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ON GROUPS WELL REPRESENTED AS AUTOMORPHISM GROUPS OF GROUPS

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Dedicated to László Fuchs for his 100th birthday.

Assuming Gödel's axiom of constructibility $V = L$, we present a characterization of those groups K for which there exist arbitrarily large groups H such that $\mathrm{Aut}(H) \cong K$. In particular, we show that it suffices to have one such group H such that the size of its center is bigger than $2^{|K|+\aleph_0}$.

0. Introduction

The representation problem from group theory asks:

Problem 0.1. For a given group K, is there a group H such that $\mathrm{Aut}(H) \cong K$?

There are a lot of interesting research papers in this area. Here, we recall only a short list of them. The study of automorphism groups of finite abelian groups started with Shoda [11]. Hallett and Hirsch [6] and Corner [3] classified finite groups which are the automorphism group of some torsion-free group. For infinite groups, Beaumont and Pierce [2] studied automorphism groups of torsion-free groups of rank two. May [8] has more results on abelian automorphism groups of torsion-free groups of countable rank.

A related problem is the classical problem of establishing whether two algebraic structures are isomorphic when their automorphism groups are isomorphic. Thanks to [7], the answer is yes for abelian p-groups, when $p \neq 2$. Also, by the work of Corner and Goldsmith [4], the answer for reduced torsion-free modules over the ring J_p of p-adic integers, where $p \neq 2$, is affirmative. The question of whether two abelian 2-groups are isomorphic when their automorphism groups are isomorphic is still an open problem. For an excellent survey on these and related topics; see Fuchs [5].

Despite lots of work, the study of the automorphism group of groups is not documented very well, and there is no universal solution to Problem 0.1. There are several examples of groups that can never be the automorphism group of some abelian group, so that Problem 0.1 reduces to characterizing those groups K that can be the automorphism group of another group.

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Throughout this paper, we work with arbitrary groups, so they are not required to be abelian. In order to study Problem 0.1, we study the structure of the automorphism group and inner automorphisms of a given group. It may be worth noting that Baer [1] studied the normal subgroup structure of $\mathrm{Aut}(H)$ for infinite p-groups H. For more details on this and further achievements; see again the book of Fuchs [5].

We suggest a version of the problem:

Problem 0.2. Characterize the groups K such that for arbitrarily large cardinals λ there is a group H of cardinal λ such that $\mathrm{Aut}(H) \cong K$.

We approach Problem 0.2 as follows. Given groups K and H and a monomorphism $F : K \to \mathrm{Aut}(H)$, our aim is to find a group $H' \supseteq H$ and an isomorphism $F' : K \to \mathrm{Aut}(H')$ in such a way that for each $k \in K$, $F'(k)$ is an automorphism of H' which extends $F(k)$. Thus, we have to find H' in such a way that, for each $k \in K$, $F(k)$ extends to an automorphism of H', and such that if $h \in \mathrm{Aut}(H)$ is not in the range of F, then h cannot be extended to an automorphism of H'; furthermore, any automorphism of H' must result from extending an automorphism of H. We show that, under some extra assumptions on our initial setup, this is always possible.

Let us recall that the group H in general is not assumed to be abelian. So, we consider its center $\mathcal{Z}(H)$, which leads to the following exact sequence:

$$\begin{array}{ccccccccc} & & & & & & N & & \\ & & & & & \swarrow^{h^*} & \uparrow = & & \\ 0 & \longrightarrow & \mathcal{Z}(H) & \longrightarrow & H & \xrightarrow{h} & N & \longrightarrow & 0 \end{array}$$

We also consider a map h^*, which assigns to each $b \in N := H/\mathcal{Z}(H)$ a preimage under h, but we do not require it to be a homomorphism. We assume that N is a normal subgroup of K, and that it determines the inner automorphisms of H in the sense that, for $a \in H$, if we define a^* as $a^* := F(h(a))$, then $a^*(x) = axa^{-1}$ for all $x \in H$, and $N \cong F(N) = \mathrm{Inn}(H) \subseteq \mathrm{Aut}(H)$. Putting all these things together will lead to the definition of a class $\boldsymbol{C}_{\mathrm{aut}}$ of tuples

$$\boldsymbol{c} := (K_{\boldsymbol{c}}, N_{\boldsymbol{c}}, H_{\boldsymbol{c}}, h_{\boldsymbol{c}}, h^*_{\boldsymbol{c}}, F_{\boldsymbol{c}}).$$

Our main purpose is to furnish $\boldsymbol{c}$ with some additional structures. This leads us to defining various versions of $\boldsymbol{C}_{\mathrm{aut}}$, which may serve as a framework for solving Problem 0.2. In Section 1, we point out the right version to use. In fact, the most advanced versions of $\boldsymbol{C}_{\mathrm{aut}}$ which appears in this work is denoted by $\boldsymbol{C}^6_{\mathrm{aut}}$, which consists of elements $(\boldsymbol{c}, \mathcal{G})$. In particular, the nonexplained notion $\mathcal{G}$ consists of a family of certain tuples $\boldsymbol{g}$ which assign some torsion-free abelian groups $\mathbb{G}_{\boldsymbol{g}}$. For more details, we refer the reader to see Definition 1.8(4). One version of our main conclusion is:

Theorem 3.6. *Assume* $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\mathrm{aut}}$, $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}}) = 0$ *for all* $\boldsymbol{g} \in \mathcal{G}$, *and* $F_{\boldsymbol{c}} : K_{\boldsymbol{c}} \xrightarrow{\cong} \mathrm{Aut}(H_{\boldsymbol{c}})$. *Let* $\lambda = \mathrm{cf}(\lambda) > 2^{\|\boldsymbol{c}\|}$ *and suppose* $\diamondsuit_\lambda(S)$ *holds for some* $S \subseteq S^{\lambda}_{\aleph_0}$ *which is stationary and nonreflecting. Then there is* $\boldsymbol{m} \in \boldsymbol{C}^6_{\mathrm{aut}}$ *such that the following hold*:

(a) $H_{\boldsymbol{m}} \supseteq H_{\boldsymbol{c}}$ *has size* λ *and* $H_{\boldsymbol{m}}/H_{\boldsymbol{c}}$ *is* λ*-free.*

(b) *If $g \in \mathrm{Aut}(H_{\boldsymbol{m}})$, then for some $k \in K_{\boldsymbol{c}}$ and $f \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$, we have $g = F^k_{\boldsymbol{m}} \circ g_f$, where the automorphism g_f of $H_{\boldsymbol{m}}$ is defined by*

$$g_f(x) := f(h_{\boldsymbol{m}}(x)) \cdot x.$$

Here, $S^{\lambda}_{\aleph_0} := \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \aleph_0\}$, and $\diamondsuit_\lambda(S)$ is the Jensen's diamond; see Definition 3.5. Also, the mysterious condition $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}}) = 0$ is slightly stronger than the trivial dual condition $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z}) = 0$; see Remark 3.4(2). In fact, modulo the consistency of the existence of large cardinals, it is proved in [10] that consistently $\boxtimes_\lambda$ may hold for $\lambda = \aleph_{\omega_1 \cdot \omega}$, where

$(\boxtimes_\lambda)$ If $G \neq 0$ is any λ-free abelian group, then $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$.

According to [9], $\boxtimes_{\aleph_k}$ fails for $k < \omega$, and, by [10], $\boxtimes_{\aleph_{\omega_1 \cdot n}}$ fails as well for all $n < \omega$. So, we expect that Theorem 3.6 holds in ZFC for $\lambda < \aleph_{\omega_1 \cdot \omega}$, but this result has to wait.

In Section 1, we give a systematic study of the representation framework $\boldsymbol{C}_{\mathrm{aut}}$. We start by defining $\boldsymbol{C}^+_{\mathrm{aut}}$, the class of tuples (K, H, F) where $F : K \xrightarrow{\cong} \mathrm{Aut}(H)$. We introduce seven other versions and compare them, as a weak version of $\boldsymbol{C}^+_{\mathrm{aut}}$. Their relation to each other is as follows:

$$\boldsymbol{C}^0_{\mathrm{aut}} \preceq \boldsymbol{C}^1_{\mathrm{aut}} \preceq \boldsymbol{C}^2_{\mathrm{aut}} \preceq \boldsymbol{C}^3_{\mathrm{aut}} \preceq \boldsymbol{C}^4_{\mathrm{aut}} \preceq \boldsymbol{C}^5_{\mathrm{aut}} \preceq \boldsymbol{C}^6_{\mathrm{aut}},$$

with the convention that $\boldsymbol{C}^i_{\mathrm{aut}} \preceq \boldsymbol{C}^{i+1}_{\mathrm{aut}}$ means $\boldsymbol{C}^{i+1}_{\mathrm{aut}}$ is constructed from $\boldsymbol{C}^i_{\mathrm{aut}}$. In Section 2, we present a systematic study of explicit automorphisms in $\mathrm{Aut}(H)$. This includes g_f and its variations. In Section 3, we prove Theorem 3.6 via presenting a connection between Problem 0.1 and the trivial dual conjecture, which searches for the existence of almost free groups with trivial dual; see [9; 10]. Namely, we show that if $\boldsymbol{m} \in \boldsymbol{M_c}$ is auto-rigid, i.e., $F_{\boldsymbol{m}} : K_{\boldsymbol{m}} \to \mathrm{Aut}(H_{\boldsymbol{m}})$ is an isomorphism, and $|H_{\boldsymbol{m}}| > |K_{\boldsymbol{c}}|$, then $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z})$ is trivial, see Lemma 3.1. We then use Jensen's diamond principle to complete the proof of Theorem 3.6. This enables us to present the following solution to Problem 0.2:

Corollary 3.16. *Assume Gödel's axiom of constructibility $\boldsymbol{V} = \boldsymbol{L}$, and let K be any group. Then the following are equivalent*:

(a) *For every cardinal $\lambda = \mathrm{cf}(\lambda) > 2^{|K|+\aleph_0}$, there is a group H such that $|H| = |\mathcal{Z}(H)| = \lambda$ and $\mathrm{Aut}(H) \cong K$.*

(b) *There is $\boldsymbol{c} \in \boldsymbol{C}^+_{\mathrm{aut}}$ such that $K_{\boldsymbol{c}} \cong K$, $\mathrm{Aut}(H_{\boldsymbol{c}}) \cong K$ and $|\mathcal{Z}(H_{\boldsymbol{c}})| > 2^{|K|+\aleph_0}$.*

1. Finding the right framework

In this section, we introduce several classes of objects which will serve as a formal framework for realizing a fixed group K as an automorphism group.

Notation 1.1. For a group K, by e_K we mean the unit element. We denote the group operation by $\cdot$, so that for two elements $k_1, k_2 \in K$, their product is denoted by $k_1 \cdot k_2$ or simply $k_1 k_2$. By k^{-1} we mean the inverse of $k \in K$. As usual, if K is abelian we use the additive notation $(K, +, -, 0)$.

By a p-group, where p is a prime number, we mean an abelian p-group, i.e., a group G such that, for all $x \in G$, there exists some $n \geq 0$ with $p^n x = 0$.

Notation 1.2. Let H be a group which is not necessarily abelian:

(1) By $\mathcal{Z}(H)$ we mean the center of H.

(2) Suppose $a \in H$. This induces an inner automorphism $a^* \in \mathrm{Inn}(H)$ defined by $a^*(x) := axa^{-1}$ for all $x \in H$. Thus $\mathrm{Inn}(H) = \{a^* \mid a \in H\} \subseteq \mathrm{Aut}(H)$.

(3) The notation $\mathrm{tor}(H)$ stands for the full torsion subgroup of $\mathcal{Z}(H)$.

(4) A group G is pure in an abelian group H if $G \subseteq H$ and $nG = nH \cap G$ for every $n \in \mathbb{Z}$. The common notation for this notion is $G \subseteq_* H$.

We will pay respect to the fact that the center $\mathcal{Z}(G)$ of a group G is abelian by giving preference to the additive notation on this specific subgroup of G.

Definition 1.3. Let $\boldsymbol{C}^+_{\mathrm{aut}}$ be the class of all tuples $\boldsymbol{c} = (K_{\boldsymbol{c}}, H_{\boldsymbol{c}}, F_{\boldsymbol{c}})$ such that $K_{\boldsymbol{c}}, H_{\boldsymbol{c}}$ are groups and $F_{\boldsymbol{c}} : K_{\boldsymbol{c}} \to \mathrm{Aut}(H_{\boldsymbol{c}})$ is an isomorphism. Let also $\|\boldsymbol{c}\| := |K_{\boldsymbol{c}}| + |H_{\boldsymbol{c}}|$.

Given a suitable group K, we are going to find when suitable arbitrarily large groups H exist such that for some isomorphism $F : K \to \mathrm{Aut}(H)$, the triple (K, H, F) is in $\boldsymbol{C}^+_{\mathrm{aut}}$. For this reason, we define several new classes of objects, which in a sense weaken the above notion of $\boldsymbol{C}^+_{\mathrm{aut}}$ and will be better understood in the sequel. The main object in the following definition is the class $\boldsymbol{C}^2_{\mathrm{aut}}$.

Definition 1.4. (1) Let $\boldsymbol{C}^0_{\mathrm{aut}}$ be the class of all triples $\boldsymbol{c} = (K_{\boldsymbol{c}}, H_{\boldsymbol{c}}, F_{\boldsymbol{c}})$ such that

- (a) $K_{\boldsymbol{c}}, H_{\boldsymbol{c}}$ are groups, and
- (b) $F_{\boldsymbol{c}} : K_{\boldsymbol{c}} \to \mathrm{Aut}(H_{\boldsymbol{c}})$ is a group embedding.

We set $F^k_{\boldsymbol{c}} := F_{\boldsymbol{c}}(k)$ for $k \in K_{\boldsymbol{c}}$ and $N_{\boldsymbol{c}} := H_{\boldsymbol{c}}/\mathcal{Z}(H_{\boldsymbol{c}})$.

(2) Let $\boldsymbol{C}^1_{\mathrm{aut}}$ be the class of all tuples $\boldsymbol{c} = (K_{\boldsymbol{c}}, N_{\boldsymbol{c}}, H_{\boldsymbol{c}}, h_{\boldsymbol{c}}, F_{\boldsymbol{c}}, Q^{\bar{s}}_{\boldsymbol{c}})$ with $(K_{\boldsymbol{c}}, H_{\boldsymbol{c}}, F_{\boldsymbol{c}}) \in \boldsymbol{C}^0_{\mathrm{aut}}$ such that:

- (a) $N_{\boldsymbol{c}}$ is a normal subgroup of $K_{\boldsymbol{c}}$.
- (b) $h_{\boldsymbol{c}} : H_{\boldsymbol{c}} \to N_{\boldsymbol{c}}$ is an epimorphism with $\mathrm{Ker}(h_{\boldsymbol{c}}) = \mathcal{Z}(H_{\boldsymbol{c}})$.
- (c) If $a \in H_{\boldsymbol{c}}$, then $F_{\boldsymbol{c}}(h_{\boldsymbol{c}}(a)) = a^*$ is the inner automorphism of $H_{\boldsymbol{c}}$ defined by $a^*(x) = axa^{-1}$ for all $x \in H_{\boldsymbol{c}}$, thus $N_{\boldsymbol{c}} \cong F_{\boldsymbol{c}}(N_{\boldsymbol{c}}) = \mathrm{Inn}(H_{\boldsymbol{c}}) \subseteq \mathrm{Aut}(H_{\boldsymbol{c}})$.
- (d) For all $n > 0$, $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n$, $Q^{\bar{s}}_{\boldsymbol{c}}$ is an n-ary relation on $K_{\boldsymbol{c}}$.
- (e) For $\bar{s} \in \mathbb{Z}^n$, we have $(b_1, \ldots, b_n) \in Q^{\bar{s}}_{\boldsymbol{c}}$ if and only if $\sum_{\ell=1}^n s_\ell F^{b_\ell}_{\boldsymbol{c}} \restriction_{\mathcal{Z}(H_{\boldsymbol{c}})} = 0$.

