Stable amalgamation over a predicate and the Gaifman property

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Abstract

We consider the following property of a first order theory T with a distinguished unary predicate P: every model of the theory of P occurs as the P-part of some model of T. We call this property the Gaifman property. Gaifman conjectured that if T is relatively categorical over P, then it has the Gaifman property. We propose a generalized version of this conjecture: if T fails the Gaifman property, then it exhibits non-structure over P, i.e., has many non-isomorphic models over P in many cardinalities. We address this conjecture for countable theories. Motivated by ideas from Classification Theory, we separate this conjecture into two parts: 1) stability over P (a structure property of theories) implies the Gaifman property, and 2) instability over P implies non-structure. In this paper prove the first part of this conjecture. In fact, we prove a stronger statement: an appropriate version of stability implies higher stable amalgamation properties.

1 Introduction

In [Gai74], Gaifman has conjectured that, if a countable theory T is categorical over a unary predicate P, then every model of the theory of P "occurs" as the P-part of some model of T. We will refer to the latter property of theories as the *Gaifman property* over P; that is, we say that T has the Gaifman property over P (or just T *has the Gaifman property*, when P is clear from the context) when every model of the theory of P is the P-part of some model of T. Gaifman [Gai] proved this conjecture in case T is *rigid* over P (see also [BPW23] for a recent "effective" version of this result). Hodges et al (e.g., [Hod99, HY09, Hod02]) investigated Gaifman's conjecture and related

questions for specific types of theories (an abelian group with a predicate picking out a subgroup, a pair of linear orders).

The first author has proven an "absolute" version of this conjecture in [She86]. Specifically, it follows from results there that if T is *absolutely categorical* over P (the categoricity is preserved in forcing extensions), then T has the Gaifman property over P. As far as we know, the general statement is still open.

In this paper we formulate and address a much stronger version of Gaifman's Conjecture (which we refer to as the "Generalized Gaifman's Conjecture"):

Conjecture 1.1. *Let T be a countable complete theory, P a distinguished unary predicate in its vocabulary. Assume that T fails the Gaifman property. That is, assume that there exists a model N of the theory of P such that for no $M \models T$ is it the case that $P^M = N$. Then for every regular cardinal λ big enough, and every $\mu \geq \lambda$, T has 2^λ models of cardinality μ, which are non-isomorphic over P.*

This conjecture states that, if a countable theory T fails the Gaifman property, then it exhibits "non-structure over P". In other words, we believe that a much weaker version of "structure" for T over P (than categoricity) is enough for the Gaifman property.

In the present paper, we break the above conjecture into two parts, and prove one of them. The motivation for our approach (and indeed for the above statement of the conjecture) comes from *Classification theory*, a study of dichotomies in model theory, developed by the first author (e.g., [She90]). Broadly speaking, classification theory is a programme of search for *dividing lines*. A dividing line is a property of first order theories (or, more generally, of classes of models), such that theories that have this property (the ones falling on the "structure" side of the dividing line) have various "positive structure" properties, whereas theories that fail the property in questions (fall on the "non-structure" side of the dividing line) exhibit "non-structure". The intuition is such dividing lines separate, in some sense, between "classifiable" theories (whose models can be described using a "reasonable" collection of invariants, e.g., dimensions), and "non-classifiable" ones, where such classification is impossible.

One of the most fundamental and best studied dividing lines is *stability*, introduced (first in classical model theory) by the first author, generalizing the work and ideas of Morley [Mor65]. If (a countable theory) T is unstable, it has the maximal number of non-isomorphic models (and even $\aleph_1$-saturated models) in any uncountable cardinality. On the other hand, if T is stable, it has a very well-behaved notion of independence, that leads to good notions of dimension, and many other positive structural properties. These classical results appear in [She90].

It is our hypothesis that the appropriate notion of stability also plays a central role in the study of model theory over a predicate. However, we believe that, unlike in the classical (first order) context, stability over a predicate

gives rise to more than one (indeed, infinitely many) dividing lines. We will make this idea of "ω levels of classification (introduced by the first author in [She86]) of "n-stability over P" for $n < \omega$ more precise in the next section.

One goal of this work is to investigate *structure* consequences for theories that fall on the positive side of all these dividing lines. In particular, we prove the following:

Theorem 1.2. *Let T be a countable complete theory, P a distinguished unary predicate in its vocabulary, such that P is "very stably embedded" (by which we mean that P is stably embedded, and every subset of P definable in T, is definable already in T^P, the theory of P). Assume that T is "n-stable" for all $n < \omega$. Then T has the Gaifman property.*

The next section we discuss the setting of this paper, the notion of n-stability mentioned in the statement of the Theorem above, our general approach, and the main technical tools. At the end, we will be able to state a stronger version of Theorem 1.2 that is actually proven in this paper, and explain the connections with previous work.

2 Background: the existence property and higher amalgamation

2.1 Background and setting

From now on, T will be a (fixed) complete first order theory, and P a distinguished unary predicate in the vocabulary of T. For simplicity, we assume that the vocabulary of T has no function symbols, and that T implies that P is infinite.

Let $\mathcal{C}$ be the monster model of T. From now on, we assume that all models of T are elementary submodels of $\mathcal{C}$, and all sets are subsets of $\mathcal{C}$.

For a set A, we denote $P^A = A \cap P^{\mathcal{C}}$. We also denote by $\mathcal{C}^P$ the substructure of $\mathcal{C}$, which (as a set) equals $P^{\mathcal{C}}$, and we let T^P be the theory of $\mathcal{C}^P$.

Throughout the paper, we make the following "structure" assumptions:

Hypothesis 2.1. *(Hypothesis 2).*

(i) *P is stably embedded.*

(ii) *Every definable subset of $P^{\mathcal{C}}$ is already definable in T^P.*

We may assume these without loss of generality while investigating theories that do not exhibit "non-structure". See a detailed discussion on this in [SU22]. The proofs of the relevant non-structure results appear in [PS85]. Because of clause (ii) of Hypothesis 2.1 above, we may also assume that T has been Morley-ized (hence has quantifier elimination, even down to the level of predicates). Again, see [PS85] or [SU22] for a more detailed discussion.

Clearly, $\mathcal{C}^P$ can be viewed as the monster model of T^P.

When no confusion should arise, we will write P for $P^{\mathcal{C}}$. Also, for a set A, we will often denote by A both the set and the substructure of $\mathcal{C}$ with universe A. So for example, when we write that $A \cap P^{\mathcal{C}}$ is λ-saturated, or just that $A \cap P$ is λ-saturated, we mean that the substructure of $\mathcal{C}$ with universe P^A is a λ-saturated model of the appropriate theory (if $A \cap P \prec P$, which will be the case in this paper, then the appropriate theory is T^P).

For two sets A, A', we will write $A \equiv A'$ if A and A' are (universes of) elementarily equivalent substructures of $\mathcal{C}$.

2.2 Basic definitions

We recall some basic concepts from [PS85], [SU22], and [Usv24].

Definition 2.2.   (i) We say that a set A has the *existence property over P*, or simply the *existence property* if there exists $M \models T$ such that $A \subseteq M$ and $P^M = P^A$.

 (ii) We say that T has the *Gaifman property* if every $N \models T^P$ has the existence property.

In order for a set A to have the existence property, P^A should be "suitable" for being the P-part of a model of T. For example, P^A should obviously be itself a model of T^P; moreover, P^A has to be "closed enough".

This closedness condition is summarized in the following definition.

Definition 2.3. $A \subseteq \mathcal{C}$ is called *complete* if for every formula $\psi(\bar{x}, \bar{y})$ and $\bar{b} \subseteq A, \models (\exists \bar{x} \in P)\psi(\bar{x}, \bar{b})$ implies $(\exists \bar{a} \subseteq P \cap A) \models \psi(\bar{a}, \bar{b})$.

