

A UNIQUE Q -POINT AND INFINITELY MANY NEAR-COHERENCE CLASSES OF ULTRAFILTERS

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Abstract. We show that in the model obtained by iteratively pseudo-intersecting a Ramsey ultrafilter via a length- ω_2 countable support iteration of restricted Mathias forcing over a ground model satisfying CH, there is a unique Q -point up to isomorphism. In particular, it is consistent that there is only one Q -point while there are $2^{\mathfrak{c}}$ -many near-coherence classes of ultrafilters.

key-words: Q -point, Ramsey ultrafilter, Mathias forcing

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1 Introduction

Throughout this paper, read *ultrafilter* as *non-principal ultrafilter on ω* . For $x \subseteq \omega$, we denote by $[x]^\omega$ the set of infinite subsets of x and by $\text{fin}(x)$ the set of finite subsets of x .

Recall that an ultrafilter E is a *Q -point* if and only if for every interval partition $\{[k_i, k_{i+1}) : i \in \omega\}$ of ω , there exists some $x \in E$ such that $\forall i \in \omega : |x \cap [k_i, k_{i+1})| \leq 1$. Furthermore, an ultrafilter $\mathcal{U}$ is a *Ramsey ultrafilter* if and only if the *Maiden* has no winning strategy in the *ultrafilter game for $\mathcal{U}$* , played between the Maiden and *Death*:

DEFINITION 1.1. *Let $\mathcal{U}$ be an ultrafilter. The ultrafilter game for $\mathcal{U}$ proceeds as follows:*

The Maiden opens the game and plays some $y_0 \in \mathcal{U}$. Death responds by playing some $n_0 \in y_0$. In the $(k+1)$ -th move, the Maiden having played $y_0 \supseteq y_1 \supseteq \dots \supseteq y_k$, and Death

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having played $n_0 < n_1 < \dots < n_k$, the Maiden plays some $y_{k+1} \in [y_k]^\omega \cap \mathcal{U}$, and Death responds by playing some $n_{k+1} \in y_{k+1}$, $n_{k+1} > n_k$.

Death wins if and only if $\{n_i : i \in \omega\} \in \mathcal{U}$.

It is well-known that every Ramsey ultrafilter is a Q -point. Canjar [5] showed that the existence of $2^{\mathfrak{c}}$ -many Ramsey ultrafilters follows from the assumption $\text{cov}(\mathcal{M}) = \mathfrak{c}$. The weaker assumption $\text{cov}(\mathcal{M}) = \mathfrak{d}$ implies the existence of $2^{\mathfrak{c}}$ Q -points, as was shown by Millán [6]. It is well-known that in the Mathias model – the model obtained by a length- ω_2 countable support iteration of unrestricted Mathias forcing over a ground model satisfying **CH** – there are no Q -points (see [1, Proposition 26.23]). In fact, the Mathias model contains no *rapid* ultrafilters, where an ultrafilter E is rapid iff for every $f \in {}^\omega\omega$ there exists some $x \in E$ such that $\forall n \in \omega : |x \cap f(n)| \leq n$ (note that every Q -point is rapid). It follows that both the Mathias model and the model considered in this paper satisfy $\text{cov}(\mathcal{M}) = \omega_1 < \mathfrak{d} = \mathfrak{c} = \omega_2$.

In contrast to the Mathias model, our model contains $2^{\mathfrak{c}}$ -many rapid ultrafilters: It follows from an observation of Millán [6, page 222] that the existence of a single rapid ultrafilter E implies the existence of $2^{\mathfrak{c}}$ of them, by considering the products $U \otimes E$ for different ultrafilters U .²

While the consistency of the non-existence of Q -points is a well-established fact with a variety of witnesses apart from the Mathias model³, the construction of models containing only ‘few’ Q -points seems to have received less attention. However, such models do arise naturally as models containing only few near-coherence classes of ultrafilters⁴: Indeed, Mildner [10] has constructed models with exactly two and exactly three near-coherence classes, and it is easy to see that these contain exactly one and exactly two Q -points, respectively: In her model with exactly two near-coherence classes, one class contains a Ramsey ultrafilter, while the other class contains an ultrafilter that is ω_1 -generated. Hence, this latter class cannot contain a Q -point, since her models satisfy $\mathfrak{d} = \mathfrak{c} = \omega_2$ and such a Q -point would thus have to be $<\mathfrak{d}$ -generated, which is impossible. Analogously, in Mildner’s model with exactly three near-coherence classes, two classes are represented by Ramsey ultrafilters, while the third contains an ω_1 -generated ultrafilter – giving exactly two Q -points in total.

