RANDOM GRAPH: STRONGER LOGIC BUT WITH THE ZERO-ONE LAW
SH1077

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Abstract. We like to find a logic really stronger than first order for the random graph with edge probability $\frac{1}{2}$ but satisfies the 0-1 law. The "really stronger" means that on the one hand it satisfies the 0-1 law, e.g. for the random graph $\mathscr{G}_{n,1/2}$ and on the other hand there is a formula $\varphi(x)$ such that for no first order $\psi(x)$ do we have: for every random enough $\mathscr{G}_{n,1/2}$ the formulas $\varphi(x), \psi(x)$ are equivalent in it. We do it adding a quantifier on graphs called $\mathbb{Q}_{\mathbf{t}}$, i.e. have a class of finite graphs closed under isomorphisms and being able to say that: if $(\varphi_0(x;\bar{y}), \varphi_1(x_0, x_1;\bar{y}))$ is a pair of formulas with parameters defining a graph in $\mathscr{G}_{n,1/2}$, then we can form a formula $\psi(\bar{y})$ such that for random enough $\mathscr{G} = \mathscr{G}_{n,1/2}$ and $\bar{c} \in {}^{\lg(\bar{y})}\mathscr{G}$, we have: $\psi(\bar{c})$ says that the graph belongs to $K_{\bar{\mathbf{t}}}$. Presently we do it for random enough $\bar{\mathbf{t}}$.

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Annotated Content

[We choose a function $\mathbf{h} : \mathbb{N} \to (0,1)_{\mathbb{R}}$ going to zero slowly enough. We intend to add to first-order logic a quantifier describing random properties of a graph but excluding some "low", and "explicitly not random" graphs. Those are graphs such that for any appropriate quantifier-free first-order formula $\varphi(\bar{x}_0, \bar{x}_1, \bar{z})$ for some k, for random enough $G = \mathscr{G}_{n,1/2}$ (or $\mathscr{G}_{n,p}$ for a given $p \in (0,1)_{\mathbb{R}}$), if $\bar{c} \in {}^{\ell g(\bar{z})}G$ and $\varphi(\bar{x}_0, \bar{x}_1, \bar{c})$ define in G a graph $H_{\varphi(G)}$ with $> k$ nodes then $H_{G,\varphi(x,y,\bar{c})}$ is "very not random" or so-called low. See Remark 1.5, where we actually explain that we cover the most general case. This will be used in §2 to find a logic as desired.]

[Adding a quantifier, we have new formulas. We imitate this by adding new relations which are also drawn randomly with appropriate probabilities.]

[We try to eliminate the graphs which may lead to the random graph $\mathscr{G}_{n,1/2}$ having a complicated theory; i.e. the low. For the $\mathbf{h}$-low version, those are the finite graphs which contains e.g. a too large bipartite graph which is complete or empty. Elaborating further, we need that, on the one hand, for the graph $\mathscr{G}_{n,1/2}$ itself, the probability of satisfying the quantifiers goes to zero as n grows, but on the other hand, considering omitting one node x of $\mathscr{G}_{n,1/2}$, almost surely there is x such that $\mathcal{G}_{n,1/2} \setminus \{x\}$ will satisfy the quantifiers. So we have a definable set of nodes whose size goes to infinity with x, but still is $<< n$. However, we exclude the definable graphs which are not random enough because otherwise we lose control.]

[We choose randomly enough a set $\mathbf{K}$ of (isomorphism types of) finite non-$\mathbf{h}$-low graphs and show that adding a quantifier for it preserves the zero-one law. So, the probability of H, a non-$\mathbf{h}$-low graph to be in the class is $\mathbf{h}(|H|)$, hence go to zero slowly with $|H|$, the number of nodes of H. Why $\mathbf{h}$ is not constant? Because we like that on the one hand, $\text{Prob}(\mathscr{G}_{n,p} \in \mathbf{K})$ converge to 0 (or to 1) so that a sentence saying "the graph $\mathscr{G}_{n,p}$ belongs to $\mathbf{K}$" converge to 0 (or to 1), and similarly for any graph definable in $\mathscr{G}_{n,p}$ by a first order formula without parameters. On the other hand, the probability of e.g. "there is $a \in \mathscr{G}_{n,p}$ such that $\mathscr{G}_{n,p} \restriction \{b \in [n] : b \neq a\}$ belongs to K" will go to 1, as there will be $\ll n$ such nodes but still many.]

[We discuss how we may replace the "random quantifier" by a definable one.]

§ 0. Introduction

The investigations of 0-1 *laws,* i.e., investigating when a sentence in random finite models was initiated by [GKLT69] and independently, Fagin [Fag76]. They showed that every first-order sentence satisfies the zero-one law in the classical case: the random graph with "being edge" has probability 1/2. On this subject history see e.g. Alon and Spencer book [AS08].

Our aim is to find a logic $\mathscr{L}$ stronger than first-order such that: for $p \in (0,1)_{\mathbb{R}}$, the p-random graph $\mathscr{G} = \mathscr{G}_{n,p}$ (i.e. with edge probability p) satisfies the 0-1-law but some formula $\varphi(x) \in \mathbb{L}$(graphs) defines in random enough graph $\mathscr{G}_{n,p}$ a set of nodes not definable by any first-order logic formula (of course, φ should be small enough compared to n, even with parameters).

The logic is gotten from first order $\mathbb{L}$ by adding a (Lindström) quantifier $\mathbf{Q}_{\bar{\mathbf{t}}} = \mathbf{Q}_{\mathbf{K}_{\bar{\mathbf{t}}}}$ gotten from a "random enough" $\bar{\mathbf{t}} \in {}^{\mathbb{N}}\{0,1\}$; on quantifiers see [Be85]. We may wonder, can we replace $\mathbf{Q}$ by a "reasonably defined quantifier"? We may from the proof see what we need from $\mathbf{K}$, the class defining the quantifier $\mathbf{Q}_{\mathbf{K}}$, i.e. a class of (finite) graphs closed under isomorphisms. Excluding some graphs which we call low, the membership in $\mathbf{K}$ should be random enough in the sense that if we consider only random enough $\mathscr{G}_{n,p}$, the non-trivial $\mathbb{L}(\mathbf{Q}_{\mathbf{K}})$-formulas with parameters will define graphs which are not $\mathbf{h}$-low and are pairwise non-isomorphic except in trivial cases. So we just need a definition satisfying this; we hope to try to do it in a work in preparation.

How does the randomness of $\bar{\mathbf{t}}$ help us to get the zero-one law? The idea is that for the quantifier $\mathbf{Q}_{\bar{t}}$ (see §2) used here, if we expand $\mathscr{G}_{n,p}$ by finitely many relations definable by formulas from $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$, we get a random structure with more relations essentially with constant probabilities, i.e. is interpretable in a suitable $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$, see §1, it look like $\mathscr{G}_{p,n}$ (but with some relations of suitable kinds as we sort out), with, e.g. $\bar{p} = \langle p_n : n < \omega \rangle$ with p_n going slowly to zero.

That is, fixing formulas $\varphi_\ell(\bar{x}_\ell) \in \mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$ starting with $\mathbf{Q}_{\bar{\mathbf{t}}}, \ell < k$ with no obvious connections we decide a priory that for a random enough $\mathscr{G}_{n,\bar{p}}$ the structure $([n], R_\ell^{\mathscr{G}_{n,p}})_{\ell<k} = (\varphi(\mathscr{G}_{p.n}), \dots, \varphi_\ell(\mathscr{G}_{p,n}), \dots)_\ell$ for suitable formulas $\varphi(x), \varphi_\ell(\bar{x}_\ell)$, will look like $\mathscr{M}$ above.

The decision is the simplest one: look as if truth values of $R_\ell^{\mathscr{G}_{n,p}}(\bar{a})$ were drawn independently, with probability p_n. This is an over simplification! We need a more involved such drawing, reflecting the original $\bar{\varphi}_\ell$ to some extent, see below.

We may replace $\mathscr{M}_{\mathbf{s},\bar{p},n}$ by using (for some irrational $\alpha \in (0,1)$) $\bar{p}_n = (p, p_n)$, such that $p_n = \frac{1}{n^\alpha}$, except the original drawing of the graphs as in [SS88]. We can also analyze $\mathscr{G}_{n,rn^\alpha}$ and use several pairs (r,α) in the analysis (as long as the sets of α's is linearly independent over the rationals). We hope later to show that for some such version there is a more natural definable $\mathbf{Q}_{\mathbf{K}}$ which imitate its behavior.

So in the proof we have two questions to address: first fixing $G = ([n], R_\ell)_{\ell<k}$, drawing the quantifiers, how $([n], R_\ell^G, \dots)$ look like. Second, we need to consider all the G's on $[n]$. For the first stage, the main problem is: the existence of two definably derived graphs which are isomorphic.

We do some kind of elimination of quantifiers: essentially if $\mathscr{M}_n$ is a τ-structure (τ relational and finite) drawn randomly according to the sequence $\langle p_{\tau,R} : R \in \tau \rangle$ of fixed probabilities, applying $\mathbf{Q}_{\bar{\mathbf{t}}}$ to some finitely many schemes $\langle \mathbf{s}_0, \dots, \mathbf{s}_k \rangle$ of interpreting graphs, define a random $\mathscr{M}'_n$ for τ'-structures by expanding $\mathscr{M}_n$ by

$R_\ell = \{\bar{c} : \ell g(\bar{c}) = \ell g(\bar{z}_{\mathbf{s}_\ell})$ and the graph $H_{\mathbf{s}_\ell,\bar{c}}$ interpreted by $\mathbf{s}_\ell$ for the parameter $\bar{c}$ is in the class $\mathbf{Q}_{\bar{\mathbf{t}}}\}$.

Our use of vocabulary and structure deviates a little from the standard, but fits with the use in graph theory and is natural here. In graph theory, the edge relation R is assumed to be symmetric and irreflexive. So we use (say k_t-place predicate) R_t such that it is always irreflexive (fails for k_t-tuples with a repetition) and K_t-invariant for some group K_t of permutation of $\{0,\dots,k_t-1\}$, i.e. if $\langle a_\ell : \ell < k_t\rangle$ satisfies it then so does $\langle \bar{a}_{\pi(\ell)} : \ell < k_t\rangle$ for every $\pi \in K_t$. This is natural because when the pair of formulas $\bar{\varphi}(\bar{c})$ defines a graph $H = H_{M,\bar{\varphi},\bar{c}}$ in the structure M (e.g. a graph) and we like to draw a truth value for "$H \in \mathbf{K}_{\bar{\mathbf{t}}}$", such a group of permutation of $\ell g(\bar{c})$ is dictated by $\bar{\varphi}$.

Why the random auxiliary structures are better defined in a different way? Recall the truth value of "$H \in \mathbf{K}_{\bar{\mathbf{t}}}$" is chosen randomly, but if H is definable in the graph G, say is $H_{G,\bar{\varphi},\bar{c},\bar{\mathbf{t}}}$ then the probability of "$H \in \mathbf{K}_{\mathbf{t}}$" depends on H, and in natural cases, on $|H|$, the number of nodes of H. But if $\mathscr{M} = ([n],\dots,R^0_\ell,\dots)$ is random, the standard way to make the probability of $\bar{c} \in R^G_\ell$ naturally depend on n and in many cases $n \neq |H|$.

We could have allowed using the quantifiers only on graphs H definable in $\mathscr{G}_{n,q}$ with set of nodes $[n]$ but this seems to me quite undesirable, restricting our logic too much. We restrict ourselves to the class of graphs - twice, we consider $\mathscr{G}_{n,q}$ and the quantifier $\mathbf{Q}_{\bar{\mathbf{t}}}$ is on graphs. But in both cases this is not really needed.

We thank Simi Haber for raising again the problem and for some stimulating discussions and Noga Alon for asking during a lecture in the Noga-fest, January 2011, why we ignore the weak graphs; a reasonable interpretation is: why we do not draw a truth value for "G is green" for G an empty graph. One problem is that for the sentence ψ saying "the graph with all nodes (is $[n]$) and no edges" the probability that $\mathscr{G}_{p,n}$ satisfies it is always zero or one and in non-trial cases is not eventually constant; see more in §3

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§ 1. Identifying the low graphs

We like to add a quantifier $\mathbf{Q}$ on finite graphs, which gives a property of finite graphs respecting isomorphism (i.e. a subset closed under automorphisms). The aim is that for e.g. for the random graph $\mathscr{G}_{n,p}$, the 0-1 law holds for $\mathbb{L}(\mathbf{Q})$ but there is an $\mathbb{L}(\mathbf{Q})$-formula $\varphi(x)$ such that for no first order $\psi(x)$ are $\varphi(x), \psi(x)$ equivalent in $\mathscr{G}_{n,p}$.

More specifically, we better make the quantifier trivial on too simple graphs, then we intend that for any fix finite set of formulas from $\mathbb{L}(\mathbf{Q})$, for random enough $G_{n,p}$ the structure $(G, \varphi^G(-))_{\varphi \in \Delta}$ is a random structure excluding the "problematic" graphs, we shall call them "low".

§ 1(A). Interpretations.

Convention 1.1. 1) $\mathbf{h} : \mathbb{N} \to (0, \frac{1}{2})_{\mathbb{R}}$ goes to zero slowly enough, e.g. $\mathbf{h}(n) = 1/\log_2 \log_2(n)$ for $n > 16$ and $= 1$ if $n \leq 16$; "slowly enough" actually means:

- (a) $\alpha \in (0,1)_{\mathbb{R}} \Rightarrow \infty = \lim\langle \mathbf{h}(n) n^\alpha : n < \omega\rangle$,
- (b) $0 = \lim\langle \mathbf{h}(n) : n < \omega\rangle$,
- (c) $n^{\mathbf{h}(n)}$ is non-decreasing (for simplicity).

2) $\mathbf{g} : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be $\mathbf{g}(n) = n^{\mathbf{h}(n)}$ hence $\mathbf{g}(1+n) \geq 1$ and so $\mathbf{g}$ go to infinity slowly enough.
3) Such $\mathbf{g}$ is called *reasonable*.

