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Maximal models up to the first measurable in ZFC

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Theorem: There is a *complete sentence* ϕ of $L_{\omega_1,\omega}$ such that ϕ has maximal models in a set of cardinals λ that is cofinal in the first measurable μ while ϕ has no maximal models in any $\chi \geq \mu$.

Keywords: $L_{\omega_1,\omega}$; Hanf number; measurable cardinal.

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In this paper, we prove in ZFC the existence of a *complete sentence* ϕ of $L_{\omega_1,\omega}$ such that ϕ has maximal models (i.e. no $L_{\omega_1,\omega}$-elementary extension satisfies ϕ) in a set of cardinals λ that is cofinal in the first measurable μ while ϕ has no maximal models in any $\chi \geq \mu$. In [4], we proved a theorem with a similar result; the earlier proof required that $\lambda = \lambda^{<\lambda}$, and extended ZFC by requiring an $S \subseteq S^{\lambda}_{\aleph_0}$, that is stationary non-reflecting, and $\diamond_S$ holds. Here, we show *in ZFC* that the sentence ϕ defined in [4] has maximal models cofinally in μ. The additional hypotheses in [4] allow one to demand that if N is a submodel with cardinality $< \lambda$ of the P_0-maximal model, N is $\boldsymbol{K}_1$-free (see Remark 4.1); that property fails for the example here. The existence of such a ϕ which is *not complete* is well known (e.g. [11]).

This paper contributes to the study of Hanf numbers for infinitary logics. Works such as [2, 3, 5, 10] study the spectrum of maximal models in the context where the class has a bounded number of models. We list now some properties that are true in every cardinality for first-order logic but are true only eventually for complete sentences of $L_{\omega_1,\omega}$ or, more generally, for abstract elementary classes, and compare the cardinalities (the Hanf number) at which the cofinal behavior must begin. Every infinite model of a first-order theory has a proper elementary extension and so each theory has arbitrarily large models. Morley [12] showed that every sentence of $L_{\omega_1,\omega}$ that has models up to $\beth_{\omega_1}$ has arbitrarily large models and provided counterexamples showing that cardinal was minimal. Thus, he showed the Hanf number for existence of $L_{\omega_1,\omega}$-sentences in a countable vocabulary is $\beth_{\omega_1}$. Hjorth [9], by a much more complicated argument, showed there are *complete* sentences ϕ_α for $\alpha < \omega_1$ such that ϕ_α has a model in $\aleph_\alpha$ and no larger so the Hanf number for complete sentences is $\aleph_{\omega_1}$. The amalgamation property holds for every complete first-order theory. However, paper [1] shows that an upper bound on the Hanf number for amalgamation is the first strongly compact; the actual value remains open. Boney and Unger [6], building on [14], show that the Hanf number 'for all AEC's are tame' is the first strongly compact cardinal. They also show the analogous property for various variants on tameness is equivalent to the existence of almost (weakly) compact, measurable, strongly compact). The result here shows in ZFC that the Hanf number for extendability (every model of a complete sentence has a proper $L_{\omega_1,\omega}$-elementary extension) is the first measurable cardinal.

Section 1 provides some background information on Boolean algebras. Section 2 is a set theoretic argument for the existence of a Boolean algebra with certain specified properties in any cardinal λ of the form $\lambda = 2^\mu$ that is less than the first measurable; this construction is completely independent of the model theoretic results. Then we make the connection with model theory. In particular, we link the construction here with the complete sentence ϕ from [4]. Section 3 builds several approximations to the counterexample. Section 3.1 introduces the most basic class of models $\boldsymbol{K}_{-1}$ and explains the connections with [4]. Section 3.2 builds on this result to find a P_0-maximal model in $\boldsymbol{K}_{-1}$ with cardinality λ satisfying certain further restrictions. We recall in Sec. 3.3 the class $\boldsymbol{K}_2$ of models of the complete sentence from [4]. In Sec. 4, the P_0-maximal model from Sec. 3.2 is converted to a P_0-maximal model in $\boldsymbol{K}_2$. From this, it is easy to find a maximal model in $\boldsymbol{K}_2$ of roughly the same cardinality.

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1. Preliminaries

This paper depends heavily on [4] which contains a fuller background and essential material on Boolean algebras. In particular, the incomplete sentence with maximal models cofinal in the first measurable and the construction of the desired complete sentence are described there; in this paper we show in ZFC that sentence has maximal models below the first measurable. We repeat in this section the main slightly nonstandard definitions from Boolean algebra that appear in [4] and some immediate consequences.

Definition 1.1. (1) A Boolean polynomial $p(v_0, \ldots, v_k)$ is a term formed by the compositions of the $\wedge, \vee, ^{-1}, 0, 1$ on the variables v_i; a polynomial over X arises when elements of X are substituted for some of the v_i.

(2) For $X \subseteq B$ and B a Boolean algebra, $\overline{X} = X_B = \langle X \rangle_B$ denotes the subalgebra of B generated by X.

(3) A set Y is *independent* (or *free*) over X *modulo* an ideal $\mathcal{I}$ (with domain I) in a Boolean algebra B if and only if for any Boolean polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0, i.e. nontrivial), and any $a \in \langle X \rangle_B - \mathcal{I}$, and distinct $y_i \in Y$, $p(y_0, \ldots, y_k) \wedge a \notin \mathcal{I}$.

(4) A Y which is independent over X modulo I is called a *basis* for $\langle X \cup Y \cup I \rangle$ over $\langle X \cup I \rangle$.

In this context, 'independent from' may sometimes be written 'independent over'. This notion of independence is distinct from each of (i) a family X of sets is independent if every finite boolean combination of members X is nonempty and (ii) from forking independence.

Observation 1.2. If $\mathcal{I}$ is the 0 ideal, (i.e. Y is independent over X),

(1) the condition becomes: for any $b \in \langle X \rangle_B - \{0\}$, $B \models p(y_0, \ldots, y_k) \wedge b > 0$. That is, every finite Boolean combination of elements of Y has nonempty meet with each nonzero $b \in \langle X \rangle_B$.

(2) or, there is no nontrivial polynomial $q(\mathbf{y}, \mathbf{x})$ and $\mathbf{b} \subseteq X$ such that $q(\mathbf{y}, \mathbf{b}) = 0$.

That (2) implies (1) is obvious. For the converse, put a counterexample $q(\mathbf{y}, \mathbf{b}) = 0$ in disjunctive normal form. Then for each disjunct (i.e. each constituent conjunction) $q'(\mathbf{y}, \mathbf{b}) = 0$ (some variables of q may not appear in q'.) We can replace those b's that appear in q' by a single element b of $\langle X \rangle$ to get a $q''(\mathbf{y}, b) = 0$; q'' contradicts condition (1).

With Observation 1.2 we obtain an analog for Boolean algebras of the notion of dependence in vector spaces in rings or fields: $\{y_0, \ldots, y_k\}$ are dependent over X if some nontrivial polynomial $p(v_0, \ldots, v_k, w_0, \ldots, w_m)$ and some $\mathbf{b}$ from X, $p(\mathbf{y}, \mathbf{b}) = 0$. This yields that if B_2 is freely generated over B_1, all atoms in B_1 remain atoms in B_2. If not, there would be an atom a of B_1 and a term $\sigma(\mathbf{b}_2, \mathbf{b}_1)$ with $0_{B_1} < \sigma(\mathbf{b}_2, \mathbf{b}_1) < a$ and $\sigma(\mathbf{b}_2, \mathbf{b}_1) \in B_1$. But then $B_2 \models \sigma(\mathbf{b}_2, \mathbf{b}_1) \wedge a = 0$; this contradicts the freeness assumption. This notion of dependence (a depends on X if and only if $a \in \langle X \rangle$) does not satisfy the exchange axiom. See [7, Chap. 5] for the strong consequences if this dependence relation satisfies exchange.

There is no requirement that $\mathcal{I}$ be contained in X. Observe the following.

Observation 1.3. Let $\mathcal{I}$ be an ideal in a Boolean algebra B.

(1) Let π map B to $B/\mathcal{I}$. If 'Y is independent from X over $\mathcal{I}$' then the image of Y is free from the image of X (over $\emptyset$) in $B/\mathcal{I}$. Conversely, if $\pi(Y)$ is independent

over $\pi(X)$ in $B/\mathcal{I}$, for any Y' mapping by π to $\pi(Y)$, Y' is independent from X over $\mathcal{I}$.

So, if X is empty, the condition 'Y is independent over $\mathcal{I}$' implies the image of Y is an independent subset of $B/\mathcal{I}$.

(2) If a set Y is *independent* (or *free*) from X over $\mathcal{I}$ in B and Y_0 is a subset of Y, then $Y - Y_0$ is *independent* (or *free*) from $X \cup Y_0$ ($\langle X \cup Y_0 \rangle_B$) over the ideal $\mathcal{I}$ in the Boolean algebra B.

2. Set Theoretic Construction of a Boolean Algebra

We define a property $\boxplus(\lambda)$, which asserts the existence in λ of a Boolean algebra that is 'uniformly $\aleph_1$-incomplete'. We then show certain conditions on λ imply $\boxplus(\lambda)$. So this section has no elaborate model theory. The arguments here are similar to those around [8, p. 7]. We connect this construction with our model theoretic approach in Sec. 3.

Definition 2.1 ($\boxplus(\lambda)$). denotes: There are a Boolean algebra $\mathbb{B} \subset \mathcal{P}(\lambda)$ with $|\mathbb{B}| = \lambda$ and a set $\mathcal{A} \subseteq {}^{\omega}\mathbb{B}$ such that:

(i) $\mathcal{A}$ has cardinality λ and if $\overline{A} = \{A_n : n \in \omega\} \in \mathcal{A}$ then for $\alpha < \lambda$ for all but finitely many n, $\alpha \notin A_n$.

(ii) $\mathbb{B}$ includes the finite subsets of λ; but is such that for every non-principal ultrafilter D of λ (equivalently an ultrafilter of $\mathbb{B}$ that is disjoint from $\lambda^{<\omega}$) for some sequence $\langle A_n : n \in \omega \rangle \in \mathcal{A}$, there are infinitely many n with $A_n \in D$.

We may say that $(\mathbb{B}, \mathcal{A})$ witness uniform $\aleph_1$-incompleteness.

Theorem 2.2 (ZFC). *Assume for some* μ, $\lambda = 2^\mu$ *and* λ *is less than the* first measurable, *then* $\boxplus(\lambda)$ *from Definition* 2.1 *holds.*

We need the following structure, which depends on μ and λ.

Definition 2.3. (1) Fix the vocabulary τ with unary predicates P, U, a binary predicate C, and a binary function F.

(2) Let $\langle C_\alpha : \alpha < \lambda \rangle$ list *without repetitions* $\mathcal{P}(\mu)$ such that $C_0 = \emptyset$ and also let $\langle f_\alpha : \mu \leq \alpha < \lambda \rangle$ list ${}^{\mu}\omega$.

(3) Define the τ-structure M by:
   - (a) The universe of M is λ;
   - (b) $P^M = \omega$; $U^M = \mu$;
   - (c) $C(x, y)$ is a binary relation on $U \times M$ defined by $C(x, \alpha)$ if and only $x \in C_\alpha$. Note that C is extensional. I.e. elements of M uniquely code subsets of U^M;
   - (d) Let $F_2^M(\alpha, \beta)$ map $M \times U^M \to P^M$ by $F_2^M(\alpha, \beta) = f_\alpha(\beta)$ for $\alpha < \lambda$, $\beta < \mu$;
   - (e) $F_2^M(\alpha, \beta) = 0$ for $\alpha < \lambda$ and $\beta \in [\mu, \lambda)$.

We use the following, likely well-known, fact pointed out to us by Sherwood Hachtman.

Fact 2.4. Let $D \subseteq \mathcal{P}(X)$ and suppose that for each partition $Y \subseteq \mathcal{P}(X)$ of X into at most countably many sets, $|D \cap Y| = 1$. Then, D is a countably complete ultrafilter.

We use the following lemma about M to find a Boolean algebra $\mathbb{B}$ in M that satisfies $\boxplus$. We lay the basis for the notion of P-maximality, a counterexample to maximality must occur in a given predicate P (Definition 3.2.1).

Lemma 2.5. *If λ is less than the first measurable cardinal and $\lambda = 2^\mu$ for some μ there is a model M, with $|M| = \lambda$, and a countable vocabulary with P^M denoting the natural numbers such that every first-order proper elementary extension N of M properly extends P^M.*

Proof. Fix M as in Definition 2.3. We first show that any proper elementary extension N of M extends U^M. Suppose for contradiction there exists $\alpha' \in N - M$ but $U^N = U^M$. By the full listing of the C_α, there is a $\beta \in M$ with $\{x : N \models C(x, \beta)\} = \{x : N \models C(x, \alpha')\}$. This contradicts extensionality of the relation C in N; but C is extensional in the elementary submodel M.