(3) Let $\boldsymbol{C}^2_{\mathrm{aut}}$ be the class of all tuples $\boldsymbol{c} = (K_{\boldsymbol{c}}, N_{\boldsymbol{c}}, H_{\boldsymbol{c}}, h_{\boldsymbol{c}}, h^*_{\boldsymbol{c}}, F_{\boldsymbol{c}}, Q^{\bar{s}}_{\boldsymbol{c}})$ such that $(K_{\boldsymbol{c}}, N_{\boldsymbol{c}}, H_{\boldsymbol{c}}, h_{\boldsymbol{c}}, F_{\boldsymbol{c}}, Q^{\bar{s}}_{\boldsymbol{c}}) \in \boldsymbol{C}^1_{\mathrm{aut}}$ and $h^*_{\boldsymbol{c}} : N_{\boldsymbol{c}} \to H_{\boldsymbol{c}}$ is a map with $h_{\boldsymbol{c}} \circ h^*_{\boldsymbol{c}} = \mathrm{id}$.[1]

Definition 1.5. (1) For $\boldsymbol{c} \in \boldsymbol{C}^1_{\mathrm{aut}} \cup \boldsymbol{C}^2_{\mathrm{aut}}$, let $\mathrm{res}_0(\boldsymbol{c}) := (K_{\boldsymbol{c}}, H_{\boldsymbol{c}}, F_{\boldsymbol{c}}) \in \boldsymbol{C}^0_{\mathrm{aut}}$.

(2) For $\boldsymbol{c} \in \boldsymbol{C}^2_{\mathrm{aut}}$, let $\mathrm{res}_1(\boldsymbol{c}) := (K_{\boldsymbol{c}}, N_{\boldsymbol{c}}, H_{\boldsymbol{c}}, h_{\boldsymbol{c}}, F_{\boldsymbol{c}}, Q^{\bar{s}}_{\boldsymbol{c}}) \in \boldsymbol{C}^1_{\mathrm{aut}}$. In other words, $\boldsymbol{C}^0_{\mathrm{aut}} \subseteq \boldsymbol{C}^1_{\mathrm{aut}} \subseteq \boldsymbol{C}^2_{\mathrm{aut}}$.

Notation 1.6. (1) The *m-power torsion subgroup* of G is

$$\mathrm{Tor}_m(G) := \{g \in \mathcal{Z}(G) \mid \exists n \geq 0 \text{ such that } m^n g = 0\}.$$

With the convenience that $\mathrm{Tor}_m(G) = \bigoplus\{\mathrm{Tor}_p(G) \mid p \textit{ is a prime factor of } m\}$.

(2) Let $\mathbb{P}$ denote the set of all prime numbers and let $p \in \mathbb{P}$. Recall that $J_p := \widehat{\mathbb{Z}}_p$, the ring of p-adic integers, is the completion of $\mathbb{Z}$ in the p-adic topology.

[1] So, the map $h^*_{\boldsymbol{c}}$ assigns to each $b \in N_{\boldsymbol{c}}$ a preimage under $h_{\boldsymbol{c}}$ but may not be a homomorphism.

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Definition 1.7. (1) Let $\ell \in \{0, 1, 2\}$ and H be a group. We define $\mathrm{Prime}_\ell(H)$ as the following subset of the set of prime numbers:

(a) We say $p \in \mathrm{Prime}_1(H)$ if and only if $\mathrm{Tor}_p(H) \neq 0$.
(b) We say $p \in \mathrm{Prime}_2(H)$ if and only if there is an embedding $(J_p, +) \hookrightarrow \mathcal{Z}(H)$.
(c) $\mathrm{Prime}_0(H) := \mathrm{Prime}_1(H) \cup \mathrm{Prime}_2(H)$.

(2) We say H is $\mathbb{P}_*$*-divisible*, where $\mathbb{P}_*$ is a set of prime numbers, if $\mathcal{Z}(H)$ is p-divisible, for all $p \notin \mathbb{P}_*$.

Definition 1.8. (1) Let $\boldsymbol{C}^3_{\mathrm{aut}}$ be the class of all $\boldsymbol{c} = (K_{\boldsymbol{c}}, N_{\boldsymbol{c}}, H_{\boldsymbol{c}}, h_{\boldsymbol{c}}, h^*_{\boldsymbol{c}}, F_{\boldsymbol{c}}, Q^{\bar{s}}_{\boldsymbol{c}}, H^*_{\boldsymbol{c}}, \mathbb{P}_{\boldsymbol{c}})$ such that the following properties hold:

(a) $\mathrm{res}_2(\boldsymbol{c}) := (K_{\boldsymbol{c}}, N_{\boldsymbol{c}}, H_{\boldsymbol{c}}, h_{\boldsymbol{c}}, h^*_{\boldsymbol{c}}, F_{\boldsymbol{c}}, Q^{\bar{s}}_{\boldsymbol{c}}) \in \boldsymbol{C}^2_{\mathrm{aut}}$.
(b) $\mathcal{Z}(H_{\boldsymbol{c}})$ is reduced.
(c) $H^*_{\boldsymbol{c}} \subseteq H_{\boldsymbol{c}}$ is a subgroup, $H_{\boldsymbol{c}} = \bigcup\{H^*_{\boldsymbol{c}} x \mid x \in \mathcal{Z}(H_{\boldsymbol{c}})\}$, and $\mathcal{Z}(H^*_{\boldsymbol{c}}) = \mathcal{Z}(H_{\boldsymbol{c}}) \cap H^*_{\boldsymbol{c}}$, hence $H^*_{\boldsymbol{c}}$ is a normal subgroup of $H_{\boldsymbol{c}}$.
(d) $\mathbb{P}_{\boldsymbol{c}} := \mathrm{Prime}_0(H_{\boldsymbol{c}})$, and we have:
(d_1) If $p \in \mathrm{Prime}_1(H_{\boldsymbol{c}})$, then $\mathrm{Tor}_p(H^*_{\boldsymbol{c}}) \neq 0$, and
(d_2) if $p \in \mathrm{Prime}_2(H_{\boldsymbol{c}})$, then $(J_p, +)$ embeds into $\mathcal{Z}(H^*_{\boldsymbol{c}})$.

(2) Let $\boldsymbol{C}^4_{\mathrm{aut}}$ be the class of all tuples $\boldsymbol{c} \in \boldsymbol{C}^3_{\mathrm{aut}}$ such that $H_{\boldsymbol{c}}/H^*_{\boldsymbol{c}} = \mathcal{Z}(H_{\boldsymbol{c}})/\mathcal{Z}(H^*_{\boldsymbol{c}})$ is a torsion-free abelian group which is $\mathbb{P}_{\boldsymbol{c}}$-divisible.

(3) Let $\boldsymbol{C}^5_{\mathrm{aut}}$ be the class of all tuples $(\boldsymbol{c}, \boldsymbol{g})$ such that $\boldsymbol{c} \in \boldsymbol{C}^4_{\mathrm{aut}}$, and $\boldsymbol{g} := (\mathbb{G}_{\boldsymbol{g}}, F^k_{\boldsymbol{g}})_{k \in K_{\boldsymbol{c}}}$ is defined by the following:

(a) $\mathbb{G}_{\boldsymbol{g}}$ is a reduced torsion-free abelian group.
(b) $F^\ell_{\boldsymbol{g}} \in \mathrm{Aut}(\mathbb{G}_{\boldsymbol{g}})$ such that for all $n > 0$, $\bar{s} \in \mathbb{Z}^n$ and $(b_1, \ldots, b_n) \in (K_{\boldsymbol{c}})^n$: if $(b_1, \ldots, b_n) \in Q^{\bar{s}}_{\boldsymbol{c}}$, then $\sum_{\ell=1}^n s_\ell F^{b_\ell}_{\boldsymbol{g}} \restriction_{\mathbb{G}_{\boldsymbol{g}}} = 0$.
(c) For all primes p, J_p does not embed into $\mathbb{G}_{\boldsymbol{g}}$.

(4) Let $\boldsymbol{C}^6_{\mathrm{aut}}$ be the class of all tuples $(\boldsymbol{c}, \mathcal{G})$ with $\boldsymbol{c} \in \boldsymbol{C}^4_{\mathrm{aut}}$ such that:

(a) $\mathcal{G}$ is a nonempty set, and each member $\boldsymbol{g} \in \mathcal{G}$ is such that $(\boldsymbol{c}, \boldsymbol{g}) \in \boldsymbol{C}^5_{\mathrm{aut}}$.
(b) If $\boldsymbol{g} \in \mathcal{G}$ and $x \in \mathbb{G}_{\boldsymbol{g}}$, then $\boldsymbol{g} \restriction \mathrm{cl}\{x\} \in \mathcal{G}$, where $\mathrm{cl}\{x\}$ is the smallest pure subgroup of $\mathbb{G}_{\boldsymbol{g}}$ containing x that is closed under all $F^k_{\boldsymbol{g}}$'s for $k \in K_{\boldsymbol{c}}$.
(c) If $(\boldsymbol{c}, \boldsymbol{g}) \in \boldsymbol{C}^5_{\mathrm{aut}}$, $\boldsymbol{g}$ is embeddable into $(\mathcal{Z}(H_{\boldsymbol{c}}), F^k_{\boldsymbol{c}})_{k \in K_{\boldsymbol{c}}}$, and there is some $x \in \mathbb{G}_{\boldsymbol{g}}$ so that $\boldsymbol{g} = \boldsymbol{g} \restriction \mathrm{cl}\{x\}$, then $\boldsymbol{g}$ is isomorphic to some member of $\mathcal{G}$.

(5) Let $\|(\boldsymbol{c}, \mathcal{G})\| := |K_{\boldsymbol{c}}| + |H_{\boldsymbol{c}}| + \aleph_0 + \Sigma\{\|\boldsymbol{g}\| \mid \boldsymbol{g} \in \mathcal{G}\}$.

Lemma 1.9. *The following assertions are valid*:

(1) *If* $(\boldsymbol{c}, \boldsymbol{g}) := (\boldsymbol{c}, (\mathbb{G}_{\boldsymbol{g}}, F^k_{\boldsymbol{g}})_{k \in K_{\boldsymbol{c}}}) \in \boldsymbol{C}^5_{\mathrm{aut}}$, *then* $\boldsymbol{n} := (\boldsymbol{c} \oplus (\mathbb{G}_{\boldsymbol{g}}, F^k_{\boldsymbol{g}})_{k \in K_{\boldsymbol{c}}}) \in \boldsymbol{C}^4_{\mathrm{aut}}$, *where* $\mathcal{Z}(H_{\boldsymbol{n}}) = \mathcal{Z}(H_{\boldsymbol{c}}) \oplus \mathbb{G}_{\boldsymbol{g}}$, *and* $F^k_{\boldsymbol{n}}$ *is the unique extension to an automorphism of* $H_{\boldsymbol{n}}$ *extending* $F^k_{\boldsymbol{c}} \cup F^k_{\boldsymbol{g}}$.

(2) *Similarly, if* $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\mathrm{aut}}$, *then* $\boldsymbol{n} := (\boldsymbol{c} \oplus \bigoplus_{\boldsymbol{g} \in \mathcal{G}} (\mathbb{G}_{\boldsymbol{g}}, F^k_{\boldsymbol{g}})_{k \in K_{\boldsymbol{c}}})) \in \boldsymbol{C}^4_{\mathrm{aut}}$, *where* $\mathcal{Z}(H_{\boldsymbol{n}}) = \mathcal{Z}(H_{\boldsymbol{c}}) \oplus \bigoplus_{\boldsymbol{g} \in \mathcal{G}} \mathbb{G}_{\boldsymbol{g}}$, *and* $F^k_{\boldsymbol{n}}$ *is the unique extension to an automorphism of* $H_{\boldsymbol{n}}$ *extending* $F^k_{\boldsymbol{c}} \cup \bigcup_{\boldsymbol{g} \in \mathcal{G}} F^k_{\boldsymbol{g}}$.

(3) *If* $\boldsymbol{d} \in \boldsymbol{C}^+_{\mathrm{aut}}$, *then there is* $\boldsymbol{c} \in \boldsymbol{C}^4_{\mathrm{aut}}$ *with* $\mathrm{res}_0(\boldsymbol{c}) = \boldsymbol{d}$ *and* $|H^*_{\boldsymbol{c}}| \leq (|K_{\boldsymbol{c}}| + \aleph_0)^{\aleph_0}$.

Proof. (1) It is enough to set $F^k_{\boldsymbol{n}}(a, b) := (F^k_{\boldsymbol{c}}(a), F^k_{\boldsymbol{g}}(b))$. This fits in the following commutative diagram:

$$\begin{array}{ccccc}
H_{\boldsymbol{c}} & \xrightarrow{\subseteq} & H_{\boldsymbol{n}} & \xleftarrow{\subseteq} & \mathbb{G}_{\boldsymbol{g}} \\
{\scriptstyle F^k_{\boldsymbol{c}}}\uparrow & & {\scriptstyle F^k_{\boldsymbol{n}}}\uparrow & & \uparrow{\scriptstyle F^k_{\boldsymbol{g}}} \\
H_{\boldsymbol{c}} & \xrightarrow{\subseteq} & H_{\boldsymbol{n}} & \xleftarrow{\supseteq} & \mathbb{G}_{\boldsymbol{g}},
\end{array}$$

and the desired claims follow easily.

(2)+(3) These are easy. □

Definition 1.10. (1) Let $j \in \{2, 3, 4, 5, 6\}$. For any $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$, let $\boldsymbol{M_c} \subseteq \boldsymbol{C}^j_{\text{aut}}$ be the class of all $\boldsymbol{m} \in \boldsymbol{C}^j_{\text{aut}}$ such that the following hold:

(a) $K_{\boldsymbol{m}} = K_{\boldsymbol{c}}$ and $N_{\boldsymbol{m}} = N_{\boldsymbol{c}}$.

(b) $H_{\boldsymbol{c}} \subseteq H_{\boldsymbol{m}}$ and $h_{\boldsymbol{c}} \subseteq h_{\boldsymbol{m}}$.

(c) $h^*_{\boldsymbol{m}} = h^*_{\boldsymbol{c}}$.

(d) $F^k_{\boldsymbol{c}} \subseteq F^k_{\boldsymbol{m}}$ for all $k \in K_{\boldsymbol{c}}$. Let us depict things:

$$\begin{array}{ccccccc}
H_{\boldsymbol{m}} & \xrightarrow{h_{\boldsymbol{m}}} & N_{\boldsymbol{m}} & \xrightarrow{h^*_{\boldsymbol{m}}} & H_{\boldsymbol{m}} & \xrightarrow{F^k_{\boldsymbol{m}}} & H_{\boldsymbol{m}} \\
{\scriptstyle \subseteq}\uparrow & & {\scriptstyle =}\uparrow & & {\scriptstyle \subseteq}\uparrow & & \uparrow{\scriptstyle \subseteq} \\
H_{\boldsymbol{c}} & \xrightarrow{h_{\boldsymbol{c}}} & N_{\boldsymbol{c}} & \xrightarrow{h^*_{\boldsymbol{c}}} & H_{\boldsymbol{c}} & \xrightarrow{F^k_{\boldsymbol{c}}} & H_{\boldsymbol{c}}
\end{array}$$

(e) $Q^{\bar{s}}_{\boldsymbol{m}} = Q^{\bar{s}}_{\boldsymbol{c}}$ for all $\bar{s} \in \bigcup_{n>0} \mathbb{Z}^n$.

