in \xi_2$. As $\beta \in \xi_2$ clearly $\beta \in \Xi_{\mathfrak{y}}[c\ell]$ for some $\mathfrak{y} \in \text{Fil}^4_{\partial(*)}(Y, \mu_1)$ for which $D^{\mathfrak{y}}_1 = D$ so there is $f \in {}^Y(\xi_1)$ such that $f{\restriction}Z^{\mathfrak{y}} \in \mathscr{F}_{\mathfrak{y},\beta}$. So $\text{rk}_{D+Z[\mathfrak{y}]}(f) = \beta$ hence by 0.10 there is $g \in {}^Y\lambda$ such that $g \leq f$, i.e., $(\forall y \in Y)(g(y) \leq f(y))$ and $\text{rk}_{D+Z[\mathfrak{y}]}(g) = \alpha$. By 2.5(4) there is $\mathfrak{z} \in \text{Fil}^4_{\partial(*)}(Y, \mu_1)$ such that $D^{\mathfrak{z}}_1 = D + Z[\mathfrak{y}]$ and $g{\restriction}Z^{\mathfrak{z}} \in \mathscr{F}_{\mathfrak{z},\alpha}$ so we are done proving ξ_2 is an ordinal.]

We define the function $c\ell'$ with domain $[\xi_2]^{<\text{hrtg}(Y)}$ as follows:

$\oplus_2$ $c\ell'(u) = \{0\} \cup \{\alpha$: there is $\mathfrak{y} \in \text{Fil}^4_{\partial(*)}(Y, \mu_1)$ such that $f_{\mathfrak{y},\alpha}[c\ell]$ is well defined[4] and $\text{Rang}(f_{\mathfrak{y},\alpha}[c\ell]) \subseteq c\ell(\mathbf{v}[u])\}$.

where

$\oplus_3$ $\mathbf{v}[u] := \cup\{c\ell(v) : v \subseteq \xi_1$ is of cardinality $\leq \partial$ and is $\subseteq \mathbf{w}(v)\}$.

where

$\oplus_4$ for $v \subseteq \xi_1$ we let $\mathbf{w}(v) = \cup\{\text{Rang}(f_{\mathfrak{z},\beta}[c\ell]) : \mathfrak{z} \in \text{Fil}^4_{\partial(*)}(Y, \mu_1)$ and $\beta \in u$ and $f_{\mathfrak{z},\beta}[c\ell]$ is well defined$\}$.

Note that

$\oplus_5$ $c\ell'(u) = \{0\} \cup \{\text{rk}_D(f) : D \in \text{Fil}_{\partial(*)}(Y), Z \in D^+$ and $f \in {}^Y\mathbf{v}(u)\}$.

Note that (by 2.5(1)):

$\boxtimes_1$ for each $u \subseteq \xi_1$ and $\mathfrak{x} \in \text{Fil}^4_{\partial(*)}(Y, \mu_1)$ the set $\{\alpha < \xi_2 : f_{\mathfrak{x},\alpha}[c\ell]$ is a well defined function into $u\}$ has cardinality $< \text{wlor}(T_{D^{\mathfrak{y}}_2}(u))$, that is, $\langle f_{\mathfrak{x},\alpha}[c\ell] : \alpha \in \Xi_{\mathfrak{x}} \cap \xi_2\rangle$ is a sequence of functions from $Z^{\mathfrak{x}}$ to $u \subseteq \xi_1$, any two are equal only on a set $= \emptyset$ mod $D^{\mathfrak{x}}_2$ (with choice it has cardinality $\leq {}^{|Y|}|u|$)), call this bound $\mu'_{|u,\mathfrak{x}|}$.

Note

$\boxtimes_2$ if $u_1 \subseteq u_2 \subseteq \xi_2$ <u>then</u>

(α) $\mathbf{w}(u_1) \subseteq \mathbf{w}(u_2)$ and $\mathbf{v}(u_1) \subseteq \mathbf{v}(u_2) \subseteq \xi_1$
(β) $c\ell'(u_1) \subseteq c\ell'(u_2)$
(γ) $u \subseteq \mathbf{v}(u)$ and $\mathbf{w}[u] \subseteq \mathbf{v}[u]$
(δ) $u_1 \subseteq c\ell'(u_1)$.

[4] We could have used $\{t \in Y : f_{\eta,\alpha}[c\ell](t) \in c\ell(\mathbf{v}(u))\} \neq \emptyset \bmod D^{\mathfrak{y}}_2$; also we could have added u to $c\ell'(u)$ but not necessarily by $\boxplus_2$.

[Why? E.g. for clause (δ); assume $\alpha \in u$ and let f be a unique function from Y into $\{\alpha\}$. Hence for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)$ we have $f_{\mathfrak{y},\alpha}$ is well defined. Now $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq \mathbf{w}(u)$ by the choice of $\mathbf{w}(u)$ in $\oplus_4$ and so $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq \mathbf{v}(u)$ by clause (γ) of $\boxplus_2$ hence $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq c\ell(\mathbf{v}, u)$ by the assumption on $c\ell$, see by 2.6(a),(b) and 2.2(2). So we have $f_{\mathfrak{y},\beta}$ well defined and $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq c\ell(\mathbf{v}(u))$ so by the definition of $c\ell'(u)$ in $\oplus_2$ we have $\alpha \in c\ell'(u)$ so we are done.]

$\boxtimes_3$ if $u \subseteq \xi_2$, $|u| < \mathrm{hrtg}(Y)$ then $\mathbf{w}(u) = \{f_{\mathfrak{y},\alpha}(z) : \alpha \in u, \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1), f_{\mathfrak{y},\alpha}$ is well defined and $z \in Z^{\mathfrak{y}}\}$ is a subset of ξ_1 of cardinality $< \mathrm{hrtg}(|u| \times \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)) \le \sup\{\mathrm{hrtg}(\beta) \times \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)) : \beta < \mathrm{hrtg}(Y)\}$ which was named $\mu_{2,1}$ in 2.6$(e)(\beta)$

$\boxtimes_4$ if $u \subseteq \xi_1$ and $|u| < \mu_{2,1}$ then $\cup\{c\ell(v) : v \in [u]^{\le\partial}\}$ is a subset of μ_1 of cardinality $< \mathrm{hrtg}(\mu_1 \times [u]^{\le\partial}) \le \sup_{\alpha<\mu_{2,1}} \mathrm{hrtg}(\mu_1 \times [\alpha]^{\le\partial})$ which we call $\mu_{2,2}$ in 2.6$(e)(\gamma)$

$\boxtimes_5$ if $u \subseteq \xi_2$ and $|u| < \mathrm{hrtg}(Y)$ then $\mathbf{v}(u)$ has cardinality $< \mu_{2,2}$.

[Why? By $\oplus_3$ and $\boxtimes_3$ and $\boxtimes_4$.]

$\boxtimes_6$ if $u \subseteq \xi_2$ and $|u| < \mathrm{hrtg}(Y)$ then $c\ell'(u) \subseteq \xi_2$ and has cardinality $< \mu_{2,3}$ is defined in 2.6$(e)(\delta)$ which we call μ_2.

[Why? Without loss of generality $\mathbf{v}(u) \ne \emptyset$. By $\oplus_5$ we have $|c\ell'(u)| < \mathrm{hrtg}({}^Y\mathbf{v}(u)) \times \mathrm{Fil}_{\partial(*)}(Y))$ and by $\boxplus_5$ the latter is $\le \sup\{\mathrm{hrtg}({}^Y\beta \times \mathrm{Fil}_{\partial(*)}(Y)) : \beta < \mu_{2,2}\} = \mu_{2,3}$ recalling clause $(e)(\delta)$ of the claim, so we are done.]

