

FAILURE OF SINGULAR COMPACTNESS FOR Hom

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ABSTRACT. Assuming Gödel's axiom of constructibility $\mathbf{V} = \mathbf{L}$, we construct an almost-free abelian group G of singular cardinality, such that for any nontrivial subgroup $G' \subseteq G$ of smaller size, we have $\text{Hom}(G', \mathbb{Z}) \neq 0$, while $\text{Hom}(G, \mathbb{Z}) = 0$. This provides a consistent counterexample to the singular compactness of Hom .

§ 0. INTRODUCTION

Hill [5] proved that if an abelian group G has a singular cardinality with cofinality at most ω_1 and every subgroup of smaller cardinality is free, then G is free. This result serves as a cornerstone for the *Singular Compactness* Theorem by Shelah [8], where he introduced an abstract notion of freeness and get ride of the cofinality restriction. Shelah extended this result by proving that if an abelian group has a singular cardinality with cofinality κ , and every subgroup of smaller cardinality is free, then the group itself must also be free. For more details on singular compactness, see [2, 3], and for its applications, we refer to the book [4].

Compactness (and its counterpart, incompactness) is a central theme in contemporary research. This concept broadly asserts that if every smaller subobject of a given object possesses a particular property denoted by Pr , then the object itself must also exhibit Pr . In this paper, we are interested in the compactness property for the nontrivial duality with respect to the hom-functor $\text{Hom}(-, \mathbb{Z})$ at singular cardinals. Namely, we study the following property:

Pr_λ : If G is a group of size λ , and if for any nontrivial subgroup $G' \subseteq G$ of size less than λ , $\text{Hom}(G', \mathbb{Z}) \neq 0$, then $\text{Hom}(G, \mathbb{Z}) \neq 0$.

For a given $\mu \leq \lambda$, recall that $S_\mu^\lambda = \{\alpha < \lambda \mid \text{cf}(\alpha) = \mu\}$ is a stationary subset of λ . For any stationary set $S \subseteq \lambda$, let $\diamond_S$ denote Jensen's diamond (see Definition 1.3). Now, assuming $\lambda > \aleph_0$ is a regular cardinal and $\diamond_S$ holds for some stationary, non-reflecting set $S \subseteq S_{\aleph_0}^\lambda$, one can construct a λ -free abelian group G of size λ such

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that $\text{Hom}(G, \mathbb{Z}) = 0$ (see [2]). Note that any subgroup G' of G with size less than λ is free, implying that $\text{Hom}(G', \mathbb{Z}) \neq 0$. Thus, Pr_λ fails for such λ . However, this argument does not extend to singular cardinals.

In this paper, we investigate the consistency of the failure of Pr_λ for some singular cardinal λ and show that this can occur in Gödel's constructible universe $\mathbf{L}$. The main result of this paper is as follows:

Theorem 0.1. *Suppose that:*

- (a) $\langle \lambda_i : i < \kappa \rangle$ is an increasing sequence of regular cardinals with limit λ ,
- (b) $\aleph_0 < \kappa = \text{cf}(\kappa) < \lambda_0$,
- (c) $\lambda_i = \text{cf}(\lambda_i)$, $S_i \subseteq S_{\text{cf}(\mu)}^{\lambda_i}$ is stationary and non-reflecting,
- (d) $\diamond_{S_i}$ holds,
- (e) there is no measurable cardinal $\leq \lambda$.

Then there is a λ_0 -free abelian group G of cardinality λ which is counterexample to singular compactness in λ for $\text{Hom}(-, \mathbb{Z}) \neq 0$.

Our work is closely related to the Whitehead property $\text{Ext}_{\mathbb{Z}}(G, \mathbb{Z}) = 0$, which is arguably more significant but also inherently more complex. In our forthcoming work [1], we investigate singular compactness in the context of Ext . Note that if Gödel's axiom of constructibility $\mathbf{V} = \mathbf{L}$ assumed, this has an easy solution. By Shelah's work [7], for $\lambda > \aleph_0$ and an abelian group G of size λ , the group G is free if and only if $\text{Ext}_{\mathbb{Z}}(G, \mathbb{Z}) = 0$. Hence, by Shelah's singular compactness theorem for free groups [8], singular compactness holds for the property $\text{Ext}_{\mathbb{Z}}(-, \mathbb{Z}) = 0$.

In this paper all groups are abelian. For all unexplained definitions from set theoretic algebra see the books by Eklof-Mekler [2] and Göbel-Trlifaj [4]. Also, for unexplained definitions from the group theory see Fuchs' book [3].

§ 1. PRELIMINARIES

In this section, we set out our notation and discuss some facts that will be used throughout the paper and refer to the book of Eklof and Mekler [2] for more information. For abelian groups G and H , we set $\text{Hom}(G, H) := \text{Hom}_{\mathbb{Z}}(G, H)$.

Notation 1.1. For an index set u , let $\mathbb{Z}_{[u]} := \bigoplus_{\alpha \in u} \mathbb{Z}x_\alpha$, so that $\langle x_\alpha : \alpha \in u \rangle$ is a basis for $\mathbb{Z}_{[u]}$. For $\eta \in {}^u\mathbb{Z}$, let $f_{[\eta]} \in \text{Hom}(\mathbb{Z}_{[u]}, \mathbb{Z})$ be defined as $f_{[\eta]}(\sum_{\alpha \in v} a_\alpha x_\alpha) = \sum_{\alpha \in v} a_\alpha \eta(\alpha)$, for finite $v \subseteq u$.

Definition 1.2. An abelian group G is called $\aleph_1$ -free if every subgroup of G of cardinality $< \aleph_1$, i.e., every countable subgroup, is free. More generally, an abelian group G is called λ -free if every subgroup of G of cardinality $< \lambda$ is free.

Definition 1.3. Suppose $\lambda > \mu \geq \aleph_0$ are regular and $S \subseteq \lambda$ is stationary.

- (1) The *Jensen's diamond* $\diamond_\lambda(S)$ asserts the existence of a sequence $(S_\alpha \mid \alpha \in S)$ such that for every $X \subseteq \lambda$ the set $\{\alpha \in S \mid X \cap \alpha = S_\alpha\}$ is stationary.

- (2) We use the following consequence of $\diamond_\lambda(S)$: let $A = \bigcup_{\alpha < \lambda} A_\alpha$ and $B = \bigcup_{\alpha < \lambda} B_\alpha$ be two λ -filtrations with $|A_\alpha|, |B_\alpha| < \lambda$. Then there exists a sequence $(g_\alpha : A_\alpha \rightarrow B_\alpha \mid \alpha < \lambda)$ such that, for any function $g : A \rightarrow B$, the set

$$\{\alpha \in S \mid g \upharpoonright_{A_\alpha} = g_\alpha\}$$

is stationary in λ .

- (3) S is *non-reflecting* if for any limit ordinal $\delta < \lambda$ of uncountably cofinality, the set $S \cap \delta$ is non-stationary in δ .
- (4) We set $S_\mu^\lambda = \{\alpha < \lambda \mid \text{cf}(\alpha) = \mu\}$.