(f) The group $H_{\boldsymbol{m}}$ is generated by the set $\mathcal{Z}(H_{\boldsymbol{m}}) \cup H_{\boldsymbol{c}}$.

(g) $\mathcal{Z}(H_{\boldsymbol{c}}) = \mathcal{Z}(H_{\boldsymbol{m}}) \cap H_{\boldsymbol{c}}$.

(h) If $j \geq 3$ then $\mathbb{P}_{\boldsymbol{m}} = \mathbb{P}_{\boldsymbol{c}}$ and $H^*_{\boldsymbol{m}} = H^*_{\boldsymbol{c}}$.

(i) If $j = 5$, then $\boldsymbol{g_m} = \boldsymbol{g_c}$.

(j) If $j = 6$, then $\mathcal{G}_{\boldsymbol{m}} = \mathcal{G}_{\boldsymbol{c}}$.

(2) Let $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$ and $\boldsymbol{m} \in \boldsymbol{M_c}$. We call $\boldsymbol{m}$ *auto-rigid*, if $\text{Im}(F_{\boldsymbol{m}}) = \text{Aut}(H_{\boldsymbol{m}})$.

Definition 1.11. Let $j \in \{2, 3, 4, 5, 6\}$. For $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$ let relation $\leq_{\boldsymbol{c}}$ on $\boldsymbol{M_c}$ be defined by $\boldsymbol{m}_1 \leq_{\boldsymbol{c}} \boldsymbol{m}_2$ if and only if $H_{\boldsymbol{m}_1} \subseteq H_{\boldsymbol{m}_2}$, $h_{\boldsymbol{m}_1} \subseteq h_{\boldsymbol{m}_2}$, and $F^k_{\boldsymbol{m}_1} \subseteq F^k_{\boldsymbol{m}_2}$ for all $k \in K_{\boldsymbol{c}}$.

We have the following easy observations.

Remark 1.12. Let $j \in \{2, 3, 4, 5, 6\}$ and $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$:

(1) The pair $(\boldsymbol{M_c}, \leq_{\boldsymbol{c}})$ is a poset, $\boldsymbol{c} \in \boldsymbol{M_c}$ and $\boldsymbol{c} \leq_{\boldsymbol{c}} \boldsymbol{m}$ for all $\boldsymbol{m} \in \boldsymbol{M_c}$.

(2) If $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \boldsymbol{M_c}$ with $\boldsymbol{m}_1 \leq_{\boldsymbol{c}} \boldsymbol{m}_2$, then $\mathcal{Z}(H_{\boldsymbol{m}_1}) = \mathcal{Z}(H_{\boldsymbol{m}_2}) \cap H_{\boldsymbol{m}_1}$.

(3) Suppose δ is a limit ordinal and $(\boldsymbol{m}_\alpha \mid \alpha < \delta)$ is a $\leq_{\boldsymbol{c}}$-increasing sequence from $\boldsymbol{M_c}$. Then there exists $\boldsymbol{m} = \bigcup_{\alpha<\delta} \boldsymbol{m}_\alpha \in \boldsymbol{M_c}$ which is the $\leq_{\boldsymbol{c}}$-least upper bound of the sequence $(\boldsymbol{m}_\alpha \mid \alpha < \delta)$.

2. Some explicit automorphisms

Let $j \in \{2, 3, 4, 5, 6\}$, $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$, and take $\boldsymbol{m} \in \boldsymbol{M_c}$. Also, let $f \in \text{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$. In this section we study the induced map $g_f(x)$ and its variations. This map plays a crucial role in our main theorem.

Definition 2.1. Let $j \in \{2, 3, 4, 5, 6\}$, $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$, and $\boldsymbol{m} \in \boldsymbol{M_c}$:

(1) Let $\mathcal{A}_{\boldsymbol{m}}$ be the set of all homomorphisms $g \in \text{Aut}(H_{\boldsymbol{m}})$ such that $g\restriction_{\mathcal{Z}(H_{\boldsymbol{m}})} = \text{id}$ and $g(x) \in x \cdot \mathcal{Z}(H_{\boldsymbol{m}})$ for all $x \in H_{\boldsymbol{m}}$.

(2) Set $\mathcal{F}_{\boldsymbol{m}} := \text{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$.

(3) Let $x \in H_{\boldsymbol{m}}$ and $f \in \mathcal{F}_{\boldsymbol{m}}$. The assignment $x \mapsto xf(h_{\boldsymbol{m}}(x))$ defines a map $g_f : H_{\boldsymbol{m}} \to H_{\boldsymbol{m}}$.

Proposition 2.2. *We have* $\mathcal{A}_{\boldsymbol{m}} = \{g_f \mid f \in \mathcal{F}_{\boldsymbol{m}}\}$ *for all* $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$ *and* $\boldsymbol{m} \in \boldsymbol{M_c}$.

Proof. First suppose that $f \in \mathcal{F}_{\boldsymbol{m}}$. We show that g_f is in $\mathcal{A}_{\boldsymbol{m}}$. Clearly, the map g_f is a homomorphism, as $f(h_{\boldsymbol{m}}(x)) \in \mathcal{Z}(H_{\boldsymbol{m}})$ and the equalities

$$g_f(x)g_f(y) = xf(h_{\boldsymbol{m}}(x))yf(h_{\boldsymbol{m}}(y)) = xyf(h_{\boldsymbol{m}}(x))f(h_{\boldsymbol{m}}(y)) = xyf(h_{\boldsymbol{m}}(xy)) = g_f(xy)$$

hold for all $x, y \in H_{\boldsymbol{m}}$. Furthermore, for all $x \in \mathcal{Z}(H_{\boldsymbol{m}})$, we have $h_{\boldsymbol{m}}(x) = e_{N_{\boldsymbol{c}}}$, thus $g_f(x) = xf(h_{\boldsymbol{m}}(x)) = x$, and hence $g_f\restriction_{\mathcal{Z}(H_{\boldsymbol{m}})} = \text{id}$. Finally, we note that g_f is an automorphism. To show this, we claim that its inverse is g_{-f}:

$$g_{-f}(g_f(x)) = g_{-f}(xf(h_{\boldsymbol{m}}(x)) = g_{-f}(x)g_{-f}(f(h_{\boldsymbol{m}}(x))) = g_{-f}(x)f(h_{\boldsymbol{m}}(x)) = xf(h_{\boldsymbol{m}}(x))^{-1}f(h_{\boldsymbol{m}}(x)) = x.$$

Similarly, $g_f(g_{-f}(x)) = x$ for all $x \in H_{\boldsymbol{m}}$. This proves $g_f \in \mathcal{A}_{\boldsymbol{m}}$. To see the reverse inclusion, we assume $g \in \mathcal{A}_{\boldsymbol{m}}$. We must show that $g = g_f$ for some $f \in \mathcal{F}_{\boldsymbol{m}}$. Define the map $f : N_{\boldsymbol{c}} \to \mathcal{Z}(H_{\boldsymbol{m}})$ by $f(a) := x^{-1}g(x)$, where $a = h^*_{\boldsymbol{m}}(x)$. But, this may depend to the choice of x. To show this is a well-defined map, assume that $h_{\boldsymbol{m}}(y) = a$ for any $y \in H_{\boldsymbol{m}}$, and we need to check $f(a) = y^{-1}g(y)$. Indeed, $[h_{\boldsymbol{m}}(x) = h_{\boldsymbol{m}}(y)]$ implies that $xy^{-1} \in \text{Ker}(h_{\boldsymbol{m}}) = \mathcal{Z}(H_{\boldsymbol{m}})$. Thus, in view of Definition 2.1(1) we observe that $g(xy^{-1}) = xy^{-1}$. In other words, $x^{-1}g(x) = y^{-1}g(y)$, i.e., the map f is well-define. In order to show f is a morphism, let $a, b \in N_{\boldsymbol{c}}$ and choose $x, y \in H_{\boldsymbol{m}}$ so that $h_{\boldsymbol{m}}(x) = a$ and $h_{\boldsymbol{m}}(y) = b$. Then $h_{\boldsymbol{m}}(xy) = ab$, $x^{-1}g(x) \in \mathcal{Z}(H_{\boldsymbol{m}})$ and also

$$f(ab) = (xy)^{-1}g(xy) = y^{-1} \cdot x^{-1}g(x) \cdot g(y) = x^{-1}g(x) \cdot y^{-1}g(y) = f(a)f(b).$$

Consequently, $f \in \mathcal{F}_{\boldsymbol{m}}$. Clearly, $g = g_f$. The proof is now complete. □

Notation 2.3. The notation $\boxtimes_{\boldsymbol{d}} := \boxtimes_{(\boldsymbol{c},\boldsymbol{m},\bar{s},\bar{b},\pi)}$ stands for the following hypotheses:

(1) $\boldsymbol{c} \in \boldsymbol{C}^j_{\text{aut}}$ for some $j \in \{2, 3, 4, 5, 6\}$ and $\boldsymbol{m} \in \boldsymbol{M_c}$.

(2) $\bar{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n$ and $\bar{b} = (b_1, \dots, b_n) \in (K_{\boldsymbol{c}})^n$ for some $n > 0$.

(3) $\pi \in \text{End}(N_{\boldsymbol{c}})$.

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Definition 2.4. Let $\boldsymbol{d} := (\boldsymbol{c}, \boldsymbol{m}, \bar{s}, \bar{b}, \pi)$ and assume $\boxtimes_{\boldsymbol{d}}$.

(0) We define $h'^{*}_{\boldsymbol{c},\pi} := h^*_{\boldsymbol{c}} \circ \pi$.

(1) Let $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$ be the set of all homomorphisms $g \in \operatorname{End}(H_{\boldsymbol{m}})$ such that

$$g\restriction_{\mathcal{Z}(H_{\boldsymbol{m}})} = F^{\bar{s},\bar{b}}_{\boldsymbol{m}} := \sum_{\ell=1}^{n} s_\ell(F^{b_\ell}_{\boldsymbol{m}}\restriction_{\mathcal{Z}(H_{\boldsymbol{m}})}), \quad \text{and} \quad g(x) \in h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x)) \cdot \mathcal{Z}(H_{\boldsymbol{m}})$$

for all $x \in H_{\boldsymbol{m}}$.

(2) Let $\mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$ be the set of all functions $f : N_{\boldsymbol{c}} \to \mathcal{Z}(H_{\boldsymbol{m}})$ such that for all $a, b \in N_{\boldsymbol{c}}$ we have $f(a)f(b) = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t') \cdot (t'')^{-1} \cdot f(ab)$, where $t', t'' \in \mathcal{Z}(H_{\boldsymbol{c}}) \subseteq \mathcal{Z}(H_{\boldsymbol{m}})$ are uniquely determined by $t'h^*_{\boldsymbol{c}}(ab) = h^*_{\boldsymbol{c}}(a)h^*_{\boldsymbol{c}}(b)$, and $t''h'^{*}_{\boldsymbol{c},\pi}(ab) = h'^{*}_{\boldsymbol{c},\pi}(a)h'^{*}_{\boldsymbol{c},\pi}(b)$.

(3) For any $f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$ let $g^{\bar{s},\bar{b}}_{\pi,f}(x) : H_{\boldsymbol{m}} \to H_{\boldsymbol{m}}$ be the map defined by the assignment $x \mapsto F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t) \cdot f(h_{\boldsymbol{m}}(x)) \cdot h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x))$ for all $x \in H_{\boldsymbol{m}}$, where $t \in \mathcal{Z}(H_{\boldsymbol{m}})$ is uniquely determined by $x = th^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))$.

Proposition 2.5. *For $\boldsymbol{d} := (\boldsymbol{c}, \boldsymbol{m}, \bar{s}, \bar{b}, \pi)$ let $\boxtimes_{\boldsymbol{d}}$. Then $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi} = \{g^{\bar{s},\bar{b}}_{\pi,f} \mid f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}\}$.*

Proof. First, assume $f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$. We shall prove that $g^{\bar{s},\bar{b}}_{\pi,f} \in \mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$. Let $x_1, x_2 \in H_{\boldsymbol{m}}$ and choose $t_i \in \mathcal{Z}(H_{\boldsymbol{m}})$ for $i \in \{1, 2\}$ such that $x_i = t_i h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x_i))$. We observe that $h_{\boldsymbol{m}}(x_i) \in N_{\boldsymbol{c}}$, and in view of Definition 2.4(2), there are $t', t'' \in \mathcal{Z}(H_{\boldsymbol{c}})$ such that

$$h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x_1))h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x_2)) = t'h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x_1x_2)), \tag{2-1}$$

$$h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_1))h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_2)) = t''h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_1x_2)) \tag{2-2}$$

$$f(h_{\boldsymbol{m}}(x_1))f(h_{\boldsymbol{m}}(x_2)) = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t')(t'')^{-1}f(h_{\boldsymbol{m}}(x_1x_2)). \tag{2-3}$$

$$x_1x_2 \overset{(2\text{-}1)}{=} t't_1t_2h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x_1x_2)). \tag{2-4}$$

Keeping in mind that

$$\operatorname{Rang}(F^{\bar{s},\bar{b}}_{\boldsymbol{m}}) + \operatorname{Rang}(f) \subseteq \mathcal{Z}(H_{\boldsymbol{m}}) \tag{2-5}$$

We deduce the following identities

$$\begin{aligned}
&g^{\bar{s},\bar{b}}_{\pi,f}(x_1)g^{\bar{s},\bar{b}}_{\pi,f}(x_2)\\
&\quad = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t_1)f(h_{\boldsymbol{m}}(x_1))h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_1))\cdot F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t_2)f(h_{\boldsymbol{m}}(x_2))h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_2)) && \text{Definition 2.4(3)}\\
&\quad = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t_1)F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t_2)\cdot f(h_{\boldsymbol{m}}(x_1))f(h_{\boldsymbol{m}}(x_2))\cdot h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_1))h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_2)) && \text{(2-5)}\\
&\quad = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t_1t_2)\cdot F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t')(t'')^{-1}f(h_{\boldsymbol{m}}(x_1x_2))\cdot t''h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_1x_2)) && \text{(2-3)}\\
&\quad = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t')F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t_1t_2)\cdot(t'')^{-1}t''\cdot f(h_{\boldsymbol{m}}(x_1x_2))\cdot h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_1x_2)) && \text{(2-5)}\\
&\quad = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t't_1t_2)f(h_{\boldsymbol{m}}(x_1x_2))h'^{*}_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x_1x_2))\\
&\quad = g^{\bar{s},\bar{b}}_{\pi,f}(x_1x_2). && \text{(2-4) and Definition 2.4(3)}
\end{aligned}$$

This shows that $g^{\bar{s},\bar{b}}_{\pi,f}$ is a homomorphism. Next, observe that for any $x \in H_{\boldsymbol{m}}$, we have $h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x)) = e_{H_{\boldsymbol{m}}} \cdot h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))))$ and

$$\text{(2-6)}\qquad \begin{aligned} g^{\bar{s},\bar{b}}_{\pi,f}(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))) &= F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(e_{H_{\boldsymbol{m}}}) \cdot f(h_{\boldsymbol{m}}(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x)))) \cdot h'^*_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x)))) \\ &= f(h_{\boldsymbol{m}}(x))h'^*_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x)). \end{aligned}$$