Now we show that if $U^M \subsetneq U^N$ and $P^M = P^N$, then there is a countably complete non-principal ultrafilter on μ, contradicting that μ is not measurable. Note that the sequence $\langle f_\alpha : \mu \leq \alpha < \lambda \rangle$ can be viewed as a list of all nontrivial partitions of μ into at most countably many pieces. Let $\nu^* \in U^N - U^M$. For $\alpha \in N$, denote $F_2^N(\alpha, \nu^*)$ by n_α. Since $P^M = P^N$, $n_\alpha \in M$. By elementarity, for $\alpha \in M, \eta \in U^M$, $F_2^N(\alpha, \eta) = F_2^M(\alpha, \eta) = f_\alpha(\eta)$. Now, let

$$D = \{x \subseteq U^M : x \neq \emptyset \wedge (\exists \alpha \in M) x \supseteq f_\alpha^{-1}(n_\alpha)\}.$$

We show D satisfies the conditions from Fact 2.4. Let W be a partition, indexed by f_α. Then $f_\alpha^{-1}(n_\alpha) \neq \emptyset$ and is in D. Suppose for contradiction there are $x_0 \neq x_1$ in W that are both in D. Then, there are $\alpha_i \in M$ such that $x_i \in W \cap D$ contains $f_{\alpha_i}^{-1}(n_{\alpha_i})$ for $i = 0, 1$. So, $N \models F(\alpha_i, \nu^*) = n_{\alpha_i}$ for $i = 1, 2$. Since $\alpha_i \in M$ and $M \prec N$, $M \models \exists x (F(\alpha_0, x) = n_{\alpha_0} \wedge F(\alpha_1, x) = n_{\alpha_1}$. So, by Definition 2.3(d), for any witness a in M for this formula, $a \in x_0 \cap x_1$; but $x_0 \cap x_1 = \emptyset$ since W is a partition.

Finally, D is non-principal on U^M since if it were generated by an $a \in U^M$,

$$D = \{x \subseteq U : (\exists \alpha) x \supseteq f_\alpha^{-1}(n_\alpha)\} = \{x \subseteq U : a \in x\}.$$

Since $\{a\} \in D$, for some $\alpha_0 \in M$, $\{a\} = f_{\alpha_0}^{-1}(n_{\alpha_0})$. Note that $\alpha_0 \in M$, because the definition of D is about the model M. That is, $M \models \exists! y F(\alpha_0, y) = n_{\alpha_0}$. But $N \models F(\alpha_0, a) = n_{\alpha_0} \wedge F(\alpha_0, \nu^*) = n_{\alpha_0}$. This contradicts the assumption $M \prec N$ and completes the proof. ◻

The following claim completes the proof of Theorem 2.2.

Claim 2.6. *If $\mathbb{B}$ is the Boolean algebra of definable formulas in the M defined in Definition 2.3, there is an $\mathcal{A}$ such that $(\mathbb{B}, \mathcal{A})$ is uniformly $\aleph_1$-incomplete so $\boxplus(\lambda)$ holds.*

Proof. We may assume τ has Skolem functions for M and then define $\mathbb{B}$ and $\mathcal{A}$ as follows to satisfy $\boxplus$(ii). Let $\mathbb{B}$ be the Boolean algebra of definable subsets of M. I.e.

$$\mathbb{B} = \{X \subseteq M : \text{for some } \tau\text{-formula } \phi(\mathbf{x}, \mathbf{y}) \text{ and } \mathbf{b} \in {}^{\lg(\mathbf{y})}M, \phi(M, \mathbf{b}) = X\}.$$

Note $\mathbb{B}$ is a Boolean algebra of cardinality λ with the normal operations. We define the Skolem functions a little differently than usual: as maps $\sigma_\phi = \sigma_{\phi(x,w,\mathbf{y})}$ from M^{n+1} to M for formulas $\phi(x, w, \mathbf{y})$ such that $\phi(\sigma_\phi(b, \boldsymbol{a}), b, \boldsymbol{a})$. Here $\lg(\mathbf{y}) = n$. Then, we specialize the Skolem functions by considering the unary function arising from fixing the $\mathbf{y}$ entry of $\sigma_\phi(w, \mathbf{y})$ to obtain $\sigma_\phi(w, \boldsymbol{a})$.

$$\begin{aligned} A_n^{\sigma_\phi(w,\boldsymbol{a})} &= \{\alpha < \lambda :: \phi(\sigma_\phi^M(\alpha, \boldsymbol{a}), \alpha, \boldsymbol{a}) \wedge P(\sigma_\phi^M(\alpha, \boldsymbol{a})) \wedge \sigma_\phi^M(\alpha, \boldsymbol{a}) \not\leq n\} \\ &\quad \cup \{\alpha < \lambda :: n = 0 \wedge \neg P(\sigma_\phi^M(\alpha, \boldsymbol{a})\}. \end{aligned}$$

Then let $\overline{A}_{\sigma_\phi(w,\boldsymbol{a})} = \langle A_n^{\sigma_\phi(w,\boldsymbol{a})} : n < \omega \rangle$ and

$$\mathcal{A} = \{\overline{A}_{\sigma_\phi(w,\boldsymbol{a})} : \text{for some } \tau_M\text{-term } \sigma_\phi(w, \mathbf{y}) \text{ and } \boldsymbol{a} \in {}^{\lg(\mathbf{y})}M\}. \qquad (*)$$

Note $|\mathcal{A}| = \lambda = \lambda^\omega$ as for each $\boldsymbol{a} \in M$ and each of the countably many terms $\sigma_\phi(w, \boldsymbol{a})$, $\overline{A}_{\sigma_\phi(x,w,\boldsymbol{a})}$ is a map from ω into $\mathbb{B}$. For each α, for each $0 < m < \omega$ and $\overline{A} = \overline{A}_{\sigma_\phi(\alpha,\mathbf{b})}$, the set $\{m : \alpha \in A_m\}$ is finite, bounded by $\sigma_\phi(\alpha, \boldsymbol{a})$. Thus, clause (i) of $\boxplus$ is satisfied.

We now show clause (ii) of $\boxplus$. Let D be an arbitrary non-principal ultrafilter on λ and where $\phi(v, \mathbf{y})$ varies over first-order τ-formulas such that $\mathbf{y}$ and $\boldsymbol{a}$ have the same length, define the type $p(x) = p_D(x)$ as

$$p(x) = \{\phi(x, \boldsymbol{a}) : \{\alpha \in M : M \models \phi(\alpha, \boldsymbol{a})\} \in D\}.$$

Since D is an ultrafilter, p is a complete type over M. So there is an elementary extension N of M where an element d realizes p. Let N be the Skolem hull of $M \cup \{d\}$. Since D is non-principal, so is p; thus, $N \neq M$. By Lemma 2.5, we can choose a witness $c \in P^N - P^M$. Since, N is the Skolem hull of $M \cup \{d\}$ there is a Skolem term $\sigma(w, \mathbf{y}) = \sigma_\phi(w, \mathbf{y})$ and $\boldsymbol{a} \in M$ such that $c = \sigma^N(d, \boldsymbol{a})$. Since $c \notin M$, for each $n \in P^M$, $N \models \bigwedge_{k<n} c \neq k$ so $N \models \bigwedge_{k<n} \sigma(d, \boldsymbol{a}) \neq k$ so $\bigwedge_{k<n} \sigma(x, \boldsymbol{a}) \neq k$ is in p. That is, for each σ_ϕ and each n, $A_n^{\sigma_\phi(w,\boldsymbol{a})}$ is in D. $\square$

3. Three Classes of Models and an Approximate Counterexample

In this section, we define the model theoretic classes that produce first an amalgamation class $\boldsymbol{K}_{-1}$ of finitely generated structures (Sec. 3.1), then the class $\boldsymbol{K}_2$ (Definition 3.3.2) of models of a complete $L_{\omega_1,\omega}$-sentence. Using Theorem 2.2, we build in Sec. 3.2 a model M_* in $\boldsymbol{K}_{-1}$ with cardinality λ, which is P_0-maximal. Section 3.3 defines the classes $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ which give us the complete sentence. In Sec. 4 we modify M_* to a P_0-maximal model in $\boldsymbol{K}_2$ and then construct the required maximal model in $\boldsymbol{K}_2$.

3.1. *Finitely generated models*

The class $\boldsymbol{K}^{-1}_{<\aleph_0}$ and the class of its direct limits, $\boldsymbol{K}_{-1}$ were introduced in [4].

Definition 3.1.1. τ is a vocabulary with unary predicates P_0, P_1, P_2, P_4, binary R, $\wedge, \vee, \leq$ unary functions $^-$, G_1, constants 0,1 and unary functions F_n, for $n < \omega$. $\leq$ is a partial order on P^M_1 and the Boolean algebra can be defined from it.

We occasionally use the notations $(\forall^\infty n)$ and $(\exists^\infty n)$ to mean 'for all but finitely many' and 'for infinitely many', respectively. It is easy to see that $\boldsymbol{K}_{-1}$ is $L_{\omega_1,\omega}$-axiomatizable but far from complete.

Definition 3.1.2 ($\boldsymbol{K}_{-1}$). $\boldsymbol{K}^{-1}_{<\aleph_0}$ is the class of *finitely generated* structures M satisfying the following conditions:

(1) P^M_0, P^M_1, P^M_2 partition M.
(2) $(P^M_1, 0, 1, \wedge, \vee, \leq, ^-)$ is a Boolean algebra ($^-$ is complement). We also consider ideals and restrictions to them of the relations/operations except for complement.
(3) $R \subset P^M_0 \times P^M_1$ with $R(M, b) = \{a : R^M(a, b)\}$ and the set of $\{R(M, b) : b \in P^M_1\}$ is a Boolean algebra. $f^M : P^M_1 \mapsto \mathcal{P}(P^M_0)$ by $f^M(b) = R(M, b)$ is a Boolean algebra homomorphism into $\mathcal{P}(P^M_0)$.

Note that f is not[a] in τ; it is simply a convenient abbreviation for the relation between the Boolean algebra P^M_1 and the set algebra on P_0 by the map $b \mapsto R(M, b)$.
(4) $P^M_{4,n}$ is the set containing each join of n distinct atoms from P^M_1; P^M_4 is the union of the $P^M_{4,n}$ and so is an ideal. That is, P^M_4 is the set of all finite joins of atoms.

There is an element $b^* \in P^M_1$ such that $P^M_4 = \{c : c \leq^M b_*\}$. Note that b_* is not a function symbol in τ.
(5) G^M_1 is a bijection from P^M_0 onto $P^M_{4,1}$ such that $R(M, G^M_1(a)) = \{a\}$. Note that $P^M_0 = \emptyset$ is allowed.
(6) P^M_2 is finite (and may be empty). Further, for each $c \in P^M_2$ the $F^M_n(c)$ are functions from P^M_2 into P^M_1. Note that it is allowed that for all but finitely many n, $F^M_n(c) = 0_{P^M_1}$.
(7) (*countable incompleteness*) If $a \in P^M_{4,1}$ and $c \in P^M_2$ then $(\forall^\infty n)\ a \not\leq_M F^M_n(c)$. Since $a \wedge F^M_n(c) = 0$ and a is an atom, this implies $\bigwedge_{n\in\omega}\{x : (G_1(x) \in F^M_n(c)\} = 0$.
(8) P^M_1 is generated as a Boolean algebra by $P^M_4 \cup \{F^M_n(c) : c \in P^M_2, n \in \omega\} \cup X$ where X is a finite subset of P^M_1.

Definition 3.1.3. (1) K_{-1} is the class of τ structures M such that every finitely generated substructure of M is in $\boldsymbol{K}^{-1}_{<\aleph_0}$. K^{-1}_μ is the members of K_{-1} with cardinality μ.

[a]The subsets of P^M_0 are *not* elements of M.

(2) We say $M \in K_{-1}$ is *atomic* if P_1^M is atomic as a Boolean algebra. That is, P_4^M is dense in $\boldsymbol{B}_M$.

3.2. *A* P_0*-maximal model in* $\boldsymbol{K}_{-1}$

In this section, we invoke Theorem 2.2 to show (Theorem 3.2.6) that we can construct P_0-maximal structures in the class $\boldsymbol{K}_{-1}$ of appropriate cardinality below the first measurable.

Definition 3.2.1. We say $M \in K_{-1}$ is P_0-*maximal* (in K_{-1}) if $M \subseteq N$ and $N \in K_{-1}$ implies $P_0^M = P_0^N$.

The notion uf(M) is the crucial link between Sec. 2 and P_0-maximality. Lemma 3.2.4 is central for Theorem 3.2.6 and is applied in Theorem 4.9.

Definition 3.2.2 (uf(M)). For $M \in K_{-1}$, let uf(M) be the set of ultrafilters D of the Boolean Algebra P_1^M such that $D \cap P_{4,1}^M = \emptyset$ and for each $c \in P_2^M$ only finitely many of the $F_n^M(c)$ are in D.

For applications we rephrase this notion with the following terminology. For any $M \in \boldsymbol{K}_{-1}$ and $d \in P_2^M$, let $S_d^M(D) = \{n : F_n^M(d) \in D\}$. So uf$(M) = \emptyset$ if and only if for every ultrafilter D on P_1^M, there exists a $d \in P_2^M$ such that $S_d^M(D)$ is infinite.

We use the following standard properties of a Boolean algebra B and ideal I in proving Lemma 3.2.4 and deducing Claim 3.2.9 from Definition 3.2.8.

Fact 3.2.3. (1) $b \wedge c \in I$ implies b/I and c/I are disjoint.
(2) $b \triangle c \in I$ implies $b/I = c/I$.
(3) $b - c \in I$ implies $b/I \leq c/I$.

For our collection of structures $\boldsymbol{K}_{-1}$, we can characterize P_0-maximality in terms of ultrafilters.

Lemma 3.2.4. *An* $M \in \boldsymbol{K}_{-1}$ *is* P_0*-maximal if and only if* uf$(M) = \emptyset$.

