STABLE FRAMES AND WEIGHTS
SH839

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ABSTRACT. We would like to generalize imaginary elements, weight of $\mathrm{tp}(a, M, N)$, **P**-weight, **P**-simple types, etc. from [She90, Ch.III,V,§4] to the context of good frames. This requires allowing the vocabulary to have predicates and function symbols of infinite arity, but it seems that we do not suffer any real loss.

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References like [Sh:950, Th0.2$_{=\mathrm{Ly5}}$] mean that the internal label of Theorem 0.2 is y5 in Sh:950. The reader should note that the version in my website is usually more up-to-date than the one in arXiv.

Annotated Content

§ 0. Introduction

We assume $\mathfrak{s}$ is a good λ-frame with some extra properties from [She09e][1] (e.g., as in the assumption of [She09e, §12]) so we shall assume knowledge of [She09e] and the basic facts on good λ-frames from [She09c].[2]

We can look at results from [She90] which were not regained in beautiful λ-frames (from [She09e, §12]). Well, of course, we are far from the main gap for the original $\mathfrak{s}$ ([She90, Ch.XIII]) and there are results which are obviously more strongly connected to elementary classes, particularly ultraproducts. This leaves us with parts of type theory: regular and semi-regular types, weight, **P**-simple[3] types, "hereditarily orthogonal to **P**" (the last two were defined and investigated in [She78, Ch.V, §0 + Def.4.4–Ex.4.15], [She90, Ch.V §0, pg.226; Def.4.4–Ex.4.15, pg.277-284]).

Some of Hrushovski's profound works are a continuation of [She78, Ch.V, §4] and [She90, Ch.V, §4]; note that "a type q is p-simple (or **P**-simple)" and "q is hereditarily orthogonal to p (or **P**)" here and in [She90] are essentially the[4] "internal" and "foreign" in Hrushovshi.

For more on understanding regular types in the first order case, see both [She04] and Laskowski and the author in [LS15].

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* * *

This paper was Part I of the original [Sheb], which has existed (and circulated to some extent) since 2002. The second and third parts have been split off into [Shec], [Shea]. They have been continued with Laskowski in [LS06] and [LS11], respectively.

Notation 0.1. 1) As in [She78] and [She90], M and N are models, M has vocabulary τ_M, $|M|$ is its universe and $\|M\|$ its cardinality. We write $\mathrm{ortp}(-)$ for the orbital type.

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[1] =[She09d, Ch.III].

[2] =[She09d, Ch.II]. As above, these two are the same paper, but the page numbering is different.

[3] The motivation is that for suitable **P** (e.g. a single regular type), on the one hand

$$\mathrm{stp}(a, A) \not\perp \mathbf{P} \Rightarrow \text{“}\mathrm{stp}(a/E, A) \text{ is } \mathbf{P}\text{-simple for some equivalence relation definable over } A\text{”},$$

and on the other hand if $\mathrm{stp}(a_i, A)$ is **P**-simple for $i < \alpha$ then $\sum\limits_{i<\alpha} w(a_i, A) \cup \{a_j : j < i\}$ does not depend on the order in which we list the a_i-s. Note that **P** here is $\mathscr{P}$ there.

[4] Note: "foreign to **P**" and "hereditarily orthogonal to **P**" are equivalent. Now (with $\mathbf{P} = \{p\}$ for simplicity)

(a) $q(x)$ is $p(x)$-*simple* when, in $\mathfrak{C}$, we have $q(\mathfrak{C}) \subseteq \mathrm{acl}(A \cup \bigcup p_i(\mathfrak{C}))$ for some set A.

(b) $q(x)$ is $p(x)$-*internal* when, in $\mathfrak{C}$, we have $q(\mathfrak{C}) \subseteq \mathrm{dcl}(A \cup p(\mathfrak{C}))$ for some set A.

Also note

(α) Internal implies simple.

(β) If we aim at computing weights, it is better to stress acl as it covers more (as was the preference in [She78, Ch.V, §4]).

(γ) But the difference is minor, and in existence it is better to stress dcl (as preferred by Hrushovski).

(δ) Also, it is useful that

$$\big\{F \restriction (p(\mathfrak{C}) \cup q(\mathfrak{C}) : F \text{ an automorphism of } \mathfrak{C} \text{ over } p(\mathfrak{C}) \cup \mathrm{dom}(p)\big\}$$

is trivial when $q(x)$ is p-internal but not so for p-simple (though form a pro-finite group).

2) We use $\mathfrak{k}$ to denote an AEC (or more generally, an essentially-$[\lambda, \mu)$ AEC — see Definition 2.1).

3) $\mathfrak{s}$ will be a good λ-frame (see Definition 2.11, and more fully in [She09d, Ch.II, §1].)

4) $\mathbb{E}$ will denote a smooth $\mathfrak{k}_\lambda$-equivalence relation, where $\mathfrak{k}$ is an AEC (see §2).

§ 1. Weight and P-weight

On 'good$^+$,' see [She09d, Ch.III, Definition 1.3(1), pg.382] and [She09d, Ch.III, Claim 1.5(1), pg.382], which relies on [She09d, Ch.II, §3], [She01].
On 'type-full,' see Definition [She09d, Ch.II, §6]: it means

$$\mathcal{S}_{\mathfrak{s}}(M) = \mathcal{S}^1_{\mathfrak{s}}(M) := \{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{s}} N,\ a \in N\}.$$

On primes and $K^{3,\mathrm{qr}}_{\mathfrak{s}}$, see [She09d, Ch.III, 5.15, pg.461].
On $K^{3,\mathrm{vq}}_{\mathfrak{s}} \supseteq K^{3,\mathrm{qr}}_{\mathfrak{s}}$, see [She09d, Ch.III, Definition 5.9, pg.456].
On orthogonality, see [She09e, §6].
Let $p(M_2) := \{c \in M_2 : c \text{ satisfies the type } p\}$.

Context 1.1. 1) $\mathfrak{s}$ is a type-full good$^+$ λ-frame with primes, $K^{3,\mathrm{vq}}_{\mathfrak{s}} = K^{3,\mathrm{qr}}_{\mathfrak{s}}$, $\perp = \underset{\mathrm{wk}}{\perp}$ and $p \perp M \Leftrightarrow p \underset{\mathrm{su}}{\perp} M$ (see [Shec, §3]). Note that as $\mathfrak{s}$ is full,

$$\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(M) = \{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{s}} N,\ a \in N\};$$

also, $\mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}[\mathfrak{s}] = (K^{\mathfrak{s}}, \leq_{\mathfrak{k}_{\mathfrak{s}}})$ is the AEC.

2) $\mathfrak{C}$ is an $\mathfrak{s}$-monster so it is $K^{\mathfrak{s}}_{\lambda^+}$-saturated over λ, and $M <_{\mathfrak{s}} \mathfrak{C}$ means $M \leq_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M \in K_{\mathfrak{s}}$. As $\mathfrak{s}$ is full, it has regulars (see [She09d, Ch.III, §10]).

3) Let $\mathbf{P}$ denote a subset of $\{\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) : M <_{\mathfrak{s}} \mathfrak{C}\}$.

4) Let nf($\mathbf{P}$) be

$\{p :$ there exist M_0, M_1, M_2, and $p_2 \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_2)$ such that $M_0 \leq_{\mathfrak{s}} M_2 <_{\mathfrak{s}} \mathfrak{C},\ M_1 \leq_{\mathfrak{s}} M_2,\ p_2$ does not fork over M_0 nor over M_1, $p_2 \restriction M_1 \in \mathbf{P}$, and $p_2 \restriction M_0 = p\}$.

5) Let nf$^+$($\mathbf{P}$) be defined similarly, but demanding $M_0 = M_2$.

Observation 1.2. $\mathfrak{s}^{\mathrm{reg}}$ *satisfies all the above except being full.*

Remark 1.3. Recall $\mathfrak{s}^{\mathrm{reg}}$ is derived from $\mathfrak{s}$, replacing $\mathcal{S}_{\mathfrak{s}}(M)$ by $\{p \in \mathcal{S}_{\mathfrak{s}}(M) : p \text{ regular}\}$ (see [She09d, Ch.III, 10.18, pg.573]).

Proof. See [She09d, Ch.III, $10.19_{\texttt{=L10.p19tex}}$, pg.573] and [She09d, Ch.III, Definition $10.18_{\texttt{=L10.p18tex}}$, pg.573]. $\square_{1.2}$

Claim 1.4. 1) *If* $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ *then we can find* b, N, M', *and a finite* $\mathbf{J}$ *such that:*

⊛ (*a*) $M \leq_{\mathfrak{s}} M' \leq_{\mathfrak{s}} N$
(*b*) $\mathbf{J} \subseteq N$ *is a finite independent set in* (M', N).
(*c*) $c \in \mathbf{J} \Rightarrow \mathrm{ortp}(c, M', N)$ *is regular, not forking over* M *(recalling that* ortp *stands for 'orbital type;' see* [She09c, §1]*)*.
(*d*) $(M, N, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$
(*e*) $b \in N$ *realizes* p, *and* $\mathrm{ortp}(b, M', N)$ *does not fork over* M.

2) *If* M *is brimmed, we can add*
(*f*) $(M, N, b) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ *and* $M' = M$.

3) *In* (2), $|\mathbf{J}|$ *depends only on* (p, M), *and only up to isomorphism.*

4) *If* M *is brimmed, then we can work in* $\mathfrak{s}$(brim) *and get the same* $\|\mathbf{J}\|$ *and* N *(so* $N \in K_{\mathfrak{s}}$ *is brimmed).*

Remark 1.5. We may wonder: can we get $M' = M$?

1) Even in the first-order case this is not always true. Define a model M as follows:

(A) τ_M (the vocabulary of M) will be $\{E, E_0, E_1\}$, where all three members are binary predicates.

(B) $|M|$ (the universe of M) will be
$$\{\bar{k} = \langle k_0, k_1\rangle : (\exists n < \omega)\big[k_0, k_1 \in [n^2, (n+1)^2)\big]\}.$$

(C) $E^M := \big\{(\bar{k}^1, \bar{k}^2) : (\exists n < \omega)\big[k^1_0, k^1_1, k^2_0, k^2_1 \in [n^2, (n+1)^2)\big]\big\}$

(D) For $\ell = 0, 1$, $E^M_\ell := \big\{(\bar{k}^1, \bar{k}^2) \in E^M : k^1_\ell = k^2_\ell\big\}$.

Now,

$\circledast$ (a) $T := \mathrm{Th}(M)$ is superstable (and even $\aleph_0$-stable).
(b) $I(\aleph_\alpha, T) = 2^{|\alpha|}$ for all infinite ordinals α.
(c) T has NDOP, NOTOP, and is shallow.
(d) No $p \in \mathcal{S}(M)$ is regular.

2) Recall that if we replace T by T^{eq}, the answer is yes.

Proof. 1) We try to choose N_ℓ, a_ℓ, q_ℓ by induction on $\ell < \omega$ such that:

$(*)$ (a) $N_0 = M$
(b) $N_\ell \le_{\mathfrak{s}} N_{\ell+1}$
(c) $q_\ell \in \mathcal{S}_{\mathfrak{s}}(N_\ell)$ (so possibly $q_\ell \notin \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N_\ell)$).
(d) $q_0 = p$
(e) $q_{\ell+1} \restriction N_\ell = q_\ell$
(f) $q_{\ell+1}$ forks over N_ℓ, so now <u>necessarily</u> $q_\ell \in \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N_\ell)$.
(g) $(N_\ell, N_{\ell+1}, a_\ell) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
(h) $r_\ell = \mathrm{ortp}(a_\ell, N_\ell, N_{\ell+1})$ is regular.
(i) r_ℓ either is $\perp M$ or does not fork over M.

If we succeed to carry the induction for all $\ell < \omega$, let $N := \bigcup_{\ell<\omega} N_\ell$. As this is a countable chain (recalling that $\mathfrak{K}_{\mathfrak{s}}$ has amalgamation), there is $q \in \mathcal{S}(N)$ such that $\ell < \omega \Rightarrow q \restriction N_\ell = q$ and as q is not algebraic (because each q_n is not), and $\mathfrak{s}$ is full, clearly $q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$; but q contradicts the finite character of non-forking. So for some $n \ge 0$ we are stuck, but this cannot occur if $q_n \in \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N_n)$.
[Why? Because we are assuming that $\mathfrak{s}$ is type-full. Alternatively, we can use $\mathfrak{s}^{\mathrm{reg}}$, recalling that by 1.2, we know that $\mathfrak{s}^{\mathrm{reg}}$ has enough regulars and then we can apply [She09d, Ch.III, 8.3, p.516].[5]]

So for some $b \in N_n$ we have $q_n = \mathrm{ortp}(b, N_n, N_n)$; i.e. b realizes q_n hence it realizes p.

Let
$$\mathbf{J} := \{a_\ell : \mathrm{ortp}(a_\ell, N_\ell, N_{\ell+1}) \text{ does not fork over } N_0\}.$$
By [She09d, Ch.III, 8.5, p.518] and '$K^{3,\mathrm{uq}}_{\mathfrak{s}} = K^{3,\mathrm{qr}}_{\mathfrak{s}}$,' we have $(M, N_n, \mathbf{J}) = (N_0, N_n, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$, so we are done.

2) Let $N, b, \mathbf{J}$ be as in part (1) with $|\mathbf{J}|$ minimal. We can find $N' \le_{\mathfrak{s}} N$ such that $(M, N', b) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ and we can find $\mathbf{J}'$ such that $\mathbf{J}' \subseteq N'$ is independent regular in (M, N') and maximal under those demands. Then we can find $N'' \le_{\mathfrak{s}} N'$ such that $(M, N'', \mathbf{J}') \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$. If $\mathrm{ortp}_{\mathfrak{s}}(b, N'', N') \in \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N'')$ is not orthogonal to M, we can contradict the maximality of $\mathbf{J}'$ in N' as in the proof of part (1), so

[5] =[She09e, 8.7=L6.2, p.94]

$\mathrm{ortp}_{\mathfrak{s}}(b, N'', N') \perp M$ (or $\notin \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N)$). Also without loss of generality $(N'', N', b) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$, so by [She09d, Ch.III, 8.5, p.518] we have $(M, N', \mathbf{J}') \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$. Hence there is an isomorphism f from N' onto N'' which is the identity of $M \cup \mathbf{J}'$ (by the uniqueness for $K^{3,\mathrm{qr}}_{\mathfrak{s}}$). So using $(N', f(b), \mathbf{J}')$ for $(N, b, \mathbf{J})$ we are done.