The construction of models with exactly n near-coherence classes of ultrafilters for various finite $n \geq 4$ would similarly yield the consistency of exactly m Q -points for some $m < n$.⁵

² $U \otimes E$ is an ultrafilter on $\omega \times \omega$ defined by $U \otimes E = \{x \subseteq \omega \times \omega : \{n \in \omega : (x)_n \in E\} \in U\}$, where $(x)_n = \{m \in \omega : \langle n, m \rangle \in x\}$.

³such as the Laver and Miller models (see [7] and [11], respectively).

⁴Two ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ are *nearly-coherent* iff there is some finite-to-one $f \in {}^\omega\omega$ such that $f(\mathcal{U}_1) = f(\mathcal{U}_2)$, where $f(\mathcal{U}_i) := \{X \subseteq \omega : f^{-1}[X] \in \mathcal{U}_i\}$. Note that two Q -points are nearly-coherent iff they are isomorphic.

⁵The inequality is strict since such a model must satisfy $\mathfrak{u} < \mathfrak{d}$, a result due to Banach and Blass [9]. Hence, one of the n near-coherence classes contains a $<\mathfrak{d}$ -generated ultrafilter and thus no Q -point.

The model considered in this paper is of a different nature, however: It contains only one Q -point while its number of near-coherence classes is $2^{\mathfrak{c}}$, i.e., the model's lack of Q -points is not the consequence of a lack of near-coherence classes. This follows from the fact that dominating reals are added at each of the ω_2 -many stages of the iteration, which gives $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2$ in the final extension. Since $\mathfrak{b} \leq \mathfrak{u}$ (see Solomon [8]), we have $\mathfrak{u} = \mathfrak{d} = \omega_2$ in our model, and hence there are $2^{\mathfrak{c}}$ -many near-coherence classes of ultrafilters by Banach and Blass [9].

DEFINITION 1.2. *Let $\mathcal{U}$ be a Ramsey ultrafilter. Mathias forcing restricted to $\mathcal{U}$, written $\mathbb{M}_{\mathcal{U}}$, consists of conditions $\langle s, x \rangle \in \text{fin}(\omega) \times \mathcal{U}$ with $\max s < \min x$, ordered by*

$$\langle s, x \rangle \leq_{\mathbb{M}_{\mathcal{U}}} \langle t, y \rangle : \iff s \subseteq t \wedge x \supseteq y \wedge t \setminus s \subseteq x.$$

Note that we use the convention that stronger forcing conditions are larger. The forcing notion $\mathbb{M}_{\mathcal{U}}$ clearly satisfies the c.c.c. and is therefore proper. We will need the following additional facts.

FACT 1.3 (e.g., see [1, Theorem 26.3]). *Let $\mathcal{U}$ be a Ramsey ultrafilter. The forcing notion $\mathbb{M}_{\mathcal{U}}$ has the pure decision property, i.e., for any sentence φ in the forcing language and any $\mathbb{M}_{\mathcal{U}}$ -condition $\langle s, x \rangle$, there exists $y \in [x]^{\omega} \cap \mathcal{U}$ such that either $\langle s, y \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \varphi$ or $\langle s, y \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \neg\varphi$.*

DEFINITION 1.4. *Recall that a forcing notion $\mathbb{P}$ has the Laver property iff for every $\mathbb{P}$ -name $\underline{g}$ for an element of ${}^{\omega}\omega$ such that there exists $f \in {}^{\omega}\omega \cap \mathbf{V}$ with*

$$\mathbb{P} \Vdash \forall n \in \omega : \underline{g}(n) \leq f(n),$$

we have that $\mathbb{P}$ forces that there exists $c : \omega \rightarrow \text{fin}(\omega)$ in $\mathbf{V}$ with

$$\forall n \in \omega : |c(n)| \leq 2^n \text{ and } \underline{g}(n) \in c(n).$$

FACT 1.5 (e.g., see [1, Corollary 26.8]). *Let $\mathcal{U}$ be a Ramsey ultrafilter. The forcing notion $\mathbb{M}_{\mathcal{U}}$ has the Laver property.*

FACT 1.6 (e.g., see [2, Ch. VI, 2.10D]). *The Laver property is preserved under countable support iterations of proper forcing notions.*

2 Result

MAIN THEOREM. *It is consistent that there is a unique Q -point while there are 2^c -many near-coherence classes of ultrafilters.*

Proof. Assume that the ground model $\mathbf{V}$ satisfies CH. By induction, we define:

- (i) A countable support iteration $\mathbb{P}_{\omega_2} := \langle \mathbb{P}_\xi, \dot{Q}_\xi : \xi \in \omega_2 \rangle$ of c.c.c. forcing notions,
- (ii) A sequence $\langle \mathcal{U}_\xi : \xi \in \omega_2 \rangle$, such that

$$\forall \xi \in \omega_2 : \mathbb{P}_\xi \Vdash \text{“}\mathcal{U}_\xi \text{ is a Ramsey ultrafilter extending } \bigcup_{\iota \in \xi} \mathcal{U}_\iota\text{”}$$

and $\dot{Q}_\xi$ is a $\mathbb{P}_\xi$ -name for Mathias forcing restricted to $\mathcal{U}_\xi$,

Assume that we are in step $\xi \in \omega_2$. Let G_ξ be $\mathbb{P}_\xi$ -generic over $\mathbf{V}$ and work in $\mathbf{V}[G_\xi]$. Note that since $\mathbb{P}_\xi$ is a countable support iteration of proper forcing notions that are forced to be of size $\leq \omega_1$, we have $\mathbf{V}[G_\xi] \models \text{CH}$ (e.g., see [3, Theorem 2.12]). For each $\iota \in \xi$, let η_ι be the Mathias real added at stage ι .

If $\xi = \xi' + 1$, $\eta_{\xi'}$ pseudo-intersects $\mathcal{U}_{\xi'}[G_{\xi'}]$ and we may construct a Ramsey ultrafilter on $\eta_{\xi'}$ using CH (and extend it to ω to obtain $\mathcal{U}_\xi$). Similarly, if ξ is a limit ordinal and $\text{cf}(\xi) = \omega$, we can build $\mathcal{U}_\xi$ on a pseudo-intersection of the tower $\langle \eta_\iota : \iota \in \xi \rangle$. Finally, if $\text{cf}(\xi) = \omega_1$, then $\bigcup_{\iota \in \xi} \mathcal{U}_\iota[G_\xi]$ is already a Ramsey ultrafilter, since no new reals are added at stage ξ . For the same reason we also have that $\mathcal{U}_{\omega_2} := \bigcup_{\xi \in \omega_2} \mathcal{U}_\xi[G]$ is a Ramsey ultrafilter in $\mathbf{V}[G]$, where G is $\mathbb{P}_{\omega_2}$ -generic over $\mathbf{V}$.

FACT 2.1 (e.g., see [3, Theorem 2.10]). $\mathbb{P}_{\omega_2}$ is proper and satisfies the ω_2 -c.c..

We need to show that $\mathcal{U}_{\omega_2}$ is the only Q -point in $\mathbf{V}[G]$. To see this, assume by contradiction that $\mathbf{V}[G] \models \text{“}E \text{ is a } Q\text{-point and not isomorphic to } \mathcal{U}_{\omega_2}\text{”}$.

LEMMA 2.2. *There exists $\delta \in \omega_2$ such that $E \cap \mathbf{V}[G_\delta] \in \mathbf{V}[G_\delta]$ and $\mathbf{V}[G_\delta] \models \text{“}E \cap \mathbf{V}[G_\delta] \text{ is a } Q\text{-point and not isomorphic to } \mathcal{U}_\delta\text{”}$.*

Proof. Fix $\xi \in \omega_2$ and consider names $\dot{e}_\xi, \dot{i}_\xi, \dot{s}_\xi, \dot{b}_\xi$ and $\dot{f}_\xi$ such that $\mathbb{P}_{\omega_2}$ forces that

- (i) “ $\dot{e}_\xi$ is an enumeration (in ω_1) of $\dot{E} \cap \mathbf{V}[G_\xi]$ ”. For each $\alpha \in \omega_1$ and $n \in \omega$ let $\mathcal{E}_{\xi, \alpha, n} \subseteq \mathbb{P}_{\omega_2}$ be a maximal antichain deciding “ $n \in \dot{e}_\xi(\alpha)$ ”.
- (ii) “ $\dot{i}_\xi$ is an enumeration (in ω_1) of the set of interval partitions of ω in $\mathbf{V}[G_\xi]$ ”. Note that we may assume that $\dot{i}_\xi$ is a $\mathbb{P}_\xi$ -name.
- (iii) “For all $\alpha \in \omega_1$, $\dot{s}_\xi(\alpha)$ is an element of $\dot{E}$ that intersects each interval in the interval partition $\dot{i}_\xi(\alpha)$ in at most one point”. Let $\mathcal{S}_{\xi, \alpha, n} \subseteq \mathbb{P}_{\omega_2}$ be a maximal antichain deciding “ $n \in \dot{s}_\xi(\alpha)$ ”.

