CONSISTENCY OF SQUARE BRACKET PARTITION RELATION

SAHARON SHELAH

ABSTRACT. Characteristic earlier results were of the form $\mathrm{CON}(2^{\aleph_0} \to [\lambda]^2_{n,2})$, with $2^{\aleph_0}$ an ex-large cardinal, in the best case the first weakly Mahlo cardinal.

Characteristic new results are $\mathrm{CON}\big((2^{\aleph_0} = \aleph_m) + \aleph_\ell \to [\aleph_k]^2_{n,2}\big)$, where $k < \ell < m$. So we improve in three respects: the continuum may be small (e.g. not a Mahlo weakly inaccessible), we use no large cardinal, and the cardinals λ involved are $< 2^{\aleph_0}$ after the forcing.

§ 0. INTRODUCTION

In their seminal list of problems [EH71], Erdös and Hajnal posed the question (15(a)): does $2^{\aleph_0} \not\to [\aleph_1]^2_3$? Recently, Komjáth [Kom21] provided a comprehensive update on this topic.

We continue here works which start with the problem above:[She88, §2], [She92], [She89], [She95] [She96], [She00] and the work with Rabus [RS00].

The simplest case of our result is (recall 0.3 below):

Theorem 0.1. *Assume* GCH *for transparency.* <u>*Then*</u> *for some ccc forcing notion of cardinality $\aleph_8$ in the universe $\mathbf{V}^{\mathbb{P}}$, we have $2^{\aleph_0} = \aleph_8$ and for any $n \geq 3$, $\aleph_7 \to [\aleph_2]^2_{n,2}$.*

Proof. Choose $(\mu, \theta, \partial, \lambda)$ as $(\aleph_8, \aleph_7, \aleph_2.\aleph_0)$ and apply Theorem 0.2 $\qquad\square_{0.1}$

The general case is:

Theorem 0.2. *Assume $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$ and $2^{\partial^{+\ell}} = \partial^{+\ell+1}$ for $\ell = 0, 1, 2, 3$ and $\partial^{+4} < \theta$.* <u>*Then*</u> *for some λ^+-cc, $(< \lambda)$-complete forcing notion $\mathbb{P}$ of cardinality μ (so the forcing does not collapse any cardinal and preserves cardinal arithmetic outside $[\lambda, \mu)$), in the universe $\mathbf{V}^{\mathbb{P}}$ we have, $2^\lambda = \mu$ and for every $\sigma < \lambda$, $\theta \to [\partial]^2_{\sigma,2}$*

Proof. All this paper is dedicated to proving this theorem. Pedantically Hypothesis 1.1 holds (see Fact 1.12) so we can apply Concusion 1.11. $\qquad\square_{1.11}$

We may replace $\theta \to [\partial]^2_{\sigma,2}$ by $(\forall \partial_1 < \partial)(\exists \theta_1 < \theta)[\theta_1 \to [\partial_1]^2_{\sigma,2}]$ and change the assumption on cardinal arithmetic accordingly.

Recall,

Definition 0.3. For possibly finite cardinals θ, ∂, σ and κ, let $\theta \to [\partial]^2_{\sigma,\kappa}$ mean:

- if $\mathbf{c}$ is a function from $[\theta]^2 := \{u \subseteq \theta : |u| = 2\}$ into σ, <u>then</u> there exists some subset $\mathscr{U}$ of θ of cardinality ∂ such that $\{\mathbf{c}(u) : u \in [\mathscr{U}]^2\}$ has at most κ-many members.

We thank Yair Hayut for many helpful comments.

§ 0(A). **Preliminaries.**

Notation 0.4. $\mathrm{cof}(\delta)$ is the class of ordinals of cofinality $\mathrm{cf}(\delta)$.

Notation 0.5.
 (1) $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ are forcing notions.
 (2) p, q, r called *conditions* are members of a forcing notion.
 (3) $\mathbf{q}$ is as in Definition 1.3, some kind of $(< \lambda)$-support iterated forcing with extra information.

Notation 0.6. We may write e.g. $N[\mathbf{q}, \beta, u]$ instead $N_{\mathbf{q}, \beta, u}$ to help with sub-scripts (or super-script).

Definition 0.7. Let θ, ∂, κ and λ be infinite cardinals. We say that $\theta \to_{\mathrm{sq}} (\partial)^{\lambda, 2}_\kappa$ <u>when</u> $\theta > \partial \geq \kappa \geq \lambda$ and:

 $\boxplus$ If (a) then (b), where:
  (a) $\mathscr{B}$ is an expansion of $(\mathscr{H}_{<\lambda}(\mu), \in, <_*)$, where $<_*$ is a well-ordering of $\mathscr{H}_{<\lambda}(\mu)$, $\mu^+ > \theta$, and its vocabulary $\tau_{\mathscr{B}}$ has cardinality $\leq \lambda$.
  (b) There is a tuple $\mathbf{s} = (\mathscr{U}, \bar{N}, \bar{\pi})$ solving $\mathbf{p} = (\theta, \partial, \kappa, \lambda, \mathscr{B})$, which means:
   $\boxplus_{\mathbf{p}, \mathbf{s}}$ for $u, v \in [\mathscr{U}]^{\leq 2}$,
    $\bullet_1$ $\bar{N} = \langle N_u : u \in [\mathscr{U}]^{\leq 2} \rangle$,
    $\bullet_2$ $\mathscr{U} \subseteq \theta$ is such that $\mathrm{otp}(\mathscr{U}) = \partial$,
    $\bullet_3$ $N_u \prec \mathscr{B}$, $[N_u]^{<\lambda} \subseteq N_u$,
    $\bullet_4$ $\varepsilon[\mathbf{s}] := \min(\mathscr{U})$,
    $\bullet_5$ $N_u \cap \mathscr{U} = u$,
    $\bullet_6$ $\|N_u\| = \kappa$ and $\kappa + 1 \subseteq N_u$,
    $\bullet_7$ $N_u \cap N_v \prec N_{u \cap v}$,
    $\bullet_8$ $\bar{\pi} = \langle \pi_{u,v} : u, v \in [\mathscr{U}]^{\leq 2}$ and $|u| = |v| \rangle$ such that if $|u| = |v|$, then $\pi_{u,v}$ is an isomorphism from N_v onto N_u mapping v onto u,
    $\bullet_9$ if $u_1 \subseteq u_2$ and $v_1 \subseteq v_2$ all from $[\mathscr{U}]^{\leq 2}$ and $|u_2| = |v_2|$, $\pi''_{u_2, v_2}(u_1) = v_1$ <u>then</u> π_{u_1, v_1}, π_{u_2, v_2} are compatible functions[1],
    $\bullet_{10}$ for $\ell = 1, 2$, the sets $N_u \cap \partial$ for $u \in [\mathscr{U}]^\ell$ are pairwise equal [2] and included in $N_\emptyset$.

Observation 0.8. *If $\bar{N} = \langle N_u : u \in [\mathscr{U}]^{\leq 2} \rangle$ satisfy 0.7(b)$\bullet_1 + \bullet_7$, then:*

 (*) *For every $x \in \cup\{N_u : u \in [\mathscr{U}]^{\leq 2}\}$ the set $\{u \in [\mathscr{U}]^{\leq 2} : x \in N_u\}$ has one of the following forms:*
  (a) $\{u\}$ for some $u \in [\mathscr{U}]^2$,
  (b) $\{\zeta\}$ for some $\zeta \in \mathscr{U}$,
  (c) $\{\{\zeta\}\} \cup \{\{\varepsilon, \zeta\} : \varepsilon \in \mathscr{U} \cap \zeta\}$ for some $\zeta \in \mathscr{U}$,
  (d) $\{\{\zeta\}\} \cup \{\{\zeta, \xi\} : \xi \in \mathscr{U} \setminus (\zeta + 1)\}$ for some $\zeta \in \mathscr{U}$,
  (e) $\{\emptyset\}$,
  (f) $\{\emptyset\} \cup \{\{\zeta\} : \zeta \in \mathscr{U}\}$,
  (g) $\{\emptyset\} \cup \{\{\zeta\} : \zeta \in \mathscr{U}\} \cup \{\{\varepsilon, \zeta\} : \varepsilon < \zeta \text{ are from } \mathscr{U}\}$.

---

[1] So e.g. it follows that: if $\zeta_1, \zeta_2 \in \mathscr{U}$ then $\pi_{\{\zeta_2\}, \{\zeta_2\}} \upharpoonright (N_\emptyset \cap N_{\{\zeta_2\}})$ is the identity map.
[2] Note that ∂ has two distinct roles: the size of $\mathscr{U}$ and the restriction on $N_u \cap \partial$. We may separate.

§ 1. The forcing

Our aim here is to prove the consistency of the following configuration:

$$2 < \sigma < \lambda = \lambda^{<\lambda} \leq \partial = \partial^{<\lambda} \leq \mu = \mu^{\theta} = 2^{\lambda},$$

and having $\theta \to [\partial]^2_{\sigma,2}$.

A continuation is in preparation $[S^+]$, aiming to further develop the directions explored here, particularly for the case of superscript $\mathbf{n} > 2$, as dealt within [She92]. We also show there that we can weaken the requirements on the cardinals and have more pairs.

