MAXIMAL FAILURES OF SEQUENCE LOCALITY IN AEC SH932

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Dedicated to the memory of Andrzej Mostowski

ABSTRACT. We are interested in examples of AECs $\mathfrak{k}$ with amalgamation having some (extreme) behavior concerning types. Note, we deal with $\mathfrak{k}$ being sequence-local, i.e. local for increasing chains of length a regular cardinal (for types, equality of all restrictions implies equality, some call it *tame*). For any cardinal $\theta \geq \aleph_0$ we construct an AEC $\mathfrak{k}$ with amalgamation and $\mathrm{LST}(\mathfrak{k}) = \theta, |\tau_{\mathfrak{K}}| = \theta$ such that $\{\kappa : \kappa$ is a regular cardinal and $\mathfrak{K}$ is not $(2^\kappa, \kappa)$-sequence-local$\}$ is maximal. In fact, we have a direct characterization of this class of cardinals: the regular κ such that there is no uniform κ^+-complete ultrafilter (on any $\lambda > \kappa$). We also prove a similar result to "$(2^\kappa, \kappa)$-compact for types".

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0. Introduction

Recall AECs (abstract elementary classes); were introduced in [She87a]; and their (orbital) types defined in [She87b], see on them [She09b], [Bal09]. It has seemed to me obvious that even with $\mathfrak{k}$ having amalgamation, those types in general lack some good properties of the classical types in model theory. E.g. "(λ, κ)-sequence-locality where

Definition 0.1. 1) We say that an AEC $\mathfrak{k}$ is a (λ, κ)-sequence-local (for types) when κ is regular and for every $\leq_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i \leq \kappa \rangle$ of models of cardinality λ and $p, q \in \mathscr{S}(M_\kappa)$ we have $(\forall i < \kappa)(p \restriction M_i = q \restriction M_i) \Rightarrow p = q$. We omit λ when we omit "$\|M_i\| = \lambda$".
2) We say an AEC $\mathfrak{k}$ is (λ, κ)-local when: $\kappa \geq$ LST$(\mathfrak{k})$ and if $M \in \mathfrak{k}_\lambda$ and $p_1, p_2 \in \mathscr{S}(M)$ and $N \leq_{\mathfrak{k}} M \wedge \|N\| \leq \kappa \Rightarrow p_1 \restriction N = p_2 \restriction N$ then $p_1 = p_2$.
3) We may replace λ by $\leq \lambda, < \lambda, [\mu, \lambda]$ with the obvious meaning (and allow λ to be infinity).

Of course, being sure is not a substitute for a proof; some examples were provided by Baldwin-Shelah [BS08, §2]. Also note our using: Abelian groups without zero is similar to e.g., the work [HS90]. There we give an example of the failure of (λ, κ)-sequence-locality for $\mathfrak{k}$-types in ZFC for some λ, κ, actually $\kappa = \aleph_0$. This was done by translating our problems to abelian group problems. While those problems seem reasonable by themselves, they may hide our real problem.

Here in §1 we get $\mathfrak{k}$, an AEC with amalgamation with the class $\{\kappa : (< \infty, \kappa)$-sequence-localness fail for $\mathfrak{k}\}$ being maximal; what seems to me a major missing point up to it, see Theorem 1.4. Also we deal with "compactness of types", getting unsatisfactory results - classes without amalgamation; in [BS08] this was done only in some universes of set theory, but with amalgamation; see §2.

We rely on [BS08] to get that $\mathfrak{k}$ has the JEP and amalgamation.

Question 0.2. Can $\{\kappa : \mathfrak{k}$ is $(< \kappa^+, \kappa)$-local$\}$ be "wild"? E.g. can it be all odd regular alephs? etc?

Similarly for $(< \infty, \kappa)$- sequentially local.

Note that for this, the present translation theorem of [BS08] is not suitable.

In §2 we deal with sequence-compactness of types.

Mostowski [Mos57] initiated the quest to find strengthenings of first-order logic that still have a "good model theory". Usually, one may add generalized quantifiers (e.g., $(\exists^{\geq \aleph_1} x)$) and/or allow certain infinitary operations (e.g., $\bigwedge_{\alpha < \lambda} \psi_\alpha$). There is much to be said on this topic; see the collection [Bar85] and, later, Väänänen's book [VÏ1].

In particular, Lindström proved that one cannot expect too much: either the downward Löwenheim–Skolem property to $\aleph_0$ fails, or $\aleph_0$-compactness fails.

Now, *abstract elementary classes* (AECs) continue this, trying to deal directly at the model theory. E.g. concerining "a theory T in logic $\mathbb{L}(\exists^{\geq \aleph_1})$", we define the AEC $\mathfrak{k} = \mathfrak{k}_T$, by:

- $K_{\mathfrak{k}}$ is the model of T,
- $M \leq_{\mathfrak{k}} N$ iff in addition to $M \prec_{\mathbb{L}(\exists^{\aleph_1})} N$, which is naturally defined, we demand that, if $M \models (\exists^{\aleph_1} x)\psi(x, \bar{a})$ then not only $N \models (\exists^{\leq \aleph_0})\varphi(x, \bar{a})$ but $N \models \varphi[b, a] \Rightarrow b \in M$.

Similarly, for e.g. the logic $\mathbb{L}_{\lambda^+, \aleph_0}$.

This work is part of the attempt to sort out which properties of first-order logic hold for AECs, particularly when $\mathfrak{k}$ is an AEC with amalgamation. In a subsequent paper [S+], we note that this is the case for the $\mathfrak{k}$ from §2. This work was submitted to Jouko Väänänen in October 2009 for a volume in honor of Andrzej Mostowski.

Later and independently, Boney [Bon14] investigated such things mainly for compact cardinals, in particular, has results close to 1.8 (and 2.8).

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It is my pleasure to dedicate this to the memory of Andrzej Mostowski, who contributed so much to mathematical logic and particularly to starting other logics and generalized quantifiers in [Mos57].

1. An AEC with maximal failure of being local

Claim 1.1. *Assume*

$\circledast_1$ (a) $\kappa = \mathrm{cf}(\kappa) > \theta \geq \aleph_0$ *and* $\sigma = \theta^{\aleph_0}$

(b) *there is no uniform* θ^+*-complete ultra-filter* D *on* κ

(c) τ_θ *is the vocabulary*

$$\{E_n, E'_n : n \in [2,\omega)\} \cup \{F_c : c \in [\sigma]^{<\aleph_0}\} \cup \{R_e : e \in {}^\sigma\sigma\} \cup \{R_{n,i} : i < \sigma, n < \omega\},$$

where R_e *are two-place predicates,* F_c *is an unary function symbol,* $R_{n,i}$ *is an* n*-place predicate and* E'_n, E_n *are* $(2n)$*-place predicates,*

(d) τ^*_θ *is* $\tau_\theta \setminus \{R_{n,i} : i < \sigma, n < \omega\}$.

<u>Then</u>

$\boxplus$ *there are* $I_\alpha, \pi_\alpha, M_{\ell,\alpha}$ *(for* $\ell = 1,2$ *and* $\alpha \leq \kappa$*), and* g_α *(for* $\alpha < \kappa$*) satisfying:*

(a) I_α*, a set of cardinality* θ^κ*, is* $\subseteq$*-increasing continuous with* α

(b) $M_{\ell,\alpha}$*, a* τ_θ*-model of cardinality* $\leq \theta^\kappa$*, is increasing continuous with* α

(c) $\pi_{\ell,\alpha}$ *is a function from* $M_{\ell,\alpha}$ *onto* I_α*, increasing continuous with* α

(d) $|\pi^{-1}_{\ell,\alpha}\{t\}| \leq \theta^{\aleph_0}$ *for* $t \in I_\alpha, \alpha \leq \kappa$ *and* $\ell = 1,2$

(e) *if* $t \in I_{\alpha+1} \backslash I_\alpha$ *then* $\pi^{-1}_{\ell,\alpha}\{t\} \subseteq M_{\ell,\alpha+1} \backslash M_{\ell,\alpha}$

(f) *for* $\alpha < \kappa, g_\alpha$ *is an isomorphism from* $M_{1,\alpha}$ *onto* $M_{2,\alpha}$ *respecting* π_α *which means* $a \in M_{1,\alpha} \Rightarrow \pi_{1,\alpha}(a) = \pi_{2,\alpha}(g_\alpha(a))$

(g) *for* $\alpha = \kappa$ *there is no isomorphism from* $M_{1,\alpha}$ *onto* $M_{2,\alpha}$ *respecting* π_α.

Proof. Follows from 1.2 which is just a fuller version adding to τ_θ unary function F_c for $c \in G$; anyhow we shall use 1.2. $\square_{1.1}$

Claim 1.2. *Assuming* $\circledast_1$ *of 1.1 we have:*

$\boxplus$ *there are* $I_\alpha, A_\alpha, \pi_\alpha, M_{\ell,\alpha}$ *(for* $\ell = 1,2, \alpha \leq \kappa$*) and* g_α *(for* $\alpha < \kappa$*) and* G *such that:*

(a) G *is an additive (so abelian) group of cardinality* $\theta^{\aleph_0}$

(b) I_α *is a set, increasing continuous with* $\alpha, |I_\alpha| = \theta^\kappa$

(c) $M_{\ell,\alpha}$ *is a* τ_θ*-model, increasing continuous with* α*, of cardinality* θ^κ *with universe* A_α*,*

(d) π_α *is a function from* $M_{\ell,\alpha}$ *onto* I_α *increasing continuous with* α

(e) $F_c^{M_{\ell,\alpha}}(c \in G)$ *is a permutation of* $M_{\ell,\alpha}$, *increasing continuous with* α

(f) $\pi_\alpha(a) = \pi_\alpha(F_c^{M_{\ell,\alpha}}(a))$

(g) $F_{c_1}^{M_{\ell,\alpha}}(F_{c_2}^{M_{\ell,\alpha}}(a)) = F_{c_1+c_1}^{M_{\ell,\alpha}}(a)$

(h) $\pi_\alpha(a) = \pi_\alpha(b) \Leftrightarrow \bigvee\limits_{c \in G} F_c^{M_{\ell,\alpha}}(a) = b$

(i) *for* $\alpha < \kappa, g_\alpha$ *is an isomorphism from* $M_{1,\alpha}$ *onto* $M_{2,\alpha}$ *which respects* π_α *which means* $a \in M_{1,\alpha} \Rightarrow \pi_\alpha(a) = \pi_\alpha(f_\alpha(a))$

(j) *there is no isomorphism from* $M_{1,\kappa} \restriction \tau_\theta$ *onto* $M_{2,\kappa} \restriction \tau_\theta$ *respecting* π_κ.

