

SUPER BLACK BOXES REVISITED

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ABSTRACT. Let $\lambda > \kappa, \theta$ be cardinals, with λ and κ regular. We say that the triple $(\lambda, \kappa, \theta)$ has a *Super Black Box* when the following holds.

For some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ and $\bar{C} = \langle C_\delta : \delta \in S \rangle$, where C_δ is a club of δ of order type κ , for every coloring $\bar{F} = \langle F_\delta : \delta \in S \rangle$ with $F_\delta : {}^C_\delta \lambda \rightarrow \theta$, there exists $\langle c_\delta : \delta \in S \rangle \in {}^S \theta$ such that for every $f : \lambda \rightarrow \theta$, for stationarily many $\delta \in S$, we have $F_\delta(f \upharpoonright C_\delta) = c_\delta$.

In an earlier work, it was proved (along with much more) that for a class of cardinals λ this holds for many pairs (κ, θ) . E.g. $\kappa < \aleph_\omega$ is large enough, and $\beth_\omega(\theta) < \lambda$. However, the most interesting cases (at least with regards to Abelian groups) are $\kappa = \aleph_0, \aleph_1$.

Here we restrict ourselves to the case where $\bar{F}$ is a so-called *continuous coloring*, which includes the case where F_δ just codes $f \upharpoonright C_\delta$ for some $f \in {}^\lambda \theta$. We mainly prove results without any other caveats: e.g.

- For every regular κ and θ there exists a λ .

§ 0. INTRODUCTION

We continue [She05] and [She08], while [She20] presents another direction we could pursue. Compared to [She05], we restrict the coloring (to the so-called *continuous colorings*) but the restrictions on κ are greatly weakened.

Recalling the BB Trichotomy Theorem from [She13b, 1.22_{Lh.7}], Case (B) there will be expanded upon in §2 here, and §3 will examine cases (C) and (A). Hopefully we shall apply this elsewhere.

For the Trivial Dual Conjecture on abelian groups, see [She20] and [She07].

Convention 0.1. $\mathfrak{p}$ will be as in Definition 1.1(1).

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References like (e.g.) [Sh:950, Th.0.2_{Lh.5}] mean that y5 is the internal label of Theorem 0.2 in the TeXfile of [Sh:950]. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number 1268 in Saharon Shelah's list.

§ 1. THE FRAMEWORK

We will open with definitions and basic results from [She05] and expand on them, concentrating on the main case that was presented in that paper.

Definition 1.1. Assume $\lambda > \kappa$ are regular cardinals, and let $\chi \leq \lambda$. Let

$$S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$$

be a stationary subset of λ .

1) We say $\mathbf{p} = \langle (C_\delta, C'_\delta) : \delta \in S \rangle$ is a (λ, κ, χ) -BB-parameter when

- (A) $C_\delta \subseteq \delta$ with $\sup(C_\delta) = \delta$ and $|C_\delta| < \chi$ (or just $\text{otp}(C_\delta) \leq \chi$).
- (B) $C'_\delta \subseteq \delta$, $\sup(C'_\delta) = \delta$, and $\text{otp}(C'_\delta) = \kappa$. (We do not require that C_δ or C'_δ be closed in δ .)
- (C) For all $\alpha < \lambda$, the set

$$\{C_\delta \cap \alpha \setminus \sup(C'_\delta \cap \alpha) : \delta \in S, C'_\delta \ni \alpha\}$$

has cardinality $< \lambda$.

1A) We say $\mathbf{p}$ is *good* when in addition,

- (C)⁺ For all $\alpha < \lambda$ the set¹ $\{(C_\delta \cap \alpha, C'_\delta \cap \alpha) : \delta \in S, C'_\delta \ni \alpha\}$ has cardinality $< \lambda$.

1B) We say that $\mathbf{p}$ *does D-guess clubs*, where D is a filter on λ , when for every club $E \subseteq \lambda$,

$$\{\delta \in S : C'_\delta \subseteq E\} \in D^+.$$

1C) For $\mathbf{p}$ as above,

- (a) $\langle \beta_{\delta,i} : i < \kappa \rangle$ will list the elements of C'_δ in increasing order.
- (b) $\beta_{\delta, < i} = \beta(\delta, < i) := \bigcup_{j < i} (\beta_{\delta,j} + 1)$.

1D) We will write $\lambda_{\mathbf{p}}$, $\kappa_{\mathbf{p}}$, $\beta_{\delta,i}^{\mathbf{p}}$, etc. whenever there are multiple BB_{*}-parameters under discussion, or the identity of $\mathbf{p}$ is otherwise unclear from context.

1E) If $(\forall \delta \in S)[C_\delta = C'_\delta]$, then we may write $\mathbf{p} = \langle C_\delta : \delta \in S \rangle$. We may omit χ when $\chi := \min\{\theta : \delta \in S \Rightarrow \text{otp}(C_\delta) \leq \theta\}$.

2) We say that $\bar{F} = \langle F_\delta : \delta \in S \rangle$ is a $(\mathbf{p}, \Upsilon, \theta)$ -coloring if $\theta \geq 2$, $\Upsilon \geq 2$, and $F_\delta : {}^{C_\delta}\Upsilon \rightarrow \theta$.

3) Let $\bar{F}$ be as above, and D be a filter on λ . (The default choice will be the club filter.)

We say $\bar{c} \in {}^S\theta$ (or $\in {}^\lambda\theta$) is a $\mathbf{p}$ - D - $\bar{F}$ -BB-sequence if for every $\eta \in {}^\lambda\Upsilon$ the set $\{\delta \in S : F_\delta(\eta \upharpoonright C_\delta) = c_\delta\}$ is a member of D^+ (and in the default case, a stationary subset of λ).

4) We may omit $\mathbf{p}$ if both $\bar{C}$ and $\bar{C}'$ are clear from the context.

5) We say $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is (λ, κ) -good when

¹ $C_\delta \cap \alpha$ will suffice.

- (A) S is a stationary subset of $\{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ and a member of $\check{I}_\kappa[\lambda]$.
- (B) $C_\delta \subseteq \delta = \sup(C_\delta)$
- (C) $\text{otp}(C_\delta) = \kappa$
- (D) For every $\beta < \lambda$ the set $\{C_\delta \cap \beta : \beta \in C_\delta, \delta \in S\}$ has cardinality $< \lambda$.

Claim 1.2. Assume $\lambda > \kappa$ are regular cardinals and $\chi \in [\kappa, \lambda]$.

1) If S is a stationary subset of $S_\kappa^\lambda := \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ then there exists a (λ, κ, χ) -BB-parameter $\mathbf{p}$ with $S_{\mathbf{p}} = S$.

1A) If $\chi = \lambda$ then we may set $C_{\mathbf{p}, \delta} := \delta$ for all $\delta \in S$.

2) If S is a stationary subset of S_κ^λ and a member of $\check{I}_\kappa[\lambda]$, then we may also add “ $\mathbf{p}$ is good.”

3) If $\lambda > \kappa^+$ then there exists a good (λ, κ, χ) -BB $_*$ -parameter.

Proof. Easy.

E.g. for part (1), use [She93, §2]. Part (3) follows by [She91, 4.4]. $\square_{1.2}$

Definition 1.3. Let $\kappa \leq \mu$.

1) We define $\mathbf{U}_\kappa(\mu)$ to be

$$\min\{|\mathcal{U}| : \mathcal{U} \subseteq [\mu]^\kappa, (\forall v \in [\mu]^\kappa)(\exists u \in \mathcal{U})[|u \cap v| = \kappa]\}.$$

2) Let $\mathbf{U}'_\kappa(\mu)$ mean

$$\min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\mu \text{ and } (\forall g \in {}^\kappa\mu)(\exists f \in \mathcal{F})(\exists^\kappa i < \kappa)[f(i) = g(i)]\}.$$

3) If J is an ideal on κ then we let

$$\mathbf{U}_J(\mu) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\mu \text{ and } (\forall g \in {}^\kappa\mu)(\exists f \in \mathcal{F})[\{i < \kappa : f(i) = g(i)\} \in J^+]\}.$$

Obviously,

Observation 1.4. 1) If $\mu \geq 2^\kappa$ then $\mathbf{U}_\kappa(\mu) = \mathbf{U}'_\kappa(\mu)$.

2) If $\mu = \mu^\kappa$ (or just $\alpha < \mu \Rightarrow |\alpha|^\kappa \leq \mu$) and $\text{cf}(\mu) \neq \kappa$, then $\mathbf{U}_\kappa(\mu) = \mathbf{U}'_\kappa(\mu) = \mu$.

Definition 1.5. 1) When we write $\text{Sep}(\chi, \mu, \partial, \theta, \Upsilon)$, we mean that there exists $\bar{f} = \langle f_\varepsilon : \varepsilon < \chi \rangle$ such that:

- (A) $f_\varepsilon : {}^\mu\partial \rightarrow \theta$
- (B) For every $\varrho \in {}^\chi\theta$, the set $\text{Sol}_\varrho := \{\nu \in {}^\mu\partial : (\forall \varepsilon < \chi)[f_\varepsilon(\nu) \neq \varrho(\varepsilon)]\}$ has cardinality $< \Upsilon$.

2) We write $\text{Sep}(\mu, \theta)$ to mean that $\text{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$ holds for some $\Upsilon = \text{cf}(\Upsilon) \leq 2^\mu$.