Notation 1.2. 1) Let $[n] = \{1, \dots, n\}$ or $\{0, \dots, n-1\}$ if you prefer (serve as the universe of the n-th random graph).
2) We use:

- (a) $n, m, k, \ell, i, j, \iota$ for natural numbers.
- (b) p, q for probabilities, i.e., from $(0,1)_{\mathbb{R}}$.
- (c) $\mathbf{s}$ for an *I-kind*, see 1.3(1), $\bar{p}$ as in 1.3(3)-(4A), so $(\mathbf{s}, \bar{p})$ is a random parameter, see 1.3(8).
- (d) $\mathbf{M}$, $\mathscr{M}$ as in 1.3(2)(c), 1.3(6) respectively,
- (e) $\pi, \varkappa$ for permutations.
- (f) φ, ψ, θ, Υ for formulas.
- (g) We say that two sequences are *disjoint* when their ranges are disjoint.
- (h) $\mathbf{s}$ is as in 1.3(1), $\mathbf{r} = (\bar{\mathbf{s}}, \bar{\Upsilon})$ as in 1.3(11).
- (i) $\bar{\varphi}$ is as in 1.6(1).
- (j) $\varphi^{\mathrm{if}(\mathbf{t})}$ mean φ if $\mathbf{t}$ is a true statement or $\mathbf{t} = 1$, and $\neg\varphi$ otherwise.

3) A sequence $\bar{a}$ means $\langle a_s \colon s \in I\rangle$, so a function whose domain is I mapping s to a_s; I is finite if not said otherwise; we shall write $I = \lg(\bar{a})$.
4) In the case $\lg(\bar{a}) = n$, i.e. $\lg(\bar{a}) = \{0, \dots, n-1\}$, we may write $\langle a_\ell \colon \ell < n\rangle$.
5) Let τ denote a vocabulary (e.g. $\tau = \tau_{\mathrm{gr}}$ is the vocabulary of graphs; see Definition 1.3(2)(a) below). Let $\mathbb{L}$ be first order logic so $\mathbb{L}(\tau)$ is the set of first order formulas in the vocabulary τ, but below we may write $\mathbb{L}(\mathbf{s})$ instead of $\mathbb{L}(\tau_{\mathbf{s}})$. Let φ, ψ, ϑ denote members of $\mathbb{L}(\tau)$.
6) A τ-model (also called τ-*structure*) M is defined as usual.
7) For a formula $\varphi = \varphi(\bar{x}, \bar{y})$, model M and $\bar{b} \in {}^{\ell g(\bar{y})}M$ let $\varphi(M, \bar{b}) = \{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}, \bar{b}]\}$.

8) For a truth value $\mathbf{t}$, and formula φ, let $\varphi^{\text{if}}[\mathbf{t}]$ is φ if $\mathbf{t}$ is true and $\neq \varphi$ otherwise. Similarly, replacing $\mathbf{t}$ by a statement.

The following is a central definition, explicating the restriction to what is definable.

Definition 1.3. 1) For a finite set I we say $\mathbf{s}$ is an I-kind or an I-kind sequence (of a vocabulary) and write $I_{\mathbf{s}} = I$ <u>when</u>:

(a) $\mathbf{s} = \langle (k_t, K_t) : t \in I \rangle = \langle (k_{\mathbf{s},t}, K_{\mathbf{s},t}) : t \in I \rangle$,
(b) $k_t \in \mathbb{N}$,
(c) K_t is a group of permutations of $\{0, \dots, k_t - 1\}$.

1A) Let $\mathbf{s}_{\text{gr}} = \mathbf{s}(\text{gr})$ be defined by (gr stands for graphs) $I_{\mathbf{s}} = \{s_{\text{gr}}\}, s_{\text{gr}}$ is fixed, and $s_{\text{gr}} = 0, k_{\mathbf{s},s_0} = 2$ and $K_{\mathbf{s},s_0} = \text{sym}(2)$, the group of permutations of $\{0, 1\}$.
2) For $\mathbf{s}$ an I-kind sequence we define:

(a) the $\mathbf{s}$-vocabulary $\tau_{\mathbf{s}} = \tau(\mathbf{s})$ is $\{R_t : t \in I\}, R_t$ a $k_{\mathbf{s},t}$-place predicate
(b) an $\mathbf{s}$-structure is $M = (|M|, R_t^M)_{t \in I}$ such that (here the universe $|M|$ of M may be empty):
 (α) R_t^M is a k_t-place relation on $|M|$,
 (β) R_t^M is K_t-invariant, i.e. if $\langle a_\ell : \ell < k_t \rangle \in R_t^M \wedge \bar{b} \in \bar{a}/E_{K_t} \Rightarrow \bar{b} \in R_t^M$ where $\bar{a}/E_{K_t} = \{\langle a_{\pi(\ell)} : \ell < n_t \rangle : \pi \in K_t\}$; let $E_{\mathbf{s},t} = E_{K_t}$,
 (γ) R_t^M is irreflexive, i.e. $\bar{a} \in R_t^M \Rightarrow \bar{a}$ with no repetitions.
(c) $\mathbf{M}_{\mathbf{s}} = \cup \{\mathbf{M}_{s,n} : n \in \mathbb{N}\}$ where $\mathbf{M}_{s,n} = \{M : M$ an $\mathbf{s}$-structure with set of elements $[n]\}$.

3) For an I-kind $\mathbf{s}$ let $\mathbf{P}_{\mathbf{s}}^1$ be the set of $\bar{p} = \langle p_{t,n} : t \in I, n \in \mathbb{N} \rangle, p_{t,n} \in (0,1)_{\mathbb{R}}$, so $p_{t,n} \notin \{0, 1\}$. We define the $(\mathbf{s}, \bar{p})$-random structure on $[n], \mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ as follows (see more in parts (5),(6)): for $t \in I$ and $\bar{a} \in {}^{k_t}([n])$ with no repetitions we draw a truth value for $\bar{a} \in R_t^{\mathscr{M}}$ with probability $p_{t,n}$, <u>but</u> demanding we have the same result for $\bar{a}', \bar{a}''$ when they are $E_{\mathbf{s},t}$-equivalent and independent of the other choices.
3A) Let $\mathbf{P}_{\mathbf{s}}^0$ for $\mathbf{s}$ as above be the set of $\bar{p} \in \mathbf{P}_{\mathbf{s}}^1$ such that $t \in I_{\mathbf{s}} \wedge n \in \mathbb{N} \Rightarrow p_{t,n} = p_{t,0}$, so we may write p_t instead of $p_{t,0}$. If $\mathbf{s} = \mathbf{s}_{\text{gr}}$, we may write gr instead $\mathbf{s}$.
4) Let $\mathbf{P}_{\mathbf{s}}^2$ be the set of $\bar{p} \in \mathbf{P}_{\mathbf{s}}^1$ such that for some $\bar{q} \in \mathbf{P}_{\mathbf{s}}^0$ and partition $\bar{I} = (I_0, I_1)$ of I, we have $p_{t,n}$ is q_t if $t \in I_0$ and is $q_0/\mathbf{g}(n)$ if $t \in I_1$; we denote $\bar{p}$ by $\bar{p}_{\bar{q},\bar{I}} = \bar{p}[\bar{q}, \bar{I}]$.
4A) We may write p instead of $\langle p_t : t \in I_{\text{gr}} \rangle$ when $p_{s_{\text{gr}}} = p$.
5) For $\bar{p} \in \mathbf{P}_{\mathbf{s}}^1$ let $\mu_{\mathbf{s},\bar{p},n}$ be the distribution (= probability space) on $\mathbf{M}_{\mathbf{s},n}$ corresponding to drawing the truth value of $R_t(\bar{a})$ really of $\langle \mathbf{R}_t(\bar{a}') : \bar{a}' \in \bar{a}/E_{\mathbf{s},t} \rangle$ for a sequence $\bar{a}$ with no repetitions of length $k_{\mathbf{s},t}$ with probability $p_{t,n}$, independently of the other choices.
6) Let $\mathscr{M}_{\mathbf{s},\bar{p},n}$ be the random variable for the finite probability space $(\mathbf{M}_{\mathbf{s},n}, \mu_{\mathbf{s},\bar{p},n})$. If $\mathbf{s} = \mathbf{s}_{\text{gr},q}$ let $\mathscr{G}_{q,n} = \mathscr{G}_{\text{gr},q,n} = \mathscr{M}_{\mathbf{s}_{\text{gr}},q,n}$ and $\mu_{\text{gr},q,n} = \mu_{\mathbf{s}_{\text{gr}},\bar{p}_{\text{gr},q},n}$.
7) Let $\text{Prob}(\mathscr{M}_{\mathbf{s},\bar{p},n}, \models \varphi)$ be the probability of $\mathscr{M}_{\mathbf{s},\bar{p},n} \models \varphi$, that is,

$$\Sigma\{\mu_{\mathbf{s},\bar{p},n}(M) \colon M \in \mathbf{M}_{\mathbf{s},\bar{p},n} \text{ and } M \models \varphi\}.$$

8) We call a pair $(\mathbf{s}, \bar{p})$ as above a *random parameter* and $\bar{\mathscr{M}}_{\mathbf{s},\bar{p}} = \langle \mathscr{M}_{\mathbf{s},\bar{p},n} \colon n \in \mathbb{N} \rangle$ a random context.
9) The 0-1-law for $(\mathbf{s}, \bar{p})$ is: for every $\psi \in \mathbb{L}(\tau_1)$, the sequence $\langle \text{Prob}(\mathscr{M}_{\mathbf{s},\bar{p},n}, \models \psi) \colon n < \omega \rangle$ converge to 0 or to 1.

10) We say that "for every random enough $\mathscr{M}_{\mathbf{s},\bar{p},n}$ we have φ" to mean that the sequence $\langle \text{Prob}(\mathscr{M}_{\mathbf{s},\bar{p},n} \models \varphi) \colon n \in \mathbb{N}\rangle$ converge to 1.
11) Assuming (A), we define the class $K_{\bar{\mathbf{s}},\bar{\Upsilon}}$ of $(\bar{\mathbf{s}},\bar{\Upsilon})$-structures and $\mathscr{M}_{\bar{\mathbf{s}},\bar{\Upsilon},\bar{p},n}$ or $K_{\mathbf{r}}$, $\mathscr{M}_{\mathbf{r},\bar{p},n}$, where $\mathbf{r} = (\bar{\mathbf{s}},\bar{\Upsilon}) = (\bar{\mathbf{s}}_{\mathbf{r}},\bar{\Upsilon}_{\mathbf{s}})$, as follows:

(A) (a) $\mathbf{s}_\ell$ is an I_ℓ-*kind* for $\ell \le \ell(*) < \omega$,
   (b) $\bar{\mathbf{s}} = \langle \mathbf{s}_\ell \colon \ell \le \ell(*)\rangle$ is an increasing sequence,
   (c) $\bar{\Upsilon} = \langle \Upsilon_{\ell,t}(\bar{y}_{\ell,t}) \colon t \in I_{\ell+1} \setminus I_\ell \text{ and } \ell < \ell(*)\rangle$, where:
   - $\bar{y}_{\ell,t}$ has length $k_{\mathbf{s}_{\ell+1},t}$ and is with no repetitions,
   - $\Upsilon_{\ell,t} \in \mathbb{L}(\tau_{\mathbf{s}_\ell})$ is quantifier-free.

   (d) let $\mathbf{P}^0_{\bar{\mathbf{s}}} = \mathbf{P}^0_{\mathbf{s}_{\ell(*)}}$.

(B) If $\bar{p} \in \mathbf{P}^2_{\mathbf{r}}$, then $\mathscr{M} = \mathscr{M}_{\bar{\mathbf{s}},\bar{\Upsilon},\bar{p},n} = \mathscr{M}_{\mathbf{r},\bar{p},n}$ is defined like $\mathscr{M}_{\mathbf{s}_{\ell(*)},\bar{p},n}$ with one change, we demand:
   - if $\ell < \ell(*)$, $t \in I_{\ell+1} \setminus I_\ell$ and $k = k_{\mathbf{s}_{\ell+1},t}$ and $\bar{c} \in {}^k\mathscr{M}$ then $\mathscr{M} \models \neg\Upsilon_{\ell,t}[\bar{c}] \Rightarrow \mathscr{M} \models \neg R_t[\bar{c}]$ and otherwise the probability is as in $\mathscr{M}_{\mathbf{s}_{\ell(*)}}$.

Recall that

Fact 1.4. 1) $\mathbf{P}^0_{\mathbf{s}} \subseteq \mathbf{P}^2_{\mathbf{s}} \subseteq \mathbf{P}^1_{\mathbf{s}}$.
2) For every $\bar{p} \in \mathbf{P}^0_{\mathbf{s}}$ or $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$,

(a) $\mathscr{M}_{\mathbf{s},\bar{p},n}$ satisfies the 0-1 law for first-order logic and the limit theory $T_{\mathbf{s},\bar{p}}$ has elimination of quantifiers, really is $T_{\mathbf{s}}$, i.e. does not depend on $\bar{p}$ and $\mathbf{g}$ and $\mathbf{h}$ (as long as they are as in 1.1),
(b) Similarly for $\mathscr{M}_{\bar{\mathbf{s}},\bar{\Upsilon},\bar{p},n}$ and $T_{\bar{\mathbf{s}},\bar{\Upsilon},\bar{p}} = T_{\bar{\mathbf{s}},\bar{\Upsilon}}$.

3) $\mathbf{M}_{\mathbf{s}_{\text{gr}},n}$ is the set of graphs with set of nodes $[n]$.
4) If $\mathbf{r} = (\bar{\mathbf{s}},\bar{\Upsilon})$ is as in 1.3(11) and $\ell_{\mathbf{r}} := \lg(\bar{\mathbf{s}}) = 1$, <u>then</u> $\mathbf{r}$ is essentially $\mathbf{s}_0$ because:

(a) $\bar{\Upsilon}$ is an empty sequence,
(b) $K_{\mathbf{r}} = K_{\mathbf{s}_0}$.