Hypothesis 1.1. The parameter $\mathbf{p} = (\mu, \theta, \partial, \lambda, \lambda, \mathscr{B})$ consists of the following:

   (a) $\lambda = \lambda^{<\lambda} < \partial < \theta \leq \mu = \mu^{\theta}$,

   (b) $\theta \to_{\mathrm{sq}} (\partial)^{\lambda,2}_{\lambda}$ (see Definition 0.7, a variant of [She89, 2.1]); in our case using λ twice in intentional.

   (c) σ will vary on the cardinal numbers from $(2, \lambda)$ and the "nice" μ-s are such that $\gamma < \mu \Rightarrow |\gamma|^{\theta} < \mu$.

   (d)   • $\mathscr{B}$ is a model expanding $(\mathscr{H}_{<\lambda}(\mu), \in, <_{\mathscr{B}})$, where $<_{\mathscr{B}}$ is well-ordering of $\mathscr{H}_{<\lambda}(\mu)$,

       • $\tau(\mathscr{B})$ is a vocabulary of cardinality $\leq \lambda$.

We intend to use $(<\lambda)$-support iterated forcing of quite a special kind but first, we define the iterand.

Definition 1.2.

   (1) Let $\mathbf{A}$ be the set of objects $\mathbf{a}$ consisting of:

   (a)   • $\gamma < \mu$ and $\sigma \in (2, \lambda)$,

       • $\mathbb{P}$ is a forcing notion such that:

$$p \in \mathbb{P} \Rightarrow \mathrm{dom}(p) \in [\gamma]^{<\lambda} \wedge (\forall \alpha \in \mathrm{dom}(p))(p(\alpha) \in [\omega \cup \gamma]^{<\lambda}),$$

       • $\mathbb{P}$ is λ^+-cc and $(<\lambda)$-complete,

       • the order $\leq_{\mathbb{P}}$ is: $p \leq_{\mathbb{P}} q$ iff:

$$\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \wedge (\forall \alpha \in \mathrm{dom}(p))[p(\alpha) \subseteq q(\alpha)],$$

   (b)   • $\underset{\sim}{\mathbf{c}}$ is a $\mathbb{P}$-name of a function from $[\theta]^2$ to σ, (we may write $\underset{\sim}{\mathbf{c}}(\alpha, \beta)$ instead of $\underset{\sim}{\mathbf{c}}(\{\alpha, \beta\})$ for $\alpha \neq \beta < \theta$).

   (c) We have $(\mathscr{U}, \bar{N}, \bar{\pi})$ solving $\mathbf{p} = (\theta, \partial, \lambda, \lambda, \mathscr{B})$, (with $\mathscr{B}$ as in Definition 0.7⊞(b) and 1.1) such that $\underset{\sim}{\mathbf{c}} \in N_u$ for every $u \in [\mathscr{U}]^{\leq 2}$.

   (1A) In the context of Definition 1.2(1), $\mathbf{a} = (\gamma, \mathscr{B}, \mathbb{P}, \underset{\sim}{\mathbf{c}}, \mathscr{U}, \bar{N}, \bar{\pi}) = (\gamma_{\mathbf{a}}, \mathscr{B}_{\mathbf{a}}, ...)$, so e.g. $N_{\mathbf{a},u} = N_u$.

   (2) We say that the pair $(p, \bar{\iota})$ is a *solution* of $\mathbf{a} \in \mathbf{A}$, and write $(\mathbf{a}, p, \bar{\iota}) \in \mathbf{A}^+$, <u>when</u>:

   (a) $\bar{\iota} = (\iota_1, \iota_2) \in \sigma \times \sigma$,

   (b) $p \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$, recalling $\varepsilon(\mathbf{a}) = \min(\mathscr{U})$,

   (c) if $p \leq q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ and $\zeta_1 < \zeta_2$ are from $\mathscr{U}$ <u>then</u> there are q_1, q_2, r_1, r_2 such that for $\ell = 1, 2$, we have:

       •$_0$ $q \leq_{\mathbb{P}_{\mathbf{a}}} q_\ell$,

       •$_1$ $q_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ and $q_1 \upharpoonright (N_{\mathbf{a}, \emptyset} \cap \mathrm{lg}(\mathbf{q})) = q_2 \upharpoonright (N_{\mathbf{a}, \emptyset} \cap \mathrm{lg}(\mathbf{q}))$,

       •$_2$ $r_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\zeta_1, \zeta_2\}}$,

       •$_3$ $r_\ell \Vdash ``\underset{\sim}{\mathbf{c}}(\zeta_1, \zeta_2) = \iota_\ell"$,

       •$_4$ $r_\ell \upharpoonright N_{\mathbf{a}, \{\zeta_1\}}$ is $\leq_{\mathbb{P}_{\mathbf{a}}}$-below $\pi^{\mathbf{a}}_{\{\zeta_1\}, \{\varepsilon[\mathbf{a}]\}}(q_\ell)$,

       •$_5$ $r_\ell \upharpoonright N_{\mathbf{a}, \{\zeta_2\}}$ is $\leq_{\mathbb{P}_{\mathbf{a}}}$-below $\pi^{\mathbf{a}}_{\{\zeta_2\}, \{\varepsilon[\mathbf{a}]\}}(q_{3-\ell})$.

   (3) If $\mathbf{b} = (\mathbf{a}, p, \bar{\iota}) \in \mathbf{A}^+$ then let $\underset{\sim}{\mathbb{Q}_{\mathbf{b}}}$ be the $\mathbb{P}$-name of the following forcing notion:

(∗) For $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$,
  (a) the set of elements of $\mathbb{Q}_{\mathbf{b}} = \underset{\sim}{\mathbb{Q}}_{\mathbf{b}}[\mathbf{G}]$ is:

$$\Big\{ u \in [\mathscr{U}]^{<\lambda} : \text{ if } \zeta_1 < \zeta_2 \text{ in } \mathscr{U}, \text{ then } \underset{\sim}{\mathbf{c}}\{\zeta_1, \zeta_2\}[\mathbf{G}] \in \{\iota_1, \iota_2\}, \text{ moreover}$$

for some q_1, q_2, r_1, r_2 as in Definition 1.2(1)(c)($\bullet_1$)-($\bullet_5$), we have $r_1 \in \mathbf{G}$ or $r_2 \in \mathbf{G}\Big\}$,

  (b) the order of $\mathbb{Q}_{\mathbf{b}}[\mathbf{G}]$ is inclusion,
  (c) the generic is $\underset{\sim}{\mathcal{Y}}_{\mathbf{b}} = \bigcup \mathbf{G}_{\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}}$.

Definition 1.3.
(1) Let $\mathbf{Q} := \mathbf{Q}_{\mathbf{p}}$ be the class of $\mathbf{q}$ which consist of (below, $\alpha \leq \lg(\mathbf{q})$ and $\beta < \lg(\mathbf{q})$ and e.g. $\mathbb{P}_\alpha = \mathbb{P}_{\mathbf{q},\alpha}$):
  (a) $\lg(\mathbf{q})$ is an ordinal $\leq \mu$,
  (b) $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \lg(\mathbf{q}), \beta < \lg(\mathbf{q}) \rangle$ is a $(<\lambda)$-support iteration,
  (c) $\mathbb{P}_\beta$ satisfies the λ^+-cc,
  (d) $\underset{\sim}{\mathbb{Q}}_\beta$ is $\mathbb{Q}_{\mathbf{b}_\beta}$, where:
    $\bullet_1$ $\mathbf{b}_\beta := (\mathbf{a}_\beta, p_\beta^*, \bar{\iota}_\beta^*) \in \mathbf{A}^+$,
    $\bullet_2$ $\mathbf{a}_\beta := (\gamma_\beta, \mathscr{B}_\beta, \mathbb{P}_\beta^\bullet, \underset{\sim}{\mathbf{c}}_\beta, \mathscr{U}_\beta, \bar{N}_\beta, \bar{\pi}_\beta) \in \mathbf{A}$,
    $\bullet_3$ $\mathbb{P}_\beta^\bullet$ is equal to $\mathbb{P}'_{\xi(\beta)}$ for some $\xi(\beta) = \xi_{\mathbf{q}}(\beta) \leq \beta$ (on $\mathbb{P}'_\beta$, see below),
    $\bullet_4$ The sequence $\langle (\mathbb{P}_\gamma, \mathbb{P}'_\gamma, \mathbf{a}_\gamma, \mathbf{b}_\gamma, \xi(\gamma)) : \gamma < \beta \rangle$ belongs to $N_{\beta,u}$ for every $u \in [\mathscr{U}_\beta]^{\leq 2}$.
    $\bullet_5$ Letting $\mathscr{W}_\beta = \bigcup\{N_{\beta,u} \cap \beta : u \in [\mathscr{U}_\beta]^{\leq 2}\}$, we [3] have: for every $\gamma \in \mathscr{W}_\beta$ the set $\mathscr{W}_\beta \cap \mathscr{W}_\gamma$ has cardinality $\leq \lambda$, and there is $u = u_{\beta,\gamma} \in [\mathscr{U}_\beta]^{\leq 2}$ such that $\mathscr{W}_\beta \cap \mathscr{W}_\gamma \subseteq N_{\beta,u}$ and without loss of generality u is minimal with this property.
  (e) $\mathbb{P}'_\alpha$ is a dense subset of $\mathbb{P}_\alpha$, where,
    $\bullet$ $\mathbb{P}'_\alpha$ is $\mathbb{P}_\alpha$ restricted to the set of conditions $p \in \mathbb{P}_\alpha$ such that: if $\beta \in \mathrm{dom}(p)$ then $p(\beta)$ is a member of $\mathbf{V}$ (not just a $\mathbb{P}_\alpha$-name) and if $\zeta_1 < \zeta_2$ are in $p(\beta) \subseteq \mathscr{U}_\beta$, then there are q_1, q_2, r_1, r_2 as in Definition 1.2(2)(c)($\bullet_1$)-($\bullet_5$) with $\mathbf{a}_\beta, \mathbf{b}_\beta$ here standing for $\mathbf{a}, \mathbf{b}$ there and

$$\bigvee_{\ell=1}^{2} (\forall \gamma \in \mathrm{dom}(r_\ell))[\gamma \in \mathrm{dom}(p) \wedge r_\ell(\gamma) \subseteq p(\gamma)].$$