The following is clear:

Observation 2.4. *If $M \prec \mathcal{C}$ and $P^M \subseteq A \subseteq M$, then A is complete.*

This confirms that in order for a set A to have the existence property, it has to be complete. In this paper we are interested in the converse.

The following useful characterization offers another understanding of the notion of completeness (see Observation 4.2 in [SU22]):

Fact 2.5. A set A is complete if and only if for every $\bar{a} \subseteq A$ and $\psi(\bar{x}, \bar{y})$ the ψ-type $tp_{\psi}(\bar{a}/P^{\mathcal{C}})$ is definable over $A \cap P^{\mathcal{C}}$ and $A \cap P^{\mathcal{C}} \prec P^{\mathcal{C}}$.

Now we recall the definitions of relevant types and stability. Note that these notions are only defined over complete sets.

Definition 2.6. Let A be a complete set.

 (i) Denote

$$S_*(A) = \{tp(\bar{c}/A) : P \cap (A \cup \bar{c}) = P \cap A \text{ and } A \cup \bar{c} \text{ is complete}\}$$

 (ii) A is *stable over P*, or simply *stable*, if for all A' with $A' \equiv A$, we have $|S_*(A')| \leq |A'|^{|T|}$.

Remark 2.7. (i) *Even though "stability over P" is a more appropriate and accurate name for our notion of stability of a set (and the term "stable set" exists in literature, and has a different meaning), since we have only one notion of stability in this article (stability over P), we will sometimes omit "over P" and simply write "stable".*

(ii) *We will refer to types in $S_*(A)$ as complete types over A which are weakly orthogonal to P.*

2.3 Finite independent systems of models and higher amalgamation properties

The main technical tool in this paper is so-called "good independent systems of models", or just *good systems.* Informally, such a system corresponds to a collection of models of T and T^P that are "independent from each other". A formal definition (which is somewhat long and quite technical) appears in section 5. The main idea is that, given a partial ordered set I, an I-system is a collection of sets A_u for $u \in I$, satisfying a collection of conditions (some very natural, some a bit more technical). In our case, we will always assume that $I \subseteq \mathcal{P}(n)$ for some n, where $\mathcal{P}(n) = \mathcal{P}(\{0, \ldots n-1\}) = \{u \colon u \subseteq \{0, \ldots, n-1\}\}$. This will allow us to distinguish between elements of the system that are models of T and those that are models of T^P. Specifically, we will (quite arbitrarily, but this turns out to be a convenient choice) assume that if $0 \notin u$, then $A_u \models T^P$, otherwise $A_u \models T$. Moreover, we assume that $P^{A_u} = A_{u \smallsetminus \{0\}}$.

We will be most interested in good $\mathcal{P}(n)$-systems (i.e., $I = \mathcal{P}(n)$) and $\mathcal{P}^-(n)$-systems, where $\mathcal{P}^-(n) = \mathcal{P}(n) \smallsetminus \{n\}$. For simplicity of notation, in this section, we shall refer to the former as an n-system, and the latter as an n^--system (we avoid the use of this shortcut later in the paper).

Note that:

- A 1-system consists of $A_\varnothing$ and $A_{\{0\}}$, where $A_{\{0\}} \models T$ and $A_\varnothing = P^{A_{\{0\}}}$

- A 1^--system consists only of $A_\varnothing \models T^P$

- A 2^--system consists of $A_\varnothing, A_{\{1\}} \models T^P$ and $A_{\{0\}}$ with $A_\varnothing = P^{A_{\{0\}}}$

Additional requirements on a 2^--system demand, for example, that $A_\varnothing \prec A_{\{1\}}$, and that $A_{\{0\}}$ is "independent from $A_{\{1\}}$ over $A_\varnothing$ (in this specific case this just means non-forking independence: for every $\bar{a} \in A_{\{0\}}$, the type $\mathrm{tp}(\bar{a}/A_{\{1\}})$ is definable over $A_\varnothing$, and is "automatic", since P is stably embedded, and $A_{\{0\}}$ is a model, hence a complete set). in a 2-system, one also has a model $A_2 = A_{\{0,1\}} \models T$, which contains all the above sets, and, indeed, $A_{\{0\}} \prec A_2$, and $P^{A_2} = A_{\{1\}}$.

In the case of $n = 3$, the requirements are already more technical, and we will not discuss them here (see section 5 for a general definition).

Sometimes one wants to discuss a particular type of system, where, for example, the model A_n is always atomic, constructible, or saturated over the "smaller" sets. For example, in [She86], the first author studied "u.l.a."

systems, i.e., systems in which A_n is *uniformly locally atomic* (u.l.a.) over the union of the appropriate n^--system (see subsection 2.5).

The property of n-stability mentioned in Theorem 1.2, corresponds to the fact that every n^--system (more precisely, the union of every n^--system) is a complete set stable over P. We say that T has n-existence if every n^--system can be "completed" to an n-system. What we really prove in the Main Theorem in section 6 is that n-stability for all n implies n-existence for all n. Moreover, the model A_n can, in this case, be taken to be locally constructible over the union of the n^--system.

In particular, we obtain the following:

Theorem 2.8. *Let T be a countable complete theory, P a distinguished unary predicate in its vocabulary, such that P is "very stably embedded" (as in Theorem 1.2)). Assume that T is "n-stable" over P for all $n < \omega$. Then the union of every n^--system has the existence property.*

So Theorem 1.2, as stated in the introduction, is a particular case of the Theorem 2.8 where (in the conclusion) n is taken to be equal to 1.

The property of n-existence can also be interpreted as an appropriate "stable n-amalgamation" property: every n^--system of models can be amalgamated (i.e., embedded into one big model) without increasing the P-part.

2.4 Why systems?

One could ask: why bother with n-systems, if all one is interested is the Gaifman property (the case $n = 1$)? There are several potential answers to this question.

First, we do not know a general proof of the Gaifman property that does not go through a proof of a stronger result (the existence property for a larger class of complete sets).

Second, n-systems are interesting objects in their own right, and n-existence, which (as we have mentioned above) in a way corresponds to a higher amalgamation property, can be quite useful, both in theoretical model theoretic arguments, and in applications to particular theories (in the same way as *excellence* in abstract elementary classes can be used in particular examples to obtain interesting consequence, as is done in e.g. [Kir13]). Problems related to "higher amalgamation" (n-existence) have generated much interesting research, with connections to e.g., algebraic topology ([Hru24, GK10, GKK11, GKK13], to only name a few).

Third, we believe that n-stability is a "true" dividing line for all $n < \omega$ in the following sense: we believe that one should be able to prove, assuming that T is n-stable (over P), but $n+1$-unstable, that T exhibits non-structure over P (and this will hopefully be addressed in subsequent work).

Finally, systems are a convenient and useful tool. Specifically, working with n-systems allows proofs by induction on λ, the cardinality of the system.

We would like to (informally) illustrate this on the following "baby" example.

Let's say we are interested in the Gaifman property (which is essentially equivalent to the problem of 1-existence). That is, given a model $N \models T^P$, we ask whether there is $M \models T$ with $P^M = N$. In the language of systems, given a 1^--system, we ask whether it can be extended to a 1-system.

Let's say that we somehow already know that T has 1-existence and 2-existence in cardinality λ; and let us deduce by an inductive argument that it has 1-existence in λ^+. Given a model $N \models T^P$ of cardinality λ^+, we write it as an increasing union N_i of models of T^P of cardinality λ. By 1-existence in λ, there is $M_0 \models T$ of cardinality λ whose P-part is N_0. By 2-existence in λ, there is $M_1 \models T$ of cardinality λ, containing M_0 and N_1, whose P-part is N_1. Continuing by induction, we obtain an increasing chain $M_i \models T$ whose P-part is N_i, and their union M will be the desired model of T of cardinality λ^+, whose P-part is N.