Proof. Should be clear. $\square_{1.4}$

Remark 1.5. 0) Why do we use Definition 1.3(11)? Clearly, this makes no real difference in proving the 0-1-law. However, in the proof of Theorem 2.5 this is helpful. There we try replacing the quantifiers by $\mathbf{Q}$ say $(\mathbf{Q}\ldots,\bar{x}_1,\bar{x}'_1,\ldots)\varphi(\bar{z})$ (see 2.2(3), 2.4) with drawing randomly $R[\bar{\varphi}]$. But there for some φ a restriction as above is natural.
1) We first concentrate on one application of the quantifier.
2) We are interested in interpreting graphs. We give the most general case. Note that we intend the quantifier to be a property of graphs. So we have to think of an interpretation of a graph. For such general interpretations, using quantifier-free formulas the elements may be only: a set of elements definable by a formula $\varphi(x,\bar{a})$ or $\varphi(\bar{x},a)$, $\bar{a}$ is a sequence of parameters. The elements will be $\{b \in \mathscr{M} \colon \mathscr{M} \models \varphi[b,\bar{a}]\}$ <u>or</u> $\{\bar{b} \in {}^{\lg(x)}\mathscr{M} \colon \mathscr{M} \models \varphi[\bar{b},\bar{a}]\}$ <u>or</u> more generally such a set of k-tuples, maybe modulo suitable E_K, or even a finite union of such. For each pair of the nodes (fixing from where in the union they come) we define when it is an edge by a quantifier free formula. So below $\bar{z}$ are parameters, $\mathbf{i}(\bar{\varphi})$ number of "kinds of elements", ways to define a node; $\varphi_{0,i}$ restrict the i-th kind, $\varphi_2(\bar{z})$ describes the relevant parameters, $\varphi_{1,i,j}$ describes the edges between a node of the i-th kind and a node of the j-th kind.

3) Generally in interpretations we allow the set of elements to be e.g. the set of equivalence classes of an equivalence relation defined say by $\varphi(\bar{x}', \bar{x}'', \bar{a})$, where $\lg(\bar{x}') = \lg(\bar{x}'')$ but in our case those will always be degenerated, see 1.11.
4) Defining generally interpretation we may replace the equivalence relation which K_i defined by a formula defining an equivalence relation. But in our case, this does not give more cases, and the present version seems more reader-friendly.

See justification in 1.11.

Definition 1.6. 1) For $\mathbf{s}$ an I-kind, we say $\bar{\varphi}$ is a $\mathbf{s}$-scheme (of a graph interpretation in $\mathbf{s}$-structures) when it consists of:

- (a) $\langle \varphi_{0,i}(\bar{x}_i, \bar{z}), \varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z}), \varphi_2(\bar{z}) : i, j < \mathbf{i}(\bar{\varphi}) \rangle$ such that:
  - $\bullet_1$ $\ell g(\bar{x}'_j) = \ell g(\bar{x}_j)$, it is possibly zero,
  - $\bullet_2$ $\langle \bar{x}_i, \bar{x}'_i : i < \mathbf{i}(\varphi) \rangle{}^\frown \langle \bar{z} \rangle$ are pairwise disjoint, each with no repetitions,
  - $\bullet_3$ $\mathbf{i}(\bar{\varphi})$ is a natural number; as we allow $\mathbf{i}(\bar{\varphi}) = 0$, we have to allow the empty graph.
- (b) $\varphi_{0,i}, \varphi_{1,i,j}, \varphi_2$ are formulas in the vocabulary $\tau_{\mathbf{s}}$, in this section they always are quantifier-free formulas in $\mathbb{L}(\tau_{\mathbf{s}})$, note that possibly $\varphi_{1,i} = \varphi_{1,j}$ though $i \neq j$.
- (c) $K = K_{\bar{\varphi}}$ is a group of permutations of $\{0, \ldots, \lg(\bar{z}) - 1\}$ and $K_i = K_{\bar{\varphi},i}$ is a group of permutations of $\{0, \ldots, \ell g(\bar{x}_i) - 1\}$, both not related to $K_{\mathbf{s},t} (t \in I)$!.
- (d) $\varphi_{0,i}(\bar{x}_i, \bar{z})$ is invariant under permuting $\bar{x}_i$ by any $\pi \in K_i$; that is if $\pi \in K_i$; $\bar{x}'_i = \langle x_{i,\pi(f)} : \ell < \ell g(\bar{x}_i) \rangle$ then $\varphi_2(\bar{z}) \vdash_{\mathbf{s}} (\forall x_0 \ldots x_\ell \ldots)(\varphi_{1,i}(\ldots x_\ell \ldots ; \bar{z}) \equiv \varphi_{1,i}(\ldots x_{\pi(\ell)}, \ldots, \bar{z}))$ where $\vdash_{\mathbf{s}}$ means implication in every $\mathbf{s}$-structure.
- (e) $\varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z})$ is invariant under permuting $\bar{x}_i, \bar{x}'_j$ by $\pi \in K_i, \varkappa \in K_j$ respectively, and $\vdash \varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z}) \equiv \varphi_{1,j,i}(\bar{x}'_j, \bar{x}_i, \bar{z})$ and $\vdash \neg\varphi_{1,i,i}(\bar{x}_i, \bar{x}_i, \bar{z})$.
- (f) $\varphi_2(\bar{z})$ is invariant under permuting $\bar{z}$ by $\pi \in K$.
- (g) if M is a $\tau_{\mathbf{s}}$-structure and $G \models \varphi_{0,i}[\bar{a}, \bar{c}]$, so $\ell g(\bar{c}) = \ell g(\bar{z})$ then $\bar{a}{}^\frown\bar{c}$ is with no repetitions.

1A) So if we have $\bar{\varphi} = \bar{\varphi}^\iota$ then $\varphi^\iota_{0,i} = \varphi_{0,i}$, etc. and we may write $\bar{z}_{\bar{\varphi}}, \bar{x}_{\bar{\varphi},1,i}, \bar{x}'_{\bar{\varphi},1,1}$ and if we have $\bar{\varphi} = \bar{\varphi}_r$ then $\varphi_{0,i} = \varphi_{r,0,i}$, etc. and we may write $\bar{z}_{r,\bar{\varphi}}$, $\bar{x}_{r,\bar{\varphi},1,i}$, $\bar{x}_{r,\bar{\varphi},1,1}$.
1B) We may replace first-order logic $\mathbb{L}(\tau_{\mathbf{s}})$, by others in §2.
2) If $\mathbf{s}$ and $\bar{\varphi}$ are as above, M is an $\mathbf{s}$-structure and $\bar{c} \in \varphi_2(M)$, i.e. $\bar{c} \in {}^{\ell g(\bar{z})}M$ satisfies $M \models \varphi_2[\bar{c}]$ then $H = H_{\bar{\varphi},M,\bar{c}}$ is the following graph:

- (α) the set of nodes is $\{(i, \bar{a}/E_{K_i}) : M \models \varphi_{0,i}[\bar{a}, \bar{c}]$ for some $i < \mathbf{i}(\bar{\varphi})$ and $\bar{a} \in {}^{\ell g(\bar{x}_i)}M\}$, see 1.3(2)(b)($\beta$),
- (β) $\{(i, \bar{a}/E_{K_i}), (j, \bar{b}/E_{K_j})\}$ is an edge iff $M \models \varphi_{1,i,j}[\bar{a}, \bar{b}, \bar{c}]$.

3) Let $k_{\mathbf{s}}(\bar{\varphi}) = \max(\{\ell g(\bar{x}_i) : i < \mathbf{i}(\bar{\varphi})\} \cup \{\ell g(\bar{z})\})$ and let $k_{\mathbf{s},i}(\bar{\varphi}) = \ell g(\bar{x}_i)$.
4) We say $\varphi_2(\bar{z})$ is complete when for some $\mathbf{s}$-structure M, $\varphi_2(M) \neq \emptyset$ and for any $\mathbf{s}$-structure M, if $\bar{a}_1, \bar{a}_2 \in \varphi_2(M)$ then $\bar{a}_1, \bar{a}_2$ realize the same quantifier free type in M and φ_2 is even quantifier-free.
5) We say $\bar{\varphi}$ is complete when $\varphi_2(\bar{z})$ and each $\varphi_{0,i}(\bar{x}_i, \bar{z})$ is (not contradictory and is) complete (see 1.6 part (4)) and $\varphi_{0,i}(\bar{x}_i, \bar{z}) \vdash \varphi_2(\bar{z})$ and each $\varphi_{1,i,j}(\bar{x}, \bar{x}', \bar{z})$ is quantifier-free. If not said otherwise, we assume $\bar{\varphi}$ is complete.

6) We say that $\bar\varphi$ is *correct* for the pair $(\bar{\mathbf{s}}, \bar\Upsilon)$ when it is quantifier-free complete and for any $\bar p = \langle p_s : s \in I_{\mathbf{s}}\rangle$ for random enough $\mathscr{M}_{\bar{\mathbf{s}},\bar\Upsilon,\bar p,n}$, if $\bar c \in \varphi_2(M)$ then $K_{\bar\varphi}$ is the following set:

$\{\pi : \pi$ is a permutation of $\{c_0, \ldots, c_{\lg(\bar c)-1}\}$ such that $\bar c, \pi(\bar c)$ realize some qf type$\}$.

Observation 1.7. *1) In Definition 1.6(2), $H_{\bar\varphi,M,\bar c}$ is indeed a graph (possibly empty) and is finite when M is a finite $\tau_{\mathbf{s}}$-structure.*
2) For each $\bar\varphi$ as in 1.6(1), (4) for each $i < \mathbf{i}(\bar\varphi)$ one of the following holds:

- (α) *for some $k, \varphi_2(\bar z) \vdash (\exists!^k \bar x_i)(\varphi_i(\bar x_i, \bar z))$,*
- (β) *for every k for some $\mathbf{s}$-structures M, in M we have $\varphi_2(\bar z) \vdash (\exists^{\geq k}\bar x)\varphi_{0,i}(\bar x_i, \bar z)$.*

3) Assume that $\mathbf{s}$ is an I-kind, $\bar\varphi$ is an $\mathbf{s}$-scheme. Then we can find $\langle \bar\varphi^\iota : \iota < \iota_*\rangle$ such that:

- (a) *each $\bar\varphi^\iota$ is an $\mathbf{s}$-scheme which is complete, see 1.6(5),*
- (b) *if M is a $\tau_{\mathbf{s}}$-structure, $\bar c$ is a sequence of elements of M of length $\lg(\bar z_{\bar\varphi})$ such that $M \models \varphi_2[i]$, then for some $\iota < \iota_*$ and $\bar c'$, we have $M \models \varphi_2^\iota[\bar c']$ and:*
    - $\bullet_1$ *$H_{\bar\varphi,M,\bar c}$, $H_{\bar\varphi^\iota,M,\bar c'}$ are isomorphic,*
    - $\bullet_2$ *$\mathrm{rang}(\bar c') = \mathrm{rang}(\bar c)$.*
- (c) *if M is a $\tau_{\mathbf{s}}$-structures, $\iota < \iota_*$, $\bar c'$ is a sequence of elements of M of length $\lg(\bar z_{\bar\varphi'})$ such that $M \models \varphi_2^\iota[\bar c']$, then for some $\bar c$ we have $M \models \varphi_2[\bar c]$ and $\bullet_1$, $\bullet_2$ above holds.*

Proof. Read Definition 1.6(1), (4). $\square_{1.7}$

Observation 1.8. *1) Let $\mathbf{s}$ be an I-kind and $\bar\varphi$ is a complete $\mathbf{s}$-scheme.*
The following are equivalent:

- (a) *for every $\bar p \in \mathbf{P}^2_{\mathbf{s}}$ and random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar p,n}$ we have $\varphi_2(\mathscr{M}) \neq \emptyset$,*
- (b) *for some $\bar p \in \mathbf{P}^2_{\mathbf{s}} \cup \mathbf{P}^0_s$ we have $0 < \limsup_n \mathrm{Prob}(\varphi_2(\mathscr{M}_{\mathbf{s},p,n}) \neq \emptyset)$.*

1A) Similarly for $\mathbf{r} = (\bar{\mathbf{s}}, \bar\Upsilon)$ as in Definition 1.3(11).
2) For any sentence $\psi \in \mathbb{L}(\tau_{\mathbf{s}})$, similarly replacing $\varphi_2(\mathscr{M}) \neq \emptyset$ by "for some $\bar c, H_{\bar\varphi,\mathscr{M},\bar c} \models \psi$".

Proof. Easy. $\square_{1.8}$

Definition 1.9. Let $\bar\varphi$ be an $\mathbf{s}$-scheme most properties of $\bar\varphi$ below depend on $\mathbf{s}$ or on $\mathbf{r} = (\bar{\mathbf{s}}, \bar\Upsilon)$, but we usually forgot to mention $\mathbf{s}$ as it is clear from the context.
1) We call $\bar\varphi$ trivial when for each $i < \mathbf{i}(\varphi)$ we have $\ell g(\bar x_i) = 0$.
2) We call $\bar\varphi$ degenerated when the conditions (a) equivalently (b) of 1.8(1) fail.
3) We say the $\bar\varphi$ is 1-weak when[1]. at least one of the following holds:

- (a) $\mathbf{s}$ is degenerated or $\mathbf{s}$ is trivial, i.e. $\ell g(\bar x_i) = 0$ for every $i < \mathbf{i}(\varphi)$,
- (b) for some truth value $\mathbf{t}$ and $i_1, i_2 < \mathbf{i}(\bar\varphi)$ satisfying $\ell g(\bar x_{i_1}), \ell g(\bar x_{i_2}) \geq 1$ and $v_1 \subsetneqq \ell g(\bar x_{i_1}), v_2 \subsetneqq \ell g(\bar x_{i_2})$ we have:
    - $\bullet_1$ for some (equivalently any) $\bar p \in \mathbf{P}^2_{\mathbf{s}}$, for random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar p,n}$, for some $\bar c \in \varphi_2(\mathscr{M})$ and $\bar a^*_\ell \in \varphi_{1,i_\ell}(\mathscr{M}, \bar c)$ for $\ell = 1,2$ we have,
    - $\bullet_2$ if $\bar a_\ell \in \varphi_{1,i_\ell}(\mathscr{M}, \bar c)$ and $\bar a_\ell \restriction v_\ell = \bar a^*_\ell \restriction v_\ell$ for $\ell = 1,2$ and $\mathrm{rang}(\bar a_1) \cap \mathrm{rang}(a_2) \subseteq \mathrm{rang}(\bar a^*_1 \restriction v_1) \cap \mathrm{rang}(\bar a^*_2 \restriction v_2)$ then $\mathscr{M} \models \varphi_{i_1,i_2}[\bar a_1, \bar a_2, \bar c]^{\mathrm{if}(\mathbf{t})}$.