  (f) $\gamma_{\mathbf{q}} := \gamma(\mathbf{q}) := \sup\{\gamma_{\mathbf{q},\beta} : \beta < \lg(\mathbf{q})\}$, so $\mathbb{P}'_{\gamma(\mathbf{q})} \subseteq \mathscr{H}_{<\lambda}(\gamma_{\mathbf{q}})$; let $\mathbb{P}_{\mathbf{q}} := \mathbb{P}_{\lg(\mathbf{q})}$ and $\mathbb{P}'_{\mathbf{q}} := \mathbb{P}'_{\lg(\mathbf{q})}$.

(1A) We may write either $\mathbb{P}_{\mathbf{q},\alpha}$ or $\mathbb{P}_\alpha$ whenever $\mathbf{q}$ is clear and $(\iota_{\mathbf{q},\beta,1}, \iota_{\mathbf{q},\beta,2})$ is $\bar{\iota}_{\mathbf{b}_\beta}$.

(2) Let $\leq_{\mathbf{p}}$ be the following two-place relation on $\mathbf{Q}_{\mathbf{p}}$:

$$\mathbf{q}_1 \leq_{\mathbf{p}} \mathbf{q}_2 \underline{\text{iff}} \ \mathbf{q}_1 = \mathbf{q}_2 \restriction \lg(\mathbf{q}_1), \text{ see below.}$$

(3) For $\mathbf{q}_2 \in \mathbf{Q}_{\mathbf{p}}$ and $\alpha_* \leq \lg(\mathbf{q}_2)$, we define $\mathbf{q}_1 := \mathbf{q}_2 \restriction \alpha_*$ by:
  (a) $\lg(\mathbf{q}_1) = \alpha_*$,
  (b) $(\mathbb{P}_{\mathbf{q}_1,\alpha}, \mathbb{P}'_{\mathbf{q}_1,\alpha}) = (\mathbb{P}_{\mathbf{q}_2,\alpha}, \mathbb{P}'_{\mathbf{q}_2,\alpha})$ for $\alpha \leq \alpha_*$,
  (c) $(\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_1,\beta}, \mathbf{b}_{\mathbf{q}_1,\beta}, \xi_{\mathbf{q}_1}(\beta)) = (\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_2,\beta}, \mathbf{b}_{\mathbf{q}_2,\beta}, \xi_{\mathbf{q}_2}(\beta)$ for $\beta < \alpha_*$.

(4) We say that two conditions $p, q \in \mathbb{P}'_\alpha$ are *isomorphic*, when:
  (a) $\mathrm{otp}(\mathrm{dom}(p)) = \mathrm{otp}(\mathrm{dom}(q))$, and
  (b) if $\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ then:
    $\bullet_1$ $\mathrm{otp}(p(\beta)) = \mathrm{otp}(q(\beta))$,

---

[3] Why? By 0.8(b)$\bullet_{10}$.

$\bullet_2$ if $\varepsilon \in p(\beta) \cap q(\beta)$ then $\mathrm{otp}(\varepsilon \cap p(\beta)) = \mathrm{otp}(\varepsilon \cap q(\beta))$,

$\bullet_3$ if $\varepsilon \in p(\beta), \zeta \in q(\beta)$ and $\mathrm{otp}(\varepsilon \cap p(\beta)) = \mathrm{otp}(\zeta \cap q(\beta))$ then:

$$\pi_{\beta,\{\zeta\},\{\varepsilon\}}(p \restriction N_{\beta,\{\varepsilon\}}) = q \restriction N_{\beta,\{\zeta\}}.$$

$\bullet_4$ if $\varepsilon < \varepsilon_1$ belongs to $p(\beta), \zeta < \zeta_1$ belongs to $q(\beta)$ then:

$$\pi_{\beta,\{\zeta,\zeta_1\},\{\varepsilon,\varepsilon_1\}}(p \restriction N_{\beta,\{\varepsilon,\varepsilon_1\}}) = q \restriction N_{\beta,\{\zeta,\zeta_1\}}.$$

Remark 1.4. If we prefer in clause (d) ($\bullet_3$) of Definition 1.3 (1) to have $\xi(\beta) = \beta$, i.e., $\mathbb{P}_\beta^\bullet = \mathbb{P}'_\beta$, we need to add, e.g. "$\mu$ is regular and e.g. use a preliminary forcing $(\{\mathbf{q} \in \mathbf{Q_p} \colon \mathrm{lg}(\mathbf{q}) < \mu\}, \lhd)$".

Claim 1.5.
(0) *For $\mathbf{q} \in \mathbf{Q_p}$, we have: $\mathbb{P}'_\mathbf{q} \models$ "$p \le q$" iff $\{p, q\} \subseteq \mathbb{P}'_\mathbf{q}$, $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and $\beta \in \mathrm{dom}(p) \Rightarrow p(\beta) \subseteq q(\beta)$.*

(1) *For $\mathbf{q} \in \mathbf{Q_p}$, any increasing sequence of members of length $< \lambda$ of $\mathbb{P}'_\mathbf{q}$ has a lub, in fact, if $\delta < \lambda$, $\bar{p} = \langle p_i \colon i < \delta \rangle \in {}^\delta(\mathbb{P}'_\mathbf{q})$ is increasing, then the following $p \in \mathbb{P}'_\mathbf{q}$ is a lub of $\bar{p}$; defined by: $\mathrm{dom}(p) = \bigcup\{\mathrm{dom}(p_i) \colon i < \delta\}$, and if $\beta \in \mathrm{dom}(p)$ then*

$$p(\beta) = \bigcup \{p_i(\beta) \colon i < \delta \text{ and } \beta \in \mathrm{dom}(p_i)\}.$$

We denote this p by $\lim(\bar{p})$.

(2) *For $\mathbf{q} \in \mathbf{Q_p}$, we have:*

- $p \in \mathbb{P}'_\mathbf{q}$ *iff:*
  (a) *p is a function with domain $\in [\mathrm{lg}(\mathbf{q})]^{<\lambda}$,*
  (b) *if $\beta \in \mathrm{dom}(p)$ then $p(\beta)$ belongs to $[\mathscr{U}_\beta]^{<\lambda}$.*
  (c) *If $\beta \in \mathrm{dom}(p)$ and $(\iota_1, \iota_2) = (\iota_{\mathbf{q},\beta,1}, \iota_{\mathbf{q},\beta,2})$ then for every $\zeta_1 < \zeta_2$ from $p(\beta)$, $(p \restriction \beta) \restriction N_{\mathbf{q},\beta,\{\zeta_1,\zeta_2\}} \Vdash_{\mathbb{P}_{\mathbf{q},\beta}} "\underline{c}\{\zeta_1, \zeta_2\} \in \{\iota_1, \iota_2\}$". Moreover, there are q_1, q_2, r_1, r_2 as in Definition 1.2(2)(c)($\bullet_1$)-($\bullet_5$) and*

$$\bigvee_{\ell=1}^{2}(\forall\gamma \in \mathrm{dom}(r_\ell))[\gamma \in \mathrm{dom}(p) \cap \beta \wedge r_\ell(\gamma) \subseteq p(\gamma)].$$

(3) *If $\mathbf{q} \in \mathbf{Q_p}$ and $\alpha \le \mathrm{lg}(\mathbf{q})$ then $\mathbf{q} \restriction \alpha \in \mathbf{Q_p}$.*

(4) *$\le_\mathbf{p}$ is a partial order on $\mathbf{Q_p}$.*

(5) *If $\bar{\mathbf{q}} = \langle \mathbf{q}_j \colon j < \delta \rangle$ is $\le_\mathbf{p}$-increasing then it has a $\le_\mathbf{p}$-lub, $\lim(\bar{\mathbf{q}})$, of length $\cup\{\mathrm{lg}(\mathbf{q}_j) \colon j < \delta\}$.*