3) If not, we can find $N_1, N_2, \mathbf{J}_1, \mathbf{J}_2, b$ such that $M \le_{\mathfrak{s}} N_\ell \le_{\mathfrak{s}} N$ and the quadruple $(M, N_\ell, \mathbf{J}_\ell, b)$ is as in (a)-(e)+(f) of part (1)+(2) for $\ell = 1, 2$. Assume toward contradiction that $|\mathbf{J}_1| \neq |\mathbf{J}_2|$, so without loss of generality $|\mathbf{J}_1| < |\mathbf{J}_2|$.

By "$(M, N_\ell, b) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$," without loss of generality, $N_2 \le_{\mathfrak{s}} N_1$.

By [She09d, Ch.III, 6(3), p.569], $\mathbf{J}_1 \cup \{c\}$ is independent in (M, N_1) for some $c \in \mathbf{J}_2 \backslash \mathbf{J}_1$, in contradiction to '$(M, N, \mathbf{J}_1) \in K^{3,\mathrm{vq}}_{\mathfrak{s}}$' by [She09d, Ch.III, 6(4), p.569].

4) Similarly. $\square_{1.4}$

Definition 1.6. 1) For $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, let the *weight* of p, $w(p)$, be the unique natural number such that if $M \le_{\mathfrak{s}} M'$, M' is brimmed, and $p' \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M')$ is a non-forking extension of p <u>then</u> it is the unique $|\mathbf{J}|$ from Claim 1.4(3). (It is a natural number.)

2) Let $w_{\mathfrak{s}}(a, M, N) = w(\mathrm{ortp}_{\mathfrak{s}}(a, M, N))$.

Claim 1.7. 0) *In Definition* 1.6, *the* weight $w(p)$ *of* p *is well-defined. (That is, it does not depend on* M''*.)*

Also, if $M_1 \le_{\mathfrak{s}} M_2$ *and* $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_2)$ *does not fork over* M_1, <u>*then*</u>

$$w(p) = w(p \restriction M_1).$$

1) *If* $p \in \mathcal{S}^{\mathrm{bs}}(M)$ *is regular,* <u>*then*</u> $w(p) = 1$.

2) *If* $\mathbf{J}$ *is independent in* (M, N) *and* $c \in N$, <u>*then*</u> *for some* $\mathbf{J}' \subseteq \mathbf{J}$ *with* $\le w_{\mathfrak{s}}(c, M, N)$ *elements,* $\{c\} \cup (\mathbf{J} \setminus \mathbf{J}')$ *is independent in* (M, N).

Proof. Easy by now. $\square_{1.7}$

Note that the converse of 1.7(1) may fail, even for elementary clauses. Also note that the use of $\mathfrak{C}$ in Definition 1.8 is for transparency only and can be avoided; see 1.12 below. Lastly, in clause 1.8(1)(B) below, there is no need to demand $M_* \le_{\mathfrak{s}} M$.

Definition 1.8. 1) We say that $\mathbf{P}$ is an M^*-*based family* (inside $\mathfrak{C}$) <u>when</u>:

- (A) $M^* <_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M^* \in K_{\mathfrak{s}}$.
- (B) $\mathbf{P} \subseteq \bigcup\{\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) : M <_{\mathfrak{s}} \mathfrak{C}\}$ (so $M \in K_{\mathfrak{s}}$).
- (C) $\mathbf{P}$ is preserved by automorphisms of $\mathfrak{C}$ over M^*.

2) Let $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, where $M <_{\mathfrak{s}} \mathfrak{C}$.

- (A) We say that p is *orthogonal* to $\mathbf{P}$ (or $\mathbf{P}$-foreign) <u>when</u> if $M \le_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$, $q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N) \cap \mathrm{nf}(\mathbf{P})$, and $p_1 \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ extends p *and* does not fork over M, <u>then</u> q is orthogonal to p_1.

  We say that p is *hereditarily orthogonal* to $\mathbf{P}$ when above, we allow p_1 to fork over M.
- (B) We say that p is $\mathbf{P}$-regular <u>when</u> p is regular, not orthogonal to $\mathbf{P}$ and if $q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M')$, $M \le_{\mathfrak{s}} M' <_{\mathfrak{s}} \mathfrak{C}$, and q is a forking extension of p <u>then</u> q is hereditarily orthogonal to $\mathbf{P}$.
- (C) p is weakly $\mathbf{P}$-regular <u>if</u> it is regular and is not orthogonal to some $\mathbf{P}$-regular p'.

3) $\mathbf{P}$ is normal <u>when</u> $\mathbf{P}$ is a set of regular types and each of them is $\mathbf{P}$-regular.

4) For $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $M <_{\mathfrak{s}} \mathfrak{C}$, let $w_{\mathbf{P}}(q)$ be defined as the natural number satisfying the following:

⊛ If $M \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2 \le_{\mathfrak{s}} \mathfrak{C}$, M_ℓ is $(\lambda,*)$-brimmed, $b \in M_2$, ${\rm ortp}_{\mathfrak{s}}(b, M_1, M_2)$ is a non-forking extension of q, $(M_1, M_2, b) \in K^{3,{\rm pr}}_{\mathfrak{s}}$, $(M_1, M_2, \mathbf{J}) \in K^{3,{\rm qr}}_{\mathfrak{s}}$, and $\mathbf{J}$ is regular in (M_1, M_2) (i.e. independent and $c \in \mathbf{J} \Rightarrow {\rm ortp}_{\mathfrak{s}}(c, M_1, M_2)$ is a regular type) <u>then</u> $w_{\mathbf{P}}(q) = |\mathbf{J}_1|$, where

$$\mathbf{J}_1 := \{c \in \mathbf{J} : {\rm ortp}_{\mathfrak{s}}(c, M_1, M) \text{ is weakly } \mathbf{P}\text{-regular}\}.$$

4A) If $\mathbf{P}$ is the set of regular types in $\mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$, then $w(q) = w_{\mathbf{P}}(p)$.

5) We replace $\mathbf{P}$ by p if $\mathbf{P} = \{p\}$, where $p \in \mathcal{S}^{\rm bs}(M^*)$ is regular (see 1.9(1)).

Claim 1.9. 1) *If $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is regular <u>then</u> $\{p\}$ is an M-based family and is normal.*

2) *Assume $\mathbf{P}$ is an M^*-based family. If $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $M^* \le_{\mathfrak{s}} M \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ <u>then</u> $w_{\mathbf{P}}(q)$ is well defined (and is a natural number).*

3) *Suppose $\mathbf{P} = {\rm nf}^+(\mathbf{P})$. <u>Then</u> we can find $\mathbf{J}, \mathbf{J}_1$ as in Definition 1.8(4), <u>but</u> ${\rm ortp}(c, M_1, M)$ is $\mathbf{P}$-regular for every $c \in \mathbf{J}_1$.*

4) *In 1.8(4), we have $w(q) \ge w_{\mathbf{P}}(p)$.*

Proof. Should be clear. $\square_{1.9}$

Discussion 1.10. 1) It is tempting to try to generalize the notion of $\mathbf{P}$-simple ($\mathbf{P}$-internal in Hrushovski's terminology) and semi-regular. An important property of those notions in the first order case is that: e.g.

(∗) If ${\rm stp}(\bar a, A, \mathfrak{C}) \not\perp p$ and p is regular, then for some equivalence relation E definable over A, ${\rm ortp}(\bar a/E, A) \not\perp p$ and is $\{p\}$-simple.

The aim of defining $\{p\}$-simple is:

(A) For an element $a \in \mathfrak{C}$ and $A \subseteq \mathfrak{C}^{\rm eq}_T$, we can define the p-weight $w_p(a, A)$. Moreover, if $|A| = \|M\|$ and a realizes p, then $w_p(a, M) = 1$.

(B) The p-weight of such elements behaves like finite sequences from a vector space — so their weights behave like dimensions of vector spaces.

(C) We have appropriate density results.

Discussion 1.11. 1) Assume ($\mathfrak{s}$ is full and) that to every $p \in \mathcal{S}^{\rm na}_{\mathfrak{s}}(M)$ we attach some $a_p \in M$, a so-called *base*. (E.g. in [She90], this means the canonical base ${\rm Cb}(p)$.) We can define for $\bar a, \bar b \in {}^{\omega>}\mathfrak{C}$ when ${\rm ortp}(\bar a, \bar b, \mathfrak{C})$ is stationary (and/or non-forking). We should check the basic properties. See §3.

2) Assume $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is regular, definable over $\bar a^*$ (in the natural sense). We may wonder if the niceness of the dependence relation holds for $p \restriction \bar a^*$?

If you feel that the use of a monster model is not natural in our context, how do we "translate" a set of types in $\mathfrak{C}^{\rm eq}$ preserved by every automorphism of $\mathfrak{C}$ which is the identity on A? By using a "place" defined by:

Definition 1.12. 1) A *local place* is a pair $\mathbf{a} = (M, A)$ such that $A \subseteq M \in K_{\mathfrak{s}}$ (compare with [Shea, §1]).

2) The places $(M_1, A_1), (M_2, A_2)$ are *equivalent* if $A_1 = A_2$ (call it A) and there is a sequence $\langle N_\ell : \ell \le n\rangle \subseteq K_{\mathfrak{s}}$ with $N_0 := M_1$, $N_n := M_2$, $A \subseteq \bigcap\limits_{\ell \le n} N_\ell$, and

$N_\ell \le_{\mathfrak{s}} N_{\ell+1} \vee N_{\ell+1} \le_{\mathfrak{s}} N_\ell$ for each $\ell < n$. We write $(M_1, A_1) \sim (M_2, A_2)$ or $M_1 \sim_A M_2$.

3) For a local place $\mathbf{a} = (M, A)$, let $K_{\mathbf{a}} = K_{(M,A)} = \{N : (N, A) \sim (M, A)\}$, so in $(M, A)/\sim$ we fix both A as a set and the type it realizes in M over $\varnothing$.

4) We call such class $K_{\mathbf{a}}$ a *place.*

5) We say that $\mathbf{P}$ is an invariant set[6] of types in a place $K_{(M,A)}$ <u>when</u>:

- (A) $\mathbf{P} \subseteq \{\mathcal{S}^{\rm bs}_{\mathfrak{s}}(N) : N \sim_A M\}$
- (B) Membership in $\mathbf{P}$ is preserved by isomorphism over A.
- (C) If $N_1 \le_{\mathfrak{s}} N_2$ are both in $K_{(M,A)}$ and $p_2 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_2)$ does not fork over N_1 then $p_2 \in \mathbf{P} \Leftrightarrow p_2 \restriction N_1 \in \mathbf{P}$.

6) We say $M \in K_{\mathfrak{s}}$ is brimmed over A <u>when</u> for some N we have $A \subseteq N \le_{\mathfrak{s}} M$ and M is brimmed over N.

Claim 1.13. **[Claim/Definition]**

1) *If $A \subseteq M \in K_{\mathfrak{s}}$ and $\mathbf{P}_0 \subseteq \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ <u>then</u> there is at most one invariant set $\mathbf{P}^+$ of types in the place $K_{(M,A)}$ such that $\mathbf{P}^+ \cap \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M) = \mathbf{P}_0$ and*

$$M \le_{\mathfrak{s}} N \wedge p \in \mathbf{P}^+ \cap \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N) \Rightarrow \text{“}p \text{ does not fork over } M\text{”}.$$

2) *If, in addition, M is brimmed[7] over A <u>then</u> we can omit the last demand in part* (1).

3) *If $\mathbf{a} = (M_1, A)$ and $(M_2, A) \in K_{\mathbf{a}}$, then $K_{(M_2,A)} = K_{\mathbf{a}}$.*

Proof. Easy. $\square_{1.13}$

Definition 1.14. 1) If in 1.13 there is such a $\mathbf{P}^+$, we denote it by $\text{inv}(\mathbf{P}_0, A) = \text{inv}(\mathbf{P}_0, A, M)$.

2) If $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $\mathbf{P}_0 = \{p\}$, then let $\text{inv}(p) = \text{inv}(p, M) = \text{inv}(\{p\}) := \text{inv}(\mathbf{P}_0, M)$.

3) We say $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ *does not split* (or *is definable*) over A <u>when</u> $\text{inv}(\{p\}, A, M)$ is well-defined.

Claim 1.15. *Suppose $M \in K^{\mathfrak{s}}_\lambda$, $A \subseteq M$, and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$. <u>Then</u> we have*

$$\text{‘}(a) \Leftrightarrow (b),\text{’}$$

where:

- *(a) p does not split over A.*
- *(b) If $M \le_{\mathfrak{s}} N \in K^{\mathfrak{s}}_\lambda$, $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is a non-forking extension of p, and π is an automorphism of N over A, <u>then</u> $\pi(q) = q$.*

Proof. Straightforward. $\square_{1.15}$

[6] Really a class.

[7] “M is brimmed over A” (see [She09c, §1]) means that there is a $\le_{\mathfrak{s}}$-increasing sequence $\langle M_\alpha : \alpha \le \delta \rangle$ such that $M_0 \supseteq A$, $M_\delta := M$, and $M_{\alpha+1}$ is $\le_{\mathfrak{s}}$-universal over M_α for all $\alpha < \lambda$.

§ 2. Imaginary elements, an essential-(μ,λ)-AEC, and frames

§ 2(A). **Essentially AEC.**

We consider revising the definition of an AEC $\mathfrak{k}$, by allowing function symbols in $\tau_{\mathfrak{k}}$ with infinite number of places while retaining local characters; e.g. if $M_n \le_{\mathfrak{k}} M_{n+1}$ and $M = \bigcup_{n<\omega} M_n$ is uniquely determined. Before this, we introduce the relevant equivalence relations. In this context, we can give name to equivalence classes for equivalence relations on infinite sequences.

Definition 2.1. We say that $\mathfrak{k}$ is an essentially-$[\lambda,\mu)$AEC or ess-$[\lambda,\mu)$-AEC (or $[\lambda,\mu)$-EAEC[8]) if ($\lambda < \mu$ and) it is an object consisting of:

I. (a) A vocabulary $\tau = \tau_{\mathfrak{k}}$, which has predicates and function symbols of possibly infinite arity but $\le \lambda$.
(b) A class $K = K_{\mathfrak{k}}$ of τ-models.
(c) A two-place relation $\le_{\mathfrak{k}}$ on K.
(Note that $\lambda < \mu$, and we allow $\mu = \infty$).