[1]We may define 2-weak mean we exclude also $(\exists i < i(\bar\varphi))(\lg(\bar x_i) \geq 2)$; this simplify the proof of Theorem 2.5 but make the definition of the quantifiers $\mathbf{Q}$ in 2.2(3), 2.4 less nice.

Observation 1.10. *1) In Fact 1.4(2)(a) we really have $T_{\mathbf{s},\bar{p}}$ in the same for all $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$.*

1A) Similarly for $T_{\bar{\mathbf{s}},\bar{\Upsilon}}$ for 1.4(2)(b).

2) In 1.9(3)(b)($\bullet_1$), the "equivalently" is justified.

Proof. By the requirement on $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ in Definition 1.3(4). $\square_{1.10}$

Claim 1.11. *For $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$, we have:*
1) For any k there is k_* such that if $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ is random enough (so this depends on k_*) and $\bar{c} \in {}^{k\geq}\mathscr{M}$, and there is an interpretation using as parameter $\bar{c}$ of a graph H in $\mathscr{M}$ using $(\leq k)$-tuples (in the widest sense - the elements can be equivalence classes of suitable definable equivalence relations on set of tuples satisfying a formula) by formulas of length $\leq k$, <u>then</u> there is a complete $\mathbf{s}$-scheme $\bar{\varphi}$ such that $H = H_{\bar{\varphi},\mathscr{M},\bar{c}'}$, $\mathrm{rang}(\bar{c}') = \mathrm{rang}(\bar{c})$ and $k(\bar{\varphi}) \leq k_*$.
1A) For any interpretation as above (so by first-order formulas with parameter $\bar{c}$)

- $(*)$ *there is an $\mathbf{s}$-scheme interpretation equivalent to it, and we can compute it,*
- $(*)$ *moreover we can compute a finite sequence $\langle \bar{\varphi}_i : i < i_* \rangle$, each $\bar{\varphi}_i$ is complete, all with the parameter $\bar{c}'$, a subsequence of $\bar{c}$.*

2) In fact $\bar{\varphi}$ depends just on the interpretation and the quantifier free type of $\bar{c}$ in $\mathscr{M}$, not on $\mathscr{M}$ (and even n).
3) Given $\mathbf{s}$ and k there only finitely many scheme $\bar{\varphi}$ as above.
4) Similarly for $\mathbf{r} = (\bar{\mathbf{s}}, \bar{\Upsilon})$ as in 1.3(11).

Proof. Obvious. $\square_{1.11}$

Definition 1.12. Let $\mathbf{s}, \bar{\varphi}$ be as above, $\bar{\varphi}$ is complete, $\mathbf{i}(\bar{\varphi}) > 0$, see 1.6(5).
We say $(\mathbf{s}, \bar{\varphi})$ or just $\bar{\varphi}$ (when $\mathbf{s}$ is clear from the context) is reduced <u>when</u>: (they are as above and) for every $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ and random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ and $\bar{c} \in {}^{\ell g(\bar{z}_{\bar{\varphi}})}\mathscr{M}$ satisfying $\varphi_2(\bar{z}_{\bar{\varphi}})$, the graph $H = H_{\bar{\varphi},\mathscr{M},\bar{c}}$ is not $H = H_{\bar{\varphi}',\mathscr{M},\bar{c}'}$ when $(\bar{\varphi}', \bar{c}'$ appropriate and) $\mathrm{Rang}(\bar{c}') \subsetneqq \mathrm{Rang}(\bar{c})$; recall $\bar{c}$ is without repetitions.

See more in 1.22, 1.23, 1.24, 1.26.

§ 1(B). **Simple Random Graph.** Our intention is that the behavior of $\mathscr{G}_{q,n}$ expanded by some formulas in the expanded logic will be like $\mathscr{M}_{\mathbf{s},\bar{p},n}, \bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ for appropriate $\mathbf{s}$, but we need a relative so that we can iterate.

Note that in Definition 1.14, $\mathbf{u} \in \mathbf{U}$ has $\mathbf{s}_\ell$ but not p-s, i.e., carry no probabilities except the "original" q. The point is that in 1.15 we choose them canonically.

Discussion 1.13. Concerning 1.14-1.17? they are alternative version of $(\bar{\mathbf{s}}, \bar{\Upsilon})$.

Definition 1.14. For $\iota = 1, 2$ let $\mathbf{U}_\iota$ be the set of objects $\mathbf{u}$ consisting of the following (we may add subscript $\mathbf{u}$; we may omit ι when $\iota = 1$):

- (a) $\bar{\mathbf{s}} = \langle \mathbf{s}_\ell : \ell \leq \ell(\mathbf{u}) \rangle$,
- (b) $\mathbf{s}_\ell$ is a kind sequence, $I_\ell = I_{\mathbf{s}_\ell}$,
- (c) $\mathbf{s}_0 = \mathbf{s}_{\mathrm{gr}}$, the graph kind sequence, see 1.3(1A),
- (d) $\mathbf{s}_\ell \subseteq \mathbf{s}_{\ell+1}$, i.e. $I_{\mathbf{s}_\ell} \subseteq I_{\mathbf{s}_{\ell+1}}$ and $t \in I_{\mathbf{s}_\ell} \Rightarrow (k_{\mathbf{s}_\ell,t}, K_{\mathbf{s}_\ell,t}) = (k_{\mathbf{s}_{\ell+1},t}, K_{\mathbf{s}_{\ell+1},t})$,
- (e) notation: so we may write $(k_{\mathbf{u},t}, K_{\mathbf{u},t})$ for $t \in I_{\mathbf{s}_{\ell(\mathbf{u})}}$ and $I_{\mathbf{q}} = I_{\mathbf{s}_{\ell(\mathbf{u})}}$,
- (f) for $t \in I_{\mathbf{s}_{\ell+1}} \backslash I_{\mathbf{s}_\ell}$ we have: $\bar{\varphi}_t$ is a complete reduced $\mathbf{s}_\ell$-scheme, not $\mathbf{h}$-ι-weak such that:
    - $K_t = K_{\bar{\varphi}_t}$, see Definition 1.22(2) below,

- let $\mathbf{i}_t = \mathbf{i}(t) = \mathbf{i}(\bar\varphi_t)$ and similarly $\varphi_{t,2}, \varphi_{t,0,i}, \varphi_{t,1,i,j}$ but let $\varphi_t(\bar z_t) = \varphi_{t,2}(\bar z_t)$,
- In the case $\iota = 2$, if $\bar x_{t,i} \neq \langle\rangle$ then $\bar x_{t,i}$ is a singleton so we may write $\varphi_{t,0,i}(x, \bar z_{t,i})$.

(g) $q = q_{\mathbf{u}} \in (0,1)_{\mathbb{R}}$.
(h) if $t \in I_{\ell+1} \setminus I_\ell$ then $\vartheta_t = \theta_t(\bar z_t) = (\mathbf{Q} \ldots \bar x_i, \bar x'_i, \ldots)_{i<\mathbf{i}(\bar\varphi)}\bar\varphi \in \mathbb{L}(\tau_{\mathbf{s}_\ell})$.

Definition 1.15. For $\mathbf{u} \in \mathbf{U}_\iota$ we define a random $\mathscr{M}_{\mathbf{u},n}$, i.e. a 0-1 context, as follows.

For a given $n, \mathscr{M}_{\mathbf{u},n}$ is gotten by drawing $\mathscr{M}_{\mathbf{u},n,\ell} \in \mathbf{M}_{\mathbf{s}_{\mathbf{u},\ell},n}$ by induction on $\ell \leq \ell(\mathbf{u})$ and in the end $\mathscr{M}_{\mathbf{u},n} = \mathscr{M}_{\mathbf{u},n,\ell(\mathbf{u})}$.

Now

(a) if $\ell = 0, \mathscr{M}_{\mathbf{u},n,\ell}$ is $\mathscr{G}_{q(\mathbf{u}),n}$, i.e. the random graph on n with edge probability q,
(b) if $\ell < \ell(\mathbf{u})$ and $\mathscr{M}_{\mathbf{u},n,\ell}$ has been drawn as M and $t \in I_{\mathbf{s}_{\ell+1}} \backslash I_{\mathbf{s}_\ell}$, we draws $R_t(\mathscr{M}_{\mathbf{s}_{\ell+1}})$ as follows:
  (α) if $\bar c \in \varphi_t(M)$ we draw the truth value of $\bar c \in R_t(\mathscr{M}_{\mathbf{s}_{\ell+1},n})$ with probability $\frac{1}{\mathbf{g}(\alpha)}$, where:
$$\alpha := \sum_{i<\mathbf{i}(t)} \frac{\mathrm{EXP}(|\varphi_{t,i}(\mathscr{M}_{\mathbf{s}_\ell,n}, \bar c)|)}{|K_{t,i}|} \in \mathbb{R},$$
  recalling EXP is the expected value,
  (β) if $\bar c$ is a sequence of length k_t but $\notin \varphi_t(M)$ then $\bar c \notin R_t(\mathscr{M}_{\mathbf{s}_{\ell+1},t})$.

Claim 1.16. *For $\mathbf{u} \in \mathbf{U}_\iota, \mathscr{M}_{\mathbf{u},n}$ is like $\mathscr{M}_{\mathbf{s}_\mathbf{q},\bar p}$ for any $\bar p \in \mathbf{P}^2_{\mathbf{s}_\mathbf{q}}$ (and $\mathscr{M}_{\mathbf{u},n,\ell}$ like $\mathscr{M}_{\mathbf{s}_{\mathbf{q},\ell},\bar p}$), in particular, satisfying the zero one law:*

(∗) *for any k_1 for some k_2, for any random enough $\mathscr{M}_{\mathbf{u},n}$ we have:*
- *if $\ell g(\bar y) + \ell g(\bar x) \leq k_1$, $\bar c \in \varphi(\mathscr{M}_{\mathbf{u},n})$ and $k_{t,i} \geq 1$, then the number of members of $\varphi_{t,0,i}(\mathscr{M}_{\mathbf{u},n}, \bar c)$ is similar to $m := \binom{n^{\ell g(\bar y)}}{k_t} \cdot \frac{k_t!}{|K_t|}$; fully*
- *at most[2] $\mathbf{g}(m)\left(1 + \frac{1}{k_2}\right)$,*
- *at least $\mathbf{g}(m)\left(1 + \frac{1}{k_2}\right)$,*
- *if $\iota = 2$, then $k_{t,i} = 1$, so this is simpler.*

Remark 1.17. What is the reason for our choice in Clause $(b)(\alpha)$ of Def 1.15? There are some demands pulling in different directions.

(a) This probability should be not too small (considering it belongs to $(0, 1/2)$) such that the argument "a Σ_1 formula $(\exists y)\varphi(y, \bar a)$ holds when not excluded" as in $\mathscr{M}_{\mathbf{s},\bar p,n}$,
(b) but always is not so large such that $\mathrm{Prob}((\exists \bar y)\varphi(\bar y))$ converge to zero or to one,

[2] we could have allowed, e.g. when $k_t = 1$ to be near to 1 though not too closely, but if we shall use a quantifier $\mathbf{Q}$ such that $\ll \frac{1}{2}$ of the structures are in it.

(c) The $\mathscr{M}_{\mathbf{u},n}$ are intended to imitate what we get by starting with $\mathscr{G}_{p,n}$ and expanding it by relations definable by formulas $\varphi(\bar{x})$ from our logic, so we are applying our quantifier to a definable (with parameters) graph. So such a graph even almost surely will not have exactly n nodes. In the non-degenerated case, the number will be of the order of magnitude

$(*)_1$ Cn^k for some positive real C and $k \geq 1$ in the $\mathbf{h}$-low case.

Proof. Should be clear. $\square_{1.16}$

§ 1(C). **Low/High Graphs.**
An $\mathbf{s}$'s scheme $\bar{\varphi}$ may be such that, e.g. the bi-partite graph with the i-th kind and the j-th kind is in the $\mathbf{h}$-low case, see Definition 1.9(4); so we try to single out those $\bar{\varphi}$'s. Those cases are "undesirable" for us and we shall try to discard them, that is, in Definition 2.2 not drawing the truth value $\mathbf{t}_m$ when H_m is such graph.

Definition 1.18. We say a finite graph H is $\mathbf{h}$-low (recall $\mathbf{h}$ is from 1.1 so can be omitted) <u>when</u> there are disjoint $A, B \subseteq H$ and $\iota < 2$ witnessed it, which means such that (letting $n = |H|$)

(a) $|A|, |B| \geq |H|^{\mathbf{h}(|H|)}$,
(b) if $a \in A$ and $b \in B$ <u>then</u> (a, b) is an edge of H <u>iff</u> $\iota = 1$.

Remark 1.19. The choice of being low is not unique, but first it seems nice, simple, and second it is a reasonable sufficient condition such that for a random enough graph $\mathscr{G}_{n,1/2}$ satisfies "every definable graph (with the definition $<< n$), if not $\mathbf{h}$-low, the definition has a specific reason for it".

We shall use it below and it is well known that:

Fact 1.20. If $\bar{a}_\ell = \langle a_{\ell,i} \colon i \in [k] \rangle$ is a sequence of length k for $\ell \in [n]$, <u>then</u> there are $\mathscr{U}, u, v, \bar{n}$ such that:

(a) $\mathscr{U} \subseteq [n]$ has $\geq \frac{n^{\frac{1}{2^k}}}{k^2}$ elements (even just $\geq n^{\frac{1}{2^k}} - k + 1$),
(b) u, v is a partition of $[k]$,
(c) $\langle \bar{a}_\ell \restriction u \colon \ell \in \mathscr{U} \rangle$ is constant,
(d) if $\ell \in \mathscr{U}$, $i \in v$ then $a_{\ell,i} \notin \{a_{\ell_1,i_1} \colon \ell_1 \in \mathscr{U} \setminus \{\ell\}, i_1 < k\}$.