Definition 1.4. Let $\mathcal{K}$ be the class of objects $\mathbf{k} := (\mu_{\mathbf{k}}, \theta_{\mathbf{k}}, K_{\mathbf{k}})$ consisting of:

- (a) $\mu_{\mathbf{k}}$ is a limit ordinal, and $\theta_{\mathbf{k}} < \mu_{\mathbf{k}}$,
- (b) $K_{\mathbf{k}}$ is an abelian group with the set of elements $\theta_{\mathbf{k}}$, and $0_{K_{\mathbf{k}}} = 0$,
- (c) if $0 \neq K_1 \subseteq K_{\mathbf{k}}$ is a subgroup, then we can find $(H_{\mathbf{k}, K_1}, \phi_{\mathbf{k}, K_1})$ such that:
 - (α) $H_{\mathbf{k}, K_1}$ is an abelian group of size $\mu_{\mathbf{k}}$ extending $(K_{\mathbf{k}})_{[\mu_{\mathbf{k}}]}$,
 - (β) $H_{\mathbf{k}, K_1} / (K_{\mathbf{k}})_{[\mu_{\mathbf{k}}]}$ is $\mu_{\mathbf{k}}$ -free,
 - (γ) $\phi_{\mathbf{k}, K_1} \in {}^{\mu_{\mathbf{k}}}(K_1)$,
 - (δ) there is no homomorphism $f : H_{\mathbf{k}, K_1} \rightarrow K_1$ such that $f(x_\alpha) = \phi_{\mathbf{k}, K_1}(\alpha)$ for $\alpha < \mu_{\mathbf{k}}$:

$$\begin{array}{ccc} (K_{\mathbf{k}})_{[\{\alpha\}]} & \xrightarrow{\subseteq} & (K_{\mathbf{k}})_{[\mu_{\mathbf{k}}]} \xrightarrow{\subseteq} H_{\mathbf{k}, K_1} \\ \tilde{\phi} \downarrow & \nearrow \nexists f & \\ K_1 & & \end{array}$$

where $\tilde{\phi}(x_\alpha) := \phi_{\mathbf{k}, K_1}(\alpha)$.

Let us address the existence problem of $\mathcal{K}$.

Fact 1.5. Let μ be a limit ordinal and $\phi : \mu \rightarrow \mathbb{Z}$ be such that $\phi(\xi) \neq 0$, for all $\xi < \mu$. Then there is a free abelian group H equipped with the following three properties:

- (i) $H \supseteq \mathbb{Z}_{[\mu]}$ is of size μ ,
- (ii) $H / \mathbb{Z}_{[\mu]}$ is μ -free,
- (iii) there is no homomorphism $f : H \rightarrow \mathbb{Z}$ such that $f(x_\alpha) = \phi(\alpha)$ for $\alpha < \mu$.

In particular, identifying the universe of $\mathbb{Z}$ with ω , we have $\mathbf{k} = (\mu, \omega, \mathbb{Z}) \in \mathcal{K}$.

Proof. Let $G_0 = \mathbb{Z}_{[\mu]} \oplus \mathbb{Z}z$, and let G_1 be the $\mathbb{Z}$ -adic completion of G_0 . We define $f : G_0 \rightarrow \mathbb{Z}$ by $f(x_\alpha) = \phi(\alpha)$ for $\alpha < \mu$ and $f(z) = 1$. For any $\vec{a} := \langle a_n : n < \omega \rangle \in {}^\omega \mathbb{Z}$, $\xi < \mu$ and $\ell < \omega$, we set

$$y_{\vec{a}, \xi, \ell} = \sum_{n \geq \ell} \frac{n!}{\ell!} (x_\xi - a_n z).$$

It is easily seen that for all ℓ as above,

$$(\dagger) : \quad (\ell + 1)y_{\vec{a}, \xi, \ell+1} = y_{\vec{a}, \xi, \ell} - (x_\xi - a_\ell z).$$

Let $G_{\vec{a},\xi}$ be the subgroup of G_1 generated by $G_0 \cup \{y_{\vec{a},\xi,n} : n < \omega\}$. Let $\xi < \mu$. We claim that for some $\vec{a}$, f does not extend to a homomorphism from $G_{\vec{a},\xi}$ into $\mathbb{Z}$. To this end, we look at

$$\mathcal{A}_\xi = \{\vec{a} \in {}^\omega 2 : f \text{ has an extension in } \text{Hom}(G_{\vec{a},\xi}, \mathbb{Z}), a_0 = a_1 = 0\}.$$

For $\vec{a} \in \mathcal{A}_\xi$ let $h_{\vec{a},\xi} \in \text{Hom}(G_{\vec{a},\xi}, \mathbb{Z})$ extends f . For $t \in \mathbb{Z}$ set

$$\mathcal{A}_{t,\xi} = \{\vec{a} \in \mathcal{A}_\xi : h_{\vec{a},\xi}(y_{\vec{a},\xi,0}) = t\}.$$

Clearly, $\mathcal{A}_\xi = \bigcup_{t \in \mathbb{Z}} \mathcal{A}_{t,\xi}$. Now, we bring the following claim.

Claim 1.6. *For each $t \in \mathbb{Z}$, $|\mathcal{A}_{t,\xi}| \leq 1$.*

Proof. Suppose by the way of contradiction that for some $t \in \mathbb{Z}$ we have $|\mathcal{A}_{t,\xi}| > 1$. Let $\vec{a} \neq \vec{b}$ be in $\mathcal{A}_{t,\xi}$ and let n be such that $\vec{a} \upharpoonright n = \vec{b} \upharpoonright n$ and $a_n \neq b_n$. Note that $n \geq 2$. By induction on $\ell \leq n$ we have

$$h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell}) = h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell}).$$

Indeed, the equality holds for $\ell = 0$ by the choice of $\vec{a}, \vec{b} \in \mathcal{A}_{t,\xi}$. For $\ell + 1 \leq n$, by $(\dagger)$ and $a_\ell = b_\ell$, we have

$$\begin{aligned} (\ell + 1)h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell+1}) &= h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell}) - (f(x_\xi) - a_\ell) \\ &= h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell}) - (f(x_\xi) - b_\ell) \\ &= (\ell + 1)h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell+1}). \end{aligned}$$

Hence, on the other hand, we have $h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell+1}) = h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell+1})$. On the other hand, by revisiting $(\dagger)$, and eventuating it with the maps $\{h_{\vec{a},\xi}, h_{\vec{b},\xi}\}$, we lead to the following equations:

$$\begin{aligned} (e_1): (n + 1)h_{\vec{a},\xi}(y_{\vec{a},\xi,n+1}) &= h_{\vec{a},\xi}(y_{\vec{a},\xi,n}) - (f(x_\xi) - a_n). \\ (e_2): (n + 1)h_{\vec{b},\xi}(y_{\vec{b},\xi,n+1}) &= h_{\vec{b},\xi}(y_{\vec{b},\xi,n}) - (f(x_\xi) - b_n). \end{aligned}$$

