ULTRAFILTERS AND TURING INDEPENDENCE

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Abstract. We investigate the following problem: Given an ultrafilter on the set of integers, is it generated by a Turing independent set? We show the following: (1) Assuming the continuum hypothesis, there is an ultrafilter all of whose bases are cofinal in the Turing degrees. (2) For each $n \geq 1$, every ultrafilter has an n-Turing independent $(n+1)$-basis. (3) It is consistent (relative to a Ramsey cardinal) that there is an ultrafilter that has a Turing independent basis.

1. Introduction

The aim of this paper is to study some problems about the global structure of Turing degrees that are sensitive to the ambient set theory. For examples of such results, see [2, 4, 8, 10]. For some history and motivation behind these questions, we refer the reader to [6].

In [7], some Ramsey type questions of the following type were studied: Given a large set of reals, does it have a large Turing independent subset? There, largeness was interpreted in the sense of cardinality, (Lebesgue) measure and (Baire) category. The results of this paper are concerned with these questions when largeness is interpreted to mean "generates an ultrafilter".

Definition 1.1. *Let $\mathcal{U}$ be an ultrafilter on some set, $1 \leq n < \omega$ and $\mathcal{B} \subseteq \mathcal{U}$.*

(1) *$\mathcal{B}$ is a basis for $\mathcal{U}$ iff for every $A \in \mathcal{U}$, there exists $B \in \mathcal{B}$ such that $A \subseteq B$.*
(2) *$\mathcal{B}$ is a n-basis for $\mathcal{U}$ iff for every $A \in \mathcal{U}$ there exists $\mathcal{F} \subseteq \mathcal{B}$ such that $|\mathcal{F}| \leq n$ and $\bigcap \mathcal{F} \subseteq A$.*
(3) *$\mathcal{B}$ is a subbasis for $\mathcal{U}$ iff for every $A \in \mathcal{U}$ there exists a finite $\mathcal{F} \subseteq \mathcal{B}$ such that $\bigcap \mathcal{F} \subseteq A$.*

It is clear that basis = 1-basis and every n-basis is also a subbasis.

Definition 1.2. *Let $\mathcal{A} \subseteq 2^\omega$ and $1 \leq n < \omega$.*

(1) *$\mathcal{A}$ is n-Turing independent iff for every $\mathcal{F} \in [\mathcal{A}]^{\leq n}$ and $Y \in \mathcal{A} \setminus \mathcal{F}$, the Turing join of $\mathcal{F}$ does not compute Y.*
(2) *$\mathcal{A}$ is Turing independent iff it is n-Turing independent for every $n \geq 1$.*

Note that 1-Turing independent sets are just Turing antichains (their members are pairwise Turing incomparable). Throughout this paper, we will assume that all ultrafilters are non-principal. We can now formulate the general question as follows.

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Question 1.3. *Let $m, n \geq 1$ and $\mathcal{U}$ be an ultrafilter on ω. Does $\mathcal{U}$ have an n-Turing independent/Turing independent basis/m-basis/subbasis?*

The answer will depend on the choice of $\mathcal{U}$ and the ambient set theory. We begin with a negative result.

Theorem 1.4. *Under CH, there is an ultrafilter $\mathcal{U}$ on ω all of whose bases are cofinal in the Turing degrees. Hence, $\mathcal{U}$ has no Turing antichain basis.*

A contrasting ZFC result says the following.

Theorem 1.5. *For every $X \in [\omega]^\omega$, there exists $Y \in [X]^\omega$ such that for every ultrafilter $\mathcal{U}$ on ω, if $Y \in \mathcal{U}$, then $\mathcal{U}$ has a Turing antichain basis.*

Next, we show the following (in ZFC).

Theorem 1.6. *For each $n \geq 1$, every ultrafilter $\mathcal{U}$ on ω has an n-Turing independent $(n+1)$-basis.*

It follows that for each $n \geq 1$, every ultrafilter on ω has an n-Turing independent subbasis. Note that by Theorem 1.4, we cannot replace $(n+1)$-basis by n-basis. Finally, we have the following consistency result.

Theorem 1.7. *Assume there is a Ramsey cardinal in V. Then there is a ccc forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an ultrafilter on ω that has a Turing independent basis.*

Some interesting combinatorial results also appear along the way. One such result (Theorem 2.3) says that if a family of introreducible sets (see Definition 2.1) has the FIP, then it is countable. Its proof uses forcing and absoluteness. Another one (Theorem 2.7) says that if an ultrafilter does not have a Turing antichain basis, then it cannot be generated by fewer than $\mathfrak{c}$ sets. Many variants of Question 1.3 remain open. For example,

Question 1.8. *In ZFC, can we construct an ultrafilter on ω that has a Turing independent basis?*

Notation. $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$ snd $[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\}$. A family $\mathcal{A}$ of sets has the FIP (finite intersection property) iff $\bigcap \mathcal{F}$ is infinite for every finite $\mathcal{F} \subseteq \mathcal{A}$. $A \subseteq^\star B$ iff $A \setminus B$ is finite. $(\forall^\infty x)$ means "For all but finitely many x" and $(\exists^\infty x)$ means "There are infinitely many x". For $F = \{x_0, x_2, \dots, x_{n-1}\} \subseteq 2^\omega$, the Turing join of F, denoted $\bigoplus_{k<n} x_k$, is the real $y \in 2^\omega$ satisfying $y(nj + k) = x_k(j)$ for every $k < n$ and $n, j < \omega$. $\langle \Phi_e : e < \omega \rangle$ is an effective listing of all Turing functionals. Given $y \in 2^\omega$ and $k < \omega$, we write $\Phi_e^y(k) \downarrow= n$ iff the eth Turing functional with oracle y converges on input k and outputs n. We write $\Phi_e^y(k)[s] \downarrow= n$, if $k, n < s$ and the oracle use of the computation is contained in $y \restriction s$. For $\sigma \in {}^{<\omega}2$, define $[\sigma] = \{y \in 2^\omega : \sigma \preceq y\}$. $T \subseteq 2^{<\omega}$ is a perfect tree iff every node in T has two incomparable extensions. For $T \subseteq 2^{<\omega}$, $[T] = \{y \in 2^\omega : (\forall n)(y \restriction n \in T)\}$ is the set of infinite branches through T. In forcing, $p \leq q$ means "p extends q".

2. Turing antichain basis I

We begin by showing that, under CH, there is an ultrafilter on ω with no Turing antichain basis. Our construction involves introreducible sets.

Definition 2.1. *$A \in [\omega]^\omega$ is introreducible iff every infinite subset of A computes A.*

For every $d \in 2^\omega$, there is an introreducible set of Turing degree d defined as follows. First, fix a computable bijection $h : 2^{<\omega} \to \omega$. Define $D_d = \{h(d \restriction k) : k < \omega\}$ (Dekker set for d). It is easily verified that D_d is an introreducible set of Turing degree d.

The following lemma says that for any introreducible set $A \in [\omega]^\omega$ of Turing degree d, the set of Turing degrees of introreducible subsets of A is the Turing cone above d.

Lemma 2.2. *Suppose $A \in [\omega]^\omega$ is introreducible, $X \subseteq \omega$ and $A \oplus X \leq_T d$. Then there exists an introreducible $B \in [A]^\omega$ such that $B \equiv_T d$. Furthermore, either $B \subseteq X$ or $B \cap X = \emptyset$.*

Proof. Let $\{n_k : k < \omega\}$ be an enumeration of A in increasing order. Fix a computable bijection $h : 2^{<\omega} \to \omega$. Put $S = \{h(d \restriction k) : k < \omega\}$ and $A' = \{n_k : k \in S\}$. Note that S is an introreducible set of Turing degree d. Define $B = A' \cap X$, if $A' \cap X$ is infinite. Otherwise, define $B = A' \cap (\omega \setminus X)$. In either case, it is clear that $B \in [A]^\omega$. Let us check that B is as required.

(i) $B \leq_T d$. First assume that $B = A' \cap X$. Since $A \leq_T d$ and $S \equiv_T d$, it follows that A' is computable from d. By assumption, $X \leq_T d$ and so $B = A' \cap X \leq_T d$. A similar argument works when $B = A' \cap (\omega \setminus X)$.