- (iv) “ $\underline{b}_\xi$ is an enumeration (in ω_1) of all permutations of ω in $\mathbf{V}[G_\xi]$ ”. We may again assume that $\underline{b}_\xi$ is a $\mathbb{P}_\xi$ -name.
- (v) “For all $\alpha \in \omega_1$, $\underline{f}_\xi(\alpha)$ is a pair $\text{op}(\underline{x}_\alpha, \underline{y}_\alpha)$ such that $\underline{x}_\alpha$ is in $\underline{E}$, $\underline{y}_\alpha$ is in $\underline{\mathcal{U}}_{\omega_2}$ and $\underline{b}_\xi(\alpha)[\underline{x}_\alpha]$ is disjoint from $\underline{y}_\alpha$ ”. Let $\mathcal{X}_{\xi, \alpha, n} \subseteq \mathbb{P}_{\omega_2}$ be a maximal antichain deciding “ $n \in \underline{x}_\alpha$ ”, and define $\mathcal{Y}_{\xi, \alpha, n}$ analogously.

By the ω_2 -c.c. of $\mathbb{P}_{\omega_2}$, there exists for each $\xi \in \omega_2$ some $\gamma_\xi \in \omega_2$ greater than ξ such that all the above antichains consist of $\mathbb{P}_{\gamma_\xi}$ -conditions. Recursively define $\lambda(0) = 0$, $\lambda(\xi + 1) = \gamma_{\lambda(\xi)}$ and for limit ordinals ξ : $\lambda(\xi) = \bigcup_{\iota \in \xi} \lambda(\iota)$, for $\xi \leq \omega_1$. Set $\delta := \lambda(\omega_1)$ and consider the extension $\mathbf{V}[G_\delta]$. Since $\text{cf}(\delta) = \omega_1$, we have that $E \cap \mathbf{V}[G_\delta] = \bigcup_{\iota \in \omega_1} E \cap \mathbf{V}[G_{\lambda(\iota)}]$, and since each $E \cap \mathbf{V}[G_{\lambda(\iota)}]$ is an element of $\mathbf{V}[G_\delta]$ by (i), $E \cap \mathbf{V}[G_\delta]$ is an element of $\mathbf{V}[G_\delta]$ (and an ultrafilter). Furthermore, any interval partition of ω in $\mathbf{V}[G_\delta]$ already appears in some $\mathbf{V}[G_{\lambda(\iota)}]$, $\iota \in \omega_1$, where it equals $\dot{\iota}_{\lambda(\iota)}[G_{\lambda(\iota)}](\alpha)$ for some $\alpha \in \omega_1$. Since $\dot{\iota}_{\lambda(\iota)}[G_\delta](\alpha) \in E \cap \mathbf{V}[G_\delta]$, we obtain that $E \cap \mathbf{V}[G_\delta]$ is a Q -point. Finally and analogously, any permutation of ω in $\mathbf{V}[G_\delta]$ already appears in $\mathbf{V}[G_{\lambda(\iota)}]$ for some $\iota \in \omega_1$ and hence there are witnesses $\underline{x}_\alpha[G_\delta] \in E \cap \mathbf{V}[G_\delta]$ and $\underline{y}_\alpha[G_\delta] \in \mathcal{U}_{\omega_2} \cap \mathbf{V}[G_\delta] = \mathcal{U}_\delta$ witnessing that $E \cap \mathbf{V}[G_\delta]$ and $\mathcal{U}_\delta$ are not isomorphic. $\dashv$

We now designate $\mathbf{V}[G_\delta]$ as the new ground model and rename the Q -point $E \cap \mathbf{V}[G_\delta]$ to E and the Ramsey ultrafilter $\mathcal{U}_\delta$ to $\mathcal{U}$. Note that by the Factor-Lemma (e.g., see [4, Theorem 4.6]), the quotient $\mathbb{P}_{\omega_2}/G_\delta$ is again isomorphic to a countable support iteration of restricted Mathias forcings. In particular, by Facts 1.5 and 1.6, $\mathbb{P}_{\omega_2}/G_\delta$ is isomorphic to the two-step iteration $\mathbb{M}_\mathcal{U} * \underline{R}$, where $\mathbb{M}_\mathcal{U} \Vdash$ “ $\underline{R}$ has the Laver property”.

It remains to show the following.