In fact, this is a general phenomenon that makes systems so incredibly useful: they allow inductive transfer of properties between cardinalities. It is very often the case that a property for $n+1$-systems in λ implies the same property for n-systems in λ^+. This exactly is the case for n-existence. The proof of the Main Theorem (Theorem 2.8) uses a version of an inductive argument described above (for all $n < \omega$ at once), and the definition of a system, that may appear quite technical at first (and second) glance, is designed to make arguments of this type work.

2.5 Previous work

In [She86], the first author has established a similar result under the assumption that T has "absolutely no two-cardinal models" (a condition that the second author has called *nulldimensionality* in [Usv24], since it is equivalent to: every type over a model of T which is "weakly orthogonal" to P is algebraic; i.e., there are no "free" dimensions that are orthogonal to P). This is a very strong hypothesis, that does not hold in most of the interesting examples, such as vector spaces over a field, $ACFA_0$, ECF, etc (nulldimensionality is a natural assumption, however, if one is interested in investigating theories that are *categorical over P*; in particular, if one's goal is to address the original Gaifman's Conjecture). Under these circumstances, the first author could work with a special kind of independent systems of models, in which every model A_n is uniformly locally atomic (u.l.a.) over the union $\bigcup_{u \subset n} A_u$. In the general case (where T is not nulldimensional), this notion of a system is too strong. In section 5 below, we develop a theory of *general* systems, that we use in order to prove an analogous result without the assumption of nulldimensionality.

In [Usv24], the second author proves the *full existence* property (all complete sets have the existence property) assuming *full stability over P* (all complete sets are stable).It is also observed there that some natural examples (such as $ACFA_0$) are fully stable. However, it does not seem likely that this result brings us closer to a generalized version of Gaifman's Conjecture, since we do not know how to approach the "non-structure" side of full stability,

or, indeed, whether the negation of full stability implies any kind of non-structure. On the other hand, we believe that techniques similar to those used by the first author in [She86] in order to establish non-structure assuming instability of u.l.a. n-systems, can be strengthened and generalized to the more general notion of a system defined here. We therefore believe that the results in this paper are indeed the first step towards the proof of Conjecture 1.1.

Finally, many particular cases of the Gaifman property (and, more generally, the existence property) have been proven by numerous authors. Let us just give a few examples. Hodges (partly in collaboration with Yakovlev) [Hod99, HY09, Hod02] has established the Gaifman property for certain classes of abelian groups (with a predicate for a distinguished subgroup) and pairs of linear orders. It follows from the work of Lachlan that the (full) existence property is always true for a countable stable theory T (the second author's work [Usv24] mentioned above generalizes this result). Afshordel [Afs14] proved that every pseudo-finite field occurs as the fixed field of some difference closed field. Hence (any completion of) the theory $ACFA$ (algebraically closed fields with a generic automorphism σ) has the Gaifman property, where P is interpreted as the fixed field of σ. Kirby and Zilber [KZ14] proved the existence property for the class of exponentially closed fields (where P is the kernel of the exponent function), assuming that P is $\aleph_0$-saturated.

3   Basic facts

It is our intention to make this paper relatively self content and accessible to any expert in model theory. We will therefore begin by recalling some basic properties of the notions defined in the previous section (mostly from [SU22] and [Usv24]).

First, we restate the assumption that P is stably embedded in a more uniform way (this is standard, but see Observations 4.1 in [Usv24]).

Observation 3.1. *There are $\langle \Psi_\psi(\bar{y}, \bar{z}) : \psi(\bar{x}, \bar{y}) \in L(T) \rangle$ such that for all $\bar{a} \subseteq \mathfrak{C}$, $tp_\psi(\bar{a}/P^{\mathfrak{C}})$ is definable by $\Psi_\psi(\bar{y}, \bar{c})$ for some $\bar{c} \subseteq P^{\mathfrak{C}}$.*

In other words, for every $\psi(\bar{x}, \bar{y})$ and $\bar{a} \in \mathfrak{C}$ there exists $\bar{c} \in P^{\mathfrak{C}}$ such that $\mathfrak{C} \models \forall \bar{y} \psi(\bar{a}, \bar{y}) \longleftrightarrow \Psi(\bar{y}, \bar{c})$.

Now we recall a few basic consequences of completeness.

Observation 3.2. *(Observation 3.2 in [Usv24]) For any complete A and for all $\bar{a} \subseteq A$, $tp_\psi(\bar{a}/P \cap A)$ is definable by $\Psi_\psi(\bar{y}, \bar{c})$ for some $\bar{c} \subseteq A \cap P$ (where $\Psi_\psi(\bar{y}, \bar{z})$ is as in Observation 3.1).*

Fact 3.3. (Corollary 4.7 in [SU22]) Let A be a complete set, $p(x)$ a (partial) type over A with $P(x) \subseteq p$. Then p is equivalent to a T^P-type p' over P^A with $|p'| = |p|$.

In particular, if p is finite, then it realized in P^A. Similarly, if $|p| < \lambda$ and P^A is λ-compact.

Lemma 3.4. *(Lemma 4.12 in [SU22]) (The Small Type Extension Lemma)*
If $A \prec \mathcal{C}$ is saturated (or just $A \cap P$ is $|A|$-compact) and $p(\bar{x})$ is an $L(T)$-type over A of cardinality $< |A|$, then there is some $p^*(\bar{x}) \in S_*(A)$ extending p.

Corollary 3.5. *(Proposition 4.13 in [SU22]). Suppose that $A \cap P$ is $|A|$-compact. Then A is complete if and only if A has the existence property (i.e., there exists an $M \prec \mathcal{C}$ with $P^M \subseteq A \subseteq M$). If $|A| = |A|^{<|A|} > |T|$, we can add "M saturated".*

Fact 3.6. (see Theorem 4.7 in [Usv24]) Let A be a set. Then the following are equivalent:

(i) A is complete

(ii) $P^A \prec P^{\mathcal{C}}$, and:

For every finite type $p_0(\bar{x})$ over A (not necessarily realized in A) and a formula $\psi(\bar{x}, \bar{b}, \bar{y})$ over A , there is $\bar{d} \subseteq P^A$ such that the following is a (finite) type over A:

$$p_0(\bar{x}) \cup \left\{ (\forall \bar{y} \subseteq P) \left[\psi(\bar{x}, \bar{b}, \bar{y}) \longleftrightarrow \Psi_\psi(\bar{y}, \bar{d}) \right] : i < k \right\}$$

where $\Psi_\psi(\bar{y}, \bar{z})$ is the defining formula for $\psi = \psi(\bar{x}\bar{x}', \bar{y})$.

(iii) For every finite type $p_0(\bar{x})$ over A (not necessarily realized in A) and every formula $\psi(\bar{x}, \bar{b}, \bar{z})$ over A, there is $\bar{d} \subseteq P^A$ such that the following is a (finite) type over A:

$$p_0(\bar{x}) \cup \left\{ \left[(\exists \bar{z} \subseteq P) \psi(\bar{x}, \bar{b}, \bar{z}) \right] \longrightarrow \psi(\bar{x}, \bar{b}, \bar{d}) \right\}$$

Finally, we recall that, although stability over P is not a property of a specific set A, but rather of its theory as a substructure, it is enough to consider one saturated $A' \equiv A$:

Fact 3.7. (Corollary 5.5 in [SU22]) In Definition 2.3(iv), it is not necessary to consider all $A' \equiv A$. More specifically, a complete set A is stable if and only if $|S_*(A')| \leq |A'|^{|T|}$ for some $A' \equiv A$ saturated, $|A'| > |T|$.
Moreover, it is enough that for A' as above, $|S_*(A')| < 2^{|A'|}$.

4 Local Constructibility

4.1 Isolation and atomicity

Let us recall the definitions of the notions of isolation relevant for the discussion in this article.