(ii) B is introreducible and $B \equiv_T d$. Let $C \in [B]^\omega$. We will show that $d \leq_T C$. This suffices since $B \leq_T d$. Since $C \in [A]^\omega$ and A is introreducible, we get that A and hence the function $k \mapsto n_k$ are computable from C. Define $S' = \{k : n_k \in C\}$. Then $S' \leq_T C$. Now since S is introreducible and $S' \in [S]^\omega$, we get $S \leq_T S' \leq_T C$. Finally, as $S \equiv_T d$, we get $d \leq_T C$.

(iii) Either $B \subseteq X$ or $B \cap X = \emptyset$. This is clear by the definition of B.

$\square$

In view of Lemma 2.2, it is natural to wonder if, assuming CH, one could construct a family $\mathcal{A}$ of introreducible sets satisfying the following.

(1) $\mathcal{A}$ has the FIP.
(2) Every $d \in 2^\omega$ is computable from some member of $\mathcal{A}$.

This would imply that no ultrafilter $\mathcal{U}$ extending $\mathcal{A}$ can have a Turing antichain basis. Unfortunately, this is impossible.

Theorem 2.3. *Every family of introreducible sets with the FIP is countable.*

Proof. Suppose $\mathcal{F}$ is an uncountable family of introreducible sets with the FIP. Let $\mathbb{P}$ be a ccc forcing for adding a pseudointersection $\mathring{B} \in [\omega]^\omega \cap V^{\mathbb{P}}$ for $\mathcal{F}$. $\mathbb{P}$ is defined as follows. $p \in \mathbb{P}$ iff $p = (s_p, F_p)$ where $s_p \in [\omega]^{<\omega}$ and $F_p \in [\mathcal{F}]^{<\omega}$. For $p, q \in \mathbb{P}$, $p \leq_{\mathbb{P}} q$ iff $s_q \subseteq s_p$, $F_q \subseteq F_p$ and for every $A \in F_q$, $s_p \setminus s_q \subseteq A$.

As $\mathbb{P}$ is ccc, all cardinals are preserved between V and $V^{\mathbb{P}}$. Hence $V^{\mathbb{P}} \models \mathcal{F}$ is uncountable. Next, observe that A is introreducible iff

$$(\forall X \in [\omega]^\omega)(\exists e < \omega)(X \subseteq A \implies \Phi_e^X = A)$$

which is a $\Pi^1_1(A)$-formula. By Mostowski's absoluteness theorem, it follows that for every $A \in \mathcal{F}$, $V^{\mathbb{P}} \models A$ is introreducible.

Let $\mathring{B} = \bigcup\{s_p : p \in \mathring{G}_{\mathbb{P}}\}$. A density argument shows that $V^{\mathbb{P}} \models \mathring{B} \in [\omega]^\omega$. Also, for every $A \in \mathcal{F}$, $V^{\mathbb{P}} \models A \subseteq^\star \mathring{B}$. Since $V^{\mathbb{P}} \models A$ is introreducible, it follows that $V^{\mathbb{P}} \models A \leq_T \mathring{B}$ for every $A \in \mathcal{F}$. This is impossible since no real can compute uncountably many reals. $\square$

Corollary 2.4. *There is a descending sequence $\langle A_n : n < \omega\rangle$ of introreducible sets such that there is no introreducible set B such that $B \subseteq^\star A_n$ for every n.*

Proof. Suppose not. Inductively construct $\langle A_\alpha : \alpha < \omega_1\rangle$ such that each $A_\alpha \in [\omega]^\omega$ is introreducible and for every $\alpha < \beta < \omega_1$, $A_\alpha \subseteq^\star A_\beta$ and $A_\beta \not\leq_T A_\alpha$. To carry out this construction, at successor stages, we use Lemma 2.2, and at limit stages, we use the assumption that every $\subseteq^\star$-descending sequence of introreducible sets has an introreducible pseudointersection. It follows that $\{A_\alpha : \alpha < \omega_1\}$ is an uncountable family of introreducible sets with the FIP. This is impossible by Theorem 2.3. $\square$

Lemma 2.5. *Assume CH. There is a family $\mathcal{F}$ of subsets of $\omega \times \omega$ that satisfies the following.*

(1) For every $A \in \mathcal{F}$ and $n < \omega$, $A_n = \{m : (n, m) \in A\} \in [\omega]^\omega$ is introreducible.
(2) For every $A, B \in \mathcal{F}$, either $(\forall^\infty n)(A_n \subseteq B_n)$ or $(\forall^\infty n)(B_n \subseteq A_n)$.
(3) For every $d \in 2^\omega$, there exists $A \in \mathcal{F}$ such that $(\forall n)(d \leq_T A_n)$.

Proof. Using CH, fix an enumeration $\{d_i : i < \omega_1\}$ of 2^ω and inductively construct $\langle A_i : i < \omega_1\rangle$ such that the following hold.

(i) Each $A_i \subseteq \omega \times \omega$ and for every $n < \omega$, $(A_i)_n = \{m : (n, m) \in A_i\} \in [\omega]^\omega$ is introreducible.
(ii) If $i < j < \omega_1$, then $(\forall^\infty n)((A_i)_n \subseteq (A_j)_n)$.
(iii) For every $i < \omega_1$ and $n < \omega$, $d_i \leq_T (A_{i+1})_n$.

Start by defining $A_0 = \omega \times \omega$. At successor stages, use Lemma 2.2 to obtain an introreducible $(A_{i+1})_n \subseteq (A_i)_n$ of Turing degree $\geq_T d_i$. This guarantees Clause (iii). If $i < \omega_1$ is limit, fix an increasing cofinal sequence $\langle i_k : i < \omega\rangle$ in i and define A_i by $(A_i)_n = (A_{i_k})_n$ where k is the largest $k \leq n$ satisfying

$$(A_{i_0})_n \supseteq (A_{i_1})_n \supseteq \cdots \supseteq (A_{i_k})_n$$

Let us check that $(\forall j < i)(\forall^\infty n)((A_i)_n \subseteq (A_j)_n)$. Let $j < i$. Let $m < \omega$ be least such that $j < i_m$. Using Clause (ii), choose $n_\star \geq m$ such that for every $n \geq n_\star$,

$$(A_{i_0})_n \supseteq (A_{i_1})_n \supseteq \cdots (A_{i_{m-1}})_n \supseteq (A_j)_n \supseteq (A_{i_m})_n$$

By the definition of A_i, it follows that $(A_i)_n \subseteq (A_j)_n$ for every $n \geq n_\star$. So Clauses (i)-(ii) continue to hold at stage i. Having completed the construction, we can define $\mathcal{F} = \{A_i : i < \omega_1\}$. $\square$

Theorem 2.6. *Assume CH. There is an ultrafilter $\mathcal{U}$ on ω all of whose bases are cofinal in the Turing degrees. Hence, $\mathcal{U}$ does not have a Turing antichain basis.*

Proof. Since there is a computable bijection between $\omega \times \omega$ and ω, it suffices to construct such an ultrafilter on $\omega \times \omega$. Let $\mathcal{F}$ be as in Lemma 2.5. Fix an ultrafilter $\mathcal{V}$ on ω. Using Zorn's lemma, choose a maximal filter $\mathcal{U}$ on $\omega \times \omega$ such that $\mathcal{F} \subseteq \mathcal{U}$ and for every $B \in \mathcal{U}$, $\{n : B_n \in [\omega^\omega]\} \in \mathcal{V}$. It is easy to see that $\mathcal{U}$ is an ultrafilter on $\omega \times \omega$.

Let $\mathcal{B}$ be any basis for $\mathcal{U}$ and $d \in 2^\omega$. We will find a $C \in \mathcal{B}$ that computes d. Choose $A \in \mathcal{F}$ such that $(\forall n)(d \leq_T (A)_n)$ (possible by Clause (3) of Lemma 2.5).