Subtracting (e_2) from (e_1) , and noting that $h_{\vec{a},\xi}(y_{\vec{a},\xi,n}) = h_{\vec{b},\xi}(y_{\vec{b},\xi,n})$ we get

$$(n + 1)(h_{\vec{a},\xi}(y_{\vec{a},\xi,n+1}) - h_{\vec{b},\xi}(y_{\vec{b},\xi,n+1})) = a_n - b_n.$$

In particular, $n + 1 \mid (a_n - b_n)$, contradicting the fact that $|a_n - b_n| = 1$. $\square_{1.6}$

Let us proceed the argument of Fact 1.5. In view of Claim 1.6, we deduce that $\mathcal{A}_\xi$ is countable. Take any $\vec{a} \in {}^\omega 2 \setminus \mathcal{A}_\xi$. Then $\vec{a}$ is as required. Finally, note that the group $G_{\vec{a},\xi}$ is generated by $\mathcal{B} := \{y_{\vec{a},\xi,\ell} : \ell \in \mathbb{N}\}_\xi \cup \{z\}$, because $x_\xi = y_{\vec{a},\xi,\ell} - (\ell + 1)y_{\vec{a},\xi,\ell+1} + a_\ell z$. Since no relations involved in $\{y_{\vec{a},\xi,\ell}\} \cup \{z\}$, we see $\mathcal{B}$ is a base. $\square$

We also need the following well-known result of Kurepa.

Fact 1.7. Assume $\text{cf}(\lambda) > \aleph_0$ and $\mathcal{T}$ is a tree of height λ , all of whose levels are finite. Then $\mathcal{T}$ has a cofinal branch.

§ 2. CONTROLLING $\text{Hom}(G, \mathbb{Z})$

In this section, we prove our main result (see Theorem 2.6).

Discussion 2.1. Recall that a cardinal κ is called *measurable* if it is uncountable and there exists a non-principal κ -complete ultrafilter $\mathcal{D}$ on κ , meaning that for every subset S of $\mathcal{D}$ with cardinality $< \kappa$, the intersection $\bigcap S$ belongs to $\mathcal{D}$. It is known that the existence of measurable cardinals cannot be proven from ZFC.

Definition 2.2. Let G be an abelian group. The dual of G is the abelian group $\text{Hom}(G, \mathbb{Z})$, which we denote by G^* . Let $g \in G$, and define $\psi_g : G^* \rightarrow \mathbb{Z}$ by the evaluation $\psi_g(G \xrightarrow{f} \mathbb{Z}) := f(g)$. The assignment $g \mapsto \psi_g$ defines a canonical map $\psi : G \rightarrow G^{**}$. We say that G is reflexive, if ψ is an isomorphism.

Fact 2.3. (Lös-Eda, Shelah; see [2, 9]). Let $\mu = \mu_{\text{first}}$ be the first measurable cardinal. The following hold:

- (a) For any $\theta < \mu$, $\mathbb{Z}^{(\theta)}$ is reflexive. In fact, its dual is $\mathbb{Z}^\theta$.
- (b) For any $\lambda \geq \mu$, $\mathbb{Z}^{(\lambda)}$ is not reflexive.
- (c) There exists a reflexive group $G \subset \mathbb{Z}^\mu$ of cardinality μ .

Let Pr be any property of abelian groups and λ be a cardinal. Recall that compactness for (λ, Pr) means that for any group G of cardinality λ and any “ $G' \subseteq G \cap |G'| < \lambda \Rightarrow G'$ has Pr ” then G has Pr . In this paper we are interested in the following fixed property of abelian groups:

Notation 2.4. By Pr_λ we mean the following property: If G is a group of size λ , and if for any nontrivial subgroup $G' \subseteq G$ of size less than λ , $\text{Hom}(G', \mathbb{Z}) \neq 0$, then $\text{Hom}(G, \mathbb{Z}) \neq 0$.

Let's now turn to the primary framework.

Definition 2.5. (1) Let $\mathbf{M}_{1,\theta}$ be the class of objects

$$\mathbf{m} = (\lambda_{\mathbf{m}}, \langle G_\alpha^{\mathbf{m}} : \alpha \leq \alpha_{\mathbf{m}} \rangle, \langle f_{\mathbf{m},s} : s \in S_{\mathbf{m}} \rangle)$$

consisting of:

- (a) (α) $\lambda_{\mathbf{m}} = \text{cf}(\lambda_{\mathbf{m}}) > \aleph_0$,
- (β) $\lambda_{\mathbf{m}} \geq \alpha_{\mathbf{m}} := \ell g(\mathbf{m})$, the length of $\mathbf{m}$,
- (b) (α) $\langle G_\alpha^{\mathbf{m}} : \alpha \leq \alpha_{\mathbf{m}} \rangle$ is an increasing and continuous sequence of abelian groups,
- (β) $|G_\alpha^{\mathbf{m}}| < \lambda_{\mathbf{m}}$ for $\alpha < \alpha_{\mathbf{m}}$,
- (c) $G_\alpha^{\mathbf{m}}/G_0^{\mathbf{m}}$ is free,
- (d) $\{\beta < \alpha_{\mathbf{m}} : G_{\beta+1}^{\mathbf{m}}/G_\beta^{\mathbf{m}} \text{ is not free}\}$ is a non-reflecting stationary set,
- (e) (α) $S_{\mathbf{m}}$ is a set of cardinality $\leq \theta$,
- (β) $f_{\mathbf{m},s} \in \text{Hom}(G_{\alpha_{\mathbf{m}}}^{\mathbf{m}}, \mathbb{Z})$ for $s \in S_{\mathbf{m}}$,
- (f) $\langle f_{\mathbf{m},s} : s \in S_{\mathbf{m}} \rangle$ is a free basis of a subgroup of $\text{Hom}(G_{\alpha_{\mathbf{m}}}^{\mathbf{m}}, \mathbb{Z})$.

(2) $\mathbf{M}_{2,\theta}$ is defined as above, where item (a)(β) is replaced by $\alpha_{\mathbf{m}} = \lambda_{\mathbf{m}}$, and we further require that:

(g) if $f \in \text{Hom}(G_{\alpha_{\mathbf{m}}}^{\mathbf{m}}, \mathbb{Z})$ then for some $h \in \text{Hom}(S_{\mathbf{m}}\mathbb{Z}, \mathbb{Z})$ we have

$$x \in G_{\alpha_{\mathbf{m}}}^{\mathbf{m}} \Rightarrow f(x) = h(\langle f_{\mathbf{m},s}(x) : s \in S_{\mathbf{m}} \rangle),$$

(h) the mapping $x \mapsto \langle f_{\mathbf{m},s}(x) : s \in S_{\mathbf{m}} \rangle$ defines a homomorphism from $G_{\alpha_{\mathbf{m}}}^{\mathbf{m}}$ onto $S_{\mathbf{m}}\mathbb{Z}$,

(i) for any $0 \neq G' \subseteq G_{\alpha}^{\mathbf{m}}$ with $\alpha < \alpha_{\mathbf{m}}$, we have $\text{Hom}(G', \mathbb{Z}) \neq 0$.

