Notre Dame Journal of Formal Logic
Volume 65, Number 4, 2024

On $C_n^s(\kappa)$ and the Juhász–Kunen Question

Mohammad Golshani and Saharon Shelah

Abstract We generalize the combinatorial principles $C_n(\kappa)$, $C_n^s(\kappa)$, and $Princ(\kappa)$ introduced by various authors, and prove some of their properties and connections between them. We also answer a question asked by Juhász and Kunen about the relation between these principles, by showing that $C_n(\kappa)$ does not imply $C_{n+1}(\kappa)$ for any $n > 2$. We also show the consistency of $C(\kappa) + \neg C^s(\kappa)$.

1 Introduction

In this paper we study two types of combinatorial principles which are consequences of the continuum hypothesis and all hold in the Cohen-real generic extensions. One type are homogeneity axioms which say that given a long sequence of reals, we can find many reals from the sequence which look alike. The other type are elementary submodel axioms which say that for all large enough regular cardinals χ, we can find many elementary submodels N of $H(\chi)$ of size $\aleph_1$ such that $N \cap \mathcal{P}(\omega)$ captures $\mathcal{P}(\omega)$.

Juhász, Soukup, and Szentmiklossy [5] initiated the study of such principles. Among other things, in particular they introduced the combinatorial principles $C^s(\kappa)$, $C(\kappa)$, and their restrictions $C_m^s(\kappa)$ and $C_m(\kappa)$, for $m < \omega$. They also derived several combinatorial and topological consequences from these principles.

Juhász and Kunen [4] have continued the work by introducing some extra principles. In particular they introduced the combinatorial principle SEP, and explored its connection with the above combinatorial principles by showing that SEP implies $C_2^s(\aleph_2)$, while the reverse inclusion does not hold and indeed even the stronger principle $C^s(\aleph_2)$ does not imply SEP. The question of the difference between $C_n^s(\kappa)$ and $C_{n+1}^s(\kappa)$ remained open by Juhász and Kunen [4], and was asked by Juhász during the Beer-Sheva 2001 Conference.

Received May 12, 2022; accepted August 2, 2024
First published online November 14, 2024
2020 Mathematics Subject Classification: 03E05, 03E35, 03E50
Keywords: infinite combinatorial principles, Cohen extension, Kunen–Juhasz question
 10.1215/00294527-2024-0026

In [7], Shelah introduced a new combinatorial principle $Princ(\kappa)$, which is weaker than SEP, but still enough strong to imply $C^s(\kappa)$.

It turned out that these combinatorial principles are very useful, and have many applications, in particular in topology and the study of cardinal invariants (see, e.g., Brendle and Fuchino [1], Fuchino [3], and Juhász, Soukup, and Szentmiklossy [6]).

In this paper, we present some generalizations of the above principles and prove some of their properties and the connections between them. We also address the above-mentioned question of Juhász and Kunen in Section 5, and give a complete solution to it. In the last section, we discuss the relation between $C^s(\kappa)$ and $C(\kappa)$, and show the consistency of $C(\kappa) + \neg C^s(\kappa)$.

2 On $Princ(\kappa)$ and Its Generalizations

In this section, we consider the combinatorial principle $Prin(\kappa)$ introduced by Shelah [7], and present some of its generalizations.

Definition 2.1 Let κ be regular uncountable, $A \supseteq \kappa$, and let D be a filter on $[A]^{<\kappa}$. D is called normal if

(1) for all $a \in [A]^{<\kappa}$, $\{b \in [A]^{<\kappa} : a \subseteq b\} \in D$, and
(2) if for $x \in A$, $A_x \in D$, then $\triangle_{x\in A} A_x \in D$, where

$$\triangle_{x\in A} A_x = \left\{a \in [A]^{<\kappa} : \forall x \in a, a \in A_x\right\}.$$

It is easily seen that if D is a normal filter on $[A]^{<\kappa}$, $X \neq \emptyset \bmod D$, and if $F : X \to A$ is regressive, that is, for all nonempty $a \in [A]^{<\kappa}$, $F(a) \in a$, then there are $Y \subseteq X$, $Y \neq \emptyset \bmod D$, and $x \in A$ such that for all $a \in Y$, $F(a) = x$. To see this, assume on the contrary that for each $x \in A$, there exists $Y_x \in D$ such that $Y_x \cap \{a \in X : F(a) = x\} = \emptyset$. Let $Y = \triangle_{x\in A} Y_x$. Then $Y \in D$ and so $Y \cap X \neq \emptyset$ (as $X \neq \emptyset \bmod D$). Let $a \in Y \cap X$ and $F(a) = x$. Then $a \in Y_x \cap \{a \in X : F(a) = x\}$, a contradiction.

Definition 2.2 Let κ be regular uncountable. $\mathfrak{D}$ is a κ-definition of normal filters if

(1) for each $A \supseteq \kappa$, $\mathfrak{D}(A)$ is a normal filter on $[A]^{<\kappa}$, and
(2) if $\kappa \subseteq A_1 \subseteq A_2$, then $\mathfrak{D}(A_1) = \{\{a \cap A_1 : a \in X\} : X \in \mathfrak{D}(A_2)\}$.

Definition 2.3 Let κ be regular uncountable, $\theta < \lambda \leq \kappa$, and let $\chi > \kappa$ be large enough regular. Then

(a) $\mathbf{N}^1_{\kappa,\lambda,\chi}$ consists of those $N \prec (H(\chi), \in)$ such that:
  (1) $|N| \leq N \cap \kappa \in \kappa$, and
  (2) for all $a \in P(\omega)$, there exists $P \in N$, such that $P \subseteq P(\omega)$, $|P| < \min\{|N|^+, \lambda\}$, and for all $b \in P(\omega) \cap N$, $a \subseteq b \Rightarrow \exists c \in P, a \subseteq c \subseteq b$ (such a P is called an N-witness for a).
(b) $\mathbf{N}^2_{\kappa,\lambda,\theta,\chi}$ consists of those $N \in \mathbf{N}^1_{\kappa,\lambda,\chi}$ such that for any θ-sequence $\langle a_\xi : \xi < \theta\rangle$ of subsets of ω, there is some $P \in N$, $P \subseteq P(\omega)$, $|P| < \min\{|N|^+, \lambda\}$, such that P is an N-witness for all a_ξ, $\xi < \theta$ simultaneously, namely, for any $b_\xi \in P(\omega) \cap N$, $\xi < \theta$, such that $a_\xi \subseteq b_\xi$, there are $c_\xi \in P$, $\xi < \theta$, such that $a_\xi \subseteq c_\xi \subseteq b_\xi$ for all $\xi < \theta$.

(c) $\mathbf{N}^3_{\kappa,\lambda,\theta,\chi}$ consists of those $N \in \mathbf{N}^1_{\kappa,\lambda,\chi}$ such that for each $Y \in [N]^\theta$, there exists some $Z \in N$, $|Z| < \min\{|N|^+, \lambda\}$ such that $Y \subseteq Z$.

We now state our generalization of $Princ(\kappa)$.

Definition 2.4 Let $\theta < \lambda \leq \kappa$ and $\mathfrak{D}$ be as above.

(a) $Princ_1(\kappa, \lambda, \mathfrak{D})$ states: for all large enough $\chi > \kappa$,

$$\mathbf{N}^1_{\kappa,\lambda,\chi} \neq \emptyset \bmod \mathfrak{D}\big(H(\chi)\big).$$

(b) $Princ_{l,\theta}(\kappa, \lambda, \mathfrak{D})$ (for $l = 2, 3$) states: for all large enough $\chi > \kappa$,

$$\mathbf{N}^l_{\kappa,\lambda,\theta,\chi} \neq \emptyset \bmod \mathfrak{D}\big(H(\chi)\big).$$

Remark 2.5

(a) Let κ be regular uncountable, and for $A \supseteq \kappa$, let $\mathfrak{D}(A)$ be the club filter on $[A]^{<\kappa}$. Then our $Princ_1(\kappa, \lambda, \mathfrak{D})$ is just $Princ(\kappa, \lambda)$ from [1]. Also note that Shelah's $Princ(\kappa)$ is $Princ(\kappa, \kappa)$.
(b) If $\theta < \lambda \leq \lambda' \leq \kappa$, then $Princ_1(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_1(\kappa, \lambda', \mathfrak{D})$ and $Princ_{l,\theta}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_{l,\theta}(\kappa, \lambda', \mathfrak{D})$ (for $l = 2, 3$).
(c) If $\theta \leq \theta' < \lambda \leq \kappa$, then $Princ_{l,\theta'}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_{l,\theta}(\kappa, \lambda, \mathfrak{D})$ (for $l = 2, 3$).
(d) If $\lambda = \mu^+$ is a successor cardinal, then we can replace $\min\{|N|^+, \lambda\}$ by λ.

The next lemma follows from the definition, and the fact that we can code an ω-sequence of subsets of ω into a subset of ω.

Lemma 2.6 *Let $\theta < \lambda \leq \kappa$ and $\mathfrak{D}$ be as above. Then*

(a) $Princ_{3,\theta}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_{2,\theta}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_1(\kappa, \lambda, \mathfrak{D})$.
(b) $Princ_{2,\omega}(\kappa, \lambda, \mathfrak{D}) \Leftrightarrow Princ_1(\kappa, \lambda, \mathfrak{D})$.

Proof (a) is by definition; let's prove (b). It suffices to show that

$$\mathbf{N}^2_{\kappa,\lambda,\omega,\chi} = \mathbf{N}^1_{\kappa,\lambda,\chi}.$$

Let $\Gamma : \omega \times \omega \to \omega$ be the Godel pairing function. Let $N \in \mathbf{N}^1_{\kappa,\lambda,\chi}$, and suppose that $\langle a_n : n < \omega\rangle$ is a sequence of subsets of ω. Let

$$a^* = \big\{\Gamma(i,n) : n < \omega, i \in a_n\big\}.$$

Let $P^* \in N$ be an N-witness for a^*. Let

$$P = \big\{\{i : \Gamma(i,n) \in b\} : n < \omega, b \in P^*\big\}.$$

We show that P is an N-witness for all $a_n, n < \omega$, simultaneously. Clearly $P \in N$, $P \subseteq P(\omega) \cap N$, and $|P| < \min\{|N|^+, \lambda\}$. Now let $n < \omega$, $b \in P(\omega) \cap N$ and assume $a_n \subseteq b$. Let $b^{[n]} = \{\Gamma(i,m) : i, m < \omega \text{ and } m = n \Rightarrow i \in b\}$. Clearly $b^{[n]} \in P(\omega) \cap N$ and $a^* \subseteq b^{[n]}$. Hence by the choice of P^*, there is $c \in P^*$ such that $a^* \subseteq c \subseteq b^{[n]}$. Let $c^{[n]} = \{i : \Gamma(i,n) \in c\}$. Then $c^{[n]} \in P$, and we can easily see that $a_n \subseteq c^{[n]} \subseteq b$. We are done. □

3 On $C^s(\kappa)$ and Its Generalizations

Recall that for a filter D on a set I, D^+ is defined by

$$D^+ = \{X \subseteq I : I \setminus X \notin D\}.$$

It is clear that $D \subseteq D^+$.

Definition 3.1 Suppose κ is regular uncountable, D is a filter on κ, J is an ideal on ω, and T is a subtree of $\theta^{<\omega}$.