We demand:

II. (a) If $M_1 \cong M_2$ then $M_1 \in K \Leftrightarrow M_2 \in K$.
(b) If $(N_1, M_1) \cong (N_2, M_2)$ then $M_1 \le_{\mathfrak{k}} N_1 \Leftrightarrow M_2 \le_{\mathfrak{k}} N_2$.
(c) Every $M \in K$ has cardinality $\in [\lambda, \mu)$.
(d) $\le_{\mathfrak{k}}$ is a partial order on K.

III$_1$. If $\langle M_i : i < \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing and the cardinality of $\bigcup_{i<\delta} M_i$ is less than μ, then there is a unique $M \in K$ such that $|M| = \bigcup_{i<\delta} |M_i|$ and

$$i < \delta \Rightarrow M_i \le_{\mathfrak{k}} M.$$

III$_2$. If in addition, $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} N$ then $M \le_{\mathfrak{k}} N$.

IV. If $M_1 \subseteq M_2$ and $M_\ell \le_{\mathfrak{k}} N$ for $\ell = 1, 2$ then $M_1 \le_{\mathfrak{k}} M_2$.

V. If $A \subseteq N \in K$, then there is M satisfying $A \subseteq M \le_{\mathfrak{k}} N$ and $\|M\| \le \lambda + |A|$. (Here it is enough to restrict ourselves to the case $|A| \le \lambda := \mathrm{LST}_{\mathfrak{k}}$.)

Definition 2.2. 1) We say $\mathfrak{k}$ is an ess-λ-AEC if it is an ess-$[\lambda, \lambda^+)$-AEC.

2) We say $\mathfrak{k}$ is an ess-AEC if there is λ such that it is an ess-$[\lambda, \infty)$-AEC, so $\lambda = \mathrm{LST}(\mathfrak{k})$.

3) If $\mathfrak{k}$ is an ess-$[\lambda,\mu)$-AEC and $\lambda \le \lambda_1 < \mu_1 \le \mu$ then let

$$K^{\mathfrak{k}}_{\lambda_1} = (K_{\mathfrak{k}})_{\lambda_1} = \{M \in K_{\mathfrak{k}} : \|M\| = \lambda_1\}$$

and $K^{\mathfrak{k}}_{\lambda_1,\mu_1} = \{M \in K_{\mathfrak{k}} : \lambda_1 \le \|M\| < \mu_1\}$.

4) We define $\Upsilon^{\mathrm{or}}_{\mathfrak{k}}$ as in [She09d, Ch.IV, 0.8(2), p.648-9].[9]

5) We may omit the "essentially" when arity$(\tau_{\mathfrak{k}}) = \aleph_0$, where (arity$(\mathfrak{k})$ = arity$(\tau_{\mathfrak{k}})$ and)

$$\mathrm{arity}(\tau) := \min\{\kappa : \text{ every predicate and function symbol has arity } < \kappa\}$$

for a vocabulary τ.

We now consider the claims on ess-AECs.

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[8] And we may write (μ, λ) instead of $[\lambda, \mu)$.
[9] = [She09b, 0.8(2)$_{=\mathrm{L11.1.3A}}$].

Claim 2.3. *Let $\mathfrak{k}$ be an ess-$[\lambda,\mu)$-AEC.*

1) *The parallel of* **Axs.(III)**$_1$,**(III)**$_2$ *hold with a directed family $\langle M_t : t \in I\rangle$.*

2) *If $M \in K$ we can find $\langle M_{\bar a} : \bar a \in {}^{\omega>}M\rangle$ such that:*

- (a) $\bar a \subseteq M_{\bar a} \le_{\mathfrak{k}} M$
- (b) $\|M_{\bar a}\| = \lambda$
- (c) *If $\bar b$ is a permutation of $\bar a$ then $M_{\bar a} = M_{\bar b}$.*
- (d) *if $\bar a$ is a subsequence of $\bar b$ then $M_{\bar a} \le_{\mathfrak{k}} M_{\bar b}$.*

3) *If $N \le_{\mathfrak{k}} M$ we can add in (2) that $\bar a \in {}^{\omega>}N \Rightarrow M_{\bar a} \subseteq N$.*

4) *If for simplicity*

$$\lambda_* := \lambda + \sup\Big\{ \sum_{R\in\tau_{\mathfrak{k}}} |R^M| + \sum_{F\in\tau_{\mathfrak{k}}} |F^M| : M \in K_{\mathfrak{k}} \text{ has cardinality } \lambda\Big\}$$

then $K_{\mathfrak{k}}$ and $\{(M,N) : N \le_{\mathfrak{k}} M\}$ are essentially $\mathrm{PC}_{\chi,\lambda_*}$-classes,[10] *where*

$$\chi := \big|\{M/{\cong} : M \in K^{\mathfrak{k}}_\lambda\}\big|$$

(noting that $\chi \le 2^{2^\theta}$). That is, $M \in K_{\mathfrak{k}}$ iff there is a sequence $\overline{M} = \langle M_{\bar a} : \bar a \in {}^{\omega>}A\rangle$ satisfying clauses (b)–(d) of part (2) such that $A = \bigcup_{\bar a \in {}^{\omega>}A} |M_{\bar a}|$,

$$\bar a \in {}^{\omega>}A \Rightarrow M_{\bar a} \le_{\mathfrak{k}} M,$$

and M has universe A. (Also, M is uniquely determined by $\overline{M}$.)

Similarly for $\le_{\mathfrak{k}}$. Note that if, in $\tau_{\mathfrak{k}}$, there are no two distinct symbols with the same interpretation in every $M \in K_{\mathfrak{k}}$, then $|\tau_{\mathfrak{k}}| \le 2^{2^\lambda}$.

5) *The results on omitting types in* [She99] *or* [She09d, Ch.IV, 0.9, p.649][11] *hold. I.e. if $\alpha < (2^{\lambda_*})^+ \Rightarrow K^{\mathfrak{k}}_{\sqsupseteq_\alpha} \neq \varnothing$ then $\theta \in [\lambda,\mu) \Rightarrow K_\theta \neq \varnothing$ and there is an* EM-*model: i.e. $\Phi \in \Upsilon^{\rm or}_{\mathfrak{k}}$ with $|\tau_\Phi| = |\tau_{\mathfrak{k}}| + \lambda$ and $\mathrm{EM}(I,\Phi)$ having cardinality $\lambda + |I|$ for any linear order I. We may replace λ_* by $\chi := \big|\{M/{\cong} : M \in K^{\mathfrak{k}}_\lambda\}\big|$.*

6) *The lemma on the equivalence of being universal model homogeneous and of being saturated (see* [She09f, 3.18=L3.10] *or* [She09d, Ch.II, 1.14, p.237][12]*) still holds.*

7) *We can generalize the results of* [She09d, Ch.II, §1] *on deriving an ess-(∞,λ)-AEC from an ess-λ-AEC.*

Proof. The same proofs as in [She09a][13] for AECs.

On the generalization mentioned in 2.3(7), see more in [Shec, §1]. The point is that, in the language of [Shec, §1], our $\mathfrak{k}$ is a (λ,μ,κ)-AEC (and automatically has primes). $\square_{2.3}$

Remark 2.4. 1) In 2.3(4), we can decrease the bound on χ if we have a nicer definition of K_λ: e.g. 'if $\mathrm{arity}(\tau) \le \kappa$ then $\chi = 2^{\lambda^{<\kappa}+|\tau|}$,' where

$$\mathrm{arity}(\tau) := \min\{\kappa : \text{every predicate and function symbol of } \tau \text{ has arity} < \kappa\}.$$

2) Above, we may use $|\tau_{\mathfrak{s}}| \le \lambda$, $\mathrm{arity}(\tau_{\mathfrak{k}}) = \aleph_0$ to get that

$$\big\{(M,\bar a)/{\cong} : M \in K^{\mathfrak{k}}_\lambda,\ \bar a \in {}^\lambda M \text{ lists } M\big\}$$

has cardinality $\le 2^\lambda$. (See also 2.18.)

[10] On the definition of $\mathrm{PC}_{\chi,\lambda_*}$, see [She09a, 1.4(3)].
[11] = [She09b, 0.9=L0n.8,0.2=L0n.11]
[12] = [She09c, 1.14=L0.19]
[13] =[She09d, Ch.I]

3) In 2.7 and 2.10 below, if we omit "$\mathbb{E}$ is small" and have

$$\lambda_1 := \sup\{|\mathrm{seq}(M)/\mathbb{E}_M| : M \in K^{\mathfrak{k}}_\lambda\} < \mu,$$

then $\mathfrak{k}\langle \mathbb{E}\rangle$ is an ess-$[\lambda_1, \mu)$-AEC.

4) In Definition 2.1, we may omit Axiom V and define $\mathrm{LST}(\mathfrak{k}) \in [\lambda, \mu]$ naturally, and if $M \in K^{\mathfrak{k}}_\lambda \Rightarrow \mu > |\mathrm{seq}(M)/\mathbb{E}_M|$ then in 2.10(1) below we can omit "$\mathbb{E}$ is small."

5) Can we preserve the finiteness of the arity in such a "transformation?" This is not clear. Note that a natural candidate is trying to code $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ by $\{\bar a : \bar a \in {}^{\omega >}M\}$, where there are $M_0 \le_{\mathfrak{s}} M_1$ such that $M \le_{\mathfrak{s}} M_1$, $\mathrm{ortp}(a_\ell, M_0, M_1)$ is parallel to p, and $\bar a$ is independent in (M_0, M_1). If (e.g.) $K_{\mathfrak{s}}$ is saturated this helps, but still we suspect that it may fail.

6) What is the meaning of ess-$[\lambda, \mu)$-AEC? Can we just look at $\langle M_t : t \in I\rangle$, I directed, with $t \le_I s \Rightarrow M_t \le_{\mathfrak{s}} M_{\mathfrak{s}} \in K_\lambda$? For isomorphism types we take a kind of completion and so make more pairs isomorphic, but $\bigcup_{t\in I} M_t$ does not determine $\overline{M} = \langle M_t : t \in I\rangle$, and the completion may depend on this representation.

7) If we like to avoid this and this number is λ', then we should change the definition of $\mathrm{seq}(N)$ (see 2.5(b)) to

$$\mathrm{seq}'(N) = \{\bar a : \ell g(\bar a) = \lambda,\ a_0 < \mu_*,\ \text{and for some } M \le_{\mathfrak{s}} N \text{ from } K^{\mathfrak{k}}_\lambda,$$
$$\langle a_{1+\alpha} : \alpha < \lambda\rangle \text{ lists the members of } M\}.$$

§ 2(B). Imaginary Elements and Smooth Equivalence Relations.

Now we return to our aim of getting canonical bases for orbital types.

Definition 2.5. Let $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ be a λ-AEC, or just an ess-$[\lambda, \mu)$-AEC. (If $\mathfrak{k}_\lambda = \mathfrak{k}_{\mathfrak{s}}$ we may write $\mathfrak{s}$ instead of $\mathfrak{k}_\lambda$; see 2.11.) We say that $\mathbb{E}$ is a *smooth $\mathfrak{k}$-equivalence relation* when:

- (A) $\mathbb{E}$ is a function with domain $K_{\mathfrak{k},\lambda}$ mapping M to $\mathbb{E}_M$.
- (B) For $M \in K^{\mathfrak{k}}_\lambda$, $\mathbb{E}_M$ is an equivalence relation on a subset of[14]

  $$\mathrm{seq}(M) := \{\bar a \in {}^\lambda M : M \restriction \mathrm{rang}(\bar a) \le_{\mathfrak{k}} M\}$$

  (so $\bar a$ is not necessarily without repetitions). Note that $\mathfrak{k}$ determines λ (pedantically, when it is non-empty).
- (C) If $M_1 \le_{\mathfrak{k}} M_2$ then $\mathbb{E}_{M_2} \restriction \mathrm{seq}(M_1) = \mathbb{E}_{M_1}$.
- (D) If f is an isomorphism from $M_1 \in K_{\mathfrak{s}}$ onto M_2 then f maps $\mathbb{E}_{M_1}$ onto $\mathbb{E}_{M_2}$.
- (E) If $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous then

  $$\{\bar a/\mathbb{E}_{M_\delta} : \bar a \in \mathrm{dom}(\mathbb{E}_{M_\delta}) \subseteq \mathrm{seq}(M_\delta)\} = \{\bar a/\mathbb{E}_{M_\delta} : \bar a \in \bigcup_{\alpha<\delta} \mathrm{dom}(\mathbb{E}_{M_\alpha})\}.$$

2) We say that $\mathbb{E}$ is *small* if each $\mathbb{E}_M$ has $\le \|M\|$ equivalence classes.

Remark 2.6. 1) Note that if we have $\langle \mathbb{E}_i : i < i^*\rangle$, where each $\mathbb{E}_i$ is a smooth $\mathfrak{k}_\lambda$-equivalence relation and $i^* < \lambda^+$, then we can find a smooth $\mathfrak{k}_\lambda$-equivalence relation $\mathbb{E}$ such that the $\mathbb{E}_M$-equivalence classes are essentially the $\mathbb{E}_i$-equivalence classes for $i < i^*$.

In detail: without loss of generality $i^* \le \lambda$, and $\bar a\ \mathbb{E}_M\ \bar b$ iff

[14] Of course, for $A \subseteq M$, $M \restriction A$ is the unique model with universe A such that $\tau_{M \restriction A} = \tau_M$, if such a model exists.

$\circledast_1$ $i(\bar a) = i(\bar b) < i^*$ and

$$\bar a \restriction [1 + i(\bar a) + 1, \lambda) \ \mathbb{E}_{i(\bar a)} \ \bar b \restriction [1 + i(\bar b) + 1, \lambda),$$

where $i(\bar a) = \min\big(\{j : j + 1 < i^* \wedge a_0 \neq a_{1+j}\} \cup \{\lambda\}\big)$.

2) In fact, $i^* \leq 2^\lambda$ is okay: e.g. choose a bijection $\mathbf{e}$ from the set of equivalence relations on λ onto i^*. For $\bar a, \bar b \in \mathrm{seq}(M)$ we let $i(\bar a) := \mathbf{e}\big(\{(i,j) : a_{2i+1} = a_{2j+1}\}\big)$ and

$\circledast_2$ $\bar a \ \mathbb{E}_M \ \bar b$ iff $i(\bar a) = i(\bar b)$ and $\langle a_{2i} : i < \lambda\rangle \ \mathbb{E}_{i(\bar a)} \ \langle b_{2i} : i < \lambda\rangle$.

3) We can redefine $\mathrm{seq}(M)$ as ${}^{\lambda\geq}M$, but then we have to make minor changes above.