We are now in a position to state and prove our main result:

Theorem 2.6. *Assume that:*

- (a) $\langle \lambda_i : i < \kappa \rangle$ is an increasing sequence of regular cardinals with limit λ ,
- (b) $\kappa = \text{cf}(\kappa) < \lambda_0$, and $\aleph_0 < \lambda_0$,
- (c) $\lambda_i = \text{cf}(\lambda_i)$, $S_i \subseteq S_{\text{cf}(\mu)}^{\lambda_i}$ is stationary and non-reflecting,
- (d) $\diamond_{S_i}$ holds,
- (e) there is no measurable cardinal $\leq \lambda$.

Then there is a λ_0 -free abelian group G of cardinality λ which is counterexample to singular compactness in λ for Pr_{λ} .

Proof. We are going to present a λ_0 -free abelian group G of cardinality λ so that for any nontrivial subgroup $G' \subseteq G$ of smaller size, we have $\text{Hom}(G', \mathbb{Z}) \neq 0$, while $\text{Hom}(G, \mathbb{Z}) = 0$. We present the proof in several stages.

Stage A: We define a tree $\mathcal{T}$ of height κ , whose i -th level $\mathcal{T}_i$ is defined as follows:

- (*) $_A^i$: $\mathcal{T}_i$ is the set of η such that:
 - (a) η is a sequence of length $i + 1$,
 - (b) for $j \leq i$ we have $\eta(j) = (\eta(j, 1), \eta(j, 2))$,
 - (c) for $j \leq i$, $\eta(j, 1) < \lambda_j$ and $\eta(j, 2) < \kappa$,
 - (d) if $j_1 < j_2 \leq i$ then $\eta(j_1, 1) \leq \eta(j_2, 1)$, and $\eta(j_1, 2) \leq \eta(j_2, 2)$,
 - (e) $\text{Im}(\eta)$ is finite,
 - (f) if $j_1 < j_2 \leq i$ and $\langle \eta(j, 1) : j \in [j_1, j_2] \rangle$ is constant then $j_2 < \eta(j_1, 2)$.

Let $\mathcal{T} = \bigcup_{i < \kappa} \mathcal{T}_i$, where $\mathcal{T}$ is ordered by end-extension relation $\triangleleft$. Then it is easily seen that $(\mathcal{T}, \triangleleft)$ is a tree with κ levels whose i -th level is $\mathcal{T}_i$ and that if $\eta \in \mathcal{T}_i$, $i < j < \kappa$, then there is $\nu \in \mathcal{T}_j$ such that $\eta \triangleleft \nu$ (so $\eta = \nu \upharpoonright (i + 1)$).

Also, we need to introduce the corresponding truncated trees, as follows:

$$\mathcal{T}_{i,\alpha} := \{\eta \in \mathcal{T}_i : \eta(i, 1) \leq \alpha\},$$

where $\alpha \leq \lambda_i$. In particular, $\mathcal{T}_i = \mathcal{T}_{i,\lambda_i}$.

Claim 2.7. $(\mathcal{T}, \triangleleft)$ has no κ -branches.

Proof. Assume by the way of contradiction that $b = \langle \eta_i : i < \kappa \rangle$, where $\eta_i \in \mathcal{T}_i$, is a branch of $\mathcal{T}$, hence the sequence $\langle \eta_i : i < \kappa \rangle$ is $\triangleleft$ -increasing. It follows that $\langle \eta_i(i, 1) : i < \kappa \rangle$ is a non-decreasing sequence of ordinals. As, by clause (e), every initial segment

has finitely many values and $\kappa = \text{cf}(\kappa) > \aleph_0$, necessarily $\langle \eta_i(i, 1) : i < \kappa \rangle$ is eventually constant, so for some $i_* < \kappa$, the sequence $\langle \eta_i(i, 1) : i \in [i_*, \kappa) \rangle$ is constant. By $(*)^i_A(f)$, $\eta(i_*, 2) > i$ for all $i < \kappa$, but on the other hand, $\eta(i_*, 2) < \kappa$, a contradiction. $\square$

Stage B: We shall choose $\mathbf{m}_i$ by induction on $i < \kappa$ such that:

- $(*)^i_B :$ (a) $\mathbf{m}_i = (\lambda_{\mathbf{m}_i}, \langle G_{\alpha}^{\mathbf{m}_i} : \alpha \leq \alpha_{\mathbf{m}_i} \rangle, \langle f_{\mathbf{m}_i, s} : s \in S_{\mathbf{m}_i} \rangle) \in \mathbf{M}_{1, \lambda_i}$,
 (b) $\lambda_{\mathbf{m}_i} = \alpha_{\mathbf{m}_i} = \lambda_i$, and the set of elements of $G_{\lambda_i}^{\mathbf{m}_i}$ is λ_i ,
 (c) $G_{<i}^{\mathbf{m}_i} := \bigcup \{G_{\lambda_j}^{\mathbf{m}_j} : j < i\} \cup \{0\}$,
 (d) $G_0^{\mathbf{m}_i} := G_{<i}^{\mathbf{m}_i}$,
 (e) $S_{\mathbf{m}_i} := \mathcal{T}_i = \mathcal{T}_{i, \lambda_i}$,
 (f) if $j < i$, then $\mathbf{m}_j \leq \mathbf{m}_i$ which means that

$$\eta \in \mathcal{T}_j \wedge \nu \in \mathcal{T}_i \wedge \eta \triangleleft \nu \Rightarrow f_{\mathbf{m}_j, \eta} \subseteq f_{\mathbf{m}_i, \nu}.$$

This can be expressed by the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & G_{\lambda_j}^{\mathbf{m}_j} \xrightarrow{\subseteq} G_{\lambda_i}^{\mathbf{m}_i} \\ & & \downarrow f_{\mathbf{m}_j, \eta} \quad \swarrow f_{\mathbf{m}_i, \nu} \\ & & \mathbb{Z} \end{array}$$

- (g) $\langle f_{\mathbf{m}_i, \eta} : \eta \in \mathcal{T}_i \rangle$ is an independent subset of $\text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$,
 (h) $\bigcap \{\text{Ker}(f_{\mathbf{m}_i, \eta}) : \eta \in \mathcal{T}_i\} = \{0\}$,
 (i) if $f \in \text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$, then for some h, α we have:
 (α) $\alpha < \lambda_i$,
 (β) $h \in \text{Hom}({}^{(\mathcal{T}_{i, \alpha})}\mathbb{Z}, \mathbb{Z})$,
 (γ) if $x \in G_{\lambda_i}^{\mathbf{m}_i}$, then $f(x) = h(\langle f_{\mathbf{m}_i, \eta}(x) : \eta \in \mathcal{T}_{i, \alpha} \rangle)$.

Remark 2.8. The cardinality of $\mathcal{T}_{i, \alpha}$ is less than λ_i for any $\alpha < \lambda_i$. This will be helpful to show $|G_{\alpha}^{\mathbf{m}_j}| < \lambda_{\mathbf{m}_j}$ for $\alpha < \lambda_{\mathbf{m}_j}$, see subsequent paragraph of $(*)$ below.