(a) The combinatorial principle $C^D_T(\kappa, J)$ states: for any $(\kappa \times \theta)$-matrix $\bar{A} = \langle a_{\alpha,\xi} : \alpha < \kappa, \xi < \theta\rangle$ of subsets of ω, one of the following holds:
  - (α) : There exists $S \in D^+$ such that for all $n \in \omega$, all $t \in T \cap \theta^n$, and all distinct $\alpha_0, \dots, \alpha_{n-1} \in S$, $\bigcap_{i<n} a_{\alpha_i,t(i)} \neq \emptyset \bmod J$.
  - (β) : There are $t \in T \cap \theta^n$, for some $0 < n < \omega$, and $S_0, \dots, S_{n-1} \in D^+$ such that for all distinct $\alpha_i \in S_i$, $i < n$, we have $\bigcap_{i<n} a_{\alpha_i,t(i)} = \emptyset \bmod J$.

(b) $C^D(\kappa, J)$ is $C^D_T(\kappa, J)$ for all trees $T \subseteq \theta^{<\omega}$.

(c) For $m < \omega$, the combinatorial principles $C^D_{T,m}(\kappa, J)$ and $C^D_m(\kappa, J)$ are defined similarly, where we require $T \subseteq \theta^{\leq m}$.

Remark 3.2 Suppose that κ is regular uncountable and $m < \omega$.

(a) If D is the club filter on κ, and $J = \{\emptyset\}$, then $C^D(\kappa, J)$, $C^D_m(\kappa, J)$ are respectively the principles $C^s(\kappa)$, $C^s_m(\kappa)$ from [5].

(b) If D is the filter of co-bounded subsets of κ, and $J = \{\emptyset\}$, then $C^D(\kappa, J)$, $C^D_m(\kappa, J)$ are respectively the principles $C(\kappa)$, $C_m(\kappa)$ from [5].

Theorem 3.3 *Assume $\theta < \kappa \leq 2^{\aleph_0}$, κ is regular, J is an ideal on ω, and T is a subtree of $\theta^{<\omega}$, and suppose that $Prin_{2,\theta}(\kappa, \kappa, \mathfrak{D})$ holds, where $\mathfrak{D}$ is a definition of κ-normal filters. Then $C^D_T(\kappa, J)$ holds, where D is any filter on κ satisfying: for $X \in D$ and $N \in \mathbf{N}^2_{\kappa,\lambda,\theta,\chi}$ with $D \in N$, $X \in N \Rightarrow \delta(N) = N \cap \kappa \in X$.*

Remark 3.4 If D is the club filter on κ or the filter of co-bounded subsets of κ, then D has the above-mentioned property.

Proof Let $\bar{A} = \langle a_{\alpha,\xi} : \alpha < \kappa, \xi < \theta\rangle$ be a $(\kappa \times \theta)$-matrix of subsets of ω. Let $\chi > 2^{\aleph_0}$ be large enough regular. By our assumption

$$\mathbf{N}^2_{\kappa,\kappa,\theta,\chi} \neq \emptyset \bmod \big(\mathfrak{D}\big(H(\chi)\big)\big)^+.$$

Hence by normality of the filter,

$$\mathcal{N} = \{N \in \mathbf{N}^2_{\kappa,\kappa,\theta,\chi} : D, \bar{A} \in N\} \in \mathfrak{D}\big(H(\chi)\big)^+.$$

For $N \in \mathcal{N}$, set $\delta(N) = N \cap \kappa \in \kappa$. By our assumption, for each $N \in \mathcal{N}$, we can find $P_N \in N$ such that P_N is an N-witness for each $a_{\delta(N),\xi}$, $\xi < \theta$, simultaneously. Then the map $N \mapsto P_N$ is regressive on $\mathcal{N}$, so by the normality of the filter $\mathfrak{D}(H(\chi))$, we can find $\mathcal{N}_* \subseteq \mathcal{N}$ and P_* such that $\mathcal{N}_* \in \mathfrak{D}(H(\chi))^+$, and for all $N \in \mathcal{N}_*$, $P_N = P_*$. Let

$$S = \big\{\delta(N) : N \in \mathcal{N}_*\big\}.$$

Claim 3.5 $S \in D^+$.

Proof Suppose not; so $\kappa \setminus S \in D$. But then for all $N \in \mathcal{N}_*$,

$$\kappa \setminus S \in N \Rightarrow \delta(N) \in \kappa \setminus S.$$

On the other hand, by normality of the filter $\mathfrak{D}(H(\chi))$, we have

$$\mathcal{N}_{**} = \{N \in \mathcal{N}_* : \kappa \setminus S \in N\} \in \mathfrak{D}\big(H(\chi)\big)^+,$$

in particular $\mathcal{N}_{**} \neq \emptyset$. Let $N \in \mathcal{N}_{**}$. Then we have $\kappa \setminus S \in D$, which implies $\delta(N) \in \kappa \setminus S$. But on the other hand, $N \in \mathcal{N}_*$ (as $\mathcal{N}_{**} \subseteq \mathcal{N}_*$), which implies $\delta(N) \in S$, a contradiction. □

If for all $t \in T \cap \theta^n$ and all distinct $\alpha_0, \ldots, \alpha_{n-1} \in S$, we have $\bigcap_{i<n} a_{\alpha_i, t(i)} \neq \emptyset$ mod J, then case (α) of Definition 3.1(a) holds and we are done. Otherwise, we can find $t \in T \cap \theta^n$ and distinct $\alpha_0, \ldots, \alpha_{n-1} \in S$, such that $\bigcap_{i<n} a_{\alpha_i, t(i)} = \emptyset$ mod J.

For each $i < n$, let $N_i \in \mathcal{N}_*$ be such that $\alpha_i = \delta(N_i)$. We also assume without loss of generality that $\alpha_0 < \cdots < \alpha_{n-1}$.

Claim 3.6 *There are $c_0, \ldots, c_{n-1} \in P_*$ such that:*

(1) $\bigcap_{i<n} c_i = \emptyset$ *mod* J,
(2) $i < n \Rightarrow a_{\alpha_i, t(i)} \subseteq c_i$ *mod* J.

Proof We construct the sets $c_i, i < n$ by downward induction on i, so that for all $i < n$,

$$\bigcap_{j<i} a_{\alpha_j, t(j)} \cap \bigcap_{i \leq j < n} c_j = \emptyset \text{ mod } J. \qquad (*)_i$$

For $i = n$, there is nothing to prove; thus, suppose that $i < n$ and $c_{i+1} \in P_*$ is defined, so that $(*)_{i+1}$ is satisfied. It then follows that

$$a_{\alpha_i, t(i)} \subseteq b_i = \omega \setminus \Big(\bigcap_{j<i} a_{\alpha_j, t(j)} \cap \bigcap_{i+1 \leq j < n} c_j\Big) \text{ mod } J.$$

It is easily seen that $b_i \in N_i$, so as P_* is an N_i-witness for $a_{\alpha_i, t(i)}$, we can find $c_i \in P_*$ so that

$$a_{\alpha_i, t(i)} \subseteq c_i \subseteq b_i \text{ mod } J.$$

It is easily seen that $c_0, \ldots, c_{n-1}$ are as required. □

For $i < n$, set

$$S_i = \{\alpha \in \kappa : a_{\alpha, t(i)} \subseteq c_i \text{ mod } J\} \in N_i.$$

Claim 3.7 *For each $i < n$, $S_i \in D^+$.*

Proof Suppose not; then $\kappa \setminus S_i \in D$. But as $\kappa \setminus S_i \in N_i$, we have $\alpha_i = \delta(N_i) \in \kappa \setminus S_i$, which is a contradiction. □

Now if $\beta_i \in S_i$ are distinct, then

$$\bigcap_{i<n} a_{\beta_i, t(i)} \subseteq \bigcap_{i<n} c_i = \emptyset \text{ mod } J,$$

and hence case (β) of Definition 3.1(a) holds and we are done. The theorem follows. □

Corollary 3.8 *Assume $\kappa \leq 2^{\aleph_0}$ is regular uncountable. Then $Princ(\kappa)$ implies $C^s(\kappa)$.*

4 Forcing $Princ_1(\kappa, \kappa, \mathfrak{D})$

In this section we consider the principles $Princ_1(\kappa, \lambda, \mathfrak{D})$ and $Princ_{2,\theta}(\kappa, \lambda, \mathfrak{D})$, where $\theta < \lambda \leq \kappa = cf(\kappa)$ and $\mathfrak{D}$ is a κ-definition of normal filters, and discuss their consistency. In fact, we will show that in the generic extension by the Cohen forcing $Add(\omega, \kappa)$, the above principles hold. We prove the result for $Princ(\kappa)$, as the other cases can be proved similarly.

Recall that the Cohen forcing $Add(\omega, I)$ for adding $|I|$-many new Cohen subsets of ω is defined as

$$Add(\omega, I) = \{p : \omega \times I \to 2 : |p| < \aleph_0\},$$

ordered by reverse inclusion.

For a nice name $\underset{\sim}{a} = \bigcup_{n<\omega}\{\check{n}\} \times A_n$, where each A_n is a maximal antichain in $Add(\omega, \lambda)$, set

$$\text{supp}(\underset{\sim}{a}) = \{\alpha \in \lambda : \exists n < \omega, \exists p \in A_n, \exists k \in \omega, (k, \alpha) \in \text{dom}(p)\}.$$

Note that, by the countable chain condition property of $Add(\omega, \lambda)$, $\text{supp}(\underset{\sim}{a})$ is a countable set, and $\underset{\sim}{a}$ can be considered as an $Add(\omega, \text{supp}(\underset{\sim}{a}))$-name. The following lemma follows easily by an absoluteness argument.

Lemma 4.1 *Assume $U \subseteq \lambda$, $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ are $Add(\omega, U)$-names, $\varphi(v_1, \ldots, v_n)$ is a Δ_1^{ZFC}-formula, and $p \in Add(\omega, \lambda)$. Then*

$$p \Vdash_{Add(\omega,\lambda)} \text{“}\varphi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\text{”} \iff p \restriction \omega \times U \Vdash_{Add(\omega,U)} \text{“}\varphi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\text{”}.$$

We also need the following simple observation.

Lemma 4.2 *Let $\mathfrak{D}$ be defined by $\mathfrak{D}(A) =$ the club filter on $[A]^{<\kappa}$. The following are equivalent:*

(a) $Princ(\kappa)$.
(b) *For all large enough $\chi > \kappa$ and $x \in H(\chi)$, there exists $N \in N^1_{\kappa,\kappa,\chi}$ such that $x \in N$.*

Proof It is clear that $(a) \implies (b)$. To show that (b) implies (a), let $\chi > \kappa$ be large enough regular, $x \in H(\chi)$ and let $C \subseteq [H(\chi)]^{<\kappa}$ be a club set. We need to show that $N^1_{\kappa,\kappa,\chi} \cap C \neq \emptyset$. We assume $|M| \leq M \cap \kappa$ for all $M \in C$. Take $\chi' > \chi$ large enough regular. By the assumption, we can find $N' \in N^1_{\kappa,\kappa,\chi'}$ such that $x, C \in N'$. Let $N = N' \cap H(\chi)$. Then by elementarity, $N \in N^1_{\kappa,\kappa,\chi}$ and $N = \bigcup(N' \cap C)$. Since C is closed, $N \in C$ and so $N \in N^1_{\kappa,\kappa,\chi} \cap C$, as required. $\square$

We are now ready to show that $Princ(\kappa)$ holds in the generic extension by Cohen forcing. We follow the proof in Fuchino [2].

