ON THE PROBLEM OF STABILITY OF ABSTRACT ELEMENTARY CLASSES OF MODULES

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ABSTRACT. It is an open problem of Mazari-Armida whether every abstract elementary class of R-modules $(\mathbf{K}, \leqslant_{\text{pure}})$, with $\leqslant_{\text{pure}}$ the pure submodule relation, is stable. We answer this question in the negative by constructing unstable abstract elementary classes $(\mathbf{K}, \leqslant_{\text{pure}})$ of torsion-free abelian groups. On the other hand, we prove (in ZFC) that if R is any ring and $\mathfrak{K} = (\mathbf{K}, \preccurlyeq)$ is an abstract elementary class of R-modules which is κ-local (a.k.a. κ-tame) for some $\kappa \geqslant \text{LS}(\mathfrak{K})$, then $(\mathbf{K}, \preccurlyeq)$ is *almost stable*, where almost stability is a new notion of independent interest that we introduce in this paper, and which is equivalent to the usual notion of stability under the assumption of amalgamation. As a consequence, we obtain that if there are sufficiently large cardinals, then every abstract elementary class $(\mathbf{K}, \preccurlyeq)$ of R-modules with amalgamation is stable.

1. INTRODUCTION

Following the development of classification theory for first-order logic [11], the second-named author initiated a program aimed at developing an abstract framework for model theory and classification theory. This led to the area of model theory known as Abstract Elementary Classes (AECs) [12]. A longstanding challenge in this theory has been the scarcity of new interesting examples beyond those arising from first-order logic. A significant breakthrough came through Zilber's work on complex exponentiation (cf. [17]). More recently, largely due to the work of Mazari-Armida (see e.g. [4, 5, 6]), the model theory community has recognized that module theory provides a rich source of applications for the general theory of AECs (see also the recent survey [2]). A central open problem in this area of model theory is the following question, formulated by Mazari-Armida in [6].

Question 1.1. *Let R be a ring and let $\leqslant_{\text{pure}}$ denote the pure submodule relation. If $(\mathbf{K}, \leqslant_{\text{pure}})$ is an abstract elementary class with $\mathbf{K} \subseteq R$-Mod, is $(\mathbf{K}, \leqslant_{\text{pure}})$ stable? Is this true when $R = \mathbb{Z}$? Under what conditions on R does this hold?*

We note that this question is inspired by a classical result from model theory: for any ring R, every complete first-order theory of R-modules is stable. This result, together with the well-known elimination of quantifiers down to pp-formulas, makes the first-order model theory of modules one of the most well-behaved areas of application of model theory to algebraic structures. For extensive background on the first-order model theory of modules see e.g. the classical references [7, 8].

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In this paper we answer Question 1.1 in the negative, more precisely, we prove:

Theorem 1.2. *There is a class of torsion-free abelian groups* $\mathbf{K}$ *such that* $(\mathbf{K}, \leqslant_{\mathrm{pure}})$ *is an* AEC *and* $(\mathbf{K}, \leqslant_{\mathrm{p}})$ *is unstable, where* $\leqslant_{\mathrm{pure}}$ *denotes the pure subgroup relation.*

At this point the reader might be discouraged by our results, but this is only half of the story. In fact, we will see that despite the failure of stability, AECs of R-modules are still as well-behaved as possible from the point of view of classification theory, in an appropriate sense. We first recall the notion of κ-locality (a.k.a. κ-tameness) [10, 3, 13]. In first-order logic, if two types $p, q \in \mathbf{S}(A)$ differ, then they already differ over a finite set of parameters $A_0 \subseteq A$. The notion of κ-locality imposes a similar behavior, where "finite" is now replaced by "$< \kappa^+$". In recent years, the notion of κ-locality (a.k.a. κ-tameness) has been recognized as central in the study of AECs, and this additional assumption is often made in case studies (cf. [16]).

Now, despite the unstability from Theorem 1.2, under the assumption of κ-locality for some $\kappa \geqslant \mathrm{LS}(\mathfrak{K})$, we establish the next best form of stability possible, namely what we call *almost stability* (cf. 2.4(2)(3)). This is a notion that we introduce in this paper, which is equivalent to the usual notion of stability under the assumption of amalgamation. In brief, almost stability means that there are only a few orbital types over M, once we restrict to a specific strong extension N of M (that is why stability and almost-stability coincide under the assumption of amalgamation). We believe that this notion is of independent interest and we hope that it will inspire future studies and new directions in the theory of AECs.

Theorem 1.3. *Let* $\mathfrak{K}$ *be an* AEC *of* R*-modules s.t.* $\mathfrak{K}$ *is* κ*-local for some* $\kappa \geqslant \mathrm{LS}(\mathfrak{K})$.

(1) There is $\xi > \kappa$ *such that, for every cardinal* μ *satisfying*

$$\mu = \mu^{<\xi} + \sum\{2^{2^{\sigma}} : \sigma < \xi\},$$

we have that $\mathfrak{K}$ *is* almost μ*-stable (cf. 2.4(2)(3)).*

(2) If in addition $\mathfrak{K}$ *has amalgamation and* μ *is as in (1), then* $\mathfrak{K}$ *is* μ*-stable.*

By known consistency results on locality (see e.g. [1]), we deduce:

Corollary 1.4. *If there is a strongly compact cardinal* κ *and* $\mathfrak{K}$ *is an* AEC *of* R*-modules with amalgamation such that* $\kappa > \mathrm{LS}(\mathfrak{K})$*, then* $\mathfrak{K}$ *is stable.*

Notice that AECs of R-modules which arise from first-order theories are local and have amalgamation and so our theorem can be seen as the most general form of stability for R-modules currently known in the literature. Explicitly, we deduce:

Corollary 1.5. *Let* $\mathfrak{K} = (\mathbf{K}, \preccurlyeq)$ *be such that* $\mathbf{K}$ *is a complete first-order theory of* R*-modules and* $\preccurlyeq$ *is the relation of elementary first-order substructure. Then* $(\mathbf{K}, \preccurlyeq)$ *is stable in the sense of first-order logic.*

The challenge that we leave for future studies are the following questions.

Question 1.6. *(1) Is there (consistently) a ring* R *and an* AEC *of* R*-modules* $\mathfrak{K}$ *which is not* κ*-local for unboundedly many* κ *among the cardinals* θ *such that* θ *is below the first strongly compact cardinal* $> \mathrm{LS}(\mathfrak{K})$.

(2) Is there (consistently) a ring R *and an* AEC *of* R*-modules* $\mathfrak{K}$ *which is not* κ*-local for unboundedly many* κ *among the cardinals* θ *such that* θ *is below* $\beth_{(2^{\mathrm{LS}(\mathfrak{K})})^+}$*?*

We conjecture that the answer to 1.6(2) is yes.

2. Preliminaries

Notation 2.1. Given a formula φ and ordinals α, β, γ, when we write $\varphi(\bar{x}_\alpha, \bar{y}_\beta, \bar{z}_\gamma)$ we mean that $\bar{x}_\gamma = (x_i : i < \alpha)$, $\bar{y}_\beta = (y_i : i < \beta)$ and $\bar{z}_\gamma = (z_i : i < \gamma)$.