$\boxtimes_7$ $c\ell'$ is a very weak closure operation on λ and has character $(< \mu_2, \mathrm{hrtg}(Y))$.

[Why? In Definition 0.18(1), clause (a) holds by the Definition of $c\ell'$, clause (b) holds by $\boxplus_6$ and as for clause (c), $0 \in c\ell'(u)$ by the definition of $c\ell'$ and $u \subseteq c\ell'(u)$ by clause (δ) of $\boxtimes_2$.]

Now it is enough to prove

$\boxtimes_8$ $c\ell'$ witnesses $\mathrm{Ax}^0_{\xi_2,<\mu_2,Y}$.

Recalling $\boxtimes_7$, toward contradiction assume $\bar{\mathscr{U}} = \langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ is $\subseteq$-decreasing, $\mathscr{U}_\varepsilon \in [\xi_1]^{<\mathrm{hrtg}(Y)}$ and $\varepsilon < \partial \Rightarrow \mathscr{U}_\varepsilon \nsubseteq c\ell(\mathscr{U}_{\varepsilon+1})$. We define $\bar{\gamma} = \langle \gamma_\varepsilon : \varepsilon < \partial\rangle$ by

$$\gamma_\varepsilon = \mathrm{Min}(\mathscr{U}_\varepsilon \backslash c\ell(\mathscr{U}_{\varepsilon+1})).$$

As AC_∂ follows from DC_∂, we can choose $\langle \mathfrak{y}_\varepsilon : \varepsilon < \partial\rangle$ such that $f_{\mathfrak{y}_\varepsilon,\gamma_\varepsilon}[c\ell]$ is well defined for $\varepsilon < \partial$.

Let for $\varepsilon < \partial$

$$u_\varepsilon = \{\gamma_\zeta : \zeta \in [\varepsilon, \partial)\}.$$

So

$(*)_1$ $u_\varepsilon \in [\xi_1]^{\le\partial} \subseteq [\xi_1]^{<\mathrm{hrtg}(Y)}$.

[Why? By clause (a) of the assumption of 2.6.]

$(*)_2$ u_ε is $\subseteq$-decreasing with ε.

[Why? By the definition.]

$(*)_3$ $\gamma_\varepsilon \in u_\varepsilon \backslash c\ell(u_{\varepsilon+1})$ for $\varepsilon < \partial$.

[Why? $\gamma_\varepsilon \in u_\varepsilon$ by the definition of u_ε.]

Now if $\zeta \in [\varepsilon, \gamma)$ then $f_{\eta_\zeta,\gamma_\zeta}[c\ell]$ is well defined and $\gamma_\zeta \in \mathscr{U}_\zeta \backslash c\ell(\mathscr{U}_{\zeta+1})$ (see the choice of γ_ε) but $\langle \mathscr{U}_\xi : \xi < \partial\rangle$ is $\subseteq$-decreasing hence $\gamma_\zeta \in \mathscr{U}_\zeta$, by the definition of $\mathbf{w}[u_\varepsilon]$, $\mathrm{Rang}(f_{\eta_\zeta,\gamma_\zeta}) \in \mathbf{w}(\mathscr{U}_\varepsilon)$, hence $\mathrm{Rang}(f_{\eta_\zeta,\gamma_\zeta}) \in \mathbf{v}(\mathscr{U}_\varepsilon) \subseteq c\ell(\mathbf{v}(\mathscr{U}_\varepsilon))$. As this holds for every $\zeta \in [\varepsilon, \gamma)$ we can deduce $u_\varepsilon = \{\gamma_\zeta : \zeta \in [\varepsilon, \partial)\} \subseteq c\ell'(\mathbf{v}(\mathscr{U}_\varepsilon))$.

Lastly, $\gamma_\varepsilon \notin \mathbf{v}(\mathscr{U}_{\varepsilon+1})$ by the choice of β_ε. So $\langle u_\varepsilon : \varepsilon < \partial\rangle$ contradict the assumption on $(\xi_1, c\ell)$. From the above the conclusion should be clear.

Claim 2.7 *Assume $\aleph_0 < \kappa = cf(\lambda) < \lambda$ hence κ is regular $\geq \partial$ of course, and D is the club filter on κ and $\bar\lambda = \langle \lambda_i : i < \kappa\rangle$ is increasing continuous with limit λ.*

Then $\lambda^+ \leq \{ rk_{D_\kappa}(f) : f \in \prod\limits_{i<\kappa^+} \lambda_i^+\}$.

Proof For each $\alpha < \lambda^+$ there is a one to one[5] function g from α into $|\alpha| \leq \lambda$ and we let $f_g \in \prod\limits_{i<\kappa} \lambda_i$ be

$$f(i) = \mathrm{otp}(\{\beta < \alpha : g(\beta) < \lambda_i\}.$$

Let

$$\begin{aligned}\mathscr{F}_\alpha = \{f : f &\text{ is a function with domain } \kappa \text{ satisfying } i < \kappa \Rightarrow f(i) < \lambda_i^+\\ &\text{such that for some one to one function } g \text{ from } \alpha \text{ into } \lambda\\ &\text{for each } i < \kappa \text{ we have } f(i) = \mathrm{otp}(\{\beta < \alpha : g(\beta) < \lambda_i\})\}.\end{aligned}$$

Now

$(*)_1$ (α) $\mathscr{F}_\alpha \neq \emptyset$ for $\alpha < \lambda^+$,
(β) $\langle \mathscr{F}_\alpha : \alpha < \lambda^+\rangle$ exists as it is well defined.

[Why? For clause (α) let $g : \alpha \to \lambda$ be one to one and so the f defined above belongs to $\mathscr{F}_\alpha$. For clause (β) see the definition of $\mathscr{F}_\alpha$ (for $\alpha < \lambda^+$).]

$(*)_2$ (α) if $f \in \mathscr{F}_\beta, \alpha < \beta < \lambda^+$ then for some $f' \in \mathscr{F}_\alpha$ we have $f' <_{J_\kappa^{\mathrm{bd}}} f$,
(β) $\langle \min\{\mathrm{rk}_D(f) : f \in \mathscr{F}_\alpha\} : \alpha < \lambda^+\rangle$ is strictly increasing hence $\min\{\mathrm{rk}_D(f) : f \in \mathscr{F}_\alpha\} \geq \alpha$.

[Why? For clause (α), let g witness "$f \in \mathscr{F}_\beta$" and define the function $f' \in \prod\limits_{i<\kappa} \lambda_i^+$ by $f'(i) = \mathrm{otp}\{\gamma < \alpha : g(\gamma) < \lambda_i\}$. So $g\restriction\alpha$ witness $f' \in \mathscr{F}_\alpha$, and letting $i(*) = \min\{i : g(\alpha) < \lambda_i\}$ we have $i \in [i(*), \kappa) \Rightarrow f'(i) < f(i)$ hence $f' <_{J_\kappa^{\mathrm{bd}}} f$ as promised. For clause (β) it follows.]

So we have proved 2.7.