Theorem 4.3 *Assume $\lambda \geq \kappa = cf(\kappa) > 2^{\aleph_0}$. Then $\Vdash_{Add(\omega,\lambda)}$ “$Princ(\kappa)$”.*

Proof Let $\chi > \lambda$ and $p \in Add(\omega, \lambda)$ be such that

$$p \Vdash_{Add(\omega,\lambda)} \text{“}\underset{\sim}{X} \text{ has transitive closure of cardinality } < \chi\text{”}.$$

Let $\langle N_i : i < \delta\rangle$ be a sequence of elementary submodels of $(H(\chi), \in)$ such that:

(1) $\delta < \kappa$, $cf(\delta) > \aleph_0$,
(2) $\langle N_i : i < \delta\rangle$ is increasing continuous,
(3) $N_i \cap \kappa \in \kappa$,
(4) $\langle N_j : j \leq i\rangle \in N_{i+1}$,

(5) each $N_{i+1}, i < \delta$, is closed under countable sequences,
(6) $|N_i| < \kappa$,
(7) $p, \lambda, \underset{\sim}{X} \in N_0$, and
(8) $|N_i \cap \lambda|$ is constant, for $i < \delta$.

Let $N = \bigcup_{i<\delta} N_i$. As $cf(\delta) > \aleph_0$, it follows from clause (5) that N is closed under countable sequences.

We show that $p \Vdash$"$\bigcup_{i<\delta} N_i[\dot{G}] \in N^1_{\kappa,\kappa,\chi}$". Let G be $Add(\omega, \lambda)$-generic over V with $p \in G$, $N_i^* = N_i[G]$, $i < \delta$, and $N^* = \bigcup_{i<\delta} N_i^*$. Note that N^* is closed under countable sequences and $\underset{\sim}{X}[G] \in N^*$. We show that $N^* \in (N^1_{\kappa,\kappa,\chi})^{V[G]}$. This will complete the proof by the previous lemma.

Thus assume that $a \in P(\omega)^{V[G]}$ and let $\underset{\sim}{a}$ be a nice name for a. Let $U = \mathrm{supp}(\underset{\sim}{a})$.

Set $U_1 = U \cap N$ and $U_2 = U \setminus N$. Then $U_1 \in N$. Let $i < \delta$ be sufficiently large such that $U_1 \subseteq N_i$ and $|N \cap \lambda| = |N_i \cap \lambda|$[1] and set $M = N_{i+1}$. It follows from (5) that $U_1 \in M$.

Let $\pi : \lambda \simeq \lambda$ be a bijection such that $\pi[U] \subseteq M$, $\pi \restriction U_1 = id \restriction U_1$, and $\pi[\lambda \cap N] \subseteq M$. Using the homogeneity of the forcing $Add(\omega, \lambda)$, extend π to an isomorphism $\pi : Add(\omega, \lambda) \simeq Add(\omega, \lambda)$. Note that this also induces an isomorphism of the class of all $Add(\omega, \lambda)$-names, $V^{Add(\omega,\lambda)}$, that we still denote it by π. Note that $\pi[U], \pi(\underset{\sim}{a}) \in M$, as M is closed under countable sequences. Let

$$P = \{\underset{\sim}{c}_r[G] : r \in Add(\omega, \lambda \cap M \setminus U)\},$$

where for $r \in Add(\omega, \lambda \cap M \setminus U)$,

$$\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{c}_r = \omega \setminus \{n \in \omega : \exists q \in \dot{G} \cap Add(\omega, U_1),\ r \cup q \Vdash \text{“}n \notin \pi(\underset{\sim}{a})\text{”}\}\text{”}.$$

Note that $\lambda \cap M \setminus U = \lambda \cap M \setminus U \cap M \in N$, so $Add(\omega, \lambda \cap M \setminus U) \in N$. Also $U_1 \in N$ so $Add(\omega, U_1) \in N$. It easily follows that $P \in N^*$. It is also clear that $P \subseteq P(\omega)^{V[G]}$ and $|P| \leq |Add(\omega, \lambda \cap M \setminus U)| = |M| < \kappa$.

To show that P is an N^*-witness for a, let $b \in P(\omega)^{V[G]} \cap N^*$ and $b \supseteq a$. Let $\underset{\sim}{b} \in N$ be a nice name for b and let $W = \mathrm{supp}(\underset{\sim}{b})$. Let $p^* \leq p$ be such that $p^* \Vdash$"$\underset{\sim}{b} \supseteq \underset{\sim}{a}$". By Lemma 4.1, we may suppose that $p^* \in Add(\omega, U \cup W)$. Let

$$r = \pi\big(p^* \restriction \omega \times (\lambda \setminus U_1)\big).$$

Then $r \in Add(\omega, \lambda \cap M \setminus U)$. We complete the proof by showing that $a \subseteq \underset{\sim}{c}_r[G] \subseteq b$.

<u>$a \subseteq \underset{\sim}{c}_r[G]$</u>**:** assume by contradiction that $n \in a \setminus \underset{\sim}{c}_r[G]$. Let $q \in G$, $q \leq p^*$ and

$$q \Vdash \text{“}n \in \underset{\sim}{a}\text{”}.$$

Again we can suppose that $q \in Add(\omega, U \cup W)$. As $n \notin \underset{\sim}{c}_r[G]$, we can find $q^* \in G \cap Add(\omega, U_1)$ such that

$$r \cup q^* \Vdash \text{“}n \notin \pi(\underset{\sim}{a})\text{”}.$$

This implies

$$\pi^{-1}(r) \cup \pi^{-1}(q^*) \Vdash \text{“}n \notin \underset{\sim}{a}\text{”}.$$

Note that $\pi^{-1}(r) \cup \pi^{-1}(q^*) = p^* \restriction \omega \times (\lambda \setminus U_1) \cup q^*$. As $q^*, q \restriction \omega \times U_1 \in G \cap Add(\omega, U_1)$, they are compatible, and we can easily conclude that q and $p^* \restriction \omega \times (\lambda \setminus U_1) \cup q^*$ are compatible, which is a contradiction, as they decide the statement "$n \in \underset{\sim}{a}$" in different ways.

$\underset{\sim}{c}_r[G] \subseteq b$: suppose by contradiction that there is some $n \in \underset{\sim}{c}_r[G] \setminus b$. Let $q \in G, q \leq p^*$ be such that $q \Vdash$"$n \notin \underset{\sim}{b}$". We can suppose that $q \in Add(\omega, U \cup W)$ and $q \restriction U_2 = p^* \restriction U_2$. As $p^* \Vdash$"$\underset{\sim}{b} \supseteq \underset{\sim}{a}$", we have $q \Vdash$"$n \notin \underset{\sim}{a}$", and hence $q \restriction U \Vdash$"$n \notin \underset{\sim}{a}$". Applying π, we have

$$\pi(q \restriction U) \Vdash \text{“}n \notin \pi(\underset{\sim}{a})\text{”}.$$

Hence

$$q \restriction U_1 \cup \pi(q \restriction U_2) \Vdash \text{“}n \notin \pi(\underset{\sim}{a})\text{”},$$

which implies

$$q \restriction U_1 \cup \pi(p^* \restriction U_2) \Vdash \text{“}n \notin \pi(\underset{\sim}{a})\text{”}.$$

Now observe that $r \leq \pi(p^* \restriction U_2)$ and r is compatible with $q \restriction U_1$, so

$$r \cup q \restriction U_1 \Vdash \text{“}n \notin \pi(\underset{\sim}{a})\text{”}.$$

Thus, $r \cup q \restriction U_1$ witnesses $n \notin \underset{\sim}{c}_r[G]$, which is a contradiction.

The theorem follows. □

The next theorem can be proved as in Theorem 4.3.

Theorem 4.4 *Assume $\theta < \lambda \leq \kappa = cf(\kappa)$ and $\mathfrak{D}$ is a κ-definition of normal filters. Then $\Vdash_{Add(\omega,\kappa)}$"$Princ_1(\kappa, \lambda, \mathfrak{D}) + Princ_{2,\theta}(\kappa, \lambda, \mathfrak{D})$".*

Remark 4.5 In $V[G]$, $\mathfrak{D}$ is defined as follows: for any large enough $\chi > \kappa$, $\mathfrak{D}(H^{V[G]}(\chi))$ is the filter generated by $\{N[G] : N \in X\}$, where $X \in \mathfrak{D}(H(\chi))$. Note that $N \prec H(\chi) \Rightarrow N[G] \prec H(\chi)[G] = H(\chi)^{V[G]}$, and so the above definition is well defined.

5 On a Question of Juhász and Kunen

In this section we answer a question of Juhász and Kunen [4] by showing that for $n \geq 2$, $C_n(\aleph_2) \nRightarrow C_{n+1}(\aleph_2)$. In fact we prove the following stronger result.

Remark 5.1 The results of this section are stated and proved for the ideal $J = [\omega]^{<\omega}$, but all of them are valid if we also assume $J = \{\emptyset\}$.

Theorem 5.2 *Assume:*

(1) $\aleph_0 < \theta = \theta^{<\theta} < \kappa = cf(\kappa) < \chi$ *and* $\forall \alpha < \chi(|\alpha|^{<\theta} < \chi)$,
(2) *D is the filter of co-bounded subsets of κ, or $D = \{S \subseteq \kappa : S \cup \{cf(\delta) < \theta\}$ contains a club$\}$,*
(3) $J = [\omega]^{<\omega}$, *and*
(4) $2 < n(*) < \omega$.

Then there is a cofinality preserving generic extension of the universe in which $C^D_n(\kappa, J)$ holds if $n < n(*)$, and fails if $n = n(*)$.[2]

The rest of this section is devoted to the proof of the above theorem. The forcing notion we define is of the form $\mathbb{P}_\chi * \underset{\sim}{\mathbb{Q}}_{\kappa,\mathcal{A}}$, where $\mathbb{P}_\chi$ is a suitable iteration of length χ, which adds a set $\mathcal{A} \subseteq [\kappa]^{<\aleph_0}$, which has nice enough properties. Then we use this added set $\mathcal{A}$ to define the forcing notion $\mathbb{Q}_{\kappa,\mathcal{A}}$.

In Section 5.1 we define the notion of having the Δ-system θ-property for a filter D, and show that under suitable conditions, some filters have this property. In Section 5.2 we define the forcing notion $\mathbb{P}_\chi$ and prove its basic properties. Section 5.3

is devoted to the definition of the forcing notion $\mathbb{Q}_{\kappa,\mathcal{A}}$. Finally in Section 5.4 we complete the proof of the above theorem.

5.1 Filters with the Δ-system θ-property In this subsection we prove a generalized version of Δ-system lemma that will be used several times later.

Definition 5.3 Let D be a filter on κ, and let $\theta < \kappa$ be a cardinal. D has the Δ-system θ-property if for any $Y \subseteq \kappa$, $Y \neq \emptyset \mod D$, and any sequence $\langle B_\alpha : \alpha \in Y\rangle$ of sets of cardinality $< \theta$, there exists $Z \subseteq Y$, $Z \neq \emptyset \mod D$ such that $\langle B_\alpha : \alpha \in Z\rangle$ forms a Δ-system, that is, there is B^* such that for all $\alpha \neq \beta$, both in Z, $B_\alpha \cap B_\beta = B^*$.

The following is essentially due to Erdos and Rado; we will present a proof for completeness.