Definition 2.2. Let $\mathfrak{K} = (\mathbf{K}_\mathfrak{K}, \preccurlyeq_\mathfrak{K}) = (\mathbf{K}, \preccurlyeq)$ be an AEC. Given $(\bar{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq N$, and $\bar{b}$ is a sequence in N, the orbital type (a.k.a. the Galois type) of $\bar{b}$ over A in N, denoted by $\mathrm{ortp}_\mathfrak{K}(\bar{b}/A; N)$, is the equivalence class of $(\bar{b}, A, N)$ modulo $E_\mathfrak{K}$, where $E_\mathfrak{K}$ is the transitive closure of the relation $E'_\mathfrak{K}$, where $(\bar{b}_1, A_1, N_1) E'_\mathfrak{K} (\bar{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exist $\mathfrak{K}$-embeddings $f_\ell : N_\ell \to_A N$ for $\ell \in \{1, 2\}$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$ and $N \in \mathbf{K}$. If $M \in \mathbf{K}$ and γ is an ordinal, let $\mathcal{S}^\gamma_\mathfrak{K}(M) = \{\mathrm{ortp}_\mathfrak{K}(\bar{b}/M; N) : M \preccurlyeq_\mathfrak{K} N \in K \text{ and } \bar{b} \in N^\gamma\}$. When $\gamma = 1$, we may write $\mathcal{S}_\mathfrak{K}(M)$ instead of $\mathcal{S}^1_\mathfrak{K}(M)$. We let $\mathcal{S}^{<\infty}_\mathfrak{K}(M) = \bigcup_{\gamma \in \mathrm{OR}} \mathcal{S}^\gamma_\mathfrak{K}(M)$.

Notation 2.3. Let $\mathfrak{K} = (\mathbf{K}_\mathfrak{K}, \preccurlyeq_\mathfrak{K}) = (\mathbf{K}, \preccurlyeq)$ be an AEC. For $\lambda \in \mathrm{Card}$, we let

$$\mathbf{K}_\lambda = \{M \in \mathbf{K} : |M| = \lambda\}.$$

Definition 2.4. Let $\mathfrak{K} = (\mathbf{K}_\mathfrak{K}, \preccurlyeq_\mathfrak{K}) = (\mathbf{K}, \preccurlyeq)$ be an AEC, $\lambda \in \mathrm{Card}$ and $\gamma \in \mathrm{Ord}$.

(1) We say that $\mathfrak{K}$ is (λ, γ)-stable if for any $M \in \mathbf{K}_\lambda$ we have that $|\mathcal{S}^\gamma_\mathfrak{K}(M)| \leqslant \lambda$.
(2) We say that $\mathfrak{K}$ is almost (λ, γ)-stable if for any $M, N \in \mathbf{K}_\lambda$ with $M \preccurlyeq_\mathfrak{K} N$ and $M \in \mathbf{K}_\lambda$ we have that $|\mathcal{S}^\gamma_\mathfrak{K}(M; N)| \leqslant \lambda$, where
$$\mathcal{S}^\gamma_\mathfrak{K}(M; N) = \{\mathrm{ortp}(\bar{c}/M, N) : \bar{c} \in N^\gamma\}.$$
(3) If $\alpha = 1$, then we simply say (almost) λ-stable.
(4) $\mathfrak{K}$ is (almost) stable if it is (almost) μ-stable for unboundedly many $\mu \in \mathrm{Card}$.

Remark 2.5. Notice that in some references (e.g. the recent survey [2]), stability is defined as follows: $\mathfrak{K}$ is stable if it is μ-stable for some $\mu \in \mathrm{Card}$. In some contexts the definition of stability from 2.4 and the one we just gave are equivalent.

Observation 2.6. Notice that if $\mathfrak{K}$ has the amalgamation property, then $\mathfrak{K}$ is almost (λ, γ)-stable if and only if it is (λ, γ)-stable. Furthermore, recalling the definition of $E_\mathfrak{K}$ and $E'_\mathfrak{K}$ from Notation 2.2, we have that $E_\mathfrak{K} = E'_\mathfrak{K}$.

Definition 2.7. Let $\mathfrak{K} = (\mathbf{K}_\mathfrak{K}, \preccurlyeq_\mathfrak{K}) = (\mathbf{K}, \preccurlyeq)$ be an AEC and κ an infinite cardinal.

(1) We say that $\mathfrak{K}$ is $(<\kappa)$-local (a.k.a. $(<\kappa)$-tame) if for any $M \in \mathbf{K}$ and $p \neq q \in \mathcal{S}_\mathfrak{K}(M)$, there exists $M_0 \subseteq M$ such that $|M_0| < \kappa$ and $p \restriction M_0 \neq q \restriction M_0$.
(2) When we say that $\mathfrak{K}$ is κ-local we mean that $\mathfrak{K}$ is $(<\kappa^+)$-local.

Remark 2.8. The notion of κ-locality was used in [10] under the assumption of the amalgamation property, and in [13] without this assumption.

Definition 2.9. Let $\mathfrak{K} = (\mathbf{K}_\mathfrak{K}, \preccurlyeq_\mathfrak{K}) = (\mathbf{K}, \preccurlyeq)$ be an AEC, $\lambda \in \mathrm{Card}$ and $\gamma \in \mathrm{Ord}$.

(1) We say that $\mathfrak{K}$ has the (λ, γ)-order property if there are $M \in \mathbf{K}$ and $(\bar{a}_i : i < \lambda)$ inside M with $\lg(\bar{a}_i) = \gamma$, for all $i < \lambda$, such that for any $i_0 < j_0 < \lambda$ and $i_1 < j_1 < \lambda$, $\mathrm{ortp}_\mathfrak{K}(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset; N) \neq \mathrm{ortp}_\mathfrak{K}(\bar{a}_{j_1}\bar{a}_{i_1}/\emptyset; N)$.
(2) We say that $\mathfrak{K}$ has the syntactic $(\lambda, \kappa, \gamma, \Delta)$-order property if there are $M \in \mathbf{K}$ and $(\bar{a}_i : i < \lambda)$ inside M with $\lg(\bar{a}_i) = \gamma$, for all $i < \lambda$, and contradictory $\varphi_1(\bar{x}_\gamma, \bar{y}_\gamma), \varphi_2(\bar{x}_\gamma, \bar{y}_\gamma) \in \Delta \subseteq \mathfrak{L}_{\infty, \kappa^+}(\tau_\mathfrak{K})$ (e.g. $\varphi_1(\bar{x}_\gamma, \bar{y}_\gamma)$ is $\neg\varphi_2(\bar{x}_\gamma, \bar{y}_\gamma)$) s.t.:
$$i < j < \lambda \Rightarrow M \models \varphi_1(\bar{a}_i, \bar{a}_j) \wedge \varphi_2(\bar{a}_j, \bar{a}_i).$$
(3) If $\kappa = \mathrm{LS}(\mathfrak{K})$ we simply say syntactic $(\lambda, \gamma, \Delta)$-order property. Furthermore, if $\gamma = 1$, then we simply say (syntactic) λ-order ((λ, Δ)-order) property.

(4) We say that $\mathfrak{K}$ does not have the γ-order property (resp. syntactic (λ, Δ)-order property) if it does not have the (μ, γ)-order property (resp. syntactic (μ, γ, Δ)-order property) for some $\mu \in \mathrm{Card}$.
(5) We define almost $(<\kappa)$-local and almost κ-local similarly.