(k) $M_{1,\alpha} \restriction \tau_\theta^* = M_{2,\alpha}^* \restriction \tau_\theta$, *for* $\alpha \leq \kappa$.

Discussion 1.3. We try to shed some light on the proof of 1.2 on how we shall to use it, see Claim 1.8. The models $M_{1,\alpha}$, $M_{2,\alpha}$ are almost the same.

Proof. Let

$(*)_0$ $\sigma = \theta^{\aleph_0}$ so $\sigma = \sigma^{\aleph_0}$

$(*)_1$ (a) let $G = ([\sigma]^{<\aleph_0}, \Delta)$, i.e., the family of finite subsets of σ with the operation of symmetric difference. This is an abelian group satisfying $\forall x(x+x=0)$, but we may identify $\varepsilon < \sigma$ with $\{\varepsilon\}$, so treating ordinals as above

(b) let $\langle a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < \kappa, u \in G\rangle$ be a sequence without repetitions

(c) for $\beta \leq \kappa$ let $A_\beta = \{a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < 1+\beta$ and $u \in G\}$

(d) for $\beta \leq \kappa$ let $I_\beta = ({}^\kappa\sigma) \times (1+\beta)$

(e) π_β be the function with domain A_β such that, we let $\pi_\beta(a_{f,\alpha,u}) = (f,\alpha)$ when $\alpha < 1+\beta \leq \kappa$

(f) for each $\beta < \kappa$ we define a permutation g_β (of order 2) of A_β by $g_\beta(a_{f,\alpha,u}) = a_{f,\alpha,u+_G\{f(\beta)\}}$ hence $a \in A_\beta \Rightarrow \pi_\beta(g_\beta(a)) = \pi_\beta(a)$.

Note that

$(*)_2$ (a) $|G| = \sigma$

(b) $\langle A_\beta : \beta \leq \kappa\rangle$ is a $\subseteq$-increasing continuous, each A_β a set of cardinality $\sigma^\kappa = \theta^\kappa$

(c) $\langle I_\beta : \beta \leq \kappa\rangle$ is $\subseteq$-increasing continuous, each I_β of cardinality $\sigma^\kappa = \theta^\kappa$

(d) π_β is a mapping from A_β onto I_β

(e) if $t \in I_\alpha \subseteq I_\beta$ then $\pi_\beta^{-1}\{t\} = \pi_\alpha^{-1}\{t\}$ has cardinality $|G| = \sigma$

(f) if $t \in I_{\alpha+1} \backslash I_\alpha$ then $\pi_{\alpha+1}^{-1}\{t\} \subseteq A_{\alpha+1} \backslash A_\alpha$

(g) if $\alpha \leq \beta \leq \kappa$ then g_β maps A_α onto itself and $g_\beta \circ g_\beta$ is the identity.

For each $n \in [2,\omega)$ and $\beta \leq \kappa$ we define equivalence relations $E'_{n,\beta}, E_{n,\beta}$ on ${}^n(A_\beta)$:

$(*)_3$ $\bar{a} E'_{n,\beta} \bar{b}$ iff $\pi_\beta(\bar{a}) = \pi_\beta(\bar{b})$ where of course $\pi_\beta(\langle a_\ell : \ell < n\rangle) = \langle \pi_\beta(a_\ell) : \ell < n\rangle$

$(*)_4$ $\bar{a} E_{n,\beta} \bar{b}$ <u>iff</u> $\bar{a} E'_{n,\beta} \bar{b}$ and there are $k < \omega$ and $\bar{a}_0, \ldots, \bar{a}_k$ such that

(i) $\bar{a}_\ell \in {}^n(A_\beta)$
(ii) $\bar{a} = \bar{a}_0$
(iii) $\bar{b} = \bar{a}_k$
(iv) for each $\ell < k$ for some $\alpha_1, \alpha_2 < \kappa$ we have $g^{-1}_{\alpha_2}(g_{\alpha_1}(\bar{a}_\ell))$ is well defined and equal to $\bar{a}_{\ell+1}$ <u>or</u> $g_{\alpha_2}(g^{-1}_{\alpha_1}(\bar{a}_\ell))$ is well defined and equal to $\bar{a}_{\ell+1}$.

Note:

$(*)_{4.1}$ (a) the two possibilities in $(*)_4(iv)$ are one as $g^{-1}_\alpha = g_\alpha$ so the first one is equal to the second;
(b) g_α <u>does not preserve</u> $\bar{a}/E_{n,\beta}$!, in fact, $a, g_\alpha(a)$ are never $E_{n,\beta}$ equivalent;
(c) clearly in $(*)_4$(iv) for $\ell < k$, the terms are well defined <u>iff</u> $\bar{a}_\ell \in {}^n(A_{\min\{\alpha_1,\alpha_2\}})$ because if $\alpha \leq \beta$ then g_β maps A_β onto itself,
(d) if $\alpha \leq \beta, a \in A_\alpha$, then g_β maps $a/E_{n,\beta}$ onto itself
(e) if $\alpha, \beta \leq \kappa$, <u>then</u> g_α, g_β commute (on the intersection of their domains, $A_{\min\{\alpha,\beta\}}$.

[Why? Easy, e.g.

<u>Clause b</u>: Why? Let $a = a_{f_1,\gamma_1,u_1}$, $b = a_{f_2,\gamma_2,u_2}$. Now, on the one hand, of $g_\alpha(a) = b$ then $f_1 = f_2$, $\gamma_1 = \gamma_2$ and $u_1 +_G u_2 = u_1 \triangle u_2$ has cardinality 1. On the other hand, if $\langle a \rangle E_{n,\beta} \langle b \rangle$ then (by induction on k in the definition), we can prove $f_1 = f_2$, $\gamma_1 = \gamma_2$ and $u_1 +_G u_2 = u_1 \triangle u_2$ is a set of even cardinality.

<u>Clause (e)</u>: Just recalling that G is a commutative group.]

<u>Note</u>

$(*)_5$ For $n \in [2, \omega)$, we have:
(a) $E'_{n,\beta}, E_{n,\beta}$ are indeed equivalence relations on ${}^n(A_\beta)$
(b) $E_{n,\beta}$ refine $E'_{n,\beta}$
(c) if $\bar{a} \in {}^n(A_\beta)$ then $\bar{a}/E'_{n,\beta}$ has exactly σ members
(d) if $\alpha < \beta \leq \kappa$ <u>then</u> $E'_{n,\beta} \restriction {}^n(A_\alpha) = E'_{n,\alpha}$ and $E_{n,\beta} \restriction {}^n(A_\alpha) = E_{n,\alpha}$ (read $(*)_4(iv)$ carefully!)
(e) if $\alpha < \beta \leq \kappa, \bar{a} \in {}^n(A_\alpha)$ and $\bar{b} \in \bar{a}/E'_{n,\beta}$ <u>then</u> $\bar{b} \in {}^n(A_\alpha)$
(f) if $g_\alpha(\bar{a}_\ell) = \bar{b}_\ell$ for $\ell = 1, 2$ then: $\bar{a}_1 E'_{n,\beta} \bar{a}_2$ <u>iff</u> $\bar{b}_1 E'_{n,\beta} \bar{b}_2$.

Now we choose a vocabulary τ^*_θ of cardinality 2^σ and for $\alpha \leq \kappa$ we choose a τ^*_θ-model $M_{1,\alpha}$ such that:

$(*)_6$ (a) $M_{1,\alpha}$ increasing with α and has universe A_α
(b) assume that $\bar{a}, \bar{b}$ are $E'_{n,\alpha}$-equivalent (so $\bar{a}, \bar{b} \in {}^n(A_\alpha)$ and $\pi_\alpha(\bar{a}) = \pi_\alpha(\bar{b})$); <u>then</u> $\bar{a}, \bar{b}$ realize the same quantifier-free type in $M_{1,\alpha}$ iff $\bar{a} E_{n,\alpha} \bar{b}$
(c) τ_θ is defined in $\circledast_1$(c) from 1.1
(d) for every function $e \in {}^\sigma\sigma$

$$R^{M_{1,\alpha}}_e = \{(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in A_\alpha \times A_\alpha : f_1 = e \circ f_2 \text{ and if } i < \sigma \text{ then } i \in u_1 \text{ iff } (|\{j \in u_2 : e(j) = i\}| \text{ is odd})\}$$

recalling $f_\ell \in {}^\kappa\sigma$

(e) $E_n^{M_{1,\alpha}} = E_{n,\alpha}$ and $F_c^{M_{1,\alpha}} = F_c$ is defined by $F_c : A_\alpha \to A_\alpha$ satisfies $F_c(a_{f,\alpha,u}) := a_{f,\alpha,u+_G c}$

(f) if $\alpha \le \beta_\ell < \kappa$ for $\ell = 1,2$ then $g_{\beta_2}^{-1} g_{\beta_1} \restriction A_\alpha$ is an automorphism of $M_{1,\alpha}$

(g) g_α is almost an automorphism of $M_{1,\alpha}$, it miss preserving $R_{n,i}$

[Why is this possible? For Clauses (a)-(e), just define the $M_{1,\alpha}$-s, except concerning being increasing with α, see (a) and the choice of the $R_{n,i}$ which should guarantee (b). Also, we have to prove clauses (f) and (g). Of course, every g_α (hence $g_{\beta_2}^{-1} \circ g_{\beta_1}$) is a permutation of A_α, the universe of $M_{1,\alpha}$. Of course, every g_α (hence $g_\beta^{-1} \circ g_{\beta_1}$) is a permutation of A_α, the universe of $M_{1,\alpha}$. First, we shall show that for each $\alpha < \kappa, g_\alpha$ maps $R_e^{M_{1,\alpha}}$ onto itself.

Assume we are given a pair $(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2})$ from $A_\alpha \times A_\alpha$ so $\beta_1,\beta_2 < 1+\alpha$ and $f_1 = e \circ f_2$ so

$(*)_{6.1}$ $(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in R_e^{M_{1,\alpha}}$ iff $u_1 = \{e(j) : j \in u_2$ and $(\exists^{\rm odd} \iota \in u_2)(e(\iota) = e(j))$.