Conclusion 2.8 *(1) Assume*

[5] but, of course, possibly there is no such sequence $\langle f_\alpha : \alpha < \lambda^+\rangle$

(*a*) $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$,
(*b*) $\lambda > cf(\lambda) = \kappa$ *(not really needed in part (1)).*

Then for some $\mathscr{F}_* \subseteq {}^{\kappa}\lambda =: \{f : f$ *a partial function from* κ *to* $\lambda\}$ *we have*

(α) *every* $f \in {}^{\kappa}\lambda$ *is a countable union of members of* $\mathscr{F}_*$,
(β) $\mathscr{F}_*$ *is the union of* $|\mathit{Fil}^4_{\partial(*)}(\kappa, <\mu)|$ *well ordered sets:* $\{\mathscr{F}^*_{\mathfrak{y}} : \mathfrak{y} \in \mathit{Fil}^4_{\partial(*)}(\kappa, \mu)\}$,
(γ) *moreover there is a function giving for each* $\mathfrak{y} \in \mathit{Fil}^4_{\partial(*)}(\kappa)$ *a well ordering of* $\mathscr{F}^*_{\mathfrak{y}}$.

(2) Assume in addition that $\mathrm{hrtg}(\mathit{Fil}^4_{\partial(*)}(\kappa, <\mu)) < \lambda, \mathrm{cf}(\lambda^+)$ *and* $\mathrm{hrtg}({}^{\kappa}\mu) < \lambda$ *then for some* $\mathfrak{y} \in \mathit{Fil}^4_{\partial(*)}(\kappa)$ *we have* $|\mathscr{F}^*_{\mathfrak{y}}| > \lambda$.
(3) If in part (2) we may omit the assumption on $\mathrm{cf}(\lambda^+)$ *still* $\lambda^+ = \sup\{\mathrm{otp}(\Xi_{\mathfrak{y}} \cap \lambda^+) : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa, \mu)\}$.

Proof (1) By the proof of 1.2.
(2) Assume that this fails; so for every $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa, <\mu)$, the set $S_{\mathfrak{y}} = \Xi_{\mathfrak{y}} \cap \lambda^+$ has order type $< \lambda^+$. But we are assuming $\mathrm{cf}(\lambda^+) \geq \mathrm{hrtg}(\mathscr{P}(\mathrm{Fil}^4_{\partial()*}(\kappa, \mu)))$, so there is $\gamma < \lambda^+$ such that $\gamma > \mathrm{otp}(S_{\mathfrak{y}})$ for every relevant $\mathfrak{y}$, without loss of generality $\gamma > \lambda$ and let g be a one-to-one function from γ onto λ.

We choose $f \in {}^{\kappa}\lambda$ by

$$f(i) = \mathrm{Min}(\lambda \backslash \{f_{\mathfrak{y},\alpha}(i) : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa, \mu) \; f_{\mathfrak{y},\alpha}(i) \text{ is well defined, i.e. } i \in Z[\mathfrak{y}] \text{ and } \alpha \in \Xi_{\mathfrak{y}} \text{ and } g(\mathrm{otp}(\alpha \cap \Xi_{\mathfrak{y}})) < \mu_i\}).$$

Now $f(i)$ is well defined as the minimum is taken over a non-empty set of ordinals, this holds as we substruct from λ a set which has cardinality $\leq \mu_i$ which is $< \lambda$. But f contradicts part (1). Note that in fact $f \in \prod_i \mu_i^+$.

(3) Same proof as in part (2).

Conclusion 2.9 *Assume* $Ax^0_{\lambda,<\mu,\kappa}$ *so* $\lambda > \mu$.
Then the cardinal λ^+ *is not measurable (even in cases it is regular*[6]*)* *when*

⊠ (*a*) $\lambda > cf(\lambda) = \kappa > \aleph_0$,
(*b*) $\lambda > \mathrm{hrtg}((\mathit{Fil}^4_{\partial(*)}(\kappa, \mu))$.

Proof Naturally we fix a witness $c\ell$ for $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$. Let $\mathscr{F}_{\mathfrak{y}}, \Xi_{\mathfrak{y}}, f_{\mathfrak{y},\alpha}, \mathscr{F}^{\lambda}_{\mathfrak{y},\alpha}$ be defined as in 2.5 so by claims 2.5, 2.7 we have $\cup\{\Xi_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa)\} \supseteq \lambda^+$; moreover, $\alpha \in \lambda^+ \cap \Xi_{\mathfrak{y}} \Rightarrow f_{\eta,\alpha} \in {}^{\kappa}\lambda$.

Let $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa, \mu)$ be such that $|\mathscr{F}_{\mathfrak{y}}| > \lambda$, we can find such $\mathfrak{y}$ by 2.8, as without loss of generality we can assume λ^+ is regular (or even measurable, toward contradiction). Let $Z = Z[\mathfrak{y}]$. So $\Xi_{\mathfrak{y}}$ is a set of ordinals of cardinality $> \lambda$. For $\zeta < \mathrm{otp}(\Xi_{\mathfrak{y}})$ let α_ζ be the ζ-th member of $\Xi_{\mathfrak{y}}$, so $f_{\mathfrak{y},\alpha_\zeta}$ is well defined. Toward contradiction let D be a (non-principal) ultrafilter on λ^+ which is λ^+-complete. For

[6] the regular holds many times by 2.13

$i \in Z$ let $\gamma_i < \lambda$ be the unique ordinal γ such that $\{\zeta < \lambda^+ : f_{\mathfrak{y},\alpha_\zeta}(i) = \gamma\} \in D$. As $|Z| \leq \kappa < \lambda^+$ and D is κ^+-complete clearly $\{\zeta : \bigwedge_{i \in Z} f_{\mathfrak{y},\alpha_\zeta}(i) = \gamma_i\} \in D$, so as D is a non-principal ultrafilter, for some $\zeta_1 < \zeta_2$, $f_{\mathfrak{y},\alpha_{\zeta_1}} = f_{\mathfrak{y},\alpha_{\zeta_2}}$, contradiction. So there is no such D.

Remark 2.10 Similarly if D is κ^+-complete and weakly λ^+-saturated and $\mathrm{Ax}^0_{\lambda^+,<\mu,\kappa}$ see [13].

Claim 2.11 *If* $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$, *then we can find* $\bar{C}$ *such that:*

(*a*) $\bar{C} = \langle C_\delta : \delta \in S \rangle$,
(*b*) $S = \{\delta < \lambda : \delta$ *is a limit ordinal of cofinality* $\geq \partial(*)\}$,
(*c*) C_δ *is an unbounded subset of* δ, *even a club,*
(*d*) *if* $\delta \in S$, $\mathrm{cf}(\delta) \leq \kappa$ *then* $|C_\delta| < \mu$,
(*e*) *if* $\delta \in S$, $\mathrm{cf}(\delta) > \kappa$ *then* $|C_\delta| < \mathrm{hrtg}(\mu \times [\mathit{cf}(\delta)]^\kappa)$.

Remark 2.12 (1) Recall that if we have Ax^4_λ (see 2.2(5)) then trivially there is $\langle C_\delta : \delta < \lambda, \mathrm{cf}(\delta) \leq \partial \rangle$, C_δ a club of δ of order type $\mathrm{cf}(\delta)$ as if $<_*$ well order $[\lambda]^{\leq \partial}$ we let $C_\delta :=$ be the $<_*$-minimal C which is a closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$.
(2) $\mathrm{Ax}^0_{\lambda,<\xi,\kappa}$ suffices if $\kappa < \xi < \lambda$.

Proof The "even a club" is not serious as we can replace C_δ by its closure in δ.