For $i = 0$, we set

- $\mathbf{m}_0 = (\lambda_0, \langle G_{\alpha}^{\mathbf{m}_0} : \alpha \leq \lambda_0 \rangle, \langle f_{\mathbf{m}_0, s} : s \in S_{\mathbf{m}_0} \rangle)$,
- $G_{\alpha}^{\mathbf{m}_0} = \bigoplus_{\eta \in \mathcal{T}_{0, \alpha}} \mathbb{Z}x_{\eta}$,
- $S_{\mathbf{m}_0} = \mathcal{T}_{0, \alpha}$,
- for $\eta \in \mathcal{T}_{0, \alpha}$, $f_{\mathbf{m}_0, \eta} : G_{\alpha}^{\mathbf{m}_0} \rightarrow \mathbb{Z}x_{\eta}$ is the projection map.

Note that by L\"os theorem [2, Corollary III. 1.5],

$$\text{Hom}({}^{(\mathcal{T}_0)}\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{\eta \in \mathcal{T}_0} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{\eta \in \mathcal{T}_0} \mathbb{Z}x_{\eta},$$

from which we can easily conclude $(*)^i_B(i)$. The reason we take $G_{\lambda_0}^{\mathbf{m}_0}$ free is to make sure at the end of the construction, all our groups are at least λ_0 -free, as for the next steps $i < \kappa$ of the construction, we only get a bit more than $\sum_{j < i} \lambda_j$ -freeness, which for $i = 0$ is not well-defined.

Now assume that $0 < i < \kappa$ and $\langle \mathbf{m}_j : j < i \rangle$ has been defined. Fix a diamond sequence $\langle F_{i,\delta} : \delta \in S_i \rangle$ with $F_{i,\delta} : \delta \rightarrow \mathbb{Z}$.

Notation 2.9. Let $\langle \beta_i(\gamma) : \gamma < \lambda_i \rangle$ be an increasing and continuous sequence of ordinals, cofinal in λ_i with $\beta_i(0) = 0$.

We proceed by setting:

- $G_{<i} = \bigcup_{j<i} G_{\lambda_j}^{\mathbf{m}_j} \cup \{0\}$ (so $G_{<i} = \{0\}$, if $i = 0$),
- for $\eta \in \mathcal{T}_i$ set $f_{<i,\eta} = \bigcup_{j<i} f_{\mathbf{m}_j,\eta \upharpoonright j+1}$, hence $f_{<i,\eta} : G_{<i} \rightarrow \mathbb{Z}$.

We shall choose $\mathbf{m}_{i,\gamma}$ by induction on $\gamma < \lambda_i$ such that:

- $(*)^\gamma_C$:
- (a) $\mathbf{m}_{i,0}$ is defined as
 - (α) $\ell g(\mathbf{m}_{i,0}) = 0$,
 - (β) $\lambda_{\mathbf{m}_{i,0}} = \sup_{j<i} \lambda_{\mathbf{m}_j}$,
 - (γ) $G_0^{\mathbf{m}_{i,0}} := G_{<i}$,
 - (δ) $S_{\mathbf{m}_{i,0}} = \mathcal{T}_{i,\beta_i(0)}$,
 - (ϵ) for $\eta \in \mathcal{T}_{i,\beta_i(0)}$, $f_{\mathbf{m}_{i,0},\eta} := f_{<i,\eta}$,
 - (b) $\langle \mathbf{m}_{i,\gamma} : \gamma < \lambda_i \rangle$ is an increasing and continuous sequence from $\mathbf{M}_{1,\lambda_i}$ with $S_{\mathbf{m}_{i,\gamma}} = \mathcal{T}_{i,\beta_i(\gamma)}$ and $\ell g(\mathbf{m}_{i,\gamma}) = \alpha_{i,\gamma} < \lambda_i$, which means:
 - (α) if $\rho < \gamma$, then $\mathbf{m}_{i,\rho} \leq \mathbf{m}_{i,\gamma}$,
 - (β) if γ is a limit ordinal, then $\mathbf{m}_{i,\gamma} = \bigcup_{\rho<\gamma} \mathbf{m}_{i,\rho}$, i.e.,
 - (β_1) $\alpha_{i,\gamma} = \sup_{\rho<\gamma} \alpha_{i,\rho}$,
 - (β_2) $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}} = \bigcup_{\rho<\gamma} G_{\alpha_{i,\rho}}^{\mathbf{m}_{i,\rho}}$,
 - (β_3) $S_{\mathbf{m}_{i,\gamma}} = \mathcal{T}_{i,\beta_i(\gamma)}$,
 - (β_4) if $\eta \in \mathcal{T}_{i,\beta_i(\gamma)}$, then $f_{\mathbf{m}_{i,\gamma},\eta} = f_{i,<\eta} \cup \bigcup_{\rho<\gamma} f_{\mathbf{m}_{i,\rho},\eta \upharpoonright \rho+1}$,
 - (c) if $\rho < \gamma$, and $\rho \notin S_i$, then $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}} / G_{\alpha_{i,\rho}}^{\mathbf{m}_{i,\rho}}$ is free,
 - (d) $\bigcap \{ \text{Ker}(f_{\mathbf{m}_{i,\gamma},\eta}) : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \} = \{0\}$,
 - (e) $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ has set of elements an ordinal ordinal $\delta_i(\gamma) < \lambda_i$,
 - (f) Recall that $\langle F_{i,\delta} : \delta \in S_i \rangle$ is the diamond sequence. Suppose we have the following list of notations and assumptions:
 - (α) $\gamma = \alpha_{i,\gamma} \in S_i$,
 - (β) the set of elements of $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ is γ ,
 - (γ) $\text{Im}(F_{i,\gamma}) \subseteq \mathbb{Z}$ is non-zero. In particular, $\text{Im}(F_{i,\gamma}) = n\mathbb{Z} \cong \mathbb{Z}$ for some nonzero $n \in \mathbb{Z}$,
 - (δ) $F_{i,\gamma}$ is a homomorphism from $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ onto $\text{Im}(F_{i,\gamma})$,
 - (ϵ) $F_{i,\gamma} \notin \langle f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle$, where $\langle f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle$ is the subgroup of $\text{Hom}(G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}, \mathbb{Z})$ generated by $\{f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)}\}$.

Then, we shall choose $\mathbf{m}_{i,\gamma+1}$ such that $F_{i,\gamma}$ has no extension to a homomorphism from $G_{\alpha_{i,\gamma+1}}^{\mathbf{m}_{i,\gamma+1}}$ into K_1 . Namely, we have

$$\begin{array}{ccc} G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}} & \xrightarrow{\subseteq} & G_{\alpha_{i,\gamma+1}}^{\mathbf{m}_{i,\gamma+1}} \\ F_{i,\gamma} \downarrow & & \downarrow \# \\ \text{Im}(F_{i,\gamma}) & \xrightarrow{=} & \text{Im}(F_{i,\gamma}) \end{array}$$

For notational simplicity, we set

- $G_{i,\gamma,\rho} := G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ for any $\rho \leq \alpha_{i,\gamma}$,
- $f_{i,\gamma,\eta} := f_{\mathbf{m}_{i,\gamma},\eta}$ for any $\eta \in \mathcal{T}_{i,\beta_i(\gamma)}$,
- $G_{i,\gamma} := G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$.