Definition 4.1. (i) A (partial) type p over a set A is called *locally isolated* (l.i.) if for every formula $\varphi(x, y)$ there exists a formula $\theta_\varphi(x, a_\varphi) \in p$ such that $\theta_\varphi(x, a_\varphi) \vdash p{\upharpoonright}\varphi$. We say that p is locally isolated (l.i.) over $B \subseteq A$ if for every $\varphi(x, y)$, its isolating formula θ_φ is over B (so $a_\varphi \in B$).

(ii) A (partial) type p over a set A is called λ-*isolated* if there exists a subset $r \subseteq p$ with $|r| < \lambda$ such that $r \equiv p$. We say that p is λ-isolated over $B \subseteq A$ if r as above is a partial type over B.

Remark 4.2. (i) So p is locally isolated if for every formula φ, the restriction of p to a φ-type is implied by a single formula in p (which does not itself have to be a φ-formula).

(ii) A type p is isolated iff it is $\aleph_0$-isolated.

The following Lemma (that follows from Fact 3.6) with play an important role in our constructions. It states that a locally isolated type over a complete set is *always* weakly orthogonal to P.

Lemma 4.3. *(Lemma 6.3 in [Usv24]) Let B be a complete set, $p \in S(B)$ locally isolated. Then $p \in S_*(B)$.*

Lemma 4.4. *(Lemma 7.1 in [Usv24] Assume that T is countable.*

(i) *Let B be a stable set, $p(\bar{x})$ be a finite m-type over B, $\psi(\bar{x}, \bar{y})$ a formula. Then there is a finite ψ-type $q(\bar{x})$ over B such that $p(\bar{x}) \cup q(\bar{x})$ is consistent, and it implies a complete ψ-type over B.*

(ii) *Let B be a stable set, $p(\bar{x})$ be a finite m-type over B. Then there is $q(\bar{x})$ such that $|q(\bar{x})| \leq |T| = \aleph_0$, $p(\bar{x}) \cup q(\bar{x})$ consistent, and there is $r \in S_*(B)$ such that r is locally isolated, and $p(\bar{x}) \cup q(\bar{x}) \equiv r(\bar{x})$.*

In particular, $r(\bar{x})$ is $\aleph_1$-isolated.

Definition 4.5. Let N be a model, $P^N \subseteq B \subseteq N$.

(i) We say that the sequence $\bar{d} = \{d_i : i < \alpha\} \subseteq N$ is a *local construction* over B in N if for all $i < \alpha$, the type $tp\{d_i/B \cup \{d_j : j < i\})$ is locally isolated.

(ii) We say that a set $C \subseteq N$ is N is *locally constructible* (l.c.) over B in N if there is a local construction $\bar{d}$ over B in N.

Definition 4.6. Let N be a model, $P^N \subseteq B \subseteq N$.

(i) We say that the sequence $\bar{d} = \{d_i : i < \alpha\} \subseteq N$ is a λ-construction over B in N if for all $i < \alpha$, the type $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated.

(ii) We say that a set $C \subseteq N$ is N is λ-*constructible* over B in N if there is a λ-construction $\bar{d}$ over B in N.

In particular, we say that N is λ-*constructible* over B if there is a construction $N = B \cup \{d_i : i < \alpha\}$ such that for all $i < \lambda$ the type $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated.

(iii) We say that a model N is λ-*primary* over B if it is λ-constructible and λ-saturated.

Definition 4.7. Let N be a model, $P^N \subseteq B \subseteq N$.

(i) We say that N is *locally atomic* (l.a.) over B if for every $\bar{d} \subseteq N$, $tp(\bar{d}, B)$ is locally isolated over B.

(ii) We say that N is λ-*atomic* over B if for every $\bar{d} \subseteq N$, $tp(\bar{d}, B)$ is λ-isolated over some $B_{\bar{d}} \subseteq B, |B_{\bar{d}}| < \lambda$.

Observation 4.8. *Let A be a set, and $p \in S(A)$ locally isolated. Then p is $|T|^+$-isolated.*

4.2   Locally constructible existence property

Definition 4.9. Let A be a complete set. We say that A has the *locally constructible (l.c.) existence property* if there exists $M \models T$ with $P^M \subseteq A \subseteq M$ such that M is locally constructible over A.

Lemma 4.10. *Let A be a stable set, $\bar{a}$ a finite tuple. Then the set $A\bar{a}$ is also stable.*

Proof: Let $B = A\bar{a}$. By Fact 3.7, it is enough to verify stability for $B' \equiv B$ saturated. Let M be a model containing B. Expand the language by an additional unary predicates Q for A and Q_1 for B. Let M' be saturated in the expanded language, and let $B' = Q_1^{M'}$, considered as a substructure of M' in the original language. Clearly $B' \equiv B$, and is saturated. Similarly, $A' = Q^{M'}$ is elementarily equivalent to A (in the original language). Let $\bar{a}' = B' \smallsetminus A'$, so it is still finite (of the same length as $\bar{a}$, which we will call n). Every type $tp(\bar{b}/B') \in S_*^m(B')$ gives rise to a type of a tuple $tp(\bar{a}\bar{b}/A') \in S_*^{n+m}(A')$, and two different types $tp(\bar{b}_1/A')$, $tp(\bar{b}_2/A')$ give rise to two different such types over A'. By stability of A, there are $\leq |A'|^{|T|}$ types in $S_*^{n+m}(A')$; hence the same is the case for $S_*^m(B')$. ∎

Lemma 4.11. *Assume that T is countable, and let A be a countable stable set. Then A has the locally constructible existence property.*

Proof: Let $\{\varphi_i(x, \bar{a}_i) : i < \omega\}$ list all formulae over A. Construct an increasing continuous sequence of sets A_i such that:

(i) $A_0 = A$

(ii) $A_{i+1} = A_i \cup \{b_i\}$

(iii) $\models \varphi_i(b_i, \bar{a}_i)$

(iv) $tp(b_i/A_i) \in S_*(A_i)$ and is locally isolated

This can be done by induction using Lemma 4.4 for the successor stages, since all A_i are stable by Lemma 4.10. Note that clause (iv) for b_i, A_i implies by induction that all A_j are complete, and $P^{A_j} = P^A$. ∎

5   Consequences of stability of models

Again, in order to keep the paper relatively self-contained, in this section, we recall some of the main concepts and results from our previous article [SU22],

that were proven under the assumption that every model $M \models T$ is stable over P. Since our ultimate goal in this paper is to investigate consequences of n-stability for all $n < \omega$ (and, as we shall see, stability of models is merely 1-stability), this is a "harmless" assumption for the current investigation. Moreover, in [She86], the first author has shown that the negation of this assumption implies (consistent) non-structure. So even when it comes to approaching Conjecture 1.1, at least a "consistent", or, rather, "absolute" version thereof, this hypothesis is "justified" by the non-structure side of the global picture.

So for the rest of the paper, we assume, in addition to Hypothesis 2.1:

Hypothesis 5.1. *(Hypothesis 2). Every (equivalently, some) $M \prec \mathbb{C}$ is stable over P (as in Definition 2.6(ii)).*
In other words, $|S_*(M)| \leq |M|^{|T|}$ for all $M \models T$.

All the results stated below have been proven assuming this Hypothesis.