PROPOSITION 2.3. *Let E be a Q -point and $\mathcal{U}$ a Ramsey ultrafilter such that E and $\mathcal{U}$ are not isomorphic. Let $\mathbb{M}_\mathcal{U}$ be Mathias forcing restricted to $\mathcal{U}$ and let $\underline{R}$ be a $\mathbb{M}_\mathcal{U}$ -name such that $\mathbb{M}_\mathcal{U} \Vdash$ “ $\underline{R}$ has the Laver property”. Then $\mathbb{M}_\mathcal{U} * \underline{R} \Vdash$ “ E cannot be extended to a Q -point”.*

Proof. It suffices to show that if $\langle p, \underline{q} \rangle \in \mathbb{M}_\mathcal{U} * \underline{R}$ and a $\mathbb{M}_\mathcal{U} * \underline{R}$ -name $\underline{a}$ for a strictly increasing element of ${}^\omega\omega$ are such that

$$\langle p, \underline{q} \rangle \Vdash_{\mathbb{M}_\mathcal{U} * \underline{R}} \forall n \in \omega : \underline{a}(n) \in (\eta(n-1), \eta(n)],$$

then there exists some $v \in E$ and some $\langle \bar{p}, \bar{q} \rangle$ greater than $\langle p, \underline{q} \rangle$ such that

$$\langle \bar{p}, \bar{q} \rangle \Vdash_{\mathbb{M}_\mathcal{U} * \underline{R}} |\text{range}(\underline{a}) \cap v| < \omega.$$

Recall that η is the canonical $\mathbb{M}_\mathcal{U}$ -name for the Mathias real (assume $\mathbb{M}_\mathcal{U} \Vdash \eta(-1) = -\infty$).

Note that $\underline{q}$ is forced by $\mathbb{M}_{\mathcal{U}}$ to be dominated by η . Hence, by the Laver property of $\underline{R}$, there exists a $\mathbb{M}_{\mathcal{U}}$ -name $\underline{c}$ for a function from ω to $\text{fin}(\omega)$ and some $\langle p', \underline{q}' \rangle \geq_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \langle p, \underline{q} \rangle$ such that

$$\langle p', \underline{q}' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \forall n \in \omega : \underline{q}(n) \in \underline{c}(n) \text{ and } |\underline{c}(n)| \leq 2^n.$$

We may assume without loss of generality that $p' \Vdash_{\mathbb{M}_{\mathcal{U}}} \forall n \in \omega : \underline{c}(n) \subseteq (\eta(n-1), \eta(n)]$. Let $\underline{C}$ be a $\mathbb{M}_{\mathcal{U}}$ -name for an element of $[\omega]^\omega$ such that $p' \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} = \bigcup \text{range}(\underline{c})$. Hence, we have

$$\langle p', \underline{q}' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \forall n \in \omega : \underline{q}(n) \in \underline{C} \cap (\eta(n-1), \eta(n)] \text{ and } |\underline{C} \cap (\eta(n-1), \eta(n)]| \leq 2^n.$$

LEMMA 2.4. *Write $p' = \langle s, x_0 \rangle$. There exists $x_1 \in [x_0]^\omega \cap \mathcal{U}$ such that the $\mathbb{M}_{\mathcal{U}}$ -condition $\langle s, x_1 \rangle \geq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_0 \rangle$ has the following property:*

For every $t \in \text{fin}(x_1)$, there exists $C_t \in \text{fin}(\omega)$ such that

$$\langle s \cup t, x_1 \setminus (\max t)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} \cap (\max t)^+ = C_t.$$

Proof. We define a strategy for the Maiden in the ultrafilter game for $\mathcal{U}$, which will not be a winning strategy since $\mathcal{U}$ is a Ramsey ultrafilter.

Since $\mathbb{M}_{\mathcal{U}}$ has pure decision, there exists $C_\emptyset \subseteq (\max s)^+$ and $y_0 \in [x_0]^\omega \cap \mathcal{U}$ such that $\langle s, y_0 \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} \cap (\max s)^+ = C_\emptyset$. The Maiden starts by playing y_0 .

Assume $y_0 \supseteq y_1 \supseteq \dots \supseteq y_k$ and $n_0 < n_1 < \dots < n_k$ have been played, where $\forall i \leq k : y_i \in \mathcal{U}$ and $n_i \in y_i$. Again by pure decision, for each $t \subseteq \{n_0, n_1, \dots, n_k\}$ with $\max t = n_k$, there exists $z_t \in [y_k \setminus n_k^+]^\omega \cap \mathcal{U}$ and $C_t \subseteq n_k^+$ such that $\langle s \cup t, z_t \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} \cap (n_k)^+ = C_t$. The Maiden plays

$$y_{k+1} := \bigcap_{\substack{t \subseteq \{n_i : i \leq k\} \\ \max t = n_k}} z_t.$$

Since Death wins, we have that $x_1 := \{n_i : i \in \omega\} \in \mathcal{U}$. It is easy to check that this x_1 satisfies the lemma. $\dashv$

The following lemma strengthens the previous one.