As $\mathcal{B}$ is a basis for $\mathcal{U}$, we can find $C \in \mathcal{B}$ such that $C \subseteq A$. Since $\{n : C_n \in [\omega]^\omega\} \in \mathcal{V}$, we can fix an $n_\star$ such that $C_{n_\star} \in [A_{n_\star}]^\omega$. It follows that $d \leq_T A_{n_\star} \leq_T C_{n_\star} \leq_T C$. Hence $\mathcal{B}$ is cofinal in the Turing degrees. $\square$

We do not know if the existence of an ultrafilter without a Turing antichain basis is consistent with $\mathfrak{c} > \omega_1$. The next result says that such an ultrafilter cannot be generated by fewer than continuum sets.

Theorem 2.7. *Let $\mathcal{U}$ be an ultrafilter on ω with no Turing antichain basis. Then every basis for $\mathcal{U}$ has size $\mathfrak{c}$.*

Proof. Let $\mathcal{A} \subseteq \mathcal{U}$ such that $|\mathcal{A}| = \kappa < \mathfrak{c}$. It suffices to construct $\mathcal{E} \subseteq \mathcal{U}$ such that $\mathcal{E}$ is a Turing antichain and for every $x \in \mathcal{A}$, there exists $y \in \mathcal{E}$ such that $y \subseteq x$.

For each $x \in \mathcal{A}$, fix $x' \in [x]^\omega$ such that $x' \notin \mathcal{U}$. Let $P_x = \{x \setminus y : y \subseteq x'\}$. Then P_x is a perfect subset of 2^ω and each member of P_x is in $\mathcal{U} \cap \mathcal{P}(x)$. For each $e < \omega$, fix a Borel function $f_e : 2^\omega \to 2^\omega$ such that

$$(\forall z \in 2^\omega)(\Phi_e^z \text{ is total } \implies \Phi_e^z = f_e(z))$$

Put $\mathcal{F} = \{f_e : e < \omega\}$. Since $|\mathcal{A}| < \mathfrak{c}$, we can apply Lemma 2.11 below to obtain an $\mathcal{F}$-free refinement $\langle Q_x : x \in \mathcal{A}\rangle$ of $\langle P_x : x \in \mathcal{A}\rangle$. For each $x \in \mathcal{A}$, fix $y_x \in Q_x$ and define $\mathcal{E} = \{y_x : x \in \mathcal{A}\}$. Then $\mathcal{E}$ is as required. $\square$

Definition 2.8. *Let $\mathcal{F}$ be a family of functions $f : 2^\omega \to 2^\omega$ and $\bar{A} = \langle A_\alpha : \alpha < \kappa\rangle$ be a sequence of subsets of 2^ω. We say that $\bar{A}$ is $\mathcal{F}$-free iff for every $f \in \mathcal{F}$ and for every distinct $\alpha, \beta < \kappa$, $f^{-1}[A_\alpha] \cap A_\beta = \emptyset$.*

Lemma 2.9. *Suppose $\mathcal{F}$ is a countable family of Borel functions $f : 2^\omega \to 2^\omega$ and $\mathcal{A}$ is a disjoint family of perfect subsets of 2^ω. Then for every perfect $P \subseteq 2^\omega$, there exists a perfect $Q \subseteq P$ such that for every $f \in \mathcal{F}$, for all but finitely many $R \in \mathcal{A}$, $Q \cap f^{-1}[R]$ is countable.*

Proof. Let $\{f_n : n < \omega\}$ be an enumeration of $\mathcal{A}$. For a perfect tree $T \subseteq 2^{<\omega}$, define $\mathsf{Splitnode}_k(T)$ to be the set of all kth level splitting nodes of T. So $\mathsf{Splitnode}_0(T)$ is the singleton containing the stem of T and $\mathsf{Splitnode}_k(T)$ has 2^k nodes.

Fix a perfect tree $T \subseteq 2^{<\omega}$ such that $[T] = P$ and construct a (fusion) sequence of perfect trees $\langle T_n : n < \omega\rangle$ as follows.

(i) $T_0 = T$.

(ii) Given T_n, define T_{n+1} as follows. Let $\{\sigma_j : j < 2^n\}$ list $\mathsf{Splitnode}_n(T_n)$. For each $j < 2^n$, let $T_n^j = \{\rho \in T_n : \rho \preceq \sigma_j \text{ or } \sigma_j \preceq \rho\}$ be the subtree of T_n above σ_j. Choose a perfect tree $S_j \subseteq T_n^j$ as follows.

(a) If for every $R \in \mathcal{A}$, $f_n^{-1}[R] \cap [T_n^j]$ is countable, then $S_j = T_n^j$.

(b) If for some $R \in \mathcal{A}$, $f_n^{-1}[R] \cap [T_n^j]$ is uncountable, then fix one such R and a perfect tree $S_j \subseteq T_n^j$ such that $[S_j] \subseteq f_n^{-1}[R] \cap [T_n^j]$.

Define $T_{n+1} = \bigcup_{j<2^n} S_j$.

Put $S = \bigcap_{n<\omega} T_n$ and $Q = [S]$. Then Q is a perfect subset of P and for every $n < \omega$, there are at most 2^n members $R \in \mathcal{A}$ such that $f_n^{-1}[R] \cap Q$ is uncountable. Hence Q is as required. $\square$

Fact 2.10 (Balcar-Vojtas [1]). *Suppose $\kappa < \mathfrak{c}$ and $\langle P_\alpha : \alpha < \kappa \rangle$ is a sequence of perfect subsets of 2^ω. Then there exists a sequence $\langle Q_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint perfect sets such that $Q_\alpha \subseteq P_\alpha$ for every $\alpha < \kappa$.*

Proof. Since every perfect set can be partitioned into $\mathfrak{c}$ perfect sets, we can apply Theorem 3.14 in [1] to the boolean algebra of all Borel subsets of 2^ω modulo countable sets. $\square$

Lemma 2.11. *Suppose $\mathcal{F}$ is a countable family of Borel functions $f : 2^\omega \to 2^\omega$. Let $\langle P_\alpha : \alpha < \kappa \rangle$ be a sequence of perfect subsets of 2^ω where $\kappa < \mathfrak{c}$. Then there exists an $\mathcal{F}$-free sequence $\langle D_\alpha : \alpha < \kappa \rangle$ in which each D_α is a perfect subset of P_α.*

Proof. By Fact 2.10, we can find a $\langle P'_\alpha : \alpha < \kappa \rangle$ such that each P'_α is a perfect subset of P_α and P'_α's are pairwise disjoint.

Fix a partition $P'_\alpha = \bigsqcup_{i<\mathfrak{c}} R_{\alpha,i}$ where each $R_{\alpha,i}$ is perfect. Put $\mathcal{A} = \{R_{\alpha,i} : \alpha < \kappa \text{ and } i < \mathfrak{c}\}$. Apply Lemma 2.9 to $\mathcal{A}$ and $P = R_{\alpha,i}$ to get a perfect subset $Q_{\alpha,i} \subseteq R_{\alpha,i}$ such that for every $f \in \mathcal{F}$, there are finitely many pairs $(\beta, j) \in \kappa \times \mathfrak{c}$ such that $f^{-1}[R_{\beta,j}] \cap Q_{\alpha,i}$ is uncountable. Next, inductively choose $\langle Q_\alpha : \alpha < \mathfrak{c} \rangle$ as follows.

(a) $Q_0 = Q_{0,0}$.
(b) Assume Q_β has been chosen for $\beta < \alpha$. Let X_α be the set of all $i < \mathfrak{c}$ such that for some $f \in \mathcal{F}$ and $\beta < \alpha$, $f^{-1}[R_{\alpha,i}] \cap Q_\beta$ is uncountable. Then $|X_\alpha| < \mathfrak{c}$. Choose $i < \mathfrak{c}$ such that $i \notin X_\alpha$ and define $Q_\alpha = Q_{\alpha,i}$.

Note that for every $\alpha < \beta < \kappa$, $f^{-1}[Q_\beta] \cap Q_\alpha$ is countable. Next, by induction on $\alpha < \kappa$, construct S_α and $\{S_{\alpha,i} : i < \mathfrak{c}\}$ as follows.

