TWINS: NON-ISOMORPHIC MODELS FORCED TO BE ISOMORPHIC PART I — 1261

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Abstract. For which (first-order complete, usually countable) T do there exist non-isomorphic models of T which become isomorphic after forcing with a forcing notion $\mathbb{P}$? Necessarily, $\mathbb{P}$ is non-trivial; i.e. it adds some new set of ordinals. It is best if we also demand that it collapses no cardinal. It is better if we demand on the one hand that the models are non-isomorphic, and even *far* from each other (in a suitable sense), but on the other hand, $\mathscr{L}$-equivalent in some suitable logic $\mathscr{L}$.

In this part we give sufficient conditions: for theories with the independence property, we prove this when $\mathbb{P}$ adds no new ω-sequence. We may prove it "for some $\mathbb{P}$," but better would be for some specific forcing notions, or a natural family. Best would be to characterize the pairs $(T, \mathbb{P})$ for which we have such models.

The results say (e.g.) that there are models M_1, M_2 which are not isomorphic (and even *far* from being isomorphic, in a rigorous sense) which become isomorphic when we extend the universe by adding a new branch to the tree $({}^{\theta>}2, \triangleleft)$.

We shall mention some specific choices of $\mathbb{P}$: mainly $({}^{\theta>}2, \triangleleft)$ with $\theta = \theta^{<\theta}$.

This work does not require any serious knowledge of forcings, nor of stability theory, though they form the motivation. Concerning forcing, the reader just has to agree that starting with a universe $\mathbf{V}$ of set theory (i.e. a model of ZFC) and a quasiorder $\mathbb{P}$, there are a new directed $\mathbf{G} \subseteq \mathbb{P}$ meeting every dense subset D of $\mathbb{P}$ and a universe $\mathbf{V}[\mathbf{G}]$ (so it satisfies ZFC) of which the original $\mathbf{V}$ is a transitive subclass. We may say that $\mathbf{V}[\mathbf{G}]$ (also denoted $\mathbf{V}^{\mathbb{P}}$) is the universe obtained by forcing with $\mathbb{P}$.

This is part of the classification and so-called *Main Gap* programs.

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References like [Sh:950, Th0.2$_{=\mathsf{Ly5}}$] mean that the internal label of Theorem 0.2 in Sh:950 is $\mathsf{y5}$. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number 1261 in Saharon Shelah's list.

Annotated Content

§ 0. Introduction

We are interested in classifying theories (or classes of models — i.e. structures) by the possible existence of models which are very similar but not isomorphic.

Definition 0.1. 1) For a forcing notion $\mathbb{P}$, we say the models M and N are $\mathbb{P}$-*isomorphic* when they become isomorphic after forcing with $\mathbb{P}$.

2) For $\mathbf{X}$ a set or class of forcing notions, we say M and N are *strongly* $\mathbf{X}$-isomorphic when they are $\mathbb{P}$-isomorphic for every $\mathbb{P} \in \mathbf{X}$.

3) *Weakly* $\mathbf{X}$-isomorphic (or simply '$\mathbf{X}$-isomorphic') will mean"for some $\mathbb{P} \in \mathbf{X}$." E.g. 'weakly ccc-isomorphic' means "for some ccc forcing notion."

Definition 0.2. 1) We say two models are $\mathbb{P}$-*twins* when they are $\mathbb{P}$-isomorphic but not isomorphic. We say they are $(\mathbb{P}, \mathscr{L})$-twins when they are $\mathbb{P}$-twins and $\mathscr{L}$-equivalent, for $\mathscr{L}$ a logic.

2) We say M and N are $(\mathbb{P}, \lambda)$-twins (or $(\mathbb{P}, \mathscr{L}, \lambda)$-twins) when in addition, $\|M\| = \|N\| = \lambda$.

3) Similarly for $\mathbf{X}$-twins and strong $\mathbf{X}$-twins.

4) We may say a theory T [or a class K of models] 'has $\mathbb{P}$-twins.'

Baldwin-Laskowski-Shelah [BLS93] and Laskowski-Shelah [LS96] investigated the case of weak ccc-twins (i.e. $\mathbf{X}$ is the class of ccc forcings).[1] Lately, Farah raised a similar question, for $\mathbb{P}$ the Random Real forcing and T an unstable theory.

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§ 0(A). A panoramic picture – the long-range view.

Thesis 0.3 (The Classification Thesis). We would like to classify the theories T: naturally, at first all complete first-order (maybe countable) ones, but later try for more — e.g. for every AEC.

Like Janus, the thesis has two faces:

(A) Set theoretic test questions which will shed light on the complexity of T, leading to constructing 'complicated' models of a theory T, when T itself is complicated.

(B) Finding dividing lines among the family of theories, such that
- $\bullet_1$ Above the line, we have results as in (A).
- $\bullet_2$ Below the line we develop structure theory, and can analyze models of T to some extent.

(C) The thesis is that those two sides of the program are strongly connected, because if we succeed in proving a case of the so-called *main gap*, we get complementary results. So we know that the assumptions in each are the best possible, and with this aim in mind one is driven to discover inherent properties of T.

(D) Even if you are only interested in clause (B) $\bullet_2$, this thesis tells you that having (A) *and* (B) in mind is a good way to advance each of them.

[1] There, twins were called 'potentially isomorphic.'

(E) For a given test question, a 'Main Gap' theorem will describe how the theories are divided into ones with complicated models (the *non-structure* side), and ones with a 'structure theory.'

But naturally, along the way we may come across other properties which could be of interest, possibly more than the original question (e.g. whether T is stable).

(F) Having the two sides gives us more than the sum of their parts; it proves that both are maximal (in the chosen context), and that those properties are the natural dividing line.

(Of course, not all interesting properties are like this: you may be able to say something about binary functions on a set which is not a group, but this is not the animating question on the class of groups. Closer are o-minimal theories.)

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The classical case was first-order complete countable theories, but there are others; e.g. universal classes up to AEC.

In [She78] and [She90a], the set theoretic test questions were:

- $I(\lambda, T)$:= the number of isomorphism classes of models in $\mathrm{EC}_\lambda(T)$ (= models of T of cardinality λ).
- $IE(\lambda, T)$:= the maximal number of pairwise-non-elementarily embeddable models in $\mathrm{EC}_\lambda(T)$.

In this case, the thesis was that this classification characterizes answers to the question "Is Mod_T (the class of models of T) complicated?", along a significant number of measures.

With regards to those test questions, the situation can be seen in the following trichotomy; the uninitiated reader may concentrate on $\boxplus_2, \boxplus_3$.

This theorem is the original case: the Main Gap Theorem of [She90a].

Theorem 0.4 (The main gap Trichotomy). 1)

$\boxplus_1$ *The countable complete first-order theories T can be divided into three classes:*

(A) *Unstable* <u>*or*</u> *stable but un*super*stable* <u>*or*</u> *superstable with OTOP* <u>*or*</u> *superstable with DOP. (The last two cases tell us some non-first-order formula defining many graphs in some models of T.)*

(B) *T is not in (A), but it is* deep.

(C) *Neither (A) nor (B). (The antonym of deep is* shallow.*)*

2) *The classification in part (1) is by the inside properties of these theories; this is not meaningful if you do not know them.*

Let us move to the other side of the coin.

$\boxplus_2$ *If T is of type (B) or (C) it is called* classifiable*, and satisfies the following:*

(a) *A model M of T can be described by a tree $\mathcal{T}$ with ω levels. That is, it is a set of finite sequences, closed under initial segments (and countably many unary predicates).*

(b) *More fully, there is a tree $\langle M_\eta : \eta \in \mathcal{T} \rangle$ of countable submodels, $\prec$-increasing with η,* "freely joined" *(i.e. this tree of models is non-forking), and M is prime over $\bigcup_{\eta \in \mathcal{T}} M_\eta$.*

(c) *Another aspect is: models of T can be characterized (up to isomorphism) by their theory in the logic $\mathscr{L} = \mathbb{L}_{\infty,\aleph_1}$, enriched by "cardinality quantifiers on dimension by definable dependence relations" (see* [She90a, Ch.XIII], [BS89], [She08b]*).*

3) *Continuing in this fashion:*

$\boxplus_3$ *If T satisfies (A), then a strong negation of the above holds. The class of models is* non-classifiable*: e.g. models (pedantically, isomorphism classes of models) code stationary sets. Specifically, for a model M of T of cardinality λ (λ regular uncountable) we can find an invariant* $\mathrm{inv}(M) = \mathrm{inv}(M/\cong)$ *of the form S/club, for some stationary $S \subseteq \lambda \cap \mathsf{cof}(\aleph_0)$, so that every such S/club occurs (see* [She87b, 2.4, 2.5(2), pp.296-7]*).*

4) *More on $\boxplus_1$ —*

$\boxplus_4$ (a) *If T satisfies $\boxplus_1(A)$ or (B), then for every cardinal λ, T has 2^λ-many pairwise non-isomorphic models of cardinality λ (the maximal number possible).*

(b) *If T satisfies $\boxplus_1(C)$ then $I(\aleph_\alpha, T) < \beth_{\omega_1}(|\alpha|)$ for every ordinal α. So it fails the conclusion of clause (a) when (e.g.)* GCH *holds.*

(c) *Suppose T satisfies $\boxplus_1(C)$. If $\langle M_\alpha : \alpha < \beth_{\omega_1}\rangle$ is a sequence of models of T, then for some $\alpha < \beta < \beth_{\omega_1}$ there is an elementary embedding of M_α into M_β.*

(d) *Suppose T satisfies $\boxplus_1(A)$. Then for every $\lambda > \aleph_0$ there exists a family of 2^λ-many models of T, each of cardinality λ, with no one elementarily embeddable into another.*

(e) *For T that satisfy $\boxplus_1(B)$, their behavior is in the middle — for some cardinal κ (the first so-called* beautiful *cardinal[2]) we have:*

- $\bullet_1$ *If $\lambda \in (\aleph_0, \kappa)$ it behaves as in clause (d) above.*
- $\bullet_2$ *If $\langle M_\alpha : \alpha < \kappa\rangle$ is a sequence of models of T (of any cardinality), then for some $\alpha < \beta < \kappa$ there is an elementary embedding of M_α into M_β.*

(We may say $\kappa = \infty$ when no such cardinal exists.)

But of course, there are other worthwhile measures:

Problem 0.5. What if we ask for which T-s do we have a weaker version of $0.4\boxplus_2$, where we replace $\mathcal{T}$ with a tree with ω_1 levels? I.e. $\mathcal{T}$ consists of sequences of countable length: say, subtrees of $({}^{\omega_1>}\lambda, \triangleleft)$.

This calls for a finer discussion of stable theories.

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§ 0(B). **First approach.** Very similar, but not the same.

True dividing lines (and measures of complexity) discussed above are relevant for a significant set of questions which are not *a priori* connected. A major case is the *Keisler order*, resolved for stable T. (See a recent survey by Keisler [Kei17] on this.) Another measure is the number of $\aleph_1$-*resplendent* models in $\mathrm{EC}_\lambda(T)$, up to isomorphism (see [Shee], which characterizes stable T). Still another direction is building *somewhat rigid* models (see [Shed] and references there).

[2] A *beautiful* cardinal is a large cardinal which is compatible with '$\mathbf{V} = \mathbf{L}$,' but whose existence cannot be proved in ZFC.

There are also works on unstable theories – simple, dependent, and NTP_2 – but here we concentrate on dividing lines among stable T.

We may ask for the number of $|T|^+$-saturated models of T (or complete metric spaces) – see [Shec]. But closer to our problem is the following way to strengthen our non-structure side:

Question 0.6. When do there exist models of a theory T (that is, an elementary class) which are very similar but *not* isomorphic? This question can serve as a yardstick for the complexity of T, and thus makes for a good test problem.

One interpretation of "M and N are very similar" is

- M and N are of cardinality λ, and are equivalent for a 'strong logic' $\mathscr{L}$.

We call this *the first approach.*

Discussion 0.7. We provide references for some relevant works (including those earlier ones asking a more basic question: are there such models not restricting T?).

(A)$_1$ Existence of $\mathbb{L}_{\infty,\lambda}$-equivalent but non-isomorphic models of cardinality λ:
  - For λ regular uncountable, this is an unpublished result of Morley.
  - [She84] covers singular $\lambda = \lambda^{\aleph_0}$.
  - [She94, Ch.II, 7.4-5, p.111] proves it for almost all

$$\lambda^{\aleph_0} > \lambda > \mathrm{cf}(\lambda) > \aleph_0.$$

(A)$_2$ For M_* a model of cardinality λ, what can be said about the value of

$$\mathrm{nu}(M_*) := |K_{M_*}/{\cong}|,$$

where $K_{M_*} := \{M : \|M\| = \lambda,\ M \equiv_{\mathbb{L}_{\infty,\lambda}} M_*\}$?
  - (a) Palyutin [Pal77]: If $\mathbf{V} = \mathbf{L}$ and $\lambda = \aleph_1$, then $\mathrm{nu}(M_*) \in \{1, 2^{\aleph_1}\}$.
  - (b) By [She81a]: if $\mathbf{V} = \mathbf{L}$ and λ is regular uncountable but not weakly compact, then $\mathrm{nu}(M_*) \in \{1, 2^{\lambda}\}$.
  - (c) By [She81b], the '$\mathbf{V} = \mathbf{L}$' in clause (b) is necessary. (That is, it cannot be proved in ZFC.)
  - (d) By [She82], if λ is weakly compact and $\theta \in [1, \lambda]$, <u>then</u> there exists a model M with cardinality λ and $\mathrm{nu}(M) = \theta$.

(A)$_3$ $\mathbb{L}_{\infty,\lambda}$-equivalent but not isomorphic models, for T unsuperstable; see [She87b].

(A)$_4$ Let $\mathscr{L} := \mathbb{L}^{(\mathrm{dim})}_{\infty,\lambda}$, where the '(dim)' means that we add quantifiers saying "λ is the dimension of a definable dependence relation satisfying the Steinitz axioms (e.g. like linear dependence in vector spaces)." By [She90a, Ch.XIII, Th.1.4], for a (countable complete first-order) T we have the following:
  - $\bullet_1$ If T satisfies $\boxplus_1$(B) or (C) of 0.4(1), then any $\mathscr{L}$-equivalent models of cardinality λ are isomorphic. (See also 0.4$\boxplus_2$(c).)
  - $\bullet_2$ If T satisfies 0.4(1)$\boxplus_1$(A) then the conclusion of $\bullet_1$ fails badly (see [She87b]).

To give more details, what we really have is a separation into three classes (recall 0.4(4)$\boxplus_4$(e)).

(B)$_1$ Game quantifier-equivalent but not isomorphic models of cardinality λ:[3] see [Vää95]. See earlier [HS81] with Hodges; also [She06], [HS07] with Havlin, and [She08a].

[3] So this is stronger than $\mathbb{L}_{\infty,\lambda}$-equivalence.

(B)$_2$ For τ-models M and N, let $\mathrm{EF}_{\alpha,\lambda}(M,N)$ denote the Ehrenfeucht-Fraïssé game with α-many moves (α an ordinal), each move adding $< \lambda$ elements.

Like (B)$_1$ for 'dividing line' T-s: see Hyttinen and Tuuri [HT91], and Hyttinen and the author [HS94], [HS95], [HS99].

By [HT91], if T is unstable and $\lambda = \lambda^{<\lambda}$, then there are non-isomorphic models $M, N \in \mathrm{EC}_\lambda(T)$ which are $\eth^{\mathrm{iso}}_\zeta(M,N)$-equivalent for all $\zeta < \lambda$. (I.e. the ISO player has a winning strategy: see Definition 0.20(2).) This also applies to the version using a tree $\mathcal{T} \subseteq {}^{\lambda>}\lambda$ with no λ-branches.

Moreover, "T has OTOP or is superstable with DOP" will suffice. For unsuperstable T the results are weaker.

By [HS94], if T is a complete first-order theory which is stable but not superstable and $\lambda := \mu^+$, where $\mu = \mathrm{cf}(\mu) \geq |T|$, then there are $\mathrm{EF}_{\mu\cdot\omega,\lambda}$-equivalent but non-isomorphic models of T (and even in $\mathrm{PC}(T_1, T)$) of cardinality λ.

See more in [HS95], [HS99].

(C)$_1$ The present work continues [She08b], in a sense. In the second part, we intend to deal with a logic suggested there, suitable to be an analogue of 0.4(3), towards 0.5. We also suggest that a family of stable theories strictly containing the superstables is relevant.

§ 0(C). **Second approach.** The immediate impetus for this work is

Conjecture 0.8 (**The Farah Conjecture**). For a (first-order countable) unstable T, there are non-isomorphic models M, N which become isomorphic when we extend the universe by adding a random real; that is, they are Random-twins.

Farah has proved this for linear orders.

The background behind this question can be found in Baldwin-Laskowski-Shelah [BLS93] and a work with Laskowski [LS96]. There, 'similar' was defined as "ccc-isomorphic" (see Definition 0.1).

Our aim here is to try to sort this out.

For both approaches, a natural dream is to characterize the theories (for now, first-order complete countable) for which this occurs. The Main Gap Theorem of [She90a] had done this for a different test question.

So Baldwin-Laskowski-Shelah [BLS93] and Laskowski-Shelah [LS96] pose (and partly answer) the following problems.

Problem 0.9.
- Characterize the (countable) T with no ccc-twins.
- Characterize the (countable) T such that for some $T_1 \supseteq T$, 'ccc-isomorphic implies isomorphic' holds in $\mathrm{PC}(T_1, T)$. (See 0.14(5).)

To explain the choices in [BLS93], recall the classification made in 0.4. The thesis was that this classification characterizes answers to the question "Is Mod_T (the class of models of T) complicated?", along a significant number of measures: e.g. for $\dot{I}(\lambda, T)$ (the number of isomorphism classes of models of T of cardinality λ) or $IE(\lambda, T)$ (the number of pairwise non-elementarily embeddable models of T of cardinality λ).

Clearly there is a connection between the two approaches.

$\boxtimes_1$ If M and N are non-isomorphic, have different $\mathscr{L}$-theories (for some logic $\mathscr{L}$), and forcing with $\mathbb{P}$ preserves the $\mathscr{L}$-theory of a model, then M and N cannot be $\mathbb{P}$-twins.
(I.e. $\nVdash_{\mathbb{P}}$ "$M \cong N$".)

Now the class of ccc forcings is a natural choice, as it preserves much of what we care about; e.g. if $\mathbb{P}$ collapses cardinals then every T has (trivial) twins — this motivated [BLS93].

There it is proved that:

$\boxtimes_2$ (a) If T is from subclass $\boxplus_1$(A) of 0.4(1), then it has ccc-twins.

(b) However, some theories from $\boxplus_1$(C) have ccc-twins as well.

More fully (quoting [LS96]):

> If T [is superstable and] has only countably many complete types *yet* has a type of infinite multiplicity, then there is a ccc forcing $\mathbb{Q}$ such that in any $\mathbb{Q}$-generic extension of the universe, there are non-isomorphic models M and N which can be forced to be isomorphic by a ccc forcing. We give examples showing that the hypothesis on the number of complete types is necessary.

So we still do not have the answer to our questions:

Question 0.10. 1) Can we characterize the class of T which have ccc-twins?

2) Can we characterize the class of T which have $\mathbb{P}$-twins, where $\mathbb{P}$ is $\aleph_1$-complete and collapses no cardinals?

The motivation for 0.10(2) was the following.

Observation 0.11. *If $\mathbb{P}$ is a forcing notion which collapses no cardinal and adds no ω-sequence, then forcing by $\mathbb{P}$ preserves the $\mathscr{L}$-theory of a model (where $\mathscr{L}$ is as in* 0.7$(A)_4$ *from* §0(B)*).*

Here we mainly tried to deal with 0.10(2), but after [BLS93] and [LS96] further work was delayed. However, Farah's Conjecture gives us new inspiration to look at these questions again, for one specific ccc forcing.

This conjecture remains open. In general, we may ask these questions for any fixed forcing notion $\mathbb{P}$: compare to [BLS93] and [LS96], where we asked about "there is a ccc $\mathbb{P}$." (Instead of '$(\exists\mathbb{P})$' or $\mathbb{P}$ being specified in the question, we may even try $(\forall\mathbb{P})$.)

Note that

$\boxtimes_4$ (a) If $\theta = \theta^{<\theta} > \aleph_0$ then $\mathsf{Cohen}_\theta := ({}^{\theta>}2, \triangleleft)$ is a forcing as in 0.10(2).

(b) Any Suslin tree $\mathcal{T}$ is a ccc forcing as in 0.10(2).