(6) *If $\beta < \mathrm{lg}(\mathbf{q})$, $\mathbf{a} = \mathbf{a}_{\mathbf{q},\beta}$, $u \in [\mathscr{U}_{\mathbf{a},\beta}]^{\le 2}$ and $N_u = N_{\mathbf{a},u}$, then:*

(*) *if $p \in \mathbb{P}'$ then $q = p \restriction N_{\mathbf{q},\beta,u}$ satisfies $q \in N_u$ and $q \le_{\mathbb{P}_\mathbf{q}} p$ where q is defined by:*
  $\bullet_1$ *$\mathrm{dom}(q) = \mathrm{dom}(p) \cap N_u \cap \beta$*
  $\bullet_2$ *If $\gamma \in \mathrm{dom}(q)$ then $q(\gamma) = p(\gamma) \cap N_u$.*

(7) *If (A) then (B), where:*

(A) (a) *$i_* < \lambda$,*
  (b) *$p_i \in \mathbb{P}'_\mathbf{q}$ for $i < i_*$,*
  (c) *if $i < j < i_*$, then p_i and q_i are essentially comparable, i.e.:*
    - *if $\beta \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j)$ then $p_i(\beta) \subseteq p_j(\beta)$ or $p_j(\beta) \subseteq p_i(\beta)$.*
(B) (a) *$\{p_i \colon i < i_*\}$ have a common upper bound in $\mathbb{P}'_\mathbf{q}$,*
  (b) *moreover, p is the least common upper bound when:*
    - *$\mathrm{dom}(p) = \bigcup\{\mathrm{dom}(p_i) \colon i < i_*\}$,*
    - *if $\beta \in \mathrm{dom}(p)$, then*

$$p(\beta) = \bigcup\{p_i(\beta) \colon i < i_* \text{ satisfying } \beta \in \mathrm{dom}(p_i)\}.$$

Proof. Part (2) is crucial but easy to verify. Parts (0), (1), (3), and (4) are also easy.

(5) For this, define $\mathbf{q} := \lim(\bar{\mathbf{q}})$ naturally, but we elaborate.

$(*)$ (a) $\lg(\mathbf{q}) = \bigcup\{\lg(\mathbf{q}_i): i < \delta\}$,

   (b) if $i < \delta$ and $\alpha \le \lg(\mathbf{q}_i)$, then $(\mathbb{P}_{\mathbf{q},\alpha}, \mathbb{P}'_{\mathbf{q},\alpha}) = (\mathbb{P}_{\mathbf{q}_i,\alpha}, \mathbb{P}'_{\mathbf{q}_i,\alpha})$,

   (c) if $i < \delta$ and $\beta < \lg(\mathbf{q}_i)$, then $(\mathbb{Q}_{\mathbf{q},\beta}, \mathbf{a}_{\mathbf{q},\beta}, \mathbf{b}_{\mathbf{q},\beta}) = (\mathbb{Q}_{\mathbf{q}_i,\beta}, \mathbf{a}_{\mathbf{q}_i,\beta}, \mathbf{b}_{\mathbf{q}_i,\beta})$,

   (d) $(\mathbb{P}_{\mathbf{q},\lg(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})})$ is $(\bigcup\{\mathbb{P}_{\mathbf{q}_i}: i < \delta\}, \bigcup\{\mathbb{P}'_{\mathbf{q}_i}: i < \delta\})$ when $\mathrm{cf}(\delta) \ge \lambda$,

   (e) if $\mathrm{cf}(\delta) < \lambda$, then $(\mathbb{P}_{\mathbf{q},\lg(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})})$ are defined as inverse limit. Then,
   - $\mathbb{P}'_{\mathbf{q}} := \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})}$ is dense in $\mathbb{P}_{\mathbf{q}}$ because by Definition 1.2(3), for each $\beta < \lg(\mathbf{q}_j)$ with $j < \delta$, $\mathbb{Q}_{\mathbf{b}[\beta,\mathbf{q}_j]}$ is closed under increasing unions of length $< \lambda$.

Recalling that in Definition 1.3(1)(c), we use β and not α, "$\mathbb{P}_{\mathbf{q}}$ satisfies the λ^+-cc" is not required for proving 1.5 (5), only "if $\beta < \lg(\mathbf{q})$ then $\mathbb{P}_{\mathbf{q},\beta}$ satisfies the λ^+-cc", which is clear. Note that even though we formally do not need it here, the chain condition of $\mathbb{P}_{\mathbf{q}}$ will be proved in claim 1.6.

(6) Note that:

(a) If $\gamma \in \mathrm{dom}(q)$ then $\gamma \in N_u$ and $q(\gamma) \subseteq N_u$,

(b) As $\mathrm{dom}(q)$ and $q(\gamma)$ for $\gamma \in \mathrm{dom}(q)$ has cardinality $< \lambda$ and $[N_u]^{<\lambda} \subseteq N_u$ so recalling clause (a) obviously $q \in N_u$.

(c) To prove q is in $\mathbb{P}_{\mathbf{q}}$ we need, for $\gamma \in \mathrm{dom}(q)$ and $\zeta_1 < \zeta_2$ from $q(\gamma) \subseteq \mathscr{U}_\gamma$ to verify the condition in 1.5(2)(c).

(d) But as $\gamma \in N_u$ hence $\mathbf{q} \upharpoonright (\gamma + 1)$ and ζ_1, ζ_2 belongs to N_u, also $N_{\mathbf{q},\gamma,\{\zeta_1\}}$, $N_{\mathbf{q},\gamma,\{\zeta_2\}}$, $N_{\mathbf{q},\gamma,\{\zeta_1,\zeta_2\}}$ belongs to N_u hence are included in it so we can finish easily.

(7) Follow by our definitions. $\square_{1.5}$

Still,

Crucial Claim 1.6. *If* $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$ *then* $\mathbb{P}_{\mathbf{q}}$ *satisfies* λ^+-cc.

Proof. It suffices, by 1.3(1)(e), to prove that $\mathbb{P}'_{\mathbf{q}} = \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})}$ satisfies the λ^+-cc, so assume:

$(*)_1$ (a) Let $\bar{p} = \langle p_\xi: \xi < \lambda^+ \rangle$, where $p_\xi \in \mathbb{P}'_{\mathbf{q}}$,

   (b) it suffice to prove that for some $\zeta < \xi < \lambda^+$, p_ζ and p_ξ are compatible.

[Why? By the definitions.]

$(*)_2$ For some stationary set $S \subseteq \mathrm{cof}(\lambda) \cap \lambda^+$, we have:
- $\bullet_1$ $\langle \mathrm{dom}(p_\xi): \xi \in S \rangle$ is a Δ-system with heart $w_* \in [\lg(\mathbf{q})]^{<\lambda}$, and
- $\bullet_2$ if $\beta \in w_*$ then $\langle p_\xi(\beta): \xi \in S \rangle$ is a Δ-system.

[Why? By the Delta system lemma, the proof using Fodor's lemma recalling $\lambda = \lambda^{<\lambda}$.]

$(*)_3$ Without loss of generality, $\langle p_\xi: \xi \in S \rangle$ are pairwise isomorphic (see Definition 1.3(4)).

[Why? Easy because for every $\mathbf{a}, u$ the model $N_{\mathbf{a},u}$ has cardinality λ.]

$(*)_4$ For $\gamma \in w_*$ we have:
- $\bullet_1$ Let $\mathscr{W}_\beta, u_{\beta,\gamma}$ be as in 1.3(1)(d)$\bullet_5$.
- $\bullet_2$ Without loss of generality, $u_{\beta,\gamma}$ is disjoint to $N_{\mathbf{q},\beta,\{\zeta\}} \setminus N_{\mathbf{q},\beta,\emptyset} \cap \mu$ for every $\zeta \in \mathscr{U}_\beta$ and is disjoint to $N_{\mathbf{q},\beta,\{\varepsilon,\zeta\}} \setminus N_{\mathbf{q},\beta,\emptyset} \cap \mu$ for every $\varepsilon < \zeta$ from $\mathscr{U}_\beta$.

[Why? As for any $\gamma < \beta$ from w_* we have to omit from $\mathscr{U}_\beta$ at most two elements and w_* has cardinality $< \lambda$.]

$(*)_5$ We fix $\xi(1) \ne \xi(2)$ from S and we shall prove that $p_{\xi(1)}$ and $p_{\xi(2)}$ have a common upper bound; this suffices for proving Crucial Claim 1.6.

$(*)_6$ For $\beta \in w_*$:
  (a) for $\ell \in \{1, 2\}$, consider the sequence $\langle \alpha^\beta_{\xi(\ell),\varepsilon} \colon \varepsilon < \varepsilon_\beta \rangle$ listing the set $p_{\xi(\ell)}(\beta)$ in increasing order
  (b) Why ε_β and not $\varepsilon_{\beta,\ell}$? as the two sequences have the same length because $p_{\xi(1)}, p_{\xi(2)}$ are isomorphic, see Definition 1.3(4) $\bullet_1$.
  (c) Let $\mathscr{S}_\beta := \{\varepsilon < \varepsilon_\beta \colon \alpha^\beta_{\xi(1),\varepsilon} \neq \alpha^\beta_{\xi(2),\varepsilon}\}$,
  (d) so by Definition 1.3 (4) $\bullet_2$ the sets $\{\alpha^\beta_{\xi(1),\varepsilon} \colon \varepsilon \in \mathscr{S}_\beta\}$, $\{\alpha^\beta_{\xi(2),\varepsilon} \colon \varepsilon \in \mathscr{S}_\beta\}$ are disjoint and disjoint to $\{\alpha^\beta_{\xi(1),\varepsilon} \colon \varepsilon \in \varepsilon_\beta \setminus \mathscr{S}_\beta\} = \{\alpha^\beta_{\xi(2),\varepsilon} \colon \varepsilon \in \varepsilon_\beta \setminus \mathscr{S}_\beta\}$.