Let us plugging $x := th^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))$. This implies that

$$\begin{aligned} g^{\bar{s},\bar{b}}_{\pi,f}(t)g^{\bar{s},\bar{b}}_{\pi,f}(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))) = g^{\bar{s},\bar{b}}_{\pi,f}(th^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))) = g^{\bar{s},\bar{b}}_{\pi,f}(x) = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t)f(h_{\boldsymbol{m}}(x))h'^*_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x)) \\ \overset{\text{(2-6)}}{=} F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t)g^{\bar{s},\bar{b}}_{\pi,f}(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))). \end{aligned}$$

Consequently, $g^{\bar{s},\bar{b}}_{\pi,f}(t) = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t)$. Letting t range over all of $\mathcal{Z}(H_{\boldsymbol{m}})$, we have $g^{\bar{s},\bar{b}}_{\pi,f}(t) = F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t)$ for all $t \in \mathcal{Z}(H_{\boldsymbol{m}})$. This proves $g^{\bar{s},\bar{b}}_{\pi,f} \in \mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$. For the reverse inclusion, assume $g \in \mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$. We must show that $g = g^{\bar{s},\bar{b}}_{\pi,f}$ for some $f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$. To this end, we define a map $f : N_{\boldsymbol{c}} \to \mathcal{Z}(H_{\boldsymbol{m}})$ as follows. For any $a \in N_{\boldsymbol{c}}$ set $f(a) := g(h^*_{\boldsymbol{c}}(a))h'^*_{\boldsymbol{c},\pi}(a)^{-1}$. With Definition 2.4(1) we have $f(a) \in \mathcal{Z}(H_{\boldsymbol{m}})$. Let $a, b \in N_{\boldsymbol{c}}$ and $t', t'' \in \mathcal{Z}(H_{\boldsymbol{c}})$ be chosen as in Definition 2.4(2). We then have

$$\begin{aligned} f(a)f(b) &= g(h^*_{\boldsymbol{c}}(a))h'^*_{\boldsymbol{c},\pi}(a)^{-1}g(h^*_{\boldsymbol{c}}(b))h'^*_{\boldsymbol{c},\pi}(b)^{-1} = g(h^*_{\boldsymbol{c}}(a))g(h^*_{\boldsymbol{c}}(b))h'^*_{\boldsymbol{c},\pi}(b)^{-1}h'^*_{\boldsymbol{c},\pi}(a)^{-1} \\ &= g(h^*_{\boldsymbol{c}}(a)h^*_{\boldsymbol{c}}(b))(h'^*_{\boldsymbol{c},\pi}(a)h'^*_{\boldsymbol{c},\pi}(b))^{-1} = g(t'h^*_{\boldsymbol{c}}(ab))(t''h'^*_{\boldsymbol{c},\pi}(ab))^{-1} \\ &= g(t')g(h^*_{\boldsymbol{c}}(ab)) \cdot h'^*_{\boldsymbol{c},\pi}(ab)^{-1}(t'')^{-1} = g(t')(t'')^{-1} \cdot g(h^*_{\boldsymbol{c}}(ab))h'^*_{\boldsymbol{c},\pi}(ab)^{-1} \\ &= F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t')(t'')^{-1} \cdot f(ab). \end{aligned}$$

Thanks to the definition, $f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$ follows. Next, let $x \in H_{\boldsymbol{m}}$ and $t \in \mathcal{Z}(H_{\boldsymbol{m}})$ be chosen as in Definition 2.4(3). We then have

$$\begin{aligned} g(x) = g(th^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))) = g(t)g(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x))) &= F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t) \cdot g(h^*_{\boldsymbol{c}}(h_{\boldsymbol{m}}(x)))h'^*_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x))^{-1} \cdot h'^*_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x)) \\ &= F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t) \cdot f(h_{\boldsymbol{m}}(x)) \cdot h'^*_{\boldsymbol{c},\pi}(h_{\boldsymbol{m}}(x)) = g^{\bar{s},\bar{b}}_{\pi,f}(x), \end{aligned}$$

which shows $g = g^{\bar{s},\bar{b}}_{\pi,f}$. This ends the proof. □

Proposition 2.6. *The following assertions are valid*:

(1) *Let* $f \in \mathrm{Hom}(H_{\boldsymbol{m}}, \mathcal{Z}(H_{\boldsymbol{m}}))$ *with* $\mathrm{Im}(f) \subseteq \mathrm{Ker}(f)$, *and take* $x \in H_{\boldsymbol{m}}$. *The assignment* $x \mapsto g^1_f(x) := x \cdot f(x)$ *defines an automorphism* $g^1_f : H_{\boldsymbol{m}} \to H_{\boldsymbol{m}}$.

(2) *Assume* $\boxtimes_{\boldsymbol{d}}$ *holds with* $g \in \mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$. *Then* $h \in \mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$ *if and only if* $h(x) = g(x) \cdot f(x)$ *for all* $x \in H_{\boldsymbol{m}}$ *and for some* $f \in \mathrm{Hom}(H_{\boldsymbol{m}}, \mathcal{Z}(H_{\boldsymbol{m}}))$ *with* $\mathcal{Z}(H_{\boldsymbol{m}}) \subseteq \mathrm{Ker}(f)$.

Proof. (1) The map g^1_f is a homomorphism as $f(x) \in \mathcal{Z}(H_{\boldsymbol{m}})$ and

$$g^1_f(x)g^1_f(y) = xf(x)yf(y) = xyf(x)f(y) = xyf(xy) = g^1_f(xy)$$

holds for all $x, y \in H_{\boldsymbol{m}}$. Since $\mathrm{Im}(f) \subseteq \mathrm{Ker}(f)$ we have $f(f(-)) = e_{H_{\boldsymbol{m}}}$. Furthermore, g^1_f is an automorphism with inverse g^1_{-f}:

$$\begin{aligned} g^1_f(g^1_{-f}(x)) &= g^1_f(xf(x)^{-1}) = xf(x)^{-1}f(x)f(f(x^{-1})) = xf(x)^{-1}f(xf(x)^{-1}) \\ &= xf(x)^{-1}f(x)f(f(x^{-1})) = xf(x)^{-1}f(x)e_{H_{\boldsymbol{m}}} = x. \end{aligned}$$

Similarly, $g^1_{-f}(g^1_f(x)) = x$ for all $x \in H_{\boldsymbol{m}}$.

(2) This is easy as well. □

We close this section by presenting situations for which $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m},\pi}$ is nonempty.

Definition 2.7. An abelian group $\mathbb{G}$ is called $\aleph_1$-*free* if every subgroup of $\mathbb{G}$ of cardinality $< \aleph_1$, i.e., every countable subgroup, is free. More generally, an abelian group $\mathbb{G}$ is called λ-*free* if every subgroup of $\mathbb{G}$ of cardinality $< \lambda$ is free.

Remark 2.8. Let $\boldsymbol{d} := (\boldsymbol{c}, \boldsymbol{m}, \bar{s}, \bar{b}, \pi)$, assume $\boxtimes_{\boldsymbol{d}}$, and let $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \boldsymbol{M_c}$ be such that $\boldsymbol{m}_1 \leq_{\boldsymbol{c}} \boldsymbol{m}_2$. Then the following assertions hold:

(1) $\mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m}_1,\pi} \subseteq \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m}_2,\pi}$.

(2) Let $f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m}_1,\pi}$ be given and define the following groups:

  (a) $I_1 := \langle t', F^{\bar{s},\bar{b}}_{\boldsymbol{c}}(t') \mid t' = h^*_{\boldsymbol{c}}(a_1)h^*_{\boldsymbol{c}}(a_2)h^*_{\boldsymbol{c}}(a_1a_2)^{-1} \text{ for some } a_1, a_2 \in N_{\boldsymbol{c}}\rangle_{\mathcal{Z}(H_{\boldsymbol{c}})}$.

  (b) $I_2 := \langle I_1, \mathrm{Im}(f)\rangle \subseteq \mathcal{Z}(H_{\boldsymbol{m}_1})$.

  Also, let $\varphi \in \mathrm{Hom}(I_2, \mathcal{Z}(H_{\boldsymbol{m}_2}))$ be such that $\varphi \restriction I_1 = \mathrm{id}$. Then $\varphi \circ f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m}_2,\pi}$.

(3) If $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m}_1,\pi} \neq \varnothing$, then $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m}_2,\pi} \neq \varnothing$.

(4) Suppose $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m}_2,\pi} \neq \varnothing$ and let $\mathcal{Z}(H_{\boldsymbol{m}_2})/\mathcal{Z}(H_{\boldsymbol{m}_1})$ be an $(\aleph_1 \cdot |N_{\boldsymbol{c}}|^+)$-free abelian group. Then $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m}_1,\pi} \neq \varnothing$.

Proof. (1) This follows from Definition 2.4(2) and Remark 1.12(2).

(2) We just need to check Definition 2.4(2), keeping in mind that $F^{\bar{s},\bar{b}}_{\boldsymbol{m}}(t') = F^{\bar{s},\bar{b}}_{\boldsymbol{c}}(t')$ and $F^{\bar{s},\bar{b}}_{\boldsymbol{c}}(t'), t'' \in I_1$.

(3) This is a consequence of (1) and Proposition 2.5.

(4) Let $f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m}_2,\pi}$. Then

$$|\langle \mathcal{Z}(H_{\boldsymbol{m}_1}), \mathrm{Im}(f)\rangle / \mathcal{Z}(H_{\boldsymbol{m}_1})| \leq \aleph_0 \cdot |N_{\boldsymbol{c}}| < \aleph_1 \cdot |N_{\boldsymbol{c}}|^+.$$

Thus $\langle \mathcal{Z}(H_{\boldsymbol{m}_1}), \mathrm{Im}(f)\rangle / \mathcal{Z}(H_{\boldsymbol{m}_1})$ is a free abelian group, and we define φ to be the projection onto the direct summand $\mathcal{Z}(H_{\boldsymbol{m}_1}) \subseteq \langle \mathcal{Z}(H_{\boldsymbol{m}_1}), \mathrm{Im}(f)\rangle$. We have $\varphi \circ f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m}_2,\pi}$ with (2), and even $\varphi \circ f \in \mathcal{F}^{\bar{s},\bar{b}}_{\boldsymbol{m}_1,\pi}$ as $\mathrm{Im}(\varphi) \subseteq \mathcal{Z}(H_{\boldsymbol{m}_1})$. This implies that $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m}_1,\pi} \neq \varnothing$. In view of Proposition 2.5 $\mathcal{A}^{\bar{s},\bar{b}}_{\boldsymbol{m}_1,\pi} \neq \varnothing$. □

3. The structure of large auto-rigid representations

In this section we present the proof of Theorem 3.6 and Corollary 3.16. The next lemma gives conditions under which $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z})$ is trivial.

Lemma 3.1. *Let $j \in \{2, 3, 4, 5, 6\}$. Let $\boldsymbol{c} \in \boldsymbol{C}^j_{\mathrm{aut}}$, and suppose that there is some auto-rigid $\boldsymbol{m} \in M_{\boldsymbol{c}}$ with $\|\boldsymbol{m}\| > \|\boldsymbol{c}\|$, or just $|\mathcal{Z}(H_{\boldsymbol{m}})| > |K_{\boldsymbol{c}}|$. Then* $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z}) = 0$.

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Proof. Suppose for the sake of contradiction that there is a nonzero $f_* \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z})$. For each $x \in \mathcal{Z}(H_{\boldsymbol{m}})$, the assignment $a \mapsto f_x(a) := x^{f_*(a)}$ defines a map $f_x \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$. It then follows that by Proposition 2.6, the map $g_{f_x} : H_{\boldsymbol{m}} \to H_{\boldsymbol{m}}$ defined by $g_{f_x}(t) = t \cdot f_x(h_{\boldsymbol{m}}(t))$ is in $\mathrm{Aut}(H_{\boldsymbol{m}})$. Now, by the auto-rigidity, we have $K_{\boldsymbol{c}} \cong \mathrm{Im}(F_{\boldsymbol{m}}) = \mathrm{Aut}(H_{\boldsymbol{m}})$, hence $|K_{\boldsymbol{c}}| = |\mathrm{Aut}(H_{\boldsymbol{m}})| \geq |\mathcal{Z}(H_{\boldsymbol{m}})| = |H_{\boldsymbol{m}}| > |K_{\boldsymbol{c}}|$, which is impossible. This contradiction shows that $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z})$ is trivial, which gives the desired conclusion. □

Definition 3.2. Suppose G is an abelian group and $\boldsymbol{k} = (G, (F^k_G \mid k \in K_{\boldsymbol{c}}))$ is an expansion of G, where each F^k_G is an automorphism of G. Let $\boldsymbol{c} \in \boldsymbol{C}^1_{\mathrm{aut}}$ and $\boldsymbol{m} \in \boldsymbol{M}_{\boldsymbol{c}}$:

(1) We say $\boldsymbol{k}$ is $\boldsymbol{c}$-*correct* if for all $n < \omega$, $\bar{s} \in \mathbb{Z}^n$ and $\bar{b} \in (K_{\boldsymbol{c}})^n$ with $\bar{b} \in Q^{\bar{s}}_{\boldsymbol{c}}$ holds $\sum_{\ell=1}^n F^{\ell}_G = 0 \in \mathrm{End}(G)$.

(2) If $\boldsymbol{k}$ is $\boldsymbol{c}$-correct, we define $\boldsymbol{n} = \boldsymbol{m} \oplus \boldsymbol{k} \in \boldsymbol{M}_{\boldsymbol{c}}$ so that $\mathcal{Z}(H_{\boldsymbol{n}}) = \mathcal{Z}(H_{\boldsymbol{m}}) \oplus G$, and $F^k_{\boldsymbol{n}}$ is the unique extension of $F^k_{\boldsymbol{m}} \cup F^k_G$ to an automorphism of $H_{\boldsymbol{n}}$.

(3) Suppose $(\boldsymbol{c}, \boldsymbol{g}) \in \boldsymbol{C}^5_{\mathrm{aut}}$ and $\boldsymbol{m} \in \boldsymbol{M}_{\boldsymbol{c}}$.
   (a) We say $\boldsymbol{m}$ is *free over* $(\boldsymbol{c}, \boldsymbol{g})$, if we can find $(f_\xi \mid \xi < \zeta)$ such that $\mathcal{Z}(H_{\boldsymbol{m}}) = \bigoplus_{\xi<\zeta} f_\xi(\mathbb{G}_{\boldsymbol{g}}) \oplus \mathcal{Z}(H_{\boldsymbol{c}})$, where $f_\xi : (\mathbb{G}_{\boldsymbol{g}}, (F^k_{\boldsymbol{g}})_{k\in K_{\boldsymbol{c}}}) \hookrightarrow (H_{\boldsymbol{c}}, (F^k_{\boldsymbol{m}})_{k\in K_{\boldsymbol{c}}})$ is an embedding. We assume that the embedding respects structures, which means $f_\xi F^k_{\boldsymbol{g}} = F^k_{\boldsymbol{m}} f_\xi$.
   (b) We say $\boldsymbol{m}$ is λ-*free over* $\boldsymbol{c}$, if for any subgroup G' of $\mathcal{Z}(H_{\boldsymbol{m}})/\mathcal{Z}(H_{\boldsymbol{c}})$ of size $< \lambda$, there is $\boldsymbol{m}' \in \boldsymbol{M}_{\boldsymbol{c}}$, $\boldsymbol{m}' \leq_{\boldsymbol{c}} \boldsymbol{m}$ which is free over $\boldsymbol{c}$ such that $G' \subseteq H_{\boldsymbol{m}'}$.
   (c) We say $\boldsymbol{m}$ is *strongly* λ-*free over* $\boldsymbol{c}$, if `free` wins the following game for which a play lasts ω moves: in the n-th move, `nonfree` chooses $X_n \in [H_{\boldsymbol{m}}]^{<\lambda}$, `free` chooses $\xi_n < \lambda$ and $(f_{n,\xi} \mid \xi < \xi_n)$, where each $f_{n,\xi} : \mathbb{G}_{\boldsymbol{g}} \to \mathcal{Z}(H_{\boldsymbol{m}})$ is an embedding, and $\sum_{m\leq n,\xi<\xi_n} f_{m,\xi}(\mathbb{G}_{\boldsymbol{g}}) + \mathcal{Z}(H_{\boldsymbol{m}}) = \bigoplus_{m\leq n,\xi<\xi_n} f_{m,\xi}(\mathbb{G}_{\boldsymbol{g}}) \oplus \mathcal{Z}(H_{\boldsymbol{m}})$, and it includes X_n. The `free` player wins if he always has a legal move.