(c) $S_0 = Q_0$ and $\{S_{0,i} : i < \mathfrak{c}\}$ is a partition of S_0 into perfect sets.
(d) Suppose S_β, $\{S_{\beta,i} : i < \mathfrak{c}\}$ have been defined for every $\beta < \alpha$ such that $S_\beta \subseteq Q_\beta$ is perfect and $\{S_{\beta,i} : i < \mathfrak{c}\}$ is a partition of S_β into perfect sets. Put $\mathcal{A} = \{S_{\beta,i} : \beta < \alpha \text{ and } i < \mathfrak{c}\}$ and apply Lemma 2.9 to obtain a perfect subset $S_\alpha \subseteq Q_\alpha$ such that for each $f \in \mathcal{F}$, there are only finitely many $S \in \mathcal{A}$ such that $f^{-1}[S] \cap S_\alpha$ is uncountable. Let $\{S_{\alpha,i} : i < \mathfrak{c}\}$ consist of pairwise disjoint perfect subsets of S_α.

Let Y_α consist of all $i < \mathfrak{c}$ such that for some $\gamma > \alpha$ and $f \in \mathcal{F}$, $f^{-1}[S_{\alpha,i}] \cap S_\gamma$ is uncountable. Note that $|Y_\alpha| < \mathfrak{c}$ (this is where we need $\kappa < \mathfrak{c}$) so we can choose $i(\alpha) < \mathfrak{c}$ such that $i(\alpha) \notin Y_\alpha$. Define $E_\alpha = D_{\alpha,i(\alpha)}$ and $E'_\alpha = E_\alpha \cap \bigcup\{f^{-1}[E_\beta] : \beta \neq \alpha, f \in \mathcal{F}\}$. Then $|E'_\alpha| < \mathfrak{c}$ so we can choose a perfect subset $D_\alpha \subseteq E_\alpha \setminus E'_\alpha$ and $\langle D_\alpha : \alpha < \kappa \rangle$ is as required. $\square$

3. Turing antichain basis II

Let $\mathbb{M}$ denote the Mathias forcing defined as follows. $\mathbb{M}$ consists of all pairs $p = (s_p, X_p)$ where $s_p \in [\omega]^{<\omega}$, $X_p \in [\omega]^\omega$ and $\max(s_p) < \min(X_p)$. For $p, q \in \mathbb{M}$, define $p \leq_{\mathbb{P}} q$ iff $s_q \subseteq s_p$, $X_p \subseteq X_q$ and $s_p \setminus s_q \subseteq X_q$. Let $\mathcal{D}$ be a family of dense subsets of $\mathbb{M}$. We say that $X \in [\omega]^\omega$ is a $\mathcal{D}$-generic Mathias real iff there is a filter G on $\mathbb{M}$ such that $X = \bigcup_{p \in G} s_p$ and G meets every set in $\mathcal{D}$. The following facts were proved by Soare in [11].

Fact 3.1 (Soare [11]). *Let $e < \omega$ and $p = (s_p, X_p) \in \mathbb{M}$. There exists $q = (s_q, X_q) \in \mathbb{M}$ such that $s_q = s_p$, $X_q \subseteq X_p$ and for every $n \in X_q$ and $Y_1, Y_2 \in [s_q \cup X_q]^\omega$, if $\Phi_e^{Y_1} = Y_2$ and $n \in Y_2$, then $n \in Y_1$.*

Let D_e consist of $q = (s_q, X_q) \in \mathbb{M}$ such that for every $n \in X_q$ and $Y_1, Y_2 \in [s_q \cup X_q]^\omega$, if $\Phi_e^{Y_1} = Y_2$ and $n \in Y_2$, then $n \in Y_1$. Then D_e is dense in $\mathbb{M}$ by Fact 3.1. Put $\mathcal{D} = \{D_e : e < \omega\}$ and let X be any $\mathcal{D}$-generic Mathias real. Suppose $Y_1, Y_2 \in [X]^\omega$ and $Y_2 \leq_T Y_1$. Fix $e < \omega$ such that $\Phi_e^{Y_1} = Y_2$. Choose $q = (s_q, X_q) \in D_e$ such that $s_q \subseteq X \subseteq s_q \cup X_q$. As $q \in D_e$ and $Y_1, Y_2 \in [X]^\omega \subseteq [s_q \cup X_q]^\omega$, it follows that $(\forall n \in X_q)(n \in Y_2 \implies n \in Y_1)$. Hence $Y_2 \setminus Y_1 \subseteq s_q$ is finite. So we get the following.

Fact 3.2 (Soare [11])**.** *Let $X \in [\omega]^\omega$ be a $\mathcal{D}$-generic Mathias real. Then for every $Y_1, Y_2 \in [X]^\omega$, if $Y_2 \leq_T Y_1$, then $Y_2 \subseteq^\star Y_1$ (i.e., $Y_2 \setminus Y_1$ is finite).*

Definition 3.3. *Let $\mathcal{U}$ be an ultrafilter on ω and $\mathcal{B} \subseteq \mathcal{U}$. We say that $\mathcal{B}$ is a $\subseteq^\star$-basis for $\mathcal{U}$ if for every $A \in \mathcal{U}$, there exists $B \in \mathcal{B}$ such that $B \subseteq^\star A$.*

Lemma 3.4. *Suppose $\mathcal{U}$ is an ultrafilter on ω and κ is the least cardinality of a basis for $\mathcal{U}$. Let $\mathcal{B}$ be a $\subseteq^\star$-basis for $\mathcal{U}$ of size κ. Then there exists $F : \mathcal{B} \to \omega$ such that $\{B \setminus F(B) : B \in \mathcal{B}\}$ is a basis for $\mathcal{U}$.*

Proof. Note that $\omega_1 \leq \kappa \leq \mathfrak{c}$. For each $A \in \mathcal{U}$, define $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$. Then $\mathcal{B}_A$ is also a $\subseteq^\star$-basis for $\mathcal{U}$ and hence $|\mathcal{B}_A| = \kappa$. Fix an enumeration $\{B_\alpha : \alpha < \kappa\}$ of $\mathcal{B}$ and recursively choose $\langle S_\alpha : \alpha < \kappa\rangle$ such that the following hold.

(i) Each $S_\alpha \in [\kappa]^{\aleph_0}$.
(ii) If $\beta < \alpha$, then $S_\beta \cap S_\alpha = \emptyset$.
(iii) For every $\gamma \in S_\alpha$, $B_\gamma \in \mathcal{B}_{A_\alpha}$ (so $B_\gamma \subseteq^\star A_\alpha$).

At any stage $\alpha < \kappa$, such an S_α can be chosen using the fact that $|\mathcal{B}_{A_\alpha}| = \kappa \geq \omega_1$. For each $\alpha < \kappa$, fix an enumeration $S_\alpha = \{\alpha(n) : n < \omega\}$ and define $F : \mathcal{B} \to \omega$ as follows. If $B = B_{\alpha(n)}$, then $F(B)$ is defined to be the least $N < \omega$ such that $B_{\alpha(n)} \setminus N \subseteq A_\alpha \setminus n$. Otherwise define $F(B) = 0$. It is easy to check that F is as required. □

Lemma 3.5. *Let $\mathcal{U}$ be an ultrafilter on ω and $n \geq 1$. Suppose there is an n-Turing independent (resp. Turing independent) $\subseteq^\star$-basis for $\mathcal{U}$. Then there exists an n-Turing independent (resp. Turing independent) basis for $\mathcal{U}$.*

Proof. Let $\mathcal{B}_1$ be an n-Turing independent (resp. Turing independent) $\subseteq^\star$-basis for $\mathcal{U}$. Let κ be the smallest cardinality of a $\subseteq^\star$-basis for $\mathcal{U}$ and $\mathcal{B}$ be a $\subseteq^\star$-basis for $\mathcal{U}$ of size κ. Choose $\mathcal{B}_2 \subseteq \mathcal{B}_1$ such that $|\mathcal{B}_2| = \kappa$ and for every $A \in \mathcal{B}$, there exists $B \in \mathcal{B}_2$ such that $B \subseteq^\star A$. Apply Lemma 3.4 to obtain $F : \mathcal{B}_2 \to \omega$ for $\mathcal{B}_2$ as there. Then $\{B \setminus F(B) : B \in \mathcal{B}_2\}$ is an n-Turing independent (resp. Turing independent) basis for $\mathcal{U}$. □

Theorem 3.6. *Let $\mathcal{U}$ be an ultrafilter on ω that contains a $\mathcal{D}$-generic Mathias real X. Then $\mathcal{U}$ has a Turing antichain basis.*

Proof. Let κ be the least cardinality of a basis for $\mathcal{U}$. Then $\omega_1 \leq \kappa \leq \mathfrak{c}$. Fix a basis $\mathcal{B}$ of size κ for $\mathcal{U}$. By replacing each $B \in \mathcal{B}$ with $B \cap X$, we can assume that every member of $\mathcal{B}$ is a subset of X. Fix an injective enumeration $\langle A_\alpha : \alpha < \kappa\rangle$ of $\mathcal{B}$ and define $h : \kappa \to \kappa$ recursively as follows.