Let $\bar{\beta} = \langle \beta_i \colon i \leq i_* \rangle$ list the closure of $\{\alpha, \alpha + 1 \colon \alpha \in w_*\} \cup \{0, \mathrm{lg}(\mathbf{q})\}$ in increasing order, so necessarily $i_* < \lambda$ and clearly it suffice:

$(*)_7$ To choose $q_i \in \mathbb{P}'_{\mathbf{q},\beta_i}$ a common upper bound of $\{p_{\xi(1)} \restriction \beta_i, p_{\xi(2)} \restriction \beta_i\}$ increasing with $i \leq i_*$ by induction on $i \leq i_*$ such that:
  $(*)$ If $\beta \in w_* \setminus \{\beta_j \colon j < i\}$ and $\zeta(1), \zeta(2)$ are from $\mathscr{S}_\beta$ then:
    $\bullet_1$ $\mathrm{dom}(q_j) \cap N_{\beta,\{\alpha_{\xi(1),\zeta(1)},\alpha_{\xi(2),\zeta(2)}\}}$ is a subset of
    $$N_{\beta,\{\alpha_{\xi(1),\zeta(1)}\}} \cup N_{\beta,\{\alpha_{\xi(2),\zeta(2)}\}} \cup N_{\beta,\emptyset},$$
    $\bullet_2$ if $\ell = 1, 2$ and $\gamma \in \mathrm{dom}(q_j) \cap N_{\beta,\{\alpha_{\xi(\ell),\zeta(\ell)}\}}$ then $q_i(\gamma) = p_{\xi(\ell)}(\gamma)$ or $\gamma \in N_{\beta,\emptyset}$

Let us carry the induction.

<u>Case 1</u>: $i = 0$. Clearly, this case is trivial, letting $q_0 = \emptyset$.

<u>Case 2</u>: i is a limit ordinal.

In this case, let $q_i := \lim \langle q_j \colon j < i \rangle$, so by Claim 1.5(1), q_i is well-defined and is as required by the definition of the order and satisfies $\bullet$ of $(*)_7$.

<u>Case 3</u>: $i = j + 1$ and $\beta_j \notin w_*$.

In this case, $\mathrm{dom}(p_{\xi(1)}) \cap \mathrm{dom}(p_{\xi(2)}) \cap \beta_i \subseteq \beta_j$, hence the condition
$$q_i := q_j \cup \big(p_{\xi(1)} \restriction [\beta_j, \beta_i]) \cup (p_{\xi(2)} \restriction [\beta_j, \beta_i]) \big)$$
is as promised.

<u>Case 4</u>: $i = j + 1$ and $\beta_j \in w_*$.

By the choice of $\bar{\beta}$, clearly $\beta_i = \beta_j + 1$ and let $\mathscr{S} = \mathscr{S}_\beta$.

Recalling 1.3(1)(d) and 0.7(b)($\bullet_8$), we have:

$(*)_8$ $\mathbf{a}_{\beta_j} = \mathbf{a}_{\mathbf{q},\beta_j}$ determine:
  (a) $\bar{\pi}_{\beta_j} = \langle \pi_{u,v} \colon u, v \in [\mathscr{U}_{\beta_j}]^{\leq 2}$ and $|u| = |v| \rangle$,
  (b) $\bar{N}_{\beta_j} = \langle N_u \colon u \in [\mathscr{U}_{\beta_j}]^{\leq 2} \rangle$,
  (c) for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$, let:
    • $v[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1),\varepsilon(1)}, \alpha_{\xi(1),\varepsilon(2)}\}$, and
    • $u[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1),\varepsilon(1)}, \alpha_{\xi(2),\varepsilon(2)}\}$.
  (d) for $\varepsilon \in \mathscr{S}$, let $v[\varepsilon] = \{\alpha_{\xi(1),\varepsilon}\}$ and $u[\varepsilon] = \{\alpha_{\xi(2),\varepsilon}\}$,
  (e) $\bar{\iota} = \bar{\iota}^*_{\beta_j}$, see 1.3 (1) (d) $\bullet_1$.
  (f) $\gamma_j = \bar{\xi}_{\mathbf{q}}(\beta_j)$; see 1.3(1)(d) $\bullet_3$.

We shall now define $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$ such that:

$(*)_9$ (a) $p_{\varepsilon(1),\varepsilon(2)} \in \mathbb{P}_{\gamma_j} \cap N_{u[\varepsilon(1),\varepsilon(2)]}$, hence $\mathrm{dom}(p_{\varepsilon(1),\varepsilon(2)}) \subseteq \gamma_j \cap N_{u[\varepsilon(1),\varepsilon(2)]}$,
  (b) if $\varepsilon(1) = \varepsilon(2)$, then $p_{\varepsilon(1),\varepsilon(2)} \restriction (\gamma_j \cap N_{v[\varepsilon(1)]})$, $p_{\xi(1)} \restriction N_{v[\varepsilon(1)]}$ are essentially comparable; see 1.5(7)(A)(c), moreover the first is $\leq_{\mathbb{P}_{\mathbf{q}}}$-above the second,
  (c) if $\varepsilon(1) = \varepsilon(2)$, then $p_{\varepsilon(1),\varepsilon(2)} \restriction (\gamma_j \cap N_{u[\varepsilon(2)]})$, $p_{\xi(2)} \restriction N_{u[\varepsilon(2)]}$ are essentially comparable, moreover the first is $\leq_{\mathbb{P}_{\mathbf{q}}}$-above the second,
  (d) $p_{\varepsilon(1),\varepsilon(2)}$ satisfies 1.3(1)(e)$\bullet$ with $(\gamma_j, \varepsilon(1), \varepsilon(2))$ here standing for $(\beta, \zeta_1, \zeta_2)$ there,

(e) $\{q_j \restriction N_\emptyset\} \cup \{p_{\varepsilon(1),\varepsilon(2)} \restriction N_\emptyset \colon \varepsilon(1), \varepsilon(2) \in \mathscr{S}\}$ are pairwise essentially comparable,

(f) if $\varepsilon(1) \neq \varepsilon(2)$ then $p_{\varepsilon(1),\varepsilon(2)} \restriction N_{\{\alpha_{\varepsilon(\ell)}\}} \leq p_{\xi(\ell)} \restriction N_{\{\alpha_{\varepsilon(\ell)}\}}$ for $\ell = 1, 2$.

(g) if $\mathscr{S}_* \subseteq \mathscr{S} \times \mathscr{S}$ then the lub $q_{\mathscr{S}_*}$ of $\{q_j[N_{u[\varepsilon(1),\varepsilon(2)]}] \colon \varepsilon(1), \varepsilon(2) \in \mathscr{S}_*\}$ satisfies the condition in $(*)_7$.

We have to show two things: $\boxplus_1$ and $\boxplus_2$. The first saying we can choose them (the $p_{\varepsilon(1),\varepsilon(2)}$-s), the second that this is enough.

$\boxplus_1$ we can choose $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$ as required in $(*)_7$.

We consider two possible cases:

<u>Case 4.1</u>: $\varepsilon(1) \neq \varepsilon(2)$.

Let $p_{\varepsilon(1),\varepsilon(2)} = \pi(p_{\xi(1)} \restriction N_{v[\varepsilon(1),\varepsilon(2)]})$, where $\pi = \pi_{u[\varepsilon(1),\varepsilon(2)],v[\varepsilon(1),\varepsilon(2)]}$.

Why is $(*)_9$ preserved? Most clauses are obvious, but $(*)_9(g)$ deserve elaboration, recalling that we have to satisfy $(*)_7$.

So let $\beta \in \mathscr{W}_* \setminus \{\beta_\iota \colon \iota < i\}$, hence for some $j(*) < i_*$, we have $\beta = \beta_{j(*)}$, hence we have $\beta_{j(*)} \geq \beta_i$ hence $\beta_{j(*)} > \beta_j$ and we have $\mathscr{S}_* \subseteq \mathscr{S} \times \mathscr{S}$ and deal with $q_{\mathscr{S}_*}$.

For this, it is enough to consider the cases:

$\oplus_1$ $\mathscr{S}_* = \{\zeta(1), \zeta(2)\}$, where $\zeta(1) = \varepsilon(1)$ and $\zeta(2) = \varepsilon(2)$ hence from $\mathscr{S}$, so $\zeta(1) \neq \zeta(2)$,

$\oplus_2$ $\mathscr{S}_* = \{\zeta(1), \zeta(2)\}$ where $\zeta(1) \neq \zeta(2)$ are from $\mathscr{S}$ but $(\zeta(1), \zeta(2)) \neq (\varepsilon(1), \varepsilon(2))$.