[Why? Read $(*)_6$(d) carefully, in particular note that if $i \notin \{e(j) : j \in u_2\}$ then $i \notin u_1$.]

$(*)_{6.2}$ $(g_\alpha(a_{f_1,\beta_1,u_1}), g_\alpha(a_{f_2,\beta_2,u_2})) \in R_e^{M_{1,\alpha}}$ iff
$(a_{f_1,\beta_1,u_1+\{f_1(\alpha)\}}, a_{f_2,\beta_2,u_2+_G\{f_2(\alpha)\}}) \in R_e^{M_{1,\alpha}}$ iff
$u_1 +_G \{f_1(\alpha)\} = \{e(j) : j \in u_2 +_G \{f_2(\alpha)\}$ and $(\exists^{\rm odd} \iota \in (u_2 +_G f_2(\alpha))(e(\iota) = e(j))\}$.

[Why? Inside $(*)_{6.2}$ the first "iff" holds by the definition of g_α, the second "iff" holds as in $(*)_{6.1}$.]

But $f_1 = e \circ f_2$ hence

$(*)_{6.3}$ $f_1(\alpha) = e(f_2(\alpha))$

$(*)_{6.4}$ letting $x = f_2(\alpha) < \sigma$ we have $u_1 = \{e(j) : j \in u_2$ and $(\exists^{\rm odd} \iota \in u_2)(e(\iota) = e(j))\}$ <u>iff</u> $u_1 +_G \{e(x)\} = \{e(j) : j \in u_2 +_G \{x\}$ and $\exists^{\rm odd} \iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}$.

[Why? Check by cases according to whether $x \in u_2$ and whether $e(x) \in u_1$. I.e., by "G is of order two," it suffices to prove the "only if" so assume the first equality in $(*)_{6.2}$. If $e(x) \notin u_1$, then just add $e(x)$ to both sides. Similarly if $e(x) \in u_1 \cap X \notin u_2$ and if $e(x) \in u_1 \wedge x \in u_2$.]

So together we get equivalence, so the "first" holds.

Second, for defining the $R_{n,i}^{M_{1,\alpha}}$'s

$(*)_{6.5}$ (a) for each $n \in [2,\omega)$ let E_n'' be the following equivalence relation on ${}^nG : \bar u_1 E_n'' \bar u_2$ <u>iff</u> for some $v \in G, |v|$ is even and $\bigwedge\limits_{\ell<n} u_{1,\ell} +_G v = u_{2,\ell}$

(b) let $\langle \mathcal{X}_{n,i} : i < \sigma \rangle$ list the E_n''-equivalence classes

(c) let $R_{n,i}^{M_{1,\alpha}} = \{\bar a \in {}^n(A_\alpha)$: if $\bar a = \langle a_{f_\ell,\alpha_\ell,u_\ell} : \ell < n \rangle$, <u>then</u> $\langle u_\ell : \ell < n \rangle \in \mathcal{X}_{n,i}\}$.

This completed the choice of $M_{1,\alpha}$. Third, g_α preserves "$\bar a,\bar b$ are $E_{n,\alpha}$-equivalent", "$\bar a,\bar b$ are $E'_{n,\alpha}$-equivalent" and their negations. That is, $\bar a, g_\alpha(\bar a)$ are not $E_{n,\alpha}$-equivalent, but as $(\forall\beta)(g_\beta = g_\beta^{-1}), \bar a, \bar b$ being $E_{n,\alpha}$-equivalent means that there is an even length pass from $\bar a$ to $\bar b$, in the graph $\{(\bar c, g_\beta(\bar c)) : \beta \in [\gamma,\kappa) \text{ and } \bar c \in {}^n(A_\gamma)\}$ where $\gamma = \min\{\gamma : \bar a, \bar b \in {}^n(A_\gamma)\}$. This proves part of clause (g) of $(*)_6$.

Fourth, no problem in the $M_{1,\alpha}$'s are increasing by $(*)_5(d)$, just check that.

Fifth, g_α commutes with $F_c^{M_{1,\alpha}}$ for $c \in G$ because G is an Abelian group; thus completing the proof of $(*)_6$(g).

Sixth, we should check clause $(*)_6(f)$. Now $g_{\beta_2}^{-1} g_{\beta_1}\restriction A_\alpha = (g_{\beta_2}\restriction A_\alpha)(g_{\beta_1}\restriction A_\alpha)$ by $(*)_2(g)$ and it has order 2 because G is of order 2 and it maps $E_n^{M_{1,\alpha}}$ to itself by the "third", commute with $F_c^{M_{1,\alpha}}$ by the fifth, maps $R_e^{M_{1,\alpha}}$ to itself by the "first".

Lastly, it maps $R^{M_{1,\alpha}}$ to itself by $(*)_{6.4}$. So we are done proving $(*)_6$.

- $(*)_7$ for $\alpha < \kappa$ let $M_{2,\alpha}$ be the τ^*_θ-model with universe A_α such that g_α is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$.

Now we note

- $(*)_8$ for $\alpha < \beta < \kappa, M_{2,\alpha} \subseteq M_{2,\beta}$.

[Why? By the definitions of $M_{1,\gamma}, g_\gamma, E'_{n,\gamma}, E_{n,\gamma}$, in particular, the "first" and "third", in "why $(*)_6$", fourth, i.e. $(*)_5(d)$.]

- $(*)_9$ let $M_{2,\kappa} := \cup\{M_{2,\alpha} : \alpha < \kappa\}$,
- $(*)_{10}$ $M_{2,\kappa}$ well defined by $(*)_8$,
- $(*)_{11}$ π_α is well defined by $(*)_1$(f),
- $(*)_{12}$ except clause (j) the demands in the conclusion of $\boxplus$ of 1.2 were proved.

[Why? Just check.]

Note

- $(*)_{13}$ if $(a_{f,\alpha,u_1}, a_{f,\alpha,u_2})$ is $E_{2,\alpha}$-equivalent to $(a_{f,\alpha,v_1}, a_{f,\alpha,v_2})$ then $G \models$ "$u_1 - u_2 = v_1 - v_2$".

[Why? By induction on the k from $(*)_4$.]

So, to finish, we assume toward contradiction

- $\boxtimes$ h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ which respects π_α for $\alpha < \kappa$, i.e. $h\restriction M_{1,\alpha}$ respect π_α, see clause $\boxplus$(i) of Claim 1.2.

So trivially

- $\circledast_1$ if $\alpha < \kappa$, then $h(a_{f,\gamma,u}) \in \{a_{f,\gamma,v} : v \in G\}$ for $\gamma < 1 + \alpha$, and $\bar a \in {}^n(A_\alpha) \Rightarrow h(\bar a) \in \bar a/E_{n,\alpha} \subseteq \bar a/E'_{n,\alpha}$.

[Why? As $h\restriction M_{1,\alpha}$ respect π_α see $(*)_1(e)$ and $(*)_{10}$ clearly $h(\bar a) \in \bar a/E'_{n,\alpha}$. But h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ hence by $(*)_6(b)$ we have $h(\bar a) \in (\bar a/E_{n,\alpha})$.]

- $\circledast_2$ for $f \in {}^\kappa\sigma$ and $\alpha < \kappa$ let $u_{f,\alpha} \in G$ be the $u \in G$ such that $h(a_{f,\alpha,\emptyset}) = a_{f,\alpha,u}$

- $\circledast_3$ for $f \in {}^\kappa\sigma, \alpha < \kappa$ and $v \in G$ we have $h(a_{f,\alpha,v}) = a_{f,\alpha,v+_G u_{f,\alpha}}$.

[Why? By $\circledast_1$ clearly h maps any finite sequence $\bar{b} \in {}^n(A_{1,\kappa})$ to an $E_{n,\alpha}$-equivalent sequence for each $\alpha < \kappa$. Now apply this to the pair $(a_{f,\alpha,\emptyset}, a_{f,\alpha,u})$ recalling $(*)_{13}$. Alternatively use the $F_c^{M_{1,\kappa}}$-s.]

$\circledast_4$ we define a partial order $\le$ on ${}^\kappa\sigma$ as follows:
$f_1 \le f_2$ <u>iff</u> there is a function $e \in {}^\sigma\sigma$ witnessing it; which means $f_1 = e \circ f_2$

$\circledast_5$ if $\alpha_1, \alpha_2 < \kappa$ and $f_1 \le f_2$ (are from ${}^\kappa\sigma$) then $|u_{f_1,\alpha_1}| \le |u_{f_2,\alpha_2}|$.

[Why? This follows from $\circledast_6$ below.]

$\circledast_6$ if $e \in {}^\sigma\sigma$, $f_2 \in {}^\kappa\theta$ and $f_1 = e \circ f_2 \in {}^\kappa\sigma$ and $\alpha_1, \alpha_2 < \kappa$ then
$u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$.

[Why? Choose $\alpha < \kappa$ such that $\alpha > \alpha_1, \alpha > \alpha_2$ so $a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_1,\emptyset} \in M_{\ell,\alpha}$ for $\ell = 1,2$. Recall that h maps $R_e^{M_{1,\alpha}}$ onto $R_e^{M_{2,\alpha}}$ by $\boxtimes$ and $R_e^{M_{2,\alpha}} = R_e^{M_{1,\alpha}}$ because g_α maps $R_e^{M_{1,\alpha}}$ onto itself (see the proof of $(*)_6$ above, the "first" in that proof). Now see $(*)_6(d)$, i.e. the definition of $R_e^{M_{1,\alpha}}$, i.e. obviously $(a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_2,\emptyset}) \in R_e^{M_{1,\alpha}}$ so as h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ we have $(h(a_{f_1,\alpha_1,\emptyset}), h(a_{f_2,\alpha_2,\emptyset})) \in R_e^{M_{2,\alpha}}$ so by the previous sentence and the definitions of $u_{f_\ell,\alpha_\ell}(\ell = 1,2)$ in $\circledast_2$ we have $(a_{f_1,\alpha_1,u_{f_1,\alpha_1}}, a_{f_2,\alpha_2,u_{f_2,\alpha_2}}) \in R_e^{M_{1,\alpha}}$ which by the definitions of $R_e^{M_{1,\alpha}}$ in $(*)_6(d)$ implies $u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$ as promised.]

$\circledast_7$ (a) $|u_{f,\alpha_1}| = |u_{f,\alpha_2}|$ for $\alpha_1, \alpha_2 < \kappa, f \in {}^\kappa\sigma$
(b) $\mathbf{n}(f) = |u_{f,\alpha}|$ is well defined
(c) if $f_1 \le f_2$ then $\mathbf{n}(f_1) \le \mathbf{n}(f_2)$.