(1) $h(0) = 0$.
(2) Suppose $h(\beta)$ has been defined for each $\beta < \alpha$. We define $h(\alpha)$ casewise as follows.

Case 1: There exists some $\beta < \alpha$ such that $A_{h(\beta)} \subseteq^\star A_\alpha$. In this case, we define $h(\alpha)$ to be the least such β.

Case 2: There is no $\beta < \alpha$ such that $A_{h(\beta)} \subseteq^\star A_\alpha$. In this case, we define $h(\alpha)$ to be the least $\gamma < \kappa$ satisfying the following.

(a) $A_\gamma \subseteq^\star A_\alpha$ and

(b) $A_\gamma \not\leq_T A_{h(\beta)}$ for every $\beta < \alpha$.

In Case (2), $h(\alpha)$ is well defined because $|\{\gamma < \kappa : A_\gamma \subseteq A_\alpha\}| = \kappa$ and $|\{\gamma < \kappa : (\exists \beta < \alpha)(A_\gamma \leq_T A_{h(\beta)})\}| \leq \max(|\gamma|, \omega) < \kappa$.

Claim 3.7. *Put $\mathcal{B}' = \{A_{h(\alpha)} : \alpha < \kappa\}$.*

(1) $\mathcal{B}'$ is a $\subseteq^\star$-basis for $\mathcal{U}$.

(2) $\mathcal{B}'$ is a Turing antichain.

Proof. (1) This follows from the fact that $\{A_\alpha : \alpha < \kappa\}$ is a basis for $\mathcal{U}$ and $A_{h(\alpha)} \subseteq^\star A_\alpha$ for every $\alpha < \kappa$.

(2) Put $\mathcal{W}_\alpha = \{A_{h(\beta)} : \beta < \alpha\}$. It suffices to show that for every $\alpha < \kappa$, either $A_{h(\alpha)} \in \mathcal{W}_\alpha$ or $A_{h(\alpha)}$ is Turing incomparable with every member of $\mathcal{W}_\alpha$. So assume $A_{h(\alpha)} \notin \mathcal{W}_\alpha$ and towards a contradiction fix $\beta < \alpha$ such that $A_{h(\alpha)}$ and $A_{h(\beta)}$ are Turing comparable. This means that $h(\alpha) = \gamma$ was chosen according to Case 2. So $A_\gamma \not\leq_T A_{h(\beta)}$. So we must have $A_{h(\beta)} \leq_T A_\gamma$. Since $A_{h(\beta)}, A_\gamma$ are both subsets of the Mathias generic X, Fact 3.2 implies that $A_{h(\beta)} \subseteq^\star A_\gamma$. As $A_\gamma \subseteq^\star A_\alpha$, we get $A_{h(\beta)} \subseteq^\star A_\alpha$. But this means that we are in Case 1. A contradiction. □

The theorem now follows from Lemma 3.5. □

Since every $X \in [\omega]^\omega$ contains a $\mathcal{D}$-generic Mathias real, Theorem 1.5 readily follows.

4. n-Turing independent $(n+1)$-basis

In this section, we are going to prove Theorem 1.6. Fix $n \geq 1$. Suppose $\mathcal{U}$ is an ultrafilter on ω and $\mathcal{B} \subseteq \mathcal{U}$ is a basis for $\mathcal{U}$. Assume $|\mathcal{B}| = \kappa$ and enumerate $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, we will choose $\{B_{\alpha,i} : i \leq n\} \subseteq \mathcal{U}$ such that $\bigcap_{i \leq n} B_{\alpha,i} \subseteq B_\alpha$. This would imply that $\{B_{\alpha,i} : \alpha < \kappa, i \leq n\}$ is an $(n+1)$-basis for $\mathcal{U}$. Now for Theorem 1.6, we will also like to ensure that $\{B_{\alpha,i} : \alpha < \kappa, i \leq n\}$ is n-Turing independent. One may try to do this by choosing $B_{\alpha,i}$'s by induction on $\alpha < \kappa$. But this strategy fails. For example, it may happen that $\omega_1 < \kappa$ and at the end of stage ω_1, the family $\{B_{\alpha,i} : i \leq n, \alpha < \omega_1\}$ is already a maximal n-Turing independent set (This situation is consistent by [4]). We avoid this difficulty by choosing everything simultaneously and quite independently. For this purpose, we make use of the following notion of e-splitting.

Definition 4.1. *Let $e, m, r, n_1, n_2 < \omega$, $2 \leq r \leq m$, $n_1 < n_2$ and $\bar{a} = \langle a_k : k < m \rangle$ where each $a_k \subseteq [n_1, n_2)$. Let $\bar{F} = \langle F_k : k < m \rangle$ where each $F_k \subseteq [0, n_1)$.*

(1) *Let $\bar{j} = \langle j_k : k < r \rangle$ be an injective sequence where each $j_k < m$. We say that $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a}, \bar{j})$ holds if (a) implies (b) below.*

(a) There exists $\langle X_k : 1 \leq k < r \rangle$ such that for each k, $X_k \cap [0, n_2) = F_{j_k} \cup a_{j_k}$ and letting $X = \bigoplus_{1 \leq k < r} X_k$, we have $(\exists^\infty \ell)(\Phi_e^X(\ell) \downarrow \neq 0)$.

(b) There exist $\ell, s < n_2$ such that letting $W = \bigoplus_{1 \leq k < r} F_{j_k} \cup a_{j_k}$, we have $\Phi_e^W(\ell)[s] \downarrow \neq (F_{j_0} \cup a_{j_0})(\ell)$.

(2) *We say that* $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a})$ *holds if* $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a}, \bar{j})$ *holds for every injective* $\bar{j} = \langle j_k : k < r\rangle$ *where each* $j_k < m$.

(3) *We say that* $\mathsf{Split}_e(n_1, n_2, r, \bar{a})$ *holds if* $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a})$ *holds for every* $\bar{F} = \langle F_k : k < m\rangle$ *where each* $F_k \subseteq [0, n_1)$.

Lemma 4.2. *Suppose* $e, m, r, n_1, n_2 < \omega$, $2 \leq r \leq m$, $n_1 < n_2$, $\bar{a} = \langle a_k : k < m\rangle$ *where each* $a_k \subseteq [n_1, n_2)$ *and* $\bar{F} = \langle F_k : k < m\rangle$ *where each* $F_k \subseteq [0, n_1)$. *Let* $\bar{j} = \langle j_k : k < r\rangle$ *be an injective sequence where each* $j_k < m$. *Then there exist* $n_3 \geq n_2$ *and* $\bar{b} = \langle b_k : 1 \leq k < r\rangle$ *where each* $b_k \subseteq [n_2, n_3)$ *such that* $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{c}, \bar{j})$ *holds where* $\bar{c} = \langle c_k : k < m\rangle$, $c_{j_k} = a_{j_k} \cup b_k$ *for* $1 \leq k < r$ *and* $c_k = a_k$ *if either* $k = 0$ *or* $k \notin \mathsf{range}(\bar{j})$.

Proof. We ask the following: Does there exist $\langle X_k : 1 \leq k < r\rangle$ such that for every k, $X_k \cap [0, n_2) = F_{j_k} \cup a_{j_k}$ and letting $X = \bigoplus_{1\leq k<r} X_k$, we have $(\exists^\infty \ell)(\Phi_e^X(\ell) \downarrow\neq 0)$?