The case $\gamma = 0$ is trivial, and can be defined as in $(*)_C^\gamma(a)$. Note that by the induction hypothesis and the way we defined $f_{i,0,\eta}$:

- the sequence $\langle f_{i,0,\eta} : \eta \in \mathcal{T}_{i,\beta_i(0)} \rangle$ is an independent subset of $\text{Hom}(G_{i,0}, \mathbb{Z})$,
- each $f_{i,0,\eta}$ extends $f_{<i,\eta}$ and
- $\bigcap \{ \text{Ker}(f_{i,0,\eta}) : \eta \in \mathcal{T}_{i,\beta_i(0)} \} = \{0\}$.

If γ is a limit ordinal, set $\alpha_{i,\gamma} = \sup_{\rho < \gamma} \alpha_{i,\rho}$ and define $\mathbf{m}_{i,\gamma}$ as in clause $(*)_C^\gamma(a)$.

Suppose we have defined $\mathbf{m}_{i,\gamma}$. Assume one of the following hypotheses hold:

- $(h_1) : \gamma \notin S_i$ or at least one of the hypotheses $(*)_C^\gamma(i)(\alpha)-(\delta)$ are not satisfied, or
- $(h_2) : \gamma \in S_i$, the hypotheses in $(*)_C^\gamma(i)(\alpha)-(\delta)$ are all satisfied, but either
 - $G_{i,\gamma}$ doesn't have domain γ , or
 - $F_{i,\gamma} \notin \text{Hom}(G_{i,\gamma}, \mathbb{Z})$ or
 - $F_{i,\gamma} \in \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle$.

Then, we define $\mathbf{m}_{i,\gamma+1}$ as the following table:

- (1) $\alpha_{i,\gamma+1} = \alpha_{i,\gamma} + 1$,
- (2) $\mathbf{m}_{i,\gamma} \leq \mathbf{m}_{i,\gamma+1}$,
- (3) $S_{\mathbf{m}_{i,\gamma+1}} = \mathcal{T}_{i,\beta_i(\gamma+1)}$,
- (4) $G_{i,\gamma+1} = G_{i,\gamma+1,\alpha_{i,\gamma+1}} := G_{i,\gamma} \oplus \mathbb{Z}_{[u_{i,\gamma}]}$, where $u_{i,\gamma} = \mathcal{T}_{i,\beta_i(\gamma+1)}$,
- (5) for $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$, $f_{i,\gamma+1,\eta} := f_{i,\gamma,\eta} \oplus \pi_\eta$, where $\pi_\eta : \mathbb{Z}_{[u_{i,\gamma}]} \rightarrow \mathbb{Z}x_\eta$ is the projection map, and for $\eta \notin \mathcal{T}_{i,\beta_i(\gamma)}$, we demand $f_{i,\gamma,\eta}$ is the zero-map.

Finally, suppose that $\mathbf{m}_{i,\gamma}$ is defined, $\gamma \in S_i$ and the hypotheses in $(*)_C^\gamma(i)(\alpha)-(\delta)$ are all satisfied. Also, suppose that $G_{i,\gamma}$ has domain γ , and $F_{i,\gamma} \in \text{Hom}(G_{i,\gamma}, \mathbb{Z})$ is such that $F_{i,\gamma} \notin \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle$. In this case, we define $\mathbf{m}_{i,\gamma+1}$ so that the conclusion of $(*)_B^\gamma(i)$ is satisfied.

Let $\alpha_{i,\gamma+1} := \alpha_{i,\gamma} + 1$, and naturally set

$$G_{i,\gamma,\rho} = G_{\rho}^{\mathbf{m}_{i,\gamma+1}} := G_{\rho}^{\mathbf{m}_{i,\gamma}} (= G_{i,\gamma,\rho}) \quad \forall \rho \leq \alpha_{i,\gamma}.$$

We have to define

- $G_{i,\gamma+1} = G_{\alpha_{i,\gamma+1}}^{\mathbf{m}_{i,\gamma+1}}$ and
- $f_{i,\gamma+1,\eta} = f_{\mathbf{m}_{i,\gamma+1},\eta} : G_{i,\gamma+1} \rightarrow \mathbb{Z}$ for $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$.

For every $\beta < \lambda_i$, set

$$G_{i,\gamma}^{[\beta]} = \{x \in G_{i,\gamma} : \eta \in \mathcal{T}_i \wedge \eta(i, 1) < \beta \Rightarrow f_{i,\gamma,\eta}(x) = 0\}.$$

Then the sequence $\langle G_{i,\gamma}^{[\beta]} : \beta < \lambda_i \rangle$ is increasing, hence as $|G_{i,\gamma}| < \lambda_i$, there is $\beta_{i,\gamma} < \lambda$ such that

$$G_{i,\gamma}^{[\beta]} = G_{i,\gamma}^{[\beta_{i,\gamma}]}, \quad \forall \beta \in (\beta_{i,\gamma}, \lambda_i).$$

Let $K_{i,\gamma} = \text{Im}(F_{i,\gamma} \upharpoonright G_{i,\gamma}^{[\beta_{i,\gamma}]})$, and define

$$\mu_{i,\gamma} = \sum_{j < i} \lambda_j + |\mathcal{T}_{i,\beta_i(\gamma+1)}| + \aleph_0 < \lambda_i.$$

Set $\mathbf{k}_{i,\gamma} := (\mu_{i,\gamma}, \omega, \mathbb{Z})$. According to Fact 1.5, we know $\mathbf{k}_{i,\gamma} \in \mathcal{K}$. This gives us

$$(H_*, \phi_*) := (H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}, \phi_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}),$$

so that:

- $H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}$ is a free abelian group of size $\mu_{i,\gamma}$, which extends $(K_{i,\gamma})_{[u_{i,\gamma}]}$, by recalling that $u_{i,\gamma} = \mathcal{T}_{i,\beta_i(\gamma+1)}$,
- $\phi_* = \phi_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} : u_{i,\gamma} \rightarrow K_{i,\gamma}$,
- $H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} / (K_{i,\gamma})_{[u_{i,\gamma}]}$ is $\mu_{i,\gamma}$ -free,
- there is no homomorphism $f : H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} \rightarrow K_{i,\gamma}$ such that $f(x_\eta) = \phi_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}(\eta)$ for $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$.

Also, for every β , with $\beta_{i,\gamma} \leq \beta < \lambda_i$, and $b \in K_{i,\gamma}$ there is $y_{b,\beta} \in G_{i,\gamma}$ such that

- (*) _{β, b} (a) $\eta \in \mathcal{T}_i \wedge \eta(i, 1) < \beta \Rightarrow f_{i,\gamma,\eta}(y_{b,\beta}) = 0$,
 (b) $F_{i,\gamma}(y_{b,\beta}) = b$.

Since $|G_{i,\gamma}| < \lambda_i$, for each $b \in K_{i,\gamma}$ as above, there exists some fixed $y_b \in G_{i,\gamma}$ such that the set

$$X_b = \{\beta < \lambda_i : \beta_{i,\gamma} \leq \beta \text{ and } y_{b,\beta} = y_b\}$$

is stationary in λ_i .