Proof. This is the finite Δ-system lemma. $\square_{1.20}$

Claim 1.21. *1) Assume $\mathbf{s}$ is an I-kind, (see Definition 1.3) and $\bar{\varphi}$ is a complete $\mathbf{s}$-scheme (see Definition1.9(2)). 1.6, 1.9(2))*

(A) *the following are equivalent:*
  (α) *$\bar{\varphi}$ is trivial, see Definition 1.9(1),*
  (β) *if $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ <u>then</u> for random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},n,\bar{p}}$ and $\bar{c} \in \varphi_2(\mathscr{M})$ the graph $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ has $\leq \mathbf{i}(\bar{\varphi}) \cdot (k(\bar{\varphi})!)$ nodes,*
  (γ) *if $\varepsilon > 0$ and $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ <u>then</u> $0 < \limsup_n \mathrm{Prob}$(letting $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$, for some $\bar{c} \in \varphi_2(\mathscr{M})$, the graph $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ has $\leq n^{1-\varepsilon}$ nodes).*

(B) *the following are equivalent for non-trivial $\bar{\varphi}$:*
  (α) *$\bar{\varphi}$ is 1-weak, see Def 1.9(3),*
  (β) *if $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ <u>then</u> for every random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},n,\bar{p}}$ and for every $\bar{c} \in \varphi_2(\mathscr{M})$ the graph $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ is $\mathbf{h}$-low,*

> (γ) *if* $\varepsilon > 0$ *and* $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ *<u>then</u>* $0 < \limsup_n \mathrm{Prob}$*(letting* $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$*, for some* $\bar{c} \in \varphi_2(\mathscr{M})$ *the graph* $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ *is* $\mathbf{h}$*-low).*

2) Similarly for $\mathbf{r} = (\bar{\mathbf{s}}, \bar{\Upsilon})$.

Proof. 1) <u>Clause (A)</u>:

Trivially $(A)(\alpha) \Rightarrow (A)(\beta)$ and $(A)(\beta) \Rightarrow (A)(\gamma)$.

So it suffices to assume $\bar{\varphi}$ is non-trivial, $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ and let $\varepsilon > 0$ be small enough and prove that for every random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ and $\bar{c} \in \varphi_2(\mathscr{M})$ the graph $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ has $\geq n^{1-\varepsilon}$ nodes.

Let $i < \mathbf{i}(\bar{\varphi})$ be such that $k_i = \ell g(\bar{x}_i) > 0$, so for n large enough and $\bar{c} \subseteq [n]$ of length $\ell g(\bar{z})$ let $S_{n,\bar{c}} = \{\bar{a} : \bar{a}$ is a sequence of length $\ell g(\bar{x}_i)$ with no repetition of members of $[n]$ not from $\bar{c}\}$. For every $\bar{a} \in S_{n,i}$, recalling φ_1 is complete, the real $\mathrm{Prob}(\mathscr{M}_{\mathbf{s},\bar{p},n} \models$ "if $\varphi_2(\bar{c})$ then $\varphi_{1,i}(\bar{a},\bar{c})$") is the same for every $\bar{a} \in S_{n,\bar{c}}$ and is of the form $r(1)/\mathbf{g}(n)^m$ for some $r(1) \in (0,1)_{\mathbb{R}}, m \in \mathbb{N}\backslash\{0\}$ not depending on n. Fixing $\bar{c}$ under the assumption $\mathscr{M}_{\mathbf{s},\bar{p},n} \models \varphi_2[\bar{c}]$, considering a maximal set $\mathbf{I}_{n,\bar{c}}$ of pairwise disjoint $\bar{a} \in S_{n,\bar{c}}$, the events $\mathscr{M}_{\mathbf{x},\bar{p},n} \models \varphi_{1,i}[\bar{a},\bar{c}]$ are independent, hence almost surely the number $|\{\bar{a} \in S_{n,\bar{c}} : \mathscr{M}_{\mathbf{s},\bar{p},n} \models \varphi_{1,i}[\bar{a},\bar{c})\}$ is $\geq n/(r(1)\mathbf{g}(n)^m(1-\varepsilon))$. Similarly almost surely the number of $\bar{c}$ such that $\mathscr{M} \models \varphi_2[\bar{c}]$ is large.

If $i < i(\bar{\varphi})$, $h = h_{\mathbf{s},i} > 1$ then we can prove the numbers is $\geq n^{k(1-\varepsilon)}$ and more, but this is irrelevant here.

<u>Clause (B)</u>:

<u>First why $(B)(\alpha) \Rightarrow (B)(\beta)$</u>?

Recall we are assuming $\bar{\varphi}$ is non-trivial; $(\mathbf{s},\bar{\varphi})$ cannot satisfy clause (a) of Definition 1.9 because in the present claim we are assuming $\bar{\varphi}$ is complete. So assume clause (b) of Definition 1.9(3) holds; this means that:

- there are $i_1, i_2, \mathbf{i}(\bar{\varphi}), v_1, v_2$ and truth value $\mathbf{t}$, i.e. $\ell g(\bar{x}_i), \ell g(\bar{x}_j) > 0$ as in Definition 1.9(3).

So assume n is large enough and $\mathscr{M} = M, \bar{c} \subseteq [n]$ has length $\ell g(\bar{z}_{\bar{\varphi}})$.

For $\ell = 1,2$, let $A_\ell = \{\bar{a} :$ is a sequence of member of $[n]$ of length $\ell g(\bar{x}_{i_\ell})$ with no repetition and is disjoint to $\bar{c}\}$. Choose for $\ell = 1,2$ disjoint $\bar{a}^*_\ell \in A_\ell$. So the event $\mathscr{E}_{\bar{c}}$ = "$(\bar{c}\hat{\ }\bar{a}^*_1\hat{\ }\bar{a}^*_2)$ is as in 1.9(3)(b)$\bullet_2$" has probability $\geq r_1/(\mathbf{g}(n)^{k(1)})$ for some $r_1 \in (0,1)_{\mathbb{R}}, k \in \mathbb{N}\backslash\{0\}$ not depending on n (and $\bar{c}$). Fixing $(\bar{c}, a^*_1, a^*_2)$, let $C_\ell \subseteq ([n]\backslash\mathrm{rang}(\bar{c}\hat{\ }\bar{a}^*_1\hat{\ }\bar{a}^*_2))$ such that $|C_\ell| \geq \frac{1}{2}(n - \ell g(\bar{z}\hat{\ }\bar{x}_{i_1}\hat{\ }\bar{x}'_{i_2}))$ for $\ell = 1,2$ and $C_1 \cap C_2 = \emptyset$. For $\ell = 1,2$, let

$$A'_\ell = \{\bar{a} \in A_\ell : \bar{a}{\restriction}v_\ell = \bar{a}^*_\ell{\restriction}\ell \text{ and } \{a_i : i < \lg(\bar{a}) \text{ but } i \notin v_\ell\} \subseteq C_\ell\}.$$

Easily for some $r(2), r(3) \in (0,1)_{\mathbb{R}}$ not depending on n the probability of the event $\mathscr{E}_2 = \mathscr{E}^r_{\bar{c},\bar{a}^*,\bar{a}^*_1}$ is $\geq 1 - 2^{-r(2)n}$ where

> $(*)$ $\mathscr{E}_2$ means: if $\mathscr{M} \models \varphi_2[\bar{c}] \wedge \varphi_{1,i_1}[\bar{a}^*_1] \wedge \varphi_{1,i_2}[\bar{a}^*_2]$ <u>then</u> $|\{\bar{a}_\ell \in A'_\ell : \mathscr{M} \models \varphi[\bar{a}_{\ell,m}]\}| \geq n^{|u(\ell)|}r(3)$ for $\ell = 1,2$.

If $\mathscr{E}_2$ occurs, clearly $\mathbf{t}$ and $A^*_{\mathscr{M},\ell} = \{\bar{a}/E_{K_{\mathbf{s},i,\ell}} : \bar{a} \in A'_\ell$ and $\mathscr{M} \models \varphi_{0,i_\ell}[a_{n,m},\bar{c}]\}$ for $\ell = 1,2$ exemplifies $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ is $\mathbf{h}$-low. As the number of $\bar{c}, \bar{a}^*_1, \bar{a}^*_2$ is a polynomial we can finish.

<u>Second, why $(B)(\beta) \Rightarrow (B)(\gamma)$</u>:

Read the clauses and Definition of 1.18.

<u>Third, $(B)(\gamma) \Rightarrow (B)(\alpha)$</u>: This suffices.

Why does this hold? Toward contradiction, for some $r(0) \in (0,1)_{\mathbb{R}}$ not depending on n there are

- $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ be random enough,
- $\bar{c}_2 \in \varphi_2(\mathscr{M})$,
- $A_1, A_2 \subseteq H = H_{\bar{\varphi},\mathscr{M},\bar{c}}$ witness H is $\mathbf{h}$-low, that is,
- for $\ell = 1,2$, we have $|A_\ell| \geq n^{h(n)r(0)}$. So
- $n_1^* = \min\{|A_1^*|; |A_2^*|\} \geq n^{\mathbf{h}(n)r(0)}$.

Clearly for each $\ell \in \{1,2\}$ for some $i(\ell) < \mathbf{i}(\bar{\varphi})$ we have:

$$|\bar{a}/K_{\mathbf{s},i,\ell} \in A_\ell : \bar{a} \in \varphi_{1,i}(\mathscr{M},\bar{c})\}| \geq |A_\ell|/\mathbf{i}(\bar{\varphi}) = n_2^* \geq n^{\mathbf{h}(n)}/\mathbf{i}(\bar{\varphi}).$$

So by the (finite) Δ-system lemma 1.20, for some $r \in (0,1)_{\mathbb{R}}$ not depending on n (in fact, $r = \frac{1}{2}\lg(\bar{x}_i)$ suffice), we have:

$\boxplus$ for $\ell = 1,2$ we can find $\langle \bar{a}_{\ell,m} : m < n_3^* = \langle (n_2^*)^r \rangle$ and partition v_ℓ, u_ℓ of $\ell g(\bar{x}_{i(\ell)})$ such that:

$(*)$ (a) $\bar{a}_{\ell,m} \restriction v_c = a_\ell^*$,
(b) $\mathrm{Rang}(\bar{a}_{\ell_1,m_1} \restriction u_{\ell_1}) \cap \mathrm{Rang}(\bar{a}_{\ell_2,m_2}) = \emptyset$ when $m_1, m_2 < n_3^* \wedge \ell_1, \ell_2 \in \{1,2\} \wedge (\ell_1,m_1) \neq (\ell_2,m_2)$,
(c) $\mathrm{Rang}(\bar{a}_{\ell,m} \restriction u_\ell), \mathrm{Rang}(\bar{c}\,\hat{}\,\bar{a}_1^*\,\hat{}\,\bar{a}_2^*)$ are disjoint for $\ell \in \{1,2\}, m < n_3^*$.

We draw $\mathscr{M} \restriction (\bar{c}\,\hat{}\,\bar{a}_{\ell,m})$ for every $\ell \in \{1,2\}$ and $m < n_3^*$ we get $\mathscr{M}'$. So ignoring events of very low probability ($\leq (\frac{1}{2})^{rn}$ for fix $r \in (0,1)_{\mathbb{R}}$)

$(*)$ $w_\ell := \{m < n_3^* : (\mathscr{M}' \restriction \bar{c}\,\hat{}\,\bar{a}_{\ell,m}) \models \varphi_{1,i(\ell)}[\bar{a}_{\ell,m},\bar{c}]\}$ has $\geq n_4^* := \sqrt{n_3^*}$ members.

So $n_4^* \geq n^\varepsilon$ for ε small enough but let $Y_\ell = \{\bar{a}_{\ell,m}/K_{t,i(\ell)} : m \in w_\ell\}$; it is a set of $\geq n_4^*$ nodes of $H_{\bar{\varphi},\mathscr{M},\bar{c}}$.

Now,

$\oplus$ As $i(1), i(2)$ ar not as required in 1.9(3)(b) and for $\mathbf{t} = 0,1$, we have:
- $m(1) \in w_1 \wedge m(2) \in w_2 \Rightarrow \mathrm{Prob}(\mathscr{M} \models \varphi_{1,i(1),i(2)}(\bar{a}_{1,m(1)},\bar{a}_{2,m(2)},\bar{c})) \geq r/\mathbf{g}(n)^k$ for some $r \in \mathbb{R}_{>0}, k \in \mathbb{N} \setminus \{0\}$ not depending on n.

But (considering the partial drawing done above) for $\mathscr{M}^*$ the events $\mathscr{E}^{\mathbf{t}}_{m(1),m(1)} =$"$\mathscr{M} \models \varphi_{1,i(1),i(2)}(\bar{a}_{1,m(1)},\bar{a}_{2,m(2)},\bar{c})$" are independent so by $\oplus$ almost surely

$$(\exists m(1) \in w_1)(\exists m(2) \in w_2)(\mathscr{E}^{\mathbf{t}}_{m(1),m(2)}).$$

As this holds for $\mathbf{t} = 0,1$, we are done proving that (A_1, A_2) cannot witness $\mathscr{M}$ is $\mathbf{h}$-low.

2) Similarly. $\square_{1.21}$

* * *

Definition 1.22. 1) Assume that $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are $\mathbf{s}$-schemes and $\bar{\varphi}_1, \bar{\varphi}_2$ are reduced and complete. We say $(\mathbf{s},\bar{\varphi}^1), (\mathbf{s},\bar{\varphi}^2)$ are explicitly isomorphic <u>when</u> some π and $\varkappa$ witness it which means:

(a) $\mathbf{i}(\bar{\varphi}_1) = \mathbf{i}(\bar{\varphi}_2)$ and $\ell g(\bar{z}_{\bar{\varphi}_1}) = \ell g(\bar{z}_{\bar{\varphi}_2})$, (*b*) π is a permutation of $\{0,\dots,\mathbf{i}(\bar\varphi_1)-1\}$ such that $k_{\bar\varphi_1,i} = k_{\bar\varphi_2,\pi(i)}$ and $K_{\bar\varphi_1,i} = K_{\bar\varphi_2,i}$ for $i < \mathbf{i}(\bar\varphi_1)$,

(*c*) $\varkappa$ is a permutation of $\ell g(\bar z_{\bar\varphi_1})$,

(*d*) for random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar p,n}$, if $\ell \in \{1,2\}, \mathscr{M} \models \varphi_{\ell,2}[\bar c_\ell]$ <u>then</u> letting $\bar c_{3-\ell}$ be such that $\bar c_2 = \varkappa(\bar c_1)$ we have $M \models \varphi_{3-\ell,2}[\bar c_{3-\ell}]$ and $\varphi_{1,i}(\mathscr{M},\bar c_1) = \varphi_{1,\pi(i)}(\mathscr{M},\bar c_2)$ and $\varphi_{1,i,j}(\mathscr{M},\bar c_1) = \varphi_{1,\pi(i),\pi(j)},(\mathscr{M},\bar c_2)$ for $i,j < \mathbf{i}(\bar\varphi)$.