Lemma 5.4 *Suppose κ is regular uncountable and $\forall\alpha < \kappa(|\alpha|^{<\theta} < \kappa)$.*

(a) *If D is a normal filter on κ and $\{\delta < \kappa : cf(\delta) \geq \theta\} \in D$, then D has the the Δ-system θ-property.*

(b) *If D is the filter of co-bounded subsets of κ, then D has the Δ-system θ-property.*

(c) *If $D = \{S \subseteq \kappa : S \cup \{cf(\delta) < \theta\}$ contains a club$\}$, then D has the Δ-system θ-property.*

Proof (*a*) Let $Y \subseteq \kappa$, $Y \neq \emptyset \mod D$, and suppose that $\langle B_\alpha : \alpha \in Y\rangle$ is a sequence of sets of cardinality $< \theta$. As $|\bigcup_{\alpha\in Y} B_\alpha| \leq \kappa$, we can assume that all B_α's, $\alpha \in Y$, are subsets of κ. Also as $\{\delta < \kappa : cf(\delta) \geq \theta\} \in D$, we can assume that $Y \subseteq \{\delta < \kappa : cf(\delta) \geq \theta\}$. Define the function g on Y by $g(\alpha) = \sup(B_\alpha \cap \alpha)$. Then for all $\alpha \in Y$, $g(\alpha) < \alpha$ (as $|B_\alpha| < \theta$ and $cf(\alpha) \geq \theta$), so by normality of D, we can find $Y_1 \subseteq Y$, $Y_1 \neq \emptyset \mod D$, and $\xi < \kappa$ such that for all $\alpha \in Y_1$, $g(\alpha) = \xi$. Then

$$\alpha \in Y_1 \Rightarrow B_\alpha \cap \alpha = B_\alpha \cap \xi.$$

As there are only $|\xi|^{<\theta} < \kappa$ many subset of ξ of cardinality $< \theta$, and since D is normal, there are $Y_2 \subseteq Y_1$, $Y_2 \neq \emptyset \mod D$ and a set B^* such that for all $\alpha \in Y_2$, $B_\alpha \cap \alpha = B_\alpha \cap \xi = B^*$. Let

$$X = \big\{\alpha < \kappa : \forall\xi \in Y_2 \cap \alpha\big(\sup(B_\xi) < \alpha\big)\big\}.$$

X is a club of κ, and hence $X \in D$ (as D contains the club filter by its normality). Set $Z = X \cap Y_2$. Then $Z \subseteq Y$, $Z \neq \emptyset \mod D$, and $\langle B_\alpha : \alpha \in Z\rangle$ forms a Δ-system with root B^*.

(*b*) and (*c*) follow from (*a*). □

The following lemma will be used in the proof of Theorem 5.2.

Lemma 5.5 *Let D be the filter of co-bounded subsets of κ, or $D = \{S \subseteq \kappa : S \cup \{cf(\delta) < \theta\}$ contains a club$\}$. Then "D has the Δ-system θ-property" is preserved under θ-closed θ^+-c.c. forcing notions in the following sense:*

> *Suppose $\mathbb{P}$ is a θ-closed θ^+-c.c. forcing notion and G is $\mathbb{P}$-generic over V, and let $D \in V$ be a filter on $\kappa > \theta$ which has the Δ-system θ-property. Let $\tilde{D}$ be the filter generated by D in $V[G]$. Then in $V[G]$, $\tilde{D}$ has the Δ-system θ-property.*

Proof This is trivial using Lemma 5.4 and the fact that $\tilde{D}$ will be of the same kind of filter in $V[G]$. $\square$

5.2 On the forcing notion $\mathbb{P}_\chi$ Fix $n(*)$, θ, κ, χ, and D as in Theorem 5.2. We describe a cofinality preserving forcing notion $\mathbb{P}_\chi$ which adds a set $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$ which has some nice properties.

Definition 5.6 $\mathbb{P}_\chi = \langle\langle\mathbb{P}_i : i \leq \chi\rangle, \langle\underset{\sim}{\mathbb{Q}}_i : i < \chi\rangle\rangle$ is defined as a $(<\theta)$-support iteration of forcing notions such that:

(1) $(\mathbb{Q}_0, \leq)$ is defined by:

(1-1) $p \in \mathbb{Q}_0$ iff $p = (w^p, \mathcal{A}^p)$, where $w^p \in [\kappa]^{<\theta}$ and $\mathcal{A} \subseteq [w]^{n(*)}$.

(1-2) $p \leq q \Leftrightarrow w^q \subseteq w^p$ and $\mathcal{A}^q = \mathcal{A}^p \cap [w^q]^{n(*)}$.

Also let $\underset{\sim}{\mathcal{A}} = \bigcup\{\mathcal{A}^p : p \in \dot{G}_{\mathbb{Q}_0}\}$.

(2) Assume $0 < i < \chi$, and $\mathbb{P}_i$ is defined. Then for some $\mathbb{P}_i$-names $\underset{\sim}{Y}_i$ and $\langle\underset{\sim}{w}^*_i, \langle\underset{\sim}{w}^i_\alpha : \alpha \in \underset{\sim}{Y}_i\rangle\rangle$ we have:

(2-1) $\Vdash_{\mathbb{P}_i}$"$\underset{\sim}{Y}_i$ is a subset of κ, $\underset{\sim}{Y}_i \neq \emptyset \bmod D$."

(2-2) $\Vdash_{\mathbb{P}_i}$"$\langle\underset{\sim}{w}^i_\alpha : \alpha \in \underset{\sim}{Y}_i\rangle$ is a Δ-system of subsets of κ with root $\underset{\sim}{w}^*_i$, each of cardinality $\leq \aleph_0$."

(2-3) $\Vdash_{\mathbb{P}_i}$"$\underset{\sim}{\mathbb{Q}}_i = \{u \subseteq \underset{\sim}{Y}_i : |u| < \theta$ and if $m < n(*), \alpha_0, \ldots, \alpha_{m-1} \in u$ are distinct, then for all $y \in \underset{\sim}{\mathcal{A}}$, $y \subseteq \bigcup_{l<m} \underset{\sim}{w}^i_{\alpha_l} \Rightarrow \exists l < m, y \subseteq \underset{\sim}{w}^i_{\alpha_l}\}$."

(2-4) $\Vdash_{\mathbb{P}_i}$"$\leq_{\underset{\sim}{\mathbb{Q}}_i} = \supseteq$."

(3) If $\underset{\sim}{Y}$ and $\langle\underset{\sim}{w}^*, \langle\underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y}\rangle\rangle$ are $\mathbb{P}_i$-names of objects as above, then for some $j \in (i, \chi)$, they are of the form $\underset{\sim}{Y}_j$ and $\langle\underset{\sim}{w}^*_j, \langle\underset{\sim}{w}^j_\alpha : \alpha \in \underset{\sim}{Y}_j\rangle\rangle$.

Remark 5.7 (a) (3) can be achieved by a bookkeeping argument, and using the fact that the forcing $\mathbb{P}_\chi$ satisfies the θ^+-c.c. (see below).

(b) It also follows from the θ^+-chain condition of the forcing that under the same assumptions as (3), $\Vdash_{\mathbb{P}_j}$"$\underset{\sim}{Y} \neq \emptyset \bmod D$", so $\mathbb{Q}_j$ is well defined.

Lemma 5.8 *Let $\alpha \leq \chi$.*

(a) *$\mathbb{P}^*_\alpha$ is a dense subset of $\mathbb{P}_\alpha$, where $\mathbb{P}^*_\alpha$ consists of those $p \in \mathbb{P}_\alpha$ such that:*

(1) *$i \in \mathrm{dom}(p) \Rightarrow p(i)$ is an object (and not just a $\mathbb{P}_i$-name),*

(2) *$0 \in \mathrm{dom}(p)$ and for some w, we have $w^{p(0)} = w$, and*

(3) *if $i \in \mathrm{dom}(p)$, then for all $j \in p(i)$, $p \restriction i$ decides $\underset{\sim}{w}^*_i$, $\underset{\sim}{w}^i_j$.*

(b) *If $\alpha < \chi$, then $\Vdash_{\mathbb{P}_\alpha}$"$\underset{\sim}{\mathbb{Q}}_\alpha$ is θ^+-Knaster."*

(c) *Each $\mathbb{P}_\alpha$ satisfies the θ^+-c.c.*

Proof (a) follows easily by induction on α, and using the fact that $\Vdash_{\mathbb{P}_i}$"$\underset{\sim}{\mathbb{Q}}_i$ is θ-closed." Let's present a proof for completeness.

Case 1. $\alpha = 0$**:** there is noting to prove.

Case 2. $\alpha + 1$ **is a successor ordinal:** thus assume that $\mathbb{P}^*_\alpha$ is a dense subset of $\mathbb{P}_\alpha$, and let $p \in \mathbb{P}_{\alpha+1}$. Then $p \restriction \alpha \in \mathbb{P}_\alpha$, so for some $p_1 \in \mathbb{P}^*_\alpha$, $p_1 \leq_{\mathbb{P}_\alpha} p \restriction \alpha$. Since $\Vdash_{\mathbb{P}_\alpha}$"$\underset{\sim}{\mathbb{Q}}_\alpha$ is θ-closed and $|p(\alpha)| < \theta$," we can find q_1 such that $p_1 \Vdash$"$p(\alpha) = q_1$." As $|q_1| < \theta$, and again using $\Vdash_{\mathbb{P}_\alpha}$"$\underset{\sim}{\mathbb{Q}}_\alpha$ is θ-closed," we can find $p_2 \leq_{\mathbb{P}_\alpha} p_1$, $p_2 \in \mathbb{P}^*_\alpha$, $q_2 \leq_{\mathbb{Q}_\alpha} q_1$, and w^*_α, w^α_j for $j \in q_2$ such that for all $j \in q_2$, $p_2 \Vdash$"$\underset{\sim}{w}^*_\alpha = w^*_\alpha$ and $\underset{\sim}{w}^\alpha_j = w^\alpha_j$." Then $(p_2, q_2) \in \mathbb{P}^*_{\alpha+1}$ and $(p_2, q_2) \leq_{\mathbb{P}_{\alpha+1}} p$.

Case 3. α **is a limit ordinal,** $cf(\alpha) \geq \theta$**:** let $p \in \mathbb{P}_\alpha$. Then as $|\operatorname{dom}(p)| < \theta$, we can find $\beta < \alpha$ such that $\operatorname{dom}(p) \subseteq \beta$, so $p \in \mathbb{P}_\beta$, and the induction applies.

Case 4. α **is a limit ordinal,** $cf(\alpha) < \theta$**:** let $\langle \alpha_\xi : \xi < cf(\alpha) \rangle$ be a normal sequence cofinal in α. Let $p \in \mathbb{P}_\alpha$. By induction and the θ-closure of forcings, we can find a decreasing sequence $\langle q_\xi : \xi < cf(\alpha) \rangle$ of conditions such that: $q_\xi \in \mathbb{P}^*_{\alpha_\xi}$ and $q_\xi \leq_{\mathbb{P}_{\alpha_\xi}} p \restriction \alpha_\xi$. Let $p_1 = \bigcup_{\xi < cf(\alpha)} q_\xi$. Then $p_1 \in \mathbb{P}^*_\alpha$ and $p_1 \leq_{\mathbb{P}_\alpha} p$.

(b) can be proved easily by a Δ-system argument. To prove (c), it suffices, by (a), to show that $\mathbb{P}^*_\alpha$ satisfies the θ^+-c.c. Let $\{p_\beta : \beta < \theta^+\} \subseteq \mathbb{P}^*_\alpha$. We can assume that

(1) $\langle \operatorname{dom}(p_\beta) : \beta < \theta^+ \rangle$ forms a Δ-system with root Δ, and
(2) for each $i \in \Delta$, $\langle p_\beta(i) : \beta < \theta^+ \rangle$ are pairwise compatible in $\mathbb{Q}_i$ (using the fact that $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is θ^+-Knaster").