If the answer is yes, we fix such $\langle X_k : 1 \leq k < r\rangle$ and $\ell > n_2$ such that $\Phi_e^X(\ell) \downarrow\neq 0$ where $X = \bigoplus_{1\leq k<r} X_k$. Choose $s > \ell$ such that $\Phi_e^X(\ell)[s] \downarrow$. Define $n_3 = s$ and $b_k = X_k \cap [n_2, n_3)$ for each $1 \leq k < r$. If the answer is no, then define $n_3 = n_2$ and $b_k = \emptyset$ for every $1 \leq k < r$. It should be clear that, in either case, $n_3, \bar{b}$ are as required. $\square$

Lemma 4.3. *Let* $e, m, r, n_1, n_2 < \omega$, $2 \leq r \leq m$, $n_1 < n_2$ *and* $\bar{a} = \langle a_k : k < m\rangle$ *where each* $a_k \subseteq [n_1, n_2)$. *Let* $n_3 \geq n_2$ *and* $b_k \subseteq [n_2, n_3)$ *for each* $k < m$. *Put* $c_k = a_k \cup b_k$ *and* $\bar{c} = \langle c_k : k < m\rangle$. *Let* $\bar{F} = \langle F_k : k < m\rangle$ *where each* $F_k \subseteq [0, n_1)$.

(1) *Let* $\bar{j} = \langle j_k : k < r\rangle$ *be an injective sequence where each* $j_k < m$. *Assume* $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a}, \bar{j})$ *holds. Then* $\mathsf{Split}_e(n_1, n_3, r, \bar{F}, \bar{c}, \bar{j})$ *also holds.*

(2) *If* $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a})$ *holds, then* $\mathsf{Split}_e(n_1, n_3, r, \bar{F}, \bar{c})$ *holds.*

(3) *If* $\mathsf{Split}_e(n_1, n_2, r, \bar{a})$ *holds, then* $\mathsf{Split}_e(n_1, n_3, r, \bar{c})$ *holds.*

Proof. (1) Suppose there are $\langle X_k : 1 \leq k < r\rangle$ such that for every k, $X_k \cap [0, n_3) = F_{j_k} \cup c_{j_k}$ and letting $X = \bigoplus_{1\leq k<r} X_k$, we have $(\exists^\infty \ell)(\Phi_e^X(\ell) \downarrow\neq 0)$. Fix such X_k's and note that $X_k \cap [0, n_2) = F_{j_k} \cup a_{j_k}$. As $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a}, \bar{j})$ holds, it follows that we can find $\ell, s < n_2$ such that letting $W = \bigoplus_{1\leq k<r} F_{j_k} \cup a_{j_k}$, we have $\Phi_e^W(\ell)[s] \downarrow\neq (F_{j_0} \cup a_{j_0})(\ell)$. Since $\ell, s < n_2 \leq n_3$, it follows that $\mathsf{Split}_e(n_1, n_3, r, \bar{F}, \bar{c}, \bar{j})$ holds.

Clauses (2) and (3) follow from Clause (1) and Definition 4.1. $\square$

Lemma 4.4. *Let* $r, m, n_1 < \omega$ *where* $2 \leq r \leq m$. *There exist* $n_2 > n_1$ *and* $\bar{a} = \langle a_k : k < m\rangle$ *such that the following hold.*

(a) *Each* $a_k \subseteq [n_1, n_2)$ *and* a_k*'s are pairwise distinct and nonempty.*

(b) *For every* $e \leq n_1$, $\mathsf{Split}_e(n_1, n_2, r, \bar{a})$.

(c) *If* $F \subseteq m$ *and* $|F| \geq r$, *then* $\bigcap_{k\in F} a_k = \emptyset$.

Proof. Let $\mathcal{R}$ be the set of all triplets $(\bar{F}, \bar{j}, e)$ where $\bar{j} = \langle j_k : k < r\rangle$ is an injective sequence with each $j_k < m$, $e \leq n_1$ and $\bar{F} = \langle F_i : i < m\rangle$ is a sequence of subsets of $[0, n_1)$. Put $|\mathcal{R}| = N$ and note that $N < \omega$. Fix an enumeration $\langle (\bar{F}_t, \bar{j}_t, e_t) : t < N\rangle$ of $\mathcal{R}$. Choose $\langle (\bar{a}_t, \ell_t) : t \leq N\rangle$ satisfying the following.

(i) $\ell_0 < \ell_1 < \cdots < \ell_t < \ell_{t+1} < \cdots < \ell_N$.

(ii) $\bar{a}_t = \langle a_{t,k} : k < m\rangle$ where each $a_{t,k} \subseteq [0, \ell_t)$.

(iii) $\ell_0 = n_1 + m$ and $a_{0,k}$'s are pairwise distinct subsets of $[n_1, \ell_0)$.

(iv) $(\forall k < m)(a_{t+1,k} \cap [0, \ell_t) = a_{t,k})$ and if $k = j_0$ or $k \notin \mathsf{range}(\bar{j})$, then $a_{t+1,k} = a_{t,k}$.

(v) For every $t < N$, $\mathsf{Split}_{e_t}(n_1, \ell_{t+1}, r, \bar{F}_t, \bar{a}_{t+1}, \bar{j}_t)$ holds.

Given $\ell_t, \bar{a}_t$, we use Lemma 4.2 to get $\ell_{t+1}, \bar{a}_{t+1}$. After completing the construction, define $n_2 = \ell_N$ and $a_k = a_{N,k}$ for each $k < m$. By Lemma 4.3(1), it follows that for every $(\bar{F}, \bar{j}, e) \in \mathcal{R}$, $\mathsf{Split}_e(n_1, n_2, r, \bar{F}, \bar{a}, \bar{j})$ holds. Hence, $\mathsf{Split}_e(n_1, n_2, r, \bar{a})$ holds. Item (iv) ensures that if $F \subseteq m$ and $|F| \geq r$, then $\bigcap_{k\in F} a_k = \emptyset$. So Clause (c) holds. Finally, Clause (a) holds because $a_{0,k}$'s are nonempty and pairwise distinct. $\square$

Lemma 4.5. *Assume $2 \leq r_i \leq m_i < \omega$ for $i < \omega$. There exists $\langle (n_i, \bar{a}_i) : i < \omega) \rangle$ such that the following hold.*

(1) $0 = n_0 < n_1 < \cdots < n_i < n_{i+1} < \cdots < \omega$.
(2) For each $i < \omega$, $\bar{a}_i = \langle a_{i,k} : k < m_i \rangle$ consists of pairwise distinct nonempty subsets of $[n_i, n_{i+1})$ such that for every $F \subseteq m_i$, if $|F| \geq r_i$, then $\bigcap_{k\in F} a_{i,k} = \emptyset$.
(3) If $e \leq i$, then $\mathsf{Split}_e(n_i, n_{i+1}, r_i, \bar{a}_i)$ holds.

Proof. Construct $(n_i, \bar{a}_i)$ by induction on $i < \omega$, using Lemma 4.4. $\square$

Proof of Theorem 1.6. Let $\mathcal{U}$ be an ultrafilter on ω and $r \geq 2$. For each $i < \omega$, let $r_i = r$ and fix $m_i \geq 2^i + r$. Let $\langle (n_i, \bar{a}_i) : i < \omega) \rangle$ be as in Lemma 4.5 for this $\langle r_i, m_i : i < \omega \rangle$.

Claim 4.6. *There exists a family $\mathcal{T} \subseteq \omega^\omega$ such that*

(1) $|\mathcal{T}| = \mathfrak{c}$,
(2) $(y \in \mathcal{T} \wedge i < \omega) \implies (y(i) < m_i)$ *and*
(3) for every $y \neq y'$ in $\mathcal{T}$, y and y' are eventually different: $(\forall^\infty i)(y(i) \neq y'(i))$.

Proof. For each $i < \omega$, fix a bijection $h_i : {}^i2 \to 2^i$. For each $x \in {}^\omega 2$, define $y_x : \omega \to \omega$ by $y_x(i) = h_i(x \restriction i)$. Then $\mathcal{T} = \{y_x : x \in {}^\omega 2\}$ is as required. $\square$

Since $\mathcal{U}$ is an ultrafilter, exactly one of the sets $N_0 = \bigcup_{i<\omega}[n_{2i}, n_{2i+1})$, $N_1 = \bigcup_{i<\omega}[n_{2i+1}, n_{2i+2})$ lies in $\mathcal{U}$. Without loss of generality, assume $N_1 \in \mathcal{U}$. For each $y \in \mathcal{T}$ and $S \subseteq N_1$, define

$$A_{y,S} = S \cup \bigcup_{i<\omega} a_{2i,y(i)}.$$

Note that since $a_{i,k}$'s all are nonempty, each $A_{y,S}$ must be infinite.