Anyhow, if $\mathbb{P}$ is an NNS[4] forcing then 0.12 answers 0.10(2).

$\boxplus_5$ If $\theta = \theta^{<\theta} > \aleph_0$ then after forcing with Cohen_θ, '$\theta = \theta^{<\theta} > \aleph_0$' still holds, so we have existence theorems (see below).

This is the approach taken in [LS96] for the class of ccc forcing notions.

§ 0(D). **The results.** In this part, we concentrate on independent (first-order) T and the forcing Cohen_θ (for $\theta = \theta^{<\theta} > \aleph_0$), because that is where the statements and their proofs are most transparent.

In §1-3, we will prove

[4] *NNS* means "adds no new ω-sequence."

Theorem 0.12. 1) *Assume T is a complete countable first-order theory which has the independence property, and $\mathbb{P}$ is $\aleph_1$-complete (or just is an NNS forcing.)* Then *T has models M and N which are $\mathbb{P}$-twins. (I.e. $M \not\cong N \wedge \Vdash_{\mathbb{P}}$ "$M \cong N$".)*

2) *Moreover, M and N are* far *from each other, as defined in* 1.10.

3) *If $T_1 \supseteq T$ is also first-order countable,* then *we can add* "PC(T_1, T) *has $\mathbb{P}$-twins." (Specifically, M and N can be expanded to models of T_1.)*

In Part II of this work [S+a], we shall deal with unstable T (e.g. for $\mathbb{P} := \mathsf{Cohen}_\theta$ with $\theta = \theta^{<\theta} > \aleph_0$) and with ccc forcing notions (e.g. Random Real) for some of those T-s. However, some stable but unsuperstable theories fail the conclusion of 0.12.

We will continue both approaches, using the relations from [BS85] (e.g. weaker versions of entangledness). Definitions here will be phrased so as to apply to Part II as well.

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§ 0(E). Preliminaries.

Notation 0.13. We will try to use standard notation.

1) $\theta, \kappa, \lambda, \mu, \chi$ will denote cardinals (infinite, if not stated otherwise). λ^+ will denote the successor of λ.

2) $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi$, i, and j will denote ordinals. δ will be a limit ordinal unless explicitly said otherwise.

3) k, ℓ, m, n will denote natural numbers.

(We may abuse this somewhat and use them as indices for ordinals $< \kappa$, in statements where the default case or usual formulation is $\kappa := \aleph_0$; if so, we will mention it explicitly.)

4) φ, ψ, and ϑ will be formulas; first-order, if not said otherwise.

5) For cardinals $\kappa < \lambda = \mathrm{cf}(\lambda)$, let

$$S^\lambda_\kappa := \big\{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\big\}$$

and

$$S^\lambda_{\leq\kappa} := \big\{\delta < \lambda : \mathrm{cf}(\delta) \leq \mathrm{cf}(\kappa)\big\}.$$

6) $\mathsf{cof}(\mu)$ will denote the class of ordinals with cofinality equal to $\mathrm{cf}(\mu)$.

7) λ^+ may be written $\lambda(+)$ (and e.g. a_i may be written $a[i]$) when they appear in a superscript or subscript.

8) $\bar{x}_{[u]} := \langle x_i : i \in u\rangle$

9) Forcing notions will be denoted by $\mathbb{P}$ and $\mathbb{Q}$. We adopt the Cohen convention that '$p \leq q$' means that q gives more information (as conditions in a forcing notion).

10) $\trianglelefteq$ means 'is an initial segment,' and $\triangleleft$ means it is proper.

11) $\mathcal{T}$ is a partial order or quasiorder, not necessarily a tree.

(Originally they were trees, but we later found it better to drop this — see the end of §2A. But it would be no problem to resurrect it in the future.)

12) We use $\mathbf{p}$ to denote *twinship parameters* (see Definition 2.2) and $\mathbf{m}$ for forcing examples: see §2B.

Notation 0.14. 1) τ will denote a vocabulary: that is, a set of predicates and function symbols of finite *arity* (that is, a finite number of places). Functions and individual constants are treated as predicates.

2) For models or structures, $\tau(M)$, $\tau(I)$, etc. are defined naturally, as their vocabularies.

3) $\mathscr{L}$ will denote a logic. $\mathbb{L}$ is first order logic, $\mathbb{L}_{\lambda,\mu}$ the usual infinitary logic.
$\mathscr{L}(\tau)$ is the language: that is, a set of formulas $\varphi(\bar{x})$ for the logic $\mathscr{L}$ in the vocabulary τ.

4) T will denote a theory; complete first-order in the vocabulary $\tau_T = \tau(T)$, if not said otherwise. For simplicity, it will have elimination of quantifiers. (Particularly in §0, we may forget to say 'countable.')

5) For such T,

$$\mathrm{EC}_\lambda(T) := \{M \models T : \|M\| = \lambda\}$$
$$\mathrm{EC}(T) := \bigcup_{\lambda \in \mathrm{Card}} \mathrm{EC}_\lambda(T).$$

For $T_1 \supseteq T$,

$$\mathrm{PC}_\lambda(T_1, T) := \{M \restriction \tau_T : M \in \mathrm{EC}_\lambda(T_1)\}$$
$$\mathrm{PC}(T_1, T) := \bigcup_{\lambda \in \mathrm{Card}} \mathrm{PC}_\lambda(T_1, T).$$

Definition 0.15. For $\mathbb{P}$ a forcing notion, we define:

(A) $\kappa(\mathbb{P}) := \min\{\kappa : \Vdash_{\mathbb{P}}$ "there is a new $A \subseteq \kappa$"$\}$

(B) $\mathrm{spec}(\mathbb{P}) :=$

$$\big\{(\kappa, \lambda, \mathcal{T}) : \mathcal{T} \text{ is a subtree of } {}^{\kappa>}\lambda \text{ of cardinality } \lambda \text{ such that forcing with } \mathbb{P} \text{ adds a } \underline{\text{new}}\ \eta \in \lim_\kappa(\mathcal{T})\big\}$$

(I.e. $\eta \in {}^\kappa\lambda \setminus \mathbf{V}$ with $\varepsilon < \kappa \Rightarrow \eta \restriction \varepsilon \in \mathcal{T}$.)

(C) "$(\kappa, \lambda) \in \mathrm{spec}(\mathbb{P})$" will be shorthand for $\big(\exists \mathcal{T}\big)\big[(\kappa, \lambda, \mathcal{T}) \in \mathrm{spec}(\mathbb{P})\big]$.

<u>**Question:**</u> will $\mathrm{spec}(\mathbb{P})$ be interesting? Do we use $\mathcal{T}$ or just use a "new" directed $\mathbf{G} \subseteq \mathbb{P}$?

Convention 0.16. If not stated otherwise, we assume $\mathbb{P}$ is such that

$$\big(\forall p \in \mathbb{P}\big)\big[\kappa(\mathbb{P}_{\geq p}) = \kappa(\mathbb{P})\big]$$

(where $\mathbb{P}_{\geq p} := \mathbb{P} \restriction \{q : q \geq_{\mathbb{P}} p\}$).

This choice can be justified by the following observation.

Observation 0.17. *For any forcing notion $\mathbb{P}$ there is a maximal antichain $\mathbf{I}$ such that for all $p \in \mathbf{I}$, either (a) or (b) holds:*

(*a*) *$\mathbb{P}_{\geq p}$ is a trivial forcing: i.e. it is a directed quasiorder. (This means that any two members are* compatible, *in that they have a common upper bound.)*

(*b*) *$\kappa(\mathbb{P}_{\geq p})$ is a well-defined regular uncountable cardinal $\leq \|\mathbb{P}_{\geq p}\| \leq \|\mathbb{P}\|$, and there is a minimal $\lambda \leq \|\mathbb{P}\|$ such that $(\kappa, \lambda) \in \mathrm{spec}(\mathbb{P}_{\geq p})$.*

Proof. Obvious, but we shall elaborate.

Let $\mathbf{I}$ be a maximal antichain of $\mathbb{P}$ included in

$$\{p \in \mathbb{P} : \text{either } \mathbb{P}_{\geq p} \text{ is directed } \underline{\text{or}} \text{ there does not exist a } q \geq p \text{ such that } \mathbb{P}_{\geq q} \text{ is directed}\}.$$

It will suffice to prove that this $\mathbf{I}$ is as promised. It will also suffice to deal with a non-directed $\mathbb{P}_{\geq p}$, where $p \in \mathbf{I}$.

Let $\mu := \|\mathbb{P}_{\geq p}\|$ and let $f : \mathbb{P}_{\geq p} \to \mu$ be a bijection. Let $\underset{\sim}{\eta} \in {}^{\mu}2$ be the $\mathbb{P}$-name satisfying

$$p \Vdash \text{`}\underset{\sim}{\eta}(\alpha) = 1\text{' } \underline{\text{iff}}\ p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}.$$

Let $\Lambda_1 := \big\{(\kappa, \lambda, \underset{\sim}{\nu}) : \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\nu} \in {}^{\kappa}\lambda \text{ is not from } \mathbf{V}, \text{ but } \varepsilon < \kappa \Rightarrow \underset{\sim}{\nu} \restriction \varepsilon \in \mathbf{V}\text{”}\big\}$

Clearly $(\mu, \mu, \underset{\sim}{\eta}) \in \Lambda_1$. Hence there is a minimal κ_* such that the set

$$\big\{(\lambda, \underset{\sim}{\nu}) : (\kappa_*, \lambda, \underset{\sim}{\nu}) \in \Lambda_1\big\}$$

is non-empty. Similarly, there is a minimal λ_* such that $(\kappa_*, \lambda_*, \underset{\sim}{\nu}) \in \Lambda_1$ for some $\underset{\sim}{\nu}$. Let

$$\mathcal{T} := \big\{\varrho \in {}^{\kappa_* >}\lambda_* : (\exists q \in \mathbb{P}_{\geq p})\big[q \Vdash \text{“}\varrho \lhd \underset{\sim}{\nu}_*\text{”}\big]\big\}.$$

Now $\mathcal{T}, \kappa_*, \lambda_*, \underset{\sim}{\nu}$ are as promised in 0.15. $\square_{0.17}$

Definition 0.18. The following definition will be used mainly in 3.2.

(a) $\tau(\mu, \kappa) = \tau_{\mu,\kappa}$ is the vocabulary with function symbols

$$\{F_{i,j} : i < \mu,\ j < \kappa\},$$

where $F_{i,j}$ is a j-place function symbol and κ is a regular cardinal.

(b) $\mathcal{M}_{\mu,\kappa}(I)$ is the free $\tau_{\mu,\kappa}$-algebra generated by I.

(c) We may write $\mathcal{M}_{\mu}(I)$ when $\kappa = \aleph_0$, and $\mathcal{M}(I)$ when $\mu = \kappa = \aleph_0$.

Remark 0.19. Concerning the first approach (see §0B) we will define some games which witness the equivalence of two models in some strong logic.

Definition 0.20. 1) We say the models M and N are *cofinally* (λ, ζ)*-equivalent* <u>when</u> there exist $\subseteq$-increasing sequences $\overline{M} = \langle M_\alpha : \alpha < \lambda\rangle$ and $\overline{N} = \langle N_\alpha : \alpha < \lambda\rangle$ satisfying the following.

(A) $M = \bigcup\limits_{\alpha<\lambda} M_\alpha$ and $N = \bigcup\limits_{\alpha<\lambda} N_\alpha$.

(B) The pro-isomorphism player ISO has a winning strategy in the game $\partial^{\text{iso}}_{\zeta}(\overline{M}, \overline{N})$ defined below.

A play of the game $\partial^{\text{iso}}_{\zeta}(\overline{M}, \overline{N})$ between the players ISO and ANTI lasts ζ-many moves. In the ε^{th} move, the ANTI player chooses $\alpha_\varepsilon \in \big(\bigcup\limits_{\xi<\varepsilon} \alpha_\xi, \lambda\big)$ and the ISO player responds with an isomorphism $f_\varepsilon : M_{\alpha_\varepsilon} \to N_{\alpha_\varepsilon}$ extending $\bigcup\limits_{\xi<\varepsilon} f_\xi$.

2) If $\|M\| = \|N\| = \lambda$ (and for transparency, both have universe λ) then we may demand that the isomorphism f_ε be a function from α_ε onto α_ε.

3) In part (1), we may replace the ordinal ζ by a tree $\mathcal{T}$ with λ-many levels and no λ-branch.

By this we mean: in the ε^{th} move of a play of $\partial^{\text{iso}}_{\mathcal{T}}(\overline{M}, \overline{N})$, ANTI starts by choosing t_ε, a member of the ε^{th} level of $\mathcal{T}$ which is $\lhd$-above t_ξ for all $\xi < \varepsilon$, and then $\alpha_\varepsilon \in \big(\bigcup\limits_{\xi<\varepsilon} \alpha_\xi, \lambda\big)$. Then ISO chooses $f_\varepsilon : M_{\alpha_\varepsilon} \to N_{\alpha_\varepsilon}$ as before. A player loses if they have no legal move on their turn.

Note that α_ε chosen exactly as in part (1), and does not depend on t_ε. The tree simply functions as the game's 'clock:' if ISO chooses a valid f_ε and ANTI has no valid $t_{\varepsilon+1}$, then ISO wins the play.

E.g. we have (in other variants of the game we get equivalence):

Claim 0.21. *If M and N are two models of cardinality $\lambda \in \mathrm{Reg}$ and are cofinally (λ, ω)-equivalent, then they are $\mathbb{L}_{\infty,\lambda}$-equivalent.*

The following property of linear orders will be used for proving that models are not isomorphic.

Definition 0.22. A model J (usually a linear order) has the λ-*indiscernibility* property when:

> If $\bar{t}_\varepsilon \in {}^{\omega >}J$ for $\varepsilon < \lambda$, then for some $A \in [\lambda]^\lambda$, the sequence $\langle \bar{t}_\varepsilon : \varepsilon \in A \rangle$ is indiscernible for the quantifier-free formulas.

Fact 0.23. If λ is regular uncountable, then any well-ordered set has the λ-indiscern-ibility property.

§ 1. GEM models

Below, the reader may concentrate on $K_{\rm or}$, $K_{\rm org}$, the order property, and the independence property.

Recall

Notation 1.1. 1) Let K denote a class of index models (i.e. structures) which have the Ramsey property. (See [She87c, 1.10, p.330], [Sheb, 1.15$_{=\mathsf{Lc2}}$].) Members of K will be denoted by I and J; we shall use them for constructing generalized Ehrenfeucht-Mostowski models $\mathrm{GEM}(I,\Phi)$. Φ (or Ψ) is called the *blueprint*, and $\mathbf{a} = \langle \bar{a}_s : s \in I\rangle$ will denote the *skeleton*.

2) We may write $K_{\rm x}$ for (e.g.) $\mathrm{x} \in \{\mathrm{or}, \mathrm{org}, \mathrm{org}(\mathbf{n}), \mathrm{tr}(\omega), \mathrm{tr}(\kappa), \mathrm{tr}(\bar{\mathbf{n}}), \mathrm{oi}(\partial)\}$.
In this case $K^{\rm x}_\lambda := \{I \in K_{\rm x} : \|I\| = \lambda\}$.

Now we can define GEM models (*Generalized Ehrenfeucht-Mostowski* models) for K. On this, see [She87c, Ch.III, 1.6, p.329] (revised in [Sheb, §1B, 1.8$_{=\mathsf{Lb8}}$, p.9]).

This usually requires generalizing Ramsey's theorem. Some examples of relevant classes:

Example 1.2. (A) $K = K_{\rm or}$: the class of linear orders.

(B) $K = K^{\rm tr}_\omega = K_{\mathrm{tr}(\omega)}$: trees with $\omega+1$ levels. We have P_i for $i \leq \omega$, $\lhd$ the tree order, and $<_{\rm lex}$ the lexicographical order.
(See [Sheb, 1.9(4)$_{=\mathsf{Lb11}}$].)

(B)$_\kappa$ $K = K_{\mathrm{tr}(\kappa)} = K^{\rm tr}_\kappa$: similarly, but with $\kappa+1$ levels (so we have restriction functions $\restriction_{i,j}$). (See [She87c, 1.7(4), p.328], [Sheb, 1.9(4)$_{=\mathsf{Lb11}}$].)

(C) $K_{\rm org}$: linearly ordered graphs.
(See 2.6, and more in [Sheb, 1.18(5)$_{=\mathsf{Lc14}}$].)

(D) $K_{\rm dorg}$ (*directed ordered graphs*). Like $K_{\rm org}$, but the graph is directed. (We may also consider it as an undirected graph.)

(E) $K_{\mathrm{org}(\mathbf{n})}$ for $\mathbf{n} \in [2,\omega]$ (see [S$^+$a]).

(F) $K_{\mathrm{ptr}(\sigma)} = K^\sigma_{\rm ptr}$ and $K_{\mathrm{tr}(\bar{\mathbf{n}})} = K^\sigma_{\mathrm{tr}(\bar{\mathbf{n}})}$ (for $\bar{\mathbf{n}} = \langle n_i : i < \sigma\rangle \in {}^\sigma\omega$) are as in [Shea, 1.1(1)$_{=\mathsf{L1.1}}$, 1.2$_{=\mathsf{L1.2}}$], respectively.
(Also called $K_{\mathrm{str}(\bar{\mathbf{n}})}$; see more in [S$^+$b, Def. 5.1$_{=\mathsf{Ls1}}$, p.30].)

(G) $K_{\mathrm{oi}(\gamma)}$; see [She08b, Def. 2.1$_{=\mathsf{L2b.1}}$].

(H) For any $K_{\rm x}$ such that $E, F_{\eta,\iota} \notin \tau(K_{\rm x})$ for $\eta \in \mathcal{T}_{\mathbf{p}}$ and $\iota = \pm 1$, we define $K^{\rm x}_{\mathcal{T},\iota}$ — the main case in this paper — in 2.6(1),(2).

Definition 1.3. 1) For T a theory (not necessarily first-order), K as in 1.1(1), and κ a cardinal, we define $\Upsilon_K[T,\kappa] = \Upsilon[T,\kappa,K]$ as the class of K-*GEM blueprints* Φ (see [She87c, Ch.III, 1.6, p.329], [Sheb, 1.8$_{=\mathsf{Lb8}}$]).

$\boxplus$ For $I \in K$, $M = M_I \in \mathrm{GEM}(I,\Phi)$ is a τ_Φ-model with skeleton $\mathbf{a}$.
Pedantically, $(M,\mathbf{a}) \in \mathrm{GEM}(I,\Phi)$ satisfies

(a) τ_Φ is a vocabulary, and M is a τ_Φ-model.

(b) $\tau_\Phi \supseteq \tau_T$ and $\mathrm{GEM}_{\tau_T}(I,\Phi) = \mathrm{GEM}(I,\Phi) \restriction \tau_T \models T$.

(c) τ_Φ is of cardinality $\leq \kappa$.

(d) Φ is a function with domain

$$\mathrm{QFT}_K := \big\{\mathrm{tp}(\bar{s},\varnothing,J) : J \in K,\ \bar{s} \in {}^{\omega>}J\big\},$$

and $\Phi\big(\mathrm{tp}_{\rm qf}(\bar{s},\varnothing,J)\big)$ is a complete $\mathbb{L}(\tau_\Phi)$-quantifier-free type.
('qf' means *quantifier-free.*)

(e) $\mathbf{a} = \langle \bar{a}_s : s \in I\rangle$ is the so-called *skeleton* of M.

(f) $\ell g(\bar{a}_s) = k_\Phi$ (So members of the skeleton are k-tuples. For simplicity, we will usually have $k_\Phi := 1$. If so, we may write a_s instead of $\langle a_s \rangle$.)

(g) M is the closure of $\{\bar{a}_s : s \in I\}$.

(h) $\mathbf{a}$ is *qf-indiscernible* in $\mathrm{GEM}(I, \Phi)$, where 'qf-indiscernible' is defined as in clause $\oplus_{\mathrm{qf}}$ below.

(i) If $\bar{s} \in {}^\varepsilon I$ then $\bar{\bar{a}}_{\bar{s}} = \langle \ldots {}^\frown \bar{a}_{s_\zeta} {}^\frown \ldots \rangle_{\zeta<\varepsilon}$.
(So if $k_\Phi := 1$ then $\bar{a}_{\bar{s}} = \langle a_{s_\zeta} : \zeta < \varepsilon \rangle$.)

$\oplus_{\mathrm{qf}}$ <u>If</u> $(s_0, \ldots, s_{n-1}), (t_0, \ldots, t_{n-1}) \in {}^n I$ realize the same quantifier-free type in I, <u>then</u> $\bar{a}_{s_0} {}^\frown \ldots {}^\frown \bar{a}_{s_{n-1}}$ and $\bar{a}_{t_0} {}^\frown \ldots {}^\frown \bar{a}_{t_{n-1}}$ realize the same quantifier-free type in $\mathrm{GEM}(I, \Phi)$.