Now let $\beta_1 < \beta_2 < \theta^+$. Let q be defined as follows:

- $\operatorname{dom}(q) = \operatorname{dom}(p_{\beta_1}) \cup \operatorname{dom}(p_{\beta_2})$,
- $q(0) = \langle w^{p_{\beta_1}} \cup w^{p_{\beta_2}}, \mathcal{A}^{p_{\beta_1}} \cup \mathcal{A}^{p_{\beta_2}} \rangle$, and
- for all $i \in \operatorname{dom}(q)$, $q(i) = p_{\beta_1}(i) \cup p_{\beta_2}(i)$ (where we assume $p_{\beta_k}(i) = \emptyset$, if $i \notin \operatorname{dom}(p_{\beta_k})$).

Clearly $q \in \mathbb{P}^*_\alpha$, and it extends both p_{β_1}, p_{β_2}. So $\{p_\beta : \beta < \theta^+\}$ is not an antichain. □

5.3 On the forcing notion $\mathbb{Q}_{\lambda,\mathcal{A}}$ In this subsection we describe a forcing notion $\mathbb{Q}_{\lambda,\mathcal{A}}$, which depends on a parameter $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$. For $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$, set

$$\mathcal{A}^+ = \{u \in [\lambda]^{<\aleph_0} : u \text{ includes some member of } \mathcal{A}\}.$$

Definition 5.9 Assume λ is a cardinal and $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$. We define the forcing notion $(\mathbb{Q}_{\lambda,\mathcal{A}}, \leq)$ as follows:

(a) $p \in \mathbb{Q}_{\lambda,\mathcal{A}}$ iff p is a finite partial function from λ to $2^{n(p)}$, for some $n(p) < \omega$.
(b) For $p, q \in \mathbb{Q}_{\lambda,\mathcal{A}}$, $p \leq q$ (p is stronger than q) if and only if:
  (b-1) $\operatorname{dom}(q) \subseteq \operatorname{dom}(p)$,
  (b-2) $\alpha \in \operatorname{dom}(q) \Rightarrow q(\alpha) \trianglelefteq p(\alpha)$, and
  (b-3) if $u \in \mathcal{A}$, $u \subseteq \operatorname{dom}(q)$ and $n(q) \leq k < n(p)$, then for some $\alpha \in u$, $p(\alpha)(k) = 0$.

We also define the following $\mathbb{Q}_{\lambda,\mathcal{A}}$-names:

(α) $\underset{\sim}{\eta}_\alpha = \bigcup\{p(\alpha) : \alpha \in \operatorname{dom}(p) \text{ and } p \in \dot{G}_{\mathbb{Q}_{\lambda,\mathcal{A}}}\}$,
(β) $\underset{\sim}{a}_\alpha = \{k < \omega : \underset{\sim}{\eta}_\alpha(k) = 1\}$, and
(γ) $\underset{\sim}{a}_{\alpha,n} = \underset{\sim}{a}_{\omega\cdot\alpha+n}$.

Remark 5.10 Given any $w \subseteq \lambda$, let $\mathcal{A} \restriction w = \{u \in \mathcal{A} : u \subseteq w\}$. Then we define $\mathbb{Q}_{\lambda,\mathcal{A}} \restriction w$ to be $\mathbb{Q}_{\lambda,\mathcal{A}\restriction w}$ which is defined in the natural way. Then for disjoint $w, v \subseteq \lambda$ if

$$\forall u \in \mathcal{A}(u \subseteq w \cup v \Rightarrow u \subseteq w \text{ or } u \subseteq v),$$

then we have a forcing isomorphism $\mathbb{Q}_{\lambda,\mathcal{A}} \restriction (w \cup v) \approx \{(p,q) \in (\mathbb{Q}_{\lambda,\mathcal{A}} \restriction w) \times (\mathbb{Q}_{\lambda,\mathcal{A}} \restriction v) : n(p) = n(q)\}$. But in general the above forcing isomorphism may not be true, if w, v do not satisfy the above requirement, as (b-3) may fail.

We have the following easy lemma.

Lemma 5.11 *Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\mathcal{A}}$. Then*

(a) *$\mathbb{Q}$ is a c.c.c. forcing notion,*
(b) $\Vdash_{\mathbb{Q}}$ *"$\underset{\sim}{\eta}_\alpha \in 2^\omega$ and $\underset{\sim}{a}_{\alpha,n} \subseteq \omega$," and*
(c) $\Vdash_{\mathbb{Q}}$ *"$\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m}$ is finite" if and only if $\{\omega \cdot \alpha_i + m : i < n\} \in \mathcal{A}^+$.*

Proof (a) follows by a simple Δ-system argument and (b) is clear. Let us prove (c). First assume that $\{\omega \cdot \alpha_i + m : i < n\} \in \mathcal{A}^+$. Then for some $u \in \mathcal{A}$, $u \subseteq \{\omega \cdot \alpha_i + m : i < n\}$, and since $\Vdash_{\mathbb{Q}}$"$\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m} \subseteq \bigcap_{\alpha\in u} \underset{\sim}{a}_{\alpha,m}$", we can assume without loss of generality that $\{\omega \cdot \alpha_i + m : i < n\} \in \mathcal{A}$. Now let $p \in \mathbb{Q}$. By extending p, if necessary, we can assume that $\{\omega.\alpha_i + m : i < n\} \subseteq \mathrm{dom}(p)$. But then by clause ($b$-$3$), any $q \leq p$ forces "$\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m} \subseteq n(p)$". The result follows immediately.

Conversely suppose that $\{\omega \cdot \alpha_i + m : i < n\} \notin \mathcal{A}^+$. Let $p \in \mathbb{Q}$ and $k < \omega$. We find $q \leq p$ and $k' > k$ such that $q \Vdash$"$k' \in \bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m}$". By extending p we may assume that $\mathrm{dom}(p) \supseteq \{\omega \cdot \alpha_i + m : i < n\}$ and $n(p) > k$. Now define $q \leq p$ as follows:

- $\mathrm{dom}(q) = \mathrm{dom}(p)$.
- $n(q) = n(p) + 1$.
- If $\alpha \in \mathrm{dom}(p) \setminus \{\omega \cdot \alpha_i + m : i < n\}$, then $q(\alpha) = p(\alpha)^\frown\langle(n(p), 0)\rangle$.
- If $i < n$, then $q(\omega \cdot \alpha_i + m) = p(\omega \cdot \alpha_i + m)^\frown\langle(n(p), 1)\rangle$.

q is easily seen to be well defined and clearly

$$q \Vdash \text{“}k < k' \in \bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m}\text{”},$$

where $k' = n(p)$. Let us show that $q \leq p$. It suffices to show that it satisfies clause (b-3) of Definition 5.9. Thus let $u \in \mathcal{A}$ be such that $u \subseteq \mathrm{dom}(p)$. We are going to find some $\alpha \in u$ such that $q(\alpha)(n_p) = 0$. As $\{\omega \cdot \alpha_i + m : i < n\} \notin \mathcal{A}^+$, $u \setminus \{\omega \cdot \alpha_i + m : i < n\} \neq \emptyset$. Let $\alpha \in u \setminus \{\omega \cdot \alpha_i + m : i < n\}$. Then by our definition, $q(\alpha)(n_p) = 0$, as requested. □

We now consider the combinatorial principle $C^D_T(\kappa, J)$ in the forcing extensions by $\mathbb{Q}_{\lambda,\mathcal{A}}$, and show that the truth or falsity of it depends on the choice of D and $\mathcal{A}$. For the rest of this subsection, let $J = [\omega]^{<\omega}$, the ideal of bounded subsets of ω. In the next lemma we discuss conditions on D and $\mathcal{A}$ which imply $\neg C^D_T(\kappa, J)$ in the forcing extensions by $\mathbb{Q}_{\lambda,\mathcal{A}}$.

Lemma 5.12 *Assume:*

(1) $\aleph_0 < \kappa = cf(\kappa) \leq \lambda$;
(2) *D is a filter on κ with the Δ-system $\aleph_0$-property;*
(3) *$\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$ and T is a subtree of $\omega^{<\omega}$;*
(4) *there exists some $Y^* \in D^+$ such that:*
   (a) *If $Y \subseteq Y^*$, $Y \neq \emptyset \bmod D$, then there are $t \in T \cap \omega^n$ and distinct $\alpha_0, \ldots, \alpha_{n-1} \in Y$ such that $\{\omega.\alpha_i + t(i) : i < n\} \in \mathcal{A}$.*
   (b) *If $t \in T \cap \omega^n$ and for $i < n$, $Y_i \subseteq Y^*$, $Y_i \neq \emptyset \bmod D$, then there are $\alpha_i \in Y_i$, for $i < n$ such that $\{\omega.\alpha_i + t(i) : i < n\} \notin \mathcal{A}$.*

Then $\Vdash_{\mathbb{Q}_{\lambda,\mathcal{A}}}$ "$\neg C^D_T(\kappa, J)$".

Proof Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\mathcal{A}}$, and suppose $\nVdash_{\mathbb{Q}}$"$\neg C_T^D(\kappa, J)$". By Lemma 5.4, $\Vdash_{\mathbb{Q}}$"$\langle \underset{\sim}{a}_{\alpha,n} : \alpha \in Y^*, n < \omega \rangle$ is a $(\kappa \times \omega)$-matrix for D (i.e., $Y^* \in D^+$)" so by our assumption one of the following holds.

Case 1. There are $p \in \mathbb{Q}$ and $\underset{\sim}{X}$ such that:

$p \Vdash$"$\underset{\sim}{X} \subseteq Y^*$, $\underset{\sim}{X} \neq \emptyset \bmod D$,"
$p \Vdash$"For every $t \in T \cap \omega^n$ and distinct $\alpha_0, \ldots, \alpha_{n-1} \in \underset{\sim}{X}$, $\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i, t(i)} \neq \emptyset \bmod J$."

Let $X^* = \{\alpha \in Y^* : p \nVdash$"$\alpha \notin \underset{\sim}{X}$"$\}$. Then $X^* \in V$ and $p \Vdash$"$\underset{\sim}{X} \subseteq X^* \subseteq Y^*$," so $X^* \neq \emptyset \bmod D$. For any $\alpha \in X^*$, let $p_\alpha \leq p$ be such that $p_\alpha \Vdash$"$\alpha \in \underset{\sim}{X}$." As D has the Δ-system $\aleph_0$-property, we can find $X_1 \subseteq X^*$, $X_1 \neq \emptyset \bmod D$ such that $\{\mathrm{dom}(p_\alpha) : \alpha \in X_1\}$ forms a Δ-system with some root, say, Δ. Let $\Delta = \{\beta_0, \ldots, \beta_{k^{**}-1}\}$, and for each $\alpha \in X_1$, let

$$\mathrm{dom}(p_\alpha) = \{\beta_{\alpha,j} : j < k_\alpha\},$$

where for $j < k^{**}$, $\beta_{\alpha,j} = \beta_j$. By shrinking X_1, and using the Δ-system $\aleph_0$-property of D, we can further suppose that

(1) there is some $k^* < \omega$ such that $\alpha \in X_1 \Rightarrow k_\alpha = k^*$,
(2) $\alpha, \beta \in X_1 \Rightarrow p_\alpha \restriction \Delta = p_\beta \restriction \Delta$, and
(3) $\{p_\alpha(\beta_{\alpha,j}) : \alpha \in X_1\}$ is constant, for each $j < k^*$.