Proof. Suppose M is not P_0-maximal and $M \subset N$ with $N \in \boldsymbol{K}_{-1}$ and $d^* \in P_0^N - P_0^M$. Then $\{b \in M : R^N(d^*, b)\}$ is a non-principal ultrafilter D_0 of the Boolean algebra P_1^M [4, 3.3.11].To see D_0 is non-principal suppose there is a $b_0 \in P_1^M$ such that $D_0 = \{b \in M : b_0 \leq b\}$. Note $b_0 = G_1^M(a)$ for some $a \in P_0^M$. But $N \models G_1^N(d^*) \not\geq b_0$, contradicting $\{d^*\} \in D_0$.

For each $c \in P_2^M$, since $N \in \boldsymbol{K}_{-1}$, by countable incompleteness (clause 7 of Definition 3.1.2), for all $a \in P_0^N$ and all but finitely many n, $G_1^N(a) \not\leq F_n^N(c)$. Since $F_n^N(c) = F_n^M(c)$, only finitely many of the $F_n^M(c)$ can be in D_0, which implies $D_0 \in$ uf(M). By contraposition we have the right to left.

Conversely, if $D \in$ uf(M), we can construct an extension by adding an element $d \in P_0^N$ satisfying $R^N(d, b)$ if and only if $b \in D$. Let P_1^N be the Boolean algebra generated by $P_1^M \cup \{G_1(d)\}$ modulo the ideal generated by $\{G_1^N(d) - b : b \in D\}$;

this implies that in the quotient $G_1(d) \leq b$. (Compare Fact 3.2.3). Let $P_2^N = P_2^M$ and $F_n^N(c) = F_n^M(c)$. Since $D \in \mathrm{uf}(M)$, it is easy to check that $N \in \boldsymbol{K}_{-1}$. □

We now introduce the requirement that the Boolean algebras constructed will, when the atoms are factored out, be free. Moreover, there is a set $Y \subseteq P_2^N$ with $|Y| = \lambda$ such that different $c \in Y$ generate coinitially disjoint collections of $F_n^N(c)$ as c varies. This strong requirement is used inductively in this section to construct an approximation to the counterexample. The correction in Sec. 4 loses this disjointness (and thus freeness).

Definition 3.2.5 (Nicely free). We say $M \in K_{-1}$ is *nicely free* when $|P_1^M| = \lambda$ and there is a sequence $\mathbf{b} = \langle b_\alpha : \alpha < \lambda \rangle$ such that:

(a) $b_\alpha \in P_1^M - P_4^M$;
(b) $\langle b_\alpha / P_4^M : \alpha < \lambda \rangle$ generate P_1^M / P_4^M freely;
(c) there is a set $Y \subset P_2^M$ of cardinality λ such that $\{F_n(c) : n < \omega; c \in Y\}$ without repetition is a subset of the basis $\{b_\alpha : \alpha < \lambda\}$ mod atoms. For $c \in Y$, we write $u_c = \{F_n^M(c) : n < \omega\}$.

Nicely free is quite distinct from the notion $\boldsymbol{K}_1$-*free* introduced in [4]. There are maximal nicely free models but there are no maximal $\boldsymbol{K}_1$-free models. Note that condition Definition 3.2.5(c) asserts that a *sub*set of P_2^M partitions a *sub*set of the basis.

Here is the main theorem of Sec. 3. The hypotheses $\lambda = 2^\mu$ and λ is less than the first measurable cardinal were used essentially as the hypotheses for proving $\boxplus(\lambda)$, the existence of a uniformly $\aleph_1$-incomplete Boolean algebra. But here we use $\boxplus(\lambda)$ and do not rely again on λ being less than the first measurable cardinal. The argument here does depend on $\lambda = \lambda^{\aleph_0}$, which follows from $\lambda = 2^\mu$. By constructing a nicely free model, we introduce at this stage the independence requirements, needed in Sec. 4 to satisfy Definition 3.3.1(6), on the $F_n(c)$.

Theorem 3.2.6. *If for some μ, $\lambda = 2^\mu$ and λ is less than the first measurable cardinal then there is a P_0-maximal model M_* in $\boldsymbol{K}_{-1}$ such that $|P_i^{M_*}| = \lambda$ (for $i = 0, 1, 2$), $P_1^{M_*}$ is an atomic Boolean algebra, $\mathrm{uf}(M_*) = \emptyset$ and M_* is nicely free.*

Proof. We first construct by induction a P_0-maximal model in $\boldsymbol{K}^{-1}$. The property $\boxplus(\lambda)$ (Definition 2.1) appears in the construction to satisfy specification (f) and is used in the proof that the construction works in considering possibility 2. We choose M_ϵ, D_ϵ and other auxiliaries by induction for $\epsilon \leq \omega + 1$ to satisfy the following *specifications* of the construction.

Construction 3.2.7 (Specifications). (a) For $\epsilon \leq \omega + 1$, M_ϵ is a continuous increasing chain of members of K_λ^{-1} with each $P_1^{M_\epsilon}$ atomic and $P_1^{M_{\omega+1}} = P_1^{M_\omega}$.
(b) For all $\epsilon \leq \omega$, $|P_i^{M_\epsilon}| = \lambda$ and $P_i^{M_\omega} = P_i^{M_{\omega+1}}$ for $i = 0, 1$.
(c) For all $\epsilon \leq \omega + 1$, $P_1^{M_\epsilon} / P_4^{M_\epsilon}$ is a free Boolean algebra.

(d) (i) If $\epsilon < \omega$, $D_\epsilon \in \mathrm{uf}(M_\epsilon)$.
 (ii) If $\epsilon = 0$, then $\mathbf{b}_{-1} = \langle b_{-1,\alpha} : \alpha < \lambda\rangle$ is a free basis of $P_1^{M_0}/P_4^{M_0}$, listed without repetition as $\langle F_n^{M_0}(c) : n < \omega, c \in P_2^{M_0}\rangle$.
 (iii) if $\epsilon = \zeta + 1 < \omega$ then there is a free basis $\mathbf{b}_\zeta = \langle b_{\zeta,\alpha}/P_4^{M_\zeta} : \alpha < \lambda\rangle$ of $P_1^{M_\epsilon}/P_4^{M_\epsilon}$. Note $b_{\zeta,\alpha} \in P_1^{M_\epsilon} - P_1^{M_\zeta}$.

(e) if $\epsilon = \omega + 1$, for each $\overline{d} \in {}^\omega(P_1^{M_{\omega+1}} - P_4^{M_{\omega+1}})$ such that for each $a \in P_0^{M_\omega}$ satisfying that all but finitely many n, $a \notin R(M_\omega, d_n)$, there is a $c \in P_2^{M_{\omega+1}}$, $F_n^{M_{\omega+1}}(c) = d_n$; (We will in fact have that $P_1^{M_{\omega+1}} = P_1^{M_\omega}$ and $P_4^{M_{\omega+1}} = P_4^{M_\omega}$.)

(f) $\epsilon = \zeta + 1 < \omega$:

Let $\mathbb{B}$ and $\mathcal{A}$ be as in Definition 2.1. There is a 1-1 function f_ϵ from λ onto $P_{4,1}^{M_\epsilon}$ such that:

 (i) for every $X \in \mathbb{B}$ (from $\boxplus$) there is a $b = b_X \in P_1^{M_\epsilon}$ such that

$$\{\alpha < \lambda : f_\epsilon(\alpha) \leq_{M_\epsilon} b_X\} = X,$$

 (ii) for each $\overline{A} = \langle A_n : n < \omega\rangle \in \mathcal{A}$ there is a $c \in P_2^{M_\epsilon}$ such that for each n:

$$A_n = \{\alpha < \lambda : f_\epsilon(\alpha) \leq_{M_\epsilon} F_n^{M_\epsilon}(c)\}.$$

Carrying out the construction.

Case 1. When $\epsilon = 0$, take $P_1^{M_0}$ as the Boolean algebra generated by a set $P_{4,1}^{M_0}$ of cardinality λ along with a set $\{b_{-1,\alpha} : \alpha < \lambda\}$ of independent subsets of $\mathcal{P}(\lambda)$. Let G_1 be a bijection between a set $P_0^{M_0}$ and $P_{4,1}^{M_0}$. Set $P_4^{M_0}$ as the ideal generated by the image of G_1. For $a \in P_0^{M_0}$ and $b \in P_1^M$, define $R^{M_0}(a,b)$ to hold if $G_1(a) \leq b$. Set $P_2^{M_0}$ as a set of cardinality of λ and let $\langle F_n^{M_0}(c) : n < \omega, c \in P_2^{M_0}\rangle$ list $\langle b_{-1,\alpha} : \alpha < \lambda\rangle$ without repetition. Thus, any non-principal ultrafilter on $P_1^{M_0}$ is in $\mathrm{uf}(M_0)$.

Case 2. For $\epsilon = \omega$, $M_\omega = \bigcup_{n<\omega} M_n$. Since the set of free generators is extended at each finite step, the union is also free mod P_4^M.

Case 3. If $\epsilon = \zeta + 1 < \omega$, the main effort is to verify clauses (c), (d) and (f) of Specification 3.2.7. The element b_{ζ,a_α} is the b_{A_α} from Specification 3.2.7(f)(i).

Now, to construct M_ϵ:

(i) Recall that $D_\zeta \in \mathrm{uf}(M_\zeta)$.
(ii) Choose as the new atoms introduced at this stage a set $B_\epsilon \subseteq \mathcal{P}(\lambda)$ with $B_\epsilon \cap M_\zeta = \emptyset$ and $|B_\epsilon| = \lambda$.
(iii) Let f_ϵ be a one-to-one function from λ onto $B_\epsilon \cup P_{4,1}^{M_\zeta}$.
(iv) Let $\langle X_\gamma : \gamma < \lambda\rangle$ list the elements of $\mathbb{B}$ (definable subsets of M 2.6) from $\boxplus$(ii) with $X_0 = \emptyset$.
(v) Fix a sequence $\{b_{\zeta,\alpha} : \alpha < \lambda\}$, which are distinct and not in $M_\zeta \cup B_\epsilon$, and let $\mathbb{B}'_\zeta$ be the Boolean Algebra generated freely by

$$P_1^{M_\zeta} \cup \{b_{\zeta,\alpha} : \alpha < \lambda\} \cup \{f_\epsilon(\alpha) : \alpha < \lambda\}.$$

Using Lemma 3.2.3, we apply the following definition at the successor stage. Here we take an abstract Boolean algebra $\mathbb{B}'_\zeta$ and impose relations to embed $P_1^{M_\zeta}$ in a quotient $\mathbb{B}''_\zeta$ of $\mathbb{B}'_\zeta$.

Definition 3.2.8 (Ideal). Let I_ζ be the ideal of $\mathbb{B}'_\zeta$ generated by:

(i) $\sigma(a_0, \dots, a_m)$ when $\sigma(x_0, \dots, x_m)$ is a Boolean term, $a_0, \dots, a_m \in P_1^{M_\zeta}$ and $P_1^{M_\zeta} \models \sigma(a_0, \dots, a_m) = 0$.

The next two clauses aim to show that in M_ζ/I_ζ, the element $b_{\zeta,\gamma}$ is the b_{X_γ} from Specification 3.2.7(f)(i). That is, $\{\alpha < \lambda : f_\epsilon(\alpha) \leq_{M_\epsilon} b_{\gamma,\zeta}\} = X_\gamma$. Recall (Definition 2.1) that the X_γ enumerate $\mathbb{B}$ and are subsets of λ.

(ii) $f_\epsilon(\alpha) - b_{\zeta,\gamma}$ when $\alpha \in X_\gamma$ and $\alpha, \gamma < \lambda$.

(iii) $b_{\zeta,\gamma} \wedge f_\epsilon(\alpha)$ when $\alpha \in \lambda - X_\gamma$ and $\alpha, \gamma < \lambda$.

To show the $f_\epsilon(\gamma)$ are disjoint atoms we add:

(iv) For any $f_\epsilon(\gamma)$ and any $b \in \mathbb{B}'_\zeta$ either $(f_\epsilon(\gamma) \wedge b) \in I_\zeta$ or $(f_\epsilon(\gamma) - b) \in I_\zeta$.

(v) $f_\epsilon(\gamma_1) \wedge f_\epsilon(\gamma_2)$ when $\gamma_1 < \gamma_2 < \lambda$;

(vi) $f_\epsilon(\alpha) - b$ when $\alpha < \lambda$, $f_\epsilon(\alpha) \notin P_{4,1}^{M_\zeta}$ and $b \in D_\zeta$.

This asserts: Every new atom is below each $b \in D_\zeta$ and is used at the end of Case 3 of the construction.

Let $\mathbb{B}''_\zeta = \mathbb{B}'_\zeta/I_\zeta$. Applying Fact 3.2.3, we see from Definition 3.2.8.