Theorem 5.2. *(Theorem 7.2 in [SU22]) If A is stable, $|A| \geq 2$, then for every $\psi(\bar{x}, \bar{y})$ there is a quantifier free $\Psi_\psi(\bar{y}, \bar{z}) \in L(T)$ such that whenever $B \equiv A$ and $p \in S_*(B)$ then $p|\psi$ is defined by $\Psi_\psi(\bar{y}, \bar{d})$ for some $\bar{d} \subseteq B$, i.e. $p|\psi = \{\psi(\bar{x}, \bar{a}) : \bar{a} \in B, B \models \Psi_\psi(\bar{a}, \bar{d})\}$.*

Theorem 5.3. *(Theorem 7.3 in [SU22]) Let A be complete and $\lambda = \lambda^{<\lambda}$. The following are equivalent:*

   (i) *A is stable.*

$(ii)_\lambda$ *If $A' \equiv A$ is λ-saturated, $\lambda = |A'| > |T|$, then over A' there is a λ-primary model M.*

$(iii)_\lambda$ *If $A' \equiv A$ is λ-saturated, $\lambda > |T|$, then every m-type p over A, $|p| < \lambda$ can be extended to a λ-isolated $q \in S_*(A')$.*

We now recall some new concepts introduced in [SU22], and their properties (see section 8 of [SU22]) for details).

Definition 5.4. $A \subseteq_t B$ if for every $\bar{a} \in A, \bar{b} \in B$ and $\psi \in L(T)$ such that $\models \psi(\bar{b}, \bar{a})$ there is some $\bar{b}' \subseteq A$ such that $\models \psi(\bar{b}', \bar{a})$

Definition 5.5. Suppose A is stable, $p \in S_*(A)$ and $A \subseteq_t B$. Then $q \in S(B)$ is a *stationarization* of p over B if for every $\psi \in L$ there is some definition $\Psi_\psi(\bar{y}, \bar{a}_\psi)$ with $\bar{a}_\psi \subseteq A$ that defines both p_ψ and q_ψ.

Notation 5.6.   (i) *We write $\bar{a} \underset{A}{\bigcup} B$ if A is stable and $q = \text{tp}(\bar{a}/B)$ is a stationarization of $p = \text{tp}(a/A)$ (so in particular $p \in S_*(A)$ and $A \subseteq_t B$). In this case, will also write $q = p|B$.*

   (ii) *We write $C \underset{A}{\bigcup} B$ if for every $\bar{a} \in C$ we have $\bar{a} \underset{A}{\bigcup} B$.*

Lemma 5.7. *([SU22, 8.5,8.6,8.7]) Assume A is stable, $A \subseteq_t B$ and $p \in S_*(A)$. Then:*

   (i) *p has a stationarization q over B.*

(ii) *It is unique: We can replace "some $\Psi_\psi(\bar{y}, \bar{a}_\psi)$" by "every...", so q does not depend on its choice.*

(iii) *If B is complete, $q \in S_*(B)$.*

(iv) *If $A \subseteq_t B, A$ stable, $\bar{c} \subseteq \bar{b}, tp(\bar{b}/A) \in S_*(A)$, then $tp(\bar{c}/A) \in S_*(A)$ and the stationarization of $tp(\bar{b}/A)$ over B includes the stationarization of $tp(\bar{c}/A)$ over B.*

Corollary 5.8. *Let $q \in S_*(B)$ definable over $A \subseteq_t B$, A a stable set. Then q is the stationarization of $q{\restriction}A$.*

Lemma 5.9. *(Lemma 8.10 in [SU22]) Let A, B, C be sets such that $A \subseteq_t B$, $A \subseteq C$, $C \mathop{\smile}\limits_A B$ (see Notation 5.6). Let F be an elementary map from B onto B', G be an elementary map from C onto C' such that $F{\restriction}A = G{\restriction}A$. Then $F \cup G$ is elementary.*

Lemma 5.10. *(Lemma 8.11 in [SU22]) Let A be λ-saturated and stable, $A \subseteq_t B$, N a λ-saturated model λ-atomic over A such that $N \mathop{\smile}\limits_A B$.*
    Then $tp(N/A) \vdash tp(N/B)$; so the types $tp(N/A)$ and $tp(B/A)$ are weakly orthogonal.

6 Good Systems

Finally we are ready to formally define the main tool that is used in our proofs: good systems.

Notation 6.1.   • $\mathcal{P}(Y) = \{Z : Z \subseteq Y\}, \mathcal{P}^-(Y) = \mathcal{P}(Y) \smallsetminus \{Y\}$.

• *We also use the standard set theoretic notation $n = \{0, 1, \ldots, n-1\}$.*

Definition 6.2. (1.) We say that I is *weakly nice* if for some $n, I \subseteq \mathcal{P}(n), \mathcal{P}(\{1, \ldots, n\}) \subseteq I$ and I is hereditary (i.e. $t \subseteq s$ and $s \in I$ implies $t \in I$). We say I is *nice* if in addition it is an initial segment of $\mathcal{P}(n)$ ordered lexicographically (identifying $s \in \mathcal{P}(n)$ with its characteristic function; so $\{t \in I : 0 \notin t\}$ is an initial segment and for every $s \in I$ all subsets of s precede s with respect to this ordering.) Let n_I be this n. If s is the last element on this list, $0 \in s$, then we say (I, s) is *nice*.

(2.) If I is weakly nice, we define by induction on n_I that $\mathcal{S} = \langle A_s : s \in I \rangle$ is a *good system* (or *good I-system*) if:

(i) I is nice,

(ii) $A_s \cap A_t = A_{s \cap t}$,

(iii) (a) $0 \notin s \Rightarrow A_s \prec P^{\mathfrak{C}}$,
     (b) $0 \in s \Rightarrow A_s \prec \mathfrak{C}$.

(iv) $A_s \cap P^{\mathfrak{C}} = A_{s \smallsetminus \{0\}}$,

(v) if $n \geq 2$, then $\langle A_s : s \in \mathcal{P}(\{1, \ldots, n-2\}) \rangle \prec \langle A_{s \cup \{n-1\}} : s \in \mathcal{P}(\{1, \ldots, n-2\}) \rangle$ and these are good systems,

(vi) $\bigcup_{t \in \mathcal{P}^-(s)} A_t$ is stable when $0 \in s \in I$.

(vii) For any $\phi \in L$ and $\bar{b}_s \in A_s$ for $s \in I$ such that $\models \phi(\ldots, \bar{b}_s, \ldots)$ there are $\bar{b}'_s, s \in I$, satisfying: $\bar{b}'_s \in A_{s \smallsetminus \{n-1\}}$ and $\models \phi(\ldots, \bar{b}'_s, \ldots)$, and, in addition, if $n - 1 \notin s$, then we can choose $\bar{b}'_s = \bar{b}_s$.)

(viii) $\langle A_{s \cup \{n-1\}} : s \in I_{[n-1]} \rangle$ and $\langle A_s : s \in I^{[n-1]} \rangle$ are both good systems where

   (a) $I^{[n-1]} = \{s : n - 1 \notin s \in I\}$
   (b) $I_{[n-1]} = \{s : s \cup \{n-1\} \in I, (n-1) \notin s\}$.

3. We call $\mathcal{S}$ a *weakly good* (w.g.) system if $(ii) - (iv)$ hold and a *medium good* (m.g.) system if $(v), (vi)$ hold too. We call $\mathcal{S}$ a *presystem* if it has the right form. A (λ, I)-system or λ-system means $|A_s| = \lambda$ for all $s \in I$, and the system is called *stable* if $\bigcup_{s \in I} A_s$ is a stable set.

Lemma 6.3. *(i) If $\langle A_s : s \in I \rangle \equiv \langle A'_s : s \in I \rangle$ where these presystems are treated as structures, then one is good if and only if the other one is good; similarly for w.g., m.g.*

(ii) If $\langle A_s : s \in I \rangle$ is good where (I, t) is nice, then $\langle A_s : s \in I \smallsetminus \{t\} \rangle$ is also good.

(iii) If $0 = l_0 < l_1 < \cdots < l_{m-1} < n_I$ and $J = \{v \subseteq m : \{l_i : i \in v\} \in I\}, B_v = A_{\{l_i : i \in v\}}$ and $\langle A_s : s \in I \rangle$ is a good system, then so is $\langle B_v : v \in J \rangle$.