Claim 4.7. *Suppose $\langle y_k : k < r \rangle$ consists of pairwise distinct members of $\mathcal{T}$ and $S_k \subseteq N_1$ for each $k < r$. Then A_{y_0,S_0} is not computable from the Turing join of $\langle A_{y_k,S_k} : 1 \leq k < r \rangle$.*

Proof. Let $A = \bigoplus_{1\leq k<r} A_{y_k,S_k}$. Towards a contradiction, fix $e < \omega$ such that $\Phi_e^A = A_{y_0,S_0}$. Since y_k's are eventually different, we can fix $i_\star > e$ such that $\langle y_k(i_\star) : k < r \rangle$ are pairwise distinct. Define $\bar{j}, \bar{F}, \bar{a}$ as follows.

(i) $\bar{j} = \langle j_k : k < r \rangle$ where $j_k = y_k(i_\star) < 2^{i_\star} < m_{2i_\star}$.
(ii) $\bar{F} = \langle F_\ell : \ell < m_{2i_\star} \rangle$ where
  (a) $F_{j_k} = A_{y_k,S_k} \cap [0, n_{2i_\star})$ for $k < r$ and
  (b) $F_\ell = \emptyset$ if $\ell \notin \mathsf{range}(j)$.
(iii) $\bar{a} = \langle a_\ell : \ell < m_{2i_\star} \rangle$ where $a_\ell = a_{2i_\star,\ell}$. Note that for every $k < r$, $a_{j_k} = a_{2i_\star,y_k(i_\star)} = A_{y_k,S_k} \cap [n_{2i_\star}, n_{2i_\star+1})$.

Since $e < i_\star < 2i_\star$, $\mathsf{Split}_e(n_{2i_\star}, n_{2i_\star+1}, r, \bar{a})$ holds by Lemma 4.5(3). It follows that $\mathsf{Split}_e(n_{2i_\star}, n_{2i_\star+1}, r, \bar{F}, \bar{a}, \bar{j})$ holds. Define $X_k = A_{y_k,S_k}$ for $1 \le k < r$. As $\Phi_e^A = A_{y_0,S_0}$ is an infinite subset of ω, Clause (a) of Definition 4.1(1) holds for $\langle X_k : 1 \le k < r\rangle$. Hence Clause (b) of Definition 4.1(1) must also hold which implies that for some $\ell < n_{2i_\star+1}$, $\Phi_e^A(\ell) \neq (F_{j_0} \cup a_{j_0})(\ell) = A_{y_0,S_0}(\ell)$. A contradiction. $\square$

Claim 4.8. *Suppose $\langle y_k : k < r\rangle$ consists of pairwise distinct members of $\mathcal{T}$ and $S \subseteq N_1$. Then $\bigcap_{k<r} A_{y_k,S} \subseteq^\star S$.*

Proof. Put $A = \bigcap_{k<r} A_{y_k,S}$. Since y_k's are pairwise eventually different, we can fix $i_\star < \omega$ such that for every $k < k' < r$ and $i \ge i_\star$, $y_k(i) \neq y_{k'}(i)$. By Lemma 4.5(2), for every $i \ge i_\star$,

$$A \cap [n_{2i}, n_{2i+1}) = \bigcap_{k<r} A_{y_k,S} \cap [n_{2i}, n_{2i+1}) = \bigcap_{k<r} a_{2i,y_k(i)} = \emptyset.$$

It follows that $A \subseteq S \cup [0, n_{2i_\star}) \subseteq^\star S$. $\square$

Let $\mathcal{B}$ be a basis for $\mathcal{U}$. Put $\kappa = |\mathcal{B}| \le \mathfrak{c}$ and fix an injective enumeration $\langle B_\alpha : \alpha < \kappa\rangle$ of $\mathcal{B}$.

Claim 4.9. *For each $n \ge 1$, there exists $\langle B_{\alpha,k} : \alpha < \kappa$ and $k \le n\rangle$ such that the following hold.*

(1) Each $B_{\alpha,k} \in \mathcal{U}$.
(2) $\bigcap_{k\le n} B_{\alpha,k} \subseteq B_\alpha$.
(3) The family $\{B_{\alpha,k} : \alpha < \kappa$ and $k \le n\}$ is n-Turing independent.

Proof. Put $r = n+1$ and fix $\langle (n_i, \bar{a}_i, m_i) : i < \omega\rangle$, $N_0, N_1, \mathcal{T}$ for this r as described above. Put $S_\alpha = B_\alpha \cap N_1$. Since $N_1 = \bigcup_{i<\omega}[n_{2i+1}, n_{2i+2}) \in \mathcal{U}$, each $S_\alpha \in \mathcal{U}$. Fix pairwise distinct $\{y(\alpha,k) : \alpha < \kappa$ and $k < r\} \subseteq \mathcal{T}$ and define $B'_{\alpha,k} = A_{y(\alpha,k),S_\alpha}$. By Claim 4.7, $\{B'_{\alpha,k} : \alpha < \kappa$ and $k < r\}$ is n-Turing independent and by Claim 4.8, $\bigcap\{B'_{\alpha,k} : k < r\} \subseteq^\star S_\alpha$. For each $\alpha < \kappa$, fix $N_\alpha < \omega$ such that $\bigcap\{B'_{\alpha,k} : k < r\} \subseteq S_\alpha \cup N_\alpha$ and define $B_{\alpha,k} = B'_{\alpha,k} \setminus N_\alpha$. Then $\bigcap\{B_{\alpha,k} : k < r\} \subseteq S_\alpha$ and since $B_{\alpha,k}$ differs from $B'_{\alpha,k}$ on a finite set, $\{B_{\alpha,k} : \alpha < \kappa$ and $k < r\}$ remains n-Turing independent. $\square$

It follows that $\{B_{\alpha,k} : \alpha < \kappa$ and $k \le n\}$ is an n-Turing independent $(n+1)$-basis for $\mathcal{U}$ and the proof of Theorem 1.6 is complete. $\square$

5. Turing independent basis

We are now going to prove Theorem 1.7.

Definition 5.1. *Let $\theta < \kappa$ be infinite cardinals and $h : [\kappa]^{<\omega} \to [\kappa]^{\le\theta}$.*

(1) We say that $X \subseteq \kappa$ is free for h iff for every $s \in [X]^{<\omega}$ and $\alpha \in X \setminus s$, $\alpha \notin h(s)$.
(2) We say that $\star(\kappa, \theta)$ holds iff for every $h : [\kappa]^{<\omega} \to [\kappa]^{\le\theta}$, there exists $X \in [\kappa]^\kappa$ such that X is free for h.

Recall (see [5]) that κ is a Ramsey cardinal iff for every $\gamma < \kappa$ and $c : [\kappa]^{<\omega} \to \gamma$, there exists $H \in [\kappa]^\kappa$ such that for every $n < \omega$, $c \restriction [H]^n$ is constant.