2A) $\text{Sep}(<\mu, \theta)$ will mean that $\text{Sep}(\sigma, \mu, \theta, \theta, \Upsilon)$ holds for some $\Upsilon = \text{cf}(\Upsilon) \leq 2^\mu$ and $\sigma < \mu$.

3) If $\partial := \theta$, we may omit it.

4) We may write Sep_1 instead of Sep , to distinguish it from Sep_2 in 1.7.

Quoting [She05, 1.11_{Ld.7}], $\text{Sep}(\mu, \theta)$ holds for many values of μ and θ :

Claim 1.6. *If at least one of the following holds, then we have $\text{Sep}(\mu, \theta)$:*

- (a) $\mu = \mu^\theta$
- (b) $\mathbf{U}_\theta(\mu) = \mu \geq 2^\theta$
- (c) We have $\mathbf{U}_{[\sigma]^{<\theta}}(\mu) = \mu$ for some $\sigma \geq \theta$ with $\sigma^\theta \leq \mu$.
- (d) $\theta = \text{cf}(\theta) < \mu$, and μ is strong limit of cofinality $\neq \theta$.
- (e) $\mu \geq \beth_\omega(\theta)$.

We introduce the following relative of $\text{Sep} = \text{Sep}_1$ which will be used in this work.

Definition 1.7. 1) When we write $\text{Sep}_2(\chi, \mu, \partial, \theta, \kappa)$, we mean that there exists a sequence $\bar{f} = \langle f_{\varepsilon, i} : \varepsilon < \chi, i < \kappa \rangle$ such that the following hold.

- (A) $f_{\varepsilon, i} : {}^\mu \partial \rightarrow \theta$
- (B) If $\mathcal{P}_i \subseteq {}^\mu \partial$ has cardinality $< \partial^\mu$ (for $i < \kappa$), then we can find a sequence $\bar{\varrho} = \langle \varrho_i : i < \kappa \rangle$ such that:
 - (a) $\varrho_i \in {}^\mu \theta$
 - (b) If $\bar{\nu} = \langle \nu_i : i < \kappa \rangle \in \prod_{i < \kappa} \mathcal{P}_i$ then there exist $\varepsilon < \mu$ and $u \in [\kappa]^\kappa$ such that

$$i \in u \Rightarrow f_{\varepsilon, i}(\nu_i) = \varrho_i(\varepsilon).$$

2) $\text{Sep}_2(\mu, \theta, \kappa)$ will mean $\text{Sep}_2(\mu, \mu, \theta, \theta, \kappa)$.

Recalling Definition 1.1,

Definition 1.8. 1) We say that $\mathbf{p}$ has the (D, Υ, θ) - $\bar{F}$ -BB-property when there exists a $\mathbf{p}$ - D - $\bar{F}$ -BB-sequence, where:

- (A) $\mathbf{p}$ is a (λ, κ, χ) -BB-parameter.
- (B) D is a filter on λ .
- (C) $\bar{F}$ is a $(\mathbf{p}, \Upsilon, \theta)$ -coloring.

2) We say that $\mathbf{p}$ has the (D, Υ, θ) -BB-property when it has (D, Υ, θ) - $\bar{F}$ -BB-property for every $(\mathbf{p}, \Upsilon, \theta)$ -coloring $\bar{F}$.

3) If D is the club filter on λ , we may omit it.

We now quote the main claim of the previous paper – [She05, 1.10_{Ld.6}] – but we will not use it here.

Claim 1.9. *Assume*

- (a) $\lambda := \text{cf}(2^\mu)$
- (b) D is a μ^+ -complete filter on λ extending the club filter.
- (c) $\kappa = \text{cf}(\kappa) < \chi \leq \lambda$
- (d) $\mathbf{p} = \langle (C_\delta, C'_\delta) : \delta \in S \rangle$ is a good (λ, κ, χ) -BB-parameter, where $S \in D$.
- (e) $2^{<\chi} \leq 2^\mu$ and $\theta \leq \mu$.
- (f) $\alpha < 2^\mu \Rightarrow \text{trp}_\kappa(|\alpha|) < 2^\mu$ (By this we mean that every tree with $|\alpha|$ -many nodes and κ levels has $< 2^\mu$ -many κ -branches.)
- (g) $\text{Sep}_1(\mu, \theta)$.

Then $\mathbf{p}$ has the $(D, 2^\mu, \theta)$ -BB-property. (Note that this means that possibly $\theta > 2$; i.e. we have more than two colors.)

Remark 1.10. 1) If μ is strong limit singular, $\kappa + \theta < \mu$, and $\lambda := \text{cf}(2^\mu)$, then the only assumption which does not follow is clause (f), which does hold for many regular $\kappa < \mu$ by [She00]. (For more, see [She06].)

Our aim here is to cover more cases of κ , and construct relatives of this property which are easier to use and have more applications.

2) By [She93, §1], there are many S as required: still, $S \in \check{I}_\kappa[\lambda]$.

3) ‘Good $\mathbf{p}$ ’ is a restriction on us, as the result covers fewer S -s.

4) But we would like to have parallel results using Sep_2 . (This will be done in §2.)

Claim 1.11. *Assume κ is regular and $\mu > \theta = \theta^{<\kappa}$. If at least one of the following holds then we have $\text{Sep}_2(\mu, \theta, \kappa)$.*

- (a) $\kappa \neq \text{cf}(\mu)$, $\alpha < \mu \Rightarrow |\alpha|^\kappa < \mu$, and $\text{Sep}_1(\mu, \theta)$.
- (b) $\mathbf{U}'_\kappa(\mu) = \mu \geq \beth_\omega(\theta + \kappa)$
- (c) $\mathbf{U}'_\kappa(\mu) = \mu$ and $\text{Sep}_1(\mu, \theta)$.
- (d) We have $\mathbf{U}_{[\sigma]^{<\theta}}(\mu) = \mu$ for some $\sigma \geq \theta$ with $\sigma^\theta \leq \mu$.

Proof. Case (a):

Let $\bar{f}^\circ = \langle f_\varepsilon^\circ : \varepsilon < \mu \rangle$ witness $\text{Sep}_1(\mu, \theta)$ (hence f_ε° is a function from ${}^\mu\theta$ to θ).
Let

$$\mathcal{F} := \{\nu \in {}^\kappa\mu : \text{rang}(\nu) \text{ is a bounded subset of } \mu\}.$$

Recalling Definition 1.5(2), let Υ be a regular cardinal $\leq 2^\mu$ such that $\text{Sep}_1(\mu, \mu, \theta, \Upsilon)$ holds.

By the assumption ‘ $\alpha < \mu \Rightarrow |\alpha|^\kappa < \mu$,’ clearly $|\mathcal{F}| = \mu$. Let $\langle \nu_\varepsilon : \varepsilon < \mu \rangle$ list the members of $\mathcal{F}$, and we shall define

$$\oplus_1 \bar{f} = \langle f_{\varepsilon,i} : \varepsilon < \mu, i < \kappa \rangle, \text{ where } f_{\varepsilon,i} := f_{\nu_\varepsilon(i)}.$$

It will suffice to prove that $\bar{f}$ witnesses $\text{Sep}_2(\mu, \theta, \kappa)$. So let $\mathcal{P}_i \subseteq {}^\mu\theta$ be of cardinality $< 2^\mu = \theta^\mu$ for $i < \kappa$, and we need to construct $\bar{\varrho}$ as in 1.7(1)(B).

Fix $i < \kappa$, so by 1.5(1)(B), for every $\rho \in {}^\mu\theta$ the set

$$\text{Sol}_\rho := \{\nu \in {}^\mu\theta : (\forall \varepsilon < \chi)[f_\varepsilon^\circ(\nu) \neq \rho(\varepsilon)]\}$$

has cardinality $< \Upsilon$. As $|\mathcal{P}_i| < 2^\mu$ and $\Upsilon = \text{cf}(\Upsilon) \leq 2^\mu$, the set $\Lambda_i := \bigcup_{\nu \in \mathcal{P}_i} \text{Sol}_\nu$ has cardinality $< 2^\mu$, so we can choose $\varrho_i \in {}^\mu\theta \setminus \Lambda_i$.

It will suffice to prove that $\langle \varrho_i : i < \kappa \rangle$ is as promised. So let $\bar{\nu} = \langle \nu_i : i < \kappa \rangle \in \prod_{i < \kappa} \mathcal{P}_i$, and we have to find $\xi < \mu$ and $u \in [\kappa]^\kappa$ as promised in 1.7(1)(B)(b).

For each $i < \kappa$, by our choice of ϱ_i we know $\varrho_i \notin \text{Sol}_{\nu_i}$. This means that there is $\varepsilon_i < \mu$ such that

$$f_{\varepsilon_i}^\circ(\nu_i) = \varrho_i(\varepsilon_i).$$

As $\text{cf}(\mu) \neq \text{cf}(\kappa)$, there exists $\zeta < \mu$ such that the set $u := \{i < \kappa : \varepsilon_i < \zeta\}$ has cardinality κ (and even order type κ). Let $\nu \in {}^\kappa\zeta$ list u in increasing order, and let $\xi < \mu$ be such that $\nu = \nu_\xi$. Clearly ξ is as required.

Case (b): Assume $\mathbf{U}_\kappa(\mu) = \mu \geq \chi := \beth_\omega(\theta + \kappa)$.

By Case (e) of 1.6 this implies $\text{Sep}_1(\mu, \theta)$ so we can apply case (a). Let $\mathcal{F} \subseteq {}^\kappa \mu$ be of cardinality μ witnessing $\mathbf{U}'_\kappa(\mu) = \mu$.