Let $c\ell$ witness $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$. For each $\delta \in S$ with $\mathrm{cf}(\delta) \in [\partial(*), \kappa]$ we let

$$C_\delta = \cap\{\delta \cap c\ell(C) : C \text{ a club of } \delta \text{ of order type } \mathrm{cf}(\delta)\}.$$

Now $\bar{C}' = \langle C_\delta : \delta \in S$ and $\mathrm{cf}(\delta) \in [\partial(*), \kappa]\rangle$ is well defined and exist. Clearly C_δ is a subset of δ.

For any club C of δ of order type $\mathrm{cf}(\delta) \in [\partial(*), \kappa]$ clearly $\delta \cap c\ell(C) \subseteq c\ell(C)$ which has cardinality $< \mu$.

The main point is to show that C_δ is unbounded in δ, otherwise we can choose by induction on $\varepsilon < \partial$, a club $C_{\delta,\varepsilon}$ of δ of order type $\mathrm{cf}(\delta)$, decreasing with ε such that $C_{\delta,\varepsilon} \nsubseteq c\ell(C_{\delta,\varepsilon+1})$, we use DC_∂. But this contradicts the choice of $c\ell$ recalling Definition 2.2(1).

If $\delta < \lambda$ and $\mathrm{cf}(\delta) > \kappa$ we let

$$C^*_\delta = \cap\{\cup\{\delta \cap c\ell(u) : u \subseteq C \text{ has cardinality } \leq \kappa\} : C \text{ is a club of } \delta \text{ of order type } \mathrm{cf}(\delta)\}.$$

A problem is a bound of $|C^*_\delta|$. Clearly for C a club of δ of order type $\mathrm{cf}(\delta)$ the order-type of the set $\cup\{\delta \cap c\ell(v) : v \subseteq C$ has cardinality $\leq \kappa\}$ is $< \mathrm{hrtg}(\mu \times [\mathrm{cf}(\delta)]^\kappa)$. As for "$C^*_\delta$ is a club" it is proved as above.

The following lemma gives the existence of a class of regular successor cardinals.

Lemma 2.13 *(1) Assume*

(*a*) *δ is a limit ordinal $< \lambda_*$ with* $\mathrm{cf}(\delta) = \partial$,
(*b*) *λ_i^* is a cardinal for $i < \delta$ increasing with i,*
(*c*) $\lambda_* = \Sigma\{\lambda_i^* : i < \delta\}$,
(*d*) $\lambda_{i+1}^* \geq \mathrm{hrtg}(\mu \times {}^{\kappa}(\lambda_i^*))$ *for $i < \delta$ and $(\alpha) \vee (\beta)$ hold where:*

  (α) Ax_λ^4,
  (β) $\lambda_{i+1}^* \geq \mathrm{hrtg}(\ \mathit{Fil}_{\partial(*)}^4(\lambda_i^*, \mu))$ *and* $\mathrm{hrtg}([\lambda_i^*]^{\leq\kappa}) \leq \lambda_{i+1}^*$.

(*e*) $\mathrm{Ax}_{\lambda,<\mu,\kappa}^0$ *and* $\mu < \lambda_0^*$,
(*f*) $\lambda = \lambda_*^+$.

<u>Then</u> λ is a regular cardinal.
(2) Assume $\mathrm{Ax}_\lambda^4, \lambda = \lambda_*^+, \lambda_*$ singular and $\chi < \lambda_* \Rightarrow \mathrm{hrtg}({}^{\partial}\chi) \leq \lambda_*$ <u>then</u> λ is regular.

Remark 2.14 This says that the successor of many strong limit singulars is regular.

Question 2.15 (1) Is $\mathrm{hrtg}(\mathscr{P}(\mathscr{P}(\lambda_i^*))) \geq \mathrm{hrtg}(\mathrm{Fil}_{\partial(*)}^4(\lambda_i^*))$?
(2) Is $|c\ell(f \restriction B)| \leq \mathrm{hrtg}([B]^{<\aleph_0})$ for the natural $c\ell$ and f, B as in the proof of 2.13?

Proof 2.13 (1) We can replace δ by $\mathrm{cf}(\delta)$ so without loss of generality δ is a regular cardinal so $\delta = \partial$.

So

$(*)_1$ (a) fix $c\ell : [\lambda]^{\leq\kappa} \to \mathscr{P}(\lambda)$ a witness to $\mathrm{Ax}_{\lambda,<\mu,\kappa}^0$,
(b) let $\langle C_\xi[c\ell] : \xi < \lambda, \mathrm{cf}(\xi) \geq \partial\rangle$ be as in the proof of 2.11, so $\xi < \lambda \wedge \partial \leq \mathrm{cf}(\xi) < \lambda \Rightarrow |C_\xi[c\ell]| < \lambda$.

[Why the last inequality? If $\delta < \lambda$, then there is i such that $\lambda_i^* > \mu + \mathrm{cf}(\delta)$ hence $\mathrm{otp}(C_\delta) < \mathrm{hrtg}(\mu \times [\mathrm{cf}(\delta)]^\kappa) \leq \mathrm{hrtg}([\lambda_i^*]^\kappa) < \lambda_{i+1}^*$.]

First, we shall use just $\lambda > \lambda_* \wedge (\forall\delta < \lambda)(\ \mathrm{cf}(\delta) < \lambda_*)$, a weakening of the assumption that $\lambda = \lambda_*^+$.

Now

$\boxtimes_1$ for every $i < \delta$ and $A \subseteq \lambda$ of cardinality $\leq \lambda_i^*$, we can find $B \subseteq \lambda$ of cardinality $\leq \lambda_*$ satisfying $(\forall\alpha \in A)[\alpha$ is limit $\wedge\ \mathrm{cf}(\alpha) \leq \lambda_i^* \Rightarrow \alpha = \sup(\alpha \cap B)]$.

The proof of this will take some time. By 2.11 (and 0.17) the only problem is for $Y := \{\alpha : \alpha \in A, \alpha > \sup(A \cap \alpha), \alpha$ a limit ordinal of cofinality $< \partial + \aleph_1\}$; so $|Y| \leq \lambda_i^*$. Note: if we assume Ax_λ^4 this would be immediate.

We define D as the family of sets $A \subseteq Y$ such that:

$\circledast_A^1$ for some set $C \subseteq \lambda$ of $\leq \partial$ ordinals, the set $B_C =: \cup\{\mathrm{Rang}(f_{\mathfrak{x},\zeta}) : \mathfrak{x} \in \mathrm{Fil}_{\partial(*)}^4(\lambda_i^*, \mu)$ and $\zeta \in C$ <u>or</u> for some $\xi \in C$, we have $\lambda_i^* \geq \mathrm{cf}(\xi) > \partial$ and $\zeta \in C_\xi[c\ell]\}$ satisfies $\alpha \in Y \backslash A \Rightarrow \alpha = \sup(\alpha \cap B_C)$.

Clearly

$\circledast_2$ (*a*) $Y \in D$,
(*b*) D is upward closed,
(*c*) D is closed under intersection of $\leq \partial$ hence of $< \partial(*)$ sets.

[Why? For clause (a) use $C = \emptyset$, for clause (b), note that if C witness a set $A \subseteq Y$ belongs to D then it is a witness for any $A' \subseteq Y$ such that $A \subseteq A'$. Lastly, for clause (c) if $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial^+$, as we have AC_∂, there is a sequence $\langle C_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ such that C_ε witnesses $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial^+$, then $C := \cup\{C_\varepsilon : \varepsilon < \varepsilon(*)\}$ witnesses $A := \cap\{A_\varepsilon : \varepsilon < \varepsilon(*)\} \in D$ and, again by AC_∂, we have $|C| \le \partial$.]