[Why? For clause (a) use $\circledast_6$ twice for the function $e = \mathrm{id}_\sigma$ and $f_1 = f_2 = f$. Clause (b) follows. Clause (c) holds by $\circledast_6$ equivalently by $\circledast_5$.]

$\circledast_8$ there are $f_* \in {}^\kappa\sigma$ and $\alpha_* < \kappa$ such that:
(i) if $f_* \le f \in {}^\kappa\sigma$ and $\alpha < \kappa$ then $|u_{f_*,\alpha_*}| = |u_{f,\alpha}|$
(ii) moreover if $f_* = e \circ f$ where $e \in {}^\sigma\sigma$ and $f \in {}^\kappa\sigma, \alpha < \kappa$ <u>then</u> $e \restriction u_{f,\alpha}$ is one-to-one from $u_{f,\alpha}$ onto $u_{f_*,\alpha}$ so $\mathbf{n}(f_*) = \mathbf{n}(f)$
(iii) if $\alpha < \kappa, f_1 = e \circ f_2, f_* = e_1 \circ f_1, f_* = e_2 \circ f_2$ so $e, e_1, e_2 \in {}^\sigma\sigma$, <u>then</u> $e \restriction u_{f_2,\alpha}$ is one-to-one onto $u_{f_1,\alpha}$.

[Why? First note that clause (ii), (iii) follows from clause (i). Second, if clause (i) fails, then we can find a sequence $\langle (f_n, \alpha_n, e_n) : n < \omega \rangle$ such that

(α) $\alpha_n < \kappa, f_n \in {}^\kappa\sigma$ for $n < \omega$
(β) $f_n \le f_{n+1}$ say $f_n = e_n \circ f_{n+1}$ and $e_n \in {}^\sigma\sigma$
(γ) $(e_n, f_{n+1}, \alpha_{n+1})$ witness that (f_n, α_n) does not satisfy the demand (i) on (f_*, α_*) hence $\mathbf{n}(f_n) < \mathbf{n}(f_{n+1})$.

Let $u_n = u_{f_n,\alpha_n}$ for $n < \omega$. For $n < \omega$ and $i < \sigma$ let $A_{n,i} = \langle \alpha < \kappa : f_n(\alpha) = i \rangle$, so $\langle A_{n,i} : i < \sigma \rangle$ is a partition of κ and $\alpha \in A_{n+1,i} \Rightarrow \alpha \in A_{n,e_n(i)}$. So letting $A_\eta = \cap\{A_{n,\eta(n)} : n < \omega\}$ for $\eta \in {}^\omega\sigma$ clearly $\langle A_\eta : \eta \in {}^\omega\sigma \rangle$ is a partition of κ.

As we have $\sigma = \sigma^{\aleph_0}$ by $(*)_0$, there is a sequence $\langle e^n : n < \omega \rangle$ satisfying $e^n \in {}^\sigma\sigma$ and $f \in {}^\kappa\sigma$ such that $f_n = e^n \circ f$ for each $n < \omega$. So $n < \omega \Rightarrow f_n \le f$ which by $\circledast_7(c)$ implies $\mathbf{n}(f_n) \le \mathbf{n}(f)$. As $\langle \mathbf{n}(f_n) : n < \omega \rangle$ is increasing, easily we get a contradiction.]

$\circledast_9$ $\mathbf{n}(f_*) > 0$, i.e. $\alpha < \kappa \Rightarrow u_{f_*,\alpha} \neq \emptyset$.

[Why? If $(\forall f \in {}^\kappa\sigma)(\forall \alpha < \kappa)(u_{f,\alpha} = \emptyset)$ then (by $\circledast_3$) we deduce h is the identity, a contradiction. Otherwise assume $u_{f,\alpha} \neq \emptyset$ hence as in the proof of $\circledast_8$ there is f' such that $f_* \le f' \wedge f \le f'$ so by $\circledast_5$ and $\circledast_8$ we have $0 < |u_{f,\alpha}| \le |u_{f',\alpha}| = |u_{f_*,\alpha_*}|$.]

$\circledast_{10}$ if $f \in {}^\kappa\sigma, \alpha < \kappa$ and $i \in u_{f,\alpha}$ then $\kappa = \sup\{\beta < \kappa : \alpha < \beta$ and $f(\beta) = i\}$.

[Why? If not, let $\beta(*) < \kappa$ be $> \sup\{\beta < \kappa : \alpha < \beta, f(\beta) = i\}$ and $> \omega$ and $> \alpha$. Let $Y = \{(a_{f,\alpha,u}, a_{f,\beta(*),u}) : u \in G, i \notin u\}$. Now for every $\beta \in (\beta(*), \kappa)$ the function g_β maps the set Y onto itself (see Y-s definition and g_β-s definition in $(*)_1$(f)) hence by the definition of $E_{2,\beta(*)+1}$ (in $(*)_4$) it follows that $\bar{a} \in Y \Rightarrow \bar{a}/E_{2,\beta(*)+1} \subseteq Y$ and as h respects $\pi_{\beta(*)+1}$, it follows that $h(\bar{a}) \subseteq \bar{a}/E'_{2,\beta(*)+1}$ and so $\kappa > \gamma \ge \beta(*)+1 \Rightarrow g_\gamma^{-1}(h(\bar{a})) \in \bar{a}/E'_{2,\beta(*)+1}$.

Now for $\bar{a} \in Y$, the two pairs $\bar{a}, h(\bar{a})$ realize the same quantifier free type in $M_{1,\beta(*)+1}, M_{2,\beta(*)+1}$ respectively, hence by the choice of $M_{2,\beta(*)+1}$ the two pairs $\bar{a}, g^{-1}_{\beta(*)+2}h(\bar{a}))$ realize the same quantifier free type in $M_{1,\alpha}$. By $\circledast_1^+ + (*)_6(b)$ recalling $g^{-1}_{\beta(*)+2}(h(\bar{a})) \in \bar{a}/E'_{2,\beta(*)+1}$ this implies that $\bar{a}, g^{-1}_{\beta(*)+2}(h(\bar{a}))$ are $E_{2,\beta(*)+1}$-equivalent. By the definition of $E_{2,\beta(*)+1}, g^{-1}_{\beta(*)+2}(h(\bar{a}))$ belongs to the closure of $\{\bar{a}\}$ under $\{g^{\pm 1}_\gamma : \gamma \in (\beta(*); \kappa)\}$ hence $h(\bar{a})$ belongs to it. But by an earlier sentence Y is closed under those functions so $h(\bar{a}) \in Y$. Similarly $h^{-1}(\bar{a}) \in Y$, hence h maps Y onto itself, recalling $\circledast_2$ and the definition of Y, this implies $i \notin u_{f,\alpha}$, contradicting an assumption of $\circledast_{10}$, so $\circledast_{10}$ holds.]

$\circledast_{11}$ Now fix f_*, α_* as in $\circledast_8$ for the rest of the proof, without loss of generality f_* is onto σ and let $u_{f_*,\alpha_*} = \{i^*_\ell : i < \ell(*)\}$ with $\langle i^*_\ell : \ell < \ell(*) \rangle$ increasing for simplicity. Now for every $f \in {}^\kappa\sigma$ such that $f_* \le f$ and $\alpha < \kappa$ by $\circledast_8(ii),(iii)$ we know that if $e \in {}^\sigma\sigma \wedge f_* = e \circ f$ then e is a one-to-one mapping from $u_{f,\alpha}$ onto u_{f_*,α_*}; but so $e \restriction u_{f,\alpha}$ is uniquely determined by $(f_*, \alpha_*, f, \alpha)$ so let $i_{f,\alpha,\ell} \in u_{f,\alpha}$ be the unique $i \in u_{f,\alpha}$ such that $e(i) = i^*_\ell$ (equivalently $(\exists \alpha)(f(\alpha) = i \wedge f_*(\alpha) = i^*_\ell)$).

Now if $f_* \le f \in {}^\kappa\sigma$ and $\alpha_1, \alpha_2 < \kappa$ and we choose $e = \mathrm{id}_\sigma$ so necessarily $f \restriction u_{f,\alpha_1} = e \circ f \restriction u_{f,\alpha_2}$, then $e \restriction \mathrm{Rang}(f \restriction u_{f,\alpha_2})$ map u_{f,α_2} onto u_{f,α_1} but e is the identity, so we can write u_f instead of $u_{f,\alpha}$ let $i_{f,\ell} = i_{f,\alpha,\ell}$ for $\ell < \ell(*), \alpha < \kappa$.

Let

$$\mathcal{A} = \{A \subseteq \kappa : \text{ for some } f, f_* \le f \text{ and } \alpha < \kappa \text{ we have } f^{-1}\{i_{f,0}\} \backslash \alpha \subseteq A\}$$

$\boxdot_1$ $\mathcal{A} \subseteq \mathcal{P}(\kappa) \backslash [\kappa]^{<\kappa}$.

[Why? As κ is regular, this means $A \in \mathcal{A} \Rightarrow A \subseteq \kappa \wedge \sup(A) = \kappa$ which holds by $\circledast_{10}$.]

$\boxdot_2$ $\kappa \in \mathcal{A}$.

[Why? By the definition of $\mathcal{A}$.]

$\boxdot_3$ if $A \in \mathcal{A}$ and $A \subseteq B \subseteq \kappa$ then $B \in \kappa$.

[Why? By the definition of $\mathcal{A}$.]

$\boxdot_4$ if $A_1, A_2 \in \mathcal{A}$ then $A =: A_1 \cap A_2$ belongs to $\mathcal{A}$.

[Why? Let $(f_\ell, e_\ell, \alpha_\ell)$ be such that $f_* = e_\ell \circ f_\ell$ and $f_\ell \in {}^\kappa\sigma, \alpha_\ell < \kappa$ and $f_\ell^{-1}\{i_{f_\ell,0}\} \backslash \alpha_\ell \subseteq A_\ell$ for $\ell = 1,2$. Let pr:$\sigma \times \sigma \to \sigma$ be one-to-one and onto and define $f \in {}^\kappa\sigma$ by $f(\alpha) = \mathrm{pr}(f_1(\alpha), f_2(\alpha))$. Clearly $f_\ell \le f$ for $\ell = 1,2$ hence $i_{f,0}$ is well defined and $i_{f,0} = \mathrm{pr}(i_{f_1,0}, i_{f_2,0})$. Now for every $\alpha < \kappa, f(\alpha) = i_{f,0} \Rightarrow f_1(\alpha) = i_{f_1,0} \wedge f_2(\alpha) = i \Rightarrow \alpha \in A_1 \wedge \alpha \in A_2 \Rightarrow \alpha \in A_1 \cap A_2 \Rightarrow \alpha \in A$ so $f^{-1}\{i_{f,0}\} \subseteq A$ hence $A \in \mathcal{A}$.]