The rest should be clear.

Case (c): Like Case (b).

Case (d): [End of Line]

$\square_{1.11}$

§ 2. THE BLACK BOX PROPERTY

Definition 2.1. 1) We say $\bar{F} = \langle F_{\delta,i} : \delta \in S, i < \kappa \rangle$ is a *continuous* $(\mathbf{p}, \Upsilon, \theta)$ -coloring when ($S = S_{\mathbf{p}}$ and)

- (A) $F_{\delta,i} : C_{\delta,i}\theta \rightarrow \theta$ for $\delta \in S$ and $i < \kappa$, where $C_{\delta,i} := C_{\delta} \cap \beta_{\delta,i+1} \setminus \beta_{\delta,<i}$.
- (B) For $\beta < \lambda$, the set $\mathcal{F}_{\beta} := \{F_{\delta,i} : \delta \in S, i < \kappa, \beta_{\delta,i} = \beta\}$ has cardinality $< \lambda$. (Note that $F_{\delta,i}$ depends only on $C_{\delta,i}$.)

1A) For $\bar{F}$, Υ , and θ as above, we say $\bar{c} = \langle c_{\delta,i} : \delta \in S, i < \kappa \rangle \in {}^{S \times \kappa} \theta$ is a **p-D- $\bar{F}$ -BB¹-sequence** when for every $\eta \in {}^{\lambda} \Upsilon$ the set

$$\{\delta \in S : (\exists^{\kappa} i < \kappa)[F_{\delta,i}(\eta \upharpoonright C_{\delta,i}) = c_{\delta,i}]\}$$

is a member of D^+ (and in the default case, a stationary subset of λ).

1B) For $\bar{F}$ (and Υ, θ) as above, we say $\bar{c} \in {}^{S \times \kappa} \theta$ is a **p-D- $\bar{F}$ -BB²-sequence** when for every $\eta \in {}^{\lambda} \Upsilon$ the set

$$\{\delta \in S : (\forall i < \kappa)[F_{\delta,i}(\eta \upharpoonright C_{\delta,i}) = c_{\delta,i}]\}$$

is a member of D^+ .

This is the default case; if we write ‘BB-sequence’ we mean BB².

2) We say that $\gamma, \delta \in S$ are **p-similar** when:

- $\text{otp}(C_{\gamma}) = \text{otp}(C_{\delta})$ (Recall $\text{otp}(C'_{\gamma}) = \text{otp}(C'_{\delta}) = \kappa$.)
- If $\alpha_1 \in C'_{\gamma}$, $\alpha_2 \in C'_{\delta}$, and $\text{otp}(C'_{\gamma} \cap \alpha_1) = \text{otp}(C'_{\delta} \cap \alpha_2)$, then
 $\text{otp}(C_{\gamma} \cap \alpha_1) = \text{otp}(C_{\delta} \cap \alpha_2)$.

3) We say $\bar{F} = \langle F_{\delta} : \delta \in S \rangle$ is a *uniform* $(\mathbf{p}, \Upsilon, \theta)$ -coloring when the implication ‘(A) $\Rightarrow$ (B)’ holds, where:

- (A) (a) δ_1 and δ_2 are **p-similar**.
 (b) $f_{\ell} : C_{\delta_{\ell}} \rightarrow \theta$ for $\ell = 1, 2$.
 (c) If $\gamma_{\ell} \in C_{\delta_{\ell}}$ for $\ell = 1, 2$, then
 $\text{otp}(\gamma_1 \cap C_{\delta_1}) = \text{otp}(\gamma_2 \cap C_{\delta_2}) \Rightarrow f_1(\gamma_1) = f_2(\gamma_2)$.
- (B) $F_{\delta_1}(f_1) = F_{\delta_2}(f_2)$.

4) ‘ $\langle F_{\delta,i} : \delta \in S, i < \kappa \rangle$ is uniformly continuous’ is defined similarly, but we replace (3)(B) with the demand

$$(B)' \quad i < \kappa \Rightarrow F_{\delta_1,i}(f_1 \upharpoonright C_{\delta_1,i}) = F_{\delta_2,i}(f_2 \upharpoonright C_{\delta_2,i}).$$

5) When we write σ -uniform instead of uniform, this means that in clause (3)(A)(a) we replace ‘**p-similar**’ by ‘ $\mathcal{E}$ -equivalent’ for some equivalence relation $\mathcal{E}$ with $\leq \sigma$ equivalence classes satisfying

$$\gamma \mathcal{E} \delta \Rightarrow [\gamma \text{ is } \mathbf{p}\text{-similar to } \delta].$$

Remark 2.2. 1) On the one hand, we can choose $C_{\delta} := \delta$ (in which case $\lambda = \chi$ and $\mathbf{C}_{\delta,i} = [\beta_{\delta,<i}, \beta_{\delta,i}]$). On the other hand, we may choose $C_{\delta} := C'_{\delta}$; in this case each $C_{\delta,i+1}$ is a singleton $\{\beta_{\delta,i}\}$.

In both cases

$$\beta < \lambda \Rightarrow |\{\mathbf{C}_{\delta,i} : \delta \in S, i < \kappa, \beta_{\delta,i} = \beta\}| < \lambda.$$

This will be used in clause $(*)_4(d)$ in the proof of 2.4.

2) Uniformity (defined in 2.1(3)-(5)) is only used in 2.14.

Definition 2.3. For $\iota = 1, 2$, we say that $\mathbf{p}$ has the *continuous* (D, Υ, θ) -BB $^\iota$ -property when it has the (D, Υ, θ) - $\bar{F}$ -BB $^\iota$ -property (see 2.1(1A)) for every continuous $(\mathbf{p}, \Upsilon, \theta)$ -coloring $\bar{F}$.

By this, we mean that the implication ‘(A) $\Rightarrow$ (B)’ holds, where:

- (A) (a) $\mathbf{p}$ is a (λ, κ, χ) -BB-parameter.
- (b) D is a filter on λ .
- (c) $\bar{F}$ is a *continuous* $(\mathbf{p}, \Upsilon, \theta)$ -coloring.
- (B) There exists a $\mathbf{p}$ - D - $\bar{F}$ -BB $^\iota$ -sequence.

Again, if D is the club filter on λ plus $S_{\mathbf{p}}$, then we may omit it.

The next claim is related to 1.9, modified to fit this new definition. (Note that this is in some sense a stronger result: in addition to demanding continuity, we omit demand 1.9(f). Also, if $\lambda = \lambda^{<\lambda}$ (hence $\lambda = 2^\mu$) then we may choose $\chi := \lambda$.)

Lemma 2.4. *We have ‘(A) $\Rightarrow$ (B)’, where*

- (A) (a) $\lambda := \text{cf}(2^\mu)$
- (b) D is a μ^+ -complete filter on λ extending the club filter.
- (c) $\kappa = \text{cf}(\kappa) < \chi \leq \lambda$
- (d) $\mathbf{p} = \langle (C_\delta, C'_\delta) : \delta \in S \rangle$ is a (λ, κ, χ) -BB-parameter, where $S \in D$.
- (e) $\theta^{<\chi} \leq 2^\mu$ and $\theta \leq \mu$.
- (f) $\text{Sep}_2(\mu, \theta, \kappa)$.
- (B) $\mathbf{p}$ has the continuous $(D, 2^\mu, \theta)$ -BB 1 -property.

Proof. $(*)_1$ Let $\bar{F}$ be a continuous $(\mathbf{p}, 2^\mu, \theta)$ -coloring.

Recall that assumption (f) means that we have $\text{Sep}_2(\mu, \mu, \theta, \theta, \kappa)$ (see Definition 1.7(2)).

Hence,

$(*)_2$ Let $\bar{f} = \langle f_{\xi, i} : \xi < \mu, i < \kappa \rangle$ exemplify $\text{Sep}_2(\mu, \theta, \kappa)$.

Let $h : \lambda \rightarrow 2^\mu$ be increasing and continuous (hence injective), with unbounded range, such that $\alpha < \lambda \Rightarrow \mu \mid h(\alpha)$. (If 2^μ is regular, then we may choose h to be the function $\alpha \mapsto \mu \cdot \alpha$, recalling $\lambda := \text{cf}(2^\mu)$ in assumption (A)(a)).

$(*)_3$ (a) Let $\text{cd} : {}^\mu(2^\mu) \rightarrow 2^\mu$ be a bijection satisfying

$$\text{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu \rangle) \geq \sup\{\alpha_\varepsilon : \varepsilon < \mu\}.$$

(b) For each $\varepsilon < \mu$, let $\text{cd}_\varepsilon : 2^\mu \rightarrow 2^\mu$ be defined such that the following diagram commutes:

$$\begin{array}{ccc} {}^\mu(2^\mu) & \xrightarrow{\text{cd}} & 2^\mu \\ & \searrow \pi_\varepsilon & \downarrow \text{cd}_\varepsilon \\ & & 2^\mu \end{array}$$

where π_ε is the function which sends $\langle \alpha_\zeta : \zeta < \mu \rangle \mapsto \alpha_\varepsilon$.

Now let us introduce some notation.

- (*)₄ (a) Let $C' := \bigcup \bar{C}' = \bigcup_{\delta \in S} C'_\delta$.
- (b) Let $\mathcal{C}_\beta := \{C_{\delta,i} : \delta \in S, i < \kappa, \beta_{\delta,i} = \beta\}$ for $\beta \in C'$.
- (c) Let $\mathcal{C} := \bigcup_{\beta \in C'} \mathcal{C}_\beta$.