1A) Of course, we are really interested in $\mathrm{GEM}_{\tau_T}(I, \Phi) = \mathrm{GEM}(I, \Phi) \restriction \tau_T$.

1B) As implied above, we define $\mathrm{GEM}_\tau(I, \Phi) := \mathrm{GEM}(I, \Phi) \restriction \tau$ for $\tau \subseteq \tau_\Phi$.

2) We may write $\Upsilon^K_\kappa(T)$ of $\Upsilon_K(T, \kappa)$, and we may write $\Upsilon^{\mathrm{x}}_\kappa(T)$ for $K := K_{\mathrm{x}}$ (e.g. $\Upsilon^{\mathrm{or}}_\kappa(T)$, $\Upsilon^{\mathrm{org}}_\kappa(T)$, $\Upsilon^{\mathrm{tr}(\omega)}_\kappa(T)$ for $K := K_{\mathrm{or}}, K_{\mathrm{org}}, K_{\mathrm{tr}(\omega)}$, respectively).

The following definition also applies to non-first-order T (and/or φ, or replace $\mathrm{EC}(T)$ by a class of models). When both are first-order, by the compactness theorem it suffices to use $\mu := \aleph_0$.

Definition 1.4. 1)

(A) We say that $\varphi = \varphi(\bar{x}_{[k]}, \bar{y}_{[k]})$ *witnesses that T has the $(<\lambda)$-order property* (not necessarily first-order) <u>when</u> for every $\mu < \lambda$, there is $M \in \mathrm{EC}(T)$ and $\langle \bar{a}_\alpha : \alpha < \mu \rangle \subseteq {}^\mu({}^k M)$ such that $M \models \varphi[\bar{a}_\alpha, \bar{a}_\beta]^{\mathsf{if}(\alpha<\beta)}$.

(B) Let T be first-order complete. We say φ *witnesses that T is unstable* <u>if</u> φ is first-order and T has the $\aleph_0$-order property as witnessed by φ.

2)

(A) We say that $\varphi = \varphi(\bar{x}_{[k]}, \bar{y}_{[k]})$ *witnesses that T has the $(<\lambda)$-independence property* (not necessarily first-order) <u>when</u> for every $\mu < \lambda$ and graph $G = (\mu, R)$ on μ, there are $M \in \mathrm{EC}(T)$ and $\langle \bar{a}_\alpha : \alpha < \mu \rangle \subseteq {}^k M$ such that

$$\alpha < \beta < \mu \Rightarrow \big[M \models \varphi[\bar{a}_\alpha, \bar{a}_\beta] \Leftrightarrow G \models \text{“}\alpha \ R \ \beta\text{”} \big].$$

(B) Let T be first-order complete. We say T is *independent* (or *has the independence property*) <u>when</u> some first-order $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ witnesses the $\aleph_0$-independence property.

Remark 1.5. For all the examples in 1.2, for the relevant (first-order) T, φ, or $\bar{\varphi}$, there exists a suitable Φ. (Usually we need that K satisfies the Ramsey property; see [She87c, III] or [Sheb, §1].)

E.g.

Claim 1.6. 1) *If T is first-order unstable (as witnessed by $\varphi = \varphi(\bar{x}, \bar{y})$) and $T_1 \supseteq T$, <u>then</u> there is $\Phi \in \Upsilon_{\mathrm{or}}(T_1)$ such that:*

(a) $|\tau_\Phi| = |T_1|$

(b) $k_\Phi = \ell g(\bar{x}) = \ell g(\bar{y})$

(c) $\mathrm{GEM}(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\mathsf{if}(s <_I t)}$

2) *Let $\mu \geq \beth((2^\lambda)^+)$. If $T_1 \subseteq \mathbb{L}_{\lambda^+, \aleph_0}(\tau_1)$ is of cardinality $\leq \lambda$ and has the $(<\mu)$-order property, as witnessed by $\varphi(\bar{x}_{[k]}, \bar{y}_{[k]})$, <u>then</u> the conclusion in part (1) holds. (See more in* [Sheb].*)*

Note that the definition below formalizes the statements in 1.5-1.6.

Definition 1.7. 1) For $\Phi \in \Upsilon_\kappa[K]$, we say Φ *represents* (φ, R) when:

(A) $R \in \tau(K)$ has arity n.

(B) $\varphi = \varphi(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \mathscr{L}(\tau_\Phi)$ for some logic $\mathscr{L}$, with $\ell g(\bar{x}_\ell) = k_\Phi$.

(C) If $I \in K$, $(M, \bar{\mathbf{a}}) = \mathrm{GEM}(I, \Phi)$, and $\bar{t} \in {}^n I$, then

$$M \models \varphi[\bar{a}_{t_0}, \ldots, \bar{a}_{t_{n-1}}] \Leftrightarrow (t_0, \ldots, t_{n-1}) \in R^I.$$

2) We may write φ instead of (φ, R) when R is clear from the context. (E.g. it is '$x < y$' for K_{or}, and $x\ R\ y$ in K_{org}.)

3) Similarly for "Φ represents $(\bar{\varphi}, \overline{R})$," where $\bar{\varphi} = \langle \varphi_\varepsilon(\bar{x}_0, \ldots, \bar{x}_{m_\varepsilon - 1}) : \varepsilon < \kappa \rangle$.

The following definitions are natural ways to make demands even stronger than "M_1 and M_2 are not isomorphic."

Definition 1.8. 1) For τ-models M_1, M_2 and $\varphi = \varphi(x_0, \ldots, x_{n-1}) \in \mathscr{L}(\tau)$, we say M_1 is *(λ, φ)-far from M_2* when there is a *witness* $\langle a_\alpha : \alpha < \lambda \rangle \in {}^\lambda(M_1)$.

By *this*, we mean:

If $\mathcal{U} \in [\lambda]^\lambda$ and $(\forall \alpha \in \mathcal{U})[b_\alpha \in M_2]$, then for some $\alpha_0 < \ldots < \alpha_{n-1}$ from $\mathcal{U}$, we have

$$M_1 \models \varphi[a_{\alpha_0}, \ldots, a_{\alpha_{n-1}}] \Leftrightarrow M_2 \models \neg\varphi[b_{\alpha_0}, \ldots, b_{\alpha_{n-1}}].$$

2) If $\varphi = \varphi(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \mathscr{L}(\tau)$ with $(\forall \ell < n)[\ell g(\bar{x}_\ell) = k]$ for some $k < \omega$, then above we will write $\bar{a}_\alpha \in {}^k M_1$, $\bar{b}_\alpha \in {}^k M_2$.

3) We say M_1 and M_2 *are (λ, φ)-far* when M_1 is (λ, φ)-far from M_2 and M_2 is (λ, φ)-far from M_1.

4) For a logic $\mathscr{L}$, τ-models M_1, M_2, and $\varphi = \varphi(\bar{x}) \in \mathscr{L}(\tau)$, we say f is a *φ-embedding of M_1 into M_2* when for every $\bar{a} \in {}^{\ell g(\bar{x})} M_1$ we have

$$M_1 \models \varphi[\bar{a}] \Leftrightarrow M_2 \models \varphi[f(\bar{a})].$$

5) "$\bar{\varphi}$-far" is defined similarly, for $\bar{\varphi} = \langle \varphi_n : n \in [2, \mathbf{n}] \rangle$.

Remark 1.9. 1) In 1.8(1)-(5), if M_1 and M_2 are $\mathbb{P}$-twins (see Definition 0.1(1), 0.2(1)) and

$$\|\mathbb{P}\| < \lambda = \mathrm{cf}(\lambda) \le \max(\|M_1\|, \|M_2\|),$$

then M_1 and M_2 cannot be 'far' in any of those definitions.

[Why? Let $\bar{a}_\alpha \in {}^k M_1$ for $\alpha < \lambda$; let

$$\Vdash_{\mathbb{P}} \text{"}\underset{\sim}{f} : M_1 \to M_2 \text{ is an isomorphism (or just a } \varphi\text{-embedding)"}.$$

For each $\alpha < \lambda$ we can choose $p_\alpha \in \mathbb{P}$ and $\bar{b}_\alpha \in {}^k M_2$ such that $p_\alpha \Vdash$ "$\underset{\sim}{f}(\bar{a}_\alpha) = \bar{b}_\alpha$". As $\|\mathbb{P}\| < \lambda$, there is some $p_* \in \mathbb{P}$ such that the set $\{\alpha < \lambda : p_\alpha = p_*\}$ has cardinality λ. So recalling Definition 1.8(1), $\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ cannot be a witness for "M_1 is (λ, φ)-far from M_2."]

So Definitions 1.10(1),(2) below are an attempt to formulate notions of being 'far' for which we might try to build examples.

2) In clause 1.10(3)(C), we may demand that α_i is increasing with i.

3) Definition 1.10 will be used in 3.6 and its relatives.

4) Sometimes negation is not so handy. Then we may replace φ by a pair $\langle \varphi_{\mathsf{pos}}, \varphi_{\mathsf{neg}} \rangle$ in 1.8+1.10, where $\varphi_{\mathsf{pos}} = \varphi_{\mathsf{pos}}(\bar{x}_0, \ldots, \bar{x}_{n-1})$ (and similarly for φ_{neg}).

What this means is that, in all occurrences,

- $\bullet_1$ If $M_1 \models \varphi_{\mathsf{pos}}[\bar{a}_0, \ldots]$ then $M_2 \models \varphi_{\mathsf{pos}}[f(\bar{a}_0), \ldots]$, for $\bar{a}_0, \ldots, \bar{a}_{n-1} \in {}^k M_1$.
- $\bullet_2$ If $M_1 \models \varphi_{\mathsf{neg}}[\bar{a}_0, \ldots]$ then $M_2 \models \varphi_{\mathsf{neg}}[f(\bar{a}_0), \ldots]$, for $\bar{a}_0, \ldots, \bar{a}_{n-1} \in {}^k M_1$.
- $\bullet_3$ For no $\bar{a}^\ell_0, \ldots \bar{a}^\ell_{n-1} \in {}^k M_\ell$ (with $\ell \in \{1, 2\}$) do we have

$$M_\ell \models \varphi_{\mathsf{pos}}[\bar{a}^\ell_0, \ldots, \bar{a}^\ell_{n-1}] \wedge \varphi_{\mathsf{neg}}[\bar{a}^\ell_0, \ldots, \bar{a}^\ell_{n-1}].$$

Definition 1.10. 1) For τ-models M_1, M_2 and $\varphi = \varphi(x_0, \ldots, x_{n-1}) \in \mathscr{L}(\tau)$, we say M_1 is $(\lambda, \sigma, \varphi)$-*far from* M_2 <u>when</u> there is a witness $\langle \mathbf{a}_i : i < \sigma \rangle$, where $\mathbf{a}_i = \langle a_{i,\alpha} : \alpha < \lambda \rangle \in {}^\lambda(M_1)$.

By *this*, we mean:

If $\mathcal{U}_i \in [\lambda]^\lambda$ for $i < \sigma$, <u>then</u> there does not exist a function

$$f : \{a_{i,\alpha} : i < \sigma,\ \alpha \in \mathcal{U}_i\} \to M_1$$

preserving the satisfaction of φ.

2) Similarly for $\varphi = \varphi(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \mathscr{L}(\tau)$ with $\ell g(\bar{x}_0) = \ldots = \ell g(\bar{x}_{n-1}) = k$.

3) We may replace φ by $\Delta \subseteq \mathscr{L}(\tau)$, or by $\langle \Delta_u : u \in [\sigma]^{<\aleph_0} \rangle$, with the natural meaning:

- (A) $\Delta_u \subseteq \big\{ \varphi(\ldots, \bar{x}_i, \ldots)_{i \in u} : \varphi \in \mathscr{L}(\tau) \big\}$
- (B) $\ell g(\bar{a}_{i,\alpha}) = \ell g(\bar{x}_i) = k_i$
- (C) There are $\mathcal{U}_i \in [\lambda]^\lambda$ and $\bar{b}_{i,\alpha} \in {}^{k_i}(M_2)$ (for $i < \sigma$ and $\alpha \in \mathcal{U}_i$) such that <u>if</u> $u \in [\sigma]^{<\aleph_0}$, $\varphi(\ldots, \bar{x}_i, \ldots)_{i \in u} \in \Delta_u$, and $\alpha_i \in \mathcal{U}_i$ for $i \in u$, <u>then</u>

$$M_1 \models \varphi[\ldots, \bar{a}_{i,\alpha_i}, \ldots]_{i \in u} \Leftrightarrow M_2 \models \varphi[\ldots, \bar{b}_{i,\alpha_i}, \ldots]_{i \in u}.$$

4) We may add $\langle n_i : i < \sigma \rangle$ with $n_i \leq \omega$, so $\varphi \in \Delta_u$ may have $\leq n_i$ copies of $\bar{x}_i$ in our block of arguments.

§ 2. Toward ℙ-twins

The idea below is that $\mathcal{T}_{\mathbf{p}}$ is a forcing notion. However, sometimes we do not use the forcing notion we are interested in, but rather a derived one (e.g. for Sacks forcing, we may use $\mathcal{T}_{\mathbf{p}} := ({}^{\omega>}2, \triangleleft)$). Even if $\mathcal{T}_{\mathbf{p}}$ is the forcing notion which interests us, $\mathcal{B}_{\mathbf{p}}$ (see 2.2) will not necessarily be the family of all dense open subsets of $\mathcal{T}_{\mathbf{p}}$.

§ 2(A). **The Frame.**

Convention 2.1. If not stated otherwise, $\mathbf{p}$ is a fixed weak twinship parameter (as in Definition 2.2), although we will sometimes have different definitions for the tree-like and non-tree-like version.

Earlier we thought it necessary to assume $\mathcal{T}$ is a tree; now it seems this is not necessary, but neither option would be harmful to us so far.

Definition 2.2. 1) We say $\mathbf{p}$ is a *weak twinship parameter* (or simply 'a twinship parameter') when it consists of:[5]

(A) A partial order $\mathcal{T}$ such that any two elements $\eta, \nu \in \mathcal{T}$ have a maximal lower bound[6] (call it $\eta \wedge \nu$).

(B) $\theta = \mathrm{cf}(\theta) \geq \aleph_0$

(C) $\mathcal{B}$, a family of subsets of $\mathcal{T}$ satisfying the following.

- (a) $\mathcal{B}$ is closed under finite intersections, and each $D \in \mathcal{B}$ is dense in $\mathcal{T}$.
- (b) If $\eta \in \mathcal{T}$ then $\{\nu \in \mathcal{T} : \eta <_{\mathcal{T}} \nu \text{ or } \eta \perp \nu\} \in \mathcal{B}$, or at least contains a member of $\mathcal{B}$.
- (c) If $\theta > \aleph_0$ then the intersection of $< \theta$ many members of $\mathcal{B}$ will always contain some other member of $\mathcal{B}$.

1A) We say $\mathbf{p}$ is a *strong* twinship parameter when we add the following to clause (C):

- (d) No directed $\mathbf{G} \subseteq \mathcal{T}$ meets every $D \in \mathcal{B}$.

2) We say $\mathbf{p}$ is *tree-like* when in addition,

(A) $\mathcal{T}$ is a tree with θ-many levels.

(B) $(\forall \eta \in \mathcal{T})(\forall \varepsilon < \theta)(\exists \nu \in \mathcal{T})[\eta \leq_{\mathcal{T}} \nu \wedge \mathrm{lev}_{\mathcal{T}}(\nu) \geq \varepsilon]$.

(C) If $\eta \in D \in \mathcal{B}$ then $\eta <_{\mathcal{T}} \nu \Rightarrow \nu \in D$.

3) We say $\mathbf{p}$ is *well-founded* when $\mathcal{T}_{\mathbf{p}}$ has no infinite decreasing sequence.

4) So in part (1) we may write $\mathbf{p} = (\mathcal{T}, \mathcal{B}, \theta) = (\mathcal{T}_{\mathbf{p}}, \mathcal{B}_{\mathbf{p}}, \theta_{\mathbf{p}})$, and in part (3) we may write $\mathbf{p} = (\mathcal{T}, \mathcal{B}) = (\mathcal{T}_{\mathbf{p}}, \mathcal{B}_{\mathbf{p}})$.

Remark 2.3. 0) We would like to add the following demand to 2.2(2), but it would hinder 2.24:

(D) $(\forall \varepsilon < \theta)(\exists D \in \mathcal{B})(\forall \eta \in D)[\mathrm{lev}(\eta) \geq \varepsilon]$.

1) Originally the main cases were K_{or} (the class of linear orders), $K_{\mathrm{tr}(\omega)}$ (trees with $\omega + 1$ levels), and most importantly K_{org} (ordered graphs), but now it appears $K_{\mathrm{tr}(\omega)}$ is of limited importance.

2) Our aim, for a given $\mathbf{p}$, is to find non-isomorphic models (preferably *far*, as well) for (e.g.) a suitable complete first-order theory T such that if a forcing adds a new

[5] So $\mathcal{T} = \mathcal{T}_{\mathbf{p}}$, etc. Hence we may write $\mathbf{p} = (\mathcal{T}, \mathcal{B}, \theta)$.

[6] Many of the usual forcing notions fail this, but it will not be a problem to fix this.

directed $\mathbf{G} \subseteq \mathcal{T}$ meeting every $D \in \mathcal{B}_{\mathbf{p}}$ (and if $\mathcal{T}$ is a tree, then a new $\theta_{\mathbf{p}}$-branch of $\mathcal{T}$ meeting every $D \in \mathcal{B}_{\mathbf{p}}$) then they become isomorphic.

Of course, if the forcing collapses some cardinal then this is trivially true (e.g. for first-order countable T). We can rectify this by restricting ourselves to 'interesting' $\mathbb{P}$-s (e.g. ccc, $\mathsf{Cohen}_{\aleph_1}$ assuming CH, or $(<\theta)$-complete θ^+-cc for a suitable $\theta = \theta^{<\theta}$) or requiring the models to be equivalent in some suitable logic. At any rate, to get something non-trivial, T has to be somewhat complicated. This leads us to classification theory, so this is part of the classification program.

Definition 2.4. 1) We say $\mathbf{G} \subseteq \mathcal{T}_{\mathbf{p}}$ *solves* $\mathbf{p}$ when it is a directed subset meeting every $D \in \mathcal{B}_{\mathbf{p}}$.

2) We say M and N are $\mathbf{p}$-*isomorphic* when for any forcing notion $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \text{"if some } \mathbf{G} \subseteq \mathcal{T}_{\mathbf{p}} \text{ solves } \mathbf{p}, \text{ then } M \cong N\text{"}.$$

3) We say $\mathbf{p}$ *represents* the forcing notion $\mathbb{P}$ when in any forcing extension $\mathbf{V}'$ of $\mathbf{V}$, $\mathbf{p}$ is solved in $\mathbf{V}'$ iff in $\mathbf{V}'$ there exists a subset of $\mathbb{P}$ generic over $\mathbf{V}$.

4) We say 'M and N are $\mathbf{p}$-twins' when they are $\mathbf{p}$-isomorphic but not isomorphic.

5) For $\mathbf{p}$ a weak twinship parameter, we say the models M and N are *strictly* $\mathbf{p}$-isomorphic when for any forcing notion $\mathbb{P}$ we have

$$\Vdash_{\mathbb{P}} \text{"if some downward closed } \mathbf{G} \subseteq \mathcal{T}_{\mathbf{p}} \text{ solves } \mathbf{p} \text{ and } \mathbf{G} \notin \mathbf{V}, \text{ then } M \cong N\text{"}.$$

Definition 2.5. 1) We define

$$\Omega_{\mathbf{p}} := \big\{\mathbf{o} = \big\langle(\eta_\ell, \iota_\ell) : \ell < k\big\rangle : k < \omega,\ \eta_\ell \in \mathcal{T}_{\mathbf{p}},\ \iota_\ell = \pm 1\big\}.$$

We may also denote $\mathbf{o}$ by the pair $(\bar\eta, \bar\iota) = (\bar\eta_{\mathbf{o}}, \bar\iota_{\mathbf{o}}) = \big(\langle \eta_\ell : \ell < k\rangle, \langle \iota_\ell : \ell < k\rangle\big)$.

As always, $\eta_\ell = \eta_{\mathbf{o},\ell}$, $\iota_\ell = \iota_{\mathbf{o},\ell}$, and $k = k_{\mathbf{o}} := \ell g(\mathbf{o})$.