2) For $\mathbf{s},\bar\varphi$ as above let $K_{\bar\varphi} = K_{\mathbf{s},\bar\varphi}$ be the group of permutations K of $\ell g(\bar z_{\bar\varphi})$ such that $\bar\varphi$ is explicitly isomorphic to itself using some $\pi,\varkappa$ in 1.22(1).
3) Assume that $\bar\varphi_1$ and $\bar\varphi_2$ are $\mathbf{s}$-schemes and complete. We say that $\bar\varphi_2$ is a k-extension of $\bar\varphi_1$ <u>when</u> we have:

(a) $\mathbf{i}(\bar\varphi_2) = \mathbf{i}(\bar\varphi_1) + k$,

(b) π is a function from $\mathbf{i}(\varphi_2)$ onto $\mathbf{i}(\varphi_1)$,

(c) if $\mathscr{M}_{\mathbf{s},n}$ is random enough and $\bar c_2 \in \bar c_2 \in \varphi_{2,2}(M)$, <u>then</u>:
- $\bar c_2{\restriction}\mathbf{i}(\bar\varphi_1)$ belongs to $\varphi_{1,2}(M)$,
- the graphs $H_{\bar\varphi_1,M,\bar c}$, $H_{\bar\varphi_2,M,\bar c_2}$ are not isomorphic.

4) Assume $M \in K_{\mathbf{s}}$ and $\bar\varphi'$ and $\bar\varphi''$ are $\mathbf{s}$-schemes and $M \models \varphi'_2[\bar c'] \cap \varphi''_2[\bar c'']$. We say that $(\varphi',\bar c')$ and $(\varphi'',\bar c'')$ are *k-neighbours* <u>when</u> there is a finite sequence $\langle \bar\varphi^*_i : i \le n\rangle$ such that $(\bar\varphi^*_0,\bar\varphi^*_n) = (\bar\varphi',\bar\varphi'')$ and for $i < n$, $\bar\varphi^*_i$, $\bar\varphi^*_{i+1}$ are explicitly isomorphic or $\bar\varphi_i$ is a k-extension of $\bar\varphi^*_{i+1}$ or $\bar\varphi^*_{i+1}$ is a k-extension of $\bar\varphi^*_i$ and $k \le \lg(\bar z_{\bar\varphi}) + \lg(\bar z_{\bar\varphi})$.
5) Similarly for "φ_1 and φ_2 are $\mathbf{r}$-scheme".

Claim 1.23. *1) For every $\mathbf{s}$-scheme, $\bar\varphi$, there is a reduced $\mathbf{s}$-scheme $\bar\varphi'$ which is a neighbour of $\bar\varphi$.*
2) Above $\bar\varphi'$ is unique up to explicit isomorphisms and can be computed.
3) In Definition 1.22(4) we can restrict ourselves to $n = 2$.
4) For each k, restricting ourselves to $\bar\varphi$ with $k(\bar\varphi) \le k$, see 1.6(3) being neighbours is an equivalence relation with finitely many equivalence classes.
5) The truth value of "$(\bar\varphi',\bar c')$, $(\bar\varphi'',\bar c'')$ being neighbours" is computable: is equivalent to a first order property of $\bar c' \hat{\ } \bar c''$.
6) If φ',φ'' are complete and reduced, <u>then</u> there is a quantifier-free $\vartheta = \vartheta(z',z'') \in \mathbb{L}(\tau_{\mathbf{s}})$ such that for random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\Upsilon}$ and $\bar c' \in \varphi'_2(\mathscr{M})$, $\bar c'' \in \varphi''_2(\mathscr{M})$ <u>then</u> $(\bar\varphi',\bar c'),(\bar\varphi'',\bar c'')$ are k-neighbours <u>iff</u> $\mathscr{M} \models \vartheta[\bar c',\bar c'']$.
7) Similarly for $\mathbf{r}$ instead of $\mathbf{s}$.

Proof. Straightforward; note that 1.23(3), (4), (5) is not used. $\square_{1.23}$

Claim 1.24. *1) For every $\mathbf{s}$-scheme $\bar\varphi$ we can find $\langle \bar\varphi^\iota(\bar z_\iota) : \iota < \iota(*)\rangle$ such that:*

(*a*) *$\bar\varphi^\iota(\bar z^\iota_i)$ is a complete reduced $\mathbf{s}$-scheme such that $\bar z^\iota$ is a subsequence of $\bar z$,*

(*b*) *for every $\mathbf{s}$-structure M and $\bar c \in \varphi_2(M)$ for some ι letting $\bar c_\iota = \langle c_j : j \in$ $\mathrm{dom}(\bar z)$ and z_j appears in $\bar z^\iota\}$ we have $H_{\bar\varphi,M,\bar c} \cong H_{\bar\varphi^\iota,M,\bar c^\iota}$,*

(*c*) *for every $\mathbf{s}$-structure M, $\iota < \iota(*)$ and $\bar c^\iota \in \varphi^\iota_2(M)$ there is $\bar c$ such that $(\bar c,\bar c^\iota,\bar\varphi,\bar\varphi^i)$ are as in clause (b).*

2) For complete $\bar\varphi$ in the definition of "trivial", "degenerated", "reduced" we can replace "some $\bar c$" by "$\bar c'$".
3) In the definition of $\mathbb{L}(\mathbf{Q_t})(\tau)$, see Definition 2.2, we can use $(\mathbf{Q}\dots,\bar x_{1,i},\bar x'_{1,i},\dots)\bar\varphi$ for complete reduced non-trivial, non-degenerated $\bar\varphi$.

Proof. Easy.

1) First we can find $\langle \varphi^*_{2,\iota}(\bar z_{\bar\varphi})\colon \iota < i_*\rangle$ such that $\varphi^*_{2,\iota}(\bar z_{\bar\varphi})$ is complete and $\vdash$"$\varphi_2(\bar z_{\bar\varphi}) = \bigvee\{\varphi^*_{2,\iota}(\bar z_{\bar\varphi})\colon \iota < \iota_*\}$. So, it suffices to deal with one φ^*_2. Fixing ι for each $j < \mathbf{i}(\bar\varphi)$, let $\langle \varphi^\iota_{1,j,\ell}(\bar x_j, z_{\bar\varphi})\colon \ell < \ell_1(j)\rangle$ be a sequence of complete formulas such that $\vdash \varphi_{1,j}(\bar x_j, \bar z_\varphi) \equiv \bigvee\{\varphi^\iota_{1,j,\ell}(\bar x_k, \bar z_{\bar\varphi})\colon \ell < \ell_\iota(j)\}$: we can ignore the case $\ell_i(j) = 0$. Let $\langle \ell_m = \ell^i_m\colon m < m_*\rangle$ listing $\{\bar\ell\colon \bar\ell \in \prod\{\ell_\iota(j)\colon j < \mathbf{i}(\bar\varphi)\}\}$ and again it suffice to deal with one m, let $\bar\ell_m = \langle \ell_{m,j} = \ell(m,j) = \ell^\iota_{m,j}\colon j < \mathbf{i}(\bar\varphi)\rangle$ For each m, we choose $\varphi^{\iota,m}_2(\bar z_{\bar\varphi}) = \varphi^\iota_2(\bar z_{\bar\varphi})$, $\varphi^{\iota,m} = \varphi^{\iota,m}_{1,j,\ell(m,j)}(\bar x_j, \bar z_{\bar\varphi})$ for $j < \mathbf{i}(\bar\varphi)$ Third, we discard contradictory pairs $(\iota, \bar c)$, i.e., $\varphi^{\iota,m}_{1,j}(\bar x_2, \bar z_{\bar\varphi}) \nvdash \psi^\bullet_{2,\iota}(\bar z_{\bar\varphi})$. Lastly, we apply the definition of "reduced" to get the desired conclusion. $\square_{1.24}$

Question 1.25. Used?

The Isomorphism Claim 1.26. *Assume* $\mathbf{s}$ *is an* I*-kind and* $\bar\varphi^1, \bar\varphi^2$ *are complete reduced* $\mathbf{s}$*-schemes.*
1) If $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar p,n}$ *is random enough and* $\mathscr{M} \models \varphi^1_2[\bar c^1] \wedge \varphi^2_2[\bar c^2]$ *so* $H^1 = H_{\bar\varphi^1,\mathscr{M},\bar c^1}, H^2 = H_{\bar\varphi^2,\mathscr{M},\bar c^2}$ *are well defined* <u>*then*</u> $H^1 \cong H^2$ <u>*iff*</u> $\mathrm{Rang}(\bar c^1) = \mathrm{Rang}(\bar c^2)$ *and moreover* $(\mathbf{s}, \bar\varphi^1), (\mathbf{s}, \bar\varphi^2)$ *are explicitly isomorphic, as witness by* $(\pi, \varkappa)$ *such that* π *maps* $\bar c^1$ *to* $\bar c^2$*, see Definition 1.22.*
2) Being explicitly isomorphic $\mathbf{s}$*-schemes is an equivalence relation.*
3) Similarly for $\mathbf{r} = (\bar{\mathbf{s}}, \bar\Upsilon)$.

Proof. We shall prove more in §2 and this is not used.

$\square_{1.26}$

§ 2. The random quantifier

Hypothesis 2.1. 1) For this section, let $\iota \in \{1,2\}$, $\iota = 1$ is default.
2) Let $\mathbf{g}, \mathbf{h}$ be as in Convention 1.1.

Definition 2.2. 1) We say $\mathbf{Q} = \mathbf{Q_K}$ is a $\mathbf{h}$-high-graph quantifier <u>when</u>:

(a) $\mathbf{Q}$ is a quantifier on finite graphs, i.e. it is a class of finite graphs closed under isomorphisms,

(b) if H is a finite graph and is $\mathbf{h}$-low <u>then</u> $H \notin \mathbf{Q}$.

2) We define a probability space on the set of high-graph as follows: let $\bar{H}^* = \langle H^*_m : m \in \mathbb{N}\rangle$ be a sequence of pairwise non-isomorphic finite graphs such that each finite graph is isomorphic to exactly one of them.

We let:

(a) $\mathbf{T} = \{\bar{\mathbf{t}} : \bar{\mathbf{t}} = \langle \mathbf{t}_m : m \in \mathbb{N}\rangle, \mathbf{t}_m$ a truth value, $\mathbf{t}_m = 0$ if H^*_m is $\mathbf{h}$-low$\}$,

(b) we draw the $\mathbf{t}_m$'s independently, $\mathbf{t}_m = 0$ if H^*_m is $\mathbf{h}$-low and $\mathbf{t}_m = 1$ has probability $1/\mathbf{g}(|H^*_m|)$ when H^*_m is not $\mathbf{h}$-low,

(c) Let $\mu_{\mathbf{T}}$ be the derived distribution.

2A) So the probability space is $(\mathbb{B}, \mu_{\mathbf{T}}), \mathbb{B}$ is the family of Borel subsets of ${}^{\mathbb{N}}2, \mu_{\mathbf{T}}$ the measure.
3) For $\bar{\mathbf{t}} \in \mathbf{T}$ let $\mathbf{Q}_{\bar{\mathbf{t}}}$ be the quantifier $\mathbf{Q}_{\mathbf{K}_{\bar{\mathbf{t}}}}$, where $\mathbf{K}_{\bar{\mathbf{t}}} = \{H : H$ a finite graph isomorphic to some H^*_m such that $\mathbf{t}_n = 1\}$.
4) We say H is $\mathbf{h}$-high where H is a finite graph which is not $\mathbf{h}$-low.

Claim 2.3. *For every random enough $\bar{\mathbf{t}} \in \mathbf{T}$, that is, there is a Borel subset $\mathbf{B}$ of $\mathbf{T}$ of measure 1, for every $\bar{\mathbf{t}} \in \mathbf{T} \setminus \mathbf{B}$, the following hold.*
1) $\mathbf{Q}_{\bar{\mathbf{t}}}$ is a Lindström quantifier.
2) For random enough graph $\mathscr{G}_{n,p}, \mathbf{Q}_{\bar{\mathbf{t}}}$ define non-trivial quantifier, defining non-first order definable sets.
3) More specifically, for the formula $\psi(x) =$ (the graph restricted to $\{y : yRx\}$ belongs to $\mathbf{K}_{\bar{\mathbf{t}}}$), for every k define in every random enough $\mathscr{G}_{n,p}$, a set which is not first-order definable by a formula of length k.

Proof. Straightforward. $\square_{2.3}$

So

Definition 2.4. 1) The set of formulas $\varphi(\bar{x})$ of the logic $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})(\tau_{\mathbf{s}})$ for a kind sequence $\mathbf{s}$ is the closure of the set of atomic formulas of $\mathbb{L}(\tau_{\mathbf{s}})$ by negation ($\psi(\bar{x}) = \neg\varphi(\bar{x})$), conjunction ($\psi(\bar{x})) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$), existential quantification ($\psi(\bar{x}) = (\exists y)\varphi(\bar{x}, y)$) and applying $\mathbf{Q}_{\bar{\mathbf{t}}}, \psi(\bar{z}_{\bar{\varphi}}) = (\mathbf{Q}_{\bar{\mathbf{t}}}, \dots, \bar{x}_{0,i}, \bar{x}'_{0,i}, \dots)_{i<\mathbf{i}(\bar{\varphi})}\bar{\varphi}$ where $\bar{\varphi}$ is an $\mathbf{s}$-scheme of formulas which are already in $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})(\tau_{\mathbf{s}})$, so as defined in 1.6(1) except that now the $\varphi_{\iota,i}$ are not necessarily quantifier free formulas from $\mathbb{L}(\tau_{\mathbf{s}})$. Pedantically, we may write $\mathbf{Q}^{\mathbf{s}}_{\bar{\mathbf{t}}}$ instead $\mathbf{Q}_{\bar{\mathbf{t}}}$.
2) Satisfaction, i.e. for a (finite) $\mathbf{s}$-structure M, formula $\vartheta(\bar{x})$ and sequence $\bar{a}$ of elements of M of length $\ell g(\bar{x})$, we define the truth value of $M \models \vartheta[\bar{a}]$ by induction on ϑ, the new case is when:

- $\vartheta(\bar{z}_{\bar{\varphi}}) = (\mathbf{Q}_{\bar{t}}, \dots, \bar{x}_{0,i}, \bar{x}'_{0,i}, \dots)_{i<\mathbf{i}(\bar{\varphi})}\bar{\varphi}$.