Easy to check.

<u>Case 4.2</u>: $\varepsilon(1) = \varepsilon(2)$.

In this case, we pick some sequence $\langle p_{\varepsilon,\varepsilon} \colon \varepsilon \in \mathscr{S} \rangle$ by choosing $p_{\varepsilon,\varepsilon}$ by induction on $\varepsilon \in \mathscr{S}$. Now, $p_{\varepsilon,\varepsilon} \in \mathbb{P}'_\beta \cap N_{u[\varepsilon(1),\varepsilon(2)]}$ is such that:

$(*)$ (a) $p_{\varepsilon,\varepsilon}$ is $\leq_{\mathbb{P}'_{\mathbf{q},\beta}}$-above $p_{\xi(1)} \restriction N_{v[\varepsilon]}$ and above the restriction $p_{\xi(2)} \restriction N_{u[\varepsilon]}$,

(b) $\langle p_{\zeta,\zeta} \restriction N_\emptyset \colon \zeta \in (\varepsilon + 1) \cap \mathscr{S} \rangle$ is $\leq_{\mathbb{P}_{\beta[j]}}$-increasing, and

(c) there are q_1, q_2, r_1, r_2 as in Definition 1.3(2)(c) $(\bullet_1)$-$(\bullet_5)$ with $\mathbf{b}_{\mathbf{q},\beta}$ standing here for $(\mathbf{a}, p, \bar{\iota})$ there such that:

$$\bigvee_{\ell=1}^{2} (\forall \gamma \in \mathrm{dom}(r_\ell)) \left[\gamma \in \mathrm{dom}(p_{\varepsilon,\varepsilon}) \wedge r_\ell(\gamma) \subseteq p_{\varepsilon,\varepsilon}(\gamma)\right].$$

We can choose $p_{\varepsilon,\varepsilon}$ by the properties of $\mathbf{b}_{\beta_j}$

Having defined all the $p_{\varepsilon(1),\varepsilon(2)}$-s we can proceed.

$\boxplus_2$ The following set of members of $\mathbb{P}_{\beta_i}$ has a common upper bound q_*:

• $p_{\xi(1)}, p_{\xi(2)}$, and

• $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$.

[Why? Recall Claim 1.5(2) and 1.2(1)(c)($\bullet_1$) by 1.5(7), clause (A) there holds, in particular sub-clause (A)(c). The main point is that:

$(*)$ $\langle N_{v[\varepsilon(1),\varepsilon(2)]} \cap \gamma_j \setminus (N_{v[\varepsilon(1)]} \cup N_{u[\varepsilon(1)]}) \colon \varepsilon(1), \varepsilon(2) \in \mathscr{S} \rangle$ is a sequence of pairwise disjoint sets.

Why? As "$N_u \cap N_v \subseteq N_{u \cap v}$ for $u, v \in [\mathscr{U}_{\beta_j}]^{<\partial}$ by $0.7\bullet_7$.

So q_* from $\boxplus_2$ is a common upper bound of $p_{\xi(1)}, p_{\xi(2)}$, as promised. $\square_{1.6}$

Remark 1.7. 1) No need so far, but we may add in $(*)_4$ of the proof of Crucial Claim 1.6 the following item:

(d) if $\beta \in w_*$ and $\langle \alpha_{\zeta,\beta,i} \colon i < \iota_{\zeta,\beta} \rangle$ list in increasing order the members of $p_\zeta(\beta)$ for $\zeta \in S$, then:

• $\langle \iota_{\zeta,\beta} \colon \zeta \in S \rangle$ is constant called i_β,

• for $i < i_\beta$, the sequence $\langle \alpha_{\zeta,\beta,i} \colon \zeta \in S \rangle$ is constant or increasing,

• if $i, j < i_\beta$ the sequence of truth values

$\langle \text{Truth value}(\alpha_{\zeta,\beta,i} < \alpha_{\xi,\beta,j}) \colon \zeta < \xi \text{ are from } S \rangle$

is constant, and
- if $i, j < i_\beta$, $\zeta \neq \xi$ are from S and $\alpha_{\zeta,\beta,i} = \alpha_{\xi,\beta,j}$ then $i = j$.

2) We can make our choice of q_1, q_2, r_1, r_2 canonical, that is

(A) In 1.2(2) we replace $(p, \bar\iota, (\mathbf{a}, p, \bar\iota, \mathbb{F})$ by $(p, \bar\iota, (\mathbf{a}, p, \bar\iota, \mathbb{F})$

    $\bullet_1$ $\mathbb{F}_{\zeta_1, \zeta_2}(q) = (q_1, q_2, r_1, r_2) = \langle \mathbb{F}_{\zeta_1, \zeta_2, \ell}(q) : \ell = 1, 2, 3, 4 \rangle$

    $\bullet_2$ if also $\zeta_3 < \zeta_4$ are from $\mathscr{U}$ then $\pi^{\mathbf{a}}_{\zeta_3, \zeta_4, \zeta_1, \zeta_2} \mathbb{F}_{\zeta_1, \zeta_2, \ell} = \mathbb{F}_{\zeta_3, \zeta_4, \ell}$,
where if $p \leq q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ and $\zeta_1 < \zeta_2$ are from $\mathscr{U}$,
then $\langle \mathbb{F}_{\zeta_1, \zeta_1, \ell}(p, q) : \ell < \mu \rangle$ is the quadruple (q_1, q_2, r_2, r_2) as in
1.2(1)(c)($\bullet_1$)-($\bullet_5$).

(B) In 1.2(3) similarly and in 1.3(1)(d)

(C) In 1.5(1)(d) use $\mathbb{F}_\beta$,

(D) In the proof of 1.6, in $(*)_7 \boxplus_1$, case 4.2$(*)_{4.2}$ we use $\mathbb{F}_{\beta_j}$,

(E) Update the proof of 1.8 accordingly.

Claim 1.8. *If (A) then (B), where:*

(A)   (a) $\mathbf{q} \in \mathbf{G_p}$,
      (b) $2 < \sigma < \lambda$,
      (c) $\underset{\sim}{c}$ *is a* $\mathbb{P}_{\mathbf{q}}$-*name of a function from* $[\theta]^2$ *into* σ.
      (d) $p_* \in \mathbb{P}_{\mathbf{q}}$.

(B) *There is some* $\mathbf{b} \in \mathbf{A}^+$ *such that* $\mathbb{P}_{\mathbf{b}} = \mathbb{P}'_{\mathbf{q}}$ *and* $\underset{\sim}{c}_{\mathbf{b}} = \underset{\sim}{c}$ *and* $p_* \leq_{\mathbb{P}_{\mathbf{q}}} p_{\mathbf{b}}$.

Proof. Recalling Hypothesis 1.1(b), on the one hand, it is clear how to choose $\mathbf{a} \in \mathbf{A}$ such that $\mathbb{P}_{\mathbf{a}} = \mathbb{P}'_{\mathbf{q}}$ and $\underset{\sim}{c}_{\mathbf{a}} = \underset{\sim}{c}$. On the other hand, the choice of $p_{\mathbf{b}}$ and $\bar\iota_{\mathbf{b}}$ is similar to the proof of [She88, 2.1]. We now elaborate.

First, we can find $\mathbf{a}$ such that:

$(*)^1_{\mathbf{a}}$ (a) $\mathbf{a} \in \mathbf{A}$,
      (b) $\mathbb{P}_{\mathbf{a}} = \mathbb{P}_{\mathbf{q}}$,
      (c) $\gamma = \lg(\mathbf{q})$,
      (d) $\underset{\sim}{c}_{\mathbf{a}} = \underset{\sim}{c}$.

Why can we find? Because we have chosen $\mathbb{P}_{\mathbf{a}}$ as in $(*)^1_{\mathbf{a}}$(b), it is λ^+-cc by Claim 1.6; also $\gamma, \underset{\sim}{c}_{\mathbf{a}}$ are as is required in Definition 1.2. Next, it is easy to choose $\mathscr{B}_{\mathbf{a}}$ as required satisfying $\underset{\sim}{c}, p_*$ are definable; and lastly we can choose $(\mathscr{U}_{\mathbf{a}}, \bar{N})$ as is required because $\theta \to_{\mathrm{sq}} (\partial)^{\lambda, 2}_\lambda$ holds by Hypothesis 1.1 clause (b) and 0.7 in particular clause $(b)\bullet_{10}$.

We are left with choosing some appropriate $(p, \bar\iota)$ and then let $\mathbf{b} = (\mathbf{a}, p, \bar\iota)$. Let

$$Y := \{(q_1, q_2) : q_1, q_2 \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}} \text{ are above } p_* \text{ and,}$$
$$q_1 \restriction (N_{\mathbf{a}, \emptyset} \cap \lg(\mathbf{q})) = q_2 \restriction (N_{\mathbf{a}, \emptyset} \cap \lg(\mathbf{q}))\},$$

and let $\leq_Y$ be the following two place relation on Y:

$(*)_2$ $(p_1, p_2) \leq_Y (q_1, q_2)$ iff:
     (a) $(p_1, p_2) \in Y$ and $(q_1, q_2) \in Y$,
     (b) $p_1 \leq_{\mathbb{P}_{\mathbf{q}}} q_1$ and $p_2 \leq_{\mathbb{P}_{\mathbf{q}}} q_2$.