Proof: (i): For clause (vi) of Definition 6.2 recall that stability is a property of the theory of a set. For the rest use induction on n_I.

   (ii): Straightforward checking.

   (iii) Clauses $(i) - (v)$ are straightforward, and we prove clauses $(vi) - (viii)$ by induction on n (for all possible systems, m and $l_0 < \cdots < l_{m-1}$). If $l_{m-1} < n - 1$ use the induction hypothesis applied to $\langle A_s : s \in I^{[n-1]} \rangle$, which is a good system by Definition 6.2 $(viii)$. For $l_{m-1} = n - 1$ clauses $(vi) - (viii)$ for $\langle B_v : v \in J \rangle$ follow from the corresponding clauses for $\langle A_s : S \in I \rangle$. ∎

Remark 6.4. *Note that from clause (vii) of Definition 6.2 it follows that if $\langle A_s : s \in I \rangle$ is a good system, then $\bigcup_{s \in I^{[n-1]}} A_s \subseteq_t \bigcup_{s \in I} A_s$.*

Lemma 6.5. *(i) Suppose $\langle A_v : v \in I \smallsetminus \{s\} \rangle$ is good, (I, s) nice, then $\bigcup_{v \in \mathcal{P}^-(s)} A_v \subseteq_t \bigcup_{v \in I, v \neq s} A_v$.*

(ii) Further, if $\langle A_v : v \in I \rangle$ is good, (I, s) nice, then for every $\bar{b} \in A_s$, $tp(\bar{b} / \bigcup_{v \in I, v \neq s} A_v)$ is definable over $\bigcup_{v \in \mathcal{P}^-(s)} A_v$.

Hence, combining with (i) and Lemma 5.7, $tp(\bar{b} / \bigcup_{v \in I, v \neq s} A_v)$ is the stationarization of $tp(\bar{b} / \bigcup_{v \in \mathcal{P}^-(s)} A_v)$.

Proof: By induction on $n_I = n$ and $|I|$. Let $J = I \smallsetminus \{s\}$. If $n_I = 0$, the statements are vacuous. Let $n_I = 1$, necessarily $s = \{0\}$, ($0 \in s$ as (I, s) is nice), so $I = \{\varnothing, s\}$; $\mathcal{P}^-(s) = \{\varnothing\} = \{v : v \in I, v \neq s\}$, so (i) holds trivially. As for (ii), using the same equality, the conclusion holds by Hypothesis 2.1

every type over $P^{\mathcal{C}}$ is definable as $A_s \prec \mathcal{C}, P^{A_s} = A_\varnothing$ by Definition 6.2 $(iii), (iv)$.

So let $n_I \geq 2$. We first prove (i). $\bigcup_{v \in J^{[n-1]}} A_v \subseteq_t \bigcup_{v \in J} A_v$ by (vii) of Definition 6.2 (see Remark 6.4). By the induction hypothesis applied to $\langle A_v : v \in J^{[n-1]} \rangle$ (which is a good system by $(viii)$ of the definition) we have $\bigcup_{v \in \mathcal{P}^-(s)} A_v \subseteq_t \bigcup_{v \in J^{[n-1]}} A_v$ (as $\mathcal{P}^-(s) \subseteq J^{[n-1]}$). But $\subseteq_t$ is transitive, so we get the desired conclusion of part (i).

So assume now $n - 1 \in s$. Let $t := s \smallsetminus \{n-1\}$, so $0 \in t$ and $t \subseteq t'$ implies $t' \in \{t, s\}$. As t is the second to last element in I with respect to lexicographical ordering, $(I \smallsetminus \{s, t\})$ is nice.

Suppose $\models \phi(\bar{a}, \bar{b}, \bar{c})$ where $\bar{a} \in A_t, \bar{b} \in \bigcup_{u \in \mathcal{P}^-(t)} A_{u \cup \{n-1\}}, \bar{c} \in \bigcup_{v \in I, v \neq s} A_v$, and we should find $\bar{c}^* \in \cup_{u \in \mathcal{P}^-(s)} A_u$ such that $\models \phi(\bar{a}, \bar{b}, \bar{c}^*)$. Clearly without loss of generality $\bar{c} \in \bigcup_{v \in I, v \neq s, t} A_v$. By the induction hypothesis on (ii) applied to $I \smallsetminus \{s\}$, $tp(\bar{a}/\bigcup_{v \in I, v \neq s, t} A_v)$ is definable over $\bigcup_{v \in \mathcal{P}^-(t)} A_v$. So we can choose $\Psi = \Psi_\phi(\bar{y}, \bar{z}, \bar{d})$ with $\bar{d} \in \bigcup_{v \in \mathcal{P}^-(t)} A_v$ which defines $tp_\phi(\bar{a}/\bigcup_{v \in I, v \neq s, t} A_v)$. Hence $\models \Psi(\bar{b}, \bar{c}, \bar{d})$.

Consider the system $\langle A_{u \cup \{n-1\}} : u \in I_{[n-1]} \smallsetminus \{t\} \rangle$. (Note: since I is an initial segment of $\mathcal{P}(n_I)$, s last in I and $n-1 \in s$, we have that $u \cup \{n-1\} \in I$ whenever $u \in I$.) This system is good by clause $(viii)$ of Definition 6.2 and the previous lemma clause (ii) and $\bar{b}\bar{d} \in \bigcup_{u \subset t} A_u$). So we can apply part (i) of the induction hypothesis to find $\bar{c}^* \in \cup_{u \in \mathcal{P}^-(t)} A_{u \cup \{n-1\}} \subseteq_t \bigcup_{u \in I_{n-1}, u \neq t} A_u$ so that $\models \Psi(\bar{b}, \bar{c}^*, \bar{d})$, hence by the choice of $\bar{d}$, $\models \phi(\bar{a}, \bar{b}, \bar{c}^*)$ as required.

Next we have to prove the induction step for (ii).

First assume that $n - 1 \notin s$.

$\langle A_v : v \in I^{[n-1]} \rangle$ is a good system, so by the induction hypothesis $tp(\bar{a}/\bigcup_{v \in J^{[n-1]}} A_v)$ is definable over $\bigcup_{v \in \mathcal{P}^-(s)} A_v$ (note that $\bar{a} \in A_s$ and $s \in I^{[n-1]}$ by the assumption).

We claim that these same definitions work for $tp(\bar{a}/\bigcup_{v \in J} A_v)$. Indeed, we know that for some $\bar{d} \in \bigcup_{v \in \mathcal{P}^-(s)} A_v$ for all $\bar{b} \in \bigcup_{t \in J^{[n-1]}} A_t$ we have $\varphi(\bar{a}, \bar{b}) \iff \Psi_\varphi(\bar{b}, \bar{d})$. If for some $\bar{b} \in A_v \in bigcup_{v \in J} A_v$ the above is not the case, since $\bigcup_{v \in J^{[n-1]}} A_v \subseteq_t \bigcup_{v \in J} A_v$ (see Remark 6.4), we can find a "counterexample" in $\bigcup_{v \in J^{[n-1]}} A_v$, a contradiction.

Now suppose $n - 1 \in s$. Let $\bar{a} \in A_s$ and $\phi(\bar{x}, \bar{y})$ be a formula. By Theorem 5.2 $tp_\phi(\bar{a}/\bigcup_{u \in \mathcal{P}^-(s)} A_u)$ is definable by some $\psi(\bar{y}, \bar{b})$ where $\bar{b} \in \bigcup_{u \in \mathcal{P}^-(s)} A_u$ since the domain is a stable set by part (vi) of the definition. Let $\bar{b} = \bar{b}_1 \bar{b}_2$ where $\bar{b}_1 \in A_t, \bar{b}_2 \in \bigcup_{u \in \mathcal{P}^-(t)} A_{u \cup \{n-1\}}$ where again $t := s \smallsetminus \{n-1\}$. Suppose $\psi(\bar{y}, \bar{b})$ does not define $tp_\phi(\bar{a}/\bigcup_{u \in I, u \neq s} A_u)$. Let $\bar{c}^*$ be a witness to this lack of definability and $\bar{c}^* = \bar{c}_1^* \bar{c}_2^*, \bar{c}_1^* \in A_t, \bar{c}_2^* \in \bigcup_{u \in I, u \neq s, t} A_{u \cup \{n-1\}}$.