$\boxdot_5$ if $A \subseteq \kappa$ then $A \in \mathcal{A}$ or $\kappa \backslash A \in \mathcal{A}$.

[Why? Define $f \in {}^\kappa\sigma$:

$$f(\alpha) = \begin{cases} 2f_*(\alpha) & \text{if } \alpha \in A \\ 2f_*(\alpha) + 1 & \text{if } \alpha \in \kappa \backslash A. \end{cases}$$

Let $i = i_{f,0}$ so by the definition of $\mathcal{A}$ we have $f^{-1}\{i\} = f^{-1}\{i_{f,0}\} \in \mathcal{A}$. But if i is even then $f^{-1}\{i\} \subseteq A$ and i is odd then $f^{-1}\{i\} \subseteq \kappa \backslash A$ so by $\boxdot_3$ we are done.]

$\boxdot_6$ $\mathcal{A}$ is a uniform ultrafilter on κ.

[Why? By $\boxdot_1 - \boxdot_5$.]

$\boxdot_7$ $\mathcal{A}$ is σ^+-complete.

[Why? Assume $B_\varepsilon \in \mathcal{A}$ for $\varepsilon < \sigma$ and let $B = \cap\{B_\varepsilon : \varepsilon < \sigma\}$. Define $A_\varepsilon \subseteq \kappa$ for $\varepsilon < \sigma$ as follows: $A_{1+\varepsilon} = \bigcap\limits_{\zeta < \varepsilon} B_\zeta \backslash B_\zeta$ (so is $\kappa \backslash B_0$ if $\varepsilon = 0$) for $\varepsilon < \sigma$ and $A_0 = B$. Clearly $\langle A_\varepsilon : \varepsilon < \sigma \rangle$ is a partition of κ, let $f \in {}^\kappa\sigma$ be such that $f \restriction A_\varepsilon$ is constantly ε. Let $f' \in {}^\kappa\theta$ be such that $f \le f' \wedge f_* \le f'$. Now $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$ is included in some A_ε. If $\varepsilon = 0$ this exemplifies $\bigcap\limits_{\varepsilon < \sigma} B_\varepsilon \in \mathcal{A}$ as required. If $\varepsilon = 1 + \zeta < \sigma$, then $(f')^{-1}\{i_{f',0}\} \subseteq A_\varepsilon \subseteq \kappa \backslash B_\varepsilon$, contradiction to $\boxdot_6$ because $B_\varepsilon \in \mathcal{A}$ and $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$.]

So by the assumptions of 1.2, that is, $\circledast_1(b)$ of 1.1 we get a contradiction, coming from the assumption "toward contradiction (j) of $\boxplus$ of 1.2 fails", so it holds, and the other clauses were proved so we are done. $\square_{1.2}$

Theorem 1.4. *For every θ there is an $\mathfrak{k} = \mathfrak{k}^*_\theta$ such that*

$\otimes$ (a) *$\mathfrak{k}$ is an AEC with* $\mathrm{LST}(\mathfrak{k}) = \theta, |\tau_{\mathfrak{k}}| = \theta$

(b) *$\mathfrak{k}$ has the amalgamation property*

(c) *$\mathfrak{k}$ admits intersections (see Definition 1.5 below)*

(d) *if κ is a regular cardinal and there is no uniform θ^+-complete ultrafilter on κ, <u>then</u>: $\mathfrak{k}$ is not $(\le 2^\kappa, \kappa)$-sequence-local for types, i.e., we can find an $\le_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i \le \kappa \rangle$ of models and $p \ne q \in \mathscr{S}_{\mathfrak{k}}(M_\kappa)$ such that $i < \kappa \Rightarrow p \restriction M_i = q \restriction M_i$ and M_κ is of cardinality $\le 2^\kappa$.*

We shall prove 1.4 below. As in [BS08, 1.2,§4] the aim of the definition of "admit intersections" is to ensure types behave reasonably.

Definition 1.5. We say an AEC $\mathfrak{k}$ admits intersections <u>when</u> there is a function $c\ell_{\mathfrak{k}}$ such that:

- (a) $c\ell_{\mathfrak{k}}(A, M)$ is well defined iff $M \in K_{\mathfrak{k}}$ and $A \subseteq M$
- (b) $c\ell_{\mathfrak{k}}(A, M)$ is preserved under isomorphisms and $\leq_{\mathfrak{k}}$-extensions
- (c) for every $M \in K_{\mathfrak{k}}$ and non-empty $A \subseteq M$ the set $B = c\ell_{\mathfrak{k}}(A, M)$ satisfies: $M{\restriction}B \in K_{\mathfrak{k}}, M{\restriction}B \leq_{\mathfrak{k}} M$ and $A \subseteq M_1 \leq_{\mathfrak{k}} N \wedge M \leq_{\mathfrak{k}} N \Rightarrow B \subseteq M_1$;
- (d) we may use $cl_{\mathfrak{k}}(A, M)$ for $M{\restriction}c\ell_{\mathfrak{k}}(A, M)$.

Claim 1.6. *Assume $\mathfrak{k}$ is an* AEC *admitting intersections. <u>Then</u>* $\mathrm{ortp}_{\mathfrak{k}}(a_1, M, N_1) = \mathrm{ortp}_{\mathfrak{k}}(a_2, M, N_2)$ *iff letting $M_\ell = N_\ell{\restriction}c\ell_{\mathfrak{k}}(M \cup \{a_\ell\})$, there is an isomorphism from M_1 onto M_2 over M mapping a_1 to a_2.*

Proof. It should be clear from the definition. $\square_{1.6}$

Remark 1.7. In Theorem 1.4 we can many times demand $\|M_\kappa\| = \kappa$, e.g., if $(\exists\lambda)(\kappa = 2^\lambda)$.

Note we now show that 1.4 is the best possible.

Claim 1.8. *1) If $\mathfrak{k}$ satisfies clause (a) of 1.4, (i.e. $\mathfrak{k}$ is an AEC with* LST*-number $\leq \theta$ and $|\tau_{\mathfrak{k}}| \leq \theta$) and κ fails the assumption of clause (d) of 1.4, that is there is a uniform θ^+-complete ultrafilter on κ, <u>then</u> the conclusion of clause (d) of 1.4 fails, that is $\mathfrak{k}$ is κ-sequence local for types.*
2) If D is a θ^+-complete ultrafilter on κ and $\mathfrak{k}$ is an AEC with $\mathrm{LST}(\mathfrak{k}) \leq \theta$ *<u>then</u> ultraproducts by D preserve "$M \in \mathfrak{k}$", "$M \leq_{\mathfrak{k}} N$", i.e.*

$\boxtimes$ *if $M_i, N_i (i < \kappa)$ are $\tau(\mathfrak{K})$-models and $M = \prod_{i<\kappa} M_i/D$ and $N = \prod_{i<\kappa} N_i$ <u>then</u>:*
- (a) *$M \in K$ if $\{i < \kappa : M_i \in \mathfrak{k}\} \in D$*
- (b) *$M \leq_{\mathfrak{k}} N$ if $\{i : M_i \leq_{\mathfrak{k}} N_i\} \in D$.*

Proof. Note that if D is θ^+-complete, <u>then</u> it is σ^+-complete where $\sigma = \theta^{\aleph_0}$ (and much more, it is θ'-complete for the first measurable $\theta' > \theta$).
1) So assume

$\boxplus$
- (a) $\langle M_i : i \leq \kappa\rangle$ is $\leq_{\mathfrak{k}}$-increasing
- (b) $M_\kappa = N_0 \leq_{\mathfrak{k}} N_\ell$ for $\ell = 1, 2$
- (c) $p_\ell = \mathrm{ortp}_{\mathfrak{k}}(a_\ell, N_0, N_\ell)$ for $\ell = 1, 2$
- (d) $i < \kappa \Rightarrow p_1{\restriction}M_i = p_2{\restriction}M_i$.

We shall show $p_1 = p_2$, this is enough.
Without loss of generality

$(*)_1$
- (a) $a_1 = a_2$ call it a
- (b) $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$.

By (d) of $\boxplus$ we have:

$(d)^+$ for each $i < \kappa$ there are $n_i < \omega$ and $\langle N_{i,m} : n \leq n_i\rangle$ such that
- (α) $N_{i,0} = N_1$

(β) $N_{i,m_i} = N_2$ or just h_i is an isomorphism from N_{i,m_i} onto N_2 such that $h_i{\restriction}(M_i \cup \{a\})$ is the identity

(γ) $a \in N_{i,\ell}$ and $M_i \le_{\mathfrak{k}} N_{i,\ell}$

(δ) if $m < m_i$ then $N_{i,2m+1} \le_{\mathfrak{k}} N_{i,2m}, N_{i,2m+2}$.

As $\kappa = \operatorname{cf}(\kappa) > \aleph_0$ without loss of generality $i < \kappa \Rightarrow n_i = n_*$. Let χ be such that $\langle M_i : i \le \kappa\rangle, \langle\langle N_{i,n} : n \le n_*\rangle : i < \kappa\rangle$ and $\mathfrak{k}_{\mathrm{LST}(\mathfrak{k})}$ all belongs to $\mathcal{H}(\chi)$; concerning $\mathfrak{k}_{\mathrm{LST}(\mathfrak{k})}$ this means τ_χ and $\mathrm{LST}(\mathfrak{k})$ belongs to $\mathcal{H}(\chi)$ hence $\{M \in K_{\mathfrak{k}} : M \in \mathcal{H}(\mathrm{LST}^+_{\mathfrak{k}})\}$ and $\le_{\mathfrak{k}} {\restriction}\mathcal{H}(\mathrm{LST}^+_{\mathfrak{k}})$ belongs to $\mathcal{H}(\chi)$; those hold by $(*)_1(b)$. Let $\mathfrak{B}$ be the ultrapower $(\mathcal{H}(\chi), \in)^\kappa/D$ and $\mathbf{j}_0$ the canonical embedding of $(\mathcal{H}(\chi), \in)$ into $\mathfrak{B}$ and let $\mathbf{j}_1$ be the Mostowski-Collapse of $\mathfrak{B}$ to a transitive set $\mathcal{H}$ and let $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$. So $\mathbf{j}$ is an elementary embedding of $(\mathcal{H}(\chi), \in)$ into $(\mathcal{H}, \in)$ and even an $\mathbb{L}_{\theta^+,\theta^+}$-elementary one. Recall we are assuming without loss of generality $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$ hence $\mathbf{j}(\tau_{\mathfrak{k}}) = \tau_{\mathfrak{k}}$ hence by part (2), $\mathbf{j}$ preserves "$N \in K_{\mathfrak{k}}$", "$N^1 \le_{\mathfrak{k}} N^2$". "$h$ is an isomorphism from N' onto N''.