Clearly,

$(*)_3$ $(Y, \leq_Y)$ is a $(< \lambda)$-complete partial order.

[Why? Recalling 1.5(1).]

$(*)_4$ For $(p_1, p_2) \in Y$, let
     (a) $\mathrm{solv}(p_1, p_2)$ be the set of pairs (ι_0, ι_1) such that for any $\zeta_1 < \zeta_2$ from $\mathscr{U}$, there are r_1, r_2 such that for $\ell = 1, 2$ clauses $\bullet_2$-$\bullet_5$ of Definition 1.2(2)(c) hold.
     (b) $\mathrm{solv}^+(p_1, p_2) := \bigcap \{\mathrm{solv}(q_1, q_2) : (p_1, p_2) \leq_Y (q_1, q_2) \in Y\}$.

$(*)_5$ (a) if $(p_1, p_2) \leq_Y (q_1, q_2)$ then:

$$\mathrm{solv}(p_1, p_2) \supseteq \mathrm{solv}(q_1, q_2) \supseteq \mathrm{solv}^+(q_1, q_2) \supseteq \mathrm{solv}^+(p_1, p_2),$$

   (b) if $(p_1, p_2) \in Y$ then $\mathrm{solv}(p_1, p_2) \neq \emptyset$.

[Why? The first inclusion in Clause (a) holds because $\leq_{\mathbb{P}_q}$ is transitive. The other inclusions are clear, and Clause (b) is easy too.]

$(*)_6$ If $(p_1, p_2) \in Y$ then for some (q_1, q_2) and $\bar\iota$, we have:
   (a) $(p_1, p_2) \leq_Y (q_1, q_2) \in Y$,
   (b) if $(q_1, q_2) \leq_Y (q_1', q_2')$ <u>then</u> $\bar\iota \in \mathrm{solv}(q_1', q_2')$, moreover, $\mathrm{solv}(q_1, q_2) = \mathrm{solv}(q_2', q_2') = \mathrm{solv}^+(q_1', q_2') = \mathrm{solv}^+(q_1, q_2)$.

[Why? Recalling $\sigma < \lambda$, hence $|\sigma \times \sigma| < \lambda$ and $(Y, \leq_Y)$ is λ-complete by $(*)_3$.]

$(*)_7$ For $p \in \mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$, let $\mathrm{solv}(p)$ be the set of $\bar\iota \in \sigma \times \sigma$ such that: if $q \in \mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ is $\leq_{\mathbb{P}_\mathbf{a}}$-above <u>then</u> there is (q_1, q_2):
   $\bullet_1$ $q \leq_{\mathbb{P}_q} q_1$, $q \leq_{\mathbb{P}_q} q_2$ and
   $\bullet_2$ $(q_1, q_2) \in Y$,
   $\bullet_3$ $\bar\iota \in \mathrm{solv}^+(q_1, q_2)$,
   $\bullet_4$ $\mathrm{solv}(q_1, q_2) = \mathrm{solv}^+(q_1, q_2)$.

$(*)_8$ (a) if $p \in \mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then $\mathrm{solv}(p) \neq \emptyset$,
   (b) if $p \leq_{\mathbb{P}_\mathbf{a}} q$ are from $\mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then $\mathrm{solv}(p) \supseteq \mathrm{solv}(q)$,
   (c) if $p \in \mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then for some q and $\bar\iota$, for every q', we have $q \leq \mathbb{P}_\mathbf{q} \wedge q' \in \mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}} \Rightarrow \bar\iota \in \mathrm{solv}(q')$.

[Why? Clause (a) follows by $(*)_6$, Clause (b) by the definitions, and Clause (c) holds as $\mathbb{P}_\mathbf{a}$ and even $\mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ is λ-complete and $|\sigma \times \sigma| < \lambda$.]

Now, applying $(*)_8$(c) to p_* finish the proof of 1.8. $\square_{1.8}$

Claim 1.9. *If (A) then (B), where:*
   (A) (a) $\mathbf{q} \in \mathbb{Q}_\mathbf{p}$ *and* $\mathbf{q}_0 <_\mathbf{p} \mathbf{q}$,
       (b) $\gamma(\mathbf{q}) < \mu$, *so* $\mathrm{lg}(\mathbf{q}) < \mu$,
       (c) $\mathbf{b} \in \mathbf{A}_\mathbf{p}$ *and* $\mathbb{P}_\mathbf{b} = \mathbb{P}_{\mathbf{q}_0}$.
   (B) *There exists some* $\mathbf{q}_1$ *such that:*
       (a) $\mathbf{q} \leq_\mathbf{p} \mathbf{q}_1$,
       (b) $\mathrm{lg}(\mathbf{q}_1) = \mathrm{lg}(\mathbf{q}) + 1$,
       (c) $\mathbf{b}_{\mathrm{lg}(\mathbf{q})}[\mathbf{q}_1] = \mathbf{b}$.

Proof. Easy. $\square_{1.9}$

Lastly, before arriving at the main conclusion, we have to prove the following.

Claim 1.10.
   (1) *Assume* $\mathbf{q} \in \mathbf{Q}_\mathbf{p}$, $\alpha < \mathrm{lg}(\mathbf{q})$ *and* $\mathbf{b} = \mathbf{b}_{\mathbf{q}, \alpha} = (\mathbf{a}_\alpha, p_\alpha, \bar\iota_\alpha) = (\mathbf{a}, p, \bar\iota)$, *then:*
   - $\Vdash_{\mathbb{P}_{\mathbf{q}, \alpha+1}} \text{“} \mathcal{V}_{\mathbb{Q}_\mathbf{b}} \in [\mathcal{U}_{\mathbf{a}_\alpha}]^\partial$ *and for every* $\alpha \neq \beta \in \mathcal{V}_{\mathbb{Q}_\mathbf{b}}$, $\mathbf{c}_{\mathbf{a}_\alpha}\{\alpha, \beta\} \in \{\iota_1, \iota_2\}\text{”}$.

   (2) *If* $\mathbf{b} = (\mathbf{a}, p, \bar\iota) \in \mathbf{A}^+$, $\mathrm{cf}(\partial) > \lambda$, *and in* $\mathbf{V}^{\mathbb{P}_\mathbf{a}}$, $\mathbb{Q}_\mathbf{b}$ *satisfies the* λ^+-*cc,* <u>then</u> *for some* $p \in \mathbb{Q}_\mathbf{b} \cap \mathbb{P}_\mathbf{a} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ *we have*[4] $p \Vdash_{\mathbb{Q}_\mathbf{b}} \text{“} \mathcal{V}_{\mathbb{Q}_\mathbf{b}} \in [\mathcal{U}_\mathbf{a}]^\partial$ *and for every* $\alpha \neq \beta \in \mathcal{V}_{\mathbb{Q}_\mathbf{b}}$, $\mathbf{c}_\mathbf{a}\{\alpha, \beta\} \in \{\iota_1, \iota_2\}\text{”}$.

Proof. (1) The second phrase in both conclusion holds by the definitions of $\mathbb{Q}_\mathbf{b}$.

By the proof of "$\mathbb{P}_\mathbf{q}$ satisfies the λ^+-cc", we can show for $\varepsilon < \partial$, the density of the set

$$\mathscr{I}_\varepsilon := \{p \in \mathbb{P}'_\mathbf{q} : \alpha \in \mathrm{dom}(p) \text{ and there is } \beta \in p(\alpha) \text{ such that } \varepsilon < \mathrm{otp}(\mathcal{U}_{\mathbf{a}_\alpha} \cap \beta)\}.$$

---

[4] We may omit p but it does not matter.

(2) Easily, for every $\beta \in \mathscr{U}_{\mathbf{a}}$ we can choose $p^0_\beta = \{\beta\}$, $q_\beta = \{(p, p^0_\beta)\}$. Clearly, $q_\beta \in \mathbb{P}_\alpha * \mathbb{Q}_{\mathbf{b}}$ for $\beta \in \mathscr{U}_{\mathbf{a}}$. So by the λ^+-cc for some $\beta \in \mathscr{U}_{\mathbf{a}}$, $q_\beta \Vdash ``\{\varepsilon \in \mathscr{U}_{\mathbf{a}} : q_\varepsilon \in \mathbb{Q}_{\mathbf{b}}\} \in [\mathscr{U}_{\mathbf{a}}]^\partial$; well assuming $\operatorname{cf}(\theta) > \lambda$. $\square_{1.10}$

Conclusion 1.11. *There exists a forcing notion $\mathbb{P}$ satisfying the following conditions:*

- *(a) $\mathbb{P}$ is λ^+-cc of cardinality μ.*
- *(b) $\mathbb{P}$ is $(<\lambda)$-complete; hence, it collapses no cardinals, changes no cofinalities, and preserves cardinal arithmetic outside the interval $[\lambda, \mu)$.*
- *(c) $\Vdash_{\mathbb{P}} ``2^\lambda = \mu".$*
- *(d) $\Vdash_{\mathbb{P}} ``\theta \to [\partial]^2_{\sigma,2}"$ for every $\sigma \in (2, \lambda)$.*

Proof. Choose a $\leq_{\mathbf{p}}$-increasing continuous sequence $\langle \mathbf{q}_\alpha : \alpha < \mu \rangle \in {}^\mu(\mathbf{Q_p})$ such that $\operatorname{lg}(\mathbf{q}_\alpha) = \alpha$, $\mathbb{P}_{\mathbf{q}_\alpha}$ has cardinality $\leq (|\alpha| + \lambda)^{<\lambda}$ and,

- if $\alpha < \mu$ and $\Vdash_{\mathbb{P}_{\mathbf{q}_\alpha}} ``\underset{\sim}{\mathbf{c}} : [\theta]^2 \to \sigma"$, then for unboundedly many $\beta \in [\alpha, \mu)$, $\underset{\sim}{\mathbf{c}}_{\mathbf{q}_{\beta+1,\beta}} = \underset{\sim}{\mathbf{c}}$.