Definition 2.7. Let $\mathfrak{k}$ be a λ-AEC or just ess-$[\lambda,\mu)$-AEC and $\mathbb{E}$ a small smooth $\mathfrak{k}$-equivalence relation and the reader may assume for simplicity that the vocabulary $\tau = \tau_{\mathfrak{k}}$ has only predicates. Also assume $F_*, c_*, P_* \notin \tau_{\mathfrak{k}}$. We define τ_* and $\mathfrak{k}_* = \mathfrak{k}\langle\mathbb{E}\rangle = (K_{\mathfrak{k}_*}, \leq_{\mathfrak{k}_*})$ as follows.

(A) $\tau^* = \tau \cup \{F_*, c_*, P_*\}$ with P_* a unary predicate, c_* an individual constant and F_* a λ-place function symbol.

(B) $K_{\mathfrak{k}_*}$ is the class of $\tau^*_{\mathfrak{k}}$-models M^* such that for some model $M \in K_{\mathfrak{k}}$ we have:
  - (a) $|M| = P_*^{M^*}$
  - (b) If $R \in \tau$ then $R^{M^*} = R^M$.
  - (c) If $F \in \tau$ has arity α, then $F^{M^*} \restriction M = F^M$ and for any $\bar a \in {}^\alpha(M^*) \backslash {}^\alpha M$ we have $F^{M^*}(\bar a) = c_*^{M^*}$.
    (Or allow partial functions, or use $F^{M^*}(\bar a) = a_0$ when $\alpha > 0$ and $F^{M^*}(\langle\ \rangle)$ when $\alpha = 0$ — i.e. F is an individual constant.)
  - (d) F_* is a λ-place function symbol, and
    - $\bullet_1$ If $\bar a \in \mathrm{dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ then $F_*^{M^*}(\bar a) \in \big(|M^*| \setminus |M|\big) \setminus \{c_*^{M^*}\}$.
    - $\bullet_2$ If $\bar a, \bar b \in \mathrm{dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ then $F_*^{M^*}(\bar a) = F_*^{M^*}(\bar b) \Leftrightarrow \bar a \ \mathbb{E}_M \ \bar b$.
    - $\bullet_3$ If $\bar a \in {}^\lambda(M^*)$ and $\bar a \notin \mathrm{dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ then $F_*^{M^*}(\bar a) = c_*^{M^*}$.
  - (e) $c_*^{M^*} \notin |M|$, and if $b \in \big(|M^*| \backslash |M|\big) \backslash \{c_*^{M^*}\}$ then for some $\bar a \in \mathrm{dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ we have $F_*^{M^*}(\bar a) = b$.

  Note that for every $M \in K_{\mathfrak{k}}$ there is such an M^*, and it is unique.

(C) $\leq_{\mathfrak{k}_*}$ is the two-place relation on $K_{\mathfrak{k}_*}$ defined as follows.
  $M^* \leq_{\mathfrak{k}_*} N^*$ if
  - (a) $M^* \subseteq N^*$
  - (b) For some $M, N \in \mathfrak{k}$ dependent on M^* and N^*, respectively, as in clause (B), we have $M \leq_{\mathfrak{k}} N$.

Definition 2.8. 1) We call $M \in \mathfrak{k}$ a *witness* for $M^* \in K_{\mathfrak{k}_*}$ (pedantically, $\mathbb{E}$ is the witness) if it is as in clause 2.7(B) above. Therefore M is uniquely determined by M^*.

2) We call (M, N) a witness for '$M^* \leq_{\mathfrak{k}^*_\lambda} N^*$' if M and N are witnesses for M^* and N^*, respectively, and $M \leq_{\mathfrak{k}} N$.

Discussion 2.9. Up to now we have restricted ourselves to vocabularies with each predicate and function symbol of finite arity, and this restriction seems very reasonable. Moreover, it seems *a priori* sensible that in any analogue to superstable it would be quite undesirable to have infinite arity. Still, our desire to have imaginary elements (in particular, canonical basis for types) forces us to accept them.

The price is that for a general class of τ-models, the union of increasing chains of τ-models is not a well-defined τ-model; more accurately, we can show its existence but not smoothness.[15] <u>However</u>, inside the class $\mathfrak{k}\langle\mathbb{E}\rangle$ defined above, it will be.

Claim 2.10. 0) *If $\mathbb{E}$ is a smooth $\mathfrak{k}$-equivalence relation, I is a directed partial order, and $\langle M_t : t \in I\rangle$ is $|leq_{\mathfrak{k}}$-increasing with union M, <u>then</u> for every $\bar{a} \in \mathrm{dom}(\mathbb{E}_\mu)$ there exists $s \in I$ and $\bar{b} \in \mathrm{dom}(\mathbb{E}_{M_\ell})$ such that $\bar{a}\ \mathbb{E}_M\ \bar{b}$.*

1) *If $\mathfrak{k}$ is a $[\lambda,\mu)$-AEC (or just an* ess-$[\lambda,\mu)$*-AEC) and $\mathbb{E}$ a small smooth $\mathfrak{k}$-equivalence relation, <u>then</u> $\mathfrak{k}\langle\mathbb{E}\rangle$ is an* ess-$[\lambda,\mu)$*-AEC.*

2) *If $\mathfrak{k}$ has amalgamation and $\mathbb{E}$ is a small $\mathfrak{k}$-equivalence class <u>then</u> $\mathfrak{k}\langle\mathbb{E}\rangle$ has the amalgamation property.*

3) *Similarly for the JEP (the* joint embedding property*).*

Proof. The same proofs as in [She09d, Ch.II]. Left as an exercise to the reader. $\square_{2.10}$

§ 2(C). **Good Frames.**

Now we return to good frames.

Definition 2.11. We say that $\mathfrak{s}$ is a good ess-$[\lambda,\mu)$-frame similarly to [She09d, Ch.II, Def. 2.1, p.259].[16] It consists of $\mathfrak{k}$, $\mathcal{S}_{\mathfrak{s}}$, $\bigcup_{\mathfrak{s}}$, with the obvious differences. In particular:

(a) $\mathfrak{k} = \mathfrak{k}_{\mathfrak{s}} = (K_{\mathfrak{s}}, \leq_{\mathfrak{s}})$, $\mathfrak{k}$ is an ess-$[\lambda,\mu)$-AEC.
(b) $K_{\mathfrak{s}}$ has a superlimit model in χ in every $\chi \in [\lambda,\mu)$.
(c) $\mathcal{S}_{\mathfrak{s}}(M)$ and $\bigcup_{\mathfrak{s}}$ are as there.

Discussion 2.12. We may consider other relatives as our choice and mostly have similar results. In particular:

(A) We can demand less: as in [SV24, §2] we may replace $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}$ by a formal version of $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}$.
(B) We may demand goodness only for $\mathfrak{s}_\lambda$: i.e. $\mathcal{S}_{\mathfrak{s}}$ and $\bigcup_{\mathfrak{s}}$ apply only to models in $K^{\mathfrak{s}}_\lambda$ (hence we can use their nice properties from [She09d, Ch.II, 2.1, p.259].[17]) and amalgamation and JEP are required only for models of cardinality λ.

Claim 2.13. *All the definitions and results in* [She09c], [She09e] *and* §*1 here work for good ess-$[\lambda,\mu)$-frames.*

Proof. No problem. $\square_{2.13}$

Definition 2.14. If $\mathfrak{s}$ is a $[\lambda,\mu)$-frame (see Definition 1.1) – or just an ess-$[\lambda,\mu)$-frame – and $\mathbb{E}$ is a small smooth $\mathfrak{s}$-equivalence relation, <u>then</u> let $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$ be defined as follows.

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[15] '$\mathfrak{k}$ is smooth' means **Axiom III$_2$** from Definition 2.1.
[16] = [She09c, Def. 2.1$_{=\text{L1.1tex}}$]
[17] = [She09c, Def. 2.1$_{=\text{L1.1tex}}$]

(A) $\mathfrak{k}_{\mathfrak{t}} := \mathfrak{k}_{\mathfrak{s}}\langle\mathbb{E}\rangle$

(B) $\mathcal{S}^{\rm bs}_{\mathfrak{t}}(M^*) :=$

$$\big\{\text{ortp}_{\mathfrak{k}_{\mathfrak{t}}}(a, M^*, N^*) : M^* \leq_{\mathfrak{k}_{\mathfrak{t}}} N^*, \text{ and if } M \leq_{\mathfrak{k}} N \text{ witness } M^*, N^* \in \mathfrak{k}_{\mathfrak{t}} \text{ then } a \in N \setminus M \text{ and } \text{ortp}_{\mathfrak{s}}(a, M, N) \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)\big\}$$

(C) Non-forking is defined similarly. That is,

$$\bigcup_{\mathfrak{t}} := \big\{(M^*_0, M^*_1, a, M^*_3) : M^*_\ell \in K_{\mathfrak{t}}, \text{ as witnessed by } M_\ell, \text{ for } \ell \leq 3, \text{ where } M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_3 \text{ and } (M_0, M_1, a, M_3) \in \bigcup_{\mathfrak{s}}\big\}$$

Note that we may lose fullness, because $\text{ortp}(b, M^*_1, M^*_3)$ is not in $\mathcal{S}^{\rm bs}_{\mathfrak{t}}(M^*_1)$ for $b \in M^*_3 \setminus M^*_1$.

(D) In clause (C), if $p_* := \text{ortp}_{\mathfrak{t}}(a, M^+_0, M^+_3)$ and $p := \text{ortp}_{\mathfrak{s}}(a, M_0, M_3)$ <u>then</u> we may say p_* is *projected to* p (or is $(\mathfrak{s}, \mathbb{E})$-projected to p).

Remark 2.15. We may extend this: if $\mathfrak{s}$ is[18] an NF-frame we define $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$ as an NF-frame similarly (see [Shea, §1]).

Claim 2.16. 1) *If $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame, $\mathbb{E}$ a small, smooth $\mathfrak{s}$-equivalence relation <u>then</u> $\mathfrak{s}\langle\mathbb{E}\rangle$ is a good ess-$[\lambda, \mu)$-frame.*

2) *In part (1), $\dot{I}(\kappa, K^{\mathfrak{s}\langle\mathbb{E}\rangle}) = \dot{I}(\kappa, K^{\mathfrak{s}})$ for every κ.*

3) *If $\mathfrak{s}$ has primes/regulars <u>then</u> $\mathfrak{s}\langle\mathbb{E}\rangle$ does as well.*

Remark 2.17. We may add: if $\mathfrak{s}$ is an NF-frame <u>then</u> so is $\mathfrak{s}\langle\mathbb{E}\rangle$, hence $(\mathfrak{s}\langle\mathbb{E}\rangle)^{\rm full}$ is a full NF-frame. (See [Shea, §1].)

Proof. Straightforward. $\square_{2.16}$

Our aim is to make some inconsequential changes to $\mathfrak{s}$ so that for every $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ there is a canonical base, etc. The following claim shows that in the context we have presented, this can be done.

Claim 2.18. The imaginary elements Claim

Assume $\mathfrak{s}$ a good λ-frame or just a good ess-$[\lambda, \mu)$-frame.

1) *If $M_* \in K_{\mathfrak{s}}$ and $p^* \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_*)$, <u>then</u>[19] there is a small, smooth $\mathfrak{k}_{\mathfrak{s}}$-equivalence relation $\mathbb{E} = \mathbb{E}_{\mathfrak{s}, M_*, p^*}$ and a unique function $\mathbf{F}$ satisfying $(*)$ below.*

$(*)$ (α)
- $\bullet_1$ *$\mathbf{F}(N, \bar{a})$ is well-defined <u>iff</u> $\bar{a} \in \text{dom}(\mathbb{E}_N)$.*
- $\bullet_2$ *If $\mathbf{F}(N, \bar{a})$ is well-defined <u>then</u> $\mathbf{F}(N, \bar{a}) \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$ (and it does not fork over $N \restriction \bar{a}$).*

(β) *$S \subseteq \big\{(N, \bar{a}, p) : N \in K_{\mathfrak{s}},\ \bar{a} \in \text{dom}(\mathbb{E}_N),\ p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)\big\}$ is the minimal class such that:*
- *(i) If $\bar{a} \in \text{seq}(M_*)$ and p^* does not fork over $M_* \restriction \text{rang}(\bar{a})$, then $(M_*, \bar{a}, p^*) \in S$.*
- *(ii) S is closed under isomorphisms.*
- *(iii) If $N_1 \leq_{\mathfrak{s}} N_2$ and $p_2 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_2)$ does not fork over $\bar{a} \in \text{seq}(N_1)$, then $(N_2, \bar{a}, p_2) \in S \Leftrightarrow (N_1, \bar{a}, p_2 \restriction N_1) \in S$.*

[18] The reader may ignore this version.

[19] Note that there may well be an automorphism of M^* which maps p^* to some $p^{**} \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M^*)$ such that $p^{**} \neq p^*$.

(*iv*) *If* $\bar a_1, \bar a_2 \in \mathrm{seq}(N)$ *and* $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak s}(N)$ *does not fork over* $N_\ell := N \restriction \mathrm{rang}(\bar a_\ell)$ *for* $\ell = 1, 2$, *then*

$$(N_2, \bar a_1, p) \in S \Leftrightarrow (N_2, \bar a_2, p) \in S.$$

(γ) $\mathbf{F}(N, \bar a) = p$ *iff* $(N, \bar a, p) \in S$; *and if* $\bar a, \bar b \in \mathrm{seq}(N)$ *then* $\bar a \ \mathbb{E}_N \ \bar b$ *iff* $\mathbf{F}(N, \bar a) = \mathbf{F}(N, \bar b)$.

2) *There are a unique small*[20] *smooth* $\mathfrak{k}_{\mathfrak s}$*-equivalence relation* $\mathbb{E} = \mathbb{E}_{\mathfrak s}$ *and a function* $\mathbf{F} = \mathbf{F}_{\mathfrak s}$ *such that:*

(∗∗) (α) *If* $\mathbf{F}(N, \bar a)$ *is well-defined then* $N \in K_{\mathfrak s}$ *and* $\bar a \in \mathrm{seq}(N)$.

(β) $\mathbf{F}(N, \bar a)$, *when defined, belongs to* $\mathcal{S}^{\mathrm{bs}}_{\mathfrak s}(N)$.

(γ) *If* $N \in K^{\mathfrak s}_\lambda$ *and* $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak s}(N)$ *<u>then</u> there is* $\bar a \in \mathrm{seq}(N)$ *such that* $\mathrm{rang}(\bar a) = N$ *and* $\mathbf{F}(N, \bar a) = p$.