So $\mathbf{j}(\langle M_i : i \le \kappa\rangle)$ has the form $\langle M^*_i : i \le \mathbf{j}(\kappa)\rangle$ but $\mathbf{j}(\kappa) > \kappa_* := \bigcup\limits_{i<\kappa} \mathbf{j}(i)$ by the uniformity of D and let $\mathbf{j}(\langle\langle N_{i,n} : n \le n_*\rangle : i < \kappa\rangle = \big\langle\langle N^*_{i,n} : n \le n^*\rangle : i < \mathbf{j}(\kappa)\big\rangle$ and $\mathbf{j}(\langle h_i : i < \kappa\rangle) = \langle h^*_i : i < \kappa_*\rangle$.

So

(a) $\mathbf{j}{\restriction}M_\kappa$ is a $\le_{\mathfrak{k}}$-embedding of M_κ into M^*_κ hence even into $M^*_{\kappa_*}$

(b) $M^*_{\kappa_*} \le_{\mathfrak{k}} N^*_{i,n}$ and $\mathbf{j}(a) \in N^*_{i,n}$ for $i < \kappa, n \le n_*$

(c) $N^*_{i,0} = \mathbf{j}(N_1)$

(d) h_{κ_*} is an isomorphism from N_{κ_*,n_*} onto $\mathbf{j}(N_2)$

(e) $N^*_{\kappa_*,2m+1} \le_{\mathfrak{k}} N^*_{\kappa_*,2m}, N^*_{\kappa_*,2m+2}$ for $2m+1 < n_*$

(f) $\mathbf{j}(a) \in N_{\kappa_*,m}$.

Together, we are done.

2) By the representation theorem of AEC [She09a, §1]. $\square_{1.8}$

Discussion 1.9. We try to help the reader by pointing out some things in the proof of Theorem 1.4.

(1) If the reader do not mind having $\tau_{\mathfrak{k}}$ to be of cardinality $2^{(\theta^{\aleph_2})}$ then we can replace R by reasonable R_e ($e \in {}^\sigma\sigma$) and omit S and $\{d_i : i < \theta\}$. This simplifies somewhat, so below (in 1.9) we follow it.

(2) We rely on the conclusion in 1.2. There we have two increasing continuous sequences of models $\bar{M}_e = \langle M_{e,\alpha} : \alpha \le \kappa\rangle$.

Now $M_{1,\alpha}, M_{2,\alpha}$ are very similar:

($*$) For $\alpha < \kappa$, they are not just isomorphic but have the same universe, but the difference is only in the interpretation of the $R_{n,i}$-s.

Here we define $M'_{0,\alpha}$ by adding I_α, π_α, omitting the $R_{n,i}$-s, and omitting the $R_{n,i}$-s.

Now we shall define $M'_{\ell,\alpha}$ adding a new element t^*_ℓ, which code the $R_{n,i}$, i.e.

$$R^{M_{e,\alpha}}_{n,i} = \{\bar{a}^\frown\langle t^*_\ell\rangle : \bar{a} \in R^{M_{\ell,i}}_{n,i}\}.$$

So this translate "$M_{1,\alpha} \subseteq M_{2,\alpha} \iff \alpha \le \kappa$" to

$$\mathrm{otp}(t^*_1, M'_{0,\alpha}, M'_{1,\alpha}) = \mathrm{otp}(t^*_2, M'_{0,\alpha}, M'_{2,\alpha}) \iff \alpha < \kappa$$

where $M'_{0,\alpha}$ is obtained from $M'_{\ell,\alpha}$ by omitting t^*_1, so we get the same $M'_{0,\alpha}$ from $M'_{1,\alpha}$ and $M'_{2,\alpha}$.

Proof. Proof of 1.4

Let $\sigma = \theta^{\aleph_0}$. Let $G = ([\sigma]^{<\aleph_0}, \Delta)$ and let $\langle c_i : i < \sigma\rangle$ list the members of G, let $\langle \eta_{i,j} : i, j < \sigma\rangle$ list ${}^\omega\theta$, so with no repetitions. We define $\tau = \tau_{\mathfrak{k}}$, by:

$\boxplus_1$ $\tau = \tau^\bullet_\theta \cup \{S, A, I, \pi, R\} \cup \{H_n : n < \omega\} \cup \{d_i : i < \theta\}$, so of cardinality θ, where:

- (a) $\tau^\bullet_\theta = \tau^*_\theta \setminus \{R_e : e \in {}^\sigma\sigma\}$, see 1.1 (so is $\{E_n, E'_n : n < \omega\} \cup \{F_c : c \in G\}$),
- (b) $R_{n,i}$ is an $(n+1)$-place predicate,
- (c) S, A, I are unary predicates,
- (d) π is an unary function,
- (e) R is a three place predicate,
- (f) H_n is an unary function,
- (g) d_i is an individual constant.

We define K as a class of τ-models by:

$\boxtimes_2$ $M \in K$ iff (up to isomorphism):

- (a) $\langle S^M, I^M, J^M, A^M\rangle$ is a partition of $|M|$, (recall that they are unary),
- (b) E^M_n is an equivalence relation on ${}^n(A^M)$, so a $(2n)$-place relation for $n \in [2, n)$,
- (c) $(E'_n)^M$ is an equivalence relation on ${}^n(A^M)$ refining E^M_n,
- (d) F^M_c is an unary function from A^M into itself,
- (e) $R^M_{n,i}$ is an $(n+1)$-relation $\subseteq {}^{n+1}(A^M)$,
- (f) π^M is a function from A^M into I^M,
- (g) $R^M \subseteq A^M \times A^M \times S^M$,
- (h) $\{d^M_i : i < \theta\}$ are pairwise distinct elements of S^M,
- (i) H^M_n is a function from $(S^M \setminus \{d^M_i : i < \theta\}) \times \{d^M_i : i < \theta\}$ into the set $\{d^M_i : i < \theta\}$.

We define $\leq_{\mathfrak{k}}$ as being a submodel, in particular $M \leq_{\mathfrak{k}} N \Rightarrow \pi^N{\restriction}M = \pi^M$, $F^N_c{\restriction}M = F^M_c$. Easily

$\boxtimes_3$ $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ is an AEC.

For $A \subseteq M \in K$ let,

$\boxplus_{3.1}$ (*a*) $c\ell_M(A)$ is the closure of $A \cup \{d^M_i : i < \theta\}$ under π^M and F^M_c $(c \in G)$.
(*b*) $c\ell(A, M) = c\ell_M(A) = M \restriction c\ell_M(A)$.

Now this function $c\ell(A, M)$ shows that $\mathfrak{k}$ admits intersections (see Definition 1.5) so

$\boxtimes_4$ $\mathfrak{k}$ admits intersections and $\mathrm{LST}(\mathfrak{k}) + |\tau_{\mathfrak{k}}| = \theta$.

Assume κ is as in clause (d) of 1.4, we use the $M_{\ell,\alpha}(\ell = 1, 2, \alpha \leq \kappa)$ as well as I_α, π_α constructed in 1.2 (the relevant properties are stated in 1.2). They are not in the right vocabulary and universe, so let $M'_{\ell,\alpha}$ be the following τ-model:

$\boxtimes_5$ (a) elements: $S^{M'_{\ell,\alpha}} = \sigma \cup {}^\sigma\sigma$, and

- $I^{M'_{\ell,\alpha}} = I_\alpha$,
- $J^{M'_{\ell,\alpha}} = \{t^*_\ell\}$, t^*_ℓ just a new element,

- $A^{M'_{\ell,\alpha}} = |M_{\ell,\alpha}| = A_\alpha$.
(we assume disjointness)

(b) $(M'_{\ell,\alpha} {\restriction} A_\alpha){\restriction}\tau^\bullet_\theta = M_{\ell,\alpha}{\restriction}\tau^\bullet_\theta$,
(c) $I^{M'_{\ell,\alpha}} = I_\alpha$,
(d) $\pi^{N'_{\ell\alpha}}$ is π_α,
(e) $R^{M'_{\ell,\alpha}}_{n,i}$ is $\{\bar a{}^\frown\langle t^*_\ell\rangle \colon \bar a \in R^{M_{\ell\alpha}}_{n,i}\}$,
(f) $d_i^{M'_{\ell,\alpha}} = i$ for $i < \sigma$,
(g) $H_n^{M'_{\ell,\alpha}}$ is a function from $({}^\sigma\sigma) \times \theta$ into θ, so if $i < \theta$, then the sequence $\langle H_n^{M'_{\ell,\alpha}}(e,i)\colon n < \omega\rangle \in {}^\omega\sigma$ is equal to η_{j_1,j_2} for some $j_1, j_2 < \sigma$ and $e(j_1) = j_2$,
(h) $R^{M'_{\ell,\alpha}} = \{(a,b,e)\colon e \in {}^\sigma\sigma$ and $(a,b) \in R_e^{M_{\ell,\alpha}}\}$.

Let $M'_{0,\alpha} = M'_{\ell,\alpha} \restriction (|M'_{\ell,\alpha}| \setminus \{t^*_\ell\})$ for $\ell = 1,2$ and $\alpha \leq \kappa$ (we get the same result for $\ell = 1,2$).

Note easily

$\boxtimes_6$ $M'_{0,\alpha} \leq_{\mathfrak{k}} M'_{\ell,\alpha}, \langle M'_{\ell,\alpha} : \alpha \leq \kappa\rangle$ is $\leq_{\mathfrak{k}}$-increasing and continuous for $\ell = 0,1,2$,
[Why? Easy to check.]