Fact 2.10 ([1, Theorem 1.3])**.** If $\mathfrak{K}$ is an AEC with $\mathrm{LS}(\mathfrak{K}) < \kappa$ and κ is strongly compact, then $\mathfrak{K}$ is κ-local.

Definition 2.11. Let $\mathfrak{K} = (\mathbf{K}_{\mathfrak{K}}, \preccurlyeq_{\mathfrak{K}}) = (\mathbf{K}, \preccurlyeq)$ be an AEC, $\Delta \subseteq \mathfrak{L}_{\infty,\kappa^+}(\tau_{\mathfrak{K}})$ and $\gamma < \kappa^+$.

(1) We say that $\mathfrak{K}$ is syntactically $(\lambda, \kappa, \gamma, \Delta)$-stable <u>when</u> for every $M \in \mathbf{K}_\lambda$ we have that $|\mathbf{S}^{\gamma}_{(\Delta,\mathfrak{K})}(M)| \leqslant \lambda$, where:
$$\mathbf{S}^{\gamma}_{(\Delta,\mathfrak{K})}(M) = \{\mathrm{tp}_\Delta(\bar{c}/M; N)\} : N \in \mathbf{K},\ M \preccurlyeq_{\mathfrak{K}} N,\ \bar{c} \in N^\gamma\}.$$
(2) We say that $\mathfrak{K}$ is syntactically almost $(\lambda, \kappa, \gamma, \Delta)$-stable <u>when</u> for every $M, N \in \mathbf{K}_\lambda$ with $M \preccurlyeq_{\mathfrak{K}} N$ and $M \in \mathbf{K}_\lambda$ we have that $|\mathbf{S}^{\gamma}_{(\Delta,\mathfrak{K})}(M; N)| \leqslant \lambda$, where:
$$\mathbf{S}^{\gamma}_{(\Delta,\mathfrak{K})}(M; N) = \{\mathrm{tp}_\Delta(\bar{c}/M; N)\} : \bar{c} \in N^\gamma\}.$$
(3) If κ is minimal such that $\kappa \geqslant \mathrm{LS}(\mathfrak{K})$ and $\Delta \subseteq \mathfrak{L}_{\infty,\kappa^+}(\tau_{\mathfrak{K}})$, then we may omit κ.
(4) If $\Delta = \mathfrak{L}_{\infty,\kappa^+}(\tau_{\mathfrak{K}})$, then we simply say syntactically (almost) $(\lambda, \kappa, \gamma)$-stable.

Observation 2.12. Notice that if $\mathfrak{K}$ has the amalgamation property, then:

(a) $\mathfrak{K}$ is syntactically almost (λ, γ)-stable iff it is syntactically (λ, γ)-stable.
(b) $\mathfrak{K}$ is syntactically almost $(\lambda, \kappa, \gamma, \Delta)$-stable iff it is syntactically $(\lambda, \kappa, \gamma, \Delta)$-stable.

We need the following crucial fact from [15]. Notice that despite the following fact is not explicitly stated in journal version of [15], it follows from the proof of the second main theorem (the "Tarski-Vaught" criterion for AECs); furthermore, this fact is explicitly stated in the latest arXiv version of the paper [15].

Fact 2.13 ([15])**.** Let $\mathfrak{K} = (\mathbf{K}_{\mathfrak{K}}, \preccurlyeq_{\mathfrak{K}}) = (\mathbf{K}, \preccurlyeq)$ be an AEC and let $(\lambda_{\kappa_0}, \kappa_0)$ be as in [15], i.e., $\kappa_0 = \mathrm{LS}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$ and $\lambda_{\kappa_0} = \beth_2(\kappa_0)^{++}$. More generally, for $\kappa \geqslant \mathrm{LS}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$, let $\lambda_\kappa = \beth_2(\kappa_0)^{++}$. Then there is $\varphi^\star_\kappa(\bar{x}_\kappa) \in \mathfrak{L}_{\lambda^+,\kappa^+}(\tau_{\mathfrak{K}})$ such that:

($\star$) if $N \in \mathbf{K}$, $\bar{a} \in N^\kappa$ and $N \restriction \bar{a}$ is a substructure of N, then we have:
$$N \restriction \bar{a} \preccurlyeq N \iff N \models \varphi^\star_\kappa(\bar{a}).$$

3. Almost stability for AECs of R-modules

Hypothesis 3.1. *(1)* $\mathfrak{K} = (\mathbf{K}_{\mathfrak{K}}, \preccurlyeq_{\mathfrak{K}}) = (\mathbf{K}, \preccurlyeq)$ *is a fixed* AEC*.*
(2) $\kappa \geqslant \mathrm{LS}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$ *and* $\lambda = \beth_2(\kappa)^{++}$*.*
(3) $\gamma \leqslant \kappa$ *is an ordinal.*

Notation 3.2. (1) Let $\mathscr{W}^{\mathrm{small}} = \mathscr{W}^{\mathrm{small}}_{(\mathfrak{K},\kappa,\gamma)}$ be the class of quintuples of the form
$$\mathfrak{u} = (M_{\mathfrak{u}}, N_{\mathfrak{u}}, \bar{a}_{\mathfrak{u}}, \bar{b}_{\mathfrak{u}}, \bar{c}_{\mathfrak{u}}) = (M, N, \bar{a}, \bar{b}, \bar{c})$$
such that:

(a) $M \preccurlyeq_{\mathfrak{K}} N$, $|M| \leqslant |N| \leqslant \kappa$ (we write "small" since we ask $|N| \leqslant \kappa$ here);
(b) $\bar{a}$ lists M and $\bar{b}$ lists N;
(c) $\bar{c} \in N^\gamma$;

(2) Let $\mathscr{W}^{\rm large} = \mathscr{W}^{\rm large}_{(\mathfrak{K},\kappa,\gamma)}$ be the class of quadruples of the form

$$\mathfrak{u} = (M_\mathfrak{u}, N_\mathfrak{u}, \bar{a}_\mathfrak{u}, \bar{c}_\mathfrak{u}) = (M, N, \bar{a}, \bar{c})$$

such that:
(a) $M \preccurlyeq_\mathfrak{K} N$, $|M| \leqslant \kappa$;
(b) $\bar{a}$ lists M;
(c) $\bar{c} \in N^\gamma$.