Lemma 5.2. *Suppose κ is a Ramsey cardinal and $\mathbb{P}$ is a forcing. If $\omega \leq \theta < \kappa$ and $\mathbb{P}$ satisfies θ^+-cc, then $V^{\mathbb{P}} \models \star(\kappa, \theta)$. Hence if $\mathbb{P}$ satisfies ccc, then $V^{\mathbb{P}} \models (\forall \theta < \kappa)(\star(\kappa, \theta))$.*

Proof. Assume $p \in \mathbb{P}$, $\mathring{h} \in V^{\mathbb{P}}$ and $p \Vdash \mathring{h} : [\kappa]^{<\omega} \to [\kappa]^{\leq\theta}$. Since $\mathbb{P}$ satisfies θ^+-cc, we can find $g \in V$ such that $g : [\kappa]^{<\omega} \to [\kappa]^{\leq\theta}$ and for every $s \in [\kappa]^{<\omega}$, $p \Vdash \mathring{f}(s) \subseteq g(s)$. Define $c : [\kappa]^{<\omega} \to {}^{<\omega}2$ as follows. If $|s| \leq 1$, then $c(s) = 0$. Otherwise, list $s = \{\alpha_0 < \alpha_1 < \cdots < \alpha_n\}$ in increasing order and define $c(s) = \langle i_k : k \leq n \rangle$ where $(\forall k \leq n)(i_k = 1 \iff \alpha_k \in g(s \setminus \{\alpha_k\}))$. As κ is Ramsey, we can find $X \in [\kappa]^\kappa$ such that for every $n < \omega$, $c \restriction [H]^n$ is constant. We claim that X is free for g and hence also for f. Towards a contradiction, assume this fails and fix $s = \{\beta_0 < \beta_1 < \cdots < \beta_n\} \subseteq X$ and $\ell \leq n$ such that $\beta_\ell \in g(s \setminus \{\beta_\ell\})$. This means that the ℓ^{th} entry of $c(s)$ is 1. Choose $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ in X such that each one of the sets $X \cap \alpha_0$ and $X \cap (\alpha_k, \alpha_{k+1})$ for $k < n$ has cardinality $> \theta$. This is possible because $|X| = \kappa$ is Ramsey and hence inaccessible (see [5]). Put $s' = \{\alpha_k : k \leq n\}$. Now by our choice of α_k's, and the fact that $c \restriction [X]^{n+1}$ is constant, we get that $g(s' \setminus \{\alpha_\ell\})$ must have size $> \theta$. A contradiction. ☐

Lemma 5.3. *Let $X \subseteq 2^\omega$ and $\star(|X|, \aleph_0)$ holds. Then there exists $Y \subseteq X$ such that $|Y| = |X|$ and Y is Turing independent.*

Proof. Define $h : [X]^{<\omega} \to [X]^{\leq\aleph_0}$ by $h(a)$ is the set of all $y \in X$ that are computable from the join of s. By $\star(|X|, \aleph_0)$, there exists $Y \subseteq X$ such that $|Y| = |X|$ and for every $s \in [Y]^{<\omega}$ and $y \in Y \setminus s$, $y \notin h(s)$. So Y is Turing independent. ☐

For an ultrafilter $\mathcal{U}$ on ω, $\mathbb{M}(\mathcal{U})$ denotes Mathias forcing along $\mathcal{U}$ defined as follows. $p \in \mathbb{M}(\mathcal{U})$ iff $p = (s_p, X_p)$ where $s_p \in [\omega]^{<\omega}$, $X_p \in \mathcal{U}$ and $\max(s_p) < \min(X_p)$. For $p, q \in \mathbb{M}(\mathcal{U})$, $p \leq q$ iff $s_q \subseteq s_p$, $X_p \subseteq X_q$ and $s_p \setminus s_q \subseteq X_q$. Note that $\mathbb{M}(\mathcal{U})$ is σ-centered and hence ccc since if $(s, X_1), (s, X_2) \in \mathbb{M}(\mathcal{U})$, then $(s, X_1 \cap X_2)$ is a common extension. If G is an $\mathbb{M}(\mathcal{U})$-generic filter over the ground model V, then $X = \bigcup_{p \in G} s_p$ is a Mathias generic real added by $\mathbb{M}(\mathcal{U})$. Standard genericity arguments show that $X \in [\omega]^\omega$, $X \notin V$ and $(\forall Y \in \mathcal{U})(X \subseteq^\star Y)$.

Theorem 5.4. *Let κ be a Ramsey cardinal. Then there is a ccc forcing $\mathbb{P}$ of size κ such that in $V^{\mathbb{P}}$, there is an ultrafilter on ω that has a Turing independent basis.*

Proof. Let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha, \mathring{X}_\alpha, \mathring{\mathcal{U}}_\alpha : \alpha < \kappa \rangle$ be defined as follows

(i) $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ is a finite support iteration of ccc forcings with limit $\mathbb{P}$.
(ii) $\mathbb{P}_0$ is the trivial forcing and $\mathcal{U}_0$ is an ultrafilter on ω.
(iii) $V^{\mathbb{P}_\alpha} \models \mathring{\mathcal{U}}_\alpha$ is an ultrafilter on ω that satisfies $(\forall \beta < \alpha)(\mathring{X}_\beta \in \mathring{\mathcal{U}}_\alpha)$.
(iv) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \mathring{\mathbb{Q}}_\alpha$ where $V^{\mathbb{P}_\alpha} \models \mathring{\mathbb{Q}}_\alpha = \mathbb{M}(\mathring{\mathcal{U}}_\alpha)$ and $\mathring{X}_\alpha \in V^{\mathbb{P}_{\alpha+1}}$ is the Mathias generic real added by $\mathring{\mathbb{Q}}_\alpha$.

Let G be $\mathbb{P}$-generic over V. Since $\mathbb{P}$ is a ccc forcing of size $\kappa = \kappa^{<\kappa}$, a standard name counting argument shows that $V[G] \models \mathfrak{c} \leq \kappa$. Put $X_\alpha = \mathring{X}_\alpha[G]$ and $\mathcal{U}_\alpha = \mathring{\mathcal{U}}_\alpha[G]$. Since each X_α is a pseudointersection of $\mathcal{U}_\alpha$ and $\{X_\beta : \beta < \alpha\} \subseteq \mathcal{U}_\alpha$, it follows that $\langle X_\alpha : \alpha < \kappa \rangle$ is a strictly $\subseteq^\star$-decreasing sequence of members of $[\omega]^\omega$. Hence $V[G] \models \mathfrak{c} = \kappa$.

Claim 5.5. *Put $\mathcal{U} = \{Y \subseteq \omega : (\exists \alpha < \kappa)(X_\alpha \subseteq^\star Y)\}$. Then $\mathcal{U}$ is an ultrafilter on ω that has a Turing independent basis.*

Proof. $\mathcal{U}$ is clearly a filter on ω. Let $Y \in \mathcal{P}(\omega) \cap V[G]$. Since $\mathsf{cf}(\kappa) = \kappa > \omega$, we can find $\alpha < \kappa$ and $\mathring{Y} \in V^{\mathbb{P}_\alpha}$ such that $Y = \mathring{Y}[G]$. Now either Y or $\omega \setminus Y$ is in $\mathcal{U}_\alpha$. Hence either $X_\alpha \subseteq^\star Y$ or $X_\alpha \subseteq^\star \omega \setminus Y$. So $\mathcal{U}$ is an ultrafilter. Put $\mathcal{B} = \{X_\alpha : \alpha < \kappa\}$. As $\mathbb{P}$ satisfies ccc, by Lemma 5.2, $V[G] \models \star(\kappa, \aleph_0)$. By Lemma 5.3, we can choose $\mathcal{B}' \subseteq \mathcal{B}$ such that $|\mathcal{B}'| = \kappa$ and $\mathcal{B}'$ is Turing independent. Since $|\mathcal{B}'| = \kappa$, $\{\alpha < \kappa : X_\alpha \in \mathcal{B}\}$ is cofinal in κ. It follows that $\mathcal{B}'$ is a Turing independent $\subseteq^\star$ basis for $\mathcal{U}$. So by Lemma 4.5, $\mathcal{U}$ has a Turing independent basis. □

Another model: Start with a Ramsey cardinal κ and force MA (Martin's axiom) plus $\mathfrak{c} = \kappa$ using the standard ccc forcing, call it $\mathbb{Q}$, for MA (see [9]). In $V^{\mathbb{Q}}$, using $\mathfrak{p} = \mathfrak{c} = \kappa$, construct a strictly $\subseteq^\star$-decreasing sequence $\langle X_\alpha : \alpha < \mathfrak{c}\rangle$ in $[\omega]^\omega$ such that for every $Y \subseteq \omega$, there exists $\alpha < \mathfrak{c}$ such that either $X_\alpha \subseteq^\star Y$ or $X_\alpha \subseteq^\star \omega \setminus Y$. Let $\mathcal{U}$ be the ultrafilter with $\mathcal{B} = \{X_\alpha : \alpha < \mathfrak{c}\}$ as a $\subseteq^\star$-basis. Now use Lemmas 5.2 and 5.3 to obtain a Turing independent subbasis for $\mathcal{U}$ as before and apply Lemma 3.5. □

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