Claim 3.2.9. *The structure $P_1^{M_\zeta}$ is embedded as a Boolean algebra into $\mathbb{B}''_\zeta$ by the map $b \mapsto b/I_\zeta$ and*

(1) *For $\gamma < \lambda$, $f_\zeta(\gamma)/I_\zeta$ is an atom of $\mathbb{B}''_\zeta$;*

(2) *If $b \in P_1^{M_\zeta}$ is nonzero, then $b/I_\zeta \geq_{\mathbb{B}''_\zeta} f_\epsilon(\gamma)$ for some $\gamma < \lambda$. (Since f_ϵ^{-1} induces an isomorphism of $\mathbb{B}''_\zeta$ into $\mathcal{P}(\lambda)$.)*

We take a further quotient of $\mathbb{B}'_\zeta$. Let

$$J_\zeta = \{b \in \mathbb{B}'_\zeta : b/I_\zeta \wedge_{\mathbb{B}''_\zeta} f_\epsilon(\gamma) = 0 \text{ for every } \gamma < \lambda\}.$$

Then J_ζ is an ideal of $\mathbb{B}'_\zeta$ extending I_ζ so $b \mapsto b/J_\zeta$ is a homomorphism. Further, $f_\epsilon(\gamma)$ is an atom of $\mathbb{B}'_\zeta/J_\zeta$ for $\gamma < \lambda$. These atoms are distinct and dense in $\mathbb{B}'_\zeta/J_\zeta$. That is, $\mathbb{B}_\epsilon$ is an atomic Boolean algebra.

Notation 3.2.10. Let $\mathbb{B}_\epsilon$ be $\mathbb{B}'_\zeta/J_\zeta$ with quotient map, $j_\epsilon(b) = b/J_\zeta$.

Now we define M_ϵ by setting $P_1^{M_\epsilon} = \mathbb{B}_\epsilon$ which contains $P_1^{M_\zeta}$; $P_{4,1}^{M_\epsilon}$ is the injective image in $P_1^{M_\epsilon}$ of $P_{4,1}^{M_\zeta} \cup B_\epsilon$. For $a \in P_{4,1}^{M_\epsilon}$ and $b \in P_1^{M_\epsilon}$, set $R^{M_\epsilon}(a, b)$ if for some γ, $a = f_\epsilon(\gamma)/J_\zeta$ and $f_\epsilon(\gamma)/J_\zeta \leq_{\mathbb{B}_\epsilon} b/J_\zeta$. Finally, let D_ϵ be the ultrafilter on $P_1^{M_\epsilon}$ generated by

$$D_\zeta \cup \{j_\epsilon(-b_{\zeta,\gamma}) : \gamma < \lambda\} \cup \{j_\epsilon(-f_\epsilon(\gamma)) : \gamma < \lambda\}.$$

We verify $M_\epsilon \in \boldsymbol{K}_{-1}$ below. By Claim 3.2.9, we have the cardinality and atomicity conditions of Specifications 3.2.7(a) and 3.2.7(b); the definition of I_ζ guarantees,

(c) and (d)(ii), (d)(iii). The elements $b_{\zeta,\gamma}$ along with (our later) definition of $F_n^{M_\epsilon}(c)$ show d.i), $D_\epsilon \in \mathrm{uf}(M_\epsilon)$, (as no new $F_n(c)$ is in D_ϵ); the elements of B_ϵ show D_ϵ is non-principal as each complement of an atom is in the ultrafilter. Note that Specification 3.2.7(e) does not apply except in the $\omega+1$st stage of the construction.

For Specification 3.2.7(f)(i), let $X \in \mathbb{B}$ be a set of atoms of M_ϵ and note that we can choose b_X by conditions (ii) and (iii) in Definition 3.2.8 of I_ζ.

We can choose $P_2^{M_\epsilon}$ and $F_n^{M_\epsilon}$ to satisfy Specification 3.2.7(f)(ii). Fix an $\overline{A} \in \mathcal{A}$ (as given by $\boxplus$). Fix a $c = c_{\overline{A}}$ and define, using the last paragraph, the $F_n^{M_\epsilon}(c)$ as b_{A_n}, so that for each n, $A_n = \{\alpha < \lambda : f_\epsilon(\alpha) \leq_{P_1^{M_\epsilon}} F_n^{M_\epsilon}(c)\}$. These are the only new $c \in P_2^{M_\epsilon}$.

Thus, it remains only to show that $M_\epsilon \in \boldsymbol{K}_{-1}$. Most of the cases are obvious. E.g. for Definition 3.1.2(8), just look at where the generators can be and recall countable free algebras are atomless. Showing M_ϵ satisfies countable incompleteness, Definition 3.1.2(7), is a bit more complex but we do so now.

($\blacklozenge$) If $a \in P_{4,1}^{M_\epsilon}$ and $c \in P_2^{M_\epsilon}$ then $(\forall^\infty n)\ a \not\leq_{M_\epsilon} F_n^{M_\epsilon}(c)$.

If $c \in P_2^{M_\zeta}$, $F_n^{M_\epsilon}(c) = F_n^{M_\zeta}(c) \in P_1^{M_\zeta}$ and we know by induction that $\blacklozenge$ holds for $a \in P_{4,1}^{M_\zeta}$. For $a \in P_{4,1}^{M_\epsilon} - P_{4,1}^{M_\zeta}$, Definition 3.1.2(5), and condition (vi) on I_ζ (from Definition 3.2.8) imply $a \leq_{M_\epsilon} b$ for every $b \in D_\zeta$. As $c \in P_2^{M_\zeta}$ and $D_\zeta \in \mathrm{uf}(M_\zeta)$, all but finitely many n, $e_n = F_n^\zeta(c)$, are *not* in D_ζ. So for all but finitely many n, the complement $e_n^- \in D_\zeta$. That is, $a \leq_{M_\epsilon} e_n^-$; so $a \wedge_{M_\epsilon} e_n = \emptyset$ as required.

If $c \in P_2^{M_\epsilon} - P_2^{M_\zeta}$ then by our choice of $P_2^{M_\epsilon}$ and the $F_n^{M_\epsilon}$, there is an $\overline{A}_c$ that is enumerated by the $F_n^{M_\epsilon}(c)$ and satisfies $\blacklozenge$ by (i) of $\boxplus$ (Definition 2.1(i)). This completes the verification of $\blacklozenge$ at stage ϵ and so M_ϵ satisfies all the specifications of the induction.

Case 4. $\epsilon = \omega + 1$:

Only clauses (c) and (e) of Specification 3.2.7 are relevant. Define $P_2^{M_\epsilon}$ and $F_n^{M_\epsilon}$ to satisfy clause (e). Since $P_i^{M_\epsilon} = P_i^{M_\omega}$ for $i = 0, 1$, specification (c) is immediate. This completes the construction.

The construction suffices.

Having completed the induction, let $M = M_{\omega+1}$. Using specifications (d) and (a) of 3.2.7, it is straightforward to verify that $M \in \boldsymbol{K}_{-1}$ and the Boolean algebra is atomic. By (b), $P_i^{M_\omega}$ for $i = 0, 1$ have cardinality λ. And by (f), the same holds for $P_2^{M_{\omega+1}}$.

We now show M is nicely free. Let $\mathbf{b} = \langle b'_\beta : \beta < \lambda \rangle$ enumerate $\langle b_{n,\alpha} : n < \omega, \alpha < \lambda \rangle$ without repetition and such that $\{b_{-1,\alpha} : \alpha < \lambda\} = \{b'_{2\alpha} : \alpha < \lambda\}$. So this picks out a first level of generators for P_1^M which is enumerated by the $F_n^{M_0}(c)$ for $c \in P_2^{M_0}$ and $n < \omega$ by Case 1 of the construction.

Now, $\mathbf{b}$ satisfies the requirements in Definition 3.2.5 of nicely free. As, by Specifications 3.2.7(c) and 3.2.7(d) and since P_1^M is constructed as the union of the $P_1^{M_n}$,

P_1^M/P_4^M is generated freely by $\mathbf{b}/P_4^M$. Finally, clause (c) of Definition 3.2.5 holds by clause (d)(ii) of Specification 3.2.7.

The crux is to show $M = M_{\omega+1}$ is P_0-maximal. For this, assume for a contradiction:

(*) P_0^M is not maximal; by Lemma 3.2.4, there is a $D \in \mathrm{uf}(M_{\omega+1}) = \mathrm{uf}(M_\omega)$.

For every $n < \omega$, is there a $d \in D$ such that $R(M_\omega, d) \cap M_n = \emptyset$?

Ask: **Possibility 1:** For every $n < \omega$, the answer is yes, exemplified by $d_n \in D$. Now for each $a \in P_0^{M_n}$, $a \notin R(M_\omega, d_m)$ for all $m \geq n$. So the sequence $\overline{d} = \langle d_n : n < \omega \rangle$ satisfies the hypothesis of Specification 3.2.7(e) and so there is a $c \in P_2^M$ such that for each $n < \omega$, $F_n^M(c) = d_n$. Thus, recalling Definition 3.2.2, $D \notin \mathrm{uf}(M)$.

Possibility 2: For some $n < \omega$, there is no such d_n; without loss of generality, assume $n > 0$. We apply specification f) with $\epsilon = n$. Recall that f_n is a 1-1 map from λ onto $P_{4,1}^{M_n}$. Let g_1 be the following homomorphism from the Boolean algebra $P_1^{M_{\omega+1}} = P_1^{M_\omega}$ into $\mathcal{P}(\lambda)$: $g_1(b) = \{\alpha < \lambda : f_n(\alpha) \leq_{\mathbb{B}_{M_\omega}} b\}$. By Specification 3.2.7(f)(i), the Boolean algebra $\mathbb{B}$ provided by $\boxplus$ is contained in the range of g_1.

Let $\mathcal{I}_n$ denote the ideal of P_1^M generated by $P_{4,1}^M - P_{4,1}^{M_n}$. Since D is non-principal, $\mathcal{I}_n \cap D = \emptyset$. Now, g_1 maps any $b \in P_1^{M_\omega} - P_4^{M_\omega}$ (and thus, any $b \in P_1^{M_\omega} - \mathcal{I}_n$) to a nonempty subset of λ. Recalling $\mathcal{I}_n \cap D = \emptyset$, $D_1 = g_1(D)$ is an ultrafilter of the Boolean Algebra $\mathrm{rg(g_1)}$ and so $D_2 = D_1 \cap \mathbb{B}$ is an ultrafilter of the Boolean algebra $\mathbb{B}$. We show D_2 is non-principal, i.e. for any $\alpha < \lambda$, $\{\alpha\} \notin D_2$. As, $f_n(\alpha) \in P_{4,1}^{M_\omega}$ and so $f_n(\alpha)$ is not in D. So $\{\alpha\} \notin D_1$. Thus, $\lambda - \{\alpha\} \in D_1$ and so $\lambda - \{\alpha\} \in D_2$. So $\{\alpha\} \notin D_2$ as promised.

Now we apply the second clause of $\boxplus$ to the ultrafilter D_2. Since we satisfied specification (f)(ii) in the construction, we can conclude there is $\overline{A} = \langle A_n : n < \omega \rangle \in \mathcal{A}$ such that for infinitely many k, A_k is in D_2. Thus, $u = \{k : A_k \in D\}$ is infinite. We will finish the proof by showing there is a c such that $u = u_c$ (Definition 3.2.5) is the set of images of the $F_n^M(c)$.

Since we are in possibility (2), if $A_k \in \mathbb{B}$ then $A_k \in \mathrm{rg(g_1)}$. So we can choose $d_k \in P_1^{M_\omega}$ with $g_1(d_k) = A_k$. As $A_k \in D_2$, by the choice of D_1, D_2 we have d_k is in the ultrafilter D from the hypothesis for contradiction: (*).

We show the sequence $\overline{d} = \langle d_k : k < \omega \rangle$ satisfies the hypothesis of clause e of Specification 3.2.7. First, $d_k \in P_1^{M_\omega} - P_4^{M_\omega}$ as D is a non-principal ultrafilter on $P_1^{M_\omega}$ so the first hypothesis is satisfied. Further, for every $a \in P_0^{M_\omega}$ all but finitely many k, $G_1^{M_\omega}(a) \not\leq_{M_\omega} d_k$ because $\overline{A} \in \mathcal{A}$, which implies by $\boxplus$(ii) that for every $\alpha < \lambda$, for some k_α, we have $k \geq k_\alpha$ implies $\alpha \notin A_k$. Now by the definition of g_1, recalling $g_1(d_k) = A_k$, we have $k \geq k_\alpha$ implies $f_k(\alpha) \not\leq d_k$ (in $P_1^{M_\omega}$). So by Specification 3.2.7(f)(ii), there is a $c \in P_2^{M_n}$ such that if for all $k < \omega$, $F_k^{M_n}(c) = d_k$. So, for each finite k, $d_k \in D$ and $F_k^{M_{\omega+1}}(c) = d_k$. This contradicts $D \in \mathrm{uf}(M_{\omega+1})$ and we finish. □

3.3. $\boldsymbol{K}_1$ *and* $\boldsymbol{K}_2$

We now introduce further terminology from [4]. We first describe three subclasses of $\boldsymbol{K}_{-1} : K^1_{<\aleph_0}$, the finitely generated models, their direct limits $\boldsymbol{K}_1$ and then the subclass $\boldsymbol{K}_2$, the models of the complete sentence.

Definition 3.3.1 ($\boldsymbol{K}^1_{<\aleph_0}$ defined). M is in the class of structures $\boldsymbol{K}^1_{<\aleph_0}$ if $M \in \boldsymbol{K}^{-1}_{<\aleph_0}$ and there is a *witness* $\langle n_*, \boldsymbol{B}, b_*\rangle$ such that:

(1) $b_* \in P_1^M$ is the supremum of the finite joins of atoms in P_1^M. Further, for some k, $\bigcup_{j\le k} P^M_{4,j} = \{c : c \le b_*\}$ and for all $n > k$, $P^M_{4,n} = \emptyset$.
(2) $\boldsymbol{B} = \langle B_n : n \ge n_*\rangle$ is an increasing sequence of finite Boolean subalgebras of P_1^M.
(3) $B_{n_*} \supsetneq \{a \in P_1^M : a \le b_*\} = P_4^M$; the subset

$$P_4^M \cup \{F_n^M(c) : n < n_*, c \in P_2^M\}$$

generates B_{n_*}.