(4) Similarly, if $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\mathrm{aut}}$ and $\boldsymbol{m} \in \boldsymbol{M}_{\boldsymbol{c}}$, we say $\boldsymbol{m}$ is *strongly* λ-*free over* $(\boldsymbol{c}, \mathcal{G})$, if `free` wins the following game for which a play lasts ω moves: in the n-th move, `nonfree` chooses $X_n \in [H_{\boldsymbol{m}}]^{<\lambda}$, `free` chooses $\xi_n < \lambda$ and $(f_{n,\xi} \mid \xi < \xi_n)$, where each $f_{n,\xi} : \boldsymbol{g}_{n,\xi} \to \mathcal{Z}(H_{\boldsymbol{m}})$ is an embedding, and $\sum_{m\leq n,\xi<\xi_n} f_{m,\xi}(\boldsymbol{g}_{m,\xi}) + \mathcal{Z}(H_{\boldsymbol{m}}) = \bigoplus_{m\leq n,\xi<\xi_n} f_{m,\xi}(\boldsymbol{g}_{m,\xi}) \oplus \mathcal{Z}(H_{\boldsymbol{m}})$ for some $\boldsymbol{g}_{m,\xi} \in \mathcal{G}$, and it includes X_n. The player `free` wins if he always has a legal move.

Lemma 3.3. *Suppose* $(\boldsymbol{c}, \boldsymbol{g}) \in \boldsymbol{C}^5_{\mathrm{aut}}$, $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}}) = 0$, *and* $\boldsymbol{m}$ *is* $|N_{\boldsymbol{c}}|^+$-*free over* $(\boldsymbol{c}, \boldsymbol{g})$. *Then*

$$\mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}})) \subseteq \mathrm{Hom}(N_{\boldsymbol{c}}, H_{\boldsymbol{c}}).$$

Proof. Let $f \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$. Suppose for the sake of contradiction that $f \notin \mathrm{Hom}(N_{\boldsymbol{c}}, H_{\boldsymbol{c}})$. This means that $\mathrm{Im}(f) \nsubseteq H_{\boldsymbol{c}}$. In other words, the following compositions map

$$\bar{f} := N_{\boldsymbol{c}} \xrightarrow{f} \mathcal{Z}(H_{\boldsymbol{m}}) \xrightarrow{\subseteq} H_{\boldsymbol{m}} \xrightarrow{\twoheadrightarrow} H_{\boldsymbol{m}}/H_{\boldsymbol{c}}$$

is nonzero and $\mathrm{Im}(\bar{f}) \subseteq H_{\boldsymbol{m}}/H_{\boldsymbol{c}}$ is of size at most $|N_{\boldsymbol{c}}|$. Thus, by our assumption, we can find $(f_\xi \mid \xi < \zeta)$ such that $f_\xi : \mathbb{G}_{\boldsymbol{g}} \to \mathcal{Z}(H_{\boldsymbol{c}})$ is an embedding and $\mathrm{Im}(\bar{f}) \subseteq \bigoplus_{\xi<\zeta} f_\xi(\mathbb{G}_{\boldsymbol{g}})$. Hence, for some $\xi < \zeta$, the natural projection $\pi_\xi : \bigoplus_{\xi<\zeta} f_\xi(\mathbb{G}_{\boldsymbol{g}}) \to f_\xi(\mathbb{G}_{\boldsymbol{g}})$ satisfies that $\pi_\xi \circ \bar{f} \in \mathrm{Hom}(N_{\boldsymbol{c}}, f_\xi(\mathbb{G}_{\boldsymbol{g}}))$ is nonzero. We proved that $f_\xi \circ \pi_\xi \circ \bar{f} : N_{\boldsymbol{c}} \to \mathbb{G}_{\boldsymbol{g}}$ is nonzero, a contradiction. □

Remark 3.4. The above proof shows that:

(1) Suppose $(\boldsymbol{c}, \boldsymbol{g}) \in \boldsymbol{C}^5_{\text{aut}}$, $\boldsymbol{m}$ is λ-free over $(\boldsymbol{c}, \boldsymbol{g})$, and $\mathbb{G}_{\boldsymbol{g}}$ is λ-free. Then $H_{\boldsymbol{m}}/H_{\boldsymbol{c}}$ is λ-free.

(2) Let $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\text{aut}}$. Then $\text{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}}) = 0$ for some $\boldsymbol{g} \in \mathcal{G}$ if and only if $\text{Hom}(N_{\boldsymbol{c}}, \mathbb{Z}) = 0$.

Definition 3.5. Suppose $\lambda > \aleph_0$ is regular and $S \subseteq \lambda$ is stationary:

(1) *Jensen's diamond* $\diamondsuit_\lambda(S)$ asserts the existence of a sequence $(S_\alpha \mid \alpha \in S)$ such that for every $X \subseteq \lambda$ the set $\{\alpha \in S \mid X \cap \alpha = S_\alpha\}$ is stationary.

(2) We use the following consequence of $\diamondsuit_\lambda(S)$: let $A = \bigcup_{\alpha<\lambda} A_\alpha$ and $B = \bigcup_{\alpha<\lambda} B_\alpha$ be two λ-filtrations. Then there are $\{g_\alpha \mid A_\alpha \to B_\alpha \mid \alpha < \lambda\}$ such that, for any function $g : A \to B$, the set $\{\alpha \in S \mid g\restriction_{A_\alpha} = g_\alpha\}$ is stationary in λ.

(3) S is *nonreflecting* if for any limit ordinal $\delta < \lambda$ of uncountably cofinality, the set $S \cap \delta$ is nonstationary in δ.

(4) We set $S^\lambda_{\aleph_0} = \{\alpha < \lambda \mid \text{cf}(\alpha) = \aleph_0\}$.

Recall that $g_f(x) := x \cdot f(h_{\boldsymbol{m}}(x))$. The following is the main result of this section:

Theorem 3.6. *Let* $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\text{aut}}$, $\lambda = \text{cf}(\lambda) > 2^{\|\boldsymbol{c}\|}$ *and assume that*:

(1) $\text{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}}) = 0$ *for all* $\boldsymbol{g} \in \mathcal{G}$, *and* $F_{\boldsymbol{c}} : K_{\boldsymbol{c}} \to \text{Aut}(H_{\boldsymbol{c}})$ *is an isomorphism.*

(2) $S \subseteq S^\lambda_{\aleph_0}$ *is stationary nonreflecting such that* $\diamondsuit_\lambda(S)$ *holds.*

Then there is some $\boldsymbol{m} \in \boldsymbol{C}^6_{\text{aut}} \cap \boldsymbol{M}_{\boldsymbol{c}}$ *of size* λ *such that the following holds*:

(α) $\boldsymbol{m}$ *is* λ*-free over* $\boldsymbol{c}$.

(β) $\text{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}})) \subseteq \text{Hom}(N_{\boldsymbol{c}}, H_{\boldsymbol{c}})$.

(γ) *If* $g \in \text{Aut}(H_{\boldsymbol{m}})$, *then for some* $k \in K_{\boldsymbol{c}}$ *and* $f \in \text{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$, *we have* $g = F^k_{\boldsymbol{m}} \circ g_f$.

Proof. Without loss of generality, $\mathcal{G}$ consists of pairwise disjoint elements. Let $\bar{g} := (g_\alpha \mid \alpha \in S)$ be such that $g_\alpha : \alpha \to \alpha$ and it is a diamond sequence for S, in the sense that for each $g : \lambda \to \lambda$, the set $\{\alpha \in S \mid g \restriction \alpha = g_\alpha\}$ is stationary in λ. Without loss of generality, we assume in addition that the set of elements of $H_{\boldsymbol{c}}$ is an ordinal, which, by our assumption, is $< \lambda$. For $\gamma \leq \lambda$, we define the set Λ_γ, consisting of sequences $\bar{\boldsymbol{m}} = (\boldsymbol{m}_\alpha \mid \alpha < \gamma)$ of length γ and a set $U_0(\bar{\boldsymbol{m}}) \subseteq \gamma$ such that:

$(*)_1$ (a) $\boldsymbol{m}_\alpha \in \boldsymbol{M}_{\boldsymbol{c}}$ has universe an ordinal less than λ.
(b) $\boldsymbol{m}_0 = \boldsymbol{c}$.
(c) $\boldsymbol{m}_\alpha$ is free over $\boldsymbol{c}$, in particular, $H_{\boldsymbol{m}_\alpha}/H_{\boldsymbol{c}}$ is free.
(d) The sequence $(\boldsymbol{m}_\alpha \mid \alpha < \gamma)$ is increasing and continuous at limit ordinals, i.e., $\boldsymbol{m}_\alpha = \bigcup_{\beta<\alpha} \boldsymbol{m}_\beta$, for all limit ordinals $\alpha < \gamma$.
(e) If $\beta < \alpha$ and $\beta \notin S$, then $\boldsymbol{m}_\alpha$ is free over $\boldsymbol{m}_\beta$.
(f) $U_0(\bar{\boldsymbol{m}}) \subseteq \gamma$ is defined by $\delta \in U_0(\bar{\boldsymbol{m}})$ if and only if:
(f_1) $\delta \in S$, the set of elements of $H_{\boldsymbol{m}_\delta}$ is δ, and $g_\delta \in \text{Aut}(H_{\boldsymbol{m}_\delta})$.
(f_2) $g_\delta \neq (F^b_{\boldsymbol{m}_\delta} \circ g_f) \restriction_{H_{\boldsymbol{m}_\delta}}$, for any $b \in K_{\boldsymbol{c}}$ and $f \in \text{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{c}}))$.

(g) If $\alpha = \beta + 1$, where $\beta \in \gamma \setminus U_0(\bar{\boldsymbol{m}})$, then $H_{\boldsymbol{m}_\alpha}$ is defined such that

$$\mathcal{Z}(H_{\boldsymbol{m}_\alpha}) = \mathcal{Z}(H_{\boldsymbol{m}_\beta}) \oplus \bigoplus_{\boldsymbol{g} \in \mathcal{G}} f_{\alpha,\boldsymbol{g}}(\mathbb{G}_{\boldsymbol{g}}),$$

where $f_{\alpha,\boldsymbol{g}}$ embeds $(\mathbb{G}_{\boldsymbol{g}}, (F^b_{\boldsymbol{g}})_{b\in K_{\boldsymbol{c}}})$ into $(H_{\boldsymbol{m}_\alpha}, (F^b_{\boldsymbol{m}_\alpha})_{b\in K_{\boldsymbol{c}}})$, so that Lemma 1.9(2) holds.

(h) If $\alpha = \beta + 1$, and $\beta \in U_0(\bar{\boldsymbol{m}})$, then for any $n < \omega$ and $\boldsymbol{g} \in \mathcal{G}$ we have the embeddings $f^{\boldsymbol{g}}_{\beta,n} : \mathbb{G}_{\boldsymbol{g}} \to H_{\boldsymbol{m}_\alpha}$ such that:

(h$_1$) $H_{\boldsymbol{m}_\alpha}/H_{\boldsymbol{m}_\beta} = \bigoplus\{f^{\boldsymbol{g}}_{\beta,n}(\mathbb{G}_{\boldsymbol{g}})/H_{\boldsymbol{m}_\beta} \mid \boldsymbol{g} \in \mathcal{G}\}$.

(h$_2$) $n!f^{\boldsymbol{g}}_{\beta,n+1}(y) - f^{\boldsymbol{g}}_{\beta,n}(y) \in H_{\boldsymbol{m}_\beta} \subseteq H_{\boldsymbol{m}_\alpha}$ for any $y \in \mathbb{G}_{\boldsymbol{g}}$.

(h$_3$) the following diagram commutes:

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{G}_{\boldsymbol{g}} & \xrightarrow{f^{\boldsymbol{g}}_{\beta,n}} & H_{\boldsymbol{m}_\alpha} \\
 & & \uparrow{\scriptstyle F^k_{\boldsymbol{g}}} & & \uparrow{\scriptstyle F^k_{\boldsymbol{m}_\alpha}} \\
0 & \longrightarrow & \mathbb{G}_{\boldsymbol{g}} & \xrightarrow{f^{\boldsymbol{g}}_{\beta,n}} & H_{\boldsymbol{m}_\alpha}
\end{array}$$

(i) Suppose $F^b_{\boldsymbol{c}} \circ g_f = F^d_{\boldsymbol{c}}$ for some $b, d \in K_{\boldsymbol{c}}$ and $f \in \operatorname{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{c}}))$. Then $F^b_{\boldsymbol{m}_\alpha} \circ g_f = F^d_{\boldsymbol{m}_\alpha}$, where $g_f(x) = x \cdot f(h_{\boldsymbol{m}}(x))$.

$(*)_2$ Suppose $\bar{\boldsymbol{m}} \in \Lambda_\gamma$. Let $U(\bar{\boldsymbol{m}}) \subseteq \gamma$ be the set of all $\delta \in S \cap \gamma$ such that there are $\bar{\boldsymbol{n}} := (\boldsymbol{n}_\alpha \mid \alpha < \lambda)$, $\chi, h, \mathcal{B}$ such that:

(a) $\bar{\boldsymbol{n}} \in \Lambda_\lambda$, set also $\boldsymbol{n}_\lambda = \bigcup_{\alpha<\lambda} \boldsymbol{n}_\alpha$, so that $\boldsymbol{n}_\lambda \in M_{\boldsymbol{c}}$ and its universe is λ.

(b) $\bar{\boldsymbol{n}} \restriction \delta = \bar{\boldsymbol{m}} \restriction \delta$.

(c) $h \in \operatorname{Aut}(H_{\boldsymbol{n}_\lambda})$.

(d) $h \restriction \delta \neq (F^b_{\boldsymbol{n}_\delta} \circ g_f) \restriction H_{\boldsymbol{n}_\delta}$, for any $b \in K_{\boldsymbol{c}}$ and $f \in \operatorname{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{c}}))$.

(e) $\chi > 2^\lambda$ is regular so that $\bar{\boldsymbol{n}}, h, (\boldsymbol{c}, \mathcal{G}) \in \mathscr{H}(\chi)$.