Now $M \models \vartheta[\bar{c}]$ iff $\bar{c} \in \varphi_2(M)$ and $H_{\bar{\varphi},M,\bar{c}}$ is isomorphic to some graph from $\{H^*_m : \mathbf{t}_m = 1\}$.
3) The syntax of $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$ does not depend on $\bar{\mathbf{t}}$ so we may write $\mathbb{L}(\mathbf{Q})$ that is $\mathbb{L}(\mathbf{Q})(\tau)$ is the relevant set of formulas, but the satisfaction depends so we shall write $M \models_{\bar{\mathbf{t}}} \varphi[\bar{a}]$ for $\bar{a}$ a sequence from M and formula $\varphi(\bar{x}) \in \mathbb{L}(\mathbf{Q})$; of course, such that $\ell g(\bar{a}) = \ell g(\bar{x})$.

Theorem 2.5. *1) For any $q \in (0,1)_{\mathbb{R}}$ for all but a null set of $\bar{\mathbf{t}} \in \mathbf{T}$, the random graph $\mathscr{G}_{n,q}$ satisfies the 0-1 law for the logic $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$, i.e. we may allow to apply $\mathbf{Q}_{\bar{\mathbf{t}}}$ to definitions as in Definition 1.6, see Claim 1.11.*
2)The limit theory T_* is decidable modulo an oracle for the random $\mathbf{K}_{\mathbf{t}}$.

Remark 2.6. 1) Of course, in 2.2, 2.3 we can replace the class of graphs by the class of $\mathbf{s}$-structures, $\mathbf{s}$ any kind sequence.
2) Does the limit theory depend on $\bar{\mathbf{t}}$? The problem is for when we apply the quantifier to graphs of fixed size, so use some complete $\bar{\varphi}$ with $k^*_{\mathbf{s},i}(\bar{\varphi}) = 0$ for every $i < \mathbf{i}(\bar{\varphi})$. So we have to decide if to include formulas in which this occurs.
3) The proof is devided into a series of Definitions and Claims.

Definition 2.7. We say that $\mathbf{n}$ is a *local frame* when it consists of (so $\Delta_\ell = \Delta_{\mathbf{n},\ell}$, etc.):

(A) (a) $\ell_{\mathbf{n}} := \ell(\mathbf{n}) < \omega$ and $\bar{\Delta} = \langle \Delta_\ell \colon \ell < \ell_{\mathbf{n}} \rangle$,
 (b) Δ_ℓ is a finite set of formulas from $\mathbb{L}(\mathbf{Q})(\tau_{\mathrm{gr}})$ increasing with ℓ,
 (c) let $\Delta_\ell := \{\vartheta_s(\bar{z}_s) \colon s \in I^*_\ell\}$, hence I^*_ℓ is finite and $m < \ell \Rightarrow I^*_m \subseteq I^*_\ell$,
 (d) $I_0 := I^*_0 := \{s_{\mathrm{gr}}\}$ and $\Delta_0 := \{\vartheta_{\mathrm{gr}}(x,y)\} = \{R_{s_{\mathrm{gr}}}(x,y)\}$.

(B) (a) $I_{\ell+1} := I^*_{\ell+1} \setminus I^*_\ell$,
 (b) for $t \in I_{\ell+1}$, we have:
  - $\vartheta_t(\bar{z}_t) := (\mathbf{Q} \ldots \bar{x}_i, \bar{x}'_i, \ldots)_{i < \mathbf{i}(\bar{\varphi})} \bar{\varphi}_t(\bar{z})$, where:
  - $\bar{\varphi}_t$ is an $\mathbf{s}_0$-scheme,
  - the formulas $\varphi^t_{0,i}(\bar{x},\bar{z})$, $\varphi^t_{1,i,j}(\bar{x}_i,\bar{x}_j,\bar{z})$, $\varphi^t_2(\bar{z})$ are Boolean combinations of formulas from Δ_ℓ, which are complete in this sense.
 (c)
  - $\mathbf{s}_\ell$ is an I^*_ℓ- kind for $\ell \le \ell_{\mathbf{n}}$,
  - $\mathbf{s}_\ell$ is increasing with ℓ,
  - $\mathbf{s}_0$ is a I^*_0-kind, so $n_{\mathbf{s}_0,s_{\mathrm{gr}}} = 2$, $K_{\mathbf{s}_0,s_{\mathrm{gr}}} = \mathrm{Sym}(2)$, the group of permutations on $\{0,1\}$,
  - $K_{\mathbf{s}_{\ell+1},t} = K_{\bar{\varphi}_t}$ (see Definition 1.6) for $t \in I_{\ell+1}$.

Definition 2.8. 1) If $\mathbf{n}$ is a local frame and $\ell \le \ell_{\mathbf{n}}$, we define $\mathbf{n}' := \mathbf{n} \restriction \ell$ naturally such that $\ell(\mathbf{n}') = \ell$.
2) Let $\Phi_{\mathbf{n},\ell}$ be the set of formulas from $\mathbb{L}(\mathbf{Q})(\tau_{\mathrm{pr}})$ such that for every sub-formulas of the form $\psi(\bar{z}) = (\mathbf{Q} \ldots \bar{x}_i, \bar{x}'_i \ldots)_{i < i(\bar{\varphi})}(\bar{\varphi})$ there is $t \in I_{\ell+1}$, $\ell < \ell_{\mathbf{n}}$ such that $\psi = \vartheta_t$ (so we replace $\bar{z}$ but a sequence of variables of the same length).

Definition 2.9. 1) Let $\mathbf{n}$ be a local frame. By induction on $\ell \le \ell(*)$ we choose some extra objects satisfying (so e.g. $\vartheta'_{\mathbf{n},s}(\bar{z}_{\mathbf{n},s}) = \vartheta'_s(\bar{z}_s)$, but may omit $\mathbf{n}$ when clear from the context):

(A) (a) $\bar{\vartheta}'_\ell = \langle \vartheta'_s(\bar{z}_s) \colon s \in I^*_\ell \rangle$,
 (b) $\bar{\vartheta}''_\ell = \langle \vartheta''_s(\bar{z}_s) \colon s \in I^*_\ell \rangle$,
 (c) $\mathbf{s}_\ell$ is a I^*_ℓ-kind,
 (d) $\vartheta''_s(\bar{z}_s)$ is the formula $R_s(\bar{z}_s) \in \mathbb{L}(\tau_{\mathbf{s}_\ell})$ for $s \in I^*_\ell$,

(e) if $s \in I_{\ell+1}$ then we define $\bar{\varphi}'_s(\bar{z}_s)$ by replacing in $\bar{\varphi}_s$ every ϑ_t by ϑ''_t (so $t \in I^*_\ell$ and $\bar{\varphi}'$ consists of quantifier-free formulas in $\mathbb{L}(\tau(\mathbf{s}_\ell)))$ and let $\vartheta'_s(\bar{z}_s) = (\mathbf{Q} \ldots, \bar{x}_i, \bar{x}'_i, \ldots)_{i<\mathbf{i}(\bar{\varphi}_\iota)} \bar{\varphi}'_s$,

(f) $K_{\mathbf{s}_{\ell,t}}$ is $K_{\bar{\varphi}_t}$, see 1.6(1)(c), so decided by $\bar{\varphi}_t(\bar{z}_t)$,

(g) let $\vartheta^\bullet_{s_{\rm gr}} := \vartheta(\bar{z}_{s_{\rm gr}})$

(B) (a) $\bar{\mathbf{s}}_{\le\ell} = \mathbf{s}(\le \ell) = \langle \mathbf{s}_k : k \le \ell\rangle$ and $(\bar{\mathbf{s}}_{\le\ell}, \bar{\Upsilon}_{\le\ell})$ is as in Definition 1.3.

(b) Let

- $\bullet_1$ $\mathbf{r}_\ell = (\bar{\mathbf{s}}_{\le\ell}, \bar{\Upsilon}_{\le\ell})$,
- $\bullet_2$ $\bar{\Upsilon}_{\le\ell} = \bar{\Upsilon}(\le \ell) = \langle \Upsilon_{k,t} : k < \ell$ and $t \in J^\bullet_k\rangle$, where:
- $\bullet_3$ $\Upsilon_{\ell,t}(\bar{z}_t) \in \mathbb{L}(\tau_{\mathbf{s}_\ell})$ is $\varphi'_{t,2}(\bar{z}_t)$,
- $\bullet_4$ $\bar{\mathbf{s}} = \bar{\mathbf{s}}(\le \ell_{\mathbf{n}})$ and $\bar{\Upsilon} = \bar{\Upsilon}\langle \ell(\mathbf{n})\rangle$.
- $\bullet_5$ $\mathbf{s}_{\mathbf{n}} = \mathbf{s}(\mathbf{n}) = \mathbf{s}_{\ell(\mathbf{n})}$.

Definition 2.10. 1) We say that a local frame $\mathbf{n}$ is *reasonable* when for $\ell < \ell_{\mathbf{n}}$,

- if $t \in I_{\ell+1}$ then $\bar{\varphi}'_{\mathbf{n},t}$ is complete reduced (see Definition 1.12) and not 1-weak,
- if $s \neq t \in I_{\ell+1}$, then $\bar{\varphi}'_s, \bar{\varphi}'_t$ are not an explicitly isomorphism (see 1.22(1)). For every $t \in I_{\ell+1}$, $\bar{\varphi}'_t$ is correct; see Definition 1.6(6).

Definition 2.11. 1) For a local frame $\mathbf{n}$ and $\bar{\mathbf{t}} \in \mathscr{T}$, we define the function $G \mapsto M_{G,\bar{\mathbf{t}},\mathbf{n},\ell}$ for G being a graph on $[n]$ for $n < \omega$, as follows: for any given $G \in \mathbf{M}_{\mathbf{s}_0,n}$ and $\bar{\mathbf{t}} \in \mathbf{T}$ for $\ell \le \ell(*)$, we define $M_{G,\bar{\mathbf{t}},\mathbf{n}} \in \mathbf{M}_{\mathbf{s}_\ell,n}$ by:

- if $\ell = 0$, $M_{G,\bar{\mathbf{t}},\mathbf{n},\ell}$ is G,
- if $\ell > 0$ then $M_{G,\bar{\mathbf{t}},n,\ell}$ is a $\tau_{\mathbf{s}_\ell}$-model expanding $M_{G,\bar{\mathbf{t}},\mathbf{n},m}$ for $m < \ell$ and for $s \in I_\ell$, $R_s^{M_{G,\bar{\mathbf{t}},\mathbf{n},\ell}}$ is defined by $\vartheta_s(\bar{z}_s)$.

2) $\mathbf{M}_{\mathbf{n},n}, \mathbf{M}_{\mathbf{n}}$ is the set of $M \in \mathbf{M}_{\bar{\mathbf{s}},\bar{\Upsilon}}, \mathbf{M}_{\bar{\mathbf{s}},\bar{\Upsilon},n}$, respectively. see 1.20(1)(B)(b)($\bullet_4$).

Observation 2.12. *1) If $G \in \mathbf{M}_{\mathbf{s}_0,m}$, then:*

(a) $M_{G,\bar{\mathbf{t}},\mathbf{n},\ell} = M_{G,\bar{\mathbf{t}},\mathbf{n}\restriction\ell,\ell}$,

(b) $M_{G,\bar{\mathbf{t}},\mathbf{n},\ell} \in \mathbf{M}_{\mathbf{n}\restriction\ell,m}$.

2) In Definition 2.9, $\mathbf{r}_{\mathbf{n}}$ is as in 1.3(11).

Definition 2.13. We say that $M = G \in \mathbf{M}_{\mathbf{s}_0,n}$ is $(\mathbf{g}, \mathbf{n}, \bar{k}^*)$-*good*, when:

(A) (a) $\mathbf{n}$ is a nice local frame,

(b) $\bar{k}^* = \langle k^*_\ell : \ell \le \ell_{\mathbf{n}}\rangle$,

(c) $\mathbf{g}$ as usual, see 1.1,

(d) let $\mathscr{P}_k := \mathscr{P}_{\mathbf{n},k}$ be the set of $p = p(\bar{x}_{[k]})$, quantifier-free-complete type in $\mathbb{L}(\tau_{\mathbf{s}\langle\mathbf{n}\rangle})$ realized in some member of $\mathbf{M}_{\mathbf{n}}$,

(e) let $\mathscr{P}^{1-1}_k = \{p(\bar{x}_{[k]}) \in \mathscr{P}_k : p(\bar{x}_{[k]}) \Vdash \bigwedge\{x_{\ell_1} \neq x_{\ell_2} : \ell_1 \neq \ell_2 < k\}\}$,

(f) for $p = p[\bar{x}_{[k]}] \in \mathscr{P}^{1-1}_k$, let size$(p)$ be the minimal $m < \omega$ such that for some $\varphi \in p$ of length $< m$, we have $\varphi \vdash p$ (or just large enough),

(g) if $p = p(\bar{x}_{[k]})$ is a complete quantifier-free in $\bar{\varphi}_t$ for some $t \in I_{\mathbf{n},\ell}$ (that is, $\varphi_{t,2}(\bar{z})$ or $\varphi_{t,1,i}(\bar{x}_i, \bar{z})$ or is a completion of $\varphi_{t,1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z})$ generate it) then size$(p) < k_\ell$.

(B) If $p(\bar{x}, \bar{y}) \in \mathscr{P}^{1-1}_k$, $1 \le k \le k_{\ell(\mathbf{n})^*}$ (so $\lg(\bar{x}) + \lg(\bar{y}) = k_1$), then the set $p(M, \bar{b})$ has $\mathbf{g}(n^{\frac{1}{\text{size}(p)}})$ members.