Note that $\beta < \lambda \Rightarrow |\mathcal{C}_\beta| < \lambda \leq 2^\mu$ by 1.1(1)(C).

- (*)₅ (a) Let $T_\alpha := \bigcup_{C \in \mathcal{C}_\alpha} C(2^\mu)$. (Note that $C \in \mathcal{C}_\alpha \Rightarrow |C| < \chi$ by 1.1(1)(A).)
- (b) $T := \bigcup_{\alpha \in C'} T_\alpha$
- (c) For $\delta < \lambda$, let $T_{<\delta} := \bigcup \{T_\alpha : \alpha \in C', \alpha < \delta\}$.

- (*)₆ For $\beta < \gamma < \lambda$ and $u \in \mathcal{C}_\beta$, let

$$\mathcal{P}_{\beta,\gamma,u} := {}^u h(\gamma) \cap T_\beta \cap \{\eta_\varepsilon : \varepsilon < h(\gamma)\}.$$

- (*)₇ (a) For $\delta \in S$ and $i < \kappa$, let $\beta := \beta_{\delta,i}$ and define

$$\mathcal{P}_{\delta,i} := \bigcup \{\mathcal{P}_{\beta,\gamma,u} : u \in \mathcal{C}_\beta \text{ and } \gamma \in C'_\delta \setminus \beta\}.$$

- (b) For $\delta \in S$, we define

$$\mathcal{P}_\delta := \{\eta \in {}^{C_\delta} \theta : i < \kappa \Rightarrow \eta \upharpoonright C_{\delta,i} \in \mathcal{P}_{\delta,i}\}$$

(recalling 2.1(1)(A)).

Now,

- (*)₈ $\delta \in S \wedge i < \kappa \Rightarrow |\mathcal{P}_{\delta,i}| \leq h(\delta) < 2^\mu$

[Why? Because $\mathcal{P}_{\delta,i} \subseteq \{\eta_\varepsilon : (\exists \gamma < \delta)[\varepsilon < h(\gamma)]\} = \{\eta_\varepsilon : \varepsilon < h(\delta)\}$.]

- (*)₉ For each $\eta \in \mathcal{P}_{\delta,i}$ and $\varepsilon < \mu$, we define $\nu_{\eta,\varepsilon} \in {}^{C_{\delta,i}} h(\delta)$ as the function which sends $\alpha \mapsto \text{cd}_\varepsilon(\eta(\alpha))$.

[Recall (*)₅(d) and our choice of h (between (*)₂ and (*)₃).]

- (*)₁₀ For each $\eta \in \mathcal{P}_{\delta,i}$, obviously $\rho_\eta := \langle F_{\delta,i}(\nu_{\eta,\varepsilon}) : \varepsilon < \mu \rangle \in {}^\mu \theta$.
- (*)₁₁ Clearly $\mathcal{R}_{\delta,i} := \{\rho_\eta : \eta \in \mathcal{P}_{\delta,i}\} \subseteq {}^\mu \theta$ is of cardinality $\leq |\mathcal{P}_{\delta,i}|$ (which, as said above, is $\leq |h(\delta)| < 2^\mu$).

Now, as we are assuming $\text{Sep}_2(\mu, \theta, \kappa)$ in clause (A)(f), recalling 1.7(1)(B)(b),

- (*)₁₂ For $\delta \in S$, there exist $\langle \varrho_{\delta,i}^* : i < \kappa \rangle$ such that:
 - (a) $\varrho_{\delta,i}^* \in {}^\mu \theta$

(b) If $\bar{\rho} = \langle \rho_i : i < \kappa \rangle \in \prod_{i < \kappa} \mathcal{R}_{\delta,i}$ then for some $\varepsilon < \mu$ we have

$$\{i < \kappa : f_{\varepsilon,i}(\varrho_{\delta,i}^*) = \nu_{\delta,i}(\varepsilon)\} \in [\kappa]^\kappa.$$

Let $\varepsilon < \mu$. Recall that $\varrho_{\delta,i}^* \in {}^\mu\theta$ for $\delta \in S$. Hence we can consider the sequence $\bar{c}_\varepsilon = \langle f_{\varepsilon,i}(\varrho_{\delta,i}^*) : \delta \in S, i < \kappa \rangle \in {}^{S \times \kappa}\theta$ as a candidate for being a $\mathbf{p}\text{-}D\text{-}\bar{F}\text{-BB}^1$ -sequence. If one of them is, we are done. So:

(*)₁₅ Assume towards contradiction that for each $\varepsilon < \mu$ there is a sequence $\eta_\varepsilon \in {}^\lambda(2^\mu)$ that exemplifies the failure of $\bar{c}_\varepsilon$ to be a $\mathbf{p}\text{-}D\text{-}\bar{F}\text{-BB}^1$ -sequence.

So for each $\varepsilon < \mu$ there is a $E_\varepsilon \in D$ such that:

(*)₁₆ If $\delta \in S \cap E_\varepsilon$ then $|\{i < \kappa : F_{\delta,i}(\eta_\varepsilon \restriction C_{\delta,i}) = f_{\varepsilon,i}(\varrho_{\delta,i}^*)\}| < \kappa$.

Define $\eta^* \in {}^\lambda(2^\mu)$ by $\eta^*(\alpha) := \text{cd}(\langle \eta_\varepsilon(\alpha) : \varepsilon < \mu \rangle)$. Now as λ is regular uncountable and $\beta < \lambda \Rightarrow |\mathcal{C}_\beta| < \lambda$, it follows by choice of h that

$$E := \{\delta < \lambda : (\forall \alpha < \delta) [\eta^*(\alpha) < h(\delta)] \text{ and } \gamma \in S \wedge i < \kappa \wedge \beta_{\gamma,i} < \delta \Rightarrow \eta^* \restriction C_{\gamma,i} \in T_{<\delta}\}$$

is a club of λ . (See the choice of T and $T_{<\delta}$.)

By clause (A)(b) in the assumption of our claim, the filter D includes the clubs of λ , so clearly $E \in D$. Also, D is μ^+ -complete hence $E^* := \bigcap_{\varepsilon < \mu} E_\varepsilon \cap E$ belongs to D .

Recalling $S \in D^+$, we can choose $\delta \in E^* \cap S$ and fix it for the rest of the proof. Clearly

$$\eta_{\delta,i}^* := \eta^* \restriction C_{\delta,i} \in \mathcal{P}_{\delta,i}$$

for each $i < \kappa$; just check the definitions of $\mathcal{P}_{\delta,i}$, E , and E^* .

Fix $\varepsilon < \mu$ for the next two paragraphs (until the end of (*)₁₇). Now recall that $\nu_{\eta_{\delta,i}^*,\varepsilon}$ is the function from $C_{\delta,i}$ to $h(\delta)$ defined by

$$\nu_{\eta_{\delta,i}^*,\varepsilon}(\alpha) = \text{cd}_\varepsilon(\eta^*(\alpha)).$$

But by our choice of η^* , clearly $\alpha \in C_\delta \Rightarrow \text{cd}_\varepsilon(\eta^*(\alpha)) = \eta_\varepsilon(\alpha)$, so

$$i < \kappa \wedge \alpha \in C_{\delta,i} \Rightarrow \nu_{\eta_{\delta,i}^*,\varepsilon}(\alpha) = \eta_\varepsilon(\alpha), \text{ so } \nu_{\eta_{\delta,i}^*,\varepsilon} = \eta_\varepsilon \restriction C_{\delta,i}.$$

Hence $F_\varepsilon(\nu_{\eta_{\delta,i}^*,\varepsilon}) = F_\varepsilon(\eta_\varepsilon \restriction C_{\delta,i})$. As $\delta \in E^* \subseteq E_\varepsilon$, clearly

$$|\{i < \kappa : F_{\varepsilon,i}(\eta_\varepsilon \restriction C_{\delta,i}) = f_{\varepsilon,i}(\varrho_{\delta,i}^*)\}| < \kappa,$$

and as $\eta_{\delta,i}^* \in \mathcal{P}_{\delta,i} \subseteq {}^\mu\theta$ (recalling (*)₁₀) clearly $\rho_{\eta_{\delta,i}^*} \in {}^\mu\theta$ is well defined. Now easily $\rho_{\eta_{\delta,i}^*}(\varepsilon) = F_\varepsilon(\nu_{\eta_{\delta,i}^*,\varepsilon})$ by the definition of $\rho_{\eta_{\delta,i}^*}$, so we have

(*)₁₇ $\{i < \kappa : \rho_{\eta_{\delta,i}^*}(\varepsilon) = f_{\varepsilon,i}(\varrho_{\delta,i}^*)\} \in [\kappa]^{<\kappa}$.

As this holds for every $\varepsilon < \mu$, it follows by (*)₁₁ that $\langle \rho_{\delta,i} : i < \kappa \rangle \in \prod_{i < \kappa} \mathcal{R}_{\delta,i}$, and so by the choice of $\langle \varrho_{\delta,i}^* : i < \kappa \rangle$ and (*)₁₂(b) there exists $\varepsilon < \mu$ such that

(*)₁₈ $\{i < \kappa : F_{\varepsilon,i}(\eta_\varepsilon \restriction C_{\delta,i}) = f_{\varepsilon,i}(\varrho_{\delta,i}^*)\} \in [\kappa]^\kappa$.