Now by (4-a), there are $t \in T \cap \omega^n$ and distinct $\alpha_0, \ldots, \alpha_{n-1} \in X_1$ such that $\{\omega.\alpha_i + t(i) : i < n\} \in \mathcal{A}$. Let q be a common extension of $p_{\alpha_i}, i < n$, which exists by our above assumptions. Then $q \Vdash$"$\alpha_0, \ldots, \alpha_{n-1} \in \underset{\sim}{X}$," and by Lemma 5.11($c$),

$$q \Vdash \text{“} \bigcap_{i<n} \underset{\sim}{a}_{\alpha_i, t(i)} \text{ is finite”},$$

which is a contradiction.

Case 2. There are $p \in \mathbb{Q}$, $t \in T \cap \omega^n$ and $\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_{n-1}$ such that

$p \Vdash$"$\underset{\sim}{X}_i \subseteq Y^*$, $\underset{\sim}{X}_i \neq \emptyset \bmod D$" for all $i < n$,
$p \Vdash$"If $\alpha_i \in \underset{\sim}{X}_i$ are distinct, then $\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i, t(i)} = \emptyset \bmod J$."

For $i < n$, set $X_i^* = \{\alpha \in Y^* : p \nVdash$"$\alpha \notin \underset{\sim}{X}_i$"$\}$. Then $X_i^* \in V$ and $p \Vdash$"$\underset{\sim}{X}_i \subseteq X_i^* \subseteq Y^*$," so $X_i^* \neq \emptyset \bmod D$. We now proceed by induction on $i < n$ and find $p_{i,\alpha} \leq p$ for $\alpha \in X_i^*$ so that

(1) $p_{i,\alpha} \Vdash$"$\alpha \in \underset{\sim}{X}_i$," and
(2) if $\alpha \in X_i^* \cap X_j^*$, for $i < j < n$, then $p_{i,\alpha} = p_{j,\alpha}$.

Now proceed as in case 1, and shrink each X_i^* to some $X_{i,1}$ so that

(3) $X_{i,1} \neq \emptyset \bmod D$.
(4) $\{\mathrm{dom}(p_{i,\alpha}) : \alpha \in X_{i,1}\}$ forms a Δ-system with root, say, $\Delta_i = \{\beta_{i,0}, \ldots, \beta_{i,k_i^{**}-1}\}$.
(5) For some $k_i^* < \omega$, $\mathrm{dom}(p_{i,\alpha}) = \{\beta_{i,\alpha,j} : j < k_i^*\}$, where for $j < k_i^{**}$, $\beta_{i,\alpha,j} = \beta_{i,j}$.
(6) $\alpha, \beta \in X_{i,1} \Rightarrow p_{i,\alpha} \restriction \Delta_i = p_{i,\beta} \restriction \Delta_i$.
(7) $\{p_{i,\alpha}(\beta_{i,\alpha,j}) : \alpha \in X_{i,1}\}$ is constant, for each $j < k_i^*$.

We again use the Δ-system argument successively on $X_{i,1}$'s to shrink them to some $X_{i,2}, i < n$, so that for all $i, j < n$:

(8) $X_{i,2} \neq \emptyset \bmod D$.

(9) For all $\alpha \in X_{i,2}$, $\beta \in X_{j,2}$, $\mathrm{dom}(p_{i,\alpha}) \cap \mathrm{dom}(p_{j,\beta}) = \Delta_{i,j}$, for some fixed set $\Delta_{i,j}$.

(10) For all $\alpha \in X_{i,2}$, $\beta \in X_{j,2}$, $p_{i,\alpha} \restriction \Delta_{i,j} = p_{j,\beta} \restriction \Delta_{i,j}$.

Now by (4-b), there are $\alpha_i \in X_{i,2}$, $i < n$, such that $\{\omega.\alpha_i + t(i) : i < n\} \notin \mathcal{A}$. Let q be a common extension of p_{i,α_i}, $i < n$, which exists by our above assumptions. Then $q \Vdash$"$\alpha_0, \ldots, \alpha_{n-1} \in \underset{\sim}{X}$" and by Lemma 5.11($c$),

$$q \Vdash \text{“} \bigcap_{i<n} \underset{\sim}{a}_{\alpha_i, t(i)} \text{ is infinite”},$$

which is a contradiction. The lemma follows. □

We now discuss conditions on D and $\mathcal{A}$ which imply $C_T^D(\kappa, J)$ in the forcing extensions by $\mathbb{Q}_{\lambda,\mathcal{A}}$.

Lemma 5.13 *Assume:*

(1) *D is a κ-complete filter on κ, where $\kappa = cf(\kappa) > \aleph_0$ and $\forall \alpha < \kappa(|\alpha|^{\aleph_0} < \kappa)$.*

(2) *$T \subseteq \omega^{<n(*)}$ is a subtree, where $n(*) < \omega$.*

(3) *$\lambda \geq \kappa$ and $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$.*

(4) *If $Y \subseteq \kappa$, $Y \neq \emptyset \bmod D$, and if $\langle w_\alpha : \alpha \in Y\rangle$ is such that $w_\alpha \in [\lambda]^{\leq\aleph_0}$, for $\alpha \in Y$, then there exists $X \subseteq Y$, $X \neq \emptyset \bmod D$ such that:*

   (a) *$\langle w_\alpha : \alpha \in X\rangle$ form a Δ-system with root, say, w^* such that for all $\alpha \neq \beta$ in X, $w_\alpha \cap w_\beta = w^*$, and for all $\gamma \in w^*$, $otp(w_\alpha \cap \gamma) = otp(w_\beta) \cap \gamma$, and*

   (b) *if $\alpha_0, \ldots, \alpha_{n(*)-1} \in X$ are distinct, then*

$$\forall u \Big(u \in \mathcal{A} \text{ and } u \subseteq \bigcup_{i<n(*)} w_{\alpha_i} \Rightarrow \exists i < n(*), u \subseteq w_{\alpha_i} \Big).$$

Then $\Vdash_{\mathbb{Q}_{\lambda,\mathcal{A}}}$ "$C_T^D(\kappa, J)$".

Proof Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\mathcal{A}}$, and suppose G is $\mathbb{Q}$-generic over V. Assume on the contrary that $V[G] \models$"$\neg C_T^D(\kappa, J)$". Let $p \in G$, $\underset{\sim}{X}$ and $\langle \underset{\sim}{b}_{\alpha,n} : \alpha \in \underset{\sim}{X}, n < \omega\rangle$ be such that

$p \Vdash$"$\underset{\sim}{X} \subseteq \kappa$, $\underset{\sim}{X} \neq \emptyset \bmod D$" and
$p \Vdash$"$\langle \underset{\sim}{b}_{\alpha,n} : \alpha \in \underset{\sim}{X}, n < \omega\rangle$" is a counterexample to $C_T^D(\kappa, J)$.

Let $X_1 = \{\alpha < \kappa : p \nVdash$"$\alpha \notin \underset{\sim}{X}$"$\}$. Then $X_1 \in V$ and $p \Vdash$"$\underset{\sim}{X} \subseteq X_1$", so $X_1 \neq \emptyset \bmod D$. For any $\alpha \in X_1$, let $p_\alpha \leq p$ be such that $p_\alpha \Vdash$"$\alpha \in \underset{\sim}{X}$". We may further assume that $\alpha \in \mathrm{dom}(p)$.

For each $\alpha \in X_1$, we can find $\langle q_{\alpha,n,m,k}, t_{\alpha,n,m} : n, m, k < \omega\rangle$ such that:

(1) $t_{\alpha,n,m} : \omega \to 2$,

(2) $\{q_{\alpha,n,m,k} : m < \omega\} \subseteq \mathbb{Q}$ is a maximal antichain below p, and

(3) $q_{\alpha,n,m,k} \Vdash$"$k \in \underset{\sim}{b}_{\alpha,n}$"$\Leftrightarrow t_{\alpha,n,m}(k) = 1$.

We may note that then

$$p \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{b}_{\alpha,n} = \big\{\langle q_{\alpha,n,m,k}, k\rangle : m, k < \omega, t_{\alpha,n,m}(k) = 1\big\}\text{”}$$

and so from now on we assume $\underset{\sim}{b}_{\alpha,n}$ is of the form. Let

$$w_\alpha = \mathrm{dom}(p_\alpha) \cup \bigcup \big\{\mathrm{dom}(q_{\alpha,n,m,k}) : n, m, k < \omega\big\} \cup \{\omega.\alpha + n : n < \omega\}.$$

Then each $w_\alpha \in [\lambda]^{\aleph_0}$. As D is κ-complete and $\kappa > 2^{\aleph_0}$, we can find X_2, $\bar{g}$ and $\bar{t}$ such that:

(4) $X_2 \subseteq X_1$, $X_2 \neq \emptyset \bmod D$,
(5) $\bar{t} = \langle t_{n,m} : n, m < \omega \rangle$ and $\forall \alpha \in X_2$, $t_{\alpha,n,m} = t_{n,m}$,
(6) For all $\alpha, \beta \in X_2$, $otp(w_\alpha) = otp(w_\beta)$,
(7) $\bar{g} = \langle g_{\alpha,\beta} : \alpha, \beta \in X_2 \rangle$,
(8) $g_{\alpha,\beta} : w_\beta \cong w_\alpha$ is an order-preserving bijection,
(9) $g_{\alpha,\beta}(\beta) = \alpha$, and
(10) $g_{\alpha,\beta}``[q_{\beta,n,m,k}] = q_{\alpha,n,m,k}$.

Consider $\langle w_\alpha : \alpha \in X_2 \rangle$. By our assumption, we can find X_3 and w^* such that:

(11) $X_3 \subseteq X_2$, $X_3 \neq \emptyset \bmod D$,
(12) For all $\alpha \neq \beta$ in X_3, $w_\alpha \cap w_\beta = w^*$, and for all $\gamma \in w^*$, $otp(w_\alpha \cap \gamma) = otp(w_\beta) \cap \gamma$, and
(13) if $\alpha_0, \ldots, \alpha_{n(*)-1} \in X_3$ are distinct, then

$$\forall u \Big(u \in \mathcal{A} \text{ and } u \subseteq \bigcup_{i<n(*)} w_{\alpha_i} \Rightarrow \exists i < n(*), u \subseteq w_{\alpha_i} \Big).$$

Note that for $\alpha \neq \beta$ in X_3, $g_{\alpha,\beta} \restriction w^* = id \restriction w^*$ (by (12)). As the conclusion of the lemma fails, we can find $q \leq p$, $q \in G$, $t \in T$ and $\alpha_0, \ldots, \alpha_{n(*)-1} \in X_3$ such that

$$q \Vdash \text{“} \bigcap_{i<n(*)} \underset{\sim}{b}_{\alpha_i,t(i)} = \emptyset \bmod J\text{”}.$$

We may suppose that $\mathrm{dom}(q) \subseteq \bigcup_{i<n(*)} w_{\alpha_i}$.[3]

For $\beta \in X_3$ and $i < n(*)$, set

$$q_{i,\beta} = g_{\beta,\alpha_i}(q \restriction w_{\alpha_i}) \in \mathbb{Q} \restriction w_\beta.$$

Let $\underset{\sim}{Y}_i$ be such that

$$\Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{Y}_i = \{\beta \in X_3 : q_{i,\beta} \in \dot{G}_{\mathbb{Q}}\}\text{”}.$$

Claim 5.14 $\Vdash_{\mathbb{Q}}$ *“$\underset{\sim}{Y}_i \neq \emptyset$ mod D”.*

Proof Assume not, so there are $r \in G$, $r \leq q$, and $X \in D^+$ such that $r \Vdash$“$X \cap \underset{\sim}{Y}_i = \emptyset$”. As $\mathrm{dom}(r)$ is finite, we can find $\beta \in X$ so that $\mathrm{dom}(r) \cap w_\beta \setminus w^* = \emptyset$. But then $r, q_{i,\beta}$ are compatible, and any common extension of them forces “$\beta \in X \cap \underset{\sim}{Y}_i$”, which is impossible. □

We show that if $\Vdash_{\mathbb{Q}}$“$\beta_i \in \underset{\sim}{Y}_i$” for $i < n(*)$, then $\Vdash_{\mathbb{Q}}$“$\bigcap_{i<n(*)} \underset{\sim}{b}_{\beta_i,t(i)} = \emptyset$ mod J”. So assume $V[G] \models$“$\beta_i \in Y_i = \underset{\sim}{Y}_i[G]$”. Then $g = \bigcup_{i<n(*)} g_{\alpha_i,\beta_i}$ is an order-preserving bijection from $\bigcup_{i<n(*)} w_{\beta_i}$ onto $\bigcup_{i<n(*)} w_{\alpha_i}$, and we can extend it to an automorphism of λ, in the natural way, so that its restriction to $\lambda \setminus (\bigcup_{i<n(*)} w_{\beta_i} \cup \bigcup_{i<n(*)} w_{\alpha_i})$ is identity. We denote the resulting function still by g. g easily extends to an automorphism $\hat{g} : \mathbb{Q} \cong \mathbb{Q}$ of $\mathbb{Q}$, which in turn also extends to an automorphism of nice names of $\mathbb{Q}$.