LEMMA 2.5. *Assume $\langle s, x_1 \rangle$ is as in the conclusion of the previous lemma. There exists $x_2 \in [x_1]^\omega \cap \mathcal{U}$ such that $\langle s, x_2 \rangle$ has the following property:*

For every $t \in \text{fin}(x_2)$, every $m \in x_2 \setminus \max t$ and all $n, n' \in x_2 \setminus m^+$, it holds that $C_{t \cup \{n\}} \cap m^+ = C_{t \cup \{n'\}} \cap m^+$.

Proof. We again prove this by playing the ultrafilter game for $\mathcal{U}$. Assume $y_0 := x_1 \supseteq y_1 \supseteq \dots \supseteq y_k$ and $n_0 < n_1 < \dots < n_k$ have been played. For every $t \subseteq \{n_0, n_1, \dots, n_k\}$ and every $d \subseteq n_k^+$ consider the set

$$P_{t,d} := \{n \in y_k \setminus n_k^+ : C_{t \cup \{n\}} \cap n_k^+ = d\}.$$

Note that for every $t \subseteq \{n_0, n_1, \dots, n_k\}$, the set $\{P_{t,d} : d \subseteq n_k^+\}$ is a partition of $y_k \setminus n_k^+$ into finitely many pieces. Hence, there exists one $d_t \subseteq n_k^+$ such that $P_{t,d_t} \in \mathcal{U}$.

The Maiden plays

$$y_{k+1} := \bigcap_{t \subseteq \{n_i : i \leq k\}} P_{t,d_t}.$$

Death will win and hence $x_2 := \{n_i : i \in \omega\} \in \mathcal{U}$. It is again not hard to check that x_2 satisfies the lemma. $\dashv$

The following fact will be needed later.

FACT 2.6. *Without loss of generality, we may assume that for all $n \in \{\max s\} \cup x_2$, if n is the j 'th element of $s \cup x_2$ in increasing order, then $n > 2^{j+1}$.*

Proof. Note that the conclusion of Lemmas 2.4 and 2.5 also holds for each $\langle s', x' \rangle \geq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_2 \rangle$. Hence, we simply trim x_2 such that the enumeration of $s \cup x_2$ dominates 2^{j+1} above $|s|$ and replace s with $s \cup \{\min x_2\}$ and x_2 with $x_2 \setminus \{\min x_2\}$. $\dashv$

Next, let N be a countable elementary submodel of some large enough $\mathcal{H}_{\chi}$ such that $\{\mathcal{U}, \mathbb{M}_{\mathcal{U}}, \mathcal{C}, \langle s, x_2 \rangle\} \in N$. By induction, construct a sequence $N_0 \subseteq N_1 \subseteq \dots$ of finite subsets of N such that

- (i) $\{\mathcal{U}, \mathbb{M}_{\mathcal{U}}, \mathcal{C}, \langle s, x_2 \rangle, s, x_2\} \subseteq N_0$,
- (ii) $\bigcup_{i \in \omega} N_i = N$,
- (iii) $\forall i \in \omega : k_i := N_i \cap \omega \in \omega$.
- (iv) $\forall i \in \omega : \forall t \in \text{fin}(\omega) : t \in N_i \iff t \subseteq N_i$,
- (v) If $\langle m, l, D \rangle \in (\omega \times \omega \times \text{fin}(\omega)) \cap N_i$, then $m, l, D \in N_i$ (and hence $D \subseteq N_i$ by the previous condition).
- (vi) $\forall i \in \omega : \text{If } \varphi(x, a_0, \dots, a_l) \text{ is a formula of length less than } 2025 \text{ with } a_0, \dots, a_l \in N_i \text{ and } N \models \exists x \varphi(x, a_0, \dots, a_l), \text{ then there exists } b \in N_{i+1} \text{ such that } N \models \varphi(b, a_0, \dots, a_l).$

LEMMA 2.7. $\langle s, x_2 \rangle$ forces that

$$\forall i \in \omega \setminus \{0, 1\} : \mathcal{C} \setminus (\max s)^+ \cap [k_{i-1}, k_i] \neq \emptyset \implies \begin{cases} \text{range}(\eta) \cap [k_{i-2}, k_{i-1}] \neq \emptyset, \text{ or} \\ \text{range}(\eta) \cap [k_{i-1}, k_i] \neq \emptyset, \text{ or} \\ \text{range}(\eta) \cap [k_i, k_{i+1}] \neq \emptyset. \end{cases}$$

Proof. Assume $\langle s \cup t, x' \rangle \geq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_2 \rangle$, $a \in \omega \setminus (\max s)^+$ and $i \in \omega \setminus \{0, 1\}$ are such that

$$\langle s \cup t, x' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} a \in \mathcal{C} \setminus (\max s)^+ \cap [k_{i-1}, k_i).$$

We show that $\langle s \cup t, x' \rangle$ forces one of the three possible conclusions in the statement of the lemma.