(δ) *If* $\bar a \in \mathrm{seq}(M)$, $(M, \bar a) \in \mathrm{dom}(\mathbf{F}_M)$, *and* $M \le_{\mathfrak s} N$, *<u>then</u>* $\mathbf{F}(N, \bar a)$ *(is well-defined and) is the non-forking extension of* $\mathbf{F}(M, \bar a)$.

(ε) *If* $\bar a_\ell \in \mathrm{seq}(N)$ *and* $\mathbf{F}(N, \bar a_\ell)$ *is well defined for* $\ell = 1, 2$ *<u>then</u>*

$$\bar a_1 \ \mathbb{E}_N \ \bar a_2 \Leftrightarrow \mathbf{F}(N, \bar a_1) = \mathbf{F}(N, \bar a_2).$$

(ζ) $\mathbf{F}$ *commutes with isomorphisms.*

(η) $\mathbb{E}_{\mathfrak s, N} :=$

$$\big\{(\bar a_1, \bar a_2) \in \mathrm{seq}(M) \times \mathrm{seq}(M) : (N, \bar a_1), (N, \bar a_2) \in \mathrm{dom}(\mathbf{F}) \text{ and } \mathbf{F}(N, \bar a_1) = \mathbf{F}(N, \bar a_2)\big\}.$$

3) *Let* $\mathfrak t := \mathfrak s\langle \mathbb{E} \rangle$, *where* $\mathbb{E}$ *is as in part* (2).

(*A*) **[Notation:]** *Whenever* $M^* \in K_{\mathfrak t}$ *as witnessed by* $M \in K_{\mathfrak s}$, $p^* \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak t}(M^*)$ *is projected to* $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak s}(M)$ *(see* 2.14(D)*), and* $\bar a \in \mathrm{seq}(M)$ *satisfies* $\mathbf{F}(M, \bar a) = p$, *<u>then</u> we let*

$$\mathrm{bas}(p^*) = \mathrm{bas}(p) := F^{M^*}_*(\bar a)$$

(see Definition 2.7*).*

(*B*) *If* M_ℓ *witnesses that* $M^*_\ell \in K_{\mathfrak t}$ *for* $\ell = 1, 2$ *and* $(M^*_1, M^*_2, a) \in K^{3,\mathrm{bs}}_{\mathfrak t}$, *<u>then</u>* $(M_1, M_2, a) \in K^{3,\mathrm{bs}}_{\mathfrak s}$, $p^* = \mathrm{ortp}_{\mathfrak t}(a, M^*_1, M^*_2)$, *and* $p = \mathrm{ortp}_{\mathfrak s}(a, M_1, M_2)$.

(*C*) *If* $M^*_\ell \le_{\mathfrak s} M^*$ *and* $p_\ell \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak t}(M^*_\ell)$ *<u>then</u>*

$$p^*_1 \parallel p^*_2 \Leftrightarrow \mathrm{bas}(p^*_1) = \mathrm{bas}(p^*_2).$$

(*D*) $p^* \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak t}(M^*)$ *does not split over* $\mathrm{bas}(p^*)$ *(see Definition* 1.14(3) *or* [She09e, §2 end]*).*

Proof. 1) Let $M_{**} \le_{\mathfrak s} M^*$ be of cardinality λ such that p^* does not fork over M_{**}. Choose an enumeration $\bar a^* = \langle a_\alpha : \alpha < \lambda \rangle$ of the elements of M_{**}.

We say that $p_1 \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak s}(M_1)$ is a *weak copy* of p^* when there is a witness (M_0, M_2, p_2, f), which means:

⊛$_1$ (a) $M_0 \le_{\mathfrak s} M_2$ and $M_1 \le_{\mathfrak s} M_2$.

(b) if $\|M_1\| = \lambda$ then $\|M_2\| = \lambda$.

(c) f is an isomorphism from M_{**} onto M_0.

(d) $p_2 \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak s}(M_2)$ is a non-forking extension of p_1.

(e) p_2 does not fork over M_0.

(f) $f(p^* \restriction M_{**})$ is $p_2 \restriction M_0$.

For $M_1 \in K^{\mathfrak s}_\lambda$ and $p_1 \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak s}(M_1)$ a weak copy of p^*, we say that $\bar b$ *explicates* its being a weak copy <u>when</u>, for some witness (M_0, M_2, p_2, f) and $\bar c$,[21]

[20] For 'small,' we use stability in λ.

[21] So necessarily M_2 has cardinality λ.

$\circledast_2$ (a) $\bar{b} = \langle b_\alpha : \alpha < \lambda \rangle$ lists the elements of M_1.
(b) $\bar{c} = \langle c_\alpha : \alpha < \lambda \rangle$ lists the elements of M_2.
(c) The set $\{\alpha : b_\alpha = b_0\}$ codes the following sets:
$\bullet_1$ The isomorphism type of $(M_2, \bar{c})$.
$\bullet_2$ $\{(\alpha, \beta) : b_\alpha = c_\beta\}$
$\bullet_3$ $\{(\alpha, \beta) : f(a^*_\alpha) = c_\beta\}$

Now,

$\circledast_3$ If $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is a weak copy of p^* then for some $\bar{a} \in \operatorname{seq}(M)$, there is a $M_1 \leq_{\mathfrak{s}} M$ over which p does not fork such that $\bar{a}$ lists M_1 and explicates '$p \restriction M_1$ is a weak copy of p^*.'

$\circledast_4$ (a) If $M \in K^{\mathfrak{s}}_\lambda$ and $\bar{b}$ explicates '$p_1 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is a weak copy of p^*,' <u>then</u> we can reconstruct p_1 from M and $\bar{b}$. (Call it $p_{M,\bar{b}}$.)
(b) If in addition $M \leq_{\mathfrak{s}} N$, then let $p_{N,\bar{b}}$ be its non-forking extension in $\mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$. We also call it $\mathbf{F}(N, \bar{b})$.

Now we define $\mathbb{E}$. First, for $N \in K_{\mathfrak{s}}$ we define a two-place relation $\mathbb{E}_N$.

$\circledast_5$ (α) The domain of $\mathbb{E}_N$ is

$$\{\bar{a} : \text{for some } M \leq_{\mathfrak{s}} N \text{ of cardinality } \lambda \text{ and } p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M) \text{ which is a weak copy of } p^*, \text{ the sequence } \bar{a} \text{ explicates } p \text{ being a weak copy of } p^*\}.$$

(β) $\bar{a}_1 \mathrel{\mathbb{E}_N} \bar{a}_2$ iff ($\bar{a}_1, \bar{a}_2$ are as above and) $p_{N,\bar{a}_1} = p_{N,\bar{a}_2}$.

Now

$\odot_1$ For $N \in K_{\mathfrak{s}}$, $\mathbb{E}_N$ is an equivalence relation on $\operatorname{dom}(E_N) \subseteq \operatorname{seq}(N)$.
$\odot_2$ If $N_1 \leq_{\mathfrak{s}} N_2$ and $\bar{a} \in \operatorname{seq}(N_1)$ <u>then</u> $\bar{a} \in \operatorname{dom}(\mathbb{E}_{N_1}) \Leftrightarrow \bar{a} \in \operatorname{dom}(\mathbb{E}_{N_2})$.
$\odot_3$ If $N_1 \leq_{\mathfrak{s}} N_2$ and $\bar{a}_1, \bar{a}_2 \in \operatorname{dom}(\mathbb{E}_{N_1})$ then $\bar{a}_2 \mathrel{\mathbb{E}_{N_1}} \bar{a}_2 \Leftrightarrow \bar{a}_1 \mathrel{\mathbb{E}_{N_2}} \bar{a}_2$.
$\odot_4$ If $\langle N_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and $\bar{a}_1 \in \operatorname{dom}(\mathbb{E}_{N_\delta})$ then, for some $\alpha < \delta$ and $\bar{a}_2 \in \operatorname{dom}(\mathbb{E}_{N_\alpha})$, we have $\bar{a}_1 \mathrel{\mathbb{E}_{N_\delta}} \bar{a}_2$.

[Why? Let $p = p_{N_\delta, \bar{a}_1} \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_\delta)$. Hence for some $\alpha < \delta$, p does not fork over M_α; hence for some $M'_1 \leq_{\mathfrak{s}} M_\alpha$ of cardinality λ, the type p does not fork over M'_1. Let $\bar{a}_2$ list the elements of M'_1 such that it explicates $p \restriction M'_1$ being a weak copy of p^*. So clearly $\bar{a}_2 \in \operatorname{dom}(\mathbb{E}_{N_\alpha}) \subseteq \operatorname{dom}(\mathbb{E}_{N_\delta})$ and $\bar{a}_1 \mathrel{\mathbb{E}_{N_\delta}} \bar{a}_2$.]

Clearly we are done.

2) Similar; we vary (M^*, p^*), but it suffices to consider 2^λ such pairs.

Let us elaborate. First, we shall manipulate the sequence $\bar{a}$ so that it lists not only some $M \in K^{\mathfrak{s}}_\lambda$, but also some $p \in \mathcal{S}^{\rm bs}(M)$. Towards this end,

$(*)_1$ Let $\langle (M_\alpha, p_\alpha) : \alpha < \alpha_* \rangle$ be such that:
(a) $\alpha_* \leq 2^\lambda$
(b) $M_\alpha \in K_{\mathfrak{s}}$ has universe λ.
(c) $p_\alpha \in \mathcal{S}^{\rm bs}(M_\alpha)$
(d) For every $M \in K^{\mathfrak{s}}_\lambda$, $\big|\{\alpha < \alpha_* : M_\alpha \cong M\}\big| = \big|\mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)\big|$.
(e) If $\alpha < \alpha_*$ then $\langle p_\beta : M_\beta = M_\alpha \rangle$ lists $\mathcal{S}^{\rm bs}(M_\alpha)$ without repetition.

Next, choose $\langle \bar{a}_\alpha : \alpha < \alpha_* \rangle$ such that:

$(*)_2$ (a) $\bar{a}_\alpha$ lists the elements of M_α.
(b) If $\alpha \neq \beta$ then $\{i : a_{\alpha,i} = a_{\alpha,0}\} \neq \{i : a_{\beta,i} = a_{\beta,0}\}$.

$(*)_3$ Now if $\bar{a} \in \operatorname{seq}(N)$ lists the elements of $M \leq_{\mathfrak{s}} N$, <u>then</u> we choose $\mathbf{F}(N, \bar{a}) \in \mathcal{S}^{\rm bs}(N)$ such that:

(a) If there is α such that $u_\alpha := \{i < \lambda : a_{\alpha,i} = a_{\alpha,0}\} = \{i < \lambda : a_i = a_0\}$ and $a_i \mapsto a_{\alpha,i}$ is an isomorphism from $N \restriction \bar{a}$ to M_α, then $\mathbf{F}(N, \bar{a}) \in \mathcal{S}^{\rm bs}(N)$ does not fork over M and that isomorphism maps $\mathbf{F}(N, \bar{a}) \restriction M$ to p_α.

(b) If there is no such α as in clause (a), then we let $\mathbf{F}(N, \bar{a})$ be undefined.

$(*)_4$ Let $\mathbb{E}_N$ be the set of pairs $\big((N, \bar{a}), (N, \bar{b})\big)$ from $\text{dom}(\mathbf{F})$ such that $\mathbf{F}(N, \bar{a}) = \mathbf{F}(N, \bar{b})$.

Note that

$(*)_5$ In $(*)_3$, $\mathbf{F}(N, \bar{a})$ is well-defined and the function is as required.

3) Should be clear. $\square_{2.18}$

Definition 2.19 (**Definition / Claim**)**.** Assume that $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame, so without loss of generality it is full.

1)We will choose a good ess-$[\lambda, \mu)$-frame $\mathfrak{t}_n = \mathfrak{t}_n\langle\mathfrak{s}\rangle$ by induction on $n < \omega$.

For $n = 0$ let $\mathfrak{t}_0 := \mathfrak{s}$.

If n is even we define $\mathfrak{t}_{n+1}$ as in 2.18(2), with $(\mathfrak{t}_{n+1}, \mathfrak{t}_n)$ here standing in for $(\mathfrak{t}, \mathfrak{s})$ there. That is, $\mathfrak{t}_{n+1} := \mathfrak{t}_n\langle\mathbb{E}_n\rangle$, where $\mathbb{E}_n := \mathbb{E}\langle\mathfrak{t}_n\rangle$.

If n is odd, then by [She90, Ch.III] we can choose a full $\mathfrak{t}_{n+1}$ such that $\mathfrak{k}_{\mathfrak{t}_{n+1}} = \mathfrak{k}_{\mathfrak{t}_n}$ and $\mathcal{S}^{\rm bs}_{\mathfrak{t}_{n+1}}(M) \subseteq \mathcal{S}^{\rm bs}_{\mathfrak{t}_n}(M)$ for all $M \in K_{\mathfrak{t}_n}$.

2) In the limit (so after ω times) we get a $\mathfrak{t}_\omega$ which is full[22] and has canonical type-bases, as witnessed by a function $\text{bas}_{\mathfrak{t}_\omega} := \bigcup_{n<\omega} \text{bas}_{\mathfrak{t}_{2n+1}}$ (see Definition 2.20 below).

Proof. Should be clear. $\square_{2.19}$

Definition 2.20. We say that $\mathfrak{s}$ has type bases <u>if</u> there is a function bas$(-)$ such that:

(A) If $M \in K_{\mathfrak{s}}$ and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ <u>then</u> bas(p) is (well defined and is) an element of M.

(B) p does not split over bas(p) (see Definitions 1.14, 1.15). Hence any automorphism[23] of M over bas(p) maps p to itself.

(C) If $M \leq_{\mathfrak{s}} N$ and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$ then bas$(p) \in M$ <u>iff</u> p does not fork over M.

(D) If f is an isomorphism from $M_1 \in K_{\mathfrak{s}}$ onto $M_2 \in K_{\mathfrak{s}}$ and $p_1 \in \mathcal{S}^{\rm bs}(M_1)$ then $f(\text{bas}(p_1)) = \text{bas}(f(p_1))$.

Remark 2.21. 1) In §3 we can add:

(E) Strong uniqueness: if $A \subseteq M <_{\mathfrak{s}} \mathfrak{C}$ and $p \in \mathcal{S}(A, \mathfrak{C})$ is well defined <u>then</u> bas$(p) \in A$ and there is at most one $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ such that q extends p. (Needed for non-forking extensions).

2) In 2.22 we can work in $\mathfrak{C}$.