Major Claim 3.3. *In the context of 3.2.*

(A) For $\mathfrak{w} = (M_\mathfrak{u}, N_\mathfrak{u}, \bar{a}_\mathfrak{u}, \bar{b}_\mathfrak{u}, \bar{c}_\mathfrak{u}) \in \mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$*, there is*

$$\theta(\bar{z}, \bar{y}_\kappa, \bar{x}_\kappa) = \theta_\mathfrak{w}(\bar{z}, \bar{y}_\kappa, \bar{x}_\kappa) \in \mathfrak{L}_{\lambda^+,\kappa^+}(\tau_\mathfrak{K})$$

such that:
(a) if $\mathfrak{w}_1 \cong \mathfrak{w}_2 \in \mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$ *are isomorphic (where this means what you expect), then* $\theta_{\mathfrak{w}_1}(\bar{z}_\gamma, \bar{y}_\kappa, \bar{x}_\kappa) = \theta_{\mathfrak{w}_2}(\bar{z}_\gamma, \bar{y}_\kappa, \bar{x}_\kappa)$*;*
(b) if $\mathfrak{w} \in \mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$, *then* $N_\mathfrak{w} \models \theta_\mathfrak{w}(\bar{c}_\mathfrak{w}, \bar{b}_\mathfrak{w}, \bar{a}_\mathfrak{w})$*;*
(c) if $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$ *and* $M_{\mathfrak{w}_1} = M_{\mathfrak{w}_2}$, *then*

$$\mathrm{ortp}(\bar{c}_{\mathfrak{w}_1}/M_{\mathfrak{w}_1}; N_{\mathfrak{w}_1}) = \mathrm{ortp}(\bar{c}_{\mathfrak{w}_2}/M_{\mathfrak{w}_2}; N_{\mathfrak{w}_2}) \Leftrightarrow \theta_{\mathfrak{w}_1} = \theta_{\mathfrak{w}_2}.$$

(B) For $\mathfrak{w} = (M_\mathfrak{u}, N_\mathfrak{u}, \bar{a}_\mathfrak{u}, \bar{c}_\mathfrak{u}) \in \mathscr{W}^{\rm large}_{(\mathfrak{K},\kappa,\gamma)}$*, there is*

$$\psi(\bar{z}_\gamma, \bar{x}_\kappa) = \psi_\mathfrak{w}(\bar{z}_\gamma, \bar{x}_\kappa) \in \mathfrak{L}_{\lambda^+,\kappa^+}(\tau_\mathfrak{K})$$

such that:
(a) if $\mathfrak{w}_1 \cong \mathfrak{w}_2 \in \mathscr{W}^{\rm large}_{(\mathfrak{K},\kappa,\gamma)}$ *are isomorphic (where this means what you expect), then* $\psi_{\mathfrak{w}_1}(\bar{z}_\gamma, \bar{x}_\kappa) = \psi_{\mathfrak{w}_2}(\bar{z}_\gamma, \bar{x}_\kappa)$*;*
(b) if $\mathfrak{w} \in \mathscr{W}^{\rm large}_{(\mathfrak{K},\kappa,\gamma)}$, *then* $N_\mathfrak{w} \models \psi_\mathfrak{w}(\bar{c}_\mathfrak{w}, \bar{a}_\mathfrak{w})$*;*
(c) if $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathscr{W}^{\rm large}_{(\mathfrak{K},\kappa,\gamma)}$ *and* $M_{\mathfrak{w}_1} = M_{\mathfrak{w}_2}$, *then*

$$\mathrm{ortp}(\bar{c}_{\mathfrak{w}_1}/M_{\mathfrak{w}_1}; N_{\mathfrak{w}_1}) = \mathrm{ortp}(\bar{c}_{\mathfrak{w}_2}/M_{\mathfrak{w}_2}; N_{\mathfrak{w}_2}) \Leftrightarrow \psi_{\mathfrak{w}_1} = \psi_{\mathfrak{w}_2}.$$

Proof. We prove clause (A). Given $\mathfrak{m} = (M_\mathfrak{m}, N_\mathfrak{m}, \bar{a}_\mathfrak{m}, \bar{b}_\mathfrak{m}, \bar{c}_\mathfrak{m}) \in \mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$, Let $\theta^0_\mathfrak{w}$ be the conjunction of formulas $\varphi(\bar{z}_\gamma \restriction u_3, \bar{y}_\kappa \restriction u_2, \bar{x}_\kappa \restriction u_1)$, where u_1, u_2 are finite subsets of κ, u_3 is a finite subset of γ, φ is an atomic formula or the negation of an atomic formula and $N_\mathfrak{m} \models \varphi(\bar{c}_\gamma \restriction u_3, \bar{b}_\kappa \restriction u_2, \bar{a}_\kappa \restriction u_1)$. Now, $\theta^0_\mathfrak{m}$ satisfies clauses (A)(a)(b) but not necessarily clause (A)(c). We define an equivalence relation $E^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$ on $\mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$ by requiring that $\mathfrak{m}_1 E^{\rm small}_{(\mathfrak{K},\kappa,\gamma)} \mathfrak{m}_2$ if and only there is a mapping π such that:

$(\cdot_1)$ $\pi(a_{\mathfrak{m}_1,i}) = a_{\mathfrak{m}_2,i}$ is an isomorphism from $M_{\mathfrak{m}_1}$ onto $M_{\mathfrak{m}_2}$ such that $\bar{a}_{\mathfrak{m}_1} \mapsto \bar{a}_{\mathfrak{m}_2}$;
$(\cdot_2)$ $\mathrm{ortp}(\bar{c}_{\mathfrak{m}_2}/M_{\mathfrak{m}_2}; N_{\mathfrak{m}_2}) = \pi(\mathrm{ortp}(\bar{c}_{\mathfrak{m}_1}/M_{\mathfrak{m}_1}; N_{\mathfrak{m}_1}))$.

Notice that for $\mathfrak{m} \in \mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$ the family of formulas

$$\Phi_\mathfrak{m} = \{\theta^0_{\mathfrak{m}_1} : \mathfrak{m}_1 E^{\rm small}_{(\mathfrak{K},\kappa,\gamma)} \mathfrak{m}\}$$

is a set with $\leqslant 2^\kappa$ members. Lastly, the formula $\theta_\mathfrak{m} = \bigvee \Phi_\mathfrak{m}$ is as required.

We prove clause (B). For every $\mathfrak{w} \in \mathscr{W}^{\rm large}_{(\mathfrak{K},\kappa,\gamma)}$ we define $\mathrm{nb}(\mathfrak{w})$ as follows:

$(\cdot)$ $\mathfrak{u} \in \mathrm{nb}(\mathfrak{w})$ iff $\mathfrak{u} \in \mathscr{W}^{\rm large}_{(\mathfrak{K},\kappa,\gamma)}$, $M_\mathfrak{u} = M_\mathfrak{w}$, $\bar{a}_\mathfrak{u} = \bar{a}_\mathfrak{w}$, $\bar{c}_\mathfrak{u} = \bar{c}_\mathfrak{w}$, $N_\mathfrak{u} \preccurlyeq_\mathfrak{K} N_\mathfrak{m}$ and $|N_\mathfrak{u}| \leqslant \kappa$, that is, letting $\bar{b}$ enumerate $N_\mathfrak{u}$, we have $(M_\mathfrak{u}, N_\mathfrak{u}, \bar{a}_\mathfrak{u}, \bar{b}, \bar{c}_\mathfrak{u}) \in \mathscr{W}^{\rm small}_{(\mathfrak{K},\kappa,\gamma)}$.