(C) If (a) then (b), where[3]:

[3]Concerning $\bullet_4$ and $\bullet_5$, using 1-1 is not essential, but using 1-weak $\bar{\varphi}$ make it natural.

(a) $\bullet_1$ $k_\iota < k^*_\ell$ for $\iota = 1,2,3$,
$\bullet_2$ $p_1 := p_1(\bar{x}^1_{[k_2]}, \bar{y}^1_{[k_1]}) \in \mathscr{P}^{1-1}_{\mathbf{n},k_2+k_1}$ and $p_2 = p_2(\bar{x}^2_{k_4}, \bar{y}^2_{k_3}) \in \mathscr{P}_{\mathbf{n},k_4+k_3}$,
$\bullet_3$ $\bar{b}_1 \in {}^{(k_1)}([n])$ realize $p_1{\restriction}\bar{y}_{[k_1]}$,
$\bullet_4$ $\mathscr{P}^*_1 = \{q \colon q = q(\bar{x}^1_{[k_2]}, \bar{x}^{1,*}_{k_2}, \bar{y}^1_{[k_1]}) \in \mathscr{P}^{1-1}_{\mathbf{n},2k_2+k_1}\}$,
$\bullet_5$ $\mathscr{P}^*_2 = \{q \colon q = q(\bar{x}^2_{[k_4]}, \bar{x}^{2,*}_{k_4}, \bar{y}^2_{[k_3]}) \in \mathscr{P}^{1-1}_{\mathbf{n},2k_4+k_3}\}$,
$\bullet_6$ if $\ell = 1,2$ then $f_\ell \colon \mathscr{P}^*_\ell \to \{0,1\}$ is not constant.

(b) if $\mathbf{I}_\ell \subseteq p_\ell(M, \bar{b}_\ell) \subseteq \mathrm{seq}^{1-1}_{k_2}([n])$ has $\geq \mathbf{g}(|p_\ell(M,\bar{b}_\ell)|\frac{1}{k^*_\ell})$ members for $\ell = 1,2$ and π us a 1-to-1 function from $\mathbf{I}_1$ into $\mathbf{I}_2$ and $\bar{a} \in \mathbf{I}_1 \Rightarrow \pi(\bar{a}) \neq \bar{a}$, <u>then</u> for some disjoint $\bar{a}', \bar{a}'' \in \mathbf{P}_1$ also $\pi(\bar{a}')$ and $\pi(\bar{a}'')$ are disjoint and $f_1(\mathrm{tp}_{\mathrm{qf}}(\bar{a}', \bar{a}'', \bar{b}_1)) \neq f_2(\mathrm{tp}_{\mathrm{qf}}(\bar{a}', \bar{a}'', \bar{b}_2))$.

(D) We have (a) $\Rightarrow$ (b), where:

(a) $\bullet_1$ $p(\bar{x}, \bar{y}) \in \mathscr{P}^{1-1}_k$, $1 \leq k \leq k^*_\ell$ (so $\lg(\bar{x}) + \lg(\bar{y}) = k_1$),
$\bullet_2$ $\bar{b}$ realizes $p{\restriction}\bar{y}$ in M,
$\bullet_3$ $\mathscr{P}_* = \{q \colon q = p(\bar{x}^1_{[k_1]}, \bar{x}^{1,*}_{[k_1]}, \bar{y}^1_{[k_2]})\} \in \mathscr{P}^{1-1}_{\mathbf{n},2k_2+k_1}$,
$\bullet_4$ $f \colon \mathscr{P}_* \to \{0,1\}$,
$\bullet_5$ $\bar{a}', \bar{a}'' \in p(M, \bar{b})$ and the set

$$\{\bar{a} \in p(M,\bar{b}) \colon \bar{a} \text{ disjoint to } \bar{a}'{}^\frown\bar{a}'' \wedge f(\mathrm{tp}_{\mathrm{gr}}(\bar{a}^\frown\bar{a}'{}^\frown\bar{b}, \emptyset, M)) \neq f(\mathrm{tp}_{\mathrm{gr}}(\bar{a}^\frown\bar{a}''{}^\frown\bar{b}, \emptyset, M))\}$$

$$\text{has } \leq |p(M,\bar{b})| - \mathbf{g}(|p(M,\bar{b})|\tfrac{1}{k^*_\ell}) \text{ elements,}$$

(b) we have $\mathrm{rang}(\bar{a}') = \mathrm{rang}(\bar{a}'')$ if $\bar{a} \in p(M,\bar{b})$ is disjoint to $\bar{a}$', equivalently to $\bar{a}''$, then $\bar{a}'{}^\frown\bar{a}'{}^\frown\bar{b}$, $\bar{a}^\frown\bar{a}''{}^\frown\bar{b}$ realizes the same quantifier free type in M (this is actually a condition on $\mathscr{P}$) for some disjoint $\bar{a}', \bar{a}'' \in p_1(M, \bar{b}_1)$, $f_1(\mathrm{tp}_{\mathrm{gr}}(\bar{a}'{}^\frown\bar{a}''{}^\frown\bar{b}_1, \emptyset, M)) \neq f_2(\mathrm{tp}_{\mathrm{gr}}(\pi(\bar{a}')^\frown\pi(\bar{a}'')^\frown\bar{b}_2, \emptyset, M))$

Claim 2.14. *If $M \in \mathscr{M}_{\mathbf{n},n}$ is $(\mathbf{g}, \mathbf{n}, \bar{k}^*)$-good <u>then</u> the non-isomorphism condition holds in M, that is, no two of the following graphs are isomorphic (except when the condition in 2.13(D)(b) holds).*

- *for some $\ell \leq \ell_{\mathbf{n}}$, $t \in J_\ell$ and $\bar{c} \in \varphi_{t,2}(M)$, we have $H = H_{\bar{\varphi}_t, M_\ell, \bar{c}}$.*

Proof. Straightforward. $\square_{2.14}$

Definition 2.15. We say that $\mathbf{n}$ is $(\mathbf{g}, \ell)$-*nice* <u>when</u> $\ell \leq \ell_{\mathbf{n}}$ and for $\bar{k}^* = \langle k^*_i \colon i \leq \ell \rangle$ increasing fast enough, we have for almost every pair $(G, \bar{\mathbf{t}})$ by the distribution $(\mathscr{M}_{\mathbf{s}_0,n}, \mu_{\mathbf{T}}) = (\mathscr{G}_{n,q}, \mu_{\mathbf{T}})$ the model $M_{G,\mathbf{n},\bar{k}^*}$ is $(\mathbf{g}, \mathbf{n}, \ell)$-good.

Claim 2.16. *If $\mathbf{n}$ is $(\mathbf{g}, \ell)$-nice, $\ell < \ell_{\mathbf{n}}$ <u>then</u> $\mathbf{n}$ is $(\mathbf{g}, \ell+1)$-nice.*

Proof. Straightforward. $\square_{2.16}$

Claim 2.17. *If $\psi \in \mathbb{L}(\mathbf{Q})(\tau_{\mathrm{gr}})$, then for some local frame $\mathbf{n}$, all sub-formulas of ψ of the form*

$$(\mathbf{Q} \ldots \bar{x}_{n+1} \ldots \bar{x}^{-1}_n)_{i < \mathbf{i}(\bar{\varphi})}\, \psi$$

are equivalent to a finite disjunction of formulas $\vartheta_{\mathbf{n},t}(\bar{z}_t)$.

Proof. Follows. $\square_{2.17}$

We are finally ready to conclude Theorem 2.5.

Proof. Follows. $\square_{2.5}$

Remark 2.18. To eliminate $(*)_4 M$, in the end of the proof we may complicate the drawing of $\mathscr{M}_{\mathbf{s}_{\ell+1},\bar{p},n}$ We draw $\mathscr{M}_{\mathbf{s}_m,\bar{p},n}$ by induction on m: if $m = 2j+2, M = M_{\mathbf{x}_{2j+1},\bar{p},n}$ given for $R_t(t \in I^*_m \backslash I^*_{2k+1})$ we consider only $\bar{c} \in \varphi'_{t,2}(M)$ let $m = m_t(\bar{c}) = m_t(\bar{c}, M)$ be the number of nodes of $H_{\bar{\varphi}'_t,M,t}$ and we draw a truth value of $R_t(\bar{c})$ with probability $1/g(m)$. Proving the 0-1 law for such drawing is easy.

§ 3. How to get a real quantifier, i.e. definable K

Discussion 3.1. (?) In the introduction, we have considered drawing a truth value to all graphs. So replacing "converge to zero or to one" we ask only "for every $\varepsilon > 0$ for every n large enough the probability is up to ε closed to zero or to one", The point is that otherwise, we can weakly express "$|\varphi_1(\mathscr{G}_{p,n}, \bar a_1)| = |\varphi_2(\mathscr{G}_{p,n}, \bar a_2)|$, e.g. for $\varphi(x,y) = xRy$. So we can find $\psi_1(x_1, x_2)$ implying valency$_{\mathscr{G}_{p,n}}(y_1) =$ valency$_{\mathscr{G}_{p,n}}(y_n)$, this will make us fail, see [HHMS].

In more details, let $\psi_\varphi(y)$ say "the empty graph on $\varphi(\mathscr{G}_{p,n}, y)$ is green".

Let $\psi_2(y_1, y_2)$ say:

(a) $\psi_1(y_1) \equiv \psi_2(y_2)$,
(b) for $\ell \in \{1,2\}$ and y'_ℓ there is $y_{3-\ell}$ such that $|\varphi(\mathscr{G}_{p,n}, y_1) \cap \varphi(\mathscr{G}_{n,p}, y'_1)| = |\varphi(\mathscr{G}_{p,n}, y_2) \cap \varphi(\mathscr{G}_{p,n}, y'_2)|$.

This nearly expresses $|\varphi(\mathscr{G}_{p,n}, y_1)| = |\varphi(\mathscr{G}_{p,n}, y_2)|$. We can strengthen this and find approximation to $a+1$ and cases of addition.

While the above does not suffice to prove impossibility, it suffices to show the problem is not promising and is different; maybe relevant is the late [S$^+$a].

Discussion 3.2. Can we use a quantifier $\mathbf{Q}_\mathbf{K}$ which depends just on the number of edges via the number of nodes.

1) If it depends only on the number of nodes, it seemed that this is bad for 0-1 laws.

2) Notes that surely graphs H_1, H_2 occur up to isomorphism when H_2 is gotten by omitting one edge of H_1. So we may try that it depends only the number modulo $(\lfloor \log\log(4+1) \rfloor)$! Quite reasonable choice of the quantifier but not ideal.

3) So we may try to change the logic such that essentially just changing one edge does not matter; that is excluding some family of graphs which with probability one does not occurs for a random enough $\mathscr{G}_{p,n}$. This is a reasonable logic, even without "$H \in \mathbf{K}$ depends just on the number of edges (and nodes)"

(A) if we forget this restriction, we need to change the flipping of coins for the logic, e.g. fixing size first, choose one randomly, do this for each neighborhood, choose with distorted probability; not clear if converge and there is a natural way.
(B) Here $n^{\mathbf{g}(h)}$ goes slowly to ∞ and is used how to make the results O.K.. Note: in $\mathscr{G}_n$ the size of a definable graph for some m, is $\approx \frac{n}{m}$ so the variance is $c\sqrt{\frac{n}{m}}$; still the edges have probability $\frac{1}{2}$ and so O.K.

However for later M_n^ι ($\iota <$ quantifier depth) the probability of each case of a relation is, i.e. $H \in \mathbf{K}$ for a structure with probability $\frac{1}{\mathbf{h}(n)}$ so manipulating $\mathbf{h}$ gives different results.

4) But we have a more profound problem: we have nicely definable H_1, H_2 getting H_2 from H_1 by, for some nodes $a \neq b$ omitting the edges (a,c) and adding the edges (b,c) whenever $(b,c))$ is an edge when (a,c) is not.

Alternatively omitting the edge (a,c) when (b,c) is an edge, The first does not change the number of edges, the second changes seriously. This may be close to the variance for the number of edges.

A medicine? ask: omitting $\log_*(H_)$ edges, what is the minimal number of edges?

The overcoming may cost: in how to make the probability computations right.

5) Note: from random $\mathscr{G}_{n,1/2}$ we build $\mathscr{M}^9_{n_1} = \mathscr{G}_{n,H}$ an $\mathbf{s}_0$-structure $\mathscr{M}^1_n$ expands by applying the quantifier getting an $\mathbf{s}_1$-structure. But $\mathscr{M}^{\mathbf{s}_1}_n$ is different:

- (a) for $\mathscr{M}^{\mathbf{s}_1}_n$ the cases are totally independent,
- (b) $\mathscr{M}^1_n$ is different: first we draw $R_{\rm gr}$ ($= R_0$ in the lecture) after this we draw the other relatives but their probabilities:
  - depends on the drawing of $\mathscr{M}_n = \mathscr{G}_{1/2,n}$,
  - in particular, on the sizes of the H's which are not too far from n but are different.

This complicates our work but the estimates are not so different.

Discussion 3.3. One which seems easiest while not unreasonable is: given a finite graph G, with m points, which is reasonable - defined as in [S$^+$b] and a point b in it, compute the valency minus $m/2$, divided by square root of m (or the variance of the related normal distribution) and ask if rounding to integers is odd or even.

We may replace the valency by the number of edges of G.

What are the dangers? As we may define a variant of the graph omitting one edge, in some cases this will change the truth value. For each nod the probability goes to zero but in binomial distribution the probability of e.g. getting valency exactly half of the expected value (rounded) is about 1 divided by the square root of m.

So we should divide not by the square root of m but by a larger value (maybe instead of asking on even/odd of the rounded value just ask if it can be larger than one, or absolute value) such that:

- (a) almost surely (i.e. with large probability) for some node the value is above 1,
- (b) the probability that it is exactly one for some node is negligible, and this is true even if we use a graph only definable (reversing edge/non-edge, omitting some, etc.).

So we should say that clearly by continuity considerations there are such choices. A danger is that the n being odd/even can be expressed.

Another avenue is to choose the more natural "the valency is at least half"; but then it seems we can express being even/odd: say change by one edge change the truth value and this is true even if we omit one node. So the number of neighborhoods is half in both cases.

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