(f) $\mathcal{B} \prec (\mathscr{H}(\chi), \in)$, $\|\mathcal{B}\| < \lambda$, and $\mathcal{B} \cap \lambda = \delta$.

(g) $\bar{\boldsymbol{n}}, h, (\boldsymbol{c}, \mathcal{G}) \in \mathcal{B}$ and $h \restriction \delta = g_\delta$.

We shall prove the theorem in a sequence of claims; see Claims 3.7–3.13.

Claim 3.7. *Suppose $\gamma \leq \lambda$ is an ordinal. Then the following assertions hold*:

(a) *If γ is a limit ordinal, $\bar{\boldsymbol{m}}_\alpha \in \Lambda_\alpha$ for $\alpha < \gamma$, and $(\bar{\boldsymbol{m}}_\alpha \mid \alpha < \gamma)$ is $\lhd$-increasing, then $\bar{\boldsymbol{m}}_\gamma = \bigcup_{\alpha<\gamma} \bar{\boldsymbol{m}}_\alpha \in \Lambda_\gamma$, and it end extends all $\bar{\boldsymbol{m}}_\alpha$'s, $\alpha < \gamma$.*

(b) *If $\bar{\boldsymbol{m}} \in \Lambda_\gamma$ and $\gamma' \in [\gamma, \lambda)$, then there is $\bar{\boldsymbol{n}} \in \Lambda_{\gamma'}$ such that $\bar{\boldsymbol{m}} \trianglelefteq \bar{\boldsymbol{n}}$.*

Proof. (a) It is enough to show that $\boldsymbol{m}_\gamma$ is free over $\boldsymbol{m}_{\boldsymbol{c}}$. To this end, let C be a club of γ which is disjoint to S, which exists as S is nonreflecting. By clause $(*)_1$(c), $\boldsymbol{m}_\alpha$ is free over $\boldsymbol{c}$, for all α in C, and by clause $(*)_1$(e), $\boldsymbol{m}_\alpha$ is free over $\boldsymbol{m}_\beta$, for all $\beta < \alpha$ in C. So clearly $\boldsymbol{m}_\gamma$ is free over $\boldsymbol{m}_{\boldsymbol{c}}$.

(b) We prove something stronger in the following. For $\gamma \leq \lambda$ we define Ω_γ as the class of all $\bar{\boldsymbol{m}} \in \Lambda_\gamma$ such that:

$(*)_3$ If $\alpha = \delta + 1 < \gamma$ and $\delta \in U(\bar{\boldsymbol{m}})$, then g_δ is unextendable, which means if $\bar{\boldsymbol{m}} \restriction \alpha + 1 \leq_{\boldsymbol{c}} \bar{\boldsymbol{n}} \in \Lambda_\beta$ with $\beta \in [\gamma, \lambda]$, then g_δ cannot be extended to an automorphism of $H_{\bar{\boldsymbol{n}}}$.

$(*)_4$ It is enough to prove that $\Omega_\lambda \neq \varnothing$.

In order to argue these, suppose Ω_λ is nonempty and drive the theorem. To this end, we take $\bar{\boldsymbol{m}} \in \Omega_\lambda$ and let $\boldsymbol{m} = \bigcup_{\alpha<\lambda} \boldsymbol{m}_\alpha$. We show that $\boldsymbol{m}$ is as required. It is clear that $\boldsymbol{m} \in M_{\boldsymbol{c}}$ and $\|\boldsymbol{m}\| = \lambda$. Furthermore, any subgroup of $H_{\boldsymbol{m}}/H_{\boldsymbol{c}}$ of size less than λ is included in some $H_{\boldsymbol{m}_\alpha}/H_{\boldsymbol{c}}$, for some $\alpha < \lambda$ and $H_{\boldsymbol{m}_\alpha}/H_{\boldsymbol{c}}$ is free, hence $\boldsymbol{m} \in M_{\boldsymbol{c}}$ is λ-free over $\boldsymbol{c}$. Since $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}})$ is trivial for all $\boldsymbol{g} \in \mathcal{G}$, by Lemma 3.3, clause (β) of the theorem holds. Here, we show the clause (γ) of the theorem holds as well. If $b \in K_{\boldsymbol{c}}$ and $f \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$, then $F^b_{\boldsymbol{m}}$ is an automorphism of $H_{\boldsymbol{m}}$, and by Proposition 2.2, $g_f \in \mathrm{Aut}(H_{\boldsymbol{m}})$. By combining these, we deduce that $g = F^b_{\boldsymbol{m}} \circ g_f$ is an automorphism. Now, let $g \in \mathrm{Aut}(H_{\boldsymbol{m}})$. We claim that g is of the form $g = F^b_{\boldsymbol{m}} \circ g_f$, for some $b \in K_{\boldsymbol{c}}$ and $f \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$. Suppose not, so in particular, for all $b \in K_{\boldsymbol{c}}$ and for all $f \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))$, we have: $g \neq F^b_{\boldsymbol{m}} \circ g_f$. As $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}})) \subseteq \mathrm{Hom}(N_{\boldsymbol{c}}, H_{\boldsymbol{c}})$, we have $|K_{\boldsymbol{c}}| + |\mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))| \le |K_{\boldsymbol{c}}| + |\mathrm{Hom}(N_{\boldsymbol{c}}, H_{\boldsymbol{c}})| \le 2^{\|\boldsymbol{c}\|} < \lambda$, so we can find some $\alpha_* < \lambda$ such that for all $\alpha > \alpha_*$, and all b, f as above, $g \restriction \boldsymbol{m}_\alpha \neq (F^b_{\boldsymbol{m}_\alpha} \circ g_f) \restriction \boldsymbol{m}_\alpha$. Choose χ large enough regular and let $\bar{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha < \lambda)$ be an increasing and continuous sequence of elementary submodels of $(\mathscr{H}(\chi), \in)$ such that for each $\alpha < \lambda$, $\mathcal{B}_\alpha$ has cardinality $<\lambda$, $\bar{\boldsymbol{m}}, \boldsymbol{c}, g \in \mathcal{B}_\alpha$ and $(\mathcal{B}_\gamma \mid \gamma \le \alpha) \in \mathcal{B}_{\alpha+1}$. Set $C := \{\delta > \alpha_* \mid \boldsymbol{m}_\delta$ has universe δ and $\mathcal{B}_\delta \cap \lambda = \delta\}$. Then C is a club of λ, hence by the choice of the sequence $(g_\delta \mid \delta \in S)$, the set $S' = \{\delta \in C \cap S \mid g \restriction \delta = g_\delta\}$ is stationary. Let $\delta \in S'$. We conclude that g_δ cannot be extended to an isomorphism of $H_{\boldsymbol{m}}$. But, this is a contradiction, because $g \supseteq g_\delta$ is such an extension. □

In order to prove that $\Omega_\lambda \neq \varnothing$ we define

$$U^+(\bar{\boldsymbol{m}}) := \big\{\delta \in U(\bar{\boldsymbol{m}}) \mid \text{there is } \bar{\boldsymbol{n}}_1 \text{ such that } \bar{\boldsymbol{m}} \restriction \delta \le_{\boldsymbol{c}} \bar{\boldsymbol{n}}_1 \in \Lambda_{\delta+1} \text{ and if } \bar{\boldsymbol{n}}_1 \le_{\boldsymbol{c}} \bar{\boldsymbol{n}} \in \Lambda_\beta \text{ with } \beta \in [\gamma, \lambda], \text{ then } g_\delta \text{ cannot be extended to an automorphism of } H_{\bar{\boldsymbol{n}}}\big\}.$$

We now prove the following, which in particular, implies that $\Omega_\gamma \neq \varnothing$, for all $\gamma \le \lambda$.

$(*)_5$ There exists $\bar{\boldsymbol{m}}$ of length λ such that, for all $\gamma < \lambda$, $\bar{\boldsymbol{m}} \restriction \gamma \in \Omega_\gamma$.

To this end, we define $\bar{\boldsymbol{m}} = (\boldsymbol{m}_\xi \mid \xi < \lambda) \in \Lambda_\lambda$ by defining $\boldsymbol{m}_\xi$ by induction on ξ. The only nontrivial case is when $\xi = \delta + 1 < \lambda$ and $\delta \in U(\bar{\boldsymbol{m}} \restriction \xi)$. Thus suppose that we are given ξ and δ as above and $\bar{\boldsymbol{m}} \restriction \delta$ is defined. We have to define $\boldsymbol{m}_\xi$. For $\bar{\boldsymbol{m}} \in \Lambda_\delta$, $\alpha \le \beta < \delta$ and $\boldsymbol{g} \in \mathcal{G}_{\boldsymbol{c}}$, let Υ be the family of all embedding $f : \boldsymbol{g} \hookrightarrow \mathcal{Z}(H_{\boldsymbol{m}_\beta})$ where $\boldsymbol{m}_\beta$ is λ-free over $\boldsymbol{m}_\alpha$, $f(\mathbb{G}_{\boldsymbol{g}}) \cap H^*_{\boldsymbol{c}} = 0$ and $(f(\mathbb{G}_{\boldsymbol{g}}) + H^*_{\boldsymbol{c}})/H^*_{\boldsymbol{c}} \subseteq_* H_{\boldsymbol{m}_\beta}/H^*_{\boldsymbol{c}}$. Now, we define:

(a) $\mathfrak{F}^{\bar{\boldsymbol{m}},\boldsymbol{g}}_\alpha := \{f \mid f \text{ embeds } \boldsymbol{g} \text{ into } \mathcal{Z}(H_{\boldsymbol{m}_\alpha})\}$.

(b) $\mathfrak{F}^{\bar{\boldsymbol{m}},\boldsymbol{g}}_{\alpha,\beta} := \{f \mid f \in \Upsilon\}$.

Here, by an embedding of $\boldsymbol{g}$ into $\mathcal{Z}(H_{\boldsymbol{m}_\beta})$ we mean an embedding from the structure $\mathbf{g} = (\mathbb{G}_{\boldsymbol{g}}, F^k_{\boldsymbol{g}})_{k \in K_{\boldsymbol{c}}}$ into the structure $(\mathcal{Z}(H_{\boldsymbol{m}_\beta}), F^k_{\boldsymbol{m}_\beta})_{k \in K_{\boldsymbol{c}}}$. To continue the proof, we split the argument into several subcases. Indeed, we may assume that clause $(*)_1(f_1)$ holds and $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$, as otherwise, we do nothing and let $\boldsymbol{m}_\xi = \boldsymbol{m}_\delta$. We are going to furnish any $\boldsymbol{g} \in \mathcal{G}$ with three versions of activities with respect to $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$. First of all, we say $\boldsymbol{g} \in \mathcal{G}$ is 1-*active with respect to* $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, when there are $\bar{\boldsymbol{n}}, \chi, h, \mathcal{B}$ witnessing $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$ such that for some nonzero $x \in \mathbb{G}_{\boldsymbol{g}}$, for arbitrarily large $\alpha < \delta$, there are $\beta \in (\alpha, \delta) \setminus S$ and $f \in \mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha,\beta}$ such that:

(1) $h(f(x)) \notin \mathcal{Z}(H_{\boldsymbol{m}_\alpha}) + \mathrm{Im}(f)$.

(2) $\mathcal{Z}(H_{\boldsymbol{m}_\alpha}) \cap \mathrm{Im}(f) = 0$.

(3) h maps $H_{\boldsymbol{m}_\alpha}$ (resp. $H_{\boldsymbol{m}_\beta}$) into $H_{\boldsymbol{m}_\alpha}$ (resp. $H_{\boldsymbol{m}_\beta}$).

Claim 3.8. *If* $\boldsymbol{g}$ *is* 1*-active with respect to* $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, *and* $\xi = \delta + 1$, *then for some* $\boldsymbol{m}_\xi$, $\bar{\boldsymbol{m}} \restriction \xi + 1 \in \Lambda_{\xi+1}$ *and* $\delta \in U^+(\bar{\boldsymbol{m}} \restriction \xi + 1)$.

Proof. As $\mathrm{cf}(\delta) = \aleph_0$, there is an increasing sequence $(\alpha^0_n \mid n < \omega) \in {}^\omega\delta$ with limit δ. Let $\bar{\boldsymbol{n}}, \chi, h, \mathcal{B}$ witness $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$ and let $x \in \mathbb{G}_{\boldsymbol{g}}$ be as guaranteed by definition of 1-activity. Choose (α_n, β_n, f_n) by induction on n so that:

$(\dagger)^1_n$ (a) $\alpha_n \in (\alpha^0_n, \delta) \setminus S$, and $\beta_m < \alpha_n < \beta_n$ for $m < n$.
(b) If $n = m + 1$ then $\mathrm{Im}(f_m) \subseteq \mathcal{Z}(H_{\boldsymbol{m}_{\alpha_n}})$.
(c) $(\boldsymbol{g}, x, \alpha_n, \beta_n, f_n) := (\boldsymbol{g}, x, \alpha, \beta, f)$, where $(\boldsymbol{g}, x, \alpha, \beta, f)$ is taken from the definition of 1-activity.

We are going to define $\bar{\boldsymbol{n}}_1 \in \Lambda_{\delta+2}$ and f^*_n such that:

$(\dagger)^2$ (a) $\bar{\boldsymbol{n}}_1 \restriction (\delta + 1) = \bar{\boldsymbol{m}} \restriction (\delta + 1)$.
(b) f^*_n embeds $\boldsymbol{g} = (\mathbb{G}_{\boldsymbol{g}}, (F^k_{\boldsymbol{g}})_{k \in K_c})$ into $(H_{(\boldsymbol{n}_1)_{\delta+1}}, (F^k_{(\boldsymbol{n}_1)_{\delta+1}})_{k \in K_c})$.
(c) If $y \in \mathbb{G}_{\boldsymbol{g}}$ then $f^*_{n+1}(y) - f^*_n(y) = -n! f_n(y)$.