The assignment $x_\eta \mapsto y_{\phi_*(\eta)}$ induces a morphism $g_{i,\gamma} : (K_{i,\gamma})_{[u_{i,\gamma}]} \rightarrow G_{i,\gamma}$. Recall that $\text{id} : (K_{i,\gamma})_{[u_{i,\gamma}]} \rightarrow H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}$ is the natural inclusion. Let us summarize these data with the following notation

$$\begin{array}{ccc} & H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} & \\ \text{id} \uparrow & & \\ (K_{i,\gamma})_{[u_{i,\gamma}]} & \xrightarrow{g_{i,\gamma}} & G_{i,\gamma}, \end{array}$$

The group that we were searching for it, is the pushout of the above data. Namely,

$$G_{i,\gamma+1} := G_{i,\gamma} \oplus_{(K_{i,\gamma})_{[u_{i,\gamma}]}} H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}.$$

In other words, $G_{i,\gamma+1}$ has the following presentation:

$$G_{i,\gamma+1} = \frac{G_{i,\gamma} \times H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}}{\langle (\text{id}(k), -g_{i,\gamma}(k)) : k \in (K_{i,\gamma})_{[u_{i,\gamma}]} \rangle} \quad (*)$$

Recall that $H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}$ is of size $\mu_{i,\gamma} = \sum_{j < i} \lambda_j + |\mathcal{T}_{i,\beta_i(\gamma+1)}| + \aleph_0 < \lambda_i$. We combine this with an inductive argument along with $(*)$, to concluded that group $G_{i,\gamma+1}$ is of size less than λ_i .

Notation 2.10. For $h \in H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}$ and $g \in G_{i,\gamma}$, let $[(h, g)] \in G_{i,\gamma+1}$ denote the equivalence class of $[(h, g)]$.

This push-out construction, gives us two embedding maps $h_{i,\gamma} : G_{i,\gamma} \rightarrow G_{i,\gamma+1}$ and $k_{i,\gamma} : H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} \rightarrow G_{i,\gamma+1}$ so that $h_{i,\gamma} \circ g_{i,\gamma} = k_{i,\gamma}$. Let us depict all things together:

$$\begin{array}{ccc} H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} & \xrightarrow{k_{i,\gamma}} & G_{i,\gamma+1} \\ \text{id} \uparrow & & \uparrow h_{i,\gamma} \\ (K_{i,\gamma})_{[u_{i,\gamma}]} & \xrightarrow{g_{i,\gamma}} & G_{i,\gamma} \end{array}$$

We now show that $h_{i,\gamma} : G_{i,\gamma} \rightarrow G_{i,\gamma+1}$, is an embedding. Indeed, the assignment $x \in G_{i,\gamma} \mapsto [(0, x)]$ defines $h_{i,\gamma}$. Suppose $0 = h_{i,\gamma}(x) = [(0, x)]$. By the above equivalence relation, there is a $k \in (K_{i,\gamma})_{[u_{i,\gamma}]}$ so that $(k, g_{i,\gamma}(k)) = (0, x)$. Hence $k = 0$ and $x = g_{i,\gamma}(0) = 0$. This shows that $h_{i,\gamma}$ is an embedding, as desired. Thus, by simplicity, we may assume that $G_{i,\gamma} \subseteq G_{i,\gamma+1}$ and $h_{i,\gamma}$ is the inclusion map. We now show that $F_{i,\gamma}$ does not extend to a homomorphism from $G_{i,\gamma+1}$ into $K_{i,\gamma}$. Indeed if $F : G_{i,\gamma+1} \rightarrow K_{i,\gamma}$ extends $F_{i,\gamma}$, then $f : H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} \rightarrow \mathbb{Z}$ defined by $f = F \circ k_{i,\gamma}$ satisfies

$$f(x_\eta) = F \circ k_{i,\gamma} = F \circ (h_{i,\gamma} \circ g_{i,\gamma})(x_\eta) = F_{i,\gamma} \circ g_{i,\gamma}(x_\eta) = \phi_*(\eta),$$

for all $\eta \in u_{i,\gamma}$. This contradicts the choice of $(H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}, \phi_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}})$ and Definition 1.4(δ).

Now, by the following well-known diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ & & H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} / (K_{i,\gamma})_{[u_{i,\gamma}]} & \xrightarrow{=} & G_{i,\gamma+1} / G_{i,\gamma} & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} & \longrightarrow & G_{i,\gamma+1} & \longrightarrow & \text{coker}(k_{i,\gamma}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & (K_{i,\gamma})_{[u_{i,\gamma}]} & \longrightarrow & G_{i,\gamma} & \longrightarrow & \text{coker}(g_{i,\gamma}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0, & & \end{array}$$

we are able to deduce that

$$G_{i,\gamma+1} / G_{i,\gamma} \cong H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}} / (K_{i,\gamma})_{[u_{i,\gamma}]},$$

which is $\mu_{i,\gamma}$ -free.

We next define the map $f_{i,\gamma+1,\eta}$. Take any $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$. For any $h \in H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ and $g \in G_{i,\gamma}$, the assignment $[(h, g)] \mapsto f_{i,\gamma,\eta}(g)$, defines a morphism

$$f_{i,\gamma+1,\eta} := f_{\mathbf{m}_{i,\gamma+1,\eta}} : G_{i,\gamma+1} \longrightarrow \mathbb{Z}.$$

Let us show that $f_{i,\gamma+1,\eta}$ is well-defined, by arguing that $f_{i,\gamma,\eta} \circ g_{i,\gamma} = 0$. Given any $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$, choose $\beta \in X_{\phi_*(\eta)}$ such that $\eta(i, 1) < \beta$. In view of $(*)_{\beta,\phi_*(\eta)}(a)$, we have

$$f_{i,\gamma,\eta} \circ g_{i,\gamma}(x_\eta) = f_{i,\gamma,\eta}(y_{\phi_*(\eta)}) = f_{i,\gamma,\eta}(y_{\phi_*(\eta),\beta}) = 0.$$

Clearly, $\langle f_{i,\gamma+1,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma+1)} \rangle$ are independent and also

$$\bigcap \{ \text{Ker}(f_{i,\gamma+1,\eta}) : \eta \in \mathcal{T}_{i,\beta_i(\gamma+1)} \} = \{0\}.$$

Having finished the construction, for $\eta \in \mathcal{T}_i$ we set $f_{i,\eta} := \bigcup_{\gamma < \lambda_i} f_{i,\gamma,\eta}$. Now, we are ready to bring the following claim.

Claim 2.11. $\langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle$ generates $\text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$.