$\boxtimes_7$ $\mathrm{ortp}_{\mathfrak{k}}(t^*_1, M'_{0,\alpha}, M'_{1,\alpha}) = \mathrm{ortp}_{\mathfrak{k}}(t^*_2, M'_{0,\alpha}, M'_{2,\alpha})$ for $\alpha < \kappa$.

[Why? By the isomorphism g_α from $M_{1,\alpha}$ onto $M_{2,\alpha}$ respecting π_α in 1.1.]

$\boxtimes_8$ $\mathrm{ortp}_{\mathfrak{k}}(t^*_1, M'_{0,\kappa}, M'_{1,\kappa}) \neq \mathrm{ortp}_{\mathfrak{k}}(t^*_2, M'_{0,\kappa}, M'_{2,\kappa})$.

[Why? By the non-isomorphism in 1.1; extension will not help.]

Now, by the"translation theorem" of [BS08, 4.7] we can find $\mathfrak{k}'$ which has all the needed properties, i.e., also the amalgamation and JEP. $\square_{1.4}$

2. Compactness of types in AEC

Baldwin [Bal09] asks "Can we in ZFC prove that some AEC has amalgamation and JEP but fails compactness of types?". The background is that in [BS08] we construct one using diamonds.

To me, the question is to show that this class can be very large (in ZFC).

Here we omit amalgamation and accomplish both by direct translations of problems of existence of models for theories in $\mathbb{L}_{\kappa^+,\kappa^+}$, first in the propositional logic. So whereas in [BS08] we have an original group G^M, here instead we have a set P^M of propositional "variables" and P^M, set of such sentences (and relations and functions explicating this; so really we use coding but are a little sloppy in stating this obvious translation).

In [BS08] we have I^M, set of indexes, 0 and H, set of Whitehead cases, H_t for $t \in I^M$, here we have I^M, each $t \in I^N$ representing a theory $P^M_t \subseteq P^M$ and in J^M we give each $t \in I^M$ some models $\mathcal{M}^M_s : P^M \to \{\text{true,false}\}$. This is set up so that amalgamation holds.

Notation 2.1. In this section types are denoted by $\mathbf{p}, \mathbf{q}$ because p, q are used for propositional variables.

Definition 2.2. 1) We say that an AEC $\mathfrak{k}$ has $(\leq \lambda, \kappa)$-sequence-compactness (for types) <u>when</u>: if $\langle M_i : i \leq \kappa\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and $i < \kappa \Rightarrow \|M_i\| \leq \lambda$

and $\mathbf{p}_i \in \mathscr{S}^{<\omega}(M_i)$ for $i < \kappa$ satisfying $i < j < \kappa \Rightarrow \mathbf{p}_i = \mathbf{p}_j \restriction M_i$ <u>then</u> there is $\mathbf{p}_\kappa \in \mathscr{S}^{<\omega}(M_\kappa)$ such that $i < \kappa \Rightarrow \mathbf{p}_\kappa \restriction M_i = \mathbf{p}_i$.
2) We define "$(=\lambda,\kappa)$-sequence-compactness" similarly. Let (λ,κ)-sequence-compactness mean $(\leq \lambda,\kappa)$-compactness.

Question 2.3. Can we find an AEC $\mathfrak{k}$ with amalgamation and JEP such that $\{\theta : \mathfrak{k}$ have (λ,θ)-compactness of types for every $\lambda\}$ is complicated, say:

- (a) not an end segment but with "large" members
- (b) any $\{\theta : \theta$ satisfies $\psi\}, \psi \in \mathbb{L}_{\kappa^+,\kappa^+}$ (second order).

Definition 2.4. Let $\kappa \geq \aleph_0$, we define $\mathfrak{k} = \mathfrak{k}_\kappa$ as follows:

- (A) the vocabulary $\tau_{\mathfrak{k}}$ consist of $F_i (i \leq \kappa), R_\ell (\ell = 1,2), P, \Gamma, I, J, c_i\ (i < \kappa), F_i (i \leq \kappa)$, (pedantically see later),
- (B) the universe of $M \in K_{\mathfrak{k}}$ is the disjoint union of P^M, Γ^M, I^M, J^M so P, Γ, I, J are unary predicates
- (C) (a) P^M a set of propositional variables (i.e. this is how we treat them)
  - (b) Γ^M is a set of sentences of one of the forms $\varphi = (p), \varphi = (r \equiv p \wedge q), \varphi = (q \equiv \neg p), \varphi = (q \equiv \bigwedge\limits_{i<\kappa} p_i)$, so $p, q, p_i \in P^M$
  but in the last case $\{p_i : i < \kappa\} \subseteq \{c_i^M : i \leq \kappa\}$ (or code this!) <u>but</u> we do not require that all of them appear
  - (c) for $i < \kappa$ the function $F_i^M : \Gamma^M \to P^M$ are such that for every $i < \kappa$ and $\varphi \in \Gamma^M$ we have:
    - (α) if $\varphi = (p)$ and $i < \kappa$ then $F_{1+i}(\varphi) = p, F_0(\varphi) = c_0$
    - (β) if $\varphi = (r \equiv p \wedge q)$ <u>then</u> $F_i(\varphi)$ is c_1 if $i = 0$, is p if $i = 1$, is q if $i = 2$, and is r if $i \geq 3$
    - (γ) if $\varphi = (q \equiv \neg p)$ then $F_i(\varphi)$ is c_2 if $i = 0$, is p if $i = 1$, and is q if $i \geq 2$
    - (δ) if $\varphi = (q \equiv \bigwedge\limits_{j<\kappa} p_j)$ then $F_i(\varphi)$ is c_3 if $i = 0$, is q if $i = 1$ and is p_{2+j} if $i = j+1$
  - (d) I a set of theories, i.e. $R_1^M \subseteq \Gamma \times I$ and for $t \in I$ let $\Gamma_t^M = \{\psi \in \Gamma^M : \psi R_1^M t\} \subseteq \Gamma^M$
  - (e) J is a set of models, i.e. $R_2^M \subseteq (\Gamma \cup P) \times J$ and for $s \in J$ we have $\mathcal{M}_s^M$ is the model, i.e. function giving truth values to (some) $p \in P^M$, i.e.
    - (α) $\mathcal{M}_s^M(p)$ is true if $p_i R_2^M s$; is false if $\neg p R_2^M s$
    - (β) $(\varphi, s) \in R_2^M$ iff computing the truth value of φ in $\mathcal{M}_s^M$ we get truth
  - (f) $F_\kappa^M : J^M \to I^M$ such that $s \in J^M \Rightarrow \mathcal{M}_s^M$ is a model of $\Gamma_{F_\kappa^M(s)}$
  - (g) $(\forall t \in I^M)(\exists s \in J^M)(F_\kappa^M(s) = t)$
- (D) $M \leq_{\mathfrak{k}} N$ <u>iff</u> $M \subseteq N$ are $\tau_{\mathfrak{k}}$-models from $K_{\mathfrak{k}}$.

Claim 2.5. *$\mathfrak{k}$ is an AEC and $LST(\mathfrak{k}) = \kappa$.*

Proof. Obvious. $\square_{2.5}$

Claim 2.6. *$\mathfrak{k}$ has the JEP.*

Proof. Just like disjoint unions (also of the relations and functions) except for the individual constants c_i (for $i < \kappa$). $\square_{2.6}$

Claim 2.7. *Assume $M_0 \le_{\mathfrak{k}} M_\ell$ for $\ell = 0, 1$ and $|M_0| = P^{M_0} \cup \Gamma^{M_0} = P^{M_\ell} \cup \Gamma^{M_\ell}$ for $\ell = 1, 2$ and $a_\ell \in I^{M_\ell}$ for $\ell = 1, 2$. Then $\mathrm{ortp}_{\mathfrak{k}}(a_1, M_0, M_1) = \mathrm{ortp}_{\mathfrak{k}}(a_2, M_0, M_2)$ iff $\Gamma^{M_1}_{a_1} = \Gamma^{M_2}_{a_2}$.*

Proof. The if direction, $\Leftarrow$:

Let h be a one to one mapping with domain M_1 such that $h{\restriction}M_0 =$ the identity, $h(a_1) = a_2$ and $h(M_1) \cap M_2 = M_0 \cup \{a_2\}$. Renaming without loss of generality h is the identity. Now define M_3 as $M_1 \cup M_2$, as in 2.6, now $a_1 = a_2$ does not cause trouble because $P^{M_0} = P^{M_\ell}, \Gamma^{M_0} = \Gamma^{M_\ell}$ for $\ell = 1, 2$.

The only if direction, $\Rightarrow$:

Obvious. $\square_{2.7}$

Claim 2.8. *Assume λ, θ are such that:*

- (a) *θ is regular $\le \lambda$ and $\lambda \ge \kappa$*
- (b) *$\langle \Gamma_i : i \le \theta \rangle$ is $\subseteq$-increasing continuous sequence of sets propositional sentences in $\mathbb{L}_{\kappa^+,\aleph_0}$ such that [Γ_i has a model $\Leftrightarrow i < \theta$]*
- (c) *$|\Gamma_\theta| \le \lambda$.*

Then $\mathfrak{k}$ fail (λ, θ)-sequence-compactness (for types).

Remark 2.9. We may wonder but: for $\theta = \aleph_0$, compactness holds? Yes, but only assuming amalgamation.

Proof. Without loss of generality $|\Gamma_\theta| = \lambda$. Without loss of generality $\langle p^*_\varepsilon : \varepsilon < \lambda \rangle$ are pairwise distinct propositions variables appearing in Γ_0 (but not necessarily $\in \Gamma_0$) and each $\psi \in \Gamma_i$ is of the form (p) or $r \equiv p \wedge q$ or $r \equiv \neg p$ or $r \equiv \bigwedge\limits_{i<\kappa} p_i$, where $\{p_i : i < \kappa\} \subseteq \{p^*_\varepsilon : \varepsilon < \kappa\}$, hence $\kappa \le \lambda$.

Let P_i be the set of propositional variables appearing in Γ_i without loss of generality $|P_i| = \lambda$.

We choose a model M_i for $i \le \theta$ such that:

$\boxplus_1$ (a) $|M_i| = P_i \cup \Gamma_i$, and $\tau(M_i) = \tau_{\mathfrak{k}}$
- (b) $P^M = P_i$ and $\Gamma^{M_i} = \Gamma_i$
- (c) $F^{M_i}_\varepsilon$ (for $\varepsilon < \kappa$) are defined naturally
- (d) $I^{M_i} = \emptyset = J^{M_i}$, hence $R^M_1 = R^M_2 = \emptyset = F^M_\kappa$.