$\circledast_3$ if $\emptyset \in D$ then we are done.

[Why? For $a = \emptyset \in D$ let $C \subseteq \lambda$ be as promised in $\circledast_1$ and then B_C is as required; its cardinality $\le \lambda^*_{i+1}$ by 2.11.]

So assume $\emptyset \notin D$, so D is an ∂^+-complete filter on Y. As $1 \le |Y| \le \lambda^*_i$, let g be a one to one function from $|Y| \le \lambda^*_i$ onto Y and let

$\circledast_4$ (a) $D_1 := \{B \subseteq \lambda^*_i : \{g(\alpha) : \alpha \in B \cap |Y|\} \in D\}$,
(b) $\zeta := \mathrm{rk}_{D_1}(g)$,
(c) $D_2 := \{B \subseteq \lambda^*_i : B \in D_1$ or $B \notin D_1$ and $\mathrm{rk}_{D_1+(\lambda^*_i \setminus B)}(g) > \zeta\} \cup D_1$.

So D_2 is an ∂^+-complete filter on λ^*_i extending D_1.

Let $B_* \in D_2$ be such that $(\forall B')[B' \in D_2 \wedge B' \subseteq B_* \Rightarrow c\ell(\mathrm{Rang}(g \restriction B')) \supseteq (\mathrm{Rang}(g \restriction B_*)]$. Let $\mathscr{U} = \cap\{c\ell(\mathrm{Rang}(g \restriction B') : B' \in D_2\}$, so $\mathrm{Rang}(g \restriction B_*) \subseteq \mathscr{U}$, even equal.

Let h be the function with domain B_* defined by $\alpha \in B_* \Rightarrow h(\alpha) = \mathrm{otp}(g(\alpha) \cap \mathscr{U})$.

So $\mathfrak{x} := (D_1, D_2, B_*, h) \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i, \mu)$ and for some ζ we have $g \restriction B_* = f_{\mathfrak{x},\zeta}[c\ell]$.

It suffices to consider the following two subcases.

<u>Subcase 1a</u>: $\mathrm{cf}(\zeta) > \partial$.

So recalling $(*)_1(b)$, $C_\zeta[c\ell]$ is well defined and let $C := \{\zeta\}$ hence $B_C = \cup\{\mathrm{Rang}(f_{\mathfrak{x},\varepsilon}[c\ell] : \varepsilon \in C_\zeta[c\ell]\}$ so C exemplifies that the set $X := \{\alpha \in Y : \alpha > \sup(\alpha \cap B_C)\}$ belongs to D hence $X_* = \{\alpha < |Y| : g(\alpha) \in X\}$ belongs to D_1.

Now define g', a function from λ^*_i to Ord by $g'(\alpha) = \sup(g(\alpha) \cap B_C) + 1$ if $\alpha \in X_*$ and $g'(\alpha) = 0$ otherwise. Clearly $g' < g \bmod D_1$ hence $\mathrm{rk}_{D_1}(g') < \zeta$, hence there is g'', $g' <_{D_1} g'' <_{D_1} g$ such that $\xi := \mathrm{rk}_{D_1}(g'') \in C_\zeta[c\ell]$.

Now for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i)$ we have $D^{\mathfrak{y}} = D_2$ and $g'' = f_{\mathfrak{y},\xi} \bmod D^{\mathfrak{y}}_2$.

So $B =: \{\varepsilon < |Y| : g''(\varepsilon) = f_{\mathfrak{y},\xi}(\varepsilon)\} \in D^{\mathfrak{y}}_2$ hence $B \in D^+_2$. So $B \cap B_* \cap X_* \in D^+_2$ but if $\varepsilon \in B \cap B_* \cap A_*$ then $f_{\mathfrak{y},\xi}(\varepsilon) \in B_C$ and $f_{\mathfrak{y},\xi}(\varepsilon) \in \sup((B_C \cap g(\varepsilon)), g(\varepsilon))$.

This gives contradiction.

<u>Subcase 1b</u>: $\mathrm{cf}(\zeta) \le \partial$.

We choose a $C \subseteq \zeta$ of order type $\le \partial$ unbounded in ζ and proceed as in subcase 1a.

As we have covered both subcases, we have proved $\boxtimes_1$.

Recall we are assuming $\delta = \partial$; now:

$\boxtimes_2$ for every $A \subseteq \lambda$ of cardinality $\le \lambda_*$ there is $B \subseteq \lambda$ of cardinality $\le \lambda_*$ such that:

$\oplus$ $A \subseteq B$, $[\alpha + 1 \in A \Rightarrow \alpha \in B]$ and $[\alpha \in A \wedge \aleph_0 \le \mathrm{cf}(\alpha) < \lambda_* \Rightarrow \alpha = \sup(B \cap \alpha)]$.

[Why? Choose a $\subseteq$-increasing sequence $\langle A_j : j < \delta\rangle$ such that $A = \cup\{A_i : i < \delta\}$ and $j < \delta \Rightarrow |A_j| \le \lambda^*_j$, possible as $|A| \le \lambda_*$. For each $j < \delta$ there exists B_j such that the conclusion of $\boxplus_1$ holds with (A_j, B_j, λ_j^*) here standing for (A, B, λ_i) there, so $|B_j| \le \lambda_*$. So as AC_δ holds (as $\delta \le \partial$) there is a sequence $\langle \bar{B}_j : j < \delta \rangle$, each $\bar{B}_j$ as above.

Lastly, let $B = \cup\{B_j : j < \delta\}$, it is as required.]

$\boxtimes_3$ for every $A \subseteq \lambda$ of cardinality $\le \lambda_*$ we can find $B \subseteq \lambda$ of cardinality $\le \lambda_*$ such that $A \subseteq B$, $[\alpha + 1 \in B \Rightarrow \alpha \in B]$ and $[\alpha \in B$ is a limit ordinal $\wedge\ \mathrm{cf}(\alpha) < \lambda_* \Rightarrow \alpha = \sup(B \cap \alpha)]$.

[Why? We choose B_i by induction on $i < \omega \le \partial$ such that $|B_i| \le \lambda_*$ by $B_0 = A$, $B_{2i+1} = \{\alpha : \alpha \in B_{2i}$ or $\alpha + 1 \in B_{2i+1}\}$ and B_{2i+2} is chosen as B was chosen in $\boxtimes_2$ for i with B_{2i+1}, B_{2i+2} here in the role of A, B there. There is such $\langle B_i : i < \omega \rangle$ as $DC = DC_{\aleph_0}$ holds. So easily $B = \cup\{B_i : i < \omega\}$ is as required.]

Now return to our main case $\lambda = \lambda_*^+$

$\boxtimes_4$ λ_*^+ is regular.

[Why? Otherwise $\mathrm{cf}(\lambda_*^+) < \lambda_*^+$ hence $\mathrm{cf}(\lambda_*^+) \le \lambda_*$, but λ_* is singular so $\mathrm{cf}(\lambda_*^+) < \lambda_*$ hence there is a set A of cardinality $\mathrm{cf}(\lambda_*^+) < \lambda_*$ such that $A \subseteq \lambda_*^+ = \sup(A)$. Now choose B as in $\boxtimes_3$. So $|B| \le \lambda_*$, B is an unbounded subset of λ_*^+, $\alpha + 1 \in B \Rightarrow \alpha \in B$ and if $\alpha \in B$ is a limit ordinal then $\mathrm{cf}(\alpha) \le |\alpha| \le \lambda_*$, but $\mathrm{cf}(\alpha)$ is regular so $\mathrm{cf}(\alpha) < \lambda_*$ hence $\alpha = \sup(B \cap \alpha)$. But this trivially implies that $B = \lambda_*^+$, but $|B| \le \lambda_*$, contradiction.]