Now (*)₁₇+(*)₁₈ give a contradiction, so by (*)₁₅ we are done. □_{2.4}

Definition 2.5. 0) For $\mathbf{p}$ a (λ, μ, κ) -BB-parameter let $C_{\delta,i}^\bullet = C_{\mathbf{p},\delta,i}^\bullet := C_\delta \cap \beta_{\delta,i}$.

1) We say $\bar{F} = \langle F_{\delta,i} : \delta \in S, i < \kappa \rangle$ is a *continuous* $[\mathbf{p}, \Upsilon, \theta]$ -coloring when ($S = S_{\mathbf{p}}$ and)

- (A) $F_{\delta,i} : C_{\delta,i}^\bullet \Upsilon \rightarrow \theta$ for $\delta \in S$ and $i < \kappa$.
- (B) For $\beta < \lambda$, the set $\mathcal{F}_\beta := \{F_{\delta,i} : \delta \in S, i < \kappa, \beta_{\delta,i} = \beta\}$ has cardinality $< \lambda$.

1A) For $\bar{F}$ as in 2.1(1), we say $\bar{c} \in {}^{S \times \kappa} \theta$ is a $\mathbf{p}$ - D - $\bar{F}$ -BB³-sequence if for every $\eta \in {}^\lambda \Upsilon$ the set

$$\{\delta \in S : (\exists^\kappa i < \kappa)[F_{\delta,i}(\eta \upharpoonright C_{\delta,i}^\bullet) = c_{\delta,i}]\}$$

is a member of D^+ .

1B) For $\bar{F}$ as above, we say $\bar{c} \in {}^{S \times \kappa} \theta$ is a $\mathbf{p}$ - D - $\bar{F}$ -BB⁴-sequence if for every $\eta \in {}^\lambda \Upsilon$ the set

$$\{\delta \in S : (\forall i < \kappa)[F_{\delta,i}(\eta \upharpoonright C_{\delta,i}^\bullet) = c_{\delta,i}]\}$$

is a member of D^+ .

2) For $\iota = 3, 4$, we say that $\mathbf{p}$ has the *continuous* (D, Υ, θ) -BB ^{ι} -property when it has the (D, Υ, θ) - $\bar{F}$ -BB ^{ι} -property for every continuous $[\mathbf{p}, \Upsilon, \theta]$ -coloring $\bar{F}$.

We might like an analogue of Lemma 2.4, to replace BB¹ by BB³. (That is, replace $F_{\delta,i} : C_{\delta,i}(2^\mu) \rightarrow \theta$ by $F_{\delta,i} : C_{\delta,i}^\bullet(2^\mu) \rightarrow \theta$.)

Lemma 2.6. *Let $\mathbf{p}$ be a (λ, μ, κ) -BB-parameter. We have ‘(A) $\Rightarrow$ (B),’ where*

- (A) *As in 2.4, but we add*
 - (g) *For every $\beta < \lambda$, the set*

$$\{C_{\delta,i}^\bullet : \delta \in S, i < \kappa, \beta_{\delta,i} = \beta\}$$
has cardinality² $< \lambda$.
- (B) *$\mathbf{p}$ has the continuous $(D, 2^\mu, \theta)$ -BB³-property.*

Proof. Like the proof of 2.4, replacing $C_{\delta,i}$ by $C_{\delta,i}^\bullet$. □_{2.6}

As an alternative to replacing $C_{\delta,i}$ by $C_{\delta,i}^\bullet$, we may instead place additional demands on $\mathbf{p}$.

Claim 2.7. *Let $\mathbf{p}$ be a (λ, μ, κ) -BB-parameter.*

1) *We have ‘(A) $\Rightarrow$ (B),’ where*

- (A) *As in 2.4 (giving us the BB¹-property), but we add*
 - (g) $\theta = \theta^{<\kappa}$
 - (h) $\mathbf{p}$ *is good.*
- (B) *$\mathbf{p}$ has the continuous $(D, 2^\mu, \theta)$ -BB³-property.*

2) *For $\iota = 1, 2$, we have ‘(A) $\Rightarrow$ (B),’ where*

² Note that this clause implies the analogous statement for $C_{\delta,i}$, as for each $\beta < \lambda$ we have

$$|\{C_{\delta,i} : \delta \in S, i < \kappa, \beta_{\delta,i} = \beta\}| \leq |\{C_{\delta,i}^\bullet : \delta \in S, i < \kappa, \beta_{\delta,i} = \beta\}| \cdot |\beta|.$$

- (A) (a) $\theta = \theta^{<\kappa}$
 (b) *There exists a good (λ, μ, κ) -BB-parameter $\mathbf{p}'$ with the continuous $(D, 2^\mu, \theta)$ -BB $^{\iota}$ -property such that $(S_{\mathbf{p}'}, \bar{C}_{\mathbf{p}'}) = (S_{\mathbf{p}}, \bar{C}_{\mathbf{p}})$.*
 (B) $\mathbf{p}$ has the continuous $(D, 2^\mu, \theta)$ -BB $^{\iota+2}$ -property.

Proof. 1) Follows by 2.4 and 2.6.

2) First note

The existence of $\mathbf{p}'$ is not used anywhere in this proof.

(*)₀ If E is a club of λ then we can replace $\mathbf{p}$ by $\mathbf{p} \upharpoonright (S_{\mathbf{p}} \cap E)$ without consequence.

We define $\mathbf{p}_* = \langle C_{\mathbf{p}_*, \delta}, C'_{\mathbf{p}_*, \delta} : \delta \in S_{\mathbf{p}_*} \rangle$ as follows. First, let us write $\text{up}(\alpha) := \sum_{\beta < \alpha} \beta$. Now,

- (*)₁ (a) $S_{\mathbf{p}_*} := \{\text{up}(\delta) : \delta \in S_{\mathbf{p}}\}$ (so $S_{\mathbf{p}_*} \triangle S_{\mathbf{p}}$ is not stationary).
 (b) $C_{\mathbf{p}_*, \delta} := \{\text{up}(\alpha) + \gamma : \alpha \in C'_{\mathbf{p}, \delta}, \gamma \in C_{\mathbf{p}, \delta} \cap \alpha\}$
 (c) $C'_{\mathbf{p}_*, \delta} := \{\text{up}(\alpha) : \alpha \in C'_{\mathbf{p}, \delta}\}$
 (*)₂ $\mathbf{p}_*$ satisfies all the demands in 2.4(A)(d).

[Why? To show $\mathbf{p}_*$ is a (λ, κ, χ) -BB-parameter, we have to check Definition 1.1(1).

Clause (A): Let $\delta_* \in S_{\mathbf{p}_*}$, so for some $\delta \in S$ we have $\delta_* = \text{up}(\delta)$. Hence obviously $C_{\mathbf{p}_*, \delta_*} \subseteq \delta_*$, and easily $\sup(C_{\mathbf{p}_*, \delta_*}) = \delta_*$.

Also, $\text{otp}(C_{\mathbf{p}_*, \delta_*}) \leq \chi$ because

$$i < \kappa \Rightarrow |C_{\mathbf{p}_*, \delta_*} \cap \beta_{\delta_*, i}| < \chi$$

because

$$|C_{\mathbf{p}_*, \delta_*} \cap \beta_{\delta_*, i}| < \sum_{j \leq i} |C_{\delta} \cap \beta_{\delta, j}| \leq |i + 1| \cdot |C_{\delta} \cap \beta_{\delta, i}| < \chi.$$

Clause (B): Obviously $C'_{\mathbf{p}_*, \delta_*} \subseteq \delta_*$, $\text{otp}(C'_{\mathbf{p}_*, \delta_*}) = \text{otp}(C'_{\delta}) = \kappa$, and $\sup(C'_{\mathbf{p}_*, \delta_*}) = \delta_*$.

Also, $C_{\mathbf{p}_*, \delta_*} \subseteq \delta_*$, $\text{otp}(C_{\mathbf{p}_*, \delta_*}) = \text{otp}(C_{\delta}) \leq \chi$, and $\sup(C_{\mathbf{p}_*, \delta_*}) = \delta_*$.

Clause (C): Note that if $\beta_* < \lambda$ then

$$|\{C_{\mathbf{p}_*, \delta_*, i} : \beta_* = \beta_{\delta_*, i}\}| \leq |\{C_{\delta, i} \cap \beta : \delta_* = \text{up}(\delta), \beta_* = \text{up}(\beta), \text{ and } \beta = \beta_{\delta, i}\}| < \lambda.$$

Why? The first inequality holds by the choice of $\mathbf{p}_*$, and the second because $\mathbf{p}$ is good.

This gives the desired inequality in 1.1(1)(C).]

(*)₃ Therefore $\mathbf{p}_*$ satisfies the conclusion in 2.4(B).

Now to prove the desired conclusion,

(*)₄ Let $\bar{F} = \langle F_{\delta, i} : \delta \in S, i < \kappa \rangle$ be a $(\mathbf{p}, 2^\mu, \theta)$ -coloring.

It will suffice to prove there is a $\mathbf{p}$ - D - $\bar{F}$ -BB $^{\iota+2}$ -sequence.

We now choose a $(\mathbf{p}_*, 2^\mu, \theta)$ -coloring $\langle F_{\delta_*, i}^* : \delta_* \in S_{\mathbf{p}_*}, i < \kappa \rangle$.