Moreover, the Boolean algebra B_{n_*} is free over the ideal P_4^M (equivalently, B_{n_*}/P_4^M is a free Boolean algebra[b]).
(4) $\bigcup_{n\ge n_*} B_n = P_1^M$.
(5) P_2^M is finite and not empty. Further, for each $c \in P_2^M$ the $F_n^M(c)$ for $n < \omega$ are independent over P_4^M.
(6) The set $\{F_m^M(c) : m \ge n_*, c \in P_2^M\}$ (the enumeration is without repetition) is free from B_{n_*} over P_4^M, $B_{n_*} \supsetneq P_4^M$ and $F_m^M(c) \wedge b_* = 0$ for $m \ge n_*$. (In this definition, $0 = 0^{P_1^M}$.)

In detail, let $\sigma(\ldots x_{c_i} \ldots)$ be a Boolean algebra term in the variables x_{c_i} (where the c_i are in P_2^M) which is not identically 0. Then, for finitely many $n_i \ge n_*$ and a finite sequence of $c_i \in P_2^M$:

$$\sigma(\ldots F^M_{n_i}(c_i)\ldots) > 0.$$

Further, for any nonzero $d \in B_{n_*}$ with $d \wedge b_* = 0$, (i.e. $d \in B_n - P^4_M$),

$$\sigma(\ldots F^M_{n_i}(c_i)\ldots) \wedge d > 0.$$

(7) For every $n \ge n_*$, B_n is generated by $B_{n_*} \cup \{F_m^M(c) : n > m \ge n_*, c \in P_2^M\}$. Thus, P_1^M and so M is generated by $B_{n_*} \cup P_2^M$.

Recall some terminology from [4].

Definition 3.3.2 ($\boldsymbol{K}_1, \boldsymbol{K}_2$ defined). (1) $\boldsymbol{K}_1$ denotes the collection of all direct limits of models in $\boldsymbol{K}^1_{<\aleph_0}$.

(2) We say a model M in $\boldsymbol{K}_1$ is rich if for any $N_1, N_2 \in \boldsymbol{K}^1_{<\aleph_0}$ with $N_1 \subseteq N_2$ and $N_1 \subseteq M$, there is an embedding of N_2 into M over N_1.
(3) $\boldsymbol{K}_2 \subseteq \boldsymbol{K}_1$ is the class of rich models.

[b] A further equivalence: $|Atom(B_{n_*})|/|P^M_{4,1}|$ is a power of two.

Note that the free generation in item 6 of Definition 3.3.1 is not preserved by arbitrary direct limits and so is not a property of each model in $\boldsymbol{K}_1$. In particular, as M_* is corrected to a model of $\boldsymbol{K}_1$, we check the freeness only for finitely generated submodels as it will be false in general.

Since $\boldsymbol{K}^1_{<\aleph_0}$ has joint embedding, amalgamation and only countably many finitely generated models, we construct in the usual way a generic model; thus $\boldsymbol{K}_2$ is not empty.

Fact 3.3.3. There is a countable generic model M for $\boldsymbol{K}_1$ [4, Corollary 3.2.18]. We denote its Scott sentence by ϕ. $\boldsymbol{K}_2$ is the class of models of this ϕ.

4. Correcting M_* to a Model of $\boldsymbol{K}_2$

We now 'correct' the P_0-maximal model of $\boldsymbol{K}_{-1}$, M_*, constructed in Sec. 3, to obtain a P_0-maximal model M (Definition 3.2.1) of the complete sentence constructed in [4], i.e. $M \in \boldsymbol{K}_2$. In Theorem 4.18 we modify M_*, to construct a model $M \in \boldsymbol{K}_2$ with $P_2^M \subseteq P^{M_*}$ by redefining the F_n, but retaining $M \restriction (P_0^M \cup P_1^M) = M_* \restriction (P_0^{M_*} \cup P_1^{M_*})$. The old values of $F_n^{M_*}$ will be used to divide the work of ensuring each ultrafilter D is not in $\mathrm{uf}(M)$ by for each D, attending one by one to only those c with infinitely many $F_n^{M_*}(c)$ in D.

We now describe some of the salient properties of the model M obtained by 'correcting' the M_* of Sec. 3.

Remark 4.1 (The corrections). (1) The domains of the structures constructed in this section are subsets of M_*; the F_n are redefined so the new structures are substructures only of the reduct of M_* to $\tau - \{F_n : n < \omega\}$.

(2) In particular, for all the M considered in Sec. 4, $P_1^M = P_1^{M_*}$ and these Boolean algebras have the same set of ultrafilters. However, $\mathrm{uf}(M) \neq \mathrm{uf}(M_*)$ as the definition of uf depends on properties of the F_n.

(3) The set $\{F_n^M(c) : c \in P_2^M\}$ is not required to be an independent subset to put $M \in \boldsymbol{K}_{-1}$.

(4) Lemma 4.13 demands a sequence of finite Boolean algebras B_n to witness finitely generated substructures belong to $\boldsymbol{K}_1$ (not required for $\boldsymbol{K}_{-1}$). The stronger class of $\boldsymbol{K}_1$-free structures [4, Definition 3.2.11], which is closed under extension by members of $\boldsymbol{K}_1$ and so has no maximal models plays no active role in this paper. In particular, the final counterexample, Theorem 4.18, is in $\boldsymbol{K}_1$ but is not $\boldsymbol{K}_1$-free.

(5) The proof is in ZFC. The proof in [4] that a non-maximal model in λ makes λ measurable depends on $\diamond$.

The main task of this section is to prove the following:

Theorem 4.2. *If λ is less than the first measurable cardinal, $2^{\aleph_0} < \lambda$, and for some μ, $2^\mu = \lambda$ (whence $\lambda^\omega = \lambda$), then there is a P_0-maximal model in $\boldsymbol{K}_2$ of cardinality λ.*

Conclusion 4.3, summarizes the results of the construction in Theorem 3.2.6, specifically to fix our assumptions for this section.

Conclusion 4.3. If λ is as in Theorem 4.2 then there is a model M_* with $|M_*| = \lambda$ satisfying:

(1) $P_1^{M_*}$ is an atomic Boolean algebra and M_* is P_0-maximal. Further, $|P_i^{M_*}| = \lambda$ for $i = 0, 1$.
(2) $P_{4,1}^{M_*}$ is the set of atoms of $P_1^{M_*}$.
(3) M_* is nicely free (Definition 3.2.5); in particular, $P_1^{M_*}/P_4^{M_*}$ is a free Boolean algebra of cardinality λ.

In order to 'correct' M_* to a model in $\boldsymbol{K}_2$, we lay out some notation for the indexing of the tasks performed in the construction, the generating set of $P_1^{M_*}$, and the free basis of the Boolean algebra $P_1^{M_*}/P_4^{M_*}$.

Notation 4.4. We define a family of trees of sequences:

(1) For $\alpha < \lambda$, let $\mathcal{T}_\alpha = \{\langle\rangle\} \cup \{\alpha\hat{\ }\eta; \eta \in {}^{<\omega}3\}$ and $\mathcal{T} = \bigcup_{\alpha<\lambda} \mathcal{T}_\alpha$.
(2) $\lim(\mathcal{T}_\alpha)$ is the collection of paths through $\mathcal{T}_\alpha$.

Combining the requirements for constructing M_* (Specification 3.2.7) and the Definition 3.2.5 of nicely free, we have

Claim 4.5 (Fixing notation). *Since M_* is nicely free, without loss of generality, we may assume*:

(1) *The universe of M_* is λ and the 0 of $P_1^{M_*}$ is the ordinal 0.*
(2) *We can choose sequences of elements of $P_1^{M_*}$, $\mathbf{b} = \langle b_\eta : \eta \in \mathcal{T}\rangle$ so that their images in the natural projection of $P_1^{M_*}$ on $P_1^{M_*}/P_4^{M_*}$ freely generate $P_1^{M_*}/P_4^{M_*}$.*
(3) *For every $a \in P_{4,1}^{M_*}$ and the even ordinals $\alpha < \lambda$, there is an n such that for any $\nu \in \mathcal{T}_\alpha$, $\lg(\nu) \geq n$ implies $a \wedge b_\nu = 0$.*

Proof. The only difficulty is deducing from (c) of Definition 3.2.5 (nicely free) that (3) holds. For that, we can insist that for each even α, for some $c \in P_2^{M_*}$, $\{b'_{\omega\alpha+n} : n < \omega\}$ enumerates $u_c = \{F_n^{M_*}(c) : n < \omega\}$ (from Definition 3.2.5(c). Now for $\alpha > 0$, let $\langle b_\eta : \eta \in \mathcal{T}_\alpha \backslash \{\langle\rangle\}\rangle$ list $\{b'_{\omega\alpha+n} : n < \omega\}$ without repetition and $\langle b_\eta : \eta \in \mathcal{T}_0\rangle$ list $\{b'_n : n < \omega\}$. By Definition 3.1.2(7) ($\boldsymbol{K}_{-1}$) we have: for every $a \in P_{4,1}^{M_*}$ for all but finitely many n, $a \wedge b'_{\omega\alpha+n} = 0_{P_1^{M_*}}$; whence for even α all but finitely many of the $\nu \in \mathcal{T}_\alpha$ satisfy $a \wedge b_\nu = 0_{P_1^{M_*}}$. □

Note that Claim 4.5 provides a 1-1 map from $P_2^{M_*}$ to ordinals less than λ. We introduce the collection of models that is the starting point for the following construction.

Definition 4.6 ($\mathbb{M}_1$ defined). Let $\mathbb{M}_1 = \mathbb{M}_1(\lambda)$ be the set of $M \in \boldsymbol{K}_{-1}$ such that the universe of M is contained in λ, which is the universe of M_*, and for $i < 2$, (or

$i = 4$ or $(4,1)$) $P_i^M = P_i^{M_*}$, $M \restriction (P_0^M \cup P_1^M) = M_* \restriction (P_0^{M_*} \cup P_1^{M_*})$ while P_2^M will not equal $P_2^{M_*}$.

The posited M_* differs from any $M \in \mathbb{M}_1$ only in that P_2^M is a proper subset of P_2^{M*} and the newly defined $F_n^M(c)$ (usually) do not equal the $F_n^{M_*}(c)$. We now spell out the tasks which must be completed to correct M_* to the required member of $\boldsymbol{K}_2$. The $F_n^{M_*}(c)$ are used as oracles.

Definition 4.7 (Tasks). (1) Let $\boldsymbol{T}_1$, the set of 1-tasks, be the set of pairs (N_1, N_2) such that:

(a) $N_1 \subseteq N_2 \subseteq \lambda$,
(b) $N_1, N_2 \in \boldsymbol{K}^1_{<\aleph_0}$,
(c) $N_1 \subset M$ for some $M \in \mathbb{M}_1$. More explicitly, $P_2^M \subseteq P_2^{M_*}$ and $N_1 \restriction (P_0^M \cup P_1^M) \subseteq M_*$ and $(F_n^M \restriction P_2^{N_1}) = F_n^{N_1}$ for each n.

(2) Let $\boldsymbol{T}_2$, the set of 2-tasks, be the set of $c \in P_2^{M_*}$.
(3) $\boldsymbol{T} = \boldsymbol{T}_1 \cup \boldsymbol{T}_2$.
(4) Let $\langle \mathbf{t}_\alpha : \alpha < \lambda \rangle$ enumerate $\boldsymbol{T}$.

Note $|\boldsymbol{T}_1| = |\boldsymbol{T}_2| = |\boldsymbol{T}|$.

Definition 4.8 (Task satisfaction). The task $\mathbf{t}$ is *relevant* to the structure M if $M \in \mathbb{M}_1$ and (i) if $\mathbf{t}$ is a 1-task (N_1, N_2) and $N_1 \subseteq M$ or (ii) if $\mathbf{t}$ is a 2-task c and $c \in P_2^M$.

We say $M \in \mathbb{M}_1$ *satisfies the task* $\mathbf{t}$ if either:

(A) $\mathbf{t} = (N_1, N_2) \in \boldsymbol{T}_1$ (so $N_1 \subset M$) and there exists an embedding of N_2 into M over N_1.
(B) $\mathbf{t} = c$, where $c \in P_2^{M_*}$, is in $\boldsymbol{T}_2$ and for every ultrafilter D on P_1^M, such that for infinitely many n, $F_n^{M_*}(c) \in D$, there is a $d \in P_2^M$ such that for infinitely many n, $F_n^M(d) \in D$.

Recall Definition 3.2.2 of $\mathrm{uf}(M)$ and Lemma 3.2.4 connecting $\mathrm{uf}(M)$ with P_0-maximality of M.