(2) Similar, just easier.

Remark 2.16 Of course, if we assume Ax_λ^4 <u>then</u> the proof of 2.13 is much simpler: if $<_*$ is a well ordering of $[\lambda]^{\le\partial}$ for $\delta < \lambda$ of cofinality $\le \partial$ let $C_\delta =$ the $<_*$-first closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$, see 3.3.

Claim 2.17 *Assume*

(*a*) $\langle \lambda_i : i < \kappa \rangle$ *is an increasing continuous sequence of cardinals* $> \kappa$
(*b*) $\lambda = \lambda_\kappa = \Sigma\{\lambda_i : i < \kappa\}$
(*c*) $\kappa = cf(\kappa) > \partial$
(*d*) $\mathrm{Ax}^0_{\lambda, <\mu, \kappa}$
(*e*) $\mathrm{hrtg}(Fil^4_{\partial(*)}(\kappa, \mu)) < \lambda$ *and* $\kappa, \mu < \lambda_0$
(*f*) $S := \{i < \kappa : \lambda_i^+$ *is a regular cardinal*$\}$ *is a stationary subset of* κ
(*g*) *let* $D := D_\kappa + S$ *where* D_κ *is the club filter on* κ
(*h*) $\gamma(*) = rk_D(\langle \lambda_i^+ : i < \kappa \rangle)$.

<u>Then</u> $\gamma(*)$ *has cofinality* $> \lambda$, *so* $(\lambda, \gamma(*)] \cap Reg \neq \emptyset$.

Proof 2.17 Recall 2.5 which we shall use. Toward contradiction assume that $\mathrm{cf}(\gamma(*)) \le \lambda_\kappa$, but λ_κ is singular hence for some $i(*) < \kappa$, $\mathrm{cf}(\gamma(*)) \le \lambda_{i(*)}$. Let $c\ell$ witness $\mathrm{Ax}^0_{\lambda, <\mu, \kappa}$.

Let B be an unbounded subset of $\gamma(*)$ of order type $\mathrm{cf}(\gamma(*)) \le \lambda_{i(*)}$. By renaming without loss of generality $i(*) = 0$.

For $\alpha < \gamma(*)$ let

$$\mathscr{U}_\alpha = \cup\{\text{Rang}(f_{\mathfrak{y},\alpha}) : f_{\mathfrak{y},\alpha}[c\ell] \text{ is well defined } \in \Pi\{\lambda_i^+ : i \in Z^{\mathfrak{y}}\} \text{ and } \mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa) \text{ and } D_1^{\mathfrak{y}} = D\}.$$

Clearly $\mathscr{U}_\alpha$ is well defined by 2.5; moreover, $\langle \mathscr{U}_\alpha : \alpha < \gamma(*)\rangle$ exists and $|\mathscr{U}_\alpha| \leq \text{hrtg}(\kappa \times \text{Fil}^4_{\partial(*)}(\kappa,\mu)) = \text{hrtg}(\text{Fil}^4_{\partial(*)}(\kappa,\mu))$, even $<$ recalling 0.17(4). Let $\mathscr{U} = \cup\{\mathscr{U}_\alpha : \alpha \in B\}$ so $|\mathscr{U}| \leq \text{hrtg}(\text{Fil}^4_{\partial(*)}(\kappa,\mu)) + |B|$.

We define $f \in \prod_{i<\kappa} \lambda_i^+$ by

(α) $f(i)$ is: $\sup(\mathscr{U} \cap \lambda_i^+) + 1$ if $\text{cf}(\lambda_i^+) > |\mathscr{U}|$ and zero otherwise.

So

(β) $f \in \prod_{i<\kappa} \lambda_i^+$.

Clearly

(γ) $\{i < \kappa : f(i) = 0\} = \emptyset \mod D$.

Let $\alpha(*) = \text{rk}_D(f)$, it is $< \text{rk}_D(\langle \lambda_i^+ : i < \kappa\rangle) = \gamma(*)$, so by clause ($\gamma$) there is $\beta(*) \in B$ such that $\alpha(*) < \beta(*) < \gamma(*)$ hence for some $g \in \prod_{i<\kappa} \lambda_i^+$ we have $\text{rk}_D(g) = \beta(*)$ and $f < g \mod D$, so for some $\mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa)$ we have $D_1^{\mathfrak{y}} = D_\kappa + S$ and $g \in \mathscr{F}_{\mathfrak{y},\beta(*)}$, hence $f(i) < g(i) < f_{\mathfrak{y},\beta(*)}(i) \in \mathscr{U} \cap \lambda_i^+$ for every $i \in Z^{\mathfrak{y}} \cap S$.

So we get an easy contradiction to the choice of g.

Claim 2.18 *Assume $c\ell$ witness $\text{Ax}^0_{\alpha,<\mu,\kappa}$ and $\text{hrtg}(Y) \in [\kappa,\mu)$. The ordinals γ_ℓ, $\ell = 0, 1, 2$ are nearly equal see, i.e. ⊛ below holds where:*
$\boxtimes$

(a) $\gamma_0 = \text{hrtg}({}^Y\alpha)$*, a cardinal*
(b) $\gamma_1 = \cup\{\text{rk}_D(\gamma) : \gamma = \text{rk}_D(\alpha)$ *for some* $D \in \mathit{Fil}_{\partial(*)}(Y)\}$
(c) $\gamma_2 = \sup\{\mathit{otp}(\Xi_{\mathfrak{y}}[c\ell]) + 1 : \mathfrak{y} \in \mathit{Fil}^4_{\partial(*)}(Y)\}$

⊛ (α) $\gamma_2 \leq \gamma_1 \leq \gamma_0$
(β) *γ_0 is the union of* $\text{Fil}^4_{\partial(*)}(Y)$ *sets each of order type* $< \gamma_2$
(γ) *γ_0 is the disjoint union of* $< \text{hrtg}(\mathscr{P}(\text{Fil}^4_{\partial(*)}(Y)))$ *sets each of order type* $< \gamma_2$
(δ) *if* $\gamma_0 > \text{hrtg}(\mathscr{P}(\mathit{Fil}^4_{\partial(*)}(Y)))$ *and* $\gamma_0 \geq |\gamma_2|^+$ *then* $|\gamma_0| \leq |\gamma_2|^{++}$ *and* $\text{cf}(|\gamma_2|^+) < \text{hrtg}(\mathscr{P}(\mathit{Fil}^4_{\partial(*)}(Y)))$.

Proof 2.18 Straightforward, see 0.17.

3 Concluding remarks

In May 2010, David Aspero asked whether it is true that I have results along the following lines (or that it follows from such a result):

If GCH holds and λ is a singular cardinal of uncountable cofinality, then there is a well-order of $\mathscr{H}(\lambda^+)$ definable in $(\mathscr{H}(\lambda^+), \in)$ using a parameter.

The answer is yes by [9, 4.6,pg.117] but we elaborate this below somewhat more generally. Much earlier Gitik [4] had proved (using suitable large cardinals) the consistency of "ZF + every infinite cardinal has cofinality $\aleph_0$, i.e. $\aleph_0$ is the only regular cardinal". This naturally raises the question what suffices to have a class of regulars. Gitik told me that in Luming 2008 Woodin has conjectured:

$\boxplus$ let **V** be a model of ZF + DC, suppose that κ is a singular strong limit cardinal of cofinality ω_1 and $|\mathscr{H}(\kappa)| = \kappa$. Is then $\mathscr{P}(\kappa)$ well orderable?