Let $\widehat{\mathcal{Z}(H_{(\boldsymbol{m})_\delta})}$ denote the $\mathbb{Z}$-adic completion of $\mathcal{Z}(H_{(\boldsymbol{m})_\delta})$. Note that in $\widehat{\mathcal{Z}(H_{(\boldsymbol{m})_\delta})}$, f^*_n is defined as $f^*_n(y) = \sum_{m=n}^\infty m! f_m(y)$. Set $\xi = \sum_{n<\omega} n!$, and define $y_n, \tilde{y}_n \in \widehat{\mathcal{Z}(H_{(\boldsymbol{m})_\delta})}$ by $y_n = f^*_n(x)$ and $\tilde{y}_n = y_n + \xi f_0(x)$. Recall that $h \restriction \delta = g_\delta$ extends to an automorphism $\hat{g}_\delta$ over $\widehat{\mathcal{Z}(H_{(\boldsymbol{m})_\delta})}$. We show that either $\hat{g}_\delta(y_0) \notin \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{y_m \mid m < \omega\}\rangle$ or $\hat{g}_\delta(\tilde{y}_0) \notin \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{\tilde{y}_m \mid m < \omega\}\rangle$, and then we choose $z_0 \in \{y_0, \tilde{y}_0\}$ such that $\hat{g}_\delta(z_0) \notin \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{\tilde{z}_m \mid m < \omega\}\rangle$ and set

$$\mathcal{Z}(H_{(\boldsymbol{n}_1)_{\delta+1}}) := \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{z_m \mid m < \omega\}\rangle \subseteq \widehat{\mathcal{Z}(H_{(\boldsymbol{m})_\delta})}.$$

This easily gives us $\bar{\boldsymbol{n}}_1 \in \Lambda_{\delta+2}$, and it will be as required. Let us depict things:

$$\begin{array}{ccccc}
\mathcal{Z}(H_{\boldsymbol{m}_\delta}) & \xrightarrow{\subseteq} & \mathcal{Z}(H_{(\boldsymbol{n}_1)_{\delta+1}}) & \xrightarrow{\subseteq} & \widehat{\mathcal{Z}(H_{(\boldsymbol{m})_\delta})} \\
{\scriptstyle g_\delta}\uparrow & & {\scriptstyle \nexists \hat{g}_\delta \restriction}\uparrow & & {\scriptstyle \hat{g}_\delta}\uparrow \\
\mathcal{Z}(H_{\boldsymbol{m}_\delta}) & \xrightarrow{\subseteq} & \mathcal{Z}(H_{(\boldsymbol{n}_1)_{\delta+1}}) & \xrightarrow{\subseteq} & \widehat{\mathcal{Z}(H_{(\boldsymbol{m})_\delta})}
\end{array}$$

So, suppose by way of contradiction that $\hat{g}_\delta(y_0) \in \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{y_m \mid m < \omega\}\rangle$ and $\hat{g}_\delta(\tilde{y}_0) \in \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{\tilde{y}_m \mid m < \omega\}\rangle$, and we search for a contradiction. As $\hat{g}_\delta(y_0) \in \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{y_m \mid m < \omega\}\rangle$. There are $z \in \mathcal{Z}(H_{\boldsymbol{n}_\delta})$, $n < \omega$ and $\{\ell_i \in \mathbb{Z}\}_{i<n}$ such that $\hat{g}_\delta(y_0) \in \mathcal{Z}(H_{\boldsymbol{n}_\delta}) + \sum_{i<n} \ell_i y_i$. It is easily seen that $\sum_{i<n} \ell_i y_i \in \mathcal{Z}(H_{\boldsymbol{n}_\delta}) + (\sum_{i<n} \ell_i) y_n$. Set $\ell := \sum_{i<n} \ell_i$. Hence $\hat{g}_\delta(y_0) = \ell y_n + z$ where $z \in \mathcal{Z}(H_{\boldsymbol{n}_\delta})$. By definition,

$$\hat{g}_\delta(\tilde{y}_0) = \hat{g}_\delta(y_0 + \xi f_0(x)) = \ell y_n + z + \xi g_\delta(f_0(x)).$$

In other words, $\hat{g}_\delta(\tilde{y}_0) - (\ell \tilde{y}_n + z) = \xi(g_\delta f_0(x) - \ell f_0(x))$. We now recall that

$$\langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{\tilde{y}_m \mid m < \omega\}\rangle \cap \xi \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{\tilde{y}_m \mid m < \omega\}\rangle = 0.$$

Finally, due to our assumption, $\hat{g}_\delta(\tilde{y}_0) \in \langle \mathcal{Z}(H_{\boldsymbol{m}_\delta}) \cup \{\tilde{y}_m \mid m < \omega\}\rangle$, thus we have $g_\delta f_0(x) - \ell f_0(x) = 0$, but this is absurd by the choice of f_0. □

For $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$, we say $\boldsymbol{g} \in \mathcal{G}$ is 2*-active with respect to* $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, when there are $\bar{\boldsymbol{n}}, \chi, h, \mathcal{B}$ witnessing $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$ such that there are nonzero $x \in \mathbb{G}_{\boldsymbol{g}}$, $\alpha < \beta$ in $\delta \setminus S$, $f_1, f_2 \in \mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha,\beta}$, and $y_1 \neq y_2$ in $\mathbb{G}_{\boldsymbol{g}}$ such that:

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(a) $h(f_\ell(x)) \in \mathcal{Z}(H_{\boldsymbol{m}_\alpha}) + f_\ell(y_\ell)$, for $\ell = 1, 2$.

(b) h maps $H_{\boldsymbol{m}_\alpha}$ (resp. $H_{\boldsymbol{m}_\beta}$) into $H_{\boldsymbol{m}_\alpha}$ (resp. $H_{\boldsymbol{m}_\beta}$).

Claim 3.9. *If $\boldsymbol{g}$ is 2-active with respect to $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, and $\xi = \delta + 1$, then for some $\boldsymbol{m}_\xi$, $\bar{\boldsymbol{m}} \restriction (\xi + 1) \in \Lambda_{\xi+1}$ and $\delta \in U^+(\bar{\boldsymbol{m}} \restriction (\xi + 1))$.*

Proof. Let $\alpha, \beta, f_1, f_2, y_1, y_2$ be as above. For each α' with $\beta < \alpha' < \delta$, we can find $\beta' \in (\alpha', \delta) \setminus S$ and $f_3 \in \mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha', \beta'}$ such that $\mathcal{Z}(H_{\boldsymbol{m}_{\alpha'}}) \cap \mathrm{Im}(f_3) = 0$ and $h(f_3(x)) = f_3(y_3)$ for some $y_3 \in \mathcal{G}_{\boldsymbol{g}}$. Then for some $\ell \in \{1, 2\}$, $y_3 \neq y_\ell$. Let us suppose without loss of generality that $y_3 \neq y_1$. Define $f : \mathcal{G}_{\boldsymbol{g}} \to \mathcal{Z}(H_{\boldsymbol{m}_{\beta'}})$ as $f(z) = f_1(z) - f_3(z)$. Then $f \in \mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha', \beta'}$ is such that $h(f(x)) \notin \mathcal{Z}(H_{\boldsymbol{m}_{\alpha'}}) + \mathrm{Im}(f)$, $\mathcal{Z}(H_{\boldsymbol{m}_{\alpha'}}) \cap \mathrm{Im}(f) = 0$, and h maps $H_{\boldsymbol{m}_{\alpha'}}$ and $H_{\boldsymbol{m}_{\beta'}}$ into themselves. It follows that $\boldsymbol{g}$ is 1-active with respect to $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, and $\xi = \delta + 1$, and we are done by Claim 3.8. □

For $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$, we say $\boldsymbol{g} \in \mathcal{G}$ is 3-*active with respect to* $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, when there are $\bar{\boldsymbol{n}}, \chi, h, \mathcal{B}$ witnessing $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$ and there exist $(y, x, z) \in \mathbb{G}_{\boldsymbol{g}} \times \mathbb{G}_{\boldsymbol{g}} \times \mathcal{Z}(H_{\boldsymbol{m}_\delta})$ with $x, z \neq 0$, $\alpha < \delta$, $\beta \in [\alpha, \delta) \setminus S$ and $f \in \mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha, \beta}$ such that $h(f(x)) = z + f(y)$. Also, h maps $H_{\boldsymbol{m}_\alpha}$ (resp. $H_{\boldsymbol{m}_\beta}$) into $H_{\boldsymbol{m}_\alpha}$ (resp. $H_{\boldsymbol{m}_\beta}$).

Claim 3.10. *If $\boldsymbol{g}$ is 3-active with respect to $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, and $\xi = \delta + 1$, then for some $\boldsymbol{m}_\xi$, $\bar{\boldsymbol{m}} \restriction (\xi + 1) \in \Lambda_{\xi+1}$ and $\delta \in U^+(\bar{\boldsymbol{m}} \restriction (\xi + 1))$.*

Proof. According to Claim 3.9 it is enough to show that $\boldsymbol{g}$ is 2-active with respect to $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$. Let $\bar{\boldsymbol{n}}, \chi, h, \mathcal{B}$ witness $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$, and fix $y, x, z, \alpha < \beta$ and f as in definition of 3-activity. Thanks to the 3-activity assumption, we have $h(f(x)) = z + f(y)$. First we claim that $y \neq 0$. Otherwise, $h(f(x)) = z \in \mathcal{Z}(H_{\boldsymbol{m}_\alpha})$. Following the choice of h, we have $f(x) \in \mathcal{Z}(H_{\boldsymbol{m}_\alpha})$. But $\mathrm{Im}(f) \cap \mathcal{Z}(H_{\boldsymbol{m}_\alpha}) = 0$, hence $f(x) = 0$ which implies $x = 0$, and this contradicts the assumption $x \neq 0$. Consequently, $y \neq 0$. Now, we set $f_1 := f$, $f_2 := 2f$, $y_1 := y$ and $y_2 := 2y$. Since $y \neq 0$, $y_1 \neq y_2$. This witness $\boldsymbol{g}$ is 2-active with respect to $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, as claimed. □

We say $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$ is *nice* if there is no $\boldsymbol{g} \in \mathcal{G}$ which is ℓ-active with respect to $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, for some $\ell \in \{1, 2, 3\}$.

Claim 3.11. *Let $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$ be nice. Then there is a sequence $\bar{h} = (h_{\boldsymbol{g}} \mid \boldsymbol{g} \in \mathcal{G})$ such that for $\alpha < \delta$ large enough, and for any $f \in \bigcup\{\mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha, \beta} \mid \beta \in [\alpha, \delta)\}$, the following implication is valid:*

$$x \in \mathbb{G}_{\boldsymbol{g}} \Rightarrow g_\delta(f(x)) = f(h_{\boldsymbol{g}}(x)).$$

Proof. Let $\boldsymbol{g} \in \mathcal{G}$ and $x \in \mathbb{G}_{\boldsymbol{g}}$. Since $\boldsymbol{g}$ is not 1-active with respect to $(\delta, \bar{\boldsymbol{m}} \restriction \delta)$, and following its definition, we can find some $\alpha < \delta$ such that $g_\delta(f(x)) \in \mathcal{Z}(H_{\boldsymbol{m}_\alpha}) + \mathrm{Im}(f)$ for all $f \in \bigcup\{\mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha, \beta} \mid \beta \in [\alpha, \delta)\}$. Hence there is some $z \in \mathcal{Z}(H_{\boldsymbol{m}_\alpha})$ and $y \in \mathbb{G}_{\boldsymbol{g}}$ so that $g_\delta(f(x)) = z + f(y)$. Recall that $\boldsymbol{g}$ is not 3-active. This forces $z = 0$. Applying this in the previous formula, gives us $g_\delta(f(x)) = f(y)$. Since both of g_δ and f are injective, y is uniquely determined via h and $\boldsymbol{g}$, so let $y = h_{\boldsymbol{g}}(x)$. Then $h_{\boldsymbol{g}}$ is as required. □

Let $\alpha_* < \delta$ be such that Claim 3.11 holds for all $\alpha \geq \alpha_*$.

Claim 3.12. *If $\delta \in U(\bar{\boldsymbol{m}} \restriction \delta)$ is nice, then:*

(a) *For all $\alpha < \delta$ and $f \in \bigcup\{\mathfrak{F}^{\bar{\boldsymbol{m}} \restriction \delta, \boldsymbol{g}}_{\alpha, \beta} \mid \beta \in [\alpha, \delta)\}$, we have $g_\delta(f(x)) = f(h_{\boldsymbol{g}}(x))$ for $x \in \mathbb{G}_{\boldsymbol{g}}$.*

(b) *Let $\boldsymbol{g} \in \mathcal{G}$ and f an embedding from $(\mathbb{G}_{\boldsymbol{g}}, F^k_{\boldsymbol{g}})_{k \in K_{\boldsymbol{c}}}$ into $(\mathcal{Z}(H_{\boldsymbol{m}_\delta}), F^k_{\boldsymbol{m}_\delta})_{k \in K_{\boldsymbol{c}}}$ and let $x \in \mathbb{G}_{\boldsymbol{g}}$. Then there is a sequence $\langle k_i, s_i \mid i < n \rangle$ with $k_i \in K_{\boldsymbol{c}}$ and $s_i \in \mathbb{Z} \setminus \{0\}$ such that $g_\delta(f(x)) - \sum_{i=1}^n s_i F^{k_i}_{\boldsymbol{m}_\delta}(f(x)) \in \mathcal{Z}(H_{\boldsymbol{m}_{\alpha_*}})$.*

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Proof. (a) Suppose not. Let $\alpha < \delta$ and $f_1 \in \mathfrak{F}^{\bar{m}\restriction\delta,g}_{\alpha,\beta}$ be a counterexample, where $\beta > \alpha, \alpha_*$. Thus for some $x \in \mathbb{G}_g$, we have $g_\delta(f_1(x)) \neq f_1(h_g(x))$. We may further assume that $\alpha_* > \alpha$ (as Claim 3.11 works for any ordinal in (α_*, δ) as well). Let $f_2 \in \mathfrak{F}^{\bar{m}\restriction\delta,g}_{\alpha_*,\beta}$ be such that $\mathcal{Z}(H_{m_{\alpha_*}}) \cap \mathrm{Im}(f_2) = 0$. It turns out that $f := f_1 + f_2 \in \mathfrak{F}^{\bar{m}\restriction\delta,g}_{\alpha_*,\beta}$, thus by Claim 3.11, $g_\delta(f(x)) = f(h_g(x))$. This means that

$$g_\delta(f_1(x)) + g_\delta(f_2(x)) = f_1(h_g(x)) + f_2(h_g(x)).$$

Hence as $g_\delta(f_2(x)) = f_2(h_g(x))$, we have $g_\delta(f_1(x)) = f_1(h_g(x))$, a contradiction.

(b) This is true as $g_\delta(f(x)) \in \langle \mathcal{Z}(H_{m_{\alpha_*}}) \cup \{F^k_{m_\delta}(f(x)) \mid k \in K_c\}\rangle$. □

Claim 3.13. *Let $\delta \in U(\bar{m} \restriction \delta)$ be nice. Then there is a sequence $(b^*_i, s^*_i \mid i < n)$ where $b^*_i \in K_c$ and $s^*_i \in \mathbb{Z} \setminus \{0\}$ such that for all $g \in \mathcal{G}$, and any embedding $f : (\mathbb{G}_g, F^k_g)_{k\in K_c} \to (\mathcal{Z}(H_{m_\delta}), F^k_{m_\delta})_{k\in K_c}$ we have*

$$x \in \mathbb{G}_g \Rightarrow g_\delta(f(x)) - \sum_{i<n} s^*_i F^{b^*_i}_{m_\delta}(f(x)) \in \mathcal{Z}(H_{m_{\alpha_*}}).$$

Proof. Fix $g_* \in \mathcal{G}_g$ and $x_* \in \mathcal{G}_g$. Due to Claim 3.12(b) applied to $g_*\restriction_{\mathrm{cl}\{x_*\}}$, we can find a sequence $(b^*_i, s^*_i \mid i < n)$ as there. Let $g \in \mathcal{G}$ and suppose f is an embedding from $(\mathbb{G}_g, F^k_g)_{k\in K_c}$ into $(\mathcal{Z}(H_{m_\delta}), F^k_{m_\delta})_{k\in K_c}$ and $x \in \mathbb{G}_g$. By replacing g by $g\restriction_{\mathrm{cl}\{x\}}$, we may assume that $\mathbb{G}_g = \mathrm{cl}\{x\}$ is one-generated (see Definition 1.8(4)(b)). Let $\phi : \mathrm{cl}\{x_*\} \to \mathrm{cl}\{x\}$ be such that $\phi(x_*) = x$, and set $\tilde{f} := f \circ \phi$. Note that ϕ is an embedding from the structure $(\mathbb{G}_{g_*\restriction_{\mathrm{cl}\{x_*\}}}, F^k_{g_*\restriction_{\mathrm{cl}\{x_*\}}})_{k\in K_c}$ into $(\mathbb{G}_{g\restriction_{\mathrm{cl}\{x\}}}, F^k_{g\restriction_{\mathrm{cl}\{x\}}})_{k\in K_c}$ and $\tilde{f}$ is an embedding from the structure $(\mathbb{G}_{g_*\restriction_{\mathrm{cl}\{x_*\}}}, F^k_{g_*\restriction_{\mathrm{cl}\{x_*\}}})_{k\in K_c}$ into $(\mathcal{Z}(H_{m_\delta}), F^k_{m_\delta})_{k\in K_c}$ and $x \in \mathbb{G}_g$. Following our assumption, it implies that $g_\delta(\tilde{f}(z_e)) - \sum_{i=1}^n s^*_i F^{b^*_i}_{m_\delta}(\tilde{f}(z_e)) \in \mathcal{Z}(H_{m_{\alpha_*}})$. Recall that f, $\tilde{f}$ and ϕ are embeddings that respect the structures. So,

$$F^{b^*_i}_{m_\delta}(\tilde{f}(z_e)) = \tilde{f}(F^{b^*_i}_{g_*\restriction_{\mathrm{cl}\{x_*\}}}(z_e)) = f(\phi(F^{b^*_i}_{g_*\restriction_{\mathrm{cl}\{x_*\}}}(z_e))) = f(F^{b^*_i}_{g\restriction_{\mathrm{cl}\{x\}}}(x)) = F^{b^*_i}_{g\restriction_{\mathrm{cl}\{x\}}}(f(x)).$$