2) For $\mathbf{o} \in \Omega_{\mathbf{p}}$ and $\{F_{\eta_\ell,\iota_\ell} : \ell < k_{\mathbf{o}}\}$ a family of partial permutations of some set $\mathcal{U}$, satisfying $F_{\eta,-\iota} = F^{-1}_{\eta,\iota}$, we define $F_{\mathbf{o}}$ naturally by induction on $\ell g(\mathbf{o})$:

- $F_{\langle\ \rangle} := \mathrm{id}_{\mathcal{U}}$ (I.e. $F_{\langle\ \rangle}(a) = a$ iff $a \in \mathcal{U}$.)
- If $\mathbf{o}_2 = \mathbf{o}_1{}^\frown\big\langle(\eta,\iota)\big\rangle$ then $F_{\mathbf{o}_2}(a) := F_{\mathbf{o}_1}(F_{\eta,\iota}(a))$ (when it is well-defined).

3) For $\mathbf{o}$ and F_{η_ℓ,ι_ℓ} as above, the $\mathbf{o}$-*orbit* of $a \in \mathcal{U}$ is the sequence $\bar a = \langle a_\ell : \ell \le k\rangle$, where $a_k := a$ and $a_\ell := F_{\eta_\ell,\iota_\ell}(a_{\ell+1})$.

(Note that this sequence may not exist, as we do not require that $\mathrm{dom}(F_{\eta_\ell,\iota_\ell}) = \mathcal{U}$.)

4) We say that the orbit $\bar a$ is *reduced* when $a_\ell \neq a_m$ for all $\ell < m \le \ell g(\bar a)$.

5) We say that $\mathbf{o} = \big\langle(\eta_\ell,\iota_\ell) : \ell < \ell g(\mathbf{o})\big\rangle \in \Omega_{\mathbf{p}}$ is *formally reduced* when

- If $\ell + 1 < \ell g(\mathbf{o})$ and $\eta_\ell = \eta_{\ell+1}$, then $\iota_\ell = \iota_{\ell+1}$.

6) $\Omega_{\mathrm{fr}} = \Omega^{\mathrm{fr}}_{\mathbf{p}}$ will denote the set of $\mathbf{o} \in \Omega_{\mathbf{p}}$ which are formally reduced.

7) $\Omega_{\mathcal{S}}$ and $\Omega^{\mathrm{fr}}_{\mathcal{S}}$ are defined similarly, replacing $\mathcal{T}_{\mathbf{p}}$ by any set $\mathcal{S}$.

We first give a special case of the main definition, combining K_{org} with $\mathbf{p}$.

Definition 2.6 (**Main Definition**). 0) $\tau_{\mathrm{org}} := \{<, R\}$, where $<$ and R are two-place predicates, and

$$K_{\mathrm{org}} := \big\{I = \big(|I|, <^I, R^I\big) : <^I \text{ is a linear order and } \big(|I|, R^I\big) \text{ is a graph}\big\}.$$

1) Let $\tau^{\mathrm{org}}_{\mathbf{p}} := \{<, E, R, F_{\eta,\iota} : \eta \in \mathcal{T}_{\mathbf{p}},\ \iota = \pm 1\}$, where

- $<$ and E are two-place predicates.

- $F_{\eta,\iota}$ is a unary function symbol (interpreted as a partial function).

1A) However, "$x \in \mathrm{dom}(F_{\eta,\iota})$" is considered an atomic formula, and even "$x \in \mathrm{dom}(F_{\mathbf{o}})$" for $\mathbf{o} \in \Omega_{\mathbf{p}}$.

2) Let $K^{\mathrm{org}}_{\mathcal{T},0}$ be the class of $\tau^{\mathrm{org}}_{\mathbf{p}}$-structures I such that:

(A) $<_I$ is a linear order on I.

(B) For $\eta \in \mathcal{T}$, $F_{\eta,\iota} = F^I_{\eta,\iota}$ is a partial automorphism of $(|I|, <_I, R^I)$, increasing with η, satisfying the following:

(a) $F_{\eta,-\iota} = F^{-1}_{\eta,\iota}$.

(b) If $a \in I$ and $\iota = \pm 1$, then $D^I_{\iota,a} := \big\{\eta : \mathrm{dom}(F_{\eta,\iota}) \ni a\big\} \in \mathcal{B}$. (If $\iota = 1$ we may omit it.)

(c) $F^I_{\mathbf{o}}$ is well-defined and a partial automorphism for all $\mathbf{o} \in \Omega_{\mathbf{p}}$.

(d) If $\eta \leq_{\mathcal{T}} \nu$ and $\iota = \pm 1$ then $F^I_{\eta,\iota} \subseteq F^I_{\nu,\iota}$.

(e) **[Follows:]** If $\iota = \pm 1$, $F^I_{\eta_\ell,\iota}(s_\ell) = t_\ell$ for $\ell = 1, 2$, and η_1, η_2 are $\leq_{\mathcal{T}}$-compatible, <u>then</u>

$$s_1\, R^I\, s_2 \Leftrightarrow t_1\, R^I\, t_2.$$

(C) $(|I|, R^I)$ is a graph.

(D) E^I is the closure of

$$\big\{\big(a, F^I_{\eta,\iota}(a)\big) : a \in \mathrm{dom}(F^I_{\eta,\iota}),\ \eta \in \mathcal{T}_{\mathbf{p}},\ \iota = \pm 1\big\}$$

to an equivalence relation.

3) Let $K^{\mathrm{org}}_{\mathcal{T},1}$ be the class of $I \in K^{\mathrm{org}}_{\mathcal{T},0}$ such that <u>if</u> $\mathbf{o} \in \Omega_{\mathrm{fr}}$, $\ell g(\mathbf{o}) = k \geq 1$, $a_k \in I$, and $a_0, \ldots, a_k$ is an $\mathbf{o}$-orbit, <u>then</u> $a_k \neq a_0$.

4) For $I, J \in K^{\mathrm{org}}_{\mathcal{T},0}$, let '$I \subseteq J$' mean $<_I = <_J \restriction |I|$, $R^I = R^J \restriction |I|$, and $F^I_{\mathbf{o}} = F^J_{\mathbf{o}} \restriction I$ for all $\mathbf{o} \in \Omega_{\mathbf{p}}$.

Similarly, we can define

Definition 2.7. Let $K = K_{\mathrm{x}}$ be as in 1.1(1A) and $E, F_{\eta,\iota} \notin \tau(K_{\mathrm{x}})$, where $\{F_{\eta,\iota} : \eta \in \mathcal{T},\ \iota = \pm 1\}$ are unary function symbols and E a binary predicate. (Below, if $\mathcal{T} = \mathcal{T}_{\mathbf{p}}$ then we may replace $\mathcal{T}$ by $\mathbf{p}$.)

1) Let $\tau^{\mathrm{x}}_{\mathcal{T}} := \tau(K_{\mathrm{x}}) \cup \{F_{\eta,\iota} : \eta \in \mathcal{T},\ \iota = \pm 1\} \cup \{E\}$.

2) $K^{\mathrm{x}}_{\mathcal{T},0}$ is the class of structures J such that:

(A) (a) J is a $\tau^{\mathrm{x}}_{\mathcal{T}}$-model.

(b) $J \restriction \tau(K_{\mathrm{x}}) \in K_{\mathrm{x}}$

(c) Every $F^J_{\eta,\iota}$ is a partial automorphism of $J \restriction \tau(K_{\mathrm{x}}) \in K_{\mathrm{x}}$, increasing with $\eta \in \mathcal{T}$.

(B) Clauses 2.6(2)(B)(a)-(d) all hold, but (e) is replaced by:

(e)′ For R any predicate from $\tau(K_{\mathrm{x}})$, if $F^I_{\eta_\ell,\iota_\ell}(s_\ell) = t_\ell$ for $\ell < \mathrm{arity}(R)$ and $\{\eta_\ell : \ell < \mathrm{arity}(R)\}$ has a common upper bound, <u>then</u>

$$\big\langle s_\ell : \ell < \mathrm{arity}(R)\big\rangle \in R^I \Rightarrow \big\langle t_\ell : \ell < \mathrm{arity}(R)\big\rangle \in R^I.$$

(Recall that we are assuming $\tau(K_{\mathrm{x}})$ has only predicates: function symbols and individual constants will be treated as predicates.)

(C),(D) As in Definition 2.6(2)(C),(D).

3-4) As in 2.6(3),(4).

Definition 2.8. 1) Let $\leq_\Omega$ be the following two-place relation on $\Omega_{\mathbf{p}}$.

⊛ $\mathbf{o}_1 \le_\Omega \mathbf{o}_2$ iff ($\mathbf{o}_1, \mathbf{o}_2 \in \Omega_{\mathbf{p}}$ and)
(a) $\ell g(\mathbf{o}_1) = \ell g(\mathbf{o}_2)$
(b) $\bar\iota^1 = \bar\iota^2$ (where $\mathbf{o}_\ell = (\bar\eta^\ell, \bar\iota^\ell)$).
(c) $\eta^1_k \trianglelefteq \eta^2_k$ for all $k < \ell g(\mathbf{o}_1) = \ell g(\mathbf{o}_2)$.

2) $\mathbf{o}_1 \parallel_\Omega \mathbf{o}_2$ will be shorthand for "$\mathbf{o}_1$ and $\mathbf{o}_2$ are compatible" (i.e. have a common $\le_\Omega$-upper bound).

Definition 2.9. 1) For $J \in K^{\mathrm{x}}_{\mathcal{T},1}$ and $s \in J$, let $\Omega^J_s := \{\mathbf{o} \in \Omega^{\mathrm{fr}}_{\mathbf{p}} : \mathrm{dom}(F^J_{\mathbf{o}}) \ni s\}$.
2) For $D \in \mathcal{B}$ (see 2.2(1)(C)), let

$$\Omega_D := \big\{\mathbf{o} \in \Omega^{\mathrm{fr}}_{\mathbf{p}} : \{\eta_{\mathbf{o},\ell} : \ell < \ell g(\mathbf{o})\} \subseteq D\big\}.$$

3) Let $K^{\mathrm{x}}_{\mathcal{T},2} = K^{\mathrm{x}}_{\mathbf{p},2}$ be the class of $J \in K^{\mathrm{x}}_{\mathcal{T},1}$ such that
(A) $(\forall s \in J)(\exists D \in \mathcal{B})[\Omega^J_s = \Omega_D]$
(B) If $F^J_{\mathbf{o}}(s) = t$ then there exists $\mathbf{o}' \le_\Omega \mathbf{o}$ such that $F^J_{\mathbf{o}'}(s) = t$ and

$$(\forall \mathbf{o}'' <_\Omega \mathbf{o}')\big[s \notin \mathrm{dom}(F^J_{\mathbf{o}''})\big].$$

Note that

Fact 2.10. 1) If $\mathbf{p}$ (i.e. $\mathcal{T}_{\mathbf{p}}$) is well-founded, then clause (B) of 2.9(3) always holds.
2) If $\mathbf{p}$ it tree-like then it is well-founded.

Proof. 1) This follows immediately from $\mathbf{p}$ being well-founded — $\{\mathbf{o}' : \mathbf{o}' \le_\Omega \mathbf{o}\}$ has no infinite decreasing sequence for any $\mathbf{o} \in \Omega$.
2) Easy as well. $\square_{2.10}$

Remark 2.11. The assumption that $\mathbf{p}$ is well-founded is not a serious hindrance, thanks to 2.24 below.

Claim 2.12. 1) *$\le_\Omega$ is a partial order on $\Omega_{\mathbf{p}}$.*
2) *$\mathbf{o}_1 \parallel_\Omega \mathbf{o}_2$ iff $\ell g(\mathbf{o}_1) = \ell g(\mathbf{o}_2)$ and for $\ell < \ell g(\mathbf{o}_1)$, $\iota_{\mathbf{o}_1,\ell} = \iota_{\mathbf{o}_2,\ell}$ and $\eta_{\mathbf{o}_1,\ell}$, $\eta_{\mathbf{o}_2,\ell}$ are $\le_{\mathcal{T}}$-compatible.*
3) *Assume $\mathbf{p}$ is tree-like.*
Any $\mathbf{o}_1, \mathbf{o}_2 \in \Omega_{\mathbf{p}}$ have a maximal common $\le_\Omega$-lower bound; we will denote it by $\mathbf{o}_1 \wedge \mathbf{o}_2$.

Proof. Easy. (E.g. part (4) follows from 2.2(A).) $\square_{2.12}$

Claim 2.13. 1) *If $J \in K^{\mathrm{x}}_{\mathcal{T},0}$ and $\mathbf{V}_1 := \mathbf{V}^{\mathbb{P}}$ (or an extension), then (A) ⇒ (B), where*
(A) $\mathbf{G} \in \mathbf{V}_1$ solves $\mathbf{p}$: that is, it is a directed subset of $\mathcal{T}_{\mathbf{p}}$ such that

$$(\forall D \in \mathcal{B}_{\mathbf{p}})[\mathbf{G} \cap D \neq \varnothing].$$

(B) $F_{\mathbf{G}} := \bigcup_{\eta \in \mathbf{G}} F^J_{\eta,1}$ is an automorphism of $J \restriction \tau_{\mathrm{org}}$. (So in the proof of 3.6(1), it will induce an isomorphism from one distinguished subset of $J \restriction \tau_{\mathrm{org}}$ to another, and hence from M^+_1 to M^+_2.)

2) *If $\mathbf{p}$ is tree-like, then clause (A) is equivalent to*
(A)′ •$_1$ $\mathcal{T}_{\mathbf{p}} \subseteq {}^{\theta>}\lambda$ is a tree, $\mathbf{V}_1 \models$ "$\eta \in {}^\theta\lambda$", and $(\forall \eta < \theta)[\eta \restriction \varepsilon \in \mathcal{T}]$.
•$_2$ $\mathbf{G} := \{\eta \restriction \varepsilon : \varepsilon < \theta\}$ solves $\mathbf{p}$.

Proof. First,

$(*)_1$ $F_{\mathbf{G}}$ is a well-defined function.

[Why? As $\mathbf{G} \subseteq \mathcal{T}_{\mathbf{p}}$, each F^J_η is a function. Furthermore, if $\eta, \nu \in \mathbf{G}$ then there is $\rho \in \mathbf{G}$ such that $\eta \leq_{\mathcal{T}} \rho \wedge \nu \leq_{\mathcal{T}} \rho$ (because $\mathbf{G}$ is directed) and $F^J_\eta \subseteq F^J_\rho \wedge F^J_\nu \subseteq F^J_\rho$ (recalling $J \in K^{\mathrm{x}}_{\mathbf{p}}$ and 2.6(2)(B)(c)).]

$(*)_2$ $F_{\mathbf{G}}$ is a partial automorphism of $J \restriction \tau_{\mathbf{x}}$.

[Why? Similarly to the proof of $(*)_1$.]

$(*)_3$ $\mathrm{dom}(F_{\mathbf{G}}) = J$.

[Why? Recall 2.6(2)(B)(b), and that $\mathbf{G}$ meets every $D \in \mathcal{B}_{\mathbf{p}}$.]

Lastly,

$(*)_4$ $\mathrm{rang}(F_{\mathbf{G}}) = J$.

[Why? Like $(*)_3$, recalling $\mathrm{dom}(F^J_{\eta,1}) = \mathrm{rang}(F^J_{\eta,-1})$.]

Together we are done. $\square_{2.13}$

Observation 2.14. *For every $D \in \mathcal{B}_{\mathbf{p}}$ there exists $I \in K^{\mathrm{org}}_{\mathcal{T},2}$ such that:*

(a) $\Omega^I_s = \Omega_D$ *for every* $s \in I$.

(b) *For every* $s \in I$, $I = \{F^I_{\mathbf{o}}(s) : \mathbf{o} \in \Omega_D\}$.

Proof. Straightforward. $\square_{2.14}$

§ 2(B). **Examples.** First we present examples of $\mathbf{p}$ with $\theta_{\mathbf{p}} = \aleph_0$ (so they do not fit the theorems in §3).

Claim 2.15 (Example / Claim).

1) *The following example is a strong tree-like twinship parameter witnessing the Cohen Real forcing.*

Let $\mathbf{p} = \mathbf{p}_{\mathsf{Cohen}} = \mathbf{p}[\mathsf{Cohen}]$ *consist of:*[7]

(a) $\mathcal{T}_{\mathbf{p}} = ({}^{\omega>}2, \lhd)$ *(so it is tree-like, and $\theta = \aleph_0$).*

(b) $\mathcal{B}_{\mathbf{p}}$, *the set of open dense subsets of* ${}^{\omega>}2$.

2) *If* $\lambda := \mathrm{cov}(\mathrm{meagre})$, *then for some* $\mathcal{B} \subseteq \mathcal{B}_{\mathbf{p}[\mathsf{Cohen}]}$ *of cardinality* λ, *the pair* $\big(({}^{\omega>}2, \lhd), \mathcal{B}\big)$ *is a strong tree-like twinship parameter — that is, it satisfies Definition 2.2.*

3) *Let* $\mathcal{T}$ *be* $({}^{\omega>}2, \lhd)$, or *a subtree of* $({}^{\delta>}2, \lhd)$ *for some* δ *such that*

$$(\forall \eta \in \mathcal{T})(\forall \varepsilon < \delta)(\exists \nu \in \mathcal{T} \cap {}^{\varepsilon}2)[\nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu].$$

Define $\mathbf{p} = \mathbf{p}[\mathcal{T}]$ *by* $\mathcal{T}_{\mathbf{p}} := \mathcal{T}$ *and* $\mathcal{B}_{\mathbf{p}} := \{\mathcal{T} \setminus {}^{\varepsilon\geq}2 : \varepsilon < \delta\}$.

Then $\mathbf{p}$ *is a weak twinship parameter.*

4) *All of the examples above are well-founded.*

Proof. 1) Covered in the proof of part (2).

2) Let $\langle Z_\alpha : \alpha < \lambda \rangle$ be a sequence of meagre subsets of ${}^\omega 2$ such that $\bigcup\limits_{\alpha<\lambda} Z_\alpha = {}^\omega 2$. Let $Z_\alpha \subseteq \bigcup\limits_n Z_{\alpha,n}$, where $Z_{\alpha,n}$ is a closed nowhere dense subset of ${}^\omega 2$, $\subseteq$-increasing with n. Let $\mathcal{B} := \{D_{u,n} : u \in [\lambda]^{<\aleph_0},\ n < \omega\}$, where

$$D_{u,n} := \Big\{\eta \in {}^{\omega>}2 : \neg\big(\exists \rho \in \bigcup_{\alpha \in u} Z_{\alpha,n}\big)\big[\eta \lhd \rho\big]\Big\}.$$

[7] So $\mathcal{T} = \mathcal{T}_{\mathbf{p}}$, etc.

This obviously suffices, but let us elaborate, checking clauses (A)-(C) of Definition 2.2.

Clause (A): As $\mathcal{T} := ({}^{\omega>}2, \triangleleft)$, this is clear.

Clause (B): $\theta := \aleph_0$, so it is regular and infinite.

Clause (C):

$(*)_1$ For $\alpha < \lambda$ and $n < \omega$, $\mathcal{T}_{\alpha,\{n\}} := \{\eta \restriction n : \eta \in Z_{\alpha,n}\}$ is a nowhere dense subtree of ${}^{\omega>}2$.
(That is, $\big(\forall \eta \in {}^{\omega>}2\big)\big(\exists \nu \in {}^{\omega>}2 \setminus \mathcal{T}_{\alpha,\{n\}}\big)\big[\eta \triangleleft \nu\big]$.)

[Why? Because $Z_{\alpha,n}$ is a nowhere dense subset of ${}^{\omega}2$.]

$(*)_2$ Each $D_{u,n}$ is a nowhere dense subtree of ${}^{\omega>}2$.

[Why? By $(*)_1$.]

$(*)_3$ $\mathcal{B}$ is a family of subsets of $\mathcal{T} := {}^{\omega>}2$.

So to verify clause (C) we have to check the sub-clauses.

(C)(a): $\mathcal{B}$ is closed under finite intersections.
Let $m < \omega$ and $\{D_\ell : \ell < m\} \subseteq \mathcal{B}$. So for each $\ell < m$ there are $n_\ell < \omega$ and $u_\ell \in [\lambda]^{\aleph_0}$ such that $D_\ell = D_{u_\ell, n_\ell}$.
Let $n := \max(\{n_\ell : \ell < m\} \cup \{1\})$ and $u := \bigcup\limits_{\ell<m} u_\ell$. Easily, $D_{u,n} \in \mathcal{B}$ and $D_{u,n} = \bigcap\limits_{\ell<m} D_{u_\ell,n} \subseteq \bigcap\limits_{\ell<m} D_{u_\ell,n_\ell} = \bigcap\limits_{\ell<m} D_\ell$ are as required.