Now [9] gives some information. The results here (3.1) confirm $\boxplus$.

Claim 3.1 [DC] *Assume that μ is a singular cardinal of cofinality $\kappa > \aleph_0$ (no* GCH *needed), the parameter $X \subseteq \mu$ codes in particular the tree $\mathscr{T} = {}^{\kappa >}\mu$ and the set $\mathscr{P}(\mathscr{P}(\kappa))$, in particular, from X a well-ordering of $[\mu]^{<\kappa} \cup \mathscr{P}(\mathscr{P}(\kappa))$ is definable. Then (with this parameter) we can define a well-ordering of the set of κ-branches of the tree $({}^{\kappa >}\lambda, \triangleleft)$.*

Proof 3.1 Proof of 3.1:

Let $\langle \mathrm{cd}_i : i < \kappa \rangle$ satisfies

$\boxplus_1$ cd_i is a one-to-one function from ${}^i\mu$ into μ, (definable from X uniformly (in i))

$\boxplus_2$ let $<_\kappa$ be a well-ordering of $\mathrm{Fil}^4_\kappa(\kappa)$ definable from X.

For $\eta \in {}^\kappa\mu$ let $f_\eta : \kappa \to \mu$ be defined by $f_\eta(i) = \mathrm{cd}_i(\eta{\restriction}i)$, so $\bar{f} = \langle f_\eta : \eta \in {}^\kappa\mu \rangle$ is well defined.

Let $\bar{\mathscr{F}} = \langle \mathscr{F}_\mathfrak{y} : \mathfrak{y} \in \mathrm{Fil}^4_\kappa(\kappa) \rangle$ be as in Theorem 1.2 with μ, κ here standing for λ, Y there; there is such $\bar{\mathscr{F}}$ definable from X as X codes also a well-ordering of $[\mu]^{\aleph_0}$, see §1.

So for every $\eta \in {}^\kappa\mu$ there is $\mathfrak{y} \in \mathrm{Fil}^4_\kappa(\kappa)$ such that $f{\restriction}Z_\mathfrak{y} \in \mathscr{F}_\mathfrak{y}$ and $D_1^\mathfrak{y}$ contains all co-bounded subsets of κ so let $\mathfrak{y}(\eta)$ be the $<_\kappa$-first such $\mathfrak{y}$. Now we define a well ordering $<_*$ of ${}^\kappa\mu$: for $\eta, \nu \in {}^\kappa\mu$ let $\eta <_* \nu$ iff $\mathrm{rk}_{D_1[\mathfrak{y}(\eta)]}(f_\eta{\restriction}Z_{\mathfrak{y}(\eta)}) < \mathrm{rk}_{D_1(\mathfrak{y}(\nu))}(f_\nu{\restriction}Z_{\mathfrak{y}(\nu)})$ or equality holds and $\mathfrak{y}(\eta) < \mathfrak{y}(\nu)$.

This is O.K. because

$(*)$ if $\eta \neq \nu \in {}^\kappa\mu$ then $f_\eta(i) \neq f_\nu(i)$ for every large enough $i < \kappa$ (i.e. $i \geq \min\{j : \eta(j) \neq \nu(j)\}$.

Conclusion 3.2 [DC] Assume μ is a singular cardinal of uncountable cofinality κ and $\mathscr{H}(\mu)$ is well orderable of cardinality μ and $X \subseteq \mu$ codes $\mathscr{H}(\mu)$ and a well ordering of $\mathscr{H}(\mu)$. Then we can (with this X as parameter) define a well-ordering of $\mathscr{P}(\mu)$; hence of $\mathscr{H}(\mu^+)$.

Proof 3.2 Proof of 3.2:

Let $\langle \mu_i : i < \kappa \rangle$ be an increasing sequence of cardinals $< \mu$ with limit μ; wlog X code this sequence. Clearly $2^{\mu_i} < \mu$ (as $|{}^{\mu_i}2| \leq |\mathscr{H}(\mu)| = \mu$, and $2^{\mu_i} = \mu$ is impossible).

Let $\langle \mathrm{cd}^*_i : i < \kappa \rangle$ satisfies

$\boxplus_2$ cd_i^* is a one-to-one function from $\mathscr{P}(\mu_i)$ into μ, (definable uniformly from X).

So $\text{cd}_* : \mathscr{P}(\mu) \to {}^\kappa\mu$ defined by $(\text{cd}_*(A))(i) = \text{cd}_i^*(A \cap \mu_i)$ for $A \subseteq \mu, i < \kappa$, is a one-to-one function from $\mathscr{P}(\mu)$ into ${}^\kappa\mu$. Now use 3.1.

We return to 2.13(2)

Claim 3.3 [DC] *(1) The cardinal λ^+ is regular* <u>*when*</u>*:*

$\boxplus$ (*a*) $\text{Ax}^4_{\lambda^+}$, *i.e.* $[\lambda^+]^{\aleph_0}$ *is well orderable,*
(*b*) $|\alpha|^{\aleph_0} < \lambda$ *for* $\alpha < \lambda$*,*
(*c*) λ *is singular.*

(2) Also there is $\bar{e} = \langle e_\delta : \delta < \lambda^+\rangle$, $e_\delta \subseteq \delta = \sup(e_\delta)$, $|e_\delta| \leq cf(\delta)^{\aleph_0}$.

Remark 3.4 Compare with 2.13; we use here more choice, but cover more cardinals.

Proof 3.3 Let $<_*$ be a well ordering of the set $[\lambda^+]^{\aleph_0}$.

As earlier let $F : {}^\omega(\lambda^+) \to \lambda^+$ be such that there is no $\subset$-decreasing sequence $\langle c\ell_F(u_n) : n < \omega\rangle$ with $u_n \subseteq \lambda^+$. Let $\Omega = \{\delta \leq \lambda^+ : \delta$ a limit ordinal, $\text{cf}(\delta) < \lambda\}$, so $\text{otp}(\Omega) \in \{\lambda^+, \lambda^+ + 1\}$.

We define $\bar{e} = \langle e_\delta : \delta \in \Omega\rangle$ as follows.

<u>Case 1</u>: $\text{cf}(\delta) = \aleph_0$, e_δ is the $<_*$-minimal member of $\{u \subseteq \delta : \delta = \sup(u)$ and $\text{otp}(u) = \omega\}$.

<u>Case 2</u>: $\text{cf}(\delta) > \aleph_0$.

Let $e_\delta = \cap\{c\ell_F(C) : C$ a club of $\delta\}$.

So

$(*)_1$ e_δ is an unbounded subset of δ of order type $< \lambda$.

[Why? If $\text{cf}(\delta) = \aleph_0$ then e_δ has order type ω which is $< \lambda$ by clause (b) of the assumption.

If $\text{cf}(\delta) > \aleph_0$ then for some club C of δ, $e_\delta = c\ell_F(C)$ has $\text{otp}(e_\delta) \leq |c\ell_F(C)| \leq (\text{cf}(\delta))^{\aleph_0} < \lambda$. The last inequality holds as $\text{cf}(\delta) \leq \lambda$ as $\delta < \lambda^+$, $\text{cf}(\delta) \neq \lambda$ as λ is singular by clause (c) of the assumption, and lastly $((\text{cf}(\delta)^{\aleph_0}) < \lambda$ by clause (b) of the assumption.]