By possibly extending t , we may assume that t contains at least one element that is greater than a . Let $l_0 := \max(t \cap a)$ and $l^* := \min(t \setminus a)$. Furthermore, let $m^* := \max(x_2 \cap l^*)$. Hence, l_0 and l^* are consecutive elements of t and $l_0 \leq m^* < l^*$ and $l_0 < a \leq l^*$. We distinguish between two cases:

Case I. Assume $l_0 \leq m^* \leq a \leq l^*$.

If $l^* \in [k_{i-1}, k_i)$, we are done, since this means that $\langle s \cup t, x' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} l^* \in \text{range}(\eta) \cap [k_{i-1}, k_i)$. Hence, assume $l^* \notin [k_{i-1}, k_i)$, which means that $l^* \notin N_i$, since l^* is certainly not in N_{i-1} (if it were, a would be as well by (iii)). Note that l^* witnesses that

$$N \models \exists l : l = \min(x_2 \setminus a).$$

Hence, by (v), we have that $l^* \in N_{i+1}$ and thus $l^* \in [k_i, k_{i+1})$.

Case II. Assume $l_0 < a < m^* < l^*$.

Let $t' := t \cap a$, i.e., $l_0 := \max t'$, and let $i^* \in \omega \setminus \{0\}$ be such that $l_0 \in [k_{i^*-1}, k_{i^*})$, i.e., l_0 first appears in N_{i^*} . If $i^* = i$, we are again done, hence assume that $a \notin N_{i^*}$. We will show that $i^* = i - 1$.

Let $j \in \omega$ be such that l^* is the j 'th elements of $s \cup t$ in increasing order. By Lemmas 2.4 and 2.5, there is $C_{t' \cup \{l^*\}} \subseteq (l^*)^+$ such that

$$\langle s \cup t' \cup \{l^*\}, x_2 \setminus (l^*)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \mathcal{C} \cap (l^*)^+ = C_{t' \cup \{l^*\}}.$$

Set $D^* := C_{t' \cup \{l^*\}} \cap (l_0, m^*)$. Since

$$\langle s \cup t' \cup \{l^*\}, x_2 \setminus (l^*)^+ \rangle \leq_{\mathbb{M}_{\mathcal{U}}} \langle s \cup t, x' \rangle,$$

and since $l_0 < a < m^*$ by assumption, we must have $a \in D^*$. Furthermore, note that $D^* \subseteq C_{t' \cup \{l^*\}} \cap (l_0, l^*]$ and thus $|D^*| =: \gamma \leq 2^j$.

Now, m^* , l^* and D^* witness that

$$N \models \exists \langle m, l, D \rangle : \begin{cases} m, l \in x_2 \setminus l_0^+, m < l, \text{ and} \\ D \subseteq (l_0, m), \text{ and} \\ |D| = \gamma, \text{ and} \\ \langle s \cup t' \cup \{l\}, x_2 \setminus l^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \mathcal{C} \cap (l_0, m) = D. \end{cases}$$

Since l_0 is the $(j-1)$ 'th element of $s \cup t'$, we have $l_0 > 2^j$ by Fact 2.6.⁶ Hence, since $l_0 \in N_{i^*}$, it follows that $\gamma \in N_{i^*}$. Thus, all the parameters in the above formula lie in N_{i^*} , which implies that there exists $\langle m^\dagger, l^\dagger, D^\dagger \rangle \in N_{i^*+1}$ satisfying the formula.

Claim. $l^\dagger \geq a$

Note that the proof of this claim will finish the proof of the Lemma, since $l^\dagger \in N_{i^*+1}$ by (v) and thus $a \in N_{i^*+1} \setminus N_{i^*}$.

Proof. Assume by contradiction that $l^\dagger < a$, i.e.,

$$l_0 < m^\dagger < l^\dagger < a < m^* < l^*.$$

By Lemma 2.5, we have that

$$C_{t' \cup \{l^\dagger\}} \cap (m^\dagger) = C_{t' \cup \{l^*\}} \cap (m^\dagger).$$

Since $\langle s \cup t' \cup \{l^\dagger\}, x_2 \setminus (l^\dagger)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \mathcal{C} \cap (l_0, m^\dagger) = D^\dagger$, it follows that $C_{t' \cup \{l^*\}} \cap (m^\dagger) = D^\dagger$ and hence $D^\dagger = D^* \cap (l_0, m^\dagger)$. However, both $D^\dagger$ and D^* have size γ and thus $D^* \subseteq (l_0, m^\dagger)$, which is a contradiction to the fact that $a \in D^*$ and $a > m^\dagger$. $\blacksquare \quad \dashv$

We now only need one final lemma to finish the proof of the proposition and thus of the main theorem.