Finally, recalling the formula $\varphi^\star_\kappa(\bar{x}_\kappa)$ from 2.13, and noticing that by assumption we have that $\kappa \geqslant \kappa_0 = \mathrm{LS}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$, we define $\psi_{\mathfrak{w}}(\bar{z}_\gamma, \bar{x}_\kappa)$ as the following formula:

$$\bigvee\{\exists \bar{y}_\kappa(\varphi^\star_\kappa(\bar{y}_\kappa) \wedge \theta_{\mathfrak{u}}(\bar{z}_\gamma, \bar{y}_\kappa, \bar{x}_\kappa)) : \mathfrak{u} \in \mathrm{nb}_{\mathfrak{w}}\},$$

where $\theta_{\mathfrak{u}}(\bar{z}_\gamma, \bar{y}_\kappa, \bar{x}_\kappa)$ is as in clause (B) of this claim. Then $\psi_{\mathfrak{w}}(\bar{z}_\gamma, \bar{x}_\kappa)$ is as desired. ∎

Claim 3.4. *Suppose that $\kappa \geqslant \mathrm{LS}(\mathfrak{K}) + |\gamma|$ and that $\mathfrak{K}$ is κ-local and let*

$$\Delta = \Delta_{(\mathfrak{K},\kappa,\gamma)} = \{\psi_{\mathfrak{w}} : \mathfrak{w} \in \mathscr{W}^{\mathrm{large}}_{(\mathfrak{K},\kappa,\gamma)}\},$$

where $\psi_{\mathfrak{w}}$ is as in 3.3(B). Then the following are equivalent:
(1) $\mathfrak{K}$ is almost (μ, γ)-stable;
(2) $\mathfrak{K}$ is syntactically almost (μ, γ, Δ)-stable.

Proof. Assume that $M \in \mathbf{K}_\mu$ and $M \preccurlyeq_{\mathfrak{K}} N$.

$(*_1)$ It suffices to prove $(a) \Leftrightarrow (b)$, where:
  (a) $\{\mathrm{ortp}(\bar{b}/M; N) : \bar{b} \in N^\gamma\}$ has cardinality $\leqslant \mu$;
  (b) $\{\mathrm{tp}_\Delta(\bar{b}/M; N); \bar{b} \in N^\gamma\}$ has cardinality $\leqslant \mu$.

In fact we shall prove more. First we observe the following.

$(*_2)$ It suffices to prove that, for $\bar{c}_1, \bar{c}_2 \in N^\gamma$, $(c)_{(\bar{c}_1,\bar{c}_2)} \Leftrightarrow (d)_{(\bar{c}_1,\bar{c}_2)}$, where:
  $(c)_{(\bar{c}_1,\bar{c}_2)}$ $\mathrm{ortp}(\bar{c}_1/M; N) = \mathrm{ortp}(\bar{c}_2/M; N)$;
  $(d)_{(\bar{c}_1,\bar{c}_2)}$ $\mathrm{tp}_\Delta(\bar{c}_1/M; N) = \mathrm{tp}_\Delta(\bar{c}_2/M; N)$.

So we proceed to the proof that for $\bar{c}_1, \bar{c}_2 \in N^\gamma$ we have that $(c)_{(\bar{c}_1,\bar{c}_2)} \Leftrightarrow (d)_{(\bar{c}_1,\bar{c}_2)}$. To prove the "left-to-right" implication, first of all observe that all the formulas in Δ are formulas in the logic $\mathfrak{L}_{\lambda^+,\kappa^+}(\tau_{\mathfrak{K}})$ and so it suffices to show that for every $M' \preccurlyeq_{\mathfrak{K}} M$ with $|M'| \leqslant \kappa$ we have that:

$(d')_{(\bar{c}_1,\bar{c}_2)}$ $\mathrm{tp}_\Delta(\bar{c}_1/M'; N) = \mathrm{tp}_\Delta(\bar{c}_2/M'; N)$.

Now, to show $(d')_{(\bar{c}_1,\bar{c}_2)}$, it suffices to first define appropriate $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathscr{W}^{\mathrm{large}}_{(\mathfrak{K},\kappa,\gamma)}$ so that $\bar{c}_{\mathfrak{m}_1} = \bar{c}_1$, $\bar{c}_{\mathfrak{m}_2} = \bar{c}_2$, $M'_{\mathfrak{m}_1} = M' = M'_{\mathfrak{m}_2}$, $\bar{a}_{\mathfrak{m}_1} = \bar{a} = \bar{a}_{\mathfrak{m}_2}$ and $N_{\mathfrak{m}_1} = N = N_{\mathfrak{m}_1}$, and second to show that if $(c)_{(\bar{c}_1,\bar{c}_2)}$ holds, then $N \models \psi(\bar{c}_1, \bar{a}) \Leftrightarrow N \models \psi(\bar{c}_2, \bar{a})$, and the latter double implication holds by 3.3(B). Concerning the "right-to-left" implication, since by assumption we have that $\mathfrak{K}$ is κ-local it suffices to show that for every $M' \preccurlyeq_{\mathfrak{K}} M$ with $|M'| = \kappa$ we have that:

$(c')_{(\bar{c}_1,\bar{c}_2)}$ $\mathrm{ortp}(\bar{c}_1/M'; N) = \mathrm{ortp}(\bar{c}_2/M'; N)$.

Let $\bar{a} \in (M')^\kappa$ list M' and let, for $\ell = 1, 2$, $\mathfrak{m}_\ell = (M', N, \bar{a}, \bar{c}_\ell)$. The obviously, for $\ell = 1, 2$, $\mathfrak{m}_\ell \in \mathscr{W}^{\mathrm{large}}_{(\mathfrak{K},\kappa,\gamma)}$ and thus using 3.3(B) we conclude. ∎

Claim 3.5. *If the conditions (1)-(5) below are met, then $\mathfrak{K}$ is syntactically almost $(\mu, \gamma, \Delta_{(\mathfrak{K},\nu,\gamma)})$-stable (recall Definition 2.11), where:*
(1) $\mathrm{LS}(\mathfrak{K}) \leqslant \nu$, $\gamma < \nu^+$ and $\gamma_* = \nu + \nu + \gamma$ (notice that $\gamma_* < \nu^+$);
(2) $\Delta = \Delta_{(\mathfrak{K},\nu,\gamma)} = \{\psi_{\mathfrak{w}}(\bar{z}_\gamma, \bar{x}_\nu) : \mathfrak{w} \in \mathscr{W}^{\mathrm{large}}_{(\mathfrak{K},\nu,\gamma)}\}$;
(3) ξ is such that $\mathrm{cf}(\xi) > |\Delta|$;
(4) $\mu = \mu^{<\xi} + \sum\{2^{2^\sigma} : \sigma < \xi\}$;
(5) $\mathfrak{K}$ fails the syntactic $(\xi, \gamma_*, \Delta^+)$-order property, for Δ^+ defined as in (⊞) below

(⊞) $$\Delta^+ = \{\delta^\ell_{(\mathfrak{m}_1,\mathfrak{m}_2)} : \ell = 1, 2, \text{ and } \psi_{\mathfrak{m}_1}, \psi_{\mathfrak{m}_2} \in \Delta_{(\mathfrak{K},\nu,\gamma)} \text{ are contradictory}\},$$

where for $\mathfrak{m}_1, \mathfrak{m}_2 \in \Delta_{(\mathfrak{K},\nu,\gamma)}$ we define *(a)* $\delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}(\bar{y}^1_{\gamma_*},\bar{y}^2_{\gamma_*})$ *is the formula*

$$\psi_{\mathfrak{m}_1}((y^1_\alpha:\alpha<\gamma),(y^2_{\gamma+\alpha}:\alpha<\nu))\wedge\psi_{\mathfrak{m}_2}((y^1_\alpha:\alpha<\gamma),(y^2_{\gamma+\nu+\alpha}:\alpha<\nu))$$