Claim 4.9. *If $M \in \mathbb{M}_1$ satisfies all tasks in $\boldsymbol{T}$ and is in $\boldsymbol{K}_1$ then from satisfying the $\boldsymbol{T}_2$ tasks, M is P_0-maximal and satisfying the tasks in $\boldsymbol{T}_1$ guarantees it is in $\boldsymbol{K}_2$.*

Proof. For P_0-maximality of M, it suffices, by Lemma 3.2.4 (since $\mathbb{M}_1 \subseteq \boldsymbol{K}_{-1}$), to show $\mathrm{uf}(M) = \emptyset$. But, since $\mathrm{uf}(M_*) = \emptyset$, for every ultrafilter D on $P_1^{M_*}$ there is $c \in P_2^{M_*}$ with $S_c^{M*}(D)$ infinite (Definition 3.2.2); satisfying task c means there is $d \in P_2^M$ such that $S_d^M(D)$ is infinite and so D is not in $\mathrm{uf}(M)$. Since M and M^* have the same ultrafilters, this implies $\mathrm{uf}(M) = \emptyset$, as required. Since we have assumed $M \in \boldsymbol{K}_1$, the second assertion follows by realizing that satisfying all the tasks in $\boldsymbol{T}_1$ establishes the model is rich, which suffices by Fact 3.3.3. ☐

Definition 4.11 lays out the use of the generating elements b_η in correcting the F_n^{M*} to require independence while maintaining that infinite intersections of members of the ultrafilter under consideration are empty. The infinite sequence η_d will guide the choice of $F_n^M(d)$.

The following facts about the relation of symmetric difference and ultrafilters are central for calculations below.

Remark 4.10. Recall that the operation of symmetric difference is associative.

(1) Suppose $\mathbb{B}_1 \subseteq \mathbb{B}_2$ are Boolean algebras with $a \in \mathbb{B}_1$ and $b_1 \neq c_1$ are in $\mathbb{B}_2$ and $\{b_1, c_1\}$ is independent over $\mathbb{B}_1$ in $\mathbb{B}_2$.

The element $(b_1 \triangle c_1) \triangle a \in \mathbb{B}_2$ is independent over $\mathbb{B}_1$. More generally, if $\{b_i, c_i : i < \omega\}$ are independent over $\mathbb{B}_1$, $\{a_i : i < \omega\} \subseteq \mathbb{B}_1$, $e_i = b_i \triangle c_i \triangle a_i$, and $f_i = b_i \triangle c_i$ then each of $\{e_i : i < \omega\}$ and $\{f_i : i < \omega\}$ are independent over $\mathbb{B}_1$.

(2) Let D be an ultrafilter on a Boolean algebra $\mathbb{B}$.
   (a) For $a_0, a_1 \in D$,
   ($a_0 \in D$ if and only if $a_1 \in D$) if and only if $a_0 \triangle a_1 \notin D$.
   (b) If $a_0, a_1, a_2 \in \mathbb{B}$ are distinct then at least one of $a_i \triangle a_j \notin D$.
   (c) More importantly for our use later, it is easy to check:
   ($a_0 \in D$ if and only if $a_1 \in D$)
   if and only if

$$(a_0 \triangle a_1 \triangle a_2) \in D \leftrightarrow a_2 \in D.$$

(3) If a is an atom, $a \wedge b_0 = 0$ and $a \wedge b_1 = 0$, then $a \wedge (b_0 \triangle b_1) = 0$.

Proof. (1) If the element $(b \triangle c) \triangle a \in \mathbb{B}_2$ is not independent over $\mathbb{B}_1$ there is a polynomial p over $\mathbb{B}_1$ with $p((b \triangle c) \triangle a) \in \mathbb{B}_1$. But then, by Observation 1.2, $p(x, y) = p((x \triangle y) \triangle a)$ is also a polynomial over $\mathbb{B}_1$ witnessing $\{b, c\}$ is dependent over $\mathbb{B}_1$. In the more general case any polynomial witnessing dependence in n of the e_i (f_i) gives a polynomial in $2n$ of the a_i, b_i, c_i witnessing dependence of the original set.

(2) For (a), if, say $a_0 \in D$ and $a_1 \notin D$, then $a_0 - a_1$ and hence $a_0 \triangle a_1 \in D$ so we have 'left to right' by contraposition. If both are in D, so is their meet which is disjoint from $a_0 \triangle a_1$ so $a_0 \triangle a_1 \notin D$. Since $a_0^- \triangle a_1^- = a_0 \triangle a_1$, we have the result if neither is in D.

(b) holds since the intersection over all pairs $i, j < 3$ of the $a_i \triangle a_j$ is empty. And (c) is propositional logic from (a) and (b).

(3) $a \leq (b_0^- \wedge b_1^-) \leq (b_0^- \triangle b_1^-) \leq (b_0 \triangle b_1)^-$. As a is an atom, $a \wedge (b_0 \triangle b_1) = 0$. ◻

We define a class $\mathbb{M}_2 \subseteq \mathbb{M}_1$ such that for each $d \in P_2^M \in \mathbb{M}_2$ there is an ordinal α_d, a tree of elements of P_1^M, indexed by sequences in $(\mathcal{T}_{\alpha_d}) \subseteq {}^{<\omega}3$, a target path η_d through that tree and a sequence $a_{d,n}$, whose indices are not in $\mathcal{T}_{\alpha_d}$, but which satisfy that each $a \in P^{M_*}_{4,1} = P^M_{4,1}$ is in at most finitely many $a_{d,n}$. In the construction (Theorem 4.18) of a model in $\mathbb{M}_2$, η_d guides definition of the sequence $F^M_k(b_{\eta_d})$. The $a_{d,n}$ are introduced to make Definition 4.11(B) uniform. In Cases 2 and 3 of Theorem 4.18 $a_{d,n}$ is always 0. In Case 4, where the $F^M_n(d)$ are defined as M is corrected from M_*, $a_{d,n} = F^{M_*}_n(d)$. The result is the values of the $F^M_n(d)$ are both independent over a finite initial segment and satisfy $\bigwedge_{n<\omega} F^M_n(d) = \emptyset$. The next definition abstracts from this construction to identify the key ideas of the proof that if $M \in \mathbb{M}_2$ then $M \in \boldsymbol{K}_1$ (Lemma 4.13) and further that there are $M \in \mathbb{M}_2$ that are in $\boldsymbol{K}_2$. The notation $\langle Z \rangle$ denotes the Boolean subalgebra of P^M_1 generated by Z.

Definition 4.11 ($\mathbb{M}_2$ defined). Let $\mathbb{M}_2$ be the set of $M \in \mathbb{M}_1$ such that there is a sequence $\mathbf{w} = \langle(\alpha_d, \eta_d, a_{d,n}) : d \in P^M_2, n < \omega\rangle$ witnessing the membership, which means:

(A) (a) For each $d \in P^M_2$, $\alpha_d < \lambda$ is even and $d_1 \neq d_2$ implies $\eta_{d_1} \neq \eta_{d_2}$. (In Case 4 of Lemma 4.18, many distinct d_η have the same α_{d_η}.)
(b) $\langle \alpha_d \rangle \lhd \eta_d \in \lim(\mathcal{T}_{\alpha_d})$.

(B) For each $n < \omega$, there are[c] $a_{d,n}$ in $P^{M_*}_1 = P^M_1$ such that for each $d \in P^M_2$, there are distinct[d] $\nu_1[d,n]$ and $\nu_2[d,n]$ that extend $\eta_d \restriction n$, $\nu_i(0) = \alpha_d$, and have length $n+1$ such that:

(a) For every n,

$$F^M_n(d) = (b_{\nu_1[d,n]} \bigtriangleup b_{\nu_2[d,n]}) \bigtriangleup a_{d,n}.$$

(b) For each $a \in P^{M_*}_{4,1}$ and each $d \in P^M_2$, there are only finitely many n with $a \leq_{P^{M_*}_1} a_{d,n}$.

(C) (k^1_Y) For each finite $Y \subseteq P^M_2$ there is a list $\langle d_\ell : \ell < |Y|\rangle$ of Y such that:

(a) The d_ℓ list Y without repetition and $\alpha_\ell = \alpha_{d_\ell}$.
(b) If $i_1 < i_2 < i_3 < |Y| = n$ and $\alpha_{i_1} = \alpha_{i_3}$ then $\alpha_{i_2} = \alpha_{i_1}$.
(c) Let η_i abbreviate η_{d_i}. There is[e] a $k_1 = k^Y_1$ such that:

(i) For $i \neq j$, both less than $|Y|$, $\eta_i \restriction k^Y_1 \neq \eta_j \restriction k^Y_1$.
(ii) Set $W \subseteq P^{M_*}_1$ as

$$W = \{a_{d_k,n} : k < |Y| \wedge n < \omega\} \cup \{F^M_i(d_k) : k < |Y|, i < k^Y_1\}. \tag{1}$$

Then W is included in the subalgebra $\mathbb{B}^0_Y$ of P^M_1 generated by

$$\left\{ b_\nu : \bigwedge_{i<|Y|} (\eta_i \restriction k^Y_1) \ntrianglelefteq \nu \right\} \cup \{b_{\langle\rangle}\} \cup P^M_{4,1}.$$

[c]In applications, the $a_{d,n}$ are either 0 or $F^{M_*}_n(c)$ (for an appropriate $c \in P^{M_*}_2$).
[d]I.e. $\nu_1[d,n]$ depends on d and n.
[e]See proof of *goal* in Lemma 4.18.

Note that the B^0_Y is a cocountable subset of P^M_1 (the countable complement is generated by b_ν where $\nu \in \bigcup_{i<|Y|}\{\nu : \nu \geq \eta_i \restriction k^Y_1\}$).

We will apply the following lemma three times to show that for $M \in \mathbb{M}_2$, for each the set $\{F^M_n(c) : n < \omega\}$ is countably incomplete (witnessing Definition 3.1.2(7)). It is a straightforward application of Remark 4.10 to Definition 4.11(2).

Lemma 4.12. *Let $M \in \mathbb{M}_1$. For any $\langle \alpha_d, \eta_d, a_{d,n}\rangle$ as in Definition* 4.11 (*in particular α_d is even*) *and any atom $a \in P^{M_*}_{4,1}$, for all but finitely many n,*

$$a \wedge (b_\nu \vartriangle b_\rho \vartriangle a_{d,n}) = 0.$$

Proof. Recall from 4.5(3), that for every $a \in P^{M_*}_{4,1}$ and the even ordinals $\alpha < \lambda$, there is an n, such that for any $\nu, \rho \in \mathcal{T}_\alpha$ with $\lg(\nu) \geq n$ and $\lg(\rho) \geq n$, $a \wedge b_\nu = 0$ and $a \wedge b_\rho = 0$. Definition 4.11(B)(b) asserts each d and for sufficiently large n, $a \wedge a_{d,n} = 0$. Apply Remark 4.10(3) twice. □

We will show in Lemma 4.13 that all members of $\mathbb{M}_2$ are in $\boldsymbol{K}_1$ and then in Theorem 4.18 that there are structures in $\mathbb{M}_2$ that are in $\boldsymbol{K}_2$. Two main features distinguish $\boldsymbol{K}_1$ from $\boldsymbol{K}_{-1}$. The $F_n(d)$ retain the 'countable incompleteness' property from $\boldsymbol{K}_{-1}$ but also must be independent; $M \in \boldsymbol{K}_1$ when M is a direct limit of members of $\boldsymbol{K}^1_{<\aleph_0}$.

Lemma 4.13. *If $M \in \mathbb{M}_2$, then $M \in \boldsymbol{K}_1$.*

Proof. Suppose $M \in \mathbb{M}_2$. Let $Y \subset P^M_2$ and $X \subset P^M_1$ be finite; we shall find $N = N_{XY} \in \boldsymbol{K}^1_{<\aleph_0}$ such that $Y \cup X \subseteq N \subseteq M$; this suffices. As, $\boldsymbol{K}_1$ is defined to be the collection of direct limits of finitely generated structures[f] in $\boldsymbol{K}^1_{<\aleph_0}$.

Our two main jobs in proving Lemma 4.13 are to find an N, n_*, b_* in which

Job (1) the $F^M_k \restriction N$ satisfy property 6 (independence) of Definition 3.3.1 over a B_{n_*} and property 7 of Definition 3.1.2 and then

Job (2) construct $N = \bigcup_{n<\omega} B_n$ for finite Boolean algebras $\langle B_n : n \geq n_*\rangle$ that witness 2 and 3 of Definition 3.3.1.

The finite $k_1 = k^Y_1$ specified in Definition 4.11 depends only on Y; in the next definition we increase k_1 to a $k^X_1 = k^{XY}_1$ and using the definition of $\mathbb{M}_2$ show the $F^M_k(d)$ are independent over X for $k \geq k^{XY}_1$. We need k^{XY}_1 only to prove Lemma 4.13.

We build two increasing chains of length $|Y|$ of subsets of boolean algebras satisfying the conditions described in Definition 4.14. The $\mathbb{B}^\ell_{XY}$ will be cocountable, while the $\mathbb{F}_\ell$ will be countable. The existence of k^{XY}_1 satisfying the conditions of Definition 4.14 is proved in Fact 4.15.

[f]The proof of Lemma 4.13 shows there is a common substructure of M containing any finite collection of finitely generated (as in this argument) substructures of M.