For $i < n(*)$, $q_{i,\beta_i} \in G_{\mathbb{Q}}$, so

$$\hat{g}(q_{i,\beta_i}) = q \restriction w_{\alpha_i} \in \hat{g}``[G].$$

Hence, $q = \bigcup_{i<n(*)} q \restriction w_{\alpha_i} \in \hat{g}``[G]$. But it is easily seen that $\hat{g}(\underset{\sim}{b}_{\beta_i,t(i)}) = \underset{\sim}{b}_{\alpha_i,t(i)}$, and so

$$\hat{g}^{-1}(q) \Vdash \text{“} \bigcap_{i<n(*)} \underset{\sim}{b}_{\beta_i,t(i)} = \emptyset \bmod J\text{”}.$$

On the other hand $\hat{g}^{-1}(q) \in G$, and the result follows. □

Remark 5.15 Condition (4-b) is implicitly used in the argument to guarantee that the restricted conditions and their union which we defined are well defined. See also Remark 5.10.

5.4 Proof of Theorem 5.2 Finally in this subsection we present the proof of Theorem 5.2. Thus let $n(*)$, θ, κ, χ, and D be as above and $J = [\omega]^{<\omega}$. Consider the forcing notion $\mathbb{P} = \mathbb{P}_\chi * \underset{\sim}{\mathbb{Q}}_{\kappa,\mathcal{A}}$, where $\mathcal{A} \subseteq [\kappa]^{<\aleph_0}$ is the set added by $\mathbb{P}_\chi$. It follows from Lemmas 5.8 and 5.11 that $\mathbb{P}$ is a cofinality-preserving forcing notion.

Firs we show that $C_n^D(\kappa, J)$ holds for $n < n(*)$. It suffices to show that in $V^{\mathbb{P}_\chi}$, the pair $(n(*), D)$ satisfies the the demands in Lemma 5.12. Conditions (1)–(3) from the lemma are clear. To prove (4-a), let $\Vdash_{\mathbb{P}_\chi}$“$\underset{\sim}{Y} \subseteq \kappa$, $\underset{\sim}{Y} \neq \emptyset \bmod D$, and $\langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y} \rangle$ is a sequence of countable subsets of λ.” Let $i < \chi$ be such that $\underset{\sim}{Y}$ and $\langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y} \rangle$ are $\mathbb{P}_i$-names. By the fact that in $V^{\mathbb{P}_i}$, D has has the Δ-system θ-property (see Lemma 5.5), we can find $\mathbb{P}_i$-names $\underset{\sim}{X}$ and $\underset{\sim}{w}^*$ such that:

(1) $\Vdash_{\mathbb{P}_i}$“$\underset{\sim}{X} \subseteq \underset{\sim}{Y}$, $\underset{\sim}{X} \neq \emptyset \bmod D$” and
(2) $\Vdash_{\mathbb{P}_i}$“$\langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{X} \rangle$ forms a Δ-system with root $\underset{\sim}{w}^*$.”

Then for some $j \in (i, \chi)$, $\underset{\sim}{X} = \underset{\sim}{Y}_j$ and $\langle \underset{\sim}{w}^*, \langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y} \rangle\rangle = \langle \underset{\sim}{w}^*_j, \langle \underset{\sim}{w}^j_\alpha : \alpha \in \underset{\sim}{Y}_j \rangle\rangle$. Now by our definition of $\mathbb{Q}_j$, we can find $\underset{\sim}{Z} \in V^{\mathbb{P}_{j+1}}$ such that:

(3) $\Vdash_{\mathbb{P}_{j+1}}$“$\underset{\sim}{Z} \subseteq \underset{\sim}{Y}_j$, $\underset{\sim}{Z} \neq \emptyset \bmod D$” and
(4) $\Vdash_{\mathbb{P}_{j+1}}$“If $\alpha_0, \dots \alpha_{n(*)-1} \in \underset{\sim}{Z}$ are distinct, then for all $u \in \mathcal{A}$, $u \subseteq \bigcup_{l<n(*)} \underset{\sim}{w}^j_{\alpha_l} \Rightarrow \exists l < n(*), u \subseteq \underset{\sim}{w}^j_{\alpha_l}$.”

The result follows immediately, as then the above are also forced to be true by $\mathbb{P}_\chi$.

Now we show that $C^D_{n(*)}(\kappa, J)$ fails. Let $T = \omega^{n(*)}$. We show that

$$\Vdash_{\mathbb{P}_\chi * \underset{\sim}{\mathbb{Q}}_{\lambda,\mathcal{A}}} \text{“}\langle \underset{\sim}{a}_{\alpha,n} : \alpha < \kappa, n < \omega \rangle \text{ exemplify } \neg C^D_{n(*)}(\kappa, J)\text{”},$$

where the names $\underset{\sim}{a}_{\alpha,n}$ are defined just after Definition 5.9. To this end, we check conditions in Lemma 5.11. Conditions (1)–(3) from the lemma are clear. For (4-a), assume on the contrary that in $V^{\mathbb{P}_\chi}$, $Y \neq \emptyset$, mode D is given and $p \in \mathbb{P}_\chi$, $p \Vdash$“$\underset{\sim}{Y}$ is a counterexample for (4-a).” In V, let $X_1 = \{\delta < \kappa : p \nVdash \text{“}\delta \notin \underset{\sim}{Y}\text{”}\}$. Then $X_1 \in V$ and $X_1 \neq \emptyset \bmod D$. For any $\delta \in X_1$, let $p_\delta \leq p$ be such that $p_\delta \Vdash$“$\delta \in \underset{\sim}{Y}$.”

As in the proof of Lemma 5.8(b), and using the fact that D has the Δ-system property, we can find $X_2 \subseteq X_1$, $X_2 \neq \emptyset \bmod D$ such that $\langle \mathrm{dom}(p_\delta) : \delta \in X_2 \rangle$ form a Δ-system with root Δ and for all $\delta, \gamma \in X_2$, $p_\delta \restriction \Delta \parallel p_\gamma \restriction \Delta$ ($p_\delta \restriction \Delta$ is compatible with $p_\gamma \restriction \Delta$). Now let $t \in \omega^{n(*)}$ and let $\delta_0, \dots, \delta_{n(*)-1}$ be in X_2 such that for each l, $\delta_{l+1} > \sup\{w^{p_{\delta_j}(0)} : j \leq l\}$. Let q be an extension of all $p_{\delta_l}, l < n(*)$ such that:

$$t \in \omega^{n(*)} \Rightarrow \left\{\omega.\delta_l + t(l) : l < n(*)\right\} \in \mathcal{A}^{q(0)},$$

and

$$u \in \mathcal{A}^{q(0)}, k < n(*), \quad v \subseteq \bigcup \left\{w^{p_{\delta_l}} : l < n(*), l \neq k\right\} \Rightarrow (\exists l) v \subseteq w^{p_{\delta_l}}.$$

For example, we can set

$$q(0) = \Big\langle \bigcup_{l<n(*)} w^{p_{\delta_l}(0)} \cup \{\omega \cdot \delta_l + t(l) : l < n(*)\},$$
$$\bigcup_{l<n(*)} \mathcal{A}^{p_{\delta_l}(0)} \cup \{\{\omega.\delta_l + t(l)\} : t \in \omega^{n(*)}\}\Big\rangle.$$

Then $q \le p$ and $q \Vdash$"$\underset{\sim}{Y}$ can not be a counterexample to (4-a)," a contradiction.

For (4-b), again assume for some $p \in \mathbb{P}_\chi$, $t \in T$, and $\underset{\sim}{Y}_i, i < n(*)$, we have $p \Vdash$"$t, \langle \underset{\sim}{Y}_i : i < n(*)\rangle$ are counterexample to (4-b)." For each $i < n(*)$, let $X_i^* = \{\delta < \kappa : p \nVdash$"$\delta \notin \underset{\sim}{Y}_i$"$\}$. Then $X_i^* \in V$ and $X_i^* \neq \emptyset \bmod D$. For any $\delta \in X_i^*$, let $p_{i,\delta} \le p$ be such that $p_{i,\delta} \Vdash$"$\delta \in \underset{\sim}{Y}_i$." Now proceed as in the proof of Lemma 5.11, case 2, to shrink each X_i^* to some $X_{i,2}$, such that

(5) $X_{i,2} \neq \emptyset \bmod D$,
(6) $\langle \mathrm{dom}(p_{i,\delta}) : \delta \in X_{i,2}\rangle$ form a Δ-system with root Δ_i,
(7) $\delta, \gamma \in X_{i,2}$, $p_{i\delta} \restriction \Delta \parallel p_{i,\gamma} \restriction \Delta$,
(8) for all $\delta \in X_{i,2}$, $\gamma \in X_{j,2}$, $\mathrm{dom}(p_{i,\delta}) \cap \mathrm{dom}(p_{j,\gamma}) = \Delta_{i,j}$, for some fixed set $\Delta_{i,j}$, and
(9) for all $\delta \in X_{i,2}$, $\gamma \in X_{j,2}$, $p_{i,\delta} \restriction \Delta_{i,j} \parallel p_{j,\gamma} \restriction \Delta_{i,j}$.

Let $\delta_l \in X_{l,2}$, $l < n(*)$. Let q be an extension of all p_{l,δ_l}, $l < n(*)$, such that

$$q(0) = \Big\langle \bigcup_{l<n(*)} w^{p_{\delta_l}} \cup \{\omega \cdot \delta_l + n : n < \omega, l < n(*)\}, \bigcup_{l<n(*)} \mathcal{A}^{p_{\delta_l}}\}\Big\rangle.$$

Then $q \le p$ and $\{\omega.\delta_l + t(l) : l < n(*)\} \notin \mathcal{A}^q$, so

$$q \Vdash \text{“}\{\omega.\delta_l + t(l) : l < n(*)\} \notin \underset{\sim}{\mathcal{A}}\text{”}.$$

So $q \Vdash$"$t, \langle \underset{\sim}{Y}_i : i < n(*)\rangle$ can not be counterexamples to (4-b)," a contradiction.