(b) $\delta^2_{(\mathfrak{m}_1,\mathfrak{m}_2)}(\bar{y}^1_{\gamma_*},\bar{y}^2_{\gamma_*})$ *is the formula*

$$\neg\psi_{\mathfrak{m}_1}((y^1_\alpha:\alpha<\gamma),(y^2_{\gamma+\alpha}:\alpha<\nu))\vee\neg\psi_{\mathfrak{m}_2}((y^1_\alpha:\alpha<\gamma),(y^2_{\gamma+\nu+\alpha}:\alpha<\nu)).$$

Remark 3.6. (A) The proof of 3.5 is similar to [9, Chapter I, Th. 1.10, pg. 277].
(B) If we want to use Δ instead of Δ^+ in 3.5, then we have to change item (5) to:

(4') $\mathfrak{K}$ fails the syntactic (ξ_1,γ_*,Δ)-order property, for some ξ_1 such that

$$\xi\to(\xi_1)^2_{|\Delta|}.$$

Proof. So we are given $M\in\mathbf{K}_\mu$ and $M\preccurlyeq_{\mathfrak{K}}N\in\mathbf{K}$ and we want to prove that

$$|\{\mathrm{tp}_\Delta(\bar{c}/M;N)\}:\bar{c}\in N^\gamma\}|\leqslant\mu.$$

Toward contradiction, for $\alpha<\mu^+$, let $\bar{c}_\alpha\in N^\gamma$ be such that the types $(\mathrm{tp}_\Delta(\bar{c}_\alpha/M;N):\alpha<\mu^+)$ are pairwise distinct. Recall that M and N are fixed, but first we observe:

($\star_1$) W.l.o.g. we can assume that if $\alpha<\mu^+$ and $p(\bar{z}_\gamma)\subseteq\mathrm{tp}_\Delta(\bar{c}_\alpha/M;N)$ has cardinality $<\xi$, <u>then</u> $p(\bar{z}_\gamma)$ is realized in M.

[Why? As by assumption $\mu=\mu^{<\xi}$, $\nu<\xi$, $|\Delta|\leqslant 2^\nu$, and so clearly $\bigcup\{\mathrm{tp}_\Delta(\bar{c}_\alpha/M;N):\alpha<\mu^+\}$ (which is simply a set of formulas) has size $\leqslant\mu$. Notice that of course we can replace M by M' if $M\preccurlyeq_{\mathfrak{K}}M'\preccurlyeq_{\mathfrak{K}}N$ and $|M'|=\mu$.]

($\star_2$) For each $\alpha<\mu^+$, we try to choose $(\bar{a}^\alpha_{(i,1)},\bar{a}^\alpha_{(i,2)},\bar{c}^\alpha_i,\mathfrak{m}^\alpha_{(i,1)},\mathfrak{m}^\alpha_{(i,2)},\psi^\alpha_{(i,1)},\psi^\alpha_{(i,2)})$, by induction on $i<\xi$, such that the following happens:
  (a) $\bar{a}^\alpha_{(i,\ell)}\in M^\nu$, for $\ell=1,2$;
  (b) for $\ell=1,2$, $\psi^\alpha_{(i,\ell)}(\bar{z}_\gamma,\bar{x}_\nu)=\psi_{\mathfrak{m}^\alpha_{(i,\ell)}}(\bar{z}_\gamma,\bar{x}_\nu)$, where

$$\mathfrak{m}^\alpha_{(i,\ell)}=(M\restriction\bar{a}^\alpha_{(i,\ell)},N,\bar{a}^\alpha_{(i,\ell)},\bar{c}^\alpha_i)\in\mathscr{W}^{\mathrm{large}}_{(\mathfrak{K},\nu,\gamma)};$$

  (c) $N\models\psi^\alpha_{(i,\ell)}(\bar{c}^\alpha_i,\bar{a}^\alpha_{(i,\ell)})$, for $\ell=1,2$;
  (d) $\psi^\alpha_{(i,1)}$ and $\psi^\alpha_{(i,2)}$ are contradictory;
  (e) $\bar{c}^\alpha_i\in N^\gamma$;
  (f) if $\ell=1,2$ and $j\leqslant i<\xi$, then

$$N\models\psi^\alpha_{(j,\ell)}(\bar{c}^\alpha_i,\bar{a}^\alpha_{(j,\ell)})$$

  (g) if $\ell=1,2$ and $j<i<\xi$, then we have that

$$N\models\psi^\alpha_{(j,\ell)}(\bar{c}^\alpha_j,\bar{a}^\alpha_{(i,1)})\leftrightarrow\psi^\alpha_{(j,\ell)}(\bar{c}^\alpha_j,\bar{a}^\alpha_{(i,2)}).$$

($\star_3$) Let $i(\alpha)$ be the minimal $i\leqslant\xi$ such that the induction from ($\star_2$) stops, so $i(\alpha)\leqslant\xi$ (recall that the induction from ($\star_2$) is on $i<\xi$).

($\star_4$) If for some $\alpha<\mu^+$ we have that $i(\alpha)=\xi$, then we get a contradiction to the assumption (4) which says that $\mathfrak{K}$ fails the syntactic (ξ,γ_*,Δ^+)-order property.

Why ($\star_4$)? Suppose that the assumption of ($\star_4$) holds, i.e., $i(\alpha)=\xi$. As by assumption we have that $\mathrm{cf}(\xi)>|\Delta|$, then, for some ψ_1,ψ_2, the order type of $\mathcal{U}$ is equal to ξ, where:

$$\mathcal{U}=\{i<\xi:(\psi_1,\psi_2)=(\psi^\alpha_{(i,1)},\psi^\alpha_{(i,2)})\}.$$

Thus, letting, for $i\in\mathcal{U}$, $\bar{b}^\alpha_i:=(\bar{a}^\alpha_{(i,1)})^\frown(\bar{a}^\alpha_{(i,2)})^\frown\bar{c}^\alpha_i$, which has length $\gamma_*=\nu+\nu+\gamma$, we have $(\bar{b}^\alpha_i:i\in\mathcal{U})$ exemplifies the syntactic (ξ,γ_*,Δ^+)-order property. To see this, let $\psi_1=\psi_{\mathfrak{m}_1}$ and $\psi_2=\psi_{\mathfrak{m}_2}$. Notice now that

$(\star_{4.1})$ If $j \leqslant i$ and $i, j \in \mathcal{U}$, then $N \models \delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}(\bar{b}_i, \bar{b}_j)$.

[Why? Note now the following.

$(\cdot_1)$ If $\ell = 1, 2$, then $N \models \psi_{\mathfrak{m}_\ell}(\bar{c}^\alpha_i, \bar{a}^\alpha_{(j,\ell)})$.
[This is by $(\star_2)$(f).]

$(\cdot_2)$ If $\ell = 1$, then $N \models \psi_{\mathfrak{m}_\ell}((\bar{b}_i(\beta) : \beta < \gamma), (\bar{b}_j(\gamma + \beta) : \beta < \nu))$,
[This is by $(\cdot_1)$ and the choice of $\bar{b}_i$ and $\bar{b}_j$.]

$(\cdot_3)$ If $\ell = 2$, then $N \models \psi_{\mathfrak{m}_\ell}((\bar{b}_i(\beta) : \beta < \gamma), (\bar{b}_j(\gamma + \nu + \beta) : \beta < \nu))$.
[This is by $(\cdot_1)$ and the choice of $\bar{b}_i$ and $\bar{b}_j$.]