Hence we have $\models \phi(\bar{a}, \bar{c}_1^*, \bar{c}_2^*)$ if and only if $\models \neg\psi(\bar{c}_1^*, \bar{c}_2^*, \bar{b}_1, \bar{b}_2)$. By part $(viii)$ of the definition $\langle A_{u \cup \{n-1\}} : u \in I_{[n-1]} \rangle$ is a good system, so by part (vi) applied to the type $tp|\phi_1(\bar{a}\bar{c}_1^*/\bigcup_{u \in \mathcal{P}^-(t)} A_{u \cup \{n-1\}})$ (where $\phi_1 = \phi(\bar{x}, \bar{y}_1, \bar{y}_2)$) is defined by some $\theta(\bar{y}_2, \bar{d})$ for some $\bar{d} \in \bigcup_{u \in \mathcal{P}^-(t)} A_{u \cup \{n-1\}}$ and by $(viii)$ of the definition and the induction hypothesis this definition works for $\bigcup_{u \in I, u \neq s, t} A_u = \bigcup_{u \in I_{[n-1]}, u \neq t} A_{u \cup \{n-1\}}$.

As we can replace ϕ by $\neg\phi$, without loss of generality we have $\models \phi(\bar{a}, \bar{c}_1^*, \bar{c}_2^*) \wedge \neg\psi(\bar{c}_1^*, \bar{c}_2^*, \bar{b}_1, \bar{b}_2)$. So $\models \theta(\bar{c}_2^*, \bar{d})$. As we have already proved part (i) for (I, s), we can apply it to find $\bar{c}_2^{**} \in \bigcup_{u \in \mathcal{P}^-(s)} A_u$ satisfying $\models \theta(\bar{c}_2^{**}, \bar{d}) \wedge \neg\psi(\bar{c}_1^*\bar{c}_2^{**}, \bar{b}_1, \bar{b}_2)$. Now by the choice of $\theta(\bar{y}_2, \bar{d})$ this says $\models \phi(\bar{a}, \bar{c}_1^*, \bar{c}_2^{**})$. But $\bar{c}_1^*, \bar{c}_2^* \in \bigcup_{v \in \mathcal{P}^-(s)} A_v$, hence by the choice of ψ we have $\models \psi(\bar{c}_1^*, \bar{c}_2^*, \bar{b})$. But $\bar{b} = \bar{b}_1 \bar{b}_2$, so the last statement contradicts the second conjunct above, and we finish. ∎

We can finally conclude that the union of a good system is a complete set:

Corollary 6.6.   *(i) If $\langle A_s : s \in I \rangle$ is a good system, then $\bigcup_{s \in I} A_s$ is complete.*

(ii) If $\mathcal{S}^l = \langle A_t^l : t \in I \rangle$ are good systems for $l = 1, 2$, (I, s) is nice, F is and elementary mapping from $\bigcup_{t \neq s} A_t^1$ onto $\bigcup_{t \neq s} A_t^2$, G is an elementary mapping from A_s^1 onto A_s^2 and $F|\bigcup_{t \subset s} A_t^1 = G|\bigcup_{t \subset s} A_t^1$, then $G \cup F$ is an elementary mapping.

Proof: By induction on $|I|$; without loss of generality $I \neq \mathcal{P}(\{1, \ldots, n_I - 1\})$ (that is, $I \supset \mathcal{P}(\{1, \ldots, n_I - 1\})$) since for $I = \mathcal{P}(\{1, \ldots n_I - 1\})$ the claim follows from condition (iii) (b) of the definition. Let $\lambda = \lambda^{<\lambda} > |T|$ and let $\mathcal{S}' = \langle A_t' : t \in I \rangle$ be such that:

(a) $\langle A_t' : t \in I \rangle \equiv \langle A_t : t \in I \rangle$

(b) $\mathcal{S}'$ is λ-saturated

(c) $\mathcal{S}'$ has cardinality λ

It suffices to prove the statement for $\mathcal{S}'$. Let $s \in I$ be the last member of I. By our "wlog" assumption, $0 \in s$, that is, (I, s) is nice. By condition (vi) of the definition, $\bigcup_{t \subset s} A_t'$ is a stable set, so by Theorem 5.3 we can find $M_s, \bigcup_{t \subset s} A_t' \subseteq M_s \subseteq A_s', M_s$ λ-primary over $\bigcup_{t \subset s} A_t'$. For $\bar{a} \in M_s, tp(\bar{a}/\bigcup_{t \subset s} A_t'$ is λ-isolated. As $\bigcup_{t \in I, t \neq s} A_t'$ is complete (by the induction hypothesis) there is a λ-saturated model $M, P^M \subseteq_{t \in I, t \neq s} A_t' \subseteq M$. So for $\bar{a} \in M_s$ we have $tp(\bar{a}/\bigcup_{t \subset s} A_t')$ implies $tp(\bar{a}/\bigcup_{t \in I, t \neq s} A_t')$ (see Lemma 5.10), and by the choice of M_s this type is realized in M. Similarly So without loss of generality, $M_s \subseteq M$. Hence $\bigcup_{t \in I, t \neq s} A_t' \cup M_s$ is complete.

Now $A_{s \smallsetminus \{0\}}' = P^{A_s'} \subseteq M_s \prec A_s'$, hence for every $\bar{a} \in A_s', tp(\bar{a}/M_s)$ is stationary and definable. By the first part of the previous lemma (as for every $\bar{a} \in A_s', tp(\bar{a}/M_s) \in S_*(M_s)$ since $P^{A_s'} \subseteq M_s$) the same definition works for $tp(\bar{a}/M_s \cup \bigcup_{t \in I, t \neq s} A_t')$. As $M_s \cup \bigcup_{t \in I, t \neq s} A_t'$ is complete, there is a λ-saturated $M', A_{n_I \smallsetminus \{0\}}' = P^{M'} \subseteq M_s \cup \bigcup_{t \in I, t \neq s} A_t' \subseteq M'$. We can extend $tp(A_s'/M_s)$ to a complete type over M' using the same definitions. So it necessarily extends $tp(A_s'/M_s \bigcup_{t \in I, t \neq s} A_t')$. So without loss of generality $tp(A_s'/M')$ is definable over M_s.

(ii): Follows from Lemma 5.9. ∎

Definition 6.7. We say that T is *n-stable* over P if for every good system $\langle A_s : s \in \mathcal{P}^-(n) \rangle$, its union $B = \bigcup_{s \in \mathcal{P}^-(n)} A_s$ is stable over P.

Definition 6.8. We say that T has the n-existence property if for every good system $\langle A_s : s \in \mathcal{P}^-(n) \rangle$, there is A_n such that $\langle A_s : s \in \mathcal{P}(n) \rangle$ is a good system.

7   Existence property

<u>Hypothesis:</u> Every good system $\langle A_s : s \in \mathcal{P}^-(n) \rangle$ is stable. That is, T is n-stable over P for all $n < \omega$.

<u>Conclusion:</u> For every nice I and good system $\langle A_s : s \in I \rangle$, $\bigcup_{s \in I} A_s$ is stable.

The conclusion follows from the Hypothesis by clause (vi) of the definition of a good system. So in particular, Hypothesis 5.1 (every model is stable over P) follows from the Hypothesis of this section.