(C)(b): Let $m < \omega$ and $\eta \in \mathcal{T}$. So for every $\nu \in {}^{n}2$ we can find $\rho_\nu \in {}^{\omega}2$ such that $\nu \triangleleft \rho_\nu$. By the choice of $\langle Z_\alpha : \alpha < \lambda\rangle$, there exists an $\alpha_\nu < \lambda$ such that $\rho_\nu \in Z_{\alpha_\nu}$. Hence $\rho_\nu \in Z_{\alpha_\nu, m_\nu}$ for some $m_\nu < \omega$.
Let $m := \max\{m_\nu : \nu \in {}^{n}2\}$, so

$$\nu \in {}^{n}2 \Rightarrow \rho_\nu \in Z_{\alpha_\nu, m_\nu} \subseteq Z_{\alpha_\nu, m} \subseteq \bigcap_{\nu' \in {}^{n}2} Z_{\alpha_{\nu'}, m} = D_{u,m},$$

where $u := \{\alpha_{\nu'} : \nu' \in {}^{n}2\}$.
Obviously $D_{u,m} \in \mathcal{B}$ and is disjoint to ${}^{n\geq}2$, so it is as required.

(C)(c): Vacuous.

Clause (C)(d): (of 2.2(1A))
If $\mathbf{G}$ solves $\mathbf{p}$, then there exists $\eta \in {}^{\omega}2$ such that

$$(\forall \alpha < \lambda)(\forall m < \omega)(\forall^\infty n)[\eta \restriction n \notin D_{\{\alpha\},m}],$$

but this contradicts our choice of $\langle Z_\alpha : \alpha < \lambda\rangle$.

3-4) Easy. $\square_{2.15}$

Claim 2.16 (**Example / Claim**). *The following example is a strong well-founded tree-like twinship parameter which witnesses the Random Real forcing.*
Let $\mathbf{T}_*$ be the set of sequences $\overline{\mathcal{T}} = \langle \mathcal{T}_n : n < \omega\rangle$ such that:

(A) *$\mathcal{T}_n$ is a subtree of ${}^{\omega>}2$.*

(B) $\mathrm{Leb}(\bigcup\limits_{n\in[m,\omega)} \lim \mathcal{T}_n) = 1$ *for every $m < \omega$.*

(C) *Let $\eta_n := \mathrm{tr}(\mathcal{T}_n)$ (the* trunk *of $\mathcal{T}_n$). The sequence $\langle \eta_n : n < \omega\rangle$ is without repetition, and*

$$\eta_n \triangleleft \eta_m \notin \mathcal{T}_n \Rightarrow (\exists \rho \in {}^{\omega>}2 \setminus \mathcal{T}_n)[\eta_n \triangleleft \rho \triangleleft \eta_m].$$

Let $\mathbf{T} := \big\{\langle \mathcal{T}_{\alpha,n} : n < \omega\rangle : \alpha < \lambda\big\} \subseteq \mathbf{T}_*$ *be such that*

$$\bigcap_{\alpha<\lambda}\bigcup_{n<\omega} \lim \mathcal{T}_{\alpha,n} = \varnothing$$

(Clearly there is such a sequence of length λ *when* $\lambda := \mathrm{cov}(\mathrm{null})$*.)*

Let $\mathbf{p} = \mathbf{p}_{\mathsf{Random}} = \mathbf{p}_{\mathsf{Random}(\mathbf{T})}$ *consist of:*

(a) $\mathcal{T}_{\mathbf{p}} = ({}^{\omega>}2, \lhd)$ *(so* $\theta = \aleph_0$*).*

(b) $\mathcal{B}_{\mathbf{p}} := \{D_{u,n} : u \in [\lambda]^{<\aleph_0},\ n < \omega\}$*, where*

$$D_{u,n} := \Big\{\eta \in {}^{\omega>}2 : \ell g(\eta) \geq n,\ (\forall \alpha \in u)(\exists m)\big[\mathrm{tr}(\mathcal{T}_{\alpha,m}) \trianglelefteq \eta \in \mathcal{T}_{\alpha,m}\big]\Big\}$$

Proof. Similar to the proof of 2.15. $\square_{2.16}$

The following wide family of examples cover Random Real forcing, Cohen forcing, and virtually every forcing which adds a new set of ordinals (even e.g. Prikry forcing).

Definition 2.17. 1) We say $\mathbf{m} = (\lambda, \theta, \mathcal{T}, \mathbb{P}, \underset{\sim}{\eta}) = (\lambda_{\mathbf{m}}, \theta_{\mathbf{m}}, \ldots)$ is a *forcing example* when:

(A) $\lambda \geq 2$ and $\theta = \mathrm{cf}(\theta) \geq \aleph_0$.

(B) $\mathcal{T}$ is a subtree of $({}^{\theta>}\lambda, \lhd)$ (so it is closed under initial segments).

(C) $\mathbb{P}$ is a forcing notion which preserves "θ is regular."

(D) $\underset{\sim}{\eta}$ is a $\mathbb{P}$-name of a member of ${}^{\theta}\lambda$.

(E) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta} \notin \mathbf{V}$ and $(\forall \varepsilon < \theta)[\underset{\sim}{\eta}_{\varepsilon} \in \mathcal{T}]$", where $\underset{\sim}{\eta}_{\varepsilon} := \underset{\sim}{\eta} \restriction \varepsilon \in {}^{\varepsilon}\lambda$.

(F) For transparency, we demand $(\forall \nu \in \mathcal{T})(\exists p \in \mathbb{P})[p \Vdash$ "$\nu \lhd \underset{\sim}{\eta}$"$]$.

2) For $\mathbf{m}$ as above, let $\mathbf{p} = \mathbf{p}_{\mathbf{m}} = (\mathcal{T}_{\mathbf{m}}, \mathcal{B}_{\mathbf{m}}, \theta_{\mathbf{m}})$ be defined as follows.

(A) $\mathcal{T}_{\mathbf{p}} := \mathcal{T}_{\mathbf{m}}$, $\theta_{\mathbf{p}} := \theta_{\mathbf{m}}$.

(B) $\mathcal{B}_{\mathbf{m}} := \big\{D_{\mathbf{I},f} \subseteq \mathcal{T} : (\mathbf{I}, f) \in \mathrm{set}(\mathbb{P})\big\}$, where

  (a) $\mathrm{set}(\mathbb{P})$ is the set of pairs $(\mathbf{I}, f)$ such that:

   $\bullet_1$ $\mathbf{I}$ is a maximal antichain of $\mathbb{P}$, or its completion.

   $\bullet_2$ $f : \mathbf{I} \to \mathcal{T}$

   $\bullet_3$ $f(p) = \nu \Rightarrow p \Vdash_{\mathbb{P}}$ "$\nu \lhd \underset{\sim}{\eta}$"

  (b) $D_{\mathbf{I},f} := \big\{\nu \in \mathcal{T} : (\exists p \in \mathbf{I})[f(p) \trianglelefteq \nu]\big\}$.

3) For a forcing notion $\mathbb{P}$, we define $\mathbf{p}[\mathbb{P}]$ as $\mathcal{T}_{\mathbf{p}} := \mathbb{P}$, $\mathcal{B}_{\mathbf{p}}$ the family of dense open subsets of $\mathbb{P}$, and $\theta_{\mathbf{p}} := \{\theta :$ forcing with $\mathbb{P}$ adds a new function $f \in {}^{\theta}\mathrm{Ord}\}$.

Remark 2.18. In 2.17(1), we can use $\mathbf{p}'_{\mathbf{m}} = (\mathcal{T}_{\mathbf{m}}, \mathcal{B}'_{\mathbf{m}}, \theta_{\mathbf{m}})$ (so $\mathcal{T}_{\mathbf{m}} \subseteq {}^{\theta>}\lambda$), where $\mathcal{B}'_{\mathbf{m}}$ is a subset of $\mathcal{B}_{\mathbf{m}}$ containing $\{\mathcal{T} \setminus {}^{\varepsilon\geq}\lambda : \varepsilon < \delta\}$, closed under finite intersections, with no solution. (See 2.4.)

Then there would be no $\eta \in {}^{\theta}\lambda$ such that

$$(\forall D \in \mathcal{B})(\forall^{\theta} \varepsilon < \theta)[\eta \restriction \varepsilon \in D].$$

Claim 2.19. 0) *If* $\mathbb{P}$ *is a forcing notion, and* λ *and* κ *are minimal such that*

$$\Vdash_{\mathbb{P}} \text{"there is a new } \eta \in {}^{\kappa}\lambda\text{"},$$

then there exists $\mathcal{T} \subseteq {}^{\kappa>}\lambda$ *of cardinality* $\leq \|\mathbb{P}\|$ *and a forcing example* $\mathbf{m}$ *such that*

$$(\lambda_{\mathbf{m}}, \theta_{\mathbf{m}}, \mathcal{T}_{\mathbf{m}}, \mathbb{P}_{\mathbf{m}}) = (\lambda, \kappa, \mathcal{T}, \mathbb{P}).$$

1) *If* $\mathbf{m}$ *is a forcing example then* $\mathbf{p}_{\mathbf{m}}$ *is a tree-like strong twinship parameter.*

2) *If* $\mathbf{m}$ *is a forcing example and* $\mathbb{P}_{\mathbf{m}}$ *is* $(<\aleph_1)$*-complete (or at least adds no new* ω*-sequence of ordinals), then necessarily* $\theta_{\mathbf{m}} > \aleph_0$.

3) *If* $\mathbf{m} = (\lambda, \theta, \mathcal{T}, \mathbb{P}, \underset{\sim}{\eta})$ *is a forcing example and* $D \in \mathcal{B}_{\mathbf{p_m}}$*, then*

- $\bullet_1$ $\Vdash_{\mathbb{P}} (\forall^\infty \varepsilon < \theta)[\underset{\sim}{\eta} \restriction \varepsilon \in D]$
- $\bullet_2$ $\mathbf{p_m}$ *has a solution in* $\mathbf{V}^{\mathbb{P}}$.

4) *If* $\mathbb{P}$ *is a non-trivial forcing then* $\mathbf{p}[\mathbb{P}]$ *(see* 2.17(3)*) is a strong twinship parameter and* $\Vdash_{\mathbb{P}}$ "$\mathbf{p}[\mathbb{P}]$ has a solution".

5) *If* $\mathcal{T}$ *is a Suslin tree, then there exists a forcing example* $\mathbf{m}$ *with* $\mathbb{P}_{\mathbf{m}} := \mathcal{T}$ *and* $\theta_{\mathbf{m}} := \aleph_1$.

6) *All of the* $\mathbf{p}$*-s defined above are well-founded.*

Proof. Easy.

E.g. in part (1), to verify Definition 2.2(1)(C)(c), use

$$\Vdash_{\mathbb{P}} \text{“}\theta \in \text{Reg”}$$

from 2.17(1)(C). $\square_{2.19}$

Remark 2.20. In 2.19(3) $\bullet_2$, we may use a smaller $\mathcal{B}' \subseteq \mathcal{B}_{\mathbf{p}[\mathbb{P}]}$ as in previous examples. (The gain from this is that if $\mathcal{B}'$ has a smaller cardinality, then 3.4 and 3.5 apply to more values of λ.)

Claim 2.21. 1) *Assume* $\mathbf{m}$ *is a forcing example.*

- (a) *If* $\mathbb{P}_{\mathbf{m}}$ *is Cohen forcing then* $\theta_{\mathbf{m}} = \aleph_0$.
- (b) *If* $\mathbb{P}_{\mathbf{m}}$ *is* $(<\aleph_1)$*-complete then* $\theta_{\mathbf{m}} \geq \aleph_1$.
- (c) *If* $\mathbb{P}_{\mathbf{m}}$ *adds no new* ω*-sequence of ordinals then* $\theta_{\mathbf{m}} \geq \aleph_1$.

2) *Assume* $\mathbb{P}$ *is a non-trivial forcing (i.e. above every* $p \in \mathbb{P}$ *there are two incompatible members). Let* $\kappa(\mathbb{P})$ *be as in* 0.15, 0.16.

Then for some $\lambda \leq |\mathbb{P}|$*, there is a forcing example* $\mathbf{m}$ *with* $\mathbb{P}_{\mathbf{m}} = \mathbb{P}$, $\theta_{\mathbf{m}} = \kappa(\mathbb{P})$, *and* $\lambda_{\mathbf{m}} = \lambda$.

Proof. Easy, but we elaborate.

1) **Clause** (a)**:** Follows from the definition of Cohen forcing, which is $({}^{\omega>}2, \triangleleft)$ or something equivalent.

Clause (b)**:** By 2.17(1)(B),(D), because forcing by $\mathbb{P}$ adds no new ω-sequence of members of $\mathbf{V}$.

Clause (c)**:** Similarly.

2) Follows from Observation 0.17. $\square_{2.21}$

Claim 2.22. *Assume* $\mathbf{m}$ *is a forcing example with* $\theta_{\mathbf{m}} > \aleph_0$.

1) $\mathbb{P}_{\mathbf{m}} \neq$ Sacks. *In fact, Sacks forcing adds no solution to* $\mathbf{p}$.

2) *If* $\mathbb{Q}$ *is a definition of a forcing notion, is non-trivial and ccc, and the truth values of* "$p \leq_{\mathbb{Q}} q$", "$p \perp_{\mathbb{Q}} q$", *and* "$\mathbb{Q}$ *is ccc*" *are preserved in* $\mathbf{V}^{\mathbb{Q}}$*, then* $\mathbb{P}_{\mathbf{m}} \neq \mathbb{Q}$. *Moreover,* $\mathbb{Q}$ *adds no solution to* $\mathbf{p}$.

2A) *If* $\theta_{\mathbf{m}} > \aleph_1$ *then the preservation assumption in part* (2) *may be omitted.*

3) $\mathbb{P}_{\mathbf{m}}$ *fails the* $\theta_{\mathbf{m}}$*-Knaster condition (e.g.* Random*).*

Remark 2.23. Similarly for the forcing notions from Rosłanowski-Shelah [RS99] (and [GS12]) — see more in [S$^+$a]..

Proof. Let $\mathbf{p} := \mathbf{p_m}$, $\mathbb{P} := \mathbb{P}_\mathbf{m}$, etc.

Towards contradiction, suppose

$$\Vdash_\mathbb{P} \text{“}\underset{\sim}{\nu} \in \lim(\mathcal{T}_\mathbf{p}) \text{ is a new } \theta_\mathbf{p}\text{-branch”}.$$

1) Let $\underset{\sim}{\nu} \in {}^\omega 2$ be the generic real for Sacks forcing, and $\underset{\sim}{\eta} := \underset{\sim}{\eta}_\mathbf{m}$. Clearly for a dense set of $p \in \mathsf{Sacks}$, p forces that for some $u = u_p \in [\theta]^{\aleph_0}$ we can compute $\underset{\sim}{\nu}$ from $\underset{\sim}{\eta} \restriction u_p$.

So for every p in this dense set, as u is uncountable and θ is a regular uncountable cardinal, clearly $\zeta_p := \bigcup\limits_{\alpha \in u_p} \alpha + 1 < \theta$. As $\underset{\sim}{\eta}_\mathbf{m} \restriction n = (\underset{\sim}{\eta} \restriction \zeta_p) \restriction u$, clearly we have $p \Vdash_\mathbb{P}$ “$\underset{\sim}{\eta} \restriction \zeta_p$ is a new sequence”. This gives us our contradiction.

2) Toward contradiction, assume $p \in \mathbb{Q}$ and $p \Vdash_\mathbb{Q}$ “$\underset{\sim}{\eta}$ solves $\mathbf{p_m}$” (so it is a $\theta_\mathbf{m}$-branch of $\mathcal{T}_{\mathbf{p_m}}$ meeting every $D \in \mathcal{B}_{\mathbf{p_m}}$).

For every $\varepsilon < \theta$, let

$$\Lambda_\varepsilon := \{\nu \in \mathcal{T}_{\mathbf{p_m}} : \ell g(\nu) = \varepsilon, \text{ and some } q \in \mathbb{Q} \text{ above } p \text{ forces “}\nu \triangleleft \underset{\sim}{\eta}\text{”}\}.$$

As $\mathbb{Q}$ satisfies the countable chain condition, clearly Λ_ε is countable and

$(*)_1$ $(\forall \varepsilon < \zeta < \theta)[\nu \in \Lambda_\zeta \Rightarrow \nu \restriction \varepsilon \in \Lambda_\varepsilon]$.

Also,

$(*)_2$ $(\forall \varepsilon < \zeta < \theta_{\mathbf{p_m}})(\forall \nu \in \Lambda_\varepsilon)(\exists \rho \in \Lambda_\zeta)[\nu \triangleleft \rho]$.

Moreover, as $\underset{\sim}{\eta}$ solves $\mathbf{p_m}$, we have $\Vdash_\mathbb{Q}$ “$\underset{\sim}{\eta} \notin \mathbf{V}$”, hence

$(*)_3$ $\big(\forall \varepsilon < \theta_{\mathbf{p_m}}\big)\big(\forall \nu \in \Lambda_\varepsilon\big)\big(\exists \zeta \in [\varepsilon, \theta_{\mathbf{p_m}})\big)\big(\exists \rho_1, \rho_2 \in \Lambda_\zeta\big)\big[\rho_1 \neq \rho_2 \wedge \nu \triangleleft \rho_1 \wedge \nu \triangleleft \rho_2\big]$.

(Recall that $\mathbb{Q}$ satisfies the ccc.) This implies $\Lambda = \bigcup\limits_{\varepsilon < \theta_\mathbf{p}} \Lambda_\varepsilon$ is a tree with no $\theta_\mathbf{p}$-branches.

Let $\mathbf{G} \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$ and contain p. Now in $\mathbf{V}[\mathbf{G}]$ we have a $\theta_\mathbf{p}$-branch η of Λ, and for every $\varepsilon < \theta_\mathbf{p}$ there exist $\xi = \xi_\varepsilon \in (\varepsilon, \theta_\mathbf{p})$ and $\varrho_\varepsilon \in \Lambda_\xi \setminus \{\eta \restriction \xi\}$.

$(*)_4$ So for some unbounded $A \subseteq \theta_\mathbf{p}$ we have

$$(\forall \varepsilon, \zeta \in A)[\varepsilon < \zeta \Rightarrow \xi_\varepsilon < \zeta].$$

For every $\varepsilon \in A$, there exists $p_\varepsilon \in \mathbf{G}$ such that

$(*)_5$ $\mathbf{V} \models \big[p_\varepsilon \Vdash \text{“}\varrho_\varepsilon \triangleleft \underset{\sim}{\eta}\text{”}\big]$.

So for all $\varepsilon \neq \zeta \in A$,

$(*)_6$ (a) $\mathbf{V} \models$ “ϱ_ε and ϱ_ζ are $\triangleleft$-incomparable in $\mathcal{T}_\mathbf{m}$”

(b) $\mathbf{V} \models$ “p_ε and p_ζ are incompatible in $\mathbb{Q}$”.

[Why? Note that $\ell g(\varrho_\varepsilon) = \xi_\varepsilon < \zeta \leq \ell g(\varrho_\zeta)$, so if ϱ_ε and ϱ_ζ are $\triangleleft$-compatible (in $\mathcal{T}_\mathbf{p}$) then necessarily $\varrho_\varepsilon \triangleleft \varrho_\zeta$. But $\ell g(\varrho_\varepsilon) = \xi_\varepsilon$ and $\varrho_\varepsilon \neq \eta \restriction \xi_\varepsilon \triangleleft \varrho_\zeta$, so if $q \in \mathbb{Q}$ is above p_ε and p_ζ, then we get a contradiction.]

Note that each p_ε belongs to $\mathbb{Q}^\mathbf{V} \subseteq \mathbf{V}$, but $\bar{p}_A = \langle p_\varepsilon : \varepsilon \in A\rangle$ may not be in $\mathbf{V}$ (just in $\mathbf{V}[\mathbf{G}]$). So $\bar{p}_A$ contradicts our assumption that forcing with $\mathbb{Q}$ preserves “$\mathbb{Q}$ satisfies the ccc and $p \perp q$” (i.e. p and q are incompatible).

2A) Using the choices in the proof of part (2), consider $\Lambda = \bigcup\limits_{\varepsilon < \theta} \Lambda_\varepsilon$. So it is a subtree of ${}^{\theta >}\lambda$, and each level is countable and non-empty. It is known that no such tree exists.