Now, motivated by 2.19, we can define

Definition 2.22. We say that $\mathfrak{s}$ is *equivalence-closed* <u>when</u>:

(A) $\mathfrak{s}$ has type bases $p \mapsto \text{bas}(p)$.

[22] That is, $\mathcal{S}^{\rm bs}_{\mathfrak{t}_\omega}(M^\omega) = \mathcal{S}^{\rm na}_{\mathfrak{t}_\omega}(M^\omega)$.

[23] There are reasonable stronger versions, but it follows that the function bas$(-)$ satisfies them.

(B) If (for each $M \in K_{\mathfrak{s}}$) $\mathbb{E}_M$ is a definition of an equivalence relation on ${}^{\omega >}M$ preserved by isomorphisms and $\leq_{\mathfrak{s}}$-extensions (i.e. $M \leq_{\mathfrak{s}} N \Rightarrow \mathbb{E}_M = \mathbb{E}_N \restriction {}^{\omega >}M$) then there is a definable function F from ${}^{\omega >}M$ to M such that $F^M(\bar{a}) = F^M(\bar{b})$ iff $\bar{a}\ \mathbb{E}_M\ \bar{b}$.

To phrase the relation between (e.g.) $\mathfrak{k}$ and $\mathfrak{k}\langle\mathbb{E}\rangle$, we define the following.

Definition 2.23. Assume $\mathfrak{k}_2$ is a ess-$[\lambda, \mu)$-AEC.

1) We say $\mathbf{i}$ is an *interpretation candidate in* $\mathfrak{k}_2$ when $\mathbf{i}$ consists of

(A) A predicate $P^*_{\mathbf{i}}$.

(B) A subset $\tau_{\mathbf{i}}$ of $\tau_{\mathfrak{k}_2}$.

2) In this case, for $M_2 \in K_{\mathfrak{k}_2}$, define the $\tau_{\mathbf{i}}$-model $M_2^{[\mathbf{i}]}$ as follows:

- $|M_2^{[\mathbf{i}]}| := P_{\mathbf{i}}^{M_2}$
- $R^{M_2^{[\mathbf{i}]}} := R^{M_2} \restriction |M_2^{[\mathbf{i}]}|$ for $R \in \tau_{\mathbf{i}}$.
- $F^{M_2^{[\mathbf{i}]}}$ is defined similarly, so it can be a partial function even if F^{M_2} is full.

3) We say that the ess-$[\lambda, \mu)$-AEC $\mathfrak{k}_1$ is $\mathbf{i}$-interpreted (or interpreted by $\mathbf{i}$) in $\mathfrak{k}_2$ when:

(A) $\mathbf{i}$ is an interpretation candidate in $\mathfrak{k}_2$.

(B) $\tau_{\mathfrak{k}_1} = \tau_{\mathbf{i}}$

(C) $K_{\mathfrak{k}_1} = \{M_2^{[\mathbf{i}]} : M_2 \in K_{\mathfrak{k}_2}\}$

(D) If $M_2 \leq_{\mathfrak{k}_2} N_2$ then $M_2^{[\mathbf{i}]} \leq_{\mathfrak{k}_1} N_2^{[\mathbf{i}]}$.

(E) If $M_1 \leq_{\mathfrak{k}_1} N_1$ and $N_1 = N_2^{[\mathbf{i}]}$ (so $N_2 \in K_{\mathfrak{k}_2}$) then for some $M_2 \leq_{\mathfrak{k}} N_2$ we have $M_1 = M_2^{[\mathbf{i}]}$.

(F) If $M_1 \leq_{\mathfrak{k}_1} N_1$ and $M_1 = M_2^{[\mathbf{i}]}$ (so $M_2 \in K_{\mathfrak{k}_2}$) then (possibly replacing M_2 by a model isomorphic to it over M_1) there is $N_2 \in K_{\mathfrak{k}_2}$ such that $M_2 \leq_{\mathfrak{k}_2} N_2$ and $N_1 = N_2^{[\mathbf{i}]}$.

Definition 2.24. 1) Assume $\mathfrak{k}_1$ is interpreted by $\mathbf{i}$ in $\mathfrak{k}_2$. We say *strictly* interpreted when $M_2^{[\mathbf{i}]} = N_2^{[\mathbf{i}]}$ implies that M_2 and N_2 are isomorphic over $M_2^{[\mathbf{i}]}$.

2) We say $\mathfrak{k}_1$ is equivalent to $\mathfrak{k}_2$ if there are n and $\mathfrak{k}'_0, \ldots, \mathfrak{k}'_n$ such that $\mathfrak{k}_1 = \mathfrak{k}'_0$, $\mathfrak{k}_2 = \mathfrak{k}'_n$, and for each $\ell < n$, $\mathfrak{k}_\ell$ is strictly interpreted in $\mathfrak{k}_{\ell+1}$ or *vice versa.*

(Actually, we can demand $n = 2$ and that k_ℓ is strictly interpreted in $\mathfrak{k}'_1$ for $\ell = 1, 2$.)

Definition 2.25. As in 2.23, 2.24 above, for good ess-$[\lambda, \mu)$-frames.

Claim 2.26. 1) *If a good* ess-(λ, μ)*-frame* $\mathfrak{t}$ *is derived from a good* ess-(λ, μ)*-frame* $\mathfrak{s}$ *as in* 2.19*, then* $\mathfrak{k}_{\mathfrak{s}}$ *and* $\mathfrak{k}_{\mathfrak{t}}$ *are equivalent as AECs.*

1A) *Similarly to part* (1)*, omitting "good." (Use* 2.18(2),(3).*)*

2) *Assume* $\mathfrak{s}$ *is a good* ess-$[\lambda, \mu)$*-frame. Then there exists* $\mathfrak{C}$ *(called a* μ-saturated monster *for* $K_{\mathfrak{s}}$*) such that:*

(*a*) $\mathfrak{C}$ *is a* $\tau_{\mathfrak{s}}$*-model of cardinality* $\leq \mu$*.*

(*b*) $\mathfrak{C}$ *is a union of some* $\leq_{\mathfrak{s}}$*-increasing continuous sequence* $\langle M_\alpha : \alpha < \mu\rangle$*.*

(c) *if* $M \in K_{\mathfrak{s}}$ *(so* $\lambda \leq \|M\| < \mu$*)* *then* M *is* $\leq_{\mathfrak{s}}$*-embeddable into some* M_α *from clause* (b).

(d) $M_{\alpha+1}$ *is brimmed over* M_α *for* $\alpha < \mu$.

Proof. Easy. $\square_{2.26}$

§ 3. P-SIMPLE TYPES

We define the basic types over sets not necessary models. Note that in Definition 3.5(2) there is no real loss using C of cardinality $\in (\lambda, \mu)$, as we can replace λ by $\lambda_1 := \lambda + |C|$ and so replace $K_{\mathfrak{k}}$ to $K^{\mathfrak{k}}_{[\lambda_1,\mu)}$.

Hypothesis 3.1. 1) $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame (see Definition 2.11).

2) $\mathfrak{s}$ is full and has type bases (see Definition 2.20).

3) $\mathfrak{C}$ will denote some μ-saturated model for $K_{\mathfrak{s}}$ of cardinality $\leq \mu$; see 2.26.

4) Without loss of generality, $\bar{a} \in {}^{\omega>}M$ can be treated as elements, but $M, A, \ldots$ will be $<_{\mathfrak{s}} \mathfrak{C}$ and $\subseteq \mathfrak{C}$, respectively. However, they will be of cardinality $< \mu$.

Definition 3.2. Let $A \subseteq M \in K_{\mathfrak{s}}$.

1) $\mathrm{dcl}(A, M) = \{a \in M :$ if $M' \leq_{\mathfrak{s}} M''$, $M \leq_{\mathfrak{s}} M''$, and $A \subseteq M'$ then $a \in M'$ and for every automorphism f of M', $f \restriction A = \mathrm{id}_A \Rightarrow f(a) = a\}$.

2) $\mathrm{acl}(A, M)$ is defined similarly, but only with the first demand.

Definition 3.3. 1) For $A \subseteq M \in K_{\mathfrak{s}}$ let

$$\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M) := \big\{q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) : \mathrm{bas}(q) \in \mathrm{dcl}(A, \mathfrak{C})\big\}.$$

2) We call $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M)$ regular <u>if</u> p as a member of $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is regular.

Definition 3.4. 1) $\mathbb{E}_{\mathfrak{s}}$ is as in Claim 2.18(2).

2) If $A \subseteq M \in K_{\mathfrak{s}}$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, then '$p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M)$' means that p is definable over A (see 1.14(3)).

Definition 3.5. Let $A \subseteq \mathfrak{C}$.

1) We define a dependency relation on

$$\mathrm{good}(A, \mathfrak{C}) := \{c \in \mathfrak{C} : \text{for some } M <_{\mathfrak{s}} \mathfrak{C} \text{ with } c \notin M,\ A \subseteq M \text{ and } \mathrm{ortp}(c, M, \mathfrak{C}) \text{ is definable over some finite } \bar{a} \subseteq A\}$$

as follows.[24]

⊛ c depends on $\mathbf{J}$ in $(A, \mathfrak{C})$ <u>iff</u> there is no $M <_{\mathfrak{s}} \mathfrak{C}$ such that $A \cup \mathbf{J} \subseteq M$ and $\mathrm{ortp}(c, M, \mathfrak{C})$ is the non-forking extension of $\mathrm{ortp}(c, \bar{a}, \mathfrak{C})$, where $\bar{a}$ witnesses $c \in \mathrm{good}(A, \mathfrak{C})$.

2) We say $\langle M_\alpha : \alpha < \alpha^* \rangle$ is *independent over M inside N* <u>when</u> for some $\overline{N} = \langle N_\alpha : \alpha < \alpha^* \rangle$ and $\overline{\mathbf{J}} = \langle \mathbf{J}_\alpha : \alpha < \alpha^* \rangle$, we have

(A) $M \leq_{\mathfrak{s}} M_\alpha \leq_{\mathfrak{s}} N_\alpha \leq_{\mathfrak{s}} N$ for all $\alpha < \alpha^*$.

(B) $(M, N_\alpha, \mathbf{J}_\alpha) \in K^{3,\mathrm{vq}}_{\mathfrak{s}}$

(C) $\mathbf{J} := \bigcup\limits_{\alpha<\alpha^*} \mathbf{J}_\alpha$ is independent inside (M, N).

(D) The $\mathbf{J}_\alpha$-s are pairwise disjoint.

[An alternative definition can be found in [She09d, Ch.III, 8.8, p.520], but they are equivalent when $\mu = \lambda^+$ — see [She09d, Ch.III, §8,10].]

3) We say $\langle A_\alpha : \alpha < \alpha^* \rangle$ is *independent over* (M, A) *in* $\mathfrak{C}$ <u>if</u> we can find $\langle M_\alpha : \alpha < \alpha^* \rangle$ such that:

[24] For the definition of $<_{\mathfrak{s}}$, see 1.1(2).

⊛ (a) $A \subseteq M \leq_{\mathfrak{s}} M_\alpha <_{\mathfrak{s}} \mathfrak{C}$ for $\alpha < \alpha^*$.
 (b) A_α is good over (A, M). (See part (5) below.)
 (c) $A_\alpha \subseteq M_\alpha$
 (d) $\mathrm{ortp}(A_\alpha, M, \mathfrak{C})$ definable over A (equivalently, 'does not split over A').
 (e) $\langle M_\alpha : \alpha < \alpha^* \rangle$ is independent over M (inside $\mathfrak{C}$).

4) We define 'locally independent' naturally; that is, every finite subfamily is independent.

5) For $A, B \subseteq \mathfrak{C}$, we say B *is good over* (A, M) (or '$\mathrm{ortp}(B, M)$ is definable over A') when

(A) $A \subseteq M <_{\mathfrak{s}} \mathfrak{C}$
(B) There exist M_1, M_2 such that
 (a) $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2$
 (b) $B \subseteq M_2$
 (c) M_1 is brimmed over M_2.
 (d) M_2 is brimmed over M_1.
 (e) $\mathrm{ortp}(B, M_1, M_2)$ does not split over A.

(In the first-order context, we would say $\mathrm{tp}(B, M)$ does not fork over A and is stationary. Here this definition is problematic, as $\mathrm{ortp}(B, A, \mathfrak{C})$ is not necessarily basic.)

Claim 3.6. 1) *If $p := \mathrm{ortp}(b, M, N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ and $a := \mathrm{base}(p) \in M$, then $\{b\}$ is good over $(\{a\}, M)$.*

2) *For any A and M, A is good over (M, M).*

3) **[Monotonicity:]**

(a) *Whether $\langle M_\alpha : \alpha < \alpha^* \rangle$ is independent over M inside N does not depend on the order. That is, if π is a one-to-one function from an ordinal β^* onto α^* then $\langle M_{\pi(\beta)} : \beta < \beta^* \rangle$ is independent over M inside N.*

(b) *If $\langle M_\alpha : \alpha < \alpha^* \rangle$ is independent over M inside N, then*

$$\bigwedge_{\alpha<\alpha^*} \left[M \leq_{\mathfrak{s}} M^*_\alpha \leq_{\mathfrak{s}} M_\alpha\right] \Rightarrow \langle M^*_\alpha : \alpha < \alpha^* \rangle \text{ is independent over } M \text{ inside } N.$$

(c) *If M, $\langle N_\alpha, \mathbf{J}_\alpha : \alpha < \alpha^* \rangle$, and N are as in 3.5(2), $N \leq_{\mathfrak{s}} N_*$,*

$$N_\alpha \leq_{\mathfrak{s}} N^*_\alpha \leq_{\mathfrak{s}} N_*,$$

and $(M, N^*_\alpha, \mathbf{J}_\alpha) \in K^{3,\mathrm{vq}}_{\mathfrak{s}}$ for all $\alpha < \alpha^*$, then $\langle N^*_\alpha, \mathbf{J}_\alpha : \alpha < \alpha^* \rangle$ is independent over M inside N_*.

Claim 3.7. 1) *Assume $A_\alpha \subset \mathfrak{C}$ for all $\alpha < \alpha^*$ and $\langle M_\alpha : \alpha \leq \alpha_* \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous. If A_α is independent over (A, M_α) for all $\alpha < \alpha^*$, then $\langle A_\alpha : \alpha < \alpha^* \rangle$ is independent over (A, M).*

2) *If $\langle A_\alpha : \alpha < \alpha^* \rangle$ is independent over (A, M), $A \subseteq A' \subseteq M$, and $A'_\alpha \subseteq A_{h(\alpha)}$ for all $\alpha < \alpha'_*$ (where $h : \alpha'_* \to \alpha^*$ is one-to-one) then $\langle A'_\alpha : \alpha < \alpha^* \rangle$ is independent over (A', M).*

3) *If $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ is regular, then, on $p(\mathfrak{C}) := \{c : c \text{ realizes } p\}$, the independence relation satisfies:*

(a) *If* c *depends on* $\{b_0, \ldots, b_n\}$ *but not on* $\{b_0, \ldots, b_{n-1}\}$, *then* b_n *depends on* $\{b_0, \ldots, b_{n-1}, c\}$.