The existence of $\mathbf{b}_\beta[\mathbf{q}_{\beta+1}]$ with $\underset{\sim}{\mathbf{c}}[\mathbf{b}_\beta[\mathbf{q}_{\beta+1}]] = \underset{\sim}{\mathbf{c}}$ as required hold by Claim 1.8 and Claim 1.9.

Clearly $\bigcup\{\mathbb{P}_{\mathbf{q}_\beta} : \beta < \mu\}$ is a forcing notion as is required. $\square_{1.11}$

Conclusion 1.11 is meaningful because:

Fact 1.12. Assume that $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$, $\alpha < \mu \Rightarrow |\alpha|^\lambda < \mu$, $\theta > \beth_4(\partial)$ and $\partial = \partial^{<\lambda}$. <u>Then</u> the demands in Hypothesis 1.1 hold.

Remark 1.13. To justify the assumption, notice that:

- (A) Omitting $\partial = \partial^{<\lambda}$ does not help.
- (B) $\theta \to_{\mathrm{sq}} (\partial)^{\lambda, \leq 2}_\partial$ implies $\theta \to (\partial)^2_{2^\partial}$, hence $\theta > 2^{2^\partial}$.

With stronger lower bound on θ, see [She89] and anyhow just $\theta < \partial^{+\omega}$ and GCH in $[\partial, \partial^{+\omega}]$ would suffice for me.

The main point is proving $\theta \to_{\mathrm{sq}} (\partial)^{\leq \lambda, 2}_\partial$. For this, see [She89], $\theta = \beth_m(\partial)$ for some small m suffice, we intend to return for better bound, see [S$^+$].

Proof. The point is to prove $\theta \to_{\mathrm{sq}} (\partial)^{\lambda, 2}_\partial$. Let $\mathscr{B}$ be as in 0.7(a).

For $u \subseteq \mathscr{B}$, let N^*_u be the minimal $N \prec \mathscr{B}$ such that $u \subseteq N$, $\partial + 1 \subseteq N$ and $[N]^{<\lambda} \subseteq N$, which exists as $<^{\mathscr{B}}_*$ is a well-ordering of $\mathscr{H}(\chi)$.

We define $\mathbf{c} : [\theta]^3 \to 2^\partial$ such that from $\mathbf{c}(u)$ we can compute the isomorphism type of $(N_u, \alpha)_{\alpha \in u}$. By Erdös-Rado theorem, $\beth_4(\partial)^+ \to ((2^\theta)^+)^3_{2^\partial}$, hence there is $\mathscr{U}_1 \subseteq \theta$ of order type $(2^\partial)^+$ such that $\mathbf{c} \restriction [\mathscr{U}_1]^3$ is constant and $\operatorname{otp}(\mathscr{U}_1) = (2^\partial)^+$.

Clearly, if $\alpha < \beta < \gamma$ are from $\mathscr{U}_1$, then $x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}$ implies $\beta < \gamma(1) \in \mathscr{U}_1 \Rightarrow x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma(1)\}}$, and

$$\alpha < \beta(1) < \gamma(1) \wedge \{\beta(1), \gamma(1)\} \subseteq \mathscr{U}_1 \Rightarrow x \in N^*_{\{\alpha,\beta(1)\}} \cap N^*_{\{\alpha,\gamma(1)\}}.$$

So, for $\alpha \in \mathscr{U}_1$, we let:

- $X^\alpha_0 := \{x : \text{for some } \beta, \gamma \text{ from } \mathscr{U}_1,\ \alpha < \beta < \gamma \text{ and } x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}\}$ have cardinality ∂ and it includes $N^*_{\{\alpha\}}$.

Similarly, X^α_1, X^α_2 has cardinality ∂, where:

- $X^\alpha_1 := \{x : \text{for some } \beta, \gamma \in \mathscr{U}_1, \text{ we have } \beta < \alpha < \gamma \text{ and } x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}\}$, and
- $X^\alpha_2 := \{x : \text{for some } \beta, \gamma \in \mathscr{U}_1, \text{ we have } \beta < \gamma < \alpha \text{ and } x \in N^*_{\{\alpha,\beta\}} \cap N^*_{\{\alpha,\gamma\}}\}$.

For $\alpha \in \mathscr{U}_1$, let $X_\alpha := \bigcup_{\ell=0}^2 X_\ell^\alpha$ and let $N_{\{\alpha\}} := N_{X_\alpha}^*$, so $N_{\{\alpha\}}^*$ has cardinality ∂.

Now,

$(*)_1$ The sets $\langle N_{\{\alpha,\beta\}}^* \setminus (N_{\{\alpha\}} \cup N_{\{\beta\}}) \colon \alpha < \beta$ are from $\mathscr{U}_1 \rangle$ is a sequence of pairwise disjoint sets.

$(*)_2$ If $\gamma \in \mathscr{U}_1$ then $\Lambda_\gamma := \{\{\alpha,\beta\} \colon \alpha < \beta$ are from $\mathscr{U}_1$ and $(N_{\{\alpha,\beta\}}^* \setminus (N_{\{\alpha\}} \cup N_{\{\beta\}})) \cap N_{\{\gamma\}} \neq \emptyset\}$ has cardinality $\leq \partial$.

So for some $\mathscr{U}_2 \subseteq \mathscr{U}_1$ of cardinality $(2^\partial)^+$, we have:

$(*)_3$ (a) $\langle N_{\{\gamma\}} \colon \gamma \in \mathscr{U}_2 \rangle$ is a Δ-system with heart called $N_\emptyset$,
   (b) The $N_{\{\gamma\}}$ for $\gamma \in \mathscr{U}_2$ are pairwise isomorphic over $N_\emptyset$.

Lastly choose $N_{\{\alpha,\beta\}}$ for $\alpha \neq \beta \in \mathscr{U}_2$ as $N_{\{\alpha,\beta\}}^*$. Replacing $\mathscr{U}_2$ by $\mathscr{U} \subseteq \mathscr{U}_2$ of order type ∂, we are done. $\square_{1.12}$

References

[EH71] Paul Erdős and Andras Hajnal, *Unsolved problems in set theory*, Axiomatic Set Theory (Providence, R.I.), Proc. of Symp. in Pure Math., vol. XIII Part I, AMS, 1971, pp. 17–48.

[Kom21] Péter Komjáth, *Notes on some Erdős-Hajnal problems*, Journal of Symbolic Logic **86** (2021), no. 3, 1116–1123.

[RS00] Mariusz Rabus and Saharon Shelah, *Covering a function on the plane by two continuous functions on an uncountable square—the consistency*, Ann. Pure Appl. Logic **103** (2000), no. 1-3, 229–240, arXiv: math/9706223. MR 1756147

[S+] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F2407.

[She88] Saharon Shelah, *Was Sierpiński right? I*, Israel J. Math. **62** (1988), no. 3, 355–380. MR 955139

[She89] _____, *Consistency of positive partition theorems for graphs and models*, Set theory and its applications (Toronto, ON, 1987), Lecture Notes in Math., vol. 1401, Springer, Berlin, 1989, pp. 167–193. MR 1031773

[She92] _____, *Strong partition relations below the power set: consistency; was Sierpiński right? II*, Sets, graphs and numbers (Budapest, 1991), Colloq. Math. Soc. János Bolyai, vol. 60, North-Holland, Amsterdam, 1992, arXiv: math/9201244, pp. 637–668. MR 1218224

[She95] _____, *Possibly every real function is continuous on a non-meagre set*, Publ. Inst. Math. (Beograd) (N.S.) **57(71)** (1995), 47–60, arXiv: math/9511220. MR 1387353

[She96] _____, *Was Sierpiński right? III. Can continuum-c.c. times c.c.c. be continuum-c.c.?*, Ann. Pure Appl. Logic **78** (1996), no. 1-3, 259–269, arXiv: math/9509226. MR 1395402

[She00] _____, *Was Sierpiński right? IV*, J. Symbolic Logic **65** (2000), no. 3, 1031–1054, arXiv: math/9712282. MR 1791363

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 9190401, Jerusalem, Israel; and, Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

URL: https://shelah.logic.at/