(Alternatively: clearly without loss of generality $\|\mathbb{P}\| \geq \theta$. Let $\mathbb{Q} \in \mathcal{H}(\chi)$; let $N \prec (\mathcal{H}(\chi), \in)$ be of cardinality $< \theta$, $\bar{\Lambda} = \langle \Lambda_\varepsilon : \varepsilon < \theta\rangle \in N$, and demand that

$\delta := N \cap \theta$ have uncountable cofinality. Choose $\eta \in \Lambda_\delta$ and continue as in the proof of part (2).)

3) For $\varepsilon < \theta_{\mathbf{m}}$, let $p_\varepsilon \in \mathbb{P}_{\mathbf{m}}$ force a value to $\underset{\sim}{\eta}_\varepsilon$ (call it ν_ε). If $\mathbb{P}_{\mathbf{m}}$ satisfies the $\theta_{\mathbf{m}}$-Knaster condition, then there exists a set $\mathcal{U} \in [\theta_{\mathbf{m}}]^{\theta_{\mathbf{m}}}$ such that $\langle p_\varepsilon : \varepsilon \in \mathcal{U}\rangle$ are pairwise compatible. Hence

$$\varepsilon < \zeta \in \mathcal{U} \Rightarrow \nu_\varepsilon \triangleleft \nu_\zeta$$

and $\nu := \bigcup_{\varepsilon \in \mathcal{U}} \nu_\varepsilon$ is a branch of $\mathcal{T}_{\mathbf{p}_{\mathbf{m}}}$.

Replacing p_ε by $p_{\min(\mathcal{U} \setminus \varepsilon)}$ for all $\varepsilon \in \theta_{\mathbf{m}} \setminus \mathcal{U}$, without loss of generality $\nu_\varepsilon = \nu \restriction \varepsilon$ for every $\varepsilon < \kappa$. It suffices to prove that

$$(\forall D \in \mathcal{B}_{\mathbf{p}_{\mathbf{m}}})\big[D \cap \{\nu_\varepsilon : \varepsilon < \kappa\} \neq \varnothing\big].$$

For every $D \in \mathcal{B}_{\mathbf{p}_{\mathbf{m}}}$ and each $\varepsilon < \kappa$, there exists $q_\varepsilon \in \mathbb{P}_{\mathbf{m}}$ above p_ε forcing $\varrho_\varepsilon \triangleleft \underset{\sim}{\eta}$ for some $\varrho_\varepsilon \in D$, and we let $\zeta_\varepsilon := \max\{\varepsilon, \ell g(\varrho_\varepsilon)\}$.

Now there necessarily exist $\varepsilon_1, \varepsilon_2 < \kappa$ such that $\zeta_{\varepsilon_1} < \varepsilon_2$ and $q_{\varepsilon_1}, q_{\varepsilon_2}$ have a common upper bound. Let r be such an upper bound. So

$$r \Vdash \text{“}\varrho_{\varepsilon_1}, \nu_{\varepsilon_2} \text{ are both } \trianglelefteq \underset{\sim}{\eta}\text{”},$$

hence ϱ_{ε_1} and ν_{ε_2} are $\trianglelefteq$-compatible. But as $\ell g(\varrho_{\varepsilon_1}) \leq \zeta_{\varepsilon_1} \leq \varepsilon_2$, necessarily $\varrho_{\varepsilon_1} \trianglelefteq \nu_{\varepsilon_2}$. But $\varrho_{\varepsilon_1} \in D \in \mathcal{B}_{\mathbf{p}_{\mathbf{m}}}$, which is an open dense subset of $\mathcal{T}_{\mathbf{p}}$, hence $\nu_{\varepsilon_2} \in D$. Therefore $D \cap \{\nu_\varepsilon : \varepsilon < \kappa\} \neq \varnothing$.

As D was an arbitrary member of $\mathcal{B}_{\mathbf{p}}$, the set $\{\nu_\varepsilon : \varepsilon < \kappa\}$ solves $\mathbf{p}_{\mathbf{m}}$ — a contradiction. $\square_{2.22}$

The demand ‘$\mathbf{p}$ is well-founded or tree-like’[8] may seem unreasonable, but

Claim 2.24. *Assume* $\mathbf{p}$ *is a [weak/strong] twinship parameter.*

(a) *For* $\eta \in \mathcal{T}_{\mathbf{p}}$, *let*

$$\delta(\eta, \mathbf{p}) := \sup\big\{\alpha + 1 : (\exists \bar{\nu} \in {}^{\alpha}\mathcal{T}_{\mathbf{p}})[\bar{\nu} \text{ is increasing} \wedge \nu_0 = \eta]\big\}.$$

(b) *Let* $\mathbf{I}$ *be a maximal antichain of* $\mathcal{T}_{\mathbf{p}}$ *such that*

$$(\forall \eta \in \mathbf{I})(\forall \nu \geq_{\mathcal{T}} \eta)\big[\delta(\eta, \mathbf{p}) = \delta(\nu, \mathbf{p})\big].$$

(c) *For* $r \in \mathbf{I}$, *let* $\mathbf{p}_r = (\mathcal{T}'_r, \mathcal{B}'_r, \theta'_r)$ *be defined as follows.*

- $\bullet_1$ $\mathcal{T}_r$ *is the set of* $<_{\mathcal{T}}$*-increasing sequences of length a successor ordinal.*
- $\bullet_2$ $<_{\mathcal{T}_r} := \triangleleft$ *(‘is an initial segment of . . .’).*
- $\bullet_3$ $\mathcal{B}_r := \{D_{[r]} : D \in \mathcal{B}\}$, *where*

$$D_{[r]} := \{\bar{p} \in \mathcal{T}_r : \bar{p} \cap D \neq \varnothing\}.$$

- $\bullet_4$ $\theta_r := \theta$.

Then for each $r \in \mathbf{I}$:

(d) $\mathbf{p}_r$ *is a [weak/strong] twinship parameter.*

(e) $\mathbf{p}_r$ *is well-founded, and even tree-like (except that the set of levels of* $\mathcal{T}_{\mathbf{p}_r}$ *is* $\delta(r, \mathbf{p})$, *which is not necessarily a cardinal).*

(f) $|\mathcal{B}_{\mathbf{p}_r}| \leq |\mathcal{B}_{\mathbf{p}}|$

(g) *For any forcing extension* $\mathbf{V}^{\mathbb{Q}}$ *of* $\mathbf{V}$, $\mathbf{p}$ *has a solution iff there exists* $r \in \mathbf{I}$ *such that* $\mathbf{p}_r$ *has a solution.*

Note: If $\delta(-, \mathbf{p})$ *is constant and* 0 *is the minimal element of* $\mathcal{T}_{\mathbf{p}}$, *then we can always choose* $\mathbf{I} := \{0\}$.

[8] See 2.2(2),(3).

Proof. Straightforward, using 0.17. $\square_{2.24}$

§ 2(C). **How 'nice' are the classes $K_{\rm x}$?** (Note that 2.25–2.27 will not be used later.)

Observation 2.25. 1) *Each of the classes from* 1.2 *is an AEC (see* 2.28*) with the JEP ("Joint Embedding Property") and amalgamation, see e.g.* [She87a, Def. 2.5]. *(For $K_{{\rm tr}(\kappa)}$, recall that the unique member of P^I_0 is an individual constant, so identified.)*

2) *This also applies for $K^{\rm or}_{\mathcal{T},\iota}$ and $K^{\rm org}_{\mathcal{T},\iota}$ (for $\iota = 0,1,2$).*

Proof. Straightforward. $\square_{2.25}$

Definition 2.26. 1) Let $\mathbf{S}^{\rm or}_{\mathcal{T}}$ be the class of $I \in K^{\rm or}_{\mathcal{T}}$ such that for any $s \in I$,

$$\{F^I_{\mathbf{o}}(s) : \mathbf{o} \in \Omega_{\mathbf{p}},\ F^I_{\mathbf{o}}(s) \text{ is well-defined}\}$$

is equal to the set of elements of I.

2) For $\mathbf{S} \subseteq \mathbf{S}^{\rm or}_{\mathcal{T}}$, let $K^{\rm or}_{\mathcal{T}}[\mathbf{S}]$ be the class of $I \in K^{\rm or}_{\mathcal{T}}$ such that for every $s \in I$,

$$I \restriction \big\{F^I_{\mathbf{o}}(s) : \mathbf{o} \in \Omega_{\mathbf{p}},\ F^I_{\mathbf{o}}(s) \text{ is well-defined}\big\}$$

is isomorphic to some member of $\mathbf{S}$.

3) $\mathbf{S}^{\rm org}_{\mathcal{T}}$ and $K^{\rm org}_{\mathcal{T}}[\mathbf{S}]$ are defined similarly.

4) For any K, we can define $K_{\mathcal{T}}$, $\mathbf{S}^K_{\mathcal{T}}$, and $K_{\mathcal{T}}[\mathbf{S}]$.

Claim 2.27. *$K_{\rm or}$, $K_{\rm org}$, $K^{\rm or}_{\mathcal{T},\ell}$, and $K^{\rm org}_{\mathcal{T},\ell}$ are universal classes. That is, if M is a $\tau(K_{\mathbf{S}})$-model and every finitely generated submodel belongs to $K_{\mathbf{S}}$, then $M \in K_{\mathbf{S}}$.*

Proof. Obvious. $\square_{2.27}$

Quoting [She09, $1.2_{=\mathsf{La5}}$]:

Definition 2.28. We say $\mathfrak{k}$ is a AEC with LST number $\lambda(\mathfrak{k}) = {\rm LST}_{\mathfrak{k}}$ <u>if</u>:

Ax.0: The truth of '$M \in K$' and '$N \le_{\mathfrak{k}} M$' depends on N and M only up to isomorphism; i.e.

$$M \in K \wedge M \cong N \Rightarrow N \in K$$

and 'if $N \le_{\mathfrak{k}} M$, f is an isomorphism from M onto the τ-model M', and $f \restriction N$ is an isomorphism from N onto N', <u>then</u> $N' \le_{\mathfrak{k}} M'$.'

Ax.I: if $M \le_{\mathfrak{k}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

Ax.II: $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_2$ implies $M_0 \le_{\mathfrak{k}} M_2$ and $M \le_{\mathfrak{k}} M$ for $M \in K$.

Ax.III: If λ is a regular cardinal, M_i is $\le_{\mathfrak{k}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \le_{\mathfrak{k}} M_j$) and continuous (i.e. for $\delta < \lambda$, $M_\delta = \bigcup\limits_{i<\delta} M_i$) for $i < \lambda$ <u>then</u>

$$M_0 \le_{\mathfrak{k}} \bigcup_{i<\lambda} M_i.$$

Ax.IV: If λ is a regular cardinal and M_i (for $i < \lambda$) is $\le_{\mathfrak{k}}$-increasing continuous and $M_i \le_{\mathfrak{k}} N$ for $i < \lambda$ <u>then</u> $\bigcup\limits_{i<\lambda} M_i \le_{\mathfrak{k}} N$.

Ax.V: If $N_0 \subseteq N_1 \le_{\mathfrak{k}} M$ and $N_0 \le_{\mathfrak{k}} M$ <u>then</u> $N_0 \le_{\mathfrak{k}} N_1$.

Ax.VI: If $A \subseteq N \in K$ and $|A| \leq \mathrm{LST}_{\mathfrak{k}}$, then for some $M \leq_{\mathfrak{k}} N$, we have $A \subseteq |M|$ and $\|M\| \leq \mathrm{LST}_{\mathfrak{k}}$ (and $\mathrm{LST}_{\mathfrak{k}}$ is the minimal infinite cardinal satisfying this axiom which is $\geq |\tau|$; the $\geq |\tau|$ is for notational simplicity).

§ 3. On Existence For independent T

Convention 3.1. $\mathbf{p}$ is a (weak) twinship parameter (that is, as in Definition 2.2).

Definition 3.2. Assume $\lambda > \kappa + \aleph_0$ and $\kappa \geq 2$ such that $\alpha < \lambda \Rightarrow |\alpha|^{<\kappa} < \lambda$.

1) We say a graph G is (λ, κ)-*entangled* when we have (A) $\Rightarrow$ (B), where

- (A) (a) $\varepsilon < \kappa$
  - (b) $\bar{a}_\alpha = \langle a_{\alpha,\zeta} : \zeta < \varepsilon \rangle \in {}^\varepsilon G$, and each $\bar{a}_\alpha$ is without repetitions (for $\alpha < \lambda$).
  - (c) For all $\alpha \neq \beta < \lambda$, the sets $\{a_{\alpha,\zeta} : \zeta < \varepsilon\}$ and $\{a_{\beta,\zeta} : \zeta < \varepsilon\}$ are disjoint.
- (B) For every $X \subseteq \varepsilon \times \varepsilon$, there exist $\alpha < \beta < \lambda$ such that

$$(\forall \zeta, \xi < \varepsilon)\big[a_{\alpha,\zeta}\, R^G\, a_{\beta,\xi} \;\Leftrightarrow\; (\zeta,\xi) \in X\big].$$

2) We say $I \in K^{\mathrm{org}}_{\mathcal{T},0}$ (of cardinality $\geq \lambda$) is (λ, κ)-entangled when we have (A) $\Rightarrow$ (B), where:

- (A) As above, but adding:
  - (d) If $\zeta, \xi < \varepsilon$, $\mathbf{o} \in \Omega_{\mathbf{p}}$, and $\alpha < \beta < \lambda$, then

$$F^I_{\mathbf{o}}(a_{\alpha,\zeta}) = a_{\alpha,\xi} \Leftrightarrow F^I_{\mathbf{o}}(a_{\beta,\zeta}) = a_{\beta,\xi}.$$

  - (e) If $\alpha < \beta < \lambda$ and $\gamma < \delta < \lambda$, then

$$(\forall \zeta, \xi < \varepsilon)[a_{\alpha,\zeta} <_I a_{\beta,\xi} \Leftrightarrow a_{\gamma,\zeta} <_I a_{\delta,\xi}].$$

- (B) For every $X \subseteq \varepsilon \times \varepsilon$, there exist $\alpha < \beta < \lambda$ such that

$$(\forall \zeta, \xi < \varepsilon)\big[a_{\alpha,\zeta}\, R^I\, a_{\beta,\xi} \;\Leftrightarrow\; (\zeta,\xi) \in X\big],$$

  provided that
  - If $\gamma < \lambda$, $\mathbf{o}_\ell \in \Omega$, $F^I_{\mathbf{o}_\ell}(a_{\gamma,\zeta_\ell}) = a_{\gamma,\xi_\ell}$ for $\ell = 1,2$, and $\mathbf{o}_1, \mathbf{o}_2$ have a common $\leq_\Omega$-upper bound, then

$$(\zeta_1, \zeta_2) \in X \Leftrightarrow (\xi_1, \xi_2) \in X.$$

3) If we omit κ and simply write 'λ-entangled,' we mean $\kappa := \aleph_0$.

We will state the following for a cardinal κ, but as in [Sheb] $\kappa := \aleph_0$ if not stated otherwise.

Definition 3.3. Assume K is as in 1.1, $\tau_{\mu,\kappa}$ is as in 0.18 (quoting from [Sheb]), and Σ is a set of $\tau_{\mu,\kappa}$-terms $\sigma(\bar{x})$, where $\bar{x} = \langle x_\zeta : \zeta < \varepsilon \rangle$ for some $\varepsilon < \kappa$. Further assume that $|\Sigma| \leq \mu$. (If $\mu = \mu^{<\kappa}$, this is automatic. If in addition Σ is the set of all $\tau_{\mu,\kappa}$-terms, then we may omit Σ.)

Then for $I, J \in K$, we say that I is *strictly* (μ, κ)-Γ-Σ-*unembeddable* into J (we may write 'μ' instead of $(\mu, \aleph_0)$) when we have '(A) $\Rightarrow$ (B),' where:

- (A) (a) $\mathcal{M}_{\mu,\kappa}(J)$ is a $\tau_{\mu,\kappa}$-structure as in 0.18.
  - (b) $F : I \to \mathcal{M}_{\mu,\kappa}(J)$
  - (c) For each $s \in I$, $F(s)$ is of the form $\sigma_s(\bar{t}_s)$, where
    - $\bullet_1$ $\sigma_s \in \Sigma$ (which is a set of $\tau_{\mu,\kappa}$-terms).
    - $\bullet_2$ $\ell g(\bar{t}_s) = \varepsilon(\sigma_s) = \varepsilon_s < \kappa$
    - $\bullet_3$ $\bar{t}_s = \langle t_{s,\varepsilon} : \varepsilon < \varepsilon_s \rangle \in {}^{\varepsilon_s}J$.
    - $\bullet_4$ If K has a linear order and

$$\kappa > \aleph_0 \Rightarrow J \text{ is well-ordered,}$$

      then $\bar{t}_s$ is $<_J$-increasing.

(d) Γ is a set of pairs $\big(p_1(\bar{x}), p_2(\bar{x})\big)$ such that for some $\varepsilon < \kappa$, $I' \in K$, and $\bar{t}_\ell \in {}^{\varepsilon}(I')$ (for $\ell = 1, 2$), we have

$$p_\ell \subseteq \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\ell, \varnothing, I').$$

(B) There exist $\varepsilon < \kappa$, $\bar{s}_1, \bar{s}_2 \in {}^{\varepsilon}I$, and $(p_1, p_2) \in \Gamma$ such that:

(a) $p_1 \subseteq \mathrm{tp}_{\mathrm{qf}}(\bar{s}_1, \varnothing, I)$ and $p_2 \subseteq \mathrm{tp}_{\mathrm{qf}}(\bar{s}_2, \varnothing, I)\big)$.

(b) $\sigma_{s_{1,\zeta}} = \sigma_{s_{2,\zeta}}$ for all $\zeta < \varepsilon$.

(c) The sequences $(\bar{t}_{s_{1,0}}{}^\frown \ldots {}^\frown \bar{t}_{s_{1,\zeta}}{}^\frown \ldots)_{\zeta<\varepsilon}$ and $(\bar{t}_{s_{2,0}}{}^\frown \ldots {}^\frown \bar{t}_{s_{2,\zeta}}{}^\frown \ldots)_{\zeta<\varepsilon}$ realize the same quantifier-free types in J.

The following claims will be used to get twins for independent theories in 3.6.

Claim 3.4. *Suppose λ is regular, $\mu \in [\aleph_0, \lambda)$, $\mathbf{p}$ is a strong twinship parameter, $\theta_{\mathbf{p}} > \aleph_0$, and $|\mathcal{T}_{\mathbf{p}}|^+ + |\mathcal{B}_{\mathbf{p}}| \le \lambda$. Let Σ be as in 3.3.*

Then we have '$\boxplus_1 \Rightarrow \boxplus_2$', where:

$\boxplus_1$ (a) *$J \in K^{\mathrm{org}}_{\mathcal{T},2}$ (see Definition 2.9(3)) is of cardinality λ.*

(b) *$\Gamma_{\mathrm{org}} := \big\{\big(p_1(x_0, x_1), p_2(x_0, x_1)\big)\big\}$, where*

$$p_1(x_0, x_1) := \big[x_0 < x_1 \ \wedge \ x_0 \, R \, x_1\big]$$

and

$$p_2(x_0, x_1) := \big[x_0 < x_1 \ \wedge \ \neg(x_0 \, R \, x_1)\big].$$

(c) *$X \in [J]^\lambda$ is well-ordered and maximal such that $x \neq y \in X$ implies $y \notin c\ell_J(\{x\})$ (equivalently, $\neg[x \ E^J \ y]$).*

(d) *$I := (J \restriction \tau_{\mathrm{org}}) \restriction X \in K_{\mathrm{org}}$ is λ-entangled.*

(e) *For $\ell = 1, 2$, we define*

$$X_\ell := \big\{F^I_{\mathbf{o}}(a) : a \in X, \ \mathbf{o} \in \Omega^J_a, \ \text{and } \ell g(\mathbf{o}) = \ell \mod 2\big\}.$$

(f) • $_1$ *If $D \in \mathcal{B}_{\mathbf{p}}$ then $|Y^0_D| \ge \lambda$, where $Y^0_D := \{s \in X : \Omega^J_s = \Omega_D\}$ (recalling Definition 2.9).*

• $_2$ *X is the disjoint union of $\langle Y^0_D : D \in \mathcal{B}_{\mathbf{p}}\rangle$.*

• $_3$ *(Note $X = X_1 \cup X_2$.)*

(g) *$I_\ell := I \restriction X_\ell$ for $\ell = 1, 2$.*

(h) $R^J = \big\{\big(F^J_{\mathbf{o}}(s), F^J_{\mathbf{o}}(t)\big) : \mathbf{o} \in \Omega, \ s \neq t \in X \cap \mathrm{dom}(F^I_{\mathbf{o}}), \ (s, t) \in R^I\big\}$

(i) *For $s_1, s_2 \in X$, if $\mathbf{o}_\ell \in \Omega^J_{s_\ell}$ for $\ell = 1, 2$, then*

$$s_1 <_J s_2 \Leftrightarrow F^J_{\mathbf{o}_1}(s_1) <_J F^J_{\mathbf{o}_2}(s_2).$$

(j) *If $t, v \in X$ and $\mathbf{o}_1, \mathbf{o}_2 \in \Omega^J_t = \Omega^J_v$, then*

$$F^J_{\mathbf{o}_1}(t) <_J F^J_{\mathbf{o}_2}(t) \Leftrightarrow F^J_{\mathbf{o}_1}(v) <_J F^J_{\mathbf{o}_2}(v).$$

$\boxplus_2$ *I_1 is strictly μ-Γ_{org}-Σ-unembeddable into I_2.*

Proof. This is a special case of 3.5 proved below, where we choose $Y := X$ and $Z := X_2$. (So $I_2 := I \restriction X_2$ here is equal to $I \restriction Z$ there, and $I_1 := I \restriction X_1$ here *contains* $I \restriction Y$ from there, hence the conclusion of 3.5 implies $\boxplus_2$ here.)