(*)₅ We define $F_{\delta_*,i}^*$ as follows:

- ₁ The domain of $F_{\delta_*,i}^*$ will be $C_{\delta_*,i}\theta$, and the range will be $\subseteq \theta$. For η_* in the domain, we define

$$F_{\delta_*,i}^*(\eta_*) := F_{\delta,i}(\eta),$$

where η is defined in the next bullet.

- ₂ $\eta : C_\delta \cap \beta_{\delta,i} \rightarrow \theta$ is the function which sends

$$\alpha \mapsto \eta_*(\text{up}(\beta_{\delta,<i}) + \alpha)$$

for all $\alpha \in C_\delta \cap \beta_{\delta,i}$.

(*)₆ (a) $\langle F_{\delta_*,i}^* : \delta_* \in S_{\mathbf{p}_*}, i < \kappa \rangle$ is indeed a $(\mathbf{p}_*, 2^\mu, \theta)$ -coloring.

(b) Hence there is a $\mathbf{p}$ - D - $\bar{F}$ -BB $^\iota$ -sequence $\bar{c}^* = \langle c_{\delta_*,i}^* : \delta_* \in S_{\mathbf{p}_*}, i < \kappa \rangle$.

[Why? Clause (a) can be checked. Now clause (b) follows by our assumptions and Theorem 2.4.]

(*)₇ Furthermore, $\bar{c}^*$ is a $\mathbf{p}_*$ - D - $\bar{F}$ -BB $^{\iota+2}$ -sequence.

Now we are done. □_{2.7}

Remark 2.8. It is nice to successfully predict the values of [_____] on some $u \in [\kappa]^\kappa$, but it would be better to succeed for $u = \kappa$.

One possibility: what if we just assume $\theta = \theta^{<\kappa}$, and for each $u \subseteq \kappa$ we define $\mathbf{p}[u]$ by $(S_{\mathbf{p}[u]}, \bar{C}_{\mathbf{p}[u]}) := (S_{\mathbf{p}}, \bar{C}_{\mathbf{p}})$, but

$$C'_{\mathbf{p}[u],\delta} := \{\alpha \in C'_{\mathbf{p},\delta} : \text{otp}(C'_{\mathbf{p},\delta} \cap \alpha) \in u\}?$$

Or use a regressive function $h : u \rightarrow \kappa$? Something close is done below.

Definition 2.9. Let $\mathbf{p}$ be a (λ, κ, χ) -BB-parameter.

For every $A \in [\kappa]^\kappa$, we define a (λ, κ, χ) -BB-parameter

$$\mathbf{p}[A] = \langle C_{A,\delta}, C'_{A,\delta} : \delta \in S \rangle$$

by

- $C_{A,\delta} := C_\delta$
- $C'_{A,\delta} := \{\beta \in C_\delta : \text{otp}(C_\delta \cap \beta) \in A\}$.

Observation 2.10. $\mathbf{p}[A]$, as defined above, is indeed a (λ, κ, χ) -BB-parameter.

Claim 2.11. Assume $\mathbf{p}$ is a (λ, κ, χ) -BB-parameter, $2^\kappa < \lambda$, $\Upsilon = \Upsilon^{2^\kappa}$, and $\theta = \theta^{2^\kappa}$.

1) $\mathbf{p}$ has the continuous (D, Υ, θ) -BB¹-property iff $\mathbf{p}[A]$ has the continuous (D, Υ, θ) -BB²-property for some $A \in [\kappa]^\kappa$.

2) $\mathbf{p}$ has the continuous (D, Υ, θ) -BB³-property iff $\mathbf{p}[A]$ has the continuous (D, Υ, θ) -BB⁴-property for some $A \in [\kappa]^\kappa$.

Proof. 1) The proof will be similar to that of part (2), but using $C_{\delta,i}$ instead of $C_{\delta,i}^\bullet$.

2) The $\Leftarrow$ implication is obvious, so we concentrate on $\Rightarrow$. Let $\mathbf{p}$ be a (λ, κ, χ) -BB-parameter and D be as usual.

- ⊗₁ Toward contradiction, assume that $\mathbf{p}[A]$ fails the continuous (D, Υ, θ) -BB⁴-property for all $A \in [\kappa]^\kappa$.
- ⊗₂ So for $A \in [\kappa]^\kappa$, let

$$\bar{F}_A = \langle F_{\delta,i}^A : \delta \in S, i < \kappa \rangle$$

be a continuous $[\mathbf{p}, \Upsilon, \theta]$ -coloring which witnesses this failure.
(I.e. there is no $\mathbf{p}$ - D - $\bar{F}_A$ -BB³-sequence $\bar{c} \in S^{\times \kappa} \theta$.)

Naturally, we choose

- ⊗₃ (a) $\text{cd} : ([\kappa]^\kappa)\theta \rightarrow \theta$ and $\text{cd}^* : ([\kappa]^\kappa)\Upsilon \rightarrow \Upsilon$, both bijections.
(b) For $B \in [\kappa]^\kappa$, let $\text{cd}_B : \theta \rightarrow \theta$ be defined as in the proof of 2.4:

$$\begin{array}{ccc} ([\kappa]^\kappa)\theta & \xrightarrow{\text{cd}} & \theta \\ & \searrow \pi_B & \downarrow \text{cd}_B \\ & & \theta \end{array}$$

where π_B is the function which sends $\langle \zeta_A : A \in [\kappa]^\kappa \rangle \mapsto \zeta_B$.

- (c) $\text{cd}_B^* : \Upsilon \rightarrow \Upsilon$ will be defined analogously.

Next,

- ⊗₄ Choose $\bar{F} = \langle F_{\delta,i} : \delta \in S, i < \kappa \rangle$ as follows.
 $\text{dom}(F_{\delta,i}) := C_{\delta,i}^\bullet \Upsilon$, and for η in the domain we define
 $F_{\delta,i}(\eta) := \text{cd}(\langle F_{\delta,i}^A(\eta) : A \in [\kappa]^\kappa \rangle)$.

By our assumption (of the BB³-property)

- ⊗₅ There exists a $\mathbf{p}$ - D - $\bar{F}$ -BB⁴-sequence $\bar{c} = \langle c_{\delta,i} : \delta \in S, i < \kappa \rangle \in S^{\times \kappa} \theta$.

Next,

- ⊗₆ For every $A \in [\kappa]^\kappa$, we choose $\bar{c}_A := \langle \text{cd}_A(c_{\delta,i}) : \delta \in S, i < \kappa \rangle$.

If $\bar{c}_A$ is a $\mathbf{p}[A]$ - D - $\bar{F}_A$ -BB³-sequence for some A , then we get our contradiction.

Therefore, assume:

- ⊗₇ For each $A \in [\kappa]^\kappa$ there exist $\eta_A \in {}^\lambda \Upsilon$ and $E_A \in D$ such that
 $(\forall \delta \in S \cap E_A)(\exists i < \kappa)[F_{\delta,i}^A(\eta_A \upharpoonright C_{\delta,i}^\bullet) \neq c_{\delta,i}^A]$.

Now,

- ⊗₈ $E := \bigcap_{A \in [\kappa]^\kappa} E_A \in D$.

[Why? Because we assumed $2^\kappa < \lambda$, and D is μ -complete.³]

Next,

- ⊗₉ Define $\eta \in ([\kappa]^\kappa)\Upsilon$ as the function
 $\alpha \mapsto \text{cd}^*(\langle \eta_A(\alpha) : A \in [\kappa]^\kappa \rangle)$.

³ Actually, it would suffice if D were $(\kappa^\kappa)^+$ -complete.

Now we can finish as in the proof of 2.4.

3)

No part (3).

The point here is that for every continuous $[\mathbf{p}, \Upsilon, \theta]$ -coloring $\bar{F}$, we define a $[\mathbf{p}_A, \Upsilon, \theta]$ -coloring $\bar{F}_A$ as follows.

$\oplus_1$ If $\delta \in A$, $i \in A$, and $\eta \in C_{\delta,i}^\bullet \Upsilon$, we let $\text{cd} : {}^\kappa > \theta \rightarrow \theta$ be a bijection, and then $\text{cd}(\langle F_{\delta,j}(\eta \upharpoonright C_{\delta,j}^\bullet) : j \leq i \rangle) \in \theta$.

The rest is clear.

$\square_{2.11}$

Conclusion 2.12. Assume all clauses of Theorem 2.4(A). Also suppose $2^\kappa \leq \mu$ and $\theta^{2^\kappa} = \theta$.

1) For some $A \in [\kappa]^\kappa$, $\mathbf{p}[A]$ has the $(D, 2^\mu, \theta)$ -BB²-property.

2) If $\mathbf{p}$ is good then for some $A \in [\kappa]^\kappa$, $\mathbf{p}[A]$ has the $(D, 2^\mu, \theta)$ -BB⁴-property.

Proof. 1) By 2.4 and 2.11(1), letting $\Upsilon := 2^\mu$. As $2^\kappa \leq \lambda$, we have $\Upsilon^{2^\kappa} = \Upsilon$.

2) By 2.4, 2.5, and 2.11(2).

$\square_{2.12}$

Claim 2.13. Let $\iota \in \{1, 2\}$, $\lambda := \min\{\partial : 2^\partial > 2^\mu\}$ (so $\lambda > \mu$ is regular), and let $\mathbf{p}$ be a (λ, μ, κ) -BB- ι -parameter with the $(D, 2^\mu, \theta)$ -BB ^{ι} -property.