LEMMA 2.8. *Let $I := \{[k_i, k_{i+1}) : i \in \omega\}$ be any interval partition of ω and E and $\mathcal{U}$ non-isomorphic Q -points. Then there exist $v \in E$ and $u \in \mathcal{U}$ such that*

$$\forall i \in \omega \setminus \{0\} : v \cap [k_i, k_{i+1}) \neq \emptyset \implies \begin{cases} u \cap [k_{i-1}, k_i) = \emptyset, \text{ and} \\ u \cap [k_i, k_{i+1}) = \emptyset, \text{ and} \\ u \cap [k_{i+1}, k_{i+2}) = \emptyset. \end{cases}$$

Proof. Say that a Q -point element *selects* from an interval partition if it intersects each interval in exactly one point. Let $v_0 \in E$ and $u_0 \in \mathcal{U}$ be such that they select from I . Let f be an order-preserving bijection from v_0 to u_0 , extended to a permutation of ω . Thus, for each $i \in \omega$, f sends the element selected by v_0 in $[k_i, k_{i+1})$ to the element selected by u_0 in $[k_i, k_{i+1})$. Since E and $\mathcal{U}$ are non-isomorphic, there exist $v_1 \in [v_0]^\omega \cap E$ and $u_1 \in [u_0]^\omega \cap \mathcal{U}$ such that $u_1 \cap f[v_1] = \emptyset$. Hence, for all $i \in \omega \setminus \{0\}$:

$$v_1 \cap [k_i, k_{i+1}) \neq \emptyset \implies u_1 \cap [k_i, k_{i+1}) = \emptyset.$$

Both E and $\mathcal{U}$ contain the set

$$y_\varepsilon := \bigcup_{\substack{i \in \omega \\ i \equiv \varepsilon \pmod{3}}} [k_i, k_{i+1}),$$

⁶Note that the additional requirement in Fact 2.6 that $\max s$ is already larger than $2^{|s|}$ is needed here, since l_0 could be $\max s$.

each for exactly one $\varepsilon = \varepsilon(E), \varepsilon(\mathcal{U}) \in 3$. Let $v_2 := v_1 \cap y_{\varepsilon(E)} \in E$ and $u_2 := u_1 \cap y_{\varepsilon(\mathcal{U})} \in \mathcal{U}$. If $\varepsilon(E) = \varepsilon(\mathcal{U})$ then v_2 and u_2 satisfy the lemma, hence assume without loss of generality that $\varepsilon(E) = 0$ and $\varepsilon(\mathcal{U}) = 1$.

Let $\bar{v}_0 \in E$ and $\bar{u}_0 \in \mathcal{U}$ be elements that select from the interval partition

$$\{[k_i, k_{i+2}) : i \in \omega, i \equiv 0 \pmod{3}\} \cup \{[k_i, k_{i+1}) : i \in \omega, i \equiv 2 \pmod{3}\}.$$

Again, by considering a permutation of ω that maps the element selected by $\bar{v}_0$ in any interval to the element selected by $\bar{u}_0$ in the same interval, we find $\bar{v}_1 \in [\bar{v}_0]^\omega \cap E$ and $\bar{u}_1 \in [\bar{u}_0]^\omega \cap \mathcal{U}$ such that $\bar{v}_1$ and $\bar{u}_1$ never select from the same interval. Now, clearly, $v_1 \cap \bar{v}_1 \in E$ and $u_1 \cap \bar{u}_1 \in \mathcal{U}$ work. $\dashv$

We can now finish the proof of the proposition and hence of the main theorem: Let $v \in E$, $u \in \mathcal{U}$ be given by the previous lemma for the interval partition $\{[k_i, k_{i+1}) : i \in \omega\} \cup \{[0, k_0)\}$ constructed in the proof of Lemma 2.7. Let $G * H$ be any $\mathbb{M}_{\mathcal{U}} * \underline{R}$ -generic filter containing $\langle \langle s, x_2 \rangle, \underline{q}' \rangle$. By Lemma 2.7, we have that in $\mathbf{V}[G * H]$, whenever $\text{range}(\underline{a}[G * H]) \setminus (\max s)^+$ intersects one of the intervals $[k_i, k_{i+1})$, then the Mathias real η intersects $[k_i, k_{i+1})$ or one of the adjacent intervals $[k_{i-1}, k_i)$ or $[k_{i+1}, k_{i+2})$. Since $\text{range}(\eta)$ is almost contained in u , the same is true for u in place of η above some $n \geq (\max s)^+$. Hence, $\text{range}(\underline{a}[G * H]) \setminus n$ is disjoint from v . $\dashv$

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