Definition 4.14 ($\boldsymbol{k^1_{XY}}$). Let the sequence $\langle(\alpha_d,\eta_d,a_{d,k}) : d \in P_2^M, k < \omega\rangle$ *witness* $M \in \mathbb{M}_2$ as in Definition 4.11. Let $X \subset P_1^M$ (as in proof of Lemma 4.13) and $\langle d_i : i < n\rangle$ enumerate $Y \subset P_2^M$ without repetition and denote, for $i < n$, η_{d_i} by η_i and α_{d_i} by α_i. Without loss, the $\langle \eta_i(0) : i < n\rangle$ are non-decreasing;

(A) Fix $k_1 = k_1^{XY}$ such that:

  (a) $k_1^{XY} \geq k_1^Y$ (see Definition 4.11(B));
  (b) $\langle \eta_i{\restriction}k_1^{XY} : i < n\rangle$ are distinct for $i < n$;
  (c) $k_1^{XY} \geq \max\{\lg(\nu) : b_\nu \in \langle X \cup \{F_k^M(d_i) : i < |Y|\}\rangle, k < k_1^Y\}$.

(B) We consider the following sets determined by $X \cup Y$ and the η_i.

  (a) $\boldsymbol{F}_{\leq 0} = \boldsymbol{F}_0 = X \cup \{F_k^M(d_i) : i < |Y|, k \leq k_1^{XY}\}$;
  (b) For $1 \leq \ell < |Y|$, $\boldsymbol{F}_\ell = \{F_k^M(d_\ell) : k \geq k_1^{XY}\}$;
  (c) $\boldsymbol{F}_{\leq \ell+1} = \boldsymbol{F}_{\leq \ell} \cup \boldsymbol{F}_\ell$;
  (d) $\mathbb{F}^\ell = \langle \boldsymbol{F}_{\leq \ell}\rangle_M$.

(C)

$$\mathbb{B}^\ell_{XY} = \left\{ b_\nu : \bigwedge_{\ell < i < n} (\eta_i{\restriction}k_1^{XY}) \ntrianglelefteq \nu \text{ for } i < \ell+1 \right\} \cup \{b_{\langle\rangle}\} \cup P^M_{4,1}.$$

For each ℓ, $\mathbb{B}^\ell_{XY} \supseteq \mathbb{B}^\ell_Y$ since $k_1^{XY} \geq k_1^Y$ and $\mathbb{B}^{\ell+1}_{XY} \supseteq \mathbb{B}^\ell_{XY}$. In the proof of Lemma 4.16 B_{n_*} will be $\mathbb{F}^0$ and N will be $\mathbb{F}^n$.

Since X and Y are finite we first choose k_1^{XY} to satisfy conditions 1–3 of Definition 4.14; we now show the other conditions are satisfied.

Fact 4.15. There is a $k_1 = k_1^{XY}$ such that for each ℓ, $\boldsymbol{F}_\ell$ is contained in $\mathbb{B}^\ell_{XY}$.

Proof. Recall (Claim 4.5) that M_* is free on the $\{b_\eta : \eta \in \mathcal{T}\}$ modulo the $P_4^{M_*}$. Choose k_1^{XY} larger than the length of any ν such that for some $x \in X$, b_ν is a generator in a minimal representation of x or $\nu(0) \in \overline{\alpha} = \{\alpha_0, \ldots, \alpha_{n-1}\}$. Then

$$\boldsymbol{F}_0 \subseteq \langle\{b_\nu : \nu \in \mathcal{T}, \lg(\nu) < k_1^{XY}\}\rangle \cup \{b_{\langle\rangle}\} \cup P_4^M \subseteq \mathbb{B}^0_{XY}.$$

Recall from Definition 4.11(D), that as ℓ increases $F_k^M(d_i)$ for $i < \ell$ and all k are admitted to $\mathbb{B}^\ell_{XY}$ and so $\boldsymbol{F}_\ell \subseteq \mathbb{B}^\ell_{XY}$. □

To establish Job (1) of Lemma 4.13 we need the following claim.

Lemma 4.16. *For each* $1 \leq \ell < n$, $\boldsymbol{F}_\ell$ *is independent over* $\mathbb{B}^0_{XY}$ *mod* P_4^M.

Proof. We prove this claim by showing by induction on $\ell \leq |Y| = n$:

$$(\oplus_\ell) \quad \boldsymbol{F}_{<\ell} = \{F_k^M(d_i) : k \geq k_1^{XY} \text{ and } i < \ell\}$$

is independent in P_1^M over $\mathbb{B}^{\ell-1}_{XY}$ mod P_4^M.

For $1 \leq \ell < |Y|$, the induction on ℓ shows incrementally, at stage $\ell + 1$, the independence of the $b_{\eta_\ell \restriction r}$ with $r \geq k_1^{XY}$ over $\mathbb{B}^\ell_{XY}$. By Claim 4.5(2) and the choice of $r \geq k_1^{XY}$, the $\{b_{\nu_1[d_\ell,r]} : r \geq k_1^{XY}\}$ are independent mod P_4^M. Thus (using the f_i from Remark 4.10) the infinite set $\{b_{\nu_1[d_\ell,n]} \vartriangle b_{\nu_2[d_\ell,n]}) : i \in \{0,1\}, n \geq k_1^{XY}\}$ is independent over $\mathbb{B}^{\ell-1}_Y$. By Definition 4.11(C)) the $\{a_{d_\ell,k} : k \geq k_1^Y\}$ are in $\mathbb{B}^0_Y \subseteq \mathbb{B}^\ell_{XY}$. Further, by Definition 4.11(B) for all n:

$$F_n^M(d_\ell) = (b_{\nu_1[d_\ell,n]} \vartriangle b_{\nu_2[d_\ell,n]}) \vartriangle a_{d_\ell,n}.$$

So, Lemma 4.10(2) (now using the e_i) implies $\boldsymbol{F}_\ell$ is independent over $\mathbb{B}^\ell_Y$. Since independence is transitive (Lemma 1.3(3)) $\boldsymbol{F}_\ell$ is independent over $\mathbb{B}^0_Y$. □

We continue the proof of Lemma 4.13. By Lemma 4.12, for sufficiently large n, $a \not\leq F_n^M(d_\ell)$. So the countable incompleteness condition in the definition of $\boldsymbol{K}_{-1}$ is satisfied. This completes Job (1). To accomplish Job (2) and finish the proof of Lemma 4.13 by satisfying conditions 2–4 of Definition 3.3.1, we must define appropriate P_i^N and find a sequence of finite Boolean algebras B_n witnessing that $N \in \boldsymbol{K}^1_{<\aleph_0}$. Let $P_1^N = \mathbb{F}^{n-1}$. We have P_1^N is freely generated (modulo the ideal generated by the atoms of B_{n_*}) by the countable set $\boldsymbol{F}_{|Y|}$ over $B_{n_*} = \mathbb{F}^0$. Let b_* be the supremum of the atoms in B_{n_*} and P_4^N the predecessors of b_*.

For $m \geq n_*$, let B_m be generated by B_{n_*} and the first m elements of this generating set. Now, $P_1^N = \mathbb{F}^{n-1}$ is equal to $\bigcup_{n_* \leq m < \omega} B_m$ and P_1^N/P_4^N is atomless. Set[g] $P_2^N = Y$ and $P_0^N = \{(G_1^M)^{-1}(a) : a \in P^M_{4,1} \cap P_1^N\}$; thus $P^N_{4,1} \subseteq B_{n_*}$. Boolean algebras are locally finite and we can recognize whether $\langle X \rangle$ is free by whether it has $2^{|X|}$ atoms. Thus, we can refine the sequence B_m to finite free algebras to witness that $N \in \boldsymbol{K}^1_{<\aleph_0}$. Since X and Y were arbitrary, $M \in \boldsymbol{K}_1$.

This completes the proof of Lemma 4.13. Now we show $\mathbb{M}_2$ is nonempty and at least one member satisfies all the tasks. In Case 4 of this argument we address the requirement that $\mathrm{uf}(M_\alpha) = \emptyset$ for each $\alpha < \lambda$ and so $\mathrm{uf}(M) = \emptyset$ as well. We need the following observation because as the construction proceeds, an N_1 may become a substructure of M_β because some value of an F_n is newly defined on a point of $P_2^{M_\beta}$.

Notation 4.17. We can enumerate $\boldsymbol{T}$ as $\langle t_\alpha : \alpha < \lambda \rangle$ such that each task appears λ times, as we assumed in Hypothesis 4.3 that $\lambda = \lambda^{\aleph_0}$.

For Theorem 4.18, to realize all the tasks, $\lambda > 2^{\aleph_0}$ would suffice; the requirement in Lemma 2.5 that $\lambda = 2^\mu$ is used to get maximal models. The object of Case 3 is to ensure that the final model is rich (existentially complete); Case 4 shows $\mathrm{uf}(M) = \mathrm{uf}(M_*) = \emptyset$. After satisfying each task a final section labeled **goal** verifies that each $M_\alpha \in \mathbb{M}_2$ and so $M \in \mathbb{M}_2$.

Theorem 4.18. *There is an $M \in \mathbb{M}_2$ and in $\boldsymbol{K}_1$ that satisfies all the tasks, Thus, by Claim 4.9 $M \in \boldsymbol{K}_2$, and is P_0-maximal.*

[g] G_1^M is from Definition 3.1.2(5).

Proof. As we construct M, we show at appropriate stages that tasks from $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are satisfied. Further, we show at each stage α the goal: $M_\alpha \in \mathbb{M}_2$. We choose M_α by induction on $\alpha \leq \lambda$ such that:

(1) $\mathbf{w}_\alpha$ witnesses $M_\alpha \in \mathbb{M}_2$ (Definition 4.11). And for $\beta < \alpha$, w_α extends w_β. That is, for $d \in P_2^{M_\beta}$, $\alpha_d[\mathbf{w}_\alpha] = \alpha_d[\mathbf{w}_\beta]$, $\eta_d[\mathbf{w}_\alpha] = \eta_d[\mathbf{w}_\beta]$ and $a_{d,n}[\mathbf{w}_\alpha] = a_{d,n}[\mathbf{w}_\beta]$.
(2) $P_2^{M_\alpha} \subseteq P_2^{M_*}$ has cardinality at most $|\alpha| + 2^{\aleph_0}$.
(3) if $\alpha = \beta + 1$ and $\mathbf{t}_\beta$ is relevant to M_β, M_α satisfies task $\mathbf{t}_\beta$.

Case 1. If $\alpha = 0$, set $M_0 = M_* \restriction (P_0^{M_*} \cup P_1^{M_*})$.

This condition will be preserved by the induction for all α.

Case 2. Take unions at limits.

At the successor stage, we now verify task t_β for each of two different types of task. Then, we will consider the two cases together to show the goal that $M = \bigcup_{\alpha<\lambda} M_\alpha \in \mathbb{M}_2$.

Case 3. $\alpha = \beta + 1$ and say, $\mathbf{t}_\beta \in \boldsymbol{T}_1$ and $\mathbf{t}_\beta = (N_1, N_2)$. (Definition 4.7)

Choose M_α**:** If N_1 is not a subset of M_β then the task is irrelevant and let $M_\alpha = M_\beta$ and $\mathbf{w}_\alpha = \mathbf{w}_\beta$. If it is, let $\langle a_\ell : \ell < m\rangle$ enumerate $P_2^{N_2} - P_2^{N_1}$ and $\langle a'_\ell : \ell < m\rangle$ enumerate the first m elements of $P_2^{M_*} - P_2^{M_\beta}$. Let M_α extend the $P_2^{M_\beta}$ by adding $\langle a'_\ell : \ell < m\rangle$ from $P_2^{M_*}$ to form $P_2^{M_\alpha}$. It remains to define the $\mathbf{w}_\alpha$ and $F_k^{M_\alpha}(a'_\ell)$.

Let $U_\alpha = \{\delta : (\exists b_\nu \in M_\beta)[\nu(0) = \delta]\}$. Clearly, $|P_2^{M_\alpha}| \leq |\alpha| + 2^{\aleph_0}$ as required for the induction. Similarly, $|U_\alpha| \leq |\alpha| + 2^{\aleph_0}$ and

$$\{a_{d,k} : k < \omega, d \in P_2^{M_\beta}\} \cup \{b_\nu : (\exists d \in P_2^{M_\beta})\nu \in \mathcal{T}_{\alpha_d}\} \cup P_{4,1}^{M_*} \tag{$*$}$$

is included in the subalgebra of M_* generated by the

$$\{b_\rho : \exists\beta \in U_\alpha, \rho(0) = \beta\} \cup \{b_{\langle\rangle}\} \cup P_{4,1}^{M_*}$$

so there is room to choose values for the $F_k^{M_\alpha}(a_\ell)$.

By induction, since $M_\beta \in \mathbb{M}_2$ there are witnesses $w_\beta = \langle \alpha_d, \eta_d, a_{d,k}\rangle$ (formally $\langle \alpha_d^\beta, \eta_d^\beta, a_{d,k}^\beta\rangle$) for each $d \in P_2^{M_\beta}$. For the new a'_ℓ, let $\mathbf{w}_\alpha(\ell) = \langle \gamma_\ell, \eta_\ell, 0^{M_*}\rangle$ be chosen with the γ_ℓ as the first m even elements of $\lambda - U_\alpha$ and with $\eta_\ell(\mathbf{w}_\alpha) = \eta_\ell$ chosen[h] so that $\eta_\ell(0) = \gamma_\ell$. We complete the definition of M_α below by choosing the new values of $F_k^{M_\alpha}$ to satisfy the task.