(Recall that dependency was defined in 3.5(1)⊛.*)*

(b) *If* b^1_ℓ *depends on* $\{b^0_0, \ldots, b^0_{n-1}\}$ *for* $\ell < k$ *and* b^2 *depends on* $\{b^1_\ell : \ell < k\}$, *then* b^2 *depends on* $\{b^0_\ell : \ell < n\}$.

(c) *If* b *depends on* $\mathbf{J}$ *and* $\mathbf{J} \subseteq \mathbf{J}'$, *then* b *depends on* $\mathbf{J}'$.

Remark 3.8. 1) We have not mentioned finite character, but the local independence satisfies it trivially.

Proof. Straightforward. $\square_{3.7}$

Definition 3.9. Assume $p, q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$, and p is regular. We say that q is *p-simple* when '(A) ⇒ (B)' holds, where

(A) (a) $M \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2$

(b) M_1 is brimmed.

(c) $b \in M_2 \setminus M_1$ and $(M_1, M_2, b) \in K^{3,\rm pr}_{\mathfrak{s}}$.

(d) $\mathrm{ortp}(b, M_1, M_2)$ is a non-forking extension of q.

(e) $p_1 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_1)$ is a non-forking extension of p.

(B) There do not exist any M_3, M_4 such that

(a) $M_2 \le_{\mathfrak{s}} M_4$ and $M_3 \le_{\mathfrak{s}} M_4$.

(b) $M_1 \cup p(M_2) \subseteq M_3$ (recalling $p(M) = \{a \in M : a \text{ realizes } p\}$).

(c) $b \notin M_3$

Claim 3.10. *Assume* $M <_{\mathfrak{s}} \mathfrak{C}$, $p, q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$, p *is regular, and* M_1, M_2, b *are as in* 3.9(A).

1) *Assume further that* $p', q' \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M')$, *are parallel to* p *and* q, *respectively. Then* q' *is* p'*-simple iff* q *is* p*-simple.*

2) *If* q *is* p*-simple then, for some* $\mathbf{J} \subseteq p(M_2)$:

(∗) (a) $\mathbf{J}$ *is finite, and independent over* M.

(b) $(M_1, M_2, \mathbf{J}) \in K^{3,\rm vq}_{\mathfrak{s}}$

(c) *We have* $b \in N$ *whenever*

•1 $M_1 \le_{\mathfrak{s}} N \le_{\mathfrak{s}} M_2$

•2 $(M_1, N, \mathbf{J}) \in K^{3,\rm vq}_{\mathfrak{s}}$

•3 $p(N) = p(M_2)$.

3) *If* $\mathbf{J}$ *satisfies* (∗) *above, then* q *is* p*-simple (recalling* b *realizes* q*).*

4) *If* $p', q' \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M')$, M'_1, M'_2, b' *are as in* 3.9(A) *(for* M', p', q'*), and* $\mathbf{J}'$ *is as in* 3.10(2), *then* $|\mathbf{J}'| = |\mathbf{J}|$.

5) *If* $\mathbf{J}_* \subseteq p(M_2)$ *is independent over* M, *and every* $c \in \mathbf{J}$ *depends on* $\mathbf{J}_*$ *over* M, *then* $(M_1, M_2, \mathbf{J}_*) \in K^{3,\rm vq}_{\mathfrak{s}}$ *(so* $\mathbf{J}_*$ *is as in clause* (2)(∗)*).*

6) *If* $\mathbf{J}$ *is as in clause* (2)(∗), $(M, N, \mathbf{J}) \in K^{3,\rm vq}_{\mathfrak{s}}$, *and* $N <_{\mathfrak{s}} \mathfrak{C}$, *then* $\bar b \subseteq N$ *or* $\mathrm{ortp}(\bar b, N, \mathfrak{C}) \perp p$.

Proof. Straightforward. $\square_{3.10}$

Definition 3.11. 1) Assume $p_\ell, q_\ell \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_\ell)$ for $\ell = 1, 2$, $p_1 \parallel p_2$, and $q_1 \parallel q_2$. We define $w_{p_1}(q_2)$ (the p_1-*weight of* q_2) as the finite number of elements of $\mathbf{J}$ as defined in 3.10(2) (with (p_2, q_2, M_2) standing in for (p, q, M) there).

Still unclear. Do you mean ". . . as the cardinality of the finite set J."? Or is the weight the set J itself?

This is well-defined by 3.10(4). We can allow q algebraic.

2) We say that $\bar b$ is p-simple *over* M when $\bar b$ is a finite sequence from $\mathfrak{C}$ such that ${\rm ortp}(\bar a, M)$ is p-simple.

3) Let **[End of Line]**

Claim 3.12. *Assume* $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ *(so* $M <_{\mathfrak{s}} \mathfrak{C}$*).*

1) *If* $\bar b_1, \bar b_2 \in \mathfrak{C}$ *are* p*-simple over* M, *then*

(a) $\bar b_1 \hat{}\, \bar b_2$ *is* p*-simple over* M.

(b) *If* $M \le_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}$ *then* $\bar b_1$ *is* p*-simple over* N.

(c) *If* $(M, N_\ell, \mathbf{J}_\ell) \in K^{3,{\rm vq}[p]}_{\mathfrak{s}}$, $\bar b_\ell \subseteq N_\ell$, *and* $|\mathbf{J}_\ell| = w_p(\bar b_\ell, M)$, *then*
$$w_p(\bar b_2, N_1) = w_p(\bar b_2, N_2).$$

(d) *Every subsequence of* $\bar b_1$ *is* p*-simple over* M.

2) *If* $\bar a_\alpha$ *is* p*-simple over* M *for all* $\alpha \le \delta$, $\langle M^\ell_\alpha : \alpha \le \delta\rangle$ *is* $\le_{\mathfrak{s}}$*-increasing continuous,* $M_0 := M$, *and* $(M_\alpha, M_{\alpha+1}, \bar a_\alpha) \in K^{3,{\rm vq}[p]}_{\mathfrak{s}}$ *for all* $\alpha < \delta$, *then*
$$w_p(\bar a_\delta, M^1_\delta) = w_p(\bar a_\delta, M^2_\delta) = \min\big\{w_p(\bar a_\delta, M \cup \bar b) : \bar b \subseteq \bigcup_{\alpha<\delta} \bar a_\alpha \text{ is finite}\big\}.$$

Proof. Straightforward. $\square_{3.12}$

Definition 3.13. Assume $M <_{\mathfrak{s}} N$, $A \subseteq N$, and $\bar b \subseteq N$ is p-simple over M. Then
$$w_p(a, A, M) = \min\big\{w_p(\bar b\hat{}\langle a\rangle, M) - w_p(\bar b, M) : \bar b \subseteq A \text{ is finite}\big\}.$$

Claim 3.14. *If* $M <_{\mathfrak{s}} N$, $\bar a \subseteq N$, $\bar b_\alpha \subseteq N$ *is* p*-simple over* M *for* $\alpha < \alpha^*$, *and* $\pi : \beta^* \to \alpha^*$ *is one-to-one and onto, then*
$$\sum_{\alpha<\alpha^*} w_p\big(b_\alpha, \bar a \cup \bigcup_{\ell<\alpha} b_\alpha\big) = \sum_{\beta<\beta^*} w\big(b_{\pi(\beta)}, \bar a \cup \bigcup_{i<\beta} \bar b_{\pi(i)}\big).$$

Proof. Straightforward. $\square_{3.14}$

Lastly, we would like to have an existence theorem for p-simple elements over M. This is done as in §2, using 3.15, 3.16.

Definition 3.15. 1) Assume $M_1 <_{\mathfrak{s}} M_2$ are brimmed models from $K_{\mathfrak{s},\lambda}$, M_2 is brimmed over M_1, and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is regular. We define an equivalence relation $\mathcal{E}_{M_1,M_2}$ on $M_2 \setminus M_1$ as follows.

$b_1\ \mathcal{E}_{M_1,M_2}\ b_2$ iff there is an automorphism $\pi : M_2 \to M_2$ over M_1 which is the identity on $M_1 \cup p(M_2)$ and maps b_1 to b_2.

2) Assume $M_1 <_{\mathfrak{s}} M_2$ are from $K_{\mathfrak{s},\lambda}$, and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is regular. We define an equivalence relation $\mathcal{E}_{M_1,M_2,p}$ as follows.

$(*)$ $b_1\ \mathcal{E}_{M_1,M_2,p}\ b_2$ iff there are M^+_1, M^+_2 such that:

(a) $M_1 \le_{\mathfrak{s}} M^+_1 \le_{\mathfrak{s}} M^+_2$ and $M_2 \le_{\mathfrak{s}} M^+_2$.

(b) M_1^+, M_2^+, p^+ are as in part (1).
(c) p^+ is a non-forking extension of p.
(d) $\{M_1^+, M_2\}$ is independent over M_1 inside M_2^+.
(e) b_1, b_2 are $\mathcal{E}_{M_1^+, M_2^+}$-equivalent.
(f) M_1^+ is brimmed, and M_2^+ is brimmed over M_1^+.

Claim 3.16. 1) *For (M_1, M_2, p) as in 3.15(1), the relation $\mathcal{E}_{M_1,M_2,p}$ is an equivalence relation on $M_2 \setminus M_1$, and is equality on $p(M_2)$.*

2) *If (M_1, M_2, p) and (M_1^+, M_2^+, p^+) are as in 3.15(2)(∗), then*

$$\mathcal{E}_{M_1,M_2,p} = \mathcal{E}_{M_1^+,M_2^+,p^+} \restriction M_2.$$

Claim 3.17. *For $M \in K_{\mathfrak{k}_{\mathfrak{s},\lambda}}$, there is $\mathbb{E} = \mathbb{E}_M$ such that*

(A) *$\mathbb{E}$ is a smooth $\mathfrak{k}_{\mathfrak{s},\lambda}$-equivalence relation (see 2.5).*
(B) *$\bar{a} \in \mathrm{dom}(\mathbb{E})$, $\bar{a}$ codes $(M_{\bar{a},2}, M_{\bar{a},1}, p_{\bar{a}}, b_{\bar{a}})$, where*
  (a) *$M_2 \leq_{\mathfrak{s}} M$ with $\|M_2\| = \lambda$, and $\bar{a}$ lists the set of elements of M_2*
  (b) *$M_1 \leq_{\mathfrak{s}} M_2$ (using $\{(i,j) \in \lambda \times \lambda : a_i = a_j\}$).*
  (c) *$p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is regular.*
  (d) *$b \in M_2 \setminus M_1$*
(C) *$\bar{a}_1 \ \mathbb{E}_M \ \bar{a}_2$ iff*
  (a) *$\bar{a}_1, \bar{a}_2 \in \mathrm{dom}(\mathbb{E}_M)$*
  (b) *For some $M_* \in K_{\mathfrak{s},\lambda}$ brimmed over M, we have $\bar{a}_1 \ \mathbb{E}'_{M_*} \ \bar{a}_2$, where $\mathbb{E}'_{M_*}$ is the closure (to an equivalence relation) of the union of the two sets below.*

$$\{(\bar{c},\bar{d}) \in \mathrm{dom}(\mathbb{E}'_{M_*}) : (M_{\bar{c},1}, M_{\bar{c},2}, p_{\bar{c}}) = (M_{\bar{d},1}, M_{\bar{d},2}, p_{\bar{d}}) \text{ and } b_{\bar{c}} \ \mathcal{E}_{M_{\bar{c},1},M_{\bar{c},2},p_{\bar{c}}} \ b_{\bar{d}}\}$$
$$\cup \{(\bar{c},\bar{d}) \in \mathrm{dom}(\mathbb{E}'_{M_*}) : b_{\bar{c}} = b_{\bar{c}},\ M_{\bar{c},1} \leq_{\mathfrak{s}} M_{\bar{d},1},\ M_{\bar{c},2} \leq_{\mathfrak{s}} M_{\bar{d},2},\ p_{\bar{d}} \text{ is a non-forking extension of } p_{\bar{c}}, \text{ and } \{M_{\bar{c},2}, M_{\bar{d},1}\} \text{ is independent over } M_{\bar{c},1} \text{ inside } M_{\bar{d},2}\}.$$

* * *

Definition 3.18. 1) Assume $q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$. We say that q is explicitly (p,n)-simple <u>when</u>:

⊛ There are $b_0, \ldots, b_{n-1}, c$ such that:[25]
  (a) b_ℓ realizes p.
  (b) c realizes q.
  (c) b_ℓ is not good[26] over $(\bar{a}, c)$ for $\ell < n$.
  (d) $\langle b_\ell : \ell < n\rangle$ is independent over $\bar{a}$.
  (e) $\langle c, b_0, \ldots, b_{n-1}\rangle$ is good over $\bar{a}$.
  (f) If[27] c' realizes q then $c = c'$ <u>iff</u> for every $b \in p(\mathfrak{C})$ we have that b is good over $(\bar{a}, c)$ iff b is good over $(\bar{a}, c')$.

[25] Clauses (c)+(e) are replacements for 'c is algebraic over $\bar{a} + \{b_\ell : \ell < n\}$' and 'each b_ℓ is necessary.'
[26] 'Not good' here is a replacement to "$\mathrm{ortp}(b_\ell, \bar{a} + c, \mathfrak{C})$ does fork over $\bar{a}$."
[27] This seems a reasonable choice here but we can take others; this is an unreasonable choice for first order.

1A) We say that a is explicitly (p,n)-simple over A <u>if</u> $\mathrm{ortp}(a,A,\mathfrak{C})$ is; similarly, in the other definitions replacing (p,n) by p will mean "for some n."

2) Assume $q\in\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar a,\mathfrak{C})$ and $\mathbf{P}$ as in Definition 1.8. We say that q is $\mathbf{P}$-simple if we can find n and explicitly $\mathbf{P}$-regular types $p_0,\dots,p_{n-1}\in\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar a,\mathfrak{C})$ such that each $c\in p(\mathfrak{C})$ is definable by its type over $\bar a\cup\bigcup\limits_{\ell<n}p_\ell(\mathfrak{C})$.