This is enough for part (2). Now we shall define a one-to-one function f_α from α into λ by induction on $\alpha \in \Omega$ as follows: let $\text{pr}_\lambda : \lambda \times \lambda \to \lambda$ be a pairing function so one to one (can add "onto λ"); if we succeed then f_{λ^+} cannot be well defined so $\lambda^+ \notin \Omega$ hence $\text{cf}(\lambda^+) \geq \lambda$, but λ is singular so $\text{cf}(\lambda^+) = \lambda^+$, i.e. λ^+ is not singular so we shall be done proving part (1).

The inductive definition is:

$\boxplus$ (*a*) <u>if $\alpha \leq \lambda$</u> <u>then</u> f_α is the identity
(*b*) <u>if $\alpha = \beta + 1 \in [\lambda, \lambda^+)$</u> <u>then</u> for $i < \alpha$ we let $f_\alpha(i)$ be
- $1 + f_\beta(i)$ if $i < \beta$
- 0 if $i = \beta$

(*c*) <u>if $\alpha \in \Omega$</u> so α is a limit ordinal, $e_\alpha \subseteq \alpha = \sup(e_\alpha)$, e_α of cardinality $< \lambda$ and we let f_α be defined by: for $i < \alpha$ we let $f_\alpha(i) = \text{pr}_\lambda(f_{\min(e_\alpha \setminus (i+1))}(i), \text{otp}(e_\alpha \cap i))$.

We later add:

Claim 3.5 *[ZFC] Assume $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$ and $\mu = \mu^{\aleph_0} + 2^{2^\kappa}$.*
(1) From some $X \subseteq \mu$ we can define a well ordering of some set $\mathscr{G} \subseteq {}^\kappa\mu$ such that ${}^\kappa\mu = \{\sup\{f_n : n < \omega\} : f_n \in \mathscr{G}$ for $n < \omega\}$.
(2) If moreover $2^{2^\theta} \le \mu$ where $\theta = \kappa^{\aleph_0}$ then from some $X \subseteq \mu$ we can define a well ordering of ${}^\kappa\mu$.

Proof 3.5 (1) Let $X \subseteq \mu$ code $\mathscr{P}(\mathscr{P}(\kappa))$ and ${}^\omega\mu$ which is as in 3.1. Unlike the proof of 3.1 we do not use the $\mathrm{cd}_i (i < \kappa)$ and we use the family of $\aleph_1$-complete filters on κ, the rest should be clear.
(2) As $\theta = \theta^{\aleph_0}$ there is a one-to-one onto function $\mathrm{cd} : {}^\omega\theta \to \theta$ onto θ, and for $i < \omega$ let $\mathrm{cd}_i : \theta \to \theta$ be such that:

$(*)_1$ if $\mathrm{cd}(\eta) = \zeta$, then $\mathrm{cd}_0(\zeta) = \ell g(\eta)$ and $\mathrm{cd}_{1+i}(\zeta) = \eta(i)$ for $i < \ell g(\eta)$.

Let D be $\{A \subseteq \theta$: for some $u \in [\theta]^{\le\aleph_0}$ we have $A \supseteq \{\varepsilon < \theta : u \subseteq \{\mathrm{cd}_i(\varepsilon) : i < \omega\}\}$, so

$(*)_2$ D is an $\aleph_1$-complete filter on θ.

[Why? Should be clear.]

$(*)_3$ for $f \in {}^\theta\mu$ let g, g_f be the unique function g with doman θ such that:

- if $\varepsilon < \kappa$ and $i < \mathrm{cd}_0(\varepsilon)$, then $\mathrm{cd}_{1+i}(\varepsilon) < \theta \Rightarrow \mathrm{cd}_{1+i}(g(\varepsilon)) = f(\mathrm{cd}_{1+i}(\varepsilon))$ and $\mathrm{cd}_0(g(\varepsilon)) = \mathrm{cd}_0(\varepsilon)$ and $f(\varepsilon) = 0$ otherwise

[Why g_f exists? Just think.]

$(*)_4$ if $f \in {}^\theta\mu, \alpha = \mathrm{rk}_D(g_f)$ and $\mathfrak{y} = \mathfrak{y}_{g_f}$ as in the proof of 3.1 for g_f, then:

(a) from $g_f \restriction Z_{\mathfrak{y}}$ we can define f (using some $Y \subseteq \kappa$ as a parameter)
(b) $\mathrm{Rang}(f) \subseteq \{\mathrm{cd}_{1+i}(g_f(\varepsilon)) : \varepsilon \in Z_{\mathfrak{y}}$ and $i < \mathrm{cd}_0(g_f(\varepsilon))\}$.

[Why? Clause (a) follows clause (b). Clause (b) holds as for every $\xi < \kappa$, the set $\{\varepsilon < \theta : \xi \in \{\mathrm{cd}_{1+i}(\varepsilon) : i < \mathrm{cd}_0(\varepsilon)\}\} \in D$.]
We continue as in the proof of 3.1.

Conclusion 3.6 [DC] *Assume $[\lambda]^{\aleph_0}$ is well ordered for every λ.*
(1) If 2^{2^κ} is well ordered then for every λ, $[\lambda]^\kappa$ is well ordered.
(2) For any set Y, there is a derived set Y_* so called $\mathrm{Fil}^4_{\aleph_1}(Y)$ of power near $\mathscr{P}(\mathscr{P}(Y))$ such that $\Vdash_{Levy(\aleph_0,Y)}$ "for every λ, ${}^Y\lambda$ is well ordered".

Proof 3.6 (1) By 3.1.
(2) Follows easily.

References

1. Apter, A., Magidor, M.: Instances of dependent choice and the measurability of $\aleph_{\omega+1}$. Ann. Pure Appl. Logic **74**, 203–219 (1995)

2. Easton, W.B.: Powers of regular cardinals. Annals Math. Logic **1**, 139–178 (1970)
3. Galvin, F., Hajnal, A.: Inequalities for cardinal powers. Ann. Math. **101**, 491–498 (1975)
4. Gitik, M.: All uncountable cardinals can be singular. Israel J. Math. **35**, 61–88 (1980)
5. Paul, B.: Larson and Saharon Shelah, Splitting stationary sets from weak forms of choice. MLQ Math. Log. Q. **55**(3), 299–306 (2009). arXiv: 1003.2477
6. Shelah, S. et al.: *Tba*, In preparation. Preliminary number: Sh:F1039
7. Shelah, S.: A note on cardinal exponentiation. J. Symbolic Logic **45**(1), 56–66 (1980)
8. Shelah, S.: Cardinal arithmetic, Oxford Logic Guides, vol. 29. The Clarendon Press, Oxford University Press, New York (1994)
9. Shelah, S.: Set theory without choice: not everything on cofinality is possible. Arch. Math. Logic **36**(2), 81–125 (1997). arXiv: math/9512227
10. Shelah, S.: Applications of PCF theory. J. Symbolic Logic **65**(4), 1624–1674 (2000). arXiv:math/9804155
11. Shelah, S.: PCF arithmetic without and with choice. Israel J. Math. **191**(1), 1–40 (2012). arXiv: 0905.3021
12. Shelah, S.: Pseudo PCF. Israel J. Math. **201**(1), 185–231 (2014). arXiv: 1107.4625
13. Shelah, S.: ZF + DC + AX_4. Arch. Math. Logic **55**(1–2), 239–294 (2016). arXiv: 1411.7164