Proof. Suppose $f \in \text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z}) \setminus \langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle$. Take $\gamma \in S$ such that $G_{i,\gamma}$ has domain γ , $f \upharpoonright \gamma = F_{i,\gamma}$, and $f \upharpoonright \gamma \notin \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_i \rangle$. Then by our construction, $f \upharpoonright \gamma$ does not extend to a homomorphism from $G_{i,\gamma+1}$ into $\mathbb{Z}$, a contradiction. $\square_{2.11}$

Also note that clause $(*)_C^\gamma(i)$ holds by Claim 2.11, indeed, given any $f \in \text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$, we can find some $\eta_0, \dots, \eta_{n-1} \in \mathcal{T}_i$ and some $\alpha_0, \dots, \alpha_{n-1}$ such that

$$f = \sum_{k < n} \alpha_k f_{i,\eta_k}.$$

Let us to pick $\alpha < \lambda_i$ large enough such that for each $k < n$, $\eta_k(i, 1) \leq \alpha$. By Löf's theorem [2, Corollary III. 1.5],

$$\text{Hom}^{(\mathcal{T}_i, \alpha)}(\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{\eta \in (\mathcal{T}_i, \alpha)} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{\eta \in (\mathcal{T}_i, \alpha)} \mathbb{Z}x_\eta.$$

In particular, there is $h \in \text{Hom}^{(\mathcal{T}_i, \alpha)}(\mathbb{Z}, \mathbb{Z})$ such that

$$h(\langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle) = \sum_{k < n} \alpha_k f_{i,\eta_k}.$$

Clause $(*)_C^\gamma(b)$ follows from the fact that S_i is non-reflecting, hence for some club $C \subseteq \gamma$ with $\min(C) = \rho$, such that $C \cap S_i = \emptyset$, hence $G_{i,\gamma}/G_{i,\rho}$ is the union of the increasing and continuous sequence $\langle G_{i,\tau}/G_{i,\rho} : \tau \in C \rangle$, and by the induction hypothesis, each $G_{i,\tau}/G_{i,\mu}$ is free for all $\mu < \tau$ from C , so easily $G_{i,\gamma}/G_{i,\rho}$ is free.

Having defined $\langle \mathbf{m}_{i,\gamma} : \gamma < \lambda_i \rangle$, set $\mathbf{m}_i = \bigcup_{\gamma < \lambda_i} \mathbf{m}_{i,\gamma}$. This completes the inductive construction of $\langle \mathbf{m}_i : i < \kappa \rangle$.

Stage C: In this step, we show that for each i , $\mathbf{m}_i \in \mathbf{M}_{2,\lambda_i}$ (see Definition 2.5(2)). Items (a)-(e) of Definition 2.5(1) and $\alpha_{\mathbf{m}_i} = \lambda_i = \lambda_{\mathbf{m}_i}$ are obvious.

For clause (i), suppose $\alpha < \lambda_i$ and $0 \neq G' \subseteq G_\alpha^{\mathbf{m}_i}$. Then for some $\gamma < \lambda_i$, $G' \subseteq G_{\alpha_i, \gamma}^{\mathbf{m}_i, \gamma}$. Let $0 \neq x \in G'$. In view of $(*)_C^\gamma(c)$, we have

$$\bigcap \{\text{Ker}(f_{i,s}) : s \in \mathcal{T}_i\} = \{0\}.$$

This in turns imply that $f_{i,s}(x) \neq 0$ for some $s \in \mathcal{T}_i$. In particular, $f_{i,s} \upharpoonright G' \in \text{Hom}(G', \mathbb{Z})$ is non-zero.

Stage D: In this stage we conclude the proof of Theorem 2.6. for each $i < \kappa$ set $G_i = G_{\mathbf{m}_i, \lambda_i}$ and $G_{<i} = \bigcup \{G_j : j < i\}$. Then the sequence $\langle G_i : i < \kappa \rangle$ is increasing continuous and for each i , $G_i/G_{<i}$ is $\sum_{j < i} \lambda_j$ -free (as for each $\gamma < \lambda_i$, $\mu_{i, \gamma} \geq \sum_{j < i} \lambda_j$ and each $G_{i, \gamma+1}/G_{i, \gamma}$ is $\mu_{i, \gamma}$ -free). Define the group

$$G := \bigcup \{G_i : i < \kappa\}.$$

From this, G is an abelian group of size λ .

We first show that G is λ_0 -free. Thus suppose that H is a subgroup of G of size less than λ_0 . Then the sequence $\langle H \cap G_i : 0 < i < \kappa \rangle$ is increasing, continuous and for each $0 < i < \kappa$,

$$(H \cap G_i)/(H \cap G_{<i}) \cong ((H \cap G_i) + G_{<i})/G_{<i}$$

is free as $G_i/G_{<i}$ is $\sum_{j < i} \lambda_j$ -free and hence λ_0 -free. It then easily follows that $H = \bigcup_{i < \kappa} (H \cap G_i)$ is free.

Next, suppose H is a non-zero subgroup of G of size less than λ . We show that $\text{Hom}(H, \mathbb{Z}) \neq 0$. Let $i < \kappa$ be such that $H \cap G_{<i} \neq \{0\}$ and $|H| < \lambda_i$. According to Definition 2.5(2)(i), we must have $\text{Hom}(H \cap G_i, \mathbb{Z}) \neq 0$. Furthermore, by an argument as above, $(H \cap G_i)/(H \cap G_{<i}) \cong ((H \cap G_i) + G_{<i})/G_{<i}$ is free. It then clearly follows that $\text{Hom}(H, \mathbb{Z}) \neq 0$.

Finally, let us show that $\text{Hom}(G, \mathbb{Z}) = 0$. Suppose, by the way of contradiction that $f \in \text{Hom}(G, \mathbb{Z})$ and f is non-zero. By $(*)_B^i(h)$, for each $i < \kappa$, we can find some $\alpha_i < \lambda_i$ and $h_i \in \text{Hom}({}^{(\mathcal{T}_i, \alpha)}\mathbb{Z}, \mathbb{Z})$ such that

$$x \in G_i \Rightarrow f(x) = h_i(\langle f_{\mathbf{m}_i, \eta}(x) : \eta \in \mathcal{T}_{i, \alpha} \rangle).$$

Thanks to Lős' theorem [2, Corollary III. 1.5], for each $i < \kappa$,

$$\text{Hom}({}^{(\mathcal{T}_i, \alpha)}\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{\eta \in (\mathcal{T}_i, \alpha)} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{\eta \in (\mathcal{T}_i, \alpha)} \mathbb{Z}x_\eta.$$

In particular, $\text{Hom}({}^{(\mathcal{T}_i, \alpha)}\mathbb{Z}, \mathbb{Z})$ is free, and it has a natural basis $\langle f_\eta : \eta \in (\mathcal{T}_i, \alpha) \rangle$. It then follows from [2, Corollary III. 3.3], that for some finite set $u_i \subseteq \mathcal{T}_{i, \alpha}$, the following holds:

$$x \in G_i \text{ and } (\forall \eta \in u_i)(f_{\mathbf{m}_i, \eta}(x) = 0) \Rightarrow f(x) = 0.$$

As $\kappa = \text{cf}(\kappa) > \aleph_0$ for some n_* the set $\mathcal{V}_1 = \{i < \kappa : |u_i| = n_*\}$ is unbounded in κ .

For any $i < j < \kappa$, we define the projection map $\text{pr}_{i,j} : \mathcal{T}_j \rightarrow \mathcal{T}_i$ in the natural way by $\text{pr}_{i,j}(\eta) = \eta \upharpoonright (i+1)$. Clearly, $\text{pr}_{i,j}$ maps u_j onto u_i . By Kurepa's theorem, see Fact 1.7, $\mathcal{T}$ has a cofinal branch, which contradicts Claim 2.7. $\square$

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