Theorem 7.1. *If $\langle A_s : s \in I \rangle$ is a good system, $\lambda = \sum_{s \in I} |A_s|, \lambda > |T|$, then we can find A_α^s for $s \in I, \alpha < \lambda$ such that*

(a) $|A_\alpha^s| \le |\alpha| + |T|$.

(b) $\langle A_\alpha^s : \alpha < \lambda \rangle$ *is increasing continuously.*

(c) *For each α, $\langle A_\alpha^s : s \in I \rangle \prec \langle A_s : s \in I \rangle$.*

(d) $\langle A_\alpha^s : s \in I \rangle$ *is a good system.*

(e) *Let $J = I \cup \{ t \cup \{n_I\} : t \in I \}$, and for $s \in J, \alpha < \beta$ let*

$$A^s{}_{\{\alpha,\beta\}} = \begin{cases} A_\alpha^s & \text{if } s \in I \\ A_\beta^t & \text{if } s \in J, \, n_I \in s, t = s \smallsetminus \{n_I\} \end{cases}$$

Then $\langle A^s_{\{\alpha,\beta\}} : s \in J \rangle$ is a good system.

Proof: We can easily define (Löwenhem-Skolem) the A_α^s such that $(a), (b)$ and (c) hold. Now (d) holds by 6.3 (i). Now, clause (e) follows:

<u>Claim</u> If $\langle A_1^s : s \in I \rangle$ is a good system, $\langle A_0^s : s \in I \rangle \prec \langle A_1^s : s \in I \rangle$, and J and $A^s_{\{0,1\}}$ are defined as in (e), then $\langle A^s_{\{0,1\}} : s \in J \rangle$ is a good system.

Proof (of claim): Let $\langle s^l : l < 2^{|I|} \rangle$ be the lexicographic enumeration of J. We prove by induction on m, $|I \cup \mathcal{P}(\{1, \ldots, n_I\})| \le m \le 2^{|I|}$ that $\langle A^{s^l}_{\{0,1\}} : l < m \rangle$ is a good system.

For $m = |I \cup \mathcal{P}(\{1, \ldots, n_I\})|$ easy. For $m+1$ check Definition 6.2.

$(i) - (v)$ are easy. (vi) holds as the induction hypothesis implies that $\langle A^t_{\{0,1\}} : t \subset s < m \rangle$ is a good system, hence (by the hypothesis of the section) $\bigcup_{t \subset s_m} A^t_{\{0,1\}}$ is stable. Next, (vii) holds as $\langle A_0^s : s \in I \rangle \prec \langle A_1^s : s \in I \rangle$. Lastly $(viii)$ holds as $\langle A_1^s : s \in I \rangle$ is a good system (by assumption) and $\langle A_0^s : s \in I \rangle$ is a good system (by 6.3 (i)). ∎

Theorem 7.2. *(T countable) If $I = \mathcal{P}^-(n), \langle A_s : s \in I \rangle$ is a good system then over $\bigcup_{s \in I} A_s$ there is an locally constructible model. Moreover, if $tp(\bar{c}/\bigcup_s A_s) \in S_*(\bigcup_s A_s)$, then we can find a locally constructible model over $\bigcup_s A_s \cup \bar{c}$.*

We have the following easy fact:

Fact 7.3. If B is locally constructible over A, $A \subseteq_t C$, then B is locally constructible over C by the same sequence, $A \cup B \subseteq_t C \cup B$, and $\mathrm{tp}(B/A) \vdash \mathrm{tp}(B/C)$.

Proof: (of Theorem 7.2) We prove the theorem by induction on $\lambda = \sum |A_s|$. Remember $I = \mathcal{P}^-(n)$.

For $\lambda \leq \aleph_0$, use Lemma 4.11. Note that the set $A_s \bar{c}$ is countable and stable by Lemma 4.10.

For $\lambda > \aleph_0$ choose $A_\alpha^s, \alpha < \lambda$ as in Theorem 7.1 such that $\mathrm{tp}(\bar{c}/\bigcup_s A_s)$ is definable over $\bigcup A_0^s$. We define $A_\alpha^{\{0,\dots,n-1\}}$ by induction on α such that it is increasing continuously, $A_0^{\{0,\dots,n-1\}}$ is locally constructible over $\bigcup\{A_0^s : s \in I\}$ and for $\alpha < \lambda, A_{\alpha+1}^{\{0,\dots,n-1\}}$ is locally constructible over $A_\alpha^{\{0,\dots,n-1\}} \cup \{A_{\alpha+1}^s : s \in I\}$. $|T|$ is countable, so use the case $\lambda = \aleph_0$

Since (Fact 7.3)

$$tp_*(A_{\alpha+1}^{\{0,\dots,n-1\}}/\bigcup_s A_{\alpha+1}^s \cup A_\alpha^{\{0,\dots,n-1\}}) \vdash tp_*((A_{\alpha+1}^{\{0,\dots,n-1\}}/\bigcup_s A_s)$$

and

$$tp_*(A_0^{\{0,\dots,n-1\}}/\bigcup_s A_0^s \cup \bar{c}) \vdash tp_*(A_0^{\{0,\dots,n-1\}}/\bigcup_s A_s \cup \bar{c})$$

we have

$$tp_*(A_\alpha^{\{0,\dots,n-1\}}/\bigcup_s A_\alpha^s \cup \bar{c}) \vdash tp_*(A_\alpha^{\{0,\dots,n-1\}}/\bigcup_s A_s \cup \bar{c})$$

.

Let for $s \in \mathcal{P}^-(n+1)$,

$$B_\alpha^s = \begin{cases} A_\alpha^s & \text{if } n \notin s, s \neq \{0,\dots,n-1\} \\ A_{\alpha+1}^{s \setminus \{n\}} & \text{if } n \in s \end{cases}$$

Note that in the notation of Theorem 7.1 for $I = \mathcal{P}^-(n)$, $B_\alpha^s = B_{\{\alpha,\alpha+1\}}^s$. So $\langle B_\alpha^s : s \in \mathcal{P}^-(n+1) \rangle$ is a good system.

We use the induction hypothesis on λ to carry the induction step from α to $\alpha + 1$. ∎

Corollary 7.4. *(T countable) Assume that T is n-stable over P for all $n < \omega$. Then T has n-existence for all $n < \omega$. Moreover: let $\langle A_s : s \in \mathcal{P}^-(n) \rangle$ be a good $\mathcal{P}^-(n)$-system. Then its union $B = \bigcup_{s \in \mathcal{P}^-(n)} A_s$ has the locally constructible existence property,*

Corollary 7.5. *(T countable). Assume that T is n-stable over P for all $n < \omega$. Then T has the Gaifman property. Moreover, every $N \models T^P$ has the locally constructible existence property.*

The last Corollary is just a particular case of the previous one with $n = 1$.

In light of the results above, we restate our generalized version of Gaifman's Conjecture:

Conjecture 7.6. *(T countable) Assume that T is n-unstable over P for some n. Then for every regular cardinal λ big enough, and every $\mu \geq \lambda$, T has 2^λ models of cardinality μ, which are non-isomorphic over P.*

Proving the last conjecture as stated may be challenging. We therefore state a weaker version of it, which, we believe, may be more attainable:

Conjecture 7.7. *(T countable) Assume that T is m-stable over P for all $m < n$, but n-unstable for some n. Then for every regular cardinal λ big enough, and every $\mu \geq \kappa = \lambda^{+n}$, T has 2^κ models of cardinality μ, which are non-isomorphic over P.*

An even more "reasonable" goal may be proving that for every regular λ (perhaps $\lambda = \lambda^{<\lambda}$), there is a forcing extension of the universe that does not collapse any cardinals, in which we have non-structure as stated in the last Conjecture. We plan to address these questions in subsequent work.

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