$(\cdot_4)$ $N \models \delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}(\bar{b}_i, \bar{b}_j)$.
[This is by the definition of $\delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}$ and $(\cdot_2)$, $(\cdot_3)$.]

So $(\star_{4.1})$ holds indeed.]

$(\star_{4.2})$ If $j < i$ and $i, j \in \mathcal{U}$, then $N \models \delta^2_{(\mathfrak{m}_1,\mathfrak{m}_2)}(\bar{b}_j, \bar{b}_i)$, i.e., $N \models \neg\delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}(\bar{b}_j, \bar{b}_i)$.

[Why? Toward contradiction assume that $N \models \delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}(\bar{b}_j, \bar{b}_i)$.

$(\cdot_1)$ for $\ell = 1, 2$, $N \models \psi_{\mathfrak{m}_\ell}(\bar{c}^\alpha_j, \bar{a}^\alpha_{(i,1)}) \leftrightarrow \psi^\alpha_{(j,\ell)}(\bar{c}^\alpha_j, \bar{a}^\alpha_{(i,2)})$.
[This is by $(\star_2)$(g).]

$(\cdot_2)$ for $\ell = 1, 2$, we have

$$N \models \psi_{\mathfrak{m}_\ell}((\bar{b}_j(\beta) : \beta < \gamma), (\bar{b}_i(\gamma + \beta) : \beta < \nu)) \leftrightarrow$$
$$\psi_{\mathfrak{m}_\ell}(((\bar{b}_j(\beta) : \beta < \gamma), (\bar{b}_i(\gamma + \nu + \beta) : \beta < \nu)))$$

[This is by $(\cdot_1)$ and the choice of $\bar{b}_i$ and $\bar{b}_j$.]

$(\cdot_3)$ $N \models \psi_{\mathfrak{m}_1}((\bar{b}_j(\beta) : \beta < \gamma), (\bar{b}_i(\gamma + \beta) : \beta < \nu))$.
[This is by our assumption toward contradiction and the definition of $\delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}$.]

$(\cdot_4)$ $N \models \psi_{\mathfrak{m}_1}(((\bar{b}_j(\beta) : \beta < \gamma), (\bar{b}_i(\gamma + \nu + \beta) : \beta < \nu)))$.
[This is by $(\cdot_2)$ and $(\cdot_3)$.]

$(\cdot_5)$ $N \models \neg\psi_{\mathfrak{m}_2}(((\bar{b}_j(\beta) : \beta < \gamma), (\bar{b}_i(\gamma + \nu + \beta) : \beta < \nu)))$.
[By $(\cdot_4)$ and $\psi_{\mathfrak{m}_1}$, $\psi_{\mathfrak{m}_2}$ being contradictory.]

But $(\cdot_5)$ contradicts our assumption toward contradiction, so $(\star_{4.2})$ holds indeed.]

$(\star_{4.3})$ $\delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}$ and $\delta^2_{(\mathfrak{m}_1,\mathfrak{m}_2)}$ are contradictory.

[Why? By thee choice of $\delta^1_{(\mathfrak{m}_1,\mathfrak{m}_2)}$ and $\delta^2_{(\mathfrak{m}_1,\mathfrak{m}_2)}$.]
Together $(\star_{4.1})$-$(\star_{4.3})$ establish $(\star_4)$, so this ends the proof of $(\star_4)$.

$(\star_5)$ Thus we have that for every $\alpha < \mu^+$ we have that $i(\alpha) < \xi$.

$(\star_7)$ For some $\alpha_* < \mu^+$, $|\mathcal{V}| = \mu^+$, where:

$$\mathcal{V} = \{\beta < \mu^+ : i(\beta) = i(\alpha_*) \text{ and } \forall i < i(\alpha_*) \forall \ell \in \{1, 2\},\ \bar{c}^{\alpha_*}_i = \bar{c}^\beta_i, \psi^\alpha_{(i,\ell)} = \psi^{\alpha_*}_{(i,\ell)}\}.$$

[Why? As the number of possible sequences

$$(i(\alpha), (\psi^\alpha_{(i,1)}, \psi^\alpha_{(i,1)}) : i < i(\alpha)), ((\bar{a}^\alpha_{(i,1)}, \bar{a}^\alpha_{(i,2)}, \bar{c}^\alpha_i) : i < i(\alpha))$$

is $\leqslant \xi \times |\Delta| \times |\Delta| \times |M|^\nu \times |M|^\nu \times |M|^\gamma \leqslant \mu^\nu = \mu$.]

$(\star_8)$ (a) The set $\{\mathrm{tp}_\Delta(\bar{c}_\alpha, M, N) : \alpha \in \mathcal{V}\}$ has size $\leqslant 2^{2^{|i(\alpha_*)|+\nu+|\Delta|}}$.
(b) as $|\mathcal{V}| = \mu^+$ and $2^{2^{|i(\alpha_*)|+\nu+|\Delta|}}$ is $\leqslant \mu$ we get a contradiction.

Why $(\star_8)$? It suffices to prove $(\star_8)$(a). Let $\mathbf{I} = \{\bar{b} : \bar{b} \in M^\nu$ and $M \restriction \mathrm{ran}(\bar{b}) \preccurlyeq_{\mathfrak{K}} M\}$. We define

$E = \{(\bar{b}_1, \bar{b}_2) \in \mathbf{I} \times \mathbf{I} :$ if $\ell = 1, 2; j < i(\alpha_*)$ then $N \models \psi^{\alpha_*}_{(j,\ell)}(\bar{c}^{\alpha_*}_j, \bar{b}_1) \leftrightarrow \psi^{\alpha_*}_{(j,\ell)}(\bar{c}^{\alpha_*}_j, \bar{b}_2)\}$.

Clearly we have that:

$(\star_{7.1})$ (a) E is an equivalence relation;
(b) E has $\leqslant 2^{2\times i(\alpha_*)}$ equivalence classes;
(c) $2 \times i(\alpha_*) < \xi$ (recalling the assumptions on ξ);
(d) if $\bar{b}_1 E \bar{b}_2$, $\psi(\bar{z}_\gamma, \bar{x}_\nu) \in \Delta$ and $\alpha \in \mathcal{V}$, then $N \models \psi(\bar{c}_\alpha, \bar{b}_1) \leftrightarrow \psi(\bar{c}_\alpha, \bar{b}_2)$.
(e) for each $\alpha \in \mathcal{V}$, let

$$\mathbf{Y}_\alpha = \{Y : Y \text{ is an } E\text{-equivalence class s.t. } \bar{b} \in Y \Rightarrow N \models \psi(\bar{c}_\alpha, \bar{b})\};$$

(f) if $\alpha \neq \beta \in \mathcal{V}$, then $\mathbf{Y}_\alpha \neq \mathbf{Y}_\beta$;
(g) $|\{\mathbf{Y}_\alpha : \alpha \in \mathcal{V}\}| \leqslant 2^{|M^\nu| \times |\Delta|}$.