Task: We now verify task $\mathbf{t}_{\beta+1}$ by showing in two stages that N_2 can be embedded over N_1 into M_α. First we show there is an embedding of the Boolean algebras; then we define the F_k on the image to put M_α in $K^1_{\aleph_0}$. Since $N_2 \in \boldsymbol{K}^1_{<\aleph_0}$, $P_1^{N_2}$ is decomposed as a union of the finite free Boolean algebras[i] $\langle B_i^{N_2} : i \geq {n_*}^{N_2}\rangle$ where,[j]

[h]In Case 3, we need choose only a single η_ℓ for each $\ell < m$. In Case 4, we choose $2^{\aleph_0}$ distinct d_η.
[i]While the domain of $N_2 \subseteq \lambda$, the N_2-interpretation of any relation symbols in τ on ordinals not in the domain of N_1 has nothing to with the interpretations in M_* or M_β.
[j]Technically, we are defining $n_*^{M_\alpha}$. But the value is set once and for all at stage α so we just call it by the final name.

writing n_* for $n_*^{N_2}$, N_2 is freely generated over $B_{n_*}^{N_2}$ mod $P_4^{N_2}$ by $\{F_k^{N_2}(f) : k \geq n_*^{N_2}, f \in P_2^{N_2}\}$. Similarly, we decompose $P_1^{N_1}$ by $\langle B_i^{N_1} : i \geq {n_*}^{N_1}\rangle$.

Since $N_1 \subseteq M_*$ and $N_1 \subseteq N_2$, for each element $e \in P_1^{N_1}$ and any s,

$$P_{4,s}^{M_*}(a) \leftrightarrow P_{4,s}^{N_1}(a) \leftrightarrow P_{4,s}^{N_2}(a).$$

So no atom in $N_2 - N_1$ is below any element of N_1.

Let $\mathbf{c} = \langle c_0, \ldots, c_{p-1}\rangle$ enumerate the atoms of N_2 with the c_i for $i < r$ enumerating those in $N_2 - N_1$; they are all in $\mathbb{B}_{n_*}^{N_2}$. We set $c_i' = c_i$ if $r \leq i < p$; for the $c_i \in N_2 - N_1$ choose any r atoms c_i' from $M_* - N_1$. By Claim 4.5, we can find a t (depending on all of the c_i') such that for all i if $\nu(0) = \gamma_\ell$ and $k > t$, $b_{\nu\restriction k} \wedge c_i' = 0$.

Each $e \in \mathbb{B}_{n_*}^{N_2} - (P_1^{N_1} \cup \mathbf{c})$ is a finite join of c_i. (Note $P_4^{N_2}$ is an alias of $\mathbb{B}_{n_*}^{N_2}$.) Recall $\{F_k^{N_2}(f) : k \geq n_*^{N_2}, f \in P_2^{N_2}\}$ is the pre-image of a basis of $P_1^{N_2}/P_4^{N_2}$. For $f \in P_2^{N_2}$, each $F_k^{N_2}(f) \wedge b_*^{N_2} = e \leq b_*^{N_2}$. Now define h_β mapping N_2 into M_α by

(1) $h_\beta \restriction P_1^{N_1}$ is the identity.
(2) $h_\beta(c_i)$ is c_i'.
(3) For $e \in \mathbb{B}_*^{N_2} - P_{4,1}^{N_2}$, $h_\beta(e) = e' = \bigvee_{c_i \leq e} c_i'$.
(4) The $b_{\eta_i}\restriction(t+k)$ for $k \geq n_*$ are independent mod $P_4^{M_*}$; for a_ℓ in $P_2^{N_2} - P_1^{N_2}$ set

$$h_\beta(F_k^{N_2}(a_\ell)) = b_{\eta_i\restriction(t+k)\frown 0} \vartriangle b_{\eta_i\restriction(t+k\frown 1)} \vee e' = F^{M_\alpha}(a_\ell'),$$

where $e' = h_\beta(e)$ and $e = F_k^{N_2}(a_\ell) \wedge b_*^{N_2}$.

(1) Since the $F_k^{N_2}(a_\ell)$ freely generate N_2/N_1 modulo the atoms, h_β extends to an embedding of N_2 into M_α.

Check using Claim 4.10(3) that step (4) is a homomorphism.

We now show $M_\alpha \in \mathbb{M}_2$. To clarify notation, by setting[k] $a_{d_\ell,k} = 0$ for $i < m$, we declared

$$F_k^{M_\alpha}(d_\ell) = (b_{\eta_i\restriction k\frown 0} \vartriangle b_{\eta_i\restriction k\frown 1}) \vartriangle a_{d_\ell,k}.$$

By Lemma 4.12, for some n, for all $k \geq n$, $a \not\leq_{P_1^{M_*}} F_k^{M_\alpha}(d_i)$ so condition 4.11(B)(2), countable incompleteness, holds.

Applying Remark 4.10(1) to $f_i = b_{\eta_i\restriction k\frown 0} \vartriangle b_{\eta_i\restriction k\frown 1}$ the $\{F_k^{M_\alpha}(d_i) : k_1^Y \leq k < \omega\}$ are independent for each i and form a basis for a subalgebra N_2' of $P_1^{M_*}$ over N_1. Thus, $N_2' \in \boldsymbol{K}_{<\aleph_0}^1$ and we have verified that task $\mathbf{t}_{\beta+1}$ is satisfied.

Case 4. $\alpha = \beta + 1$ and $\mathbf{t}_\beta \in \boldsymbol{T}_2$; say, $\mathbf{t}_\beta = c$.

We define M_α. Define U_α as in Case 3, but extending U_α to U_α' by adding the ordinal c if $c \notin M_\beta$. Now for any even ordinal γ in $\lambda - U_\alpha'$,

$$\langle\{b_\eta : \eta(0) = \gamma\}\rangle \cap \{b_\eta : \eta(0) \in U_\alpha'\} = \emptyset$$

since b_η are determined by the choice of η and $\gamma \notin U_\alpha'$. Extend $P_2^{M_\beta}$ by adding a $d_\eta \in P_2^{M_*} - P_2^{M_\beta}$ for each η with $\eta(0) = \gamma$ to form $P_2^{M_\alpha}$.

[k] The $a_{d,n}$ are dummies in this case to provide uniformity with Case 4 in proving Lemma 4.13.

To define $F_k^{M_\alpha}(d_\eta)$, for each $\eta \in \lim \mathcal{T}_\gamma$ and $k < \omega$, choose $i_0 < i_1 \leq 2$ that are different from $\eta(k)$. Recalling $c = \mathbf{t}_\beta$, let

$$F_k^{M_\alpha}(d_\eta) = (b_{\eta\restriction k\frown i_0} \triangle b_{\eta\restriction k\frown i_1}) \triangle (F_k^{M_*}(c)).$$

Since $M_* \in \boldsymbol{K}_{-1}$ for each $a \in P_1^{M_*}$ for all but finitely many n, $a \wedge F_k^{M_*}(c) = 0$. Thus, for the $d \in P_2^{M_\alpha} - P_2^{M_\beta}$, chosen towards satisfying $\mathbf{t}_\beta = c$, we have set $\langle \alpha_d, \eta_d^\alpha, a_{d,k}\rangle = \langle \gamma, d_\eta, F_k^{M_*}(c)\rangle$. That is, $a_{d,k} = F_k^{M_*}(c)$. Thus, by Lemma 4.12 for any atom a and all but finitely many n, $a \wedge F_k^{M_\alpha}(c) = 0$ and the countable incompleteness requirement is satisfied.

Task: We must show M_α satisfies task $\mathbf{t}_\beta$. Since $\mathrm{uf}(M_*) = \emptyset$, for any non-principal ultrafilter D, there is an $e \in P_2^{M_*}$ such that the set $S_e^{M_*}(D) = \{n : F_n^{M_\alpha}(e) \in D\}$ is infinite (Definition 3.2.2). By the definition of the task $\mathbf{t}_\beta = c$, there is a D where the given c witnesses for D in $\mathrm{uf}(M_*)$. We show task $\mathbf{t}_\beta$ is satisfied for D by one of the d_η, which thus is a witness to $D \notin \mathrm{uf}(M_\alpha)$.

Define $\eta^D \in \lim(\mathcal{T}_\gamma)$ by induction[l]: $\eta^D(0) = \gamma$. By Remark 4.10(2)(b) one of the three elements $b_{\langle\gamma,i\rangle} \triangle b_{\langle\gamma,j\rangle}$, for $i \neq j$ and $i, j < 3$, must not be in D. Let $\eta^D(1)$ such an element. For $k \geq 1$, suppose $\nu = \eta^D \restriction k$ has been defined. Again, by Remark 4.10(2) one of the three elements $b_{\nu\frown i} \triangle b_{\nu\frown j}$, for $i \neq j$ and $i, j < 3$, must not be in D. Again, let $\eta^D(k+1)$ be such a triple. Now for each k if $\nu = \eta^D \restriction k$ we know there are $i_0, i_1 < 3$ with $b_{\nu\frown i_0} \triangle b_{\nu\frown i_1} \notin D$. Now apply Lemmas 4.10(2)(a) and 4.10(2)(c) to conclude that with $a_0 = b_{\nu\frown i_0}$, $a_1 = b_{\nu\frown i_1}$ and a_2 as $F_k^{M_*}(c)$,

$$F^{M_\alpha}(d_{\eta^D\restriction k+1}) = b_{\nu\frown i_0} \triangle b_{\nu\frown i_1} \triangle F_k^{M_\alpha}(c) \in D$$

for the infinitely many k with $F_n^{M_\alpha}(c) \in D$.

Now we establish the goal for both cases.

Goal: $M_\alpha \in \mathbb{M}_2$**:** To show $M \in \boldsymbol{K}_{-1}$ (and so in $\mathbb{M}_1$, Definition 4.6). For $M_\alpha \in \mathbb{M}_2$, we show M_α satisfies Definition 4.11. The descriptive portions of conditions (A) and (B)(i) of Definition 4.11 are clearly satisfied by the construction; condition (B)(ii) was shown in the proof of each case.

For Condition 4.11(C) choose any finite $Y \subset P_2^{M_\alpha}$ and partition Y into $Y_1 = Y \cap P_2^{M_\beta}$ and $Y_2 = Y - Y_1$. We show every element of $W = \{a_{d_k,n} : k < |Y| \wedge n < \omega\} \cup \{F_i^M(d_k) : k < |Y|, i < k_1^Y\}$ is in the $\langle\{b_\nu; \nu(0) \in U_\alpha\}\rangle$ and so in $\mathbb{B}_Y^0$. Set $k_1 = k_Y^1$ as the least integer[m] such that for all $\eta_d \neq \eta_e$ with $d, e \in Y$, $\eta_d \restriction k^1 \neq \eta_e \restriction k^1$. For those $d \in Y_1$, we set $\mathbf{w}_\alpha = \mathbf{w}_\beta$ and the result follows since $P_1^{M^\beta} \subseteq \mathbb{B}_Y^0$. For $d \in Y_2$, the two cases[n] differ slightly.

For $d \in Y_2$ the $F_n^M(d)$ for $i < n$ and $n < \omega$ are all Boolean combinations of the $a_{d_i,n}$ with elements b_ν with $\nu \trianglelefteq \eta_i \restriction k_1$. In Case 3, we (implicitly) defined

[l]This argument is patterned on the simple black box in [13, Lemma 1.5], but even simpler.
[m]Naturally this is only relevant when $\alpha_d = \alpha_e$ but than can happen in Case 3 and must happen in Case 4.
[n]Note that in Case 3, $a_{d,n}$ is constant. In Case 4 it depends on n. We do not define the value of $F_n^{M_\alpha}$) at c; the $F_n^{M_*}(c)$ are oracles and $c \notin P_2^{M_\alpha}$. We define $F_n^{M_\alpha}$ on the d_{η_d}.

$\mathbf{w}_d(\alpha) = \langle \alpha_d, \eta_d, 0 \rangle$, so the $a_{d_i,n}$ are all 0. In Case 4, the elements of Y_2 are among the $2^{\aleph_0}$ d_η with $\eta(0) = \gamma$. For them, $\mathbf{w}_d(\alpha) = \langle \gamma, \eta_d, F_n^{M_*}(c) \rangle$. If $F_n^{M_*}(c) = b_\zeta$ then $\zeta(0) = c \in U'_\alpha$ by the definition of U'_α. Thus, $\zeta(0) \neq \gamma$ and $b_\zeta = a_{d,n} \in \mathbb{B}^0_Y$.

Now, let $M = \bigcup_{\alpha<\lambda} M_\alpha$. Then, $M \in \mathbb{M}_2$, $|P_2^M| = \lambda$. By Lemma 4.13, $M \in \boldsymbol{K}_1$ and each task has been satisfied, so by Claim 4.9, $M \in \boldsymbol{K}_2$. □

This yields.

Conclusion 4.19. The $M \in \boldsymbol{K}_2$ constructed in Theorem 4.18 is P_0-maximal and all $|P_i^M| = \lambda$. As in [4, Corollary 3.3.14], for all λ less than the first measurable, since $M \in \boldsymbol{K}_2$ implies $|M| \leq 2^{P_0^M}$, there is a maximal model $M \in \boldsymbol{K}_2$ with $2^\lambda \leq |M| < 2^{2^\lambda}$.

Question 4.20. (1) Is there a $\kappa < \mu$, where μ is the first measurable, such that if a complete sentence has a maximal model in cardinality κ, it has maximal models in cardinalities cofinal in μ?

(2) Is there a complete sentence that has maximal models cofinally in some κ with $\beth_{\omega_1} < \kappa < \mu$ where μ is the first measurable, but no larger models are maximal. Could the first inaccessible be such a κ?

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