Then (in 2.1) we can replace $C_{\delta,i}$ and $C_{\delta,i}^\bullet$ by $C_{\delta,i}^\dagger := \bigcup_{\alpha \in C_{\delta,i}^\bullet} C_{\delta,i} \cap \alpha$.

Proof. Let $\bar{F}$ be a $(\mathbf{p}_1, 2^\mu, \theta)$ -coloring (in the current sense).

We choose a bijection $\text{cd}_\beta : 2^\mu \rightarrow {}^\beta(2^\mu)$. We will define a $(\mathbf{p}, 2^\mu, \theta)$ -coloring $\bar{F}' = \langle F'_{\delta,i} : \delta \in S, i < \kappa \rangle$ as follows.

$$F'_{\delta,i}(\eta) := \begin{cases} F_{\delta,i} \cup \bigcup_{\alpha \in C_{\delta,i}^\bullet} \text{cd}_\alpha(\eta \upharpoonright \alpha) & \text{if this union is a well-defined function,} \\ 0 & \text{otherwise.} \end{cases}$$

Now there exists a $(\mathbf{p}, 2, \theta)$ - $\bar{F}'$ -sequence $\bar{c} = \langle c_{\delta,i} : \delta \in S, i < \kappa \rangle$.

We need to show that $\bar{c}$ is as required. Given an $\eta \in {}^\lambda(2^\mu)$, define $\eta' \in {}^\lambda(2^\mu)$ as the function $\alpha \mapsto \text{cd}_\alpha(\eta \upharpoonright \alpha)$. Now check.

$\square_{2.13}$

Conclusion 2.14. In 2.4, we can add the following.

If $\lambda_* = \text{cf}(\lambda_*) \geq \lambda$, then there exists a $\mathbf{p}^*$ such that

- (a) $\mathbf{p}^*$ is a good $(\lambda_*^+, \kappa, \chi)$ -BB_{*}-parameter.
- (b) $\mathbf{p}^*$ has the continuous λ -uniform $(D, 2^\mu, \theta)$ -BB-property.

Proof. By [She91, §4], as in [She05, §2].

$\square_{2.14}$

Remark 2.15. We can say more, replacing λ_*^+ by λ' weakly inaccessible or successor of singular; see [She05, §1].

§ 3. THE DBB PROPERTY

The following result relies on [She13b, 2.2=_{Ld.6}].

Theorem 3.1. *We have ‘(A) $\Rightarrow$ (B)’, where*

- (A) (a) $\lambda = \text{cf}(\lambda) \geq \lambda_* := \min\{\partial : 2^\partial > 2^\mu\}$ (so $\lambda_* > \mu$ is regular) and S is a stationary subset of λ .
- (b) For each $\delta \in S$, let D_δ be a μ^+ -complete filter on λ_* extending the co-bounded filter.
- (c) For each $\delta \in S$ we have $\bar{C}^\delta = \langle C_\gamma^\delta : \gamma < \lambda_* \rangle$, where $C_\gamma^\delta \subseteq \delta$ and $|\bigcup_{\gamma < \lambda_*} C_\gamma^\delta| = \mu$.
- (d) $\theta \in [2, \mu]$
- (e) $\text{Sep}(\mu, \theta, \Upsilon)$ for some $\Upsilon \leq \mu$.
- (B) If $\mathbf{F}_\gamma^\delta : C_\gamma^\delta(2^\mu) \rightarrow \theta$ for $\gamma < \lambda_*$ and $\delta \in S$, then we can find a

$$\bar{c}^\delta = \langle c_\gamma^\delta : \gamma < \lambda_* \rangle \in {}^{\lambda_*}\theta$$

such that for any $\delta \in S$ and $f : \delta \rightarrow 2^\mu$, for D_δ^+ -many $\gamma < \lambda_*$, we have

$$\mathbf{F}_\gamma^\delta(f \upharpoonright C_\gamma^\delta) = c_\gamma^\delta.$$

Proof. For each $\delta \in S$ we apply clause [She13b, 2.2(β)=_{Ld.6}], with $\lambda_*, D_\delta, \bar{C}^\delta$ here standing in for $\lambda, D, \bar{C}$ there. (Pedantically, we can replace μ from there by any set of equal cardinality — e.g. $\bigcup_{\gamma < \lambda} C_\gamma^\delta$.) $\square_{3.1}$

Definition 3.2. Suppose $\lambda = \text{cf}(\lambda) > \mu \geq \kappa = \text{cf}(\kappa)$ and $\mu_* \leq \mu^+$.

1) We say that $\mathbf{p}$ is a $(\lambda, \lambda_*, \mu, \mu_*, \kappa)$ -DBB-parameter⁴ when:

- (A) (a) $\lambda \geq \lambda_* > \kappa$ are regular cardinals.
- (b) $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$, a stationary subset of λ .
- (B) $\mathbf{p}$ consists of $\bar{C}_0 = \langle C_\delta : \delta \in S \rangle$, $\bar{C}_1 = \langle C_\gamma^\delta : \delta \in S, \gamma < \lambda \rangle$, and D such that
 - (a) $\langle C_\delta : \delta \in S \rangle$ is as usual (that is, $C_\delta \subseteq \delta = \sup(C_\delta)$ and $\text{otp}(C_\delta) = \kappa$) but

$$\alpha \in C_\delta \Rightarrow \alpha > \mu \wedge \mu^\omega \mid \alpha.$$
 - (b) $C_\gamma^\delta \subseteq \bigcup_{\alpha \in C_\delta} [\alpha, \alpha + \mu)$ such that

$$|C_\gamma^\delta \cap [\alpha, \alpha + \mu)| = 1$$
 for all $\delta \in S, \gamma < \lambda_*$, and $\alpha \in C_\delta$. (So $\text{otp}(C_\gamma^\delta) = \kappa$.)
 - (c) $\bar{C}_1$ is a μ_* -free sequence.
 By this we mean: if $u \subseteq S \times \lambda_*$ is of cardinality $\leq \mu_*$, then there exists some sequence $\bar{v} = \langle v_\gamma^\delta : (\delta, \gamma) \in u \rangle$ with $v_\gamma^\delta \in [C_\gamma^\delta]^{<\kappa}$ such that

$$\langle C_\gamma^\delta \setminus v_\gamma^\delta : (\delta, \gamma) \in u \rangle$$
 is a sequence of pairwise disjoint sets.
 - (d) D is a λ -complete filter on λ which includes S and the clubs of λ .
 - (e) If $\alpha < \lambda$ then $S_{\geq \alpha} := \{\delta \in S : \min C_\delta \geq \alpha\}$ is stationary.

⁴ DBB stands for *Double Black Box*.

2) We say that $\mathbf{p}$ has the $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$ -DBB-property when in addition to the above, clause 3.1(B) holds.

If we say $\mathbf{p}$ *guesses clubs*, we mean $\bar{C}_{\mathbf{p}}$ does.

3) If $\lambda_* = \lambda$ and/or $\mu_* = \mu^+$ then we may omit them, as in earlier definitions.

Claim 3.3. 1) If $\boxplus$ below holds, then there exists a $\mathbf{p}$ with the $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$ -DBB-property. Furthermore, $\langle C_{\mathbf{p}, \delta} : \delta \in S_{\mathbf{p}} \rangle$ is (λ, κ) -good (see 1.1(5)).

- $\boxplus$ (a) $\kappa = \text{cf}(\mu) < \mu$
- (b) $\lambda = \text{cf}(\lambda) > \lambda_* := \min\{\partial : 2^\partial > 2^\mu\}$
- (c) $\text{pp}_{J_{\kappa}^{\text{bd}}}(\mu) > \lambda$
- (d) $\theta \in [2, \mu]$
- (e) $\text{Sep}(\mu, \theta, \Upsilon)$ for some $\Upsilon \leq \mu$.
- (f) $\mu_* := \mu^+$.

1A) If (1) $\boxplus$ holds and $S \subseteq S_{\kappa}^{\lambda}$, then there exists a $\mathbf{p}$ with the $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$ -DBB-property and $S_{\mathbf{p}} = S$.

2) In part (1), we may replace clause $\boxplus$ (d) by

- (d)' $\bullet_1$ $\text{pp}_J(\mu) > \lambda$ for some ideal $J \supseteq [\kappa]^{<\kappa}$.
- $\bullet_2$ $\mu = \mu^{<\kappa}$

3) Alternatively,

- (d)'' $\bullet_1$ As above.
- $\bullet_2$ $2^{\mu^{<\kappa}} < 2^\mu$.

Proof. 1) First, choose a stationary $S \subseteq S_{\kappa}^{\lambda}$ such that $S \in \check{I}_{\kappa}[\lambda]$ and

$$\delta \in S \Rightarrow \mu^{\omega+1} \mid \delta.$$

Such an S exists because λ and κ are regular with $\lambda > \kappa^+$ (as $\lambda > \mu > \kappa$), and so we can apply [She93, §1].

Next, choose a (λ, κ) -good $\bar{C}$ as in 3.2(1)(B)(a); this is possible by our choice of S . This will take care of the ‘Furthermore.’ (If we replace S by $S \cap E$, where E is a club of λ , we can also demand that $\bar{C}$ guesses clubs.) Third, choose a μ^+ -free sequence

$$\langle \rho_{\gamma} : \gamma < \lambda \rangle \subseteq {}^{\kappa}\mu$$

as in [She94, Ch.II, §3]. Without loss of generality $\gamma < \lambda \Rightarrow \rho_{\gamma}(i) = i \pmod{\kappa}$. Let $\langle \rho_{\delta, \gamma}^* : \delta \in S, \gamma < \lambda \rangle$ list $\langle \rho_{\gamma} : \gamma < \lambda \rangle$ without repetition: we can do this because $|S \times \lambda| = \lambda$.