Now we have to verify that the conditions of 3.5 hold. This is straightforward, noting that in clause (d) $\bullet_2$, the λ-indiscernibility property follows from J being well-ordered, by 3.4$\boxplus_1$(a) (recalling 0.23). $\square_{3.4}$

What we really need is the following: it will be used in 3.6(2).

Claim 3.5. *Like 3.4, but*

$\boxplus_1$ (a) *$J \in K^{\mathrm{org}}_{\mathcal{T},2}$*

(b) *As there.*

(c) *$X \subseteq J$ is maximal such that $x \neq y \in X$ implies $y \notin c\ell_J(\{x\})$ (equivalently, $\neg[x\ E^J\ y]$).*[9]

(d) $\bullet_1$ *$Y \in [X]^{\geq\lambda}$ (So in clause (c) we necessarily have $|X| \geq \lambda$.)*

$\bullet_2$ *$I := (J \restriction \tau_{\mathrm{org}}) \restriction X$*

$\bullet_3$ *$I \restriction \{<\}$ has the λ-indiscernibility property. (See 0.22.)*

$\bullet_4$ *$I \in K_{\mathrm{org}}$ is λ-entangled.*

(e) *$Z :=$*

$$\{F^J_{\mathbf{o}}(a) : a \in X \cap \mathrm{dom}(F^J_{\mathbf{o}}),\ \mathbf{o} \in \Omega^J_a,\ \text{and } a \in Y \Rightarrow \ell g(\mathbf{o}) = 1 \mod 2\}$$

(f) $\bullet_1$ *If $D \in \mathcal{B}_{\mathbf{p}}$ then $|Y^0_D| \geq \lambda$, where $Y^0_D := \{s \in Y : \Omega^J_s = \Omega_D\}$ (recalling Definition 2.9).*

$\bullet_2$ *Y is the disjoint union of $\langle Y^0_D : D \in \mathcal{B}_{\mathbf{p}}\rangle$.*

(h), (i), (j) *As there.*

$\boxplus_2$ *$I \restriction Y$ is strictly μ-Γ_{or}-Σ-unembeddable into $I \restriction Z$.*

Proof. First,

$(*)_0$ Let F and $\langle \sigma_s(\bar{t}_s) : s \in Y\rangle$ (where $\bar{t}_s \in {}^{\omega>}Z$) be as in 3.3(A), with $I \restriction Y$ and $I \restriction Z$ here standing in for I, J there.

It will suffice to find $\bar{s}_1, \bar{s}_2 \in {}^2Y$ as in 3.3(B), recalling our choice of Γ in clause $\boxplus_1$(b).

Assume, for the sake of contradiction, that there are no such $\bar{s}_1, \bar{s}_2$.

$(*)_1$ For $s \in Y$, let $(\bar{t}'_s, \bar{\mathbf{o}}_s)$ be such that

(a) $\bar{t}'_s \in {}^{\omega>}X$ (Not to be confused with $\bar{t}_s \in {}^{\omega>}Z$!)

(b) $\ell g(\bar{t}_s) = \ell g(\bar{t}'_s) = \ell g(\bar{\mathbf{o}}_s) = n_s = n[s] := \ell g(\bar{t}_s)$ and $\mathbf{o}_{s,\ell} \in \Omega^J_s$.

(c) $\ell < n_s \Rightarrow t_{s,\ell} = F^J_{\mathbf{o}_{s,\ell}}(t'_{s,\ell})$

(d) Let $e_s := \{(\ell, k) : \ell, k < n_s,\ t'_{s,\ell} = t'_{s,k}\}$.

(e) $\langle \mathbf{o}_{s,\ell} : \ell < n_s\rangle \subseteq \Omega_{\mathbf{p}}$ satisfies

$$\mathbf{o} <_\Omega \mathbf{o}_{s,\ell} \Rightarrow s \notin \mathrm{dom}(F^J_{\mathbf{o}}).$$

[Why? For clause (e), recall the definition of $K^{\mathrm{org}}_{\mathbf{p},2}$ and see 2.9(3)(B). (This is guaranteed when $\mathbf{p}$ is well-founded, recalling 2.10.)]

$(*)_2$ Choose χ large enough, and choose $N \prec (\mathcal{H}(\chi), \in)$ of cardinality $< \lambda$ such that:

(a) $J, I, F, \mu, \mathbf{p}, X, Y, Z, \Phi$, and $\langle \sigma_s(\bar{t}_s) : s \in Y\rangle$ all belong to N.

(b) $\|N\| < \lambda$

(c) $N \cap \lambda \in \lambda$ (so $\mathcal{B}, \tau(\Phi), \Omega$, and Φ are all $\subseteq N$).

Next,

$(*)_3$ If $s \in Y \setminus N$ then there are sets $v_{s,1}$, $v_{s,2}$, $v_{s,3}$, and $\mathcal{U}_s$ (the first three being finite) such that:

(a) $v_{s,1} \subseteq (N \cap Z) \cup \{\infty\}$ and $v_{s,2} \subseteq n_s$.

(b) $s \in \mathcal{U}_s \in [Y]^\lambda$ and $\mathcal{U}_s \cap N = \varnothing$.

(c) The sequence $\langle(\sigma_r, \ell g(\bar{t}_r), e_r, \bar{\mathbf{o}}_r) : r \in \mathcal{U}_s\rangle$ is constant.

(d) $(\forall \ell \in v_{s,2})(\forall r \in \mathcal{U}_s)[t'_{r,\ell} \notin N \wedge t_{r,\ell} \notin N]$ and

$$(\forall \ell \in n_s \setminus v_{s,2})(\forall r \in \mathcal{U}_s)[t_{r,\ell} = t_{s,\ell} \in v_{s,1} \wedge t'_{r,\ell} = t'_{s,\ell}].$$

[9] I.e. we do not demand that X is well-ordered.

(e) $\langle t_{r,\ell} : \ell < n_s,\ r \in \mathcal{U}_s,\ t_{s,\ell} \notin N\rangle$ and $\langle t'_{r,\ell} : \ell < n_s,\ r \in \mathcal{U}_s,\ t'_{s,\ell} \notin N\rangle$ are without repetition (except for $t'_{r,\ell} = t'_{r,k}$ when $(\ell,k) \in e_s$).

(f) $t'_{s,\ell} \in v_{s,1}$ for every $\ell \in n_s \setminus v_{s,2}$.

(g) $v_{s,3} := \{\ell \in v_{s,2} : t'_{s,\ell} = s,\ \ell < n_s\} \subseteq v_{s,2}$

[Why? Easy.[10]]

$(*)_4$ Recall that we defined

$$Y^0_D := \{s \in Y : \Omega^J_s = \Omega_D\}$$

(so by assumption $\boxplus_1(f)$ we know $Y^0_D \in [Y]^{\geq\lambda}$).

$(*)_5$ For each $D \in \mathcal{B}$, we can choose σ_D, e_D, $v_{D,\iota}$ (for $\iota = 2,3$), $\langle t^*_{D,\ell} : \ell \in n_s \setminus v_{s,2}\rangle$, and $\bar{\mathbf{o}}_D = \langle \mathbf{o}_{D,\ell} : \ell < n_s\rangle$ such that $|Y^1_D| \geq \lambda$, where

$$Y^1_D := \big\{s \in Y^0_D : \sigma_D = \sigma_s,\ \bar{\mathbf{o}}_D = \bar{\mathbf{o}}_s,\ v_{D,\iota} = v_{s,\iota} \text{ for } \iota = 2,3, \text{ and } t^*_{D,\ell} = t'_{s,\ell} \text{ for } \ell \in n_s \setminus v_{s,2}\big\}.$$

[Why does this exist? As $|Y^0_D| = \lambda$ is regular and $> \|N\| \geq |\tau_J| + |\mathcal{T}_{\mathbf{p}}|$, there exists $s \in Y^0_D \cap Y \setminus N$, and we can use the same argument as for $(*)_3$.]

$(*)_6$ If $D_1, D_2 \in \mathcal{B}$, then there exist $\ell = \ell_{D_1,D_2} \in v_{D_1,3}$ and $k = k_{D_1,D_2} \in v_{D_2,3}$ such that $\mathbf{o}_{D_1,\ell}$ and $\mathbf{o}_{D_2,k}$ have a common $\leq_\Omega$-upper bound.

Why? First:

$(*)_{6.1}$ We can choose $s_{1,\varepsilon} \in Y^1_{D_1}$ and $s_{2,\varepsilon} \in Y^1_{D_2}$ for $\varepsilon < \lambda$ such that

$$\varepsilon < \zeta < \lambda \Rightarrow s_{1,\varepsilon} \neq s_{1,\zeta} \wedge s_{2,\varepsilon} \neq s_{2,\zeta}.$$

Second, by $\boxplus_1(d)\bullet_3$, without loss of generality

$(*)_{6.2}$ (a) $\big\langle \langle s_{1,\varepsilon}, s_{2,\varepsilon}\rangle \hat{}\ \bar{t}'_{s_{1,\varepsilon}} \hat{}\ \bar{t}'_{s_{2,\varepsilon}} : \varepsilon < \lambda\big\rangle$ is a qf-indiscernible sequence for the linear order $<^J$.

(b) All the individual sequences in clause (a) realize the same quantifier-free type in I.
(That is, the type $\mathrm{tp}_{\mathrm{qf}}(\langle s_{1,\varepsilon}, s_{2,\varepsilon}\rangle \hat{}\ \bar{t}'_{s_{1,\varepsilon}} \hat{}\ \bar{t}'_{s_{2,\varepsilon}}, \varnothing, I)$ does not depend on ε.)

Third,

$(*)_{6.3}$ $\big\langle \langle s_{1,\varepsilon}, s_{2,\varepsilon}\rangle \hat{}\ \bar{t}_{s_{1,\varepsilon}} \hat{}\ \bar{t}_{s_{2,\varepsilon}} : \varepsilon < \lambda\big\rangle$ is a qf-indiscernible sequence for $<^J$ as well.

[Why? By $(*)_{6.2}$ and $\boxplus_1(i),(j)$, recalling $(*)_5$.]

That is,

$(*)_{6.4}$ $\bullet_1$ For $\varepsilon < \zeta < \lambda$, let $\mathcal{W}^0_{\varepsilon,\zeta} := \big\{(\ell,k) \in n_{D_1} \times n_{D_2} : t'_{s_{1,\varepsilon},\ell}\ R^J\ t'_{s_{2,\zeta},k}\big\}$.

$\bullet_2$ $\mathcal{W}_* := \big\{(\ell,k) \in n_{D_1} \times n_{D_2} : \mathbf{o}_{D_1,\ell} = \mathbf{o}_{D_2,k}\big\}$

$(*)_{6.5}$ If $\varepsilon_1 < \zeta_1 < \lambda$ and $\varepsilon_2 < \zeta_2 < \lambda$ are such that $\mathcal{W}^0_{\varepsilon_1,\zeta_1} = \mathcal{W}^0_{\varepsilon_2,\zeta_2}$, <u>then</u>

$\bullet_1$ The sequences $\langle s_{1,\varepsilon_1}, s_{2,\varepsilon_1}\rangle \hat{}\ \bar{t}_{s_{1,\varepsilon_1}} \hat{}\ \bar{t}_{s_{2,\zeta_1}}$ and $\langle s_{1,\varepsilon_2}, s_{2,\varepsilon_2}\rangle \hat{}\ \bar{t}_{s_{1,\varepsilon_2}} \hat{}\ \bar{t}_{s_{2,\zeta_2}}$ realize the same quantifier-free type in $J \restriction \{<\}$.

$\bullet_2$ The sequences $\bar{t}_{s_{1,\varepsilon_1}} \hat{}\ \bar{t}_{s_{2,\zeta_1}}$ and $\bar{t}_{s_{1,\varepsilon_2}} \hat{}\ \bar{t}_{s_{2,\zeta_2}}$ realize the same quantifier-free type in J.

$\bullet_3$ If

$$[s_{1,\varepsilon_1}\ R^J\ s_{1,\zeta_1}] \Leftrightarrow \neg[s_{2,\varepsilon_1}\ R^J\ s_{2,\zeta_1}]$$

<u>then</u> we get the desired contradiction.

[10] Earlier, we had also demanded that

(h) For every $\ell \in v_{s,2}$ there exists $t^*_\ell \in v_{s,1}$ such that if $r \in \mathcal{U}_s$ <u>then</u> t^*_ℓ is the $<^J$-minimal member of $\{s \in X \cap N : s \geq_J t'_{r,\ell}\}$. (If there are *no* members of $X \cap N$ above $t'_{r,\ell}$, we say $t^*_\ell := \infty$. This is why we allowed $\infty \in v_{s,1}$ in clause $(*)_1$(a).)

[Why? For $\bullet_1$, first recall $(*)_{6.2}$(a) for the order on $\langle s_{1,\varepsilon}, s_{2,\varepsilon}\rangle{}^\frown\langle \bar{t}'_{s_{1,\varepsilon}}{}^\frown \bar{t}'_{s_{2,\varepsilon}} : \varepsilon < \lambda\rangle$. For $\bullet_2$, recall $(*)_{6.2}$(b) and our assumption $\mathcal{W}^0_{\varepsilon_1,\zeta_1} = \mathcal{W}^0_{\varepsilon_2,\zeta_2}$ for the graph relation. Lastly, $\bullet_3$ follows from our choices.]

$(*)_{6.6}$ $(\forall \varepsilon < \zeta < \lambda)\big[\mathcal{W}^0_{\varepsilon,\zeta} \subseteq \mathcal{W}_*\big]$

[Why? Holds by assumption $\boxplus_1(h)$.]

$(*)_{6.7}$ If $\mathcal{W}_*$ and $v_{D_1,3} \times v_{D_2,3}$ are disjoint, then we get a contradiction.

[Why? Because $I \in K_{\mathrm{org}}$ is λ-entangled, and get a contradiction by $(*)_{6.5}$+$(*)_{6.6}$.]

So for some $\mathbf{o} \in \Omega$ and $\ell < n_{D_1}$, $k < n_{D_2}$, for all $\varepsilon < \zeta < \lambda$, we have $F_{\mathbf{o}}(s_{1,\varepsilon}) = t_{s_{1,\varepsilon},\ell}$ and $F_{\mathbf{o}}(s_{2,\zeta}) = t_{s_{2,\zeta},k}$.

Now $J \in K^{\mathrm{org}}_{\mathcal{T},2} \subseteq K^{\mathrm{org}}_{\mathcal{T},1}$ (see 2.6(2),(3)). Therefore, recalling $(*)_1$(e), we conclude that $\mathbf{o}_{D_1,\ell}$ and $\mathbf{o}_{D_2,k}$ have a common $\leq_\Omega$-upper bound.

This proves $(*)_6$.

Now we recall 2.2(1A):

$\boxplus$ $\theta_{\mathbf{p}} > \aleph_0$, and the intersection of countably many members of $\mathcal{B}$ will always contain some other member of $\mathcal{B}$.

Hence

$(*)_7$ There exists $\mathbf{n}$ such that
$$\mathcal{B}_{\mathbf{n}} := \{D \in \mathcal{B} : n_D = \mathbf{n}\}$$
is $\leq_{\mathbf{p}}$-cofinal in $(\mathcal{B}, \leq_{\mathbf{p}})$.

$(*)_8$ Let $\overline{\mathbf{m}}_D := \big(\langle \ell g(\mathbf{o}_{D,\ell}) : \ell < \mathbf{n}\rangle, v_{D,2}, v_{D,3}\big)$.

$(*)_9$ Let $\mathbb{E}$ be an ultrafilter on $\mathcal{B}$ which includes $\mathcal{B}_{\mathbf{n}}$, such that for every $D \in \mathcal{B}$ we have
$$\{D' \in \mathcal{B} : D' \subseteq D\} \in \mathbb{E}.$$

[Why does such an $\mathbb{E}$ exist? Because $(\mathcal{B}_{\mathbf{p}}, \supseteq)$ is directed and $\mathcal{B}_{\mathbf{n}}$ is cofinal in it.]

$(*)_{10}$ (a) For each $D \in \mathcal{B}_{\mathbf{n}}$ there exist $\ell_D, k_D < \mathbf{n}$ such that
$$\mathcal{X}_D := \{D' \in \mathcal{B}_{\mathbf{n}} : \ell_D = \ell_{D,D'},\ k_D = k_{D,D'}\} \in \mathbb{E}.$$
(b) There exist $\ell_*, k_* < \mathbf{n}$ such that
$$\mathcal{B}_\bullet := \{D \in \mathcal{B}_{\mathbf{n}} : \ell_D = \ell_*,\ k_D = k_*\} \in \mathbb{E}.$$

[Why? Obvious: there are only finitely many possibilities (clause (a) gives $< \mathbf{n}$ and clause (b) gives $< \mathbf{n}^2$), and $\mathbb{E}$ is an ultrafilter. We may add more, but this is not necessary.]

$(*)_{11}$ $\big(\forall^{\mathbb{E}} D_1 \in \mathcal{B}_\bullet\big)\big(\forall^{\mathbb{E}} D_2 \in \mathcal{B}_\bullet\big)\big[\ell g(\mathbf{o}_{D_1,\ell_*}) \leq \ell g(\mathbf{o}_{D_2,k_*})\big]$

[Why? By clause (C)(b) of Definition 2.2(1), for any $D_1 \in \mathcal{B}_\bullet$ there is $D'_2 \in \mathcal{B}_{\mathbf{p}}$ such that
$$\{\eta : \eta \text{ appears in } \mathbf{o}_{D_1,\ell_*}\} \cap D'_2 = \varnothing.$$
(That is, $\eta \in \mathrm{rang}(\bar{\eta}_{D_1,\ell_*})$ where $\mathbf{o}_{D_1,\ell_*} = (\bar{\eta}, \bar{\iota}) = (\bar{\eta}_{D_1,\ell_*}, \bar{\iota}_{D_1,\ell_*})$.)

Now any $D_2 \subseteq D'_2$ from $\mathcal{B}_\bullet$ will work.]

$(*)_{12}$ If $k < \omega$ and $D_i \in \mathcal{B}_\bullet$ for $i < k$, <u>then</u> $\{\mathbf{o}_{D_i,\ell_*} : i < k\}$ has a common $\leq_\Omega$-upper bound.

[Why? Recalling $(*)_{10}$(a)+$(*)_{11}$, choose any $D_* \in \bigcap_{i<k} \mathcal{X}_{D_i}$ such that $\ell g(\mathbf{o}_{D_*,k_*}) \geq \ell g(\mathbf{o}_{D_i,\ell_*})$ for all $i < k$. Then $\mathbf{o}_{D_*,k_*}$ is a common upper bound.]

$(*)_{13}$ Assume $D \in \mathcal{B}_\bullet$, and let $\eta_D \in \mathcal{T}$ denote the 0^{th} η-term in $\mathbf{o}_{D,\ell_*}$.
(a) $\mathbf{G} := \{\eta_D : D \in \mathcal{B}_\bullet\}$ is directed under $\leq_{\mathcal{T}}$.
(b) $\mathbf{G} \cap D \neq \varnothing$ for all $D \in \mathcal{B}_\bullet$.

[Why? Clause (a) holds by our choices. Clause (b) holds because for every $D \in \mathcal{B}_{\mathbf{p}}$ there exists $D' \in \mathcal{B}_{\bullet}$ such that $D' \subseteq D$.]

So $\mathbf{G}$ contradicts clause 2.2(1A)(d) in the definition of 'strong twinship parameter,' and we are done. $\square_{3.5}$

Now at last we are able to prove Theorem 0.12.