6 On $C^s(\kappa)$ v.s. $C(\kappa)$

In this section, we consider the difference between the combinatorial principles $C^s(\kappa)$ and $C(\kappa)$, and prove the consistency of "$C(\kappa)$ holds but $C_T^s(\kappa)$ fails for all nontrivial T."

Lemma 6.1 *Assume that:*

(1) $\kappa = cf(\kappa) > \aleph_0$;
(2) $S^* \subseteq \kappa$ *is a stationary subset of* κ;
(3) $\bar{C} = \langle C_\delta : \delta \in S^*\rangle$ *is such that:*
  (a) *each* C_δ *is a club of* δ, *and*
  (b) *for every club* E *of* κ, *the set* $\{\delta \in S^* : \sup(C_\delta \setminus E) = \delta\}$ *is not stationary;*
(4) $2 \le n(*) < \omega$.

Then there is $\mathcal{A} \subseteq [\kappa]^{n(*)}$ *such that:*

(α) *If* $S_l \subseteq S^*$ *is stationary for* $l < n(*)$, *then we can find* $\alpha_l, \beta_l \in S_l$, *for* $l < n(*)$ *such that* $\alpha_0 < \cdots < \alpha_{n(*)-1} < \beta_0 < \cdots < \beta_{n(*)-1}$ *and* $\{\alpha_l : l < n(*)\} \in \mathcal{A}$, $\{\beta_l : l < n(*)\} \notin \mathcal{A}$.
(β) *If* $Y \subseteq \kappa$ *is unbounded, then for some unbounded subset* $Z \subseteq Y$ *we have* $[Z]^{n(*)} \cap \mathcal{A} \in \{\emptyset, [Z]^{n(*)}\}$.

Proof Let

$$\mathcal{A} = \big\{\{\alpha_0, \dots, \alpha_{n(*)-1}\} : \alpha_{n(*)-1} \in S^* \text{ and } l < n(*) - 1 \Rightarrow otp(\alpha_l \cap C_{\alpha_{n(*)-1}}) \text{ is odd}\big\}.$$

Let's show that $\mathcal{A}$ is as required:

(α) should be clear; let's prove (β). So assume $Y \subseteq \kappa$ is unbounded. So there is $Z_1 \subseteq Y$ of size κ such that $Z_1 \subseteq S^*$ or $Z_1 \cap S^* = \emptyset$. If $Z_1 \cap S^* = \emptyset$, then obviously $[Z_1]^{n(*)} \cap \mathcal{A} = \emptyset$ and we are done; so assume $Z_1 \subseteq S^*$. Define the sequence $\langle \alpha_i : i < \kappa \rangle$, by induction on $i < \kappa$, such that

(1) $\alpha_i \in Z_1, \alpha_i > \sup\{\alpha_j : j < i\}$, and
(2) $\sup(C_{\alpha_i} \cap \bigcup_{j<i} \alpha_j)$ is minimal.

Let $E = \{\delta < \kappa : \delta = \sup_{j<\delta} \alpha_j \text{ is a limit ordinal}\}$ so that E is a club of κ. Set

$$W_1 = \big\{\delta \in E \cap S^* : \delta < \sup(C_{\alpha_\delta} \cap \delta)\big\} \subseteq S^*.$$

Then W_1 is a stationary subset of κ, as otherwise we can find a club $C \subseteq E$ which is disjoint from W_1 and we get a contradiction with (3-b). It follows from Fodor's lemma that for some $\alpha^* < \kappa$, the set

$$W_2 = \big\{\delta \in W_1 : \sup(C_{\alpha_\delta} \cap \delta) = \alpha^* < \delta\big\}$$

is stationary. Again by Fodor's lemma, there exists $\delta^* < \kappa$ such that the set

$$W_3 = \big\{\delta \in W_2 : \sup(C_{\alpha_\delta} \cap \alpha^*) = \delta^*\big\}$$

is stationary. Let $Z = \{\alpha_\delta : \delta \in W_3\}$. Clearly Z is an unbounded subset of Y. We show that $[Z]^{n(*)} \cap \mathcal{A} \in \{\emptyset, [Z]^{n(*)}\}$. Thus suppose that $[Z]^{n(*)} \cap \mathcal{A} \neq \emptyset$. Let $\delta_0 < \cdots < \delta_{n(*)-1} \in W_3$. Then $\alpha_{\delta_{n(*)-1}} \in S^*$ and for $l < n(*) - 1$ we have

$$\begin{aligned} otp(C_{\alpha_{\delta_{n(*)-1}}} \cap \alpha_{\delta_l}) &= otp(C_{\alpha_{\delta_{n(*)-1}}} \cap \alpha^*) + otp\big((C_{\alpha_{\delta_{n(*)-1}}} \setminus \alpha^*) \cap \alpha_{\delta_l}\big) \\ &= \delta^* + otp\big((C_{\alpha_{\delta_{n(*)-1}}} \setminus \delta_{n(*)-1}) \cap \alpha_{\delta_l}\big) \\ &= \delta^* + 1 \ \big(\text{as } C_{\alpha_{\delta_{n(*)-1}}} \setminus \delta_{n(*)-1}) \cap \alpha_{\delta_l} = \{\alpha^*\}\big), \end{aligned}$$

which is odd. So $\{\alpha_0, \dots, \alpha_{n(*)-1}\} \in \mathcal{A}$, as required. □

Remark 6.2 We can replace (3-b) with (α) & (β), where

(α) For every club E_1 of κ, there exists a club $E_2 \subseteq E_1$ of κ, such that for every $\delta \in S^* \cap E_2$, we have $\delta = \sup\{\alpha < \delta : otp(C_\delta \cap \alpha) \text{ is even}\} = \sup\{\alpha < \delta : otp(C_\delta \cap \alpha) \text{ is odd}\}$.

(β) There is no increasing continuous sequence $\langle \alpha_i : i < \kappa \rangle$ of ordinals $< \kappa$ such that $C_{\alpha_{2i+1}} \supseteq \{\alpha_{2j} : j < i\}$ (note that this holds if $\sup\{C_\delta : \delta \in S^*\} < \kappa$).

Remark 6.3 We can force the existence of such an S^* and $\bar{C}$ by forcing.

Theorem 6.4 *Assume $\kappa = cf(\kappa) > \aleph_0$, $\forall \alpha < \kappa(|\alpha|^{\aleph_0} < \kappa)$, and let $\mathcal{A} \subseteq [\kappa]^{n(*)}$ be as in the conclusion of Lemma 6.1. Then for any nontrivial tree $T \subseteq \omega^{\leq n(*)}$, we have $V^{\mathbb{Q}_{\kappa,\mathcal{A}}} \models$ "$C_{T,n(*)}(\kappa) + \neg C^s_{T,n(*)}(\kappa)$".*

Proof That $C^s_{T,n(*)}(\kappa)$ fails in $V^{\mathbb{Q}_{\kappa,\mathcal{A}}}$ follows from Lemmas 5.11 and 6.1(α). Also, $C_{T,n(*)}(\kappa)$ holds in $V^{\mathbb{Q}_{\kappa,\mathcal{A}}}$ by Lemmas 5.12 and 6.1(β). □

The following lemma can be proved similar to the proof of Lemma 6.1.

Lemma 6.5 *Let S^* and $\bar{C}$ be as in Lemma 6.1, and assume any $\delta \in S^*$ has uncountable cofinality. Then there is $\mathcal{A} \subseteq [\kappa]^{<\omega}$ such that:*

(α) *If $n(*) < \omega$ and $S_l \subseteq S^*$ is stationary for $l < n(*)$, then we can find $\alpha_l, \beta_l \in S_l$, for $l < n(*)$ such that $\alpha_0 < \cdots < \alpha_{n(*)-1} < \beta_0 < \cdots < \beta_{n(*)-1}$ and $\{\alpha_l : l < n(*)\} \in \mathcal{A}$, $\{\beta_l : l < n(*)\} \notin \mathcal{A}$.*

(β) *If $Y \subseteq \kappa$ is unbounded, then for some unbounded subset $Z \subseteq Y$ we have $[Z]^{<\omega} \cap \mathcal{A} \in \{\emptyset, [Z]^{<\omega}\}$.*

Finally, we have the following, whose proof is the same as the proof of Theorem 6.4, using Lemma 6.5, instead of Lemma 6.1.

Theorem 6.6 *Assume $\kappa = cf(\kappa) > \aleph_0$, $\forall \alpha < \kappa (|\alpha|^{\aleph_0} < \kappa)$, and let $\mathcal{A} \subseteq [\kappa]^{<\omega}$ be as in the conclusion of Lemma 6.5. Then $V^{\mathbb{Q}_{\kappa,\mathcal{A}}} \models$"$C(\kappa)+$ For any nontrivial tree $T \subseteq \omega^{<\omega}$, $\neg C_T^s(\kappa)$."*

Notes

1. Such an i exists as $U = \text{supp}(\underset{\sim}{a})$ is a countable set and $cf(\delta) > \aleph_0$.

2. When working in a forcing extension, we use D to denote the filter generated by D in that extension.

3. This is because, by our representation of $\underset{\sim}{b}_{\alpha_i, t(i)}$, we can imagine each $\underset{\sim}{b}_{\alpha_i, t(i)}$ as a $\mathbb{Q} \restriction w_{\alpha_i}$-name.

References

[1] Brendle, J., and S. Fuchino, "Coloring ordinals by reals," *Fundamenta Mathematicae*, vol. 196 (2007), pp. 151–95. Zbl 1143.03023. MR 2342625. DOI 10.4064/fm196-2-5. 482, 483

[2] Fuchino, S., "On combinatorial principles Princ(κ, λ), $C^s(\kappa)$, $HP(\kappa)$ etc. and variants of the bounding number," *RIMS Kôkûroku* (2005), no. 1423, pp. 13–27. 486

[3] Fuchino, S., and S. Geschke, "Some combinatorial principles defined in terms of elementary submodels," *Fundamenta Mathematicae*, vol. 181 (2004), pp. 233–55. Zbl 1051.03035. MR 2099602. DOI 10.4064/fm181-3-3. 482

[4] Juhász, I., and K. Kunen, "The power set of ω elementary submodels and weakenings of CH," *Fundamenta Mathematicae*, vol. 170 (2001), pp. 257–65. Zbl 0988.03078. MR 1880903. DOI 10.4064/fm170-3-4. 481, 488

[5] Juhász, I., L. Soukup, and Z. Szentmiklossy, "Combinatorial principles from adding Cohen reals," pp. 79–103 in *Logic Colloquium '95 (Haifa)*, Springer, Berlin, 1998. Zbl 0896.03039. MR 1678356. DOI 10.1007/978-3-662-22108-2_7. 481, 484

[6] Juhász, I., L. Soukup, and Z. Szentmiklossy, "What is left of CH after you add Cohen reals?" *Topology and Its Applications*, vol. 85 (1998), no. 1–3, pp. 165–74. Zbl 0936.54005. MR 1617461. DOI 10.1016/S0166-8641(97)00148-X. 482

[7] Shelah, S., unpublished note for I. Juhász, 2002. 482

Acknowledgments

The first author's research has been supported by a grant from the Institute for Research in Fundamental Sciences (no. 96030417). The second author's research has been partially supported by European Research Council grant 338821. This article is publication 1125 on Shelah's list.

The authors thank Ashutosh Kumar for many useful comments and corrections. They also thank the referee of the paper for carefully reading the paper and making some useful suggestions.

Golshani
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran; golshani.m@gmail.com

Shelah
Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel, and Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08854, USA; shelah@math.huji.ac.il