Let $\langle \beta_i^{\delta} : i < \kappa \rangle$ list C_{δ} in increasing order, and let

$$\rho_{\delta, \gamma} := \langle \beta_i^{\delta} + \rho_{\delta, \gamma}^*(i) : i < \kappa \rangle.$$

Let D be the club filter on λ . So

$$\mathbf{p} := (\lambda, \kappa, S, \langle C_{\delta} : \delta \in S \rangle, \langle C_{\gamma}^{\delta} : \delta \in S, \gamma < \lambda \rangle)$$

is well-defined.

Now we have to check that $\mathbf{p}$ is indeed a $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$ -DBB-parameter: that is, all clauses of 3.2(1).

First, the demands on the cardinals in the beginning of the definition hold, as does clause (A).

Clause 3.2(B)(a): Holds by the choice of $\langle C_\delta : \delta \in S \rangle$.

Clause (B)(b): Holds by our choice of the C_γ^δ -s.

Clause (B)(c): Holds by our choice of D .

Clause (B)(d):

Let $u \in [S \times \lambda]^{<\mu^+}$. By the choice of $\langle \rho_\gamma : \gamma < \lambda \rangle$, we can find a function $h : u \rightarrow \kappa$ such that

(*)₁ If $(\delta_1, \gamma_1) \neq (\delta_2, \gamma_2)$ are from u and $i \geq \max(h(\delta_1, \gamma_1), h(\delta_2, \gamma_2))$, then
 $\rho_{\delta_1, \gamma_1}^*(i) \neq \rho_{\delta_2, \gamma_2}^*(i)$.

Hence

(*)₂ If $(\delta_1, \gamma_1) \neq (\delta_2, \gamma_2)$ are from u and $i_\ell \in [h(\delta_\ell, \gamma_\ell), \kappa)$ for $\ell = 1, 2$, then
 $\rho_{\delta_1, \gamma_1}^*(i_1) \neq \rho_{\delta_2, \gamma_2}^*(i_2)$.

[Why? If $i_1 \neq i_2$ then use $\gamma < \lambda \Rightarrow \rho_\gamma(i) \equiv i \pmod{\kappa}$. The $i_1 = i_2$ case is just (*)₁.]

So clause (B)(d) does indeed hold.

Together we have proved that $\mathbf{p}$ is a $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$ -DBB-parameter, and so it has the $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$ -DBB-property by Theorem 3.1.

1A) Similarly.

2) Similarly, but when choosing $\bar{\rho} = \langle \rho_\gamma : \gamma < \lambda \rangle$ we only require that it is (μ^+, J) -free.

Then we let $\text{cd} : {}^\kappa \lambda \rightarrow \lambda$ be a bijection.

3) Similarly as well. □_{3.3}

Discussion 3.4. Let μ be strong limit singular of cofinality $\kappa < \mu$, and $\lambda = \lambda_* := \min\{\partial : 2^\partial > 2^\mu\}$.

- (A) (a) If $\lambda < 2^\mu$ and $\kappa > \aleph_0$, then 3.3⊞ holds (see [She94]).
- (b) What about $\lambda < 2^\mu$ and $\kappa := \aleph_0$? Still, 3.3⊞ holds in many cases.⁵
- (c) 3.3 would seem to be helpful for constructing (e.g.) μ^+ -free Abelian groups.
- (B) But what about the $\lambda = \lambda_* = 2^\mu$ case? In this case we have $\lambda = \lambda^{<\lambda}$, a condition which is again helpful in constructions. Can we construct an entangled linear order of cardinality λ^+ ? Recall that by [She00], [She06] we have $(D\ell)_\lambda^*$. Can we use several pairwise disjoint subset of λ ?
 Alternatively, find a subset of ${}^\lambda \theta$ for some regular θ (e.g. $\text{cf}(2^{\aleph_0})$)?
- (C) Again, if $\lambda = 2^\mu$ then we may try to use
 $\mathfrak{d} := \{\theta \in \mu \cap \text{Reg} : (\exists \mu' \in (\mu, \lambda)) [\text{cf}(\mu') = \theta \wedge \text{pp}_{\theta\text{-comp}}(\mu') = {}^+ \lambda]\}$
 as in [She13b] whenever $\mathfrak{d} := \{\kappa\}$ does not work. The new proof is as in [She20], using [She13a].

⁵ E.g. for a club of μ , when $\mu = \beth_\delta > \kappa = \text{cf}(\mu) > \aleph_0$.

- (D) However, we can use BB_k in clause (C). We consider $\mu_0 < \dots < \mu_{3n}$ as above (i.e. all strong limit of cofinality $\kappa < \mu_0$). For each ℓ we choose $\mathbf{p}_\ell$ as in 3.3, except that their free-ness (in the sense of [She20]) is such that their “product” is $\aleph_{n,\kappa^+}$ -free, and they have a Black Box as there.

Definition 3.5. 1) For $\Lambda_* \subseteq \Lambda_\bullet \subseteq {}^\kappa\mu$, we say that $\Lambda_\bullet$ is (θ_2, θ_1) -free over⁶ Λ_* when $\theta_2 \geq \theta_1$ and for every $\Lambda \subseteq \Lambda_\bullet \setminus \Lambda_*$ of cardinality $< \theta_2$ there is a *witness* $(\bar{\Lambda}, h)$. By this we mean

- (A) $\bar{\Lambda} = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ is a partition of Λ into γ_* -many sets, each of cardinality $< \theta_1$ (so γ_* is an ordinal $< \theta_2$).
 (B) $h : \Lambda \rightarrow \kappa$.
 (C) If $\gamma < \gamma_*$, $\eta \in \Lambda_\gamma$, and $i \in [h(\eta), \kappa)$, then

$$\eta(i) \notin \{\rho(j) : j < \kappa, \rho \in \bigcup_{\beta < \gamma} \Lambda_\beta \cup \Lambda_*\}.$$

2) For $\Omega \subseteq \{(\theta_2, \theta_1) \in \text{Card} \times \text{Card} : \theta_2 \geq \theta_1\}$, we say $\Lambda_\bullet$ is Ω -free over Λ_* when it is (θ_2, θ_1) -free over Λ_* for every $(\theta_2, \theta_1) \in \Omega$.

Observation 3.6. Assume (for transparency) that $\Lambda_\bullet \subseteq {}^{\kappa>}\mu$ is tree-like. (That is, $\eta \neq \nu \in \Lambda_\bullet \wedge \eta(i) = \nu(j) \Rightarrow i = j \wedge \eta \restriction i = \nu \restriction i$.)

If $\Lambda_\bullet$ is of cardinality $< \theta$ and (θ, κ^+) -free over $\emptyset$, then $\Lambda_\bullet$ is free.

Proof. See [She20, §1]. □_{3.6}

Claim 3.7. 1) If $\boxplus$ holds then there exists a $\mathbf{p}$ with the $(\lambda, \mu, \Omega, \theta, \kappa)$ -DBB-property, where

- $\boxplus$ (a) $\kappa = \text{cf}(\mu) < \mu$
 (b) $\lambda = \lambda^{<\lambda} = 2^\mu$
 (c) $\text{pp}_{J_{\kappa}^{\text{bd}}}^+(\mu) > \lambda$
 (d) $\theta \in [2, \mu]$
 (e) $\text{Sep}(\mu, \theta, \Upsilon)$ for some $\Upsilon \leq \mu$.
 (f) $\Omega := \{(\kappa^{+\kappa}, \kappa^{+4})\}$

2) Like part (1), but replacing clause $\boxplus(f)$ by

$$(f)' \quad \Omega := \{(\theta^{+\kappa}, \theta^{+4}) : \theta \in [\kappa, \mu]\}$$

3) In parts (1) and (2), we may replace clause $\boxplus(c)$ by

- (c)' $\bullet_1$ $\text{pp}_J(\mu) \geq \lambda$ for some ideal $J \supseteq [\kappa]^{<\kappa}$.
 $\bullet_2$ $\mu = \mu^{<\kappa}$

as in 3.3(2).

4) If S is a stationary subset of $\{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ then we can demand $S_{\mathbf{p}} := S$, and we can add “ $\langle C_{\mathbf{p},\delta} : \delta \in S \rangle$ guesses clubs.” If $S \in \check{I}_\kappa[\lambda]$ then we can add “ $\langle C_{\mathbf{p},\delta} : \delta \in S \rangle$ is good.”

5) In part (2), we can replace $\mu = \mu^{<\kappa}$ by $2^{\mu^{<\kappa}} < 2^\lambda$.

⁶ We may omit λ_* if it is empty.

Remark 3.8. The $\lambda = \lambda^{<\lambda}$ is not necessary; just otherwise 3.3 gives us more.

Proof. 1) Like the proof of 3.3, but in the choice of $\bar{\rho}$ (at the beginning of the proof) we replace ‘ μ^+ -free’ by ‘ Ω -free.’

[Why is this possible? Use θ in the beginning of the proof of [She20, 1.26=_{La}51] (which relies on [She13a, 0.4-0.6=_{Ly}19,_y22,_y40]).]

2) As above.

3) Similarly to the proof of 2.4(2).

4-5) Clear. □_{3.7}

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