Conclusion 3.6. *Assume $\mathbb{P}$ is a forcing notion adding no new ω-sequence of ordinals, but does add some infinite sequence (so necessarily of length $\geq \omega_1$).*

Assume $T \subseteq T_1$ are complete first-order theories, and T is independent as witnessed by $\varphi = \varphi(\bar{x}_{[k]}, \bar{y}_{[k]}) \in \mathbb{L}(\tau_T)$.

1) *Then there are models M, N such that:*

(a) *M and N are models of T_1 of cardinality $\lambda := (2^{\|\mathbb{P}\|} + |T_1|)^+$ (or use any regular cardinal $\lambda' \geq \lambda$).*

(b) *$\Vdash_{\mathbb{P}}$ "$M \restriction \tau_T \cong N \restriction \tau_T$", and even $\Vdash_{\mathbb{P}}$ "$M \cong N$".*

(c) *$M \restriction \tau_T$ and $N \restriction \tau_T$ are not isomorphic.*

(d) *Moreover, M is $(\lambda, 2^{\|\mathbb{P}\|}, \varphi)$-far from N (see 1.10(1)).*

2) *We may strengthen clause (d) to*

$(d)^+$ *M and N are $(\lambda, 2^{\|\mathbb{P}\|}, \varphi)$-far from each other.*

Discussion 3.7. Recalling Definition 1.8, note that in 3.6 we cannot deduce that M and N are (λ, φ)-far, as the partial isomorphisms $F^J_{\eta,\iota}$ form a witness. Now clause $\boxplus_1$(c) in the assumptions of 3.4 is strong, *but* it is only talking about $(J \restriction \tau_{\text{org}}) \restriction X$, so the partial isomorphism $F^J_{\eta,\iota}$ disappears.

However, the possibility of being $(\lambda, |\mathcal{B}_{\mathbf{p}}|, \varphi)$-far (see Definition 1.10) is not excluded.

Remark 3.8. The following are natural extensions of Claim 3.6. (Their proofs will be delayed to [S$^+$a].)

3) In parts (1) and (2) of 3.6, we may add

(e) M and N are $\mathbb{L}_{\infty,\lambda}$-equivalent.

4) If $\lambda = \text{cf}(\lambda) > 2^{\|\mathbb{P}\|} + |T_1|$ and $\xi < \lambda$, <u>then</u> we can find models M, N of T_1 such that:

(a) $\|M\| = \|N\| = |{}^{\xi>}\lambda|$

(b), (c) As in 3.6(1).

(d) M and N are cofinally (λ, ξ)-equivalent (see Definition 0.20).

(e) M and N are $(\lambda, 2^{\|\mathbb{P}\|}, \varphi)$-*far* from each other.

5) It is enough to demand that λ is regular, $> |\mathbb{P}|$, and $\geq$ the number of maximal antichains in $\mathbb{P}$.

Proof. PROOF OF 3.6:

1) First, choose a strong twinship parameter $\mathbf{p}$ by Claim 2.19 (so $\theta_{\mathbf{p}} \leq \|\mathbb{P}\|$, $|\mathcal{T}_{\mathbf{p}}| \leq \|\mathbb{P}\|$, and $|\mathcal{B}_{\mathbf{p}}| \leq 2^{\|\mathbb{P}\|}$).

$(*)_1$ We may require

(a) $\theta_{\mathbf{p}} := \min\{\theta : \Vdash_{\mathbb{P}}$ "there is a new $\eta \in {}^{\theta}2$"$\}$

(b) $|\mathcal{B}_{\mathbf{p}}| = \big|\{|Y| : Y \text{ is a maximal antichain of } \mathbb{P}\}\big|$.

Second,

$(*)_2$ (a) Choose λ regular such that $\lambda > |\tau(T_1)| + |\mathcal{T}_{\mathbf{p}}|$, $\lambda \geq |\mathcal{B}_{\mathbf{p}}|$.

(b) $\mathbf{c} : [\lambda]^2 \to \omega$ witnesses the property called $\mathrm{Pr}_0(\lambda, \aleph_0)$ in [She90b, Th. 1.1] (later called $\mathrm{Pr}_0(\lambda, \lambda, \aleph_0, \aleph_0)$).

Recall

⊙ $\mathrm{Pr}_0(\lambda, \aleph_0)$ means that if $n < \omega$ and $\langle \bar\zeta_\alpha = \langle \zeta^\alpha_1 : i \leq n \rangle : \alpha < \lambda \rangle$ is such that $\zeta^\alpha_1 < \zeta^\alpha_2 < \ldots < \zeta^\alpha_n < \lambda$ and $\alpha < \beta < \lambda \Rightarrow \bar\zeta_\alpha \cap \bar\zeta_\beta = \varnothing$, then for any function $h : n \times n \to \mu$ there exists $\alpha < \beta < \lambda$ such that $\zeta^\alpha_n < \zeta^\beta_1$ and

$$k, \ell \in [1, n] \Rightarrow \mathbf{c}(\{\zeta^\alpha_k, \zeta^\beta_\ell\}) = h(k, \ell).$$

[Why does such a $\mathbf{c}$ exist? By [She90b, Th. 1.1].]

Now choose $J \in K^{\mathrm{org}}_{\mathcal{T},2}$ as in 3.4 $\boxplus_1$ such that:

$(*)_3$ (a) $X \subseteq J$, $(X, <_J) = (\lambda, <)$, and the pair (X, J) satisfies clauses (h),(i),(j) of 3.4 $\boxplus_1$.

(b) For $s, t \in X$, we have $(s, t) \in R^J \Leftrightarrow s, t \in J \wedge s \neq t \wedge \mathbf{c}(\{s, t\}) = 1$.

[Why? For each $D \in \mathcal{B}_{\mathbf{p}}$ we can find $I_D \in K_{\mathrm{org}}$ of cardinality $|D|$ (which is infinite, but $\leq \aleph_0 + |\mathcal{T}_{\mathbf{p}}| < \lambda$) such that $s \in I_D \Rightarrow \Omega^{I_D}_s = \Omega_D$ by 2.8. Let $s_D = s[D]$ be some member of I_D.

Let $\langle D_\alpha : \alpha < \lambda \rangle \in {}^\lambda\mathcal{B}_{\mathbf{p}}$ be such that $(\forall D \in \mathcal{B}_{\mathbf{p}})(\exists^\lambda \alpha < \lambda)[D_\alpha = D]$. We define $J \in K^{\mathrm{org}}_{\mathcal{T},2}$ as follows.

$(*)_4$ (a) $|J| := \lambda \times I_D$. (That is, $\{(\alpha, s) : \alpha < \lambda,\ s \in I_D\}$.)

(b) $(\alpha_1, s_1) <_J (\alpha_2, s_2)$ iff

$\bullet_1$ $(\alpha_1, s_1), (\alpha_2, s_2) \in J$

$\bullet_2$ $\alpha_1 < \alpha_2 \vee \big[\alpha_1 = \alpha_2 \wedge s_1 <_{I_{D_\alpha}} s_2\big]$.

(c) $F^J_{\eta,1}(\alpha_1, s_1) = (\alpha_2, s_2)$ iff $\alpha_1 = \alpha_2 \wedge D = D_{\alpha_1} \wedge F^{I_D}_{\eta,1}(s_1) = s_2$.

(d) $X := \big\{(\alpha, s_{D_\alpha}) : \alpha < \lambda\big\}$

(e) We choose

$$R^J := \big\{(F^J_{\mathbf{o}}(\alpha_1, s_1), F^J_{\mathbf{o}}(\alpha_2, s_2)) : \alpha_1 \neq \alpha_2 < \lambda, \quad \mathbf{c}(\{\alpha_1, \alpha_2\}) = 1, \quad \mathbf{o} \in \Omega_{D_{\alpha_1}} \cap \Omega_{D_{\alpha_2}}\big\}.$$

Now check. Note that $(J, <_J)$ is not $(\lambda, <)$, but we get this by renaming.]

Next,

$(*)_5$ (a) Choose $\varphi = \varphi(\bar x_{[k]}, \bar y_{[k]}) \in \mathbb{L}_{\tau_T}$ witnessing the independence property for T (see Definition 1.4(2)).

(b) Choose $\Phi \in \Upsilon_{K_{\mathrm{org}}}\big[T_1, |T_1|\big]$ such that

$$I \in K_{\mathrm{org}} \Rightarrow \big(\forall s, t \in I\big)\big[\mathrm{GEM}(I, \Phi) \models \text{“}\varphi[\bar a_s, \bar a_t] \iff s\ R^I\ t\text{”}\big].$$

[Why can we do this? By 1.6.]

Now we shall finish by 3.4. That is, for $\ell = 1, 2$:

$(*)_6$ (a) Recall that we defined

$$X_\ell := \{F^J_{\mathbf{o}}(s) : s \in X \cap \mathrm{dom}(F^J_{\mathbf{o}}),\ \mathbf{o} \in \Omega^{\mathrm{fr}}_{\mathbf{p}},\ \ell g(\mathbf{o}) \equiv \ell \mod 2\}.$$

(b) Now let M_ℓ be the submodel of $\mathrm{GEM}(J \restriction \tau_{\mathrm{org}}, \Phi)$ generated by $\{a_s : s \in X_\ell\}$, where $\langle a_s : s \in J \rangle$ is the skeleton.

Now $(M, N) := (M_1, M_2)$ are as required. Moreover, M_1 and M_2 are strictly **p**-isomorphic (see 2.4).

We have to check all four clauses of the conclusion in part (1).

Clause (a)**:** M and N are models of T_1 of cardinality λ.

Why? "Models of T_1:" this follows from our choice of Φ in $(*)_5(b)$ and of M_ℓ in $(*)_6(b)$. Their cardinalities are as required because they are the same as $|X_\ell|$, recalling $(*)_6(a)$.

Clause (b)**:** $\Vdash_{\mathbb{P}}$ "$M \restriction \tau_T \cong N \restriction \tau_T$", and even $\Vdash_{\mathbb{P}}$ "$M \cong N$".

Why? This follows from

$$\Vdash_{\mathbb{P}} \text{“}(J \restriction \tau_{\text{org}}) \restriction X_1 \cong (J \restriction \tau_{\text{org}}) \restriction X_2\text{”},$$

which holds by 2.13(1).

Clause (c)**:** $M \restriction \tau_T$ and $N \restriction \tau_T$ are not isomorphic.

This is the main point, and it follows from **??**.

Clause (d)**:** M is $(\lambda, 2^{\|\mathbb{P}\|}, \varphi)$-far from N.

Just repeat the proof of 3.5.

2) We need to prove

$(D)^+$ M and N are $(\lambda, 2^{\|\mathbb{P}\|}, \varphi)$-*far* from each other.

We intend to use 3.5 instead of 3.4 for $\ell = 1, 2$. For proving clause $(d)^+$, we need Y_1 and Y_2, hence Z_1, Z_2.

So let $Y_1, Y_2 \in [X]^\lambda$ be a partition of X and let

$$Z_\ell := \{F^J_{\mathbf{o}}(s) : s \in X \cap \operatorname{dom}(F^J_{\mathbf{o}}), \text{ and } s \in Y_\ell \Rightarrow \ell g(\mathbf{o}) \equiv \ell \mod 2\}.$$

Also choose $\eta_s \in D^J_s$ for all $s \in X$. Lastly, let

$$X_\ell := \{F^J_{\mathbf{o}}(s) : s \in X \cap \operatorname{dom}(F^J_{\mathbf{o}}), \ell g(\mathbf{o}) \equiv \ell \mod 2\}.$$

We continue as in the proof of part (1). $\square_{3.6}$

Claim 3.9. *Like* 3.6*, when T is countable and superstable, with OTOP or DOP, and $T_1 = T$.*

Remark 3.10. 1) For the existence of ccc forcings with T unsuperstable, this is proved in [BLS93, 1.1], building on [She90a, Ch.X, §2; Ch.XIII, §2]. Here we have to use $\Phi \in \Upsilon[T, K_{\text{org}}]$ (not just $\Upsilon[T, K_{\text{or}}]$).

2) For T stable but not superstable, there is a more liberal version allowing infinite sequences in the parameters; we will return to this in [S+a].

3) Concerning OTOP below, note that [Sheb, $1.28_{\text{=Ld17}}$] is quite relevant.

4) This proof will require some knowledge of stability theory.

Proof. Similar to the proof of 3.6.

The point is that those properties imply the existence of Φ as in the proof of 3.6, except that the relevant φ is not first-order.

Below, we will prove existence for the two possible cases.

Case 1: T has OTOP but fails DOP.

Recall the definition of OTOP from [She90a, Ch.XII, 4.1, p.608].

$\boxplus_{1.1}$ We say that T has the *Omitting Type Order Property* (OTOP) if there is a type $p(\bar{x}, \bar{y}, \bar{z})$ (where $\bar{x}, \bar{y}, \bar{z}$ are finite sequences with $\ell g(\bar{x}) = \ell g(\bar{y})$) such that for every λ and every two-place relation R on λ, there exists a model M of T and $\langle \bar{a}_\alpha : \alpha < \lambda \rangle \subseteq M$ such that

$$\alpha < \beta < \lambda \Rightarrow \big[\alpha\ R\ \beta \Leftrightarrow \text{The type } p(\bar{a}_\alpha, \bar{a}_\beta, \bar{z}) \text{ is realized in } M\big].$$

Everything we need has been proven in [Sheb], but let us elaborate.

For Case 1, the following claim will suffice. (Note that this is [Sheb, $1.28_{=\mathsf{Ld17}}$], for ordered graphs instead of linear orders.)

$\boxplus_{1.2}$ If T is first-order, countable, and has the OTOP, then for some sequence $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}, \bar{z}) : i < i_* \rangle$ of first-order formulas in $\mathbb{L}(\tau_T)$ and a template Φ proper for ordered graphs, we have:
(a) $\tau_T \subseteq \tau_\Phi$ and $|\tau_\Phi| = |\tau_T| + \aleph_0$.
(b) $\mathrm{GEM}_{\tau_T}(I, \Phi) \models T$ for all $I \in K_{\mathrm{org}}$.
(c) If $I \in K_{\mathrm{org}}$ and $s, t \in I$, then

$$\big[\mathrm{GEM}_{\tau_T}(I, \Phi) \models (\exists \bar{x}) \bigwedge_{i<i_*} \varphi_i(\bar{x}, \bar{a}_s, \bar{a}_t)\big] \Leftrightarrow I \models \text{`}s\ R\ t\text{'}.$$

Proof. PROOF OF $\boxplus_{1.2}$:

We would like to apply [Sheb, $1.25(\mathrm{e})_{=\mathsf{Ld8}}$], but it requires us to assume enough cases of a certain partition theorem generalizing Erdős-Rado (with K_{or} replaced by K_{org}). However, this theorem is proven to hold only after a forcing — in fact, for every infinite cardinal κ there is a κ-complete class forcing which ensures that the analogous result holds above κ (it holds by [She89]).

Moreover, by [She90a, Ch.XII, §5], assuming T is countable, superstable, and has the NDOP, if it has the $(\aleph_0, 2)$-existence property then it satisfies clause (B) or (C) of $0.4(1)\boxplus_1$. This gives us a contradiction, so T fails the $(\aleph_0, 2)$-existence property. This is preserved by any κ-complete forcing $\mathbb{P}$, even for $\kappa := \aleph_0$ (but $\aleph_1$ is more convenient here).

By [She90a, Ch.XII, 4.3, p.609] T still has the OTOP in $\mathbf{V}^{\mathbb{P}}$, so we can apply [Sheb, $1.25(\mathrm{e})_{=\mathsf{Ld8}}$] to get Φ as promised. But as we assumed $\kappa > \aleph_0$ we know $\Phi \in \mathbf{V}$, so we are done proving $\boxplus_{1.2}$, and hence we have finished the present case. $\square_{\boxplus_{1.2}}$

Case 2: T **has** DOP (*dimensional order property*)**.**

Here we shall use [She90a, Ch.X, 2.1-2, p.512] and [She90a, Ch.X, 2.4, p.515]. Without loss of generality $\kappa := |T|^+$.

Recall:

$\boxplus_{2.1}$ We say that T has the *Dimensional Order Property* (DOP) if there are κ-saturated models $M_\ell \prec \mathfrak{C}$ of cardinality $\leq 2^{|T|}$ (for $\ell = 0, 1, 2$), where $\mathfrak{C}$ is a $\varkappa$-saturated model of T for some $\varkappa > \|M_1\| + \|M_2\|$, such that $M_0 \subseteq M_1 \cap M_2$, $\{M_1, M_2\}$ is independent over M_0, and the κ-prime model M over $M_1 \cup M_2$ is not $\mathbf{F}^a_\kappa$-minimal over $M_1 \cup M_2$.

$\boxplus_{2.2}$ For any $\kappa > 2^{|T|}$ (for transparency), for any κ-complete forcing $\mathbb{P}$, the relevant properties of T are still preserved in $\mathbf{V}^{\mathbb{P}}$.

$\boxplus_{2.3}$ In $\boxplus_{2.1}$, we can find finite $\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{b}, c$ such that:
(a) $\bar{a}_\ell \subseteq M_\ell$ for $\ell = 0, 1, 2$.
(b) $\bar{b}{}^\frown\langle c \rangle \subseteq M$
(c) For $\ell = 1, 2$, $\mathrm{tp}(\bar{a}_\ell, M_0, M)$ does not fork over $\bar{a}_0$ and $\mathrm{tp}(\bar{a}_\ell, \bar{a}_0, M)$ is stationary.

(d) $\mathrm{tp}(\bar b{}^\frown\langle c\rangle, M_1 \cup M_2, M)$ does not fork over $\bar a_1{}^\frown\bar a_2$, and
$$\mathrm{tp}(\bar b{}^\frown\langle c\rangle, \bar a_1{}^\frown\bar a_2, M)$$
has a unique non-forking extension in $\mathcal{S}(M_1, M_2)$.

(e) $\mathrm{tp}(c, \bar b, M)$ is stationary and $c \in \mathbf{I}$, where $\mathbf{I} \subseteq M$ is infinite, indiscernible over $M_1 \cup M_2 \cup \bar b$, and based on $\mathrm{tp}(c, \bar b, M)$.

[Why? Because T is superstable. (For clause (d), recall [She90a, Ch.XII, §3].)]

$\boxplus_{2.4}$ For every λ and $R \subseteq \lambda \times \lambda$, we can find $\langle f^\ell_\alpha : \alpha < \lambda,\ \ell = 1, 2\rangle$, $\langle f_{\alpha,\beta}, \mathbf{I}_{\alpha,\beta} : (\alpha, \beta) \in R\rangle$, and N such that:

(a) $M_0 \prec N$

(b) f^ℓ_α is an elementary embedding of M_ℓ into N over M.
We let $\bar a_{\ell,\alpha} := f^\ell_\alpha(\bar a)$ and $M_{\ell,\alpha} := f^\ell_\alpha(M_\ell)$.

(c) $\langle M_{\ell,\alpha} : \alpha < \lambda,\ \ell = 1, 2\rangle$ is independent over M_0.

(d) $f_{\alpha,\beta}$ is an elementary embedding of M into N extending $f^1_\alpha \cup f^2_\beta$.
We let $\bar b_{\alpha,\beta} := f_{\alpha,\beta}(\bar b)$ and $c_{\alpha,\beta} := f_{\alpha,\beta}(c)$.

(e) $\mathbf{I}_{\alpha,\beta}$ is an indiscernible set over $\bar b_\alpha$ based on $\mathrm{tp}(c_{\alpha,\beta}, \bar b_\alpha)$ (equivalently, on $\mathrm{tp}(c_{\alpha,\beta}, \bar b_\alpha \cup M_{1,\alpha} \cup M_{2,\beta})$) of cardinality $|T|$.

(f) N is $|T|^+$-prime over
$$\bigcup_{\substack{\alpha<\lambda\\ \ell=1,2}} M_{\ell,\alpha} \cup \bigcup_{(\alpha,\beta)\in R} f_{\alpha,\beta}(\bar b{}^\frown\langle c\rangle) \cup \bigcup_{(\alpha,\beta)\in R} \mathbf{I}_{\alpha,\beta}.$$

(g) If $(\alpha, \beta) \in \lambda \times \lambda \setminus R$, then we cannot find $\bar b_{\alpha,\beta}, c_{\alpha,\beta}, \mathbf{I}_{\alpha,\beta}$ as above (in N).

As in Case 1, we can find a suitable Φ. $\square_{3.9}$

Problem 3.11. Can we eliminate "$\theta_{\mathbf{p}} > \aleph_0$" in §3?

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