$\boxplus_2$ (a) $M_i \in K_{\mathfrak{k}}$,
- (b) $\langle M_i \colon i \le \theta \rangle$ is $\le_{\mathfrak{k}}$-increasing and continuous

Let $\mathcal{M}_i : P_i \to \{\text{true false}\}$ be a model of Γ_i.

We define a model $N_i \in K_{\mathfrak{k}}$ for $i < \theta$ (but not for $i = \theta$!)

$\boxtimes$ (a) $M_i \le_{\mathfrak{k}} N_i$
- (b) $P^{N_i} = P^{M_i}$
- (c) $\Gamma^{N_i} = \Gamma^{M_i}$
- (d) $I^M = \{t_j \colon j < i\}$
- (e) $J^M = \{s_i : j < i\}$

(f) $F_\kappa^{N_i}(s_j) = t_j$
(g) $R_1^{N_i} = \bigcup\{\Gamma_j \times \{t_j\} : j < i\}$
(h) $R_2^{N_i}$ is chosen such that $\mathcal{M}_{s_j}^{N_j}$ is $\mathcal{M}_j$.

Now

$(*)_1$ $\mathbf{p}_i = \mathrm{ortp}_{\mathfrak{k}}(t_i, M_i, N_i) \in \mathscr{S}^1(M_i)$.

[Why? Trivial.]

$(*)_2$ $i < j < \theta \to \mathbf{p}_i = \mathbf{p}_j{\restriction}M_j$.

[Why? Let $N_{i,j} = N_j{\restriction}(M_j \cup \{s_j, t_j\})$.]
Easily $\mathrm{ortp}(t_j, M_i, N_{i,j}) \le \mathbf{p}_j$ and $\mathrm{ortp}(t_j, M_i, N_{i,j}) = \mathbf{p}_j$ by the claim 2.7 above.]

$(*)_3$ there is no $\mathbf{p} \in \mathscr{S}^1(M_\theta)$ such that $i < \theta \Rightarrow \mathbf{p}_j{\restriction}M_i = \mathbf{p}_i$.

Why? We prove more:

$(*)_4$ there is no (N, t) such that
(a) $M_\kappa \le_{\mathfrak{k}} N$
(b) $t \in I^N$
(c) $(\forall \varphi \in \Gamma^{M_\kappa})[\varphi R_1^N t]$.

[Why? As then $\Gamma_\theta = \Gamma^M$ has a model contradiction to an assumption.] $\square_{2.8}$

So e.g.

Conclusion 2.10. If $\theta > \kappa$ is regular with no κ^+-complete uniform ultrafilter on θ and $\lambda = 2^\theta$, <u>then</u> $\mathfrak{k}$ is not (λ, θ)-sequence-compact.

Remark 2.11. Recall if D is an ultrafilter on θ then $\min\{\sigma' : D \text{ is not } \sigma'\text{-complete}\}$ is $\aleph_0$ or a measurable cardinality.

Proof. (Well known).
Let M be the model with universe 2^θ, $P_0^M = \theta$, $c_i^M = i$ for $i \le \kappa$ and $R^M \subseteq \theta \times \lambda$ be such that $\{\{\alpha < \lambda : \alpha R^M \beta\} : \beta < \lambda\} = \mathcal{P}(\theta)$, and let $<^M$ the well ordering of the ordinal on λ. Moreover, the vocabulary of M has cardinality κ and elimination of quantifiers and Skolem functions.
Let $\Gamma_i =$ $\mathrm{Th}(M,\beta)_{\beta<\lambda} \cup \{\alpha < c : \alpha < \theta\} \cup \big\{(\forall x)[x < c_\kappa \equiv \bigvee_{i<\kappa} x = c_i]\big\}$, where without loss of generality, c is a new individual constant then $\langle \Gamma_i : i \le \theta\rangle$ is as[1] required in 2.12 below hence 2.8 apply. $\square_{2.10}$

Conclusion 2.12. In Claim 2.8 if $\lambda = \lambda^\kappa$ then we can allow $\langle \Gamma_i : i \le \theta\rangle$ to be a sequence of theories in $\mathbb{L}_{\kappa^+,\kappa^+}(\tau), \tau$ any vocabulary of cardinality $\le \lambda$.

Proof. Without loss of generality, we can add Skolem functions (each with $\le \kappa$ places) in particular. So Γ_i becomes universal, and adding propositional variables for each quantifier-free sentence and writing down the obvious sentences, we get a set of propositional sentences, and we get Γ_i as there. $\square_{2.12}$

Note that:

[1] or directly as Γ_i has Skolem functions

Observation 2.13. *If $\lambda \ge \kappa \ge \theta = \mathrm{cf}(\theta)$ <u>then</u> the condition in 2.8 holds.*

Proof. Just let $\Gamma_0 = \{\bigvee_{i<\theta} \neg p_i\}, \Gamma_i = \Gamma_0 \cup \{p_j : j < i\}$. $\square_{2.13}$

Conclusion 2.14. 1) $\mathbf{C}_\kappa = \{\theta : \theta =$ cf(θ) and for every λ and AEC $\mathfrak{k}$ with LST$(\mathfrak{k}) \le \kappa, |\tau_{\mathfrak{k}}| = \kappa$ have (λ,θ)-sequence-compactness of type$\}$ is the class $\{\theta : \theta =$ cf$(\theta) > \kappa$ and there is a uniform κ^+-complete ultrafilter on $\theta\}$.
2) In $\mathbf{C}_\kappa$ we can replace "every λ" by $\lambda = 2^\theta + \kappa$.

Proof. Put together 2.10 and 2.16. $\square_{2.13}$

Of course, a complementary result (showing the main claim is best possible) is:

Claim 2.15. *If $\mathfrak{k}'$ is an AEC, $\mathrm{LST}(\mathfrak{k}') \le \kappa$ and on θ there is a uniform κ^+-complete ultrafilter on θ and θ is regular and λ any cardinality <u>then</u> $\mathfrak{k}'$ has (λ,κ)-compactness of types.*

Proof. Write down a set of sentences on $\mathbb{L}_{\kappa^+,\kappa^+}(\tau^+_{\mathfrak{k}})$ expressing the demands.

Let $\langle M_i : i \le \theta\rangle$ be $<_{\mathfrak{k}}$-increasing continuous, $\|M_i\| \le \lambda, \mathbf{p}_i = \mathrm{ortp}_{\mathfrak{k}}(a_i, M_i, N_i)$ so $M_i \le_{\mathfrak{k}} N_i$ such that $i < j < \theta \Rightarrow \mathbf{p}_i = \mathbf{p}_j \restriction M_i$. Without loss of generality $\|N_i\| \le \lambda$.

Let $\langle N_{i,j,\ell} : \ell \le n_{i,j,\ell}\rangle, \pi_{i,1}$ witness $\mathbf{p}_i = \mathbf{p}_j \restriction M_i$ for $i < j < \theta$ (i.e. $M_i \le_{\mathfrak{k}} N_{i,j,\ell}$ (without loss of generality $\|N_{i,j,\ell}\| \le \lambda$), $N_{i,j,0} = N_i, a_i \in N_{i,j,\ell}$, $\bigwedge\limits_{\ell<n_{i,j,\ell}} (N_{i,j,\ell} \le_{\mathfrak{k}} N_{i,j,\ell+1} \vee N_{i,j,\ell+1} \le_{\mathfrak{k}} N_{i,j,\ell}$ and $\pi_{i,j}$ be an isomorphism from N_j onto $N_{i,j,n_{i,j}}$ over M_i mapping a_j to a_i.

Let $\tau^+ = \tau \cup \{F_{\varepsilon,n} : \varepsilon < \kappa, n < \omega\}$, arity$(F_{\varepsilon,n}) = n$. Let $\langle M^+_i : i \le \theta\rangle$ be $\subseteq$-increasing, M^+_i a τ^+-expansion of M_i such that $u \subseteq M^+_i \Rightarrow M_i \restriction c\ell_{M^+_i}(u) \le_{\mathfrak{k}} M_i$. Similarly $(N^{+,\varepsilon}_{i,j,\ell} : \ell \le n_{i,j,\ell}); \varepsilon = 1, \ell$ such that $N^{+,\varepsilon}_{i,j,\ell}$ is a τ^+-expansion of $N_{i,j,\varepsilon}$ as above such that $(\forall \ell < n_{i,j,\ell})(\exists \varepsilon \in \{1,2\})(N^{+,\varepsilon}_{i,j,\ell} \subset N^{+,\varepsilon}_{i,j,\ell+1} \vee N^{+,\varepsilon}_{i,j,\ell+1} \subseteq N^{+,\varepsilon}_{i,j,\ell})$.

Now write down a translation of the question, "is there $\mathbf{p}$ such that..." $\square_{2.15}$

Claim 2.16. *Assume that D is a uniform κ-complete ultrafilter on $\theta, \langle M_i : i \le \theta\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous, $\mathbf{p}_i \in \mathscr{S}^\alpha_{\mathfrak{k}}(M_i)$ as witnessed by (N_i, a_i) for $i < \kappa, \mathbf{p}_i = \mathbf{p}_j \restriction M_i$ for $i < j < \kappa$ as witnessed by $(\pi_i, \langle N_{i,j,\ell} : \ell \le m_{i,j}\rangle$ as in the proof above.*
1) There is $\mathbf{p}_\kappa \in \mathscr{S}^\alpha(M_\theta)$ such that $i < \theta \Rightarrow \mathbf{p}_\kappa \restriction M_i$.
2) In fact for each $i < \kappa$ let $\mathcal{U}_i \in D$ be such that $i < j \in \mathcal{U}_i \Rightarrow n_{i,j} = n^*_i$. Let $N_{i,\kappa,\ell} = \prod\limits_{j \in \mathcal{U}_i} N_{i,j,\ell}/D$. So $\langle N_{i,\kappa,\ell} : \ell \le n^*_\ell\rangle$ are as above. Let $M = \prod\limits_{i<\kappa} M_i/D, \pi_{i,\kappa} = \prod\limits_{j\in\mathcal{U}_i} \pi_{i,j}/D$, etc.

Proof. Straightforward. $\square_{2.16}$

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