3A) In part (1) we say weakly (p,n)-simple if in $\circledast$, clause (f) is replaced by

(f)$'$ If b is good over $(\bar a,a^*_m)$ then c and c' realize the same type over $\bar a\char`^\langle a^*_m,b\rangle$.

3B) In part (1) we say (p,n)-simple <u>if</u> for some $\bar a^*\in{}^{\omega>}\mathfrak{C}$ good over $\bar a$, for every $c\in q(\mathfrak{C})$, there are $b_0,\dots,b_{n-1}\in p(\mathfrak{C})$ such that $c\in\mathrm{dcl}(\bar a,\bar a^*,b_0,\dots,b_{n-1})$ and $\bar a\char`^\langle b_0,\dots,b_{n-1}\rangle$ is good over $\bar a$ if simple.

4) Similarly in (2).

5) We define $\mathrm{gw}_p(b,\bar a)$ for p regular and parallel to some $p'\in\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar a)$. (Here gw stands for 'general weight.') Similarly for $\mathrm{gw}_p(q)$.

We first list some obvious properties.

Claim 3.19. 1) *If c is $\mathbf{P}$-simple over $\bar a$, with $\bar a\subseteq A\subset\mathfrak{C}$, <u>then</u> $w_p(c,A)$ is finite.*

2) *The obvious implications.*

Claim 3.20. 1) [*Closures of the simple* bs].

2) *Assume $p\in\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar a,\mathfrak{C})$. If $\bar b_1,\bar b_2$ are p-simple over A <u>then</u>*

(*a*) *$\bar b_1\char`^\bar b_2$ is p-simple (of course, $\mathrm{ortp}_{\mathfrak{s}}(\bar b_2\bar b_2,\bar a,\mathfrak{C})$ is not necessary in $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar a,\mathfrak{C})$ even if $\mathrm{ortp}_{\mathfrak{s}}(\bar b_\ell,\bar a,\mathfrak{C})\in\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar a,\mathfrak{C})$ for $\ell=1,2$).*

(*b*) *Also, $\mathrm{ortp}(\bar b_2,\bar a b_1,\mathfrak{C})$ is p-simple.*

3) *If $\bar b_\alpha$ is p-simple over $\bar a$ for $\alpha<\alpha^*$ and $\pi:\beta^*\to\alpha^*$ one to one and onto, <u>then</u>*

$$\sum_{\alpha<\alpha^*}\mathrm{gw}_p\big(b_\alpha,\bar a_*\cup\bigcup_{\ell<\alpha}b_\alpha\big)=\sum_{\beta<\beta^*}\mathrm{gw}\big(b_{\pi(\beta)},\bar a\cup\bigcup_{i<\beta}\bar b_{\pi(i)}\big).$$

The following definition comes from [Shec, 6.9(1)$_{\mathsf{=Lg29}}$].

Definition 3.21. Assume $p_1,p_2\in\mathcal{S}^{\mathrm{bs}}(M)$. We say p_1,p_2 are weakly orthogonal (and denote it $p_1\underset{\mathrm{wk}}{\perp}p_2$) when the following implication holds: if $M_0\le_{\mathfrak{s}}M_\ell\le_{\mathfrak{s}}M_3$, $(M_0,M_\ell,a_\ell)\in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ and $\mathrm{ortp}_{\mathfrak{s}}(a_\ell,M_0,M_\ell)=p_\ell$ for $\ell=1,2$ <u>then</u> $\mathrm{ortp}_{\mathfrak{s}}(a_2,M_1,M_3)$ does not fork over M_0. (this is symmetric by $\mathbf{Ax.E}$(f).)

Claim 3.22. [$\mathfrak{s}$ *is equivalence-closed.*]

Assume that $p,q\in\mathcal{S}^{\mathrm{bs}}(M)$ are not weakly orthogonal (see 3.21*). <u>Then</u> for some $\bar a\in{}^{\omega>}M$ we have that p,q are definable over $\bar a$ (this works without being stationary) and for some $\mathfrak{k}_{\mathfrak{s}}$-definable function $\mathbf{F}$, for each $c\in q(\mathfrak{C})$, $\mathrm{ortp}_{\mathfrak{s}}(\mathbf{F}(c,\bar a),\bar a,\mathfrak{C})\in\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar a,\mathfrak{C})$ and is explicitly (p,n)-simple for some n. (If –e.g.– M is $(\lambda,*)$-brimmed, then $n=w_p(q)$.)*

Proof. We can find n and $c_1,b_0,\dots,b_{n-1}\in\mathfrak{C}$ with c realizing q, b_ℓ realizing p, $\{b_\ell,c\}$ not independent over M, and n maximal. Choose $\bar a\in{}^{\omega>}M$ such that

$$\mathrm{ortp}_{\mathfrak{s}}(\langle c,b_0,\dots,b_{n-1}\rangle,M,\mathfrak{C})$$

is definable over $\bar a$. Define $E_{\bar a}$, an equivalence relation on $q(\mathfrak{C})$: $c_1\ E_{\bar a}\ c_2$ iff for every $b\in p\mathfrak{C}$, we have "b is good over (a,c_1)" $\Rightarrow$ "b is good over $(\bar a,c_2)$." By "$\mathfrak{s}$ is eq-closed," we are done. $\square_{3.22}$

Claim 3.23. 1) *Assume $p, q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ are weakly orthogonal (see 3.21) but not orthogonal. Then we can find $\bar a \in {}^{\omega>}M$ over which p, q are definable and $r_1 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a, \mathfrak{C})$ such that (letting $p_1 := p \restriction \bar a$, $q_1 := q \restriction \bar a$, $n := w_p(q) \geq 1$) we have:*[28]

$\circledast^n_{\bar a, p_1, q_1, r_2}$ (a) *$p_1, q_1, r_1 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a)$, $\bar a \in {}^{\omega>}\mathfrak{C}$.*

(b) *p_1, q_1 are weakly orthogonal.*

(c) *If $\{a^*_n : n < \omega\} \subseteq r_1(\mathfrak{C})$ is independent over $\bar a$ and c realizes q then for infinitely many $m < \omega$ there is $b \in p(\mathfrak{C})$ such that b is good over $(\bar a, a^*_n)$ but not over $(\bar a, a^*_n, c)$.*

(d) *In (c) there really are n independent $b_0, \dots, b_{n-1}$ which are all good over $(\bar a, a^*_n)$ but not over $(\bar a, a^*_n, c)$ (but we cannot find $n+1$ such b-s.).*

2) *If $\circledast^n_{\bar a, p_1, q_1, r_1}$ then (see Definition 3.18(3) for some definable function $\mathbf{F}$, if c realizes q_1, $c^* = F(c, \bar a)$ and $\mathrm{ortp}_{\mathfrak{n}}(c^*, \bar a, \mathfrak{C})$ is (p_1, n)-simple.*

See proof below.

Claim 3.24. 0) *Assuming $A \subseteq \mathfrak{C}$ and $a \in A$, we say $\mathrm{ortp}(a, A, \mathfrak{C})$ is* finitary *when it is definable over $A \cup \{a_0, \dots, a_{n-1}\}$ for some n, where each a_ℓ is in $\mathfrak{C}$ and is good over A inside $\mathfrak{C}$.*

1) *If $a \in \mathrm{dcl}\big(\bigcup\{A_i : i < \alpha\} \cup A, \mathfrak{C}\big)$, $\mathrm{ortp}(a, A, \mathfrak{C})$ is finitary, and $\{A_i : i < \alpha\}$ is independent over A then for some finite $u \subseteq \alpha$ we have*

$$a \in \mathrm{dcl}\big(\bigcup_{i \in u} A_i \cup A, \mathfrak{C}\big).$$

2) *If $\mathrm{ortp}(b, \bar a, \mathfrak{C})$ is $\mathbf{P}$-simple, then it is finitary.*

3) *If $\{A_i : i < \alpha\}$ is independent over A and a is finitary over A then for some finite $u \subseteq \alpha$ (even $|u| < \mathrm{wg}(c, A)$), $\{A_i : i \in \alpha \setminus u\}$ is independent over $(A, A \cup \{c\})$. (Or use $(A', A''), (A', A'' \cup \{c\})$.)*

Definition 3.25. 1) $\mathrm{dcl}(A) = \{a : f(a) = a$ for every automorphism f of $\mathfrak{C}\}$.

2) $\mathrm{dcl}_{\rm fin}(A) = \bigcup\{\mathrm{dcl}(B) : B \subseteq A$ finite$\}$.

3) a is finitary over A if there are $n < \omega$ and $c_0, \dots, c_{n-1} \in \mathrm{good}(A)$ such that $a \in \mathrm{dcl}(A \cup \{c_0, \dots, c_{n-1}\})$.

4) For such A, let $\mathrm{wg}(a, A)$ be $w(\mathrm{tp}(a, A, \mathfrak{C}))$ when well defined.

5) Strongly simple implies simple.

Claim 3.26. *In Definition 3.18(3), for some $m, k < \omega$ large enough, for every $c \in q(\mathfrak{C})$ there are $b_0, \dots, b_{m-1} \in \bigcup_{\ell < n} p_\ell(\mathfrak{C})$ such that*

$$c \in \mathrm{dcl}\big(\bar a \cup \{a^*_\ell : \ell < k\} \cup \{b_\ell : \ell < m\}\big).$$

Proof. Let $M_1, M_2 \in K_{\mathfrak{s}(\rm brim)}$ be such that $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2$, M_1 is $(\lambda, *)$-brimmed over M, $p_\ell \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_\ell)$ a non-forking extension of p, $q_\ell \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_\ell)$ is a non-forking extension of q, $c \in M_2$ realizes q_1, and $(M_1, M_2, c) \in K^{3,\rm pr}_{\mathfrak{s}(\rm brim)}$. Let $b_\ell \in p_1(M_2)$ for $\ell < n^* := w_p(q)$ be such that $\{b_\ell : \ell < n^*\}$ is independent in (M_1, M_2); let $\bar a^* \in {}^{\omega>}(M_1)$ be such that $\mathrm{ortp}_{\mathfrak{s}}(\langle c, b_0, \dots, b_{n-1}\rangle, M_1, M_2)$ is definable over $\bar a^*$ and $r = \mathrm{ortp}_{\mathfrak{s}}(\bar a^*, M_1, M_2)$, $r^+ = \mathrm{ortp}(\bar a^* {}^\frown \langle b_0, \dots, b_{n-1}\rangle, M, M_2)$.

Let $\bar a \in {}^{\omega>}M$ be such that $\mathrm{ortp}_{\mathfrak{s}}(\bar a^*, \langle c, b_0, \dots, b_{n-1}\rangle, M, M_2)$ is definable over $\bar a$. As M_1 is $(\lambda, *)$-saturated over M, there is $\{\bar a^*_f : f < \omega\} \subseteq r(\mathfrak{C})$ independent

[28] We can say more concerning simple types.

in (M, M_1). Moreover, letting $a^*_\omega = \bar{a}^*$, we have $\langle a^*_\alpha : \alpha \leq \omega \rangle$ is independent in (M, M_1). Clearly $\mathrm{ortp}_{\mathfrak{s}}(c\bar{a}^*_n, M, M_2)$ does not depend on n hence we can find $\big\langle \langle b^\alpha_\ell : \ell < n \rangle : \alpha \leq \omega \big\rangle$ such that $b^\alpha_\ell \in M_2$, $b^\omega_\ell = b_\ell$, and $\{c\bar{a}^*_\alpha, b^\alpha_0 \ldots b^\alpha_{n-1} : \alpha \leq \omega\}$. (As usual, this is because the index set is independent in (M_1, M_2).)

The rest should be clear. $\square_{3.23}$

Definition 3.27. Assume $\bar{a} \in {}^{\omega >}\mathfrak{C}$, $n < \omega$, and $p, q, r \in \mathcal{S}^{\mathrm{bs}}(M)$ are as in the definition of p-simple$^{[-]}$ but p and q are weakly orthogonal (see e.g. Definition 3.21(1)). Let p be a definable related function such that for any $\bar{a}^\nu_\ell \in r(\mathfrak{C})$, $\ell < k^*$, the independent mapping $c \mapsto \big\{ b \in q(\mathfrak{C}) : R\mathfrak{C} \models R(b, c, \bar{a}^*_\ell) \big\}$ is a one-to-one function from $q(\mathfrak{C})$ into

$$\big\{ \langle J_\ell : \ell < k^* \rangle : J_\ell \subseteq p(\mathfrak{C}) \text{ is closed under dependence and has } p\text{-weight } n^* \big\}.$$

1) We can define $E = E_{p,q,r}$, a two-place relation over $r(\mathfrak{C})$: $\bar{a}^*_1 \; E \; \bar{a}^*_2$ iff $\bar{a}_1, \bar{a}_2 \in r(\mathfrak{C})$ have the same projection common to $p(\mathfrak{C})$ and $q(\mathfrak{C})$.

2) Define the unitless group on r/E and its action on $q(\mathfrak{C})$.

* * *

Remark 3.28. 1) A major point: as q is p-simple, $w_p(-)$ acts "nicely" on $p(\mathfrak{C})$, so if $c_1, c_2, c_3 \in q(\mathfrak{C})$ then $w_p(\langle c_1, c_2, c_3 \rangle \bar{a}) \leq 3n^*$. This enables us to define averages using a finite sequence in a quite satisfying way. Alternatively, look more at averages of independent sets.

2) **Silly Groups**: Concerning interpreting groups, note that in our present context, for every definable set P^M we can add the group of finite subsets of P^M with symmetric difference (as addition).

3) The axiomatization above has prototype $\mathfrak{s}$, where

$$K_{\mathfrak{s}} = \{M : M \text{ a } \kappa\text{-saturated model of } T\},$$

$\leq_{\mathfrak{s}} = \prec \restriction K_{\mathfrak{s}}$, $\bigcup_{\mathfrak{s}}$ is non-forking, T a stable first order theory with $\kappa(T) \leq \mathrm{cf}(\kappa)$. But we may prefer to formalize the pair $(\mathfrak{t}, \mathfrak{s})$ with $\mathfrak{s}$ as above, $K_{\mathfrak{t}}$ = models of T, $\leq_{\mathfrak{t}} = \prec \restriction K_{\mathfrak{t}}$, $\bigcup_{\mathfrak{t}}$ is non-forking.

From $\mathfrak{s}$ we can reconstruct a $\mathfrak{t}$ by closing $\mathfrak{k}_{\mathfrak{s}}$ under direct limits, but in interesting cases we end up with a bigger $\mathfrak{t}$.

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