But note that $F^{b^*_i}_{g\restriction_{\mathrm{cl}\{x\}}}(f(x)) = F^{b^*_i}_{g}(f(x))$, hence

$$g_\delta(f(x)) - \sum_{i=1}^n s^*_i F^{b^*_i}_{m_\delta}(f(x)) = g_\delta(\tilde{f}(z_e)) - \sum_{i=1}^n s^*_i F^{b^*_i}_{m_\delta}(\tilde{f}(z_e)) \in \mathcal{Z}(H_{m_{\alpha_*}}),$$

as claimed. □

Now, we proceed the proof of Theorem 3.6. Let us derive the desired presentation in the above nice case. Recall that each element of $\mathcal{Z}(H_{m_\delta})$ is of the form $f(x)$ for some $f \in \mathfrak{F}^{\bar{m}\restriction\delta,g}_{\alpha,\beta}$ and $x \in \mathbb{G}_g$, and also recall that $h_c : H_c \to N_c$ is an epimorphism with $\mathrm{Ker}(h_c) = \mathcal{Z}(H_c)$. These yield an automorphism $\pi \in \mathrm{Aut}(N_c)$ via the assignment $h_c(f(x)) \mapsto h_c(g_\delta(f(x)))$. In view of Claim 3.13 we set $F^{\bar{s}^*,\bar{k}^*}_c = \sum_{i=1}^n s^*_i F^{k^*_i}_c$. Thanks to clause (1) from our assumption, for some $k \in K_c$ one has $F^k_{m_\delta}\restriction_{H_c} = F^{\bar{s}^*,\bar{k}^*}_c$. By the way we extended the functions, $F^k_{m_\delta} = F^{\bar{s}^*,\bar{k}^*}_{m_\delta}$, and following its definition, we have $\pi(h_c(f(x))) = h_c(F^k_{m_\delta}(f(x)))$ for all f and x as above. Let $g_1 := (F^k_{m_\delta})^{-1} \circ g_\delta$, and recall that $g_1\restriction_{\mathcal{Z}(H_{m_\delta})} = \mathrm{id}$ and $g_1(t) \in t \cdot \mathcal{Z}(H_{m_\delta})$ for all $t \in H_{m_\delta}$. According to Proposition 2.2, there is some $f \in \mathrm{Hom}(N_c, \mathcal{Z}(H_{m_\delta}))$ so that $g_1 = g_f$. Due to clause (1) of the theorem and in the light of Lemma 3.3 we observe that $f \in \mathrm{Hom}(N_c, \mathcal{Z}(H_c))$. It follows that $h \restriction \delta = g_\delta = F^b_{m_\delta} \circ g_1 = F^b_{m_\delta} \circ g_f$. This is a contradiction to $(*)_2(d + g)$. In sum, we have proved that $\Omega_\gamma \neq \varnothing$, completing the proof of in the nice case. Recall from clause $(*)_4$ that this completes the proof of the theorem. □

Lemma 3.14. *Assume $\boldsymbol{c} \in \boldsymbol{C}^4_{\mathrm{aut}}$ is such that $|H_{\boldsymbol{c}}| > 2^{|K_{\boldsymbol{c}}|+\aleph_0}$. Then we can extend $\boldsymbol{c}$ to some $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\mathrm{aut}}$.*

Proof. It suffices to define $\mathcal{G}$. First, we say $\boldsymbol{g}$ is *one-generated*, provided there is some $x \in \mathcal{Z}(H_{\boldsymbol{c}})$ so that $\mathbb{G}_{\boldsymbol{g}}/(H^*_{\boldsymbol{c}} \cap \mathcal{Z}(H_{\boldsymbol{c}}))$ is the smallest pure subgroup of $\mathcal{Z}(H_{\boldsymbol{c}})/(H^*_{\boldsymbol{c}} \cap \mathcal{Z}(H_{\boldsymbol{c}}))$ to which x belongs and that is closed under $F^k_{\boldsymbol{g}}$'s for $k \in K_{\boldsymbol{c}}$. Let $\mathcal{G}$ consist of all $\boldsymbol{g}$ such that:

(1) $\boldsymbol{g} = (\mathbb{G}_{\boldsymbol{g}}, (F^k_{\boldsymbol{g}})_{k \in K_{\boldsymbol{c}}})$, where $\mathbb{G}_{\boldsymbol{g}} \subseteq \mathcal{Z}(H_{\boldsymbol{c}})$.

(2) $\mathbb{G}_{\boldsymbol{g}}$ is a torsion-free subgroup such that $\mathbb{G}_{\boldsymbol{g}} \cap H^*_{\boldsymbol{c}} = 0$.

(3) $\boldsymbol{g}$ is one-generated.

(4) $|\mathbb{G}_{\boldsymbol{g}}| \le |K_{\boldsymbol{c}}| + \aleph_0$.

$(*)_1$ Denoting the isomorphic structures equivalence relation with $\cong$, we have:
 (a) $|\mathcal{G}/\cong| \le 2^{|K_{\boldsymbol{c}}|+\aleph_0}$.
 (b) If $\mathcal{G} \ne \varnothing$, then $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\mathrm{aut}}$.
 (c) If $\boldsymbol{g} \in \mathcal{G}$, then $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}}) = 0$.

$(*)_2$ Assume $|H_{\boldsymbol{c}}| > 2^{|K_{\boldsymbol{c}}|+\aleph_0}$. Then $\mathcal{G} \ne \varnothing$.

Indeed, first, we note that $|\mathcal{Z}(H_{\boldsymbol{c}})| \ge 2^{|K_{\boldsymbol{c}}|+\aleph_0}$. From this, we can find a sequence $(x_\alpha \mid \alpha < 2^{|K_{\boldsymbol{c}}|+\aleph_0})$ of distinct elements of $\mathcal{Z}(H_{\boldsymbol{c}}) \setminus H^*_{\boldsymbol{c}}$. For each α, let $\mathbb{G}_\alpha$ be minimal such that:

(a) $\{x_\alpha\} \cup (H^*_{\boldsymbol{c}} \cap \mathcal{Z}(H_{\boldsymbol{c}})) \subseteq \mathbb{G}_\alpha$,

(b) $\mathbb{G}_\alpha$ is closed under the action of $F^k_{\boldsymbol{c}}$, for $k \in K_{\boldsymbol{c}}$,

(c) $\mathbb{G}_\alpha/(H^*_{\boldsymbol{c}} \cap \mathcal{Z}(H_{\boldsymbol{c}}))$ is a pure subgroup of $\mathcal{Z}(H_{\boldsymbol{c}})/(H^*_{\boldsymbol{c}} \cap \mathcal{Z}(H_{\boldsymbol{c}}))$,

(d) $|\mathbb{G}_\alpha| \le |K_{\boldsymbol{c}}| + \aleph_0$.

For each α let $(x_{\alpha,\ell} \mid \ell < |\mathbb{G}_\alpha|)$ enumerate $\mathbb{G}_\alpha$ so that the elements of $H^*_{\boldsymbol{c}} \cap \mathcal{Z}(H_{\boldsymbol{c}})$ are enumerated first. Then for some $\alpha < \beta < 2^{|K_{\boldsymbol{c}}|+\aleph_0}$ we have $|\mathbb{G}_\alpha| = |\mathbb{G}_\beta|$ and $\{(x_{\alpha,\ell}, x_{\beta,\ell}) \mid \ell < |\mathbb{G}_\alpha|\}$ is an isomorphism from $\mathbb{G}_\alpha$ onto $\mathbb{G}_\beta$ which is identity on $H^*_{\boldsymbol{c}} \cap \mathcal{Z}(H_{\boldsymbol{c}})$ and commutes under $F^k_{\boldsymbol{c}}$, for all $k \in K_{\boldsymbol{c}}$. In order to define $\boldsymbol{g}$, we set $\mathbb{G}_{\boldsymbol{g}} = \{x_{\alpha,\ell} - x_{\beta,\ell} \mid \ell < |\mathbb{G}_\alpha|\}$ and define $F^k_{\boldsymbol{g}} := F^k_{\boldsymbol{c}} \restriction_{\mathbb{G}_{\boldsymbol{g}}}$. It turns out that $\boldsymbol{g} \in \mathcal{G}$. The lemma follows. □

Corollary 3.15. *Assume Gödel's axiom of constructibility $\boldsymbol{V} = \boldsymbol{L}$, and let K be a group. Then the following are equivalent*:

(a) *For every cardinal $\lambda > 2^{|K|+\aleph_0}$ as in Theorem 3.6, there is a group H such that $|H| = |\mathcal{Z}(H)| = \lambda$ and* $\mathrm{Aut}(H) \cong K$.

(b) *There is some $\boldsymbol{c} \in \boldsymbol{C}^+_{\mathrm{aut}}$ such that $K_{\boldsymbol{c}} \cong K$,* $\mathrm{Aut}(H_{\boldsymbol{c}}) \cong K$ *and* $|\mathcal{Z}(H_{\boldsymbol{c}})| > 2^{|K|+\aleph_0}$.

Proof. $(a) \Rightarrow (b)$ This is clear, as we can define $\boldsymbol{c} \in \boldsymbol{C}^+_{\mathrm{aut}}$ by $\boldsymbol{c} = (K_{\boldsymbol{c}}, H_{\boldsymbol{c}}, F_{\boldsymbol{c}}) = (K, H, F)$, where $F : K \cong \mathrm{Aut}(H)$ is an isomorphism.

$(b) \Rightarrow (a)$ Let $\boldsymbol{c} \in \boldsymbol{C}^+_{\mathrm{aut}}$ be such that $K_{\boldsymbol{c}} \cong K$ and $|H_{\boldsymbol{c}}| > 2^{|K|+\aleph_0}$. Let also $\lambda > 2^{|K|+\aleph_0}$ be as in Theorem 3.6. We combine Lemma 1.9(3) along with Lemma 3.14, and extend $\boldsymbol{c}$ to some $(\boldsymbol{c}, \mathcal{G}) \in \boldsymbol{C}^6_{\mathrm{aut}}$. We are going to show that $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z}) = 0$. Indeed, this follows from the fact that the $\boldsymbol{c}$ we construct is auto-rigid and the center of $H_{\boldsymbol{c}}$ has size bigger than the size of K. Thanks to Lemma 3.1, with $\boldsymbol{m} = \boldsymbol{c}$, we see

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$\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{Z}) = 0$. In fact, following the previous proof, $\mathrm{Hom}(N_{\boldsymbol{c}}, \mathbb{G}_{\boldsymbol{g}}) = 0$, for all $\boldsymbol{g} \in \mathcal{G}$. Theorem 3.6 gives us an $\boldsymbol{m}$ so that $|H_{\boldsymbol{m}}| = \lambda$ and

$$\mathrm{Aut}(H_{\boldsymbol{m}}) = \{F^k_{\boldsymbol{m}} \circ g_f \mid k \in K_{\boldsymbol{c}},\ f \in \mathrm{Hom}(N_{\boldsymbol{c}}, \mathcal{Z}(H_{\boldsymbol{m}}))\}.$$

We note that for any b, f as above, $(F^b_{\boldsymbol{m}} \circ g_f) \restriction_{H_{\boldsymbol{c}}} \in \mathrm{Aut}(H_{\boldsymbol{c}})$, hence for some $d \in K_{\boldsymbol{c}}$, we have $(F^b_{\boldsymbol{m}} \circ g_f) \restriction_{H_{\boldsymbol{c}}} = F^d_{\boldsymbol{c}}$. Then $F^b_{\boldsymbol{m}} \circ g_f = F^d_{\boldsymbol{m}}$. Thus $\mathrm{Aut}(H_{\boldsymbol{m}}) \subseteq \{F^k_{\boldsymbol{m}} \mid k \in K_{\boldsymbol{c}}\} \subseteq \mathrm{Aut}(H_{\boldsymbol{m}})$, so $\mathrm{Aut}(H_{\boldsymbol{m}}) \cong K_{\boldsymbol{c}} \cong K$. □

Similarly, we can prove the following.

Corollary 3.16. *Assume Gödel's axiom of constructibility* $\boldsymbol{V} = \boldsymbol{L}$, *and let* K *be a group. Then the following are equivalent*:

(a) *For every* $\lambda > 2^{|K|+\aleph_0}$ *as in Theorem 3.6, there is a group* H *of cardinality* λ *such that* $\mathrm{Aut}(H) \cong K$, *and for some* $H_* \subseteq \mathcal{Z}(H)$ *of cardinality* $<\lambda$, $\mathcal{Z}(H)/H_*$ *is a* λ*-free abelian group.*

(b) *There is* $\boldsymbol{c} \in \boldsymbol{C}^+_{\mathrm{aut}}$ *such that* $K_{\boldsymbol{c}} \cong K$, $\mathrm{Aut}(H_{\boldsymbol{c}}) \cong K$ *and* $|\mathcal{Z}(H_{\boldsymbol{c}})| > 2^{|K|+\aleph_0}$, *and there is* $\boldsymbol{g} \in \mathcal{G}$ *such that* $\mathbb{G}_{\boldsymbol{g}}$ *is an abelian group.*

Corollary 3.17. *Assume Gödel's axiom of constructibility* $\boldsymbol{V} = \boldsymbol{L}$, *and let* K *be an abelian group such that* $F : K \xrightarrow{\cong} \mathrm{Aut}(H)$ *for some abelian group* H *and some isomorphism* F. *Then for every cardinal* $\lambda > 2^{|K|+\aleph_0}$, *there is an abelian group* H_λ *of size* λ *and some* F_λ *such that*:

(1) $F_\lambda : K \xrightarrow{\cong} \mathrm{Aut}(H_\lambda)$.

(2) *For every* $k \in K$, $F_\lambda(k)$ *is an automorphism of* H_λ *such that* $(F_\lambda(k) \restriction H) = F(k) \in \mathrm{Aut}(H)$.

Proof. Define $\boldsymbol{c} \in \boldsymbol{C}^+_{\mathrm{aut}}$ by $\boldsymbol{c} = (K, H, F)$. Since H is abelian, $N_{\boldsymbol{c}} = 0$. This shows the hypotheses of Theorem 3.6 are satisfied. The result follows immediately. □

Note that the above results give a negative answer to the natural question of whether the automorphism group of a group H can determine the group H.

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