We prove $(\star_{7.1})$. The items needing proofs are (d), (f) and (g). Item (g) follows from (f). We are left with items (d) and (f). Concerning item (d), if not then, we have that for some $\alpha \in \mathcal{V}$, $\psi \in \Delta$ and $\bar{b}_1, \bar{b}_2 \in \mathbf{I}$, which are E-equivalent, we have

$$N \models \psi(\bar{c}_\alpha, \bar{b}_1) \leftrightarrow \neg\psi(\bar{c}_\alpha, \bar{b}_2).$$

By symmetry, w.l.o.g., we have that

$$N \models \psi(\bar{c}_\alpha, \bar{b}_1) \wedge \neg\psi(\bar{c}_\alpha, \bar{b}_2).$$

Now, as $\psi \in \Delta$ there is $\mathfrak{m}_1 \in \mathcal{W}^{\mathrm{large}}_{(\mathfrak{K},\nu,\gamma)}$ such that $\psi = \psi_{\mathfrak{m}_1}$. Since $\bar{b}_2 \in \mathbf{I}$ we can find $\mathfrak{m}_2 \in \mathcal{W}^{\mathrm{large}}_{(\mathfrak{K},\nu,\gamma)}$ such that $N \models \psi_{\mathfrak{m}_2}(\bar{c}_\alpha, \bar{b}_2)$, and obviously $\psi_{\mathfrak{m}_1}$ and $\psi_{\mathfrak{m}_2}$ are contradictory, so we get a contradiction to $i(\alpha) = i(\alpha_*)$. Together we are done. ■

Proof of 1.3. This follows from 3.4, 3.5, and [14, 3.3], since it easily follows from [14, 3.3] that the syntactic order property stated in 3.5 fails for any such AEC. ■

We make the following easy observation, which is relevant to the present context.

Claim 3.7. *Assume the following conditions:*
(1) $\mathfrak{K} = (\mathbf{K}, \preccurlyeq)$ *is an* AEC*;*
(2) $\mu > \mathrm{LS}(\mathfrak{K})$ *is a weakly compact cardinal;*
(3) $\mathfrak{K}_{\leqslant\mu} = (\mathbf{K}_{\leqslant\mu}, \preccurlyeq\restriction \mathbf{K}_{\leqslant\mu})$ *has amalgamation, where* $\mathbf{K}_{\leqslant\mu} = \{M \in \mathbf{K} : |M| \leqslant \mu\}$.
Then for every $M \in \mathbf{K}_{\leqslant\mu}$ *and* $p, q \in \mathcal{S}_{\mathfrak{K}_{\leqslant\mu}}(M)$ *we have that* $p = q$ *if and only if for every* $N \preccurlyeq M$ *of cardinality* $< \mu$, $p \restriction N = q \restriction N$.

Proof. Easy. ■

4. A counterexample to Mazari-Armida's question

Notation 4.1. Let $\mathcal{P}$ be a set of primes. We denote by $R_{\mathcal{P}}$ the sub-ring of the ring $\mathbb{Q}$ generated by $\{\frac{1}{p} : p \in \mathcal{P}\}$. For $\mathcal{P} = \{p\}$ we simply write R_p.

Definition 4.2. Let $\mathcal{P}$ be a set of primes and $G \in \mathrm{AB}$. We say that G is $\mathcal{P}$-torsion when G is torsion and, for any prime p, if $px = 0$ and $x \neq 0$, then $p \in \mathcal{P}$. We say that G is $\mathcal{P}$-divisible if $p \in \mathcal{P}$ implies that $pG = 0$.

Proof of Theorem 1.2. Let $\bar{p} = (p_1, ..., p_5)$ be distinct primes.

$(\star_1)$ We define $\mathbf{K} = \mathbf{K}(\bar{p})$ as the class of G such that:
(a) $G \in \mathrm{TFAB}$;

(b) for $\ell \in \{1, ..., 5\}$, we define $G[p_\ell] = \{a \in G : p_\ell^\infty \,|\, a\}$;
(c) $G[p_1]$ is an $\aleph_1$-free R_{p_1}-module (recall 4.1);
(d) if $G \neq G[p_1]$, then for some $a_\star$ we have:
  ($\cdot_1$) $a_\star \in G[p_2] \setminus \{0\}$;
  ($\cdot_2$) $G[p_2] = \langle a_\star \rangle^*_G \cong R_{p_2} a_\star$ (where $\langle a_\star \rangle^*_G$ denotes pure closure in G);
  ($\cdot_3$) inside G the group $C := G[p_1] \oplus G[p_2] \oplus G[p_3]$ is well-defined;
  ($\cdot_4$) G/C is $\{p_4, p_5\}$-torsion;
  ($\cdot_5$) for some partial embedding h from $G[p_1]$ onto $G[p_3]$ we have:

$$\{(x, h(x)) : x \in \mathrm{dom}(h)\} = \{(x, z) : x \in G[p_1],\, z \in G[p_3],\, p_4^\infty \,|\, (x + z)\}$$

$$G[p_4] = \{(x, h(x)) : x \in \mathrm{dom}(h))\};$$

  ($\cdot_6$) for $H_1 = \{x \in G[p_1] : \exists z \in G[p_3]$ such that $p_4^\infty \,|\, (x + z)\}$ we have:
    (i) H_1 is an $\aleph_1$-free R_{p_1}-module;
    (ii) $G[p_1]/H_1$ is an $\aleph_1$-free R_{p_1}-module;
  ($\cdot_7$) for $H_3 = \{z \in G[p_3] : \exists x \in G[p_1]$ such that $p_4^\infty \,|\, (x + z)\}$ we have:
    (i) H_3 is an $\aleph_1$-free R_{p_1}-module;
    (ii) $G[p_3]/H_3$ is an $\aleph_1$-free R_{p_1}-module;
  ($\cdot_8$) $G[p_5]$ is equal to A, where:

$$A = \langle \{r(x + a_\star + h(x)) : r \in R_{p_5}, x \in H_1\} \rangle_G.$$

($\star_2$) $(\mathbf{K}, \leqslant_{\mathrm{pure}})$ is an AEC.
[Why? Easy.]
($\star_3$) Fix λ infinite and let $G_\lambda = \bigoplus \{R_{p_1} x_\alpha : \alpha < \lambda\}$. Then $G_\lambda \in \mathbf{K}$.
($\star_4$) For every $\mathcal{U} \subseteq \lambda$ we define $G^*_{\mathcal{U}} \in$ TFAB as follows:
(a) $G^0_{\mathcal{U}} = G_\lambda \oplus N \oplus H$, where:

$$N = R_{p_2} y \text{ and } H = \bigoplus \{\mathbb{Z} z_\alpha : \alpha < \lambda\};$$

(b) $G^1_{\mathcal{U}} = \mathbb{Q} G_\lambda \oplus \mathbb{Q} N \oplus \mathbb{Q} H$;
(c) $G^*_{\mathcal{U}}$ is the subgroup of $G^1_{\mathcal{U}}$ generated by:
  (i) $p_1^{-n} x_\alpha$, $\alpha < \lambda$, $n < \omega$;
  (ii) $p_2^{-n} y$, $\alpha < \lambda$;
  (iii) $p_3^{-n} z_\alpha$, $\alpha < \lambda$, $n < \omega$;
  (iv) $p_4^{-n}(x_\alpha + z_\alpha)$, $\alpha \in \mathcal{U}$, $n < \omega$;
  (v) $p_5^{-n}(x_\alpha + y + z_\alpha)$, $\alpha \in \mathcal{U}$, $n < \omega$.

($\star_5$) For every $\mathcal{U} \subseteq \lambda$, $G^*_{\mathcal{U}} \in \mathbf{K}$ and $G_\lambda \leqslant_{\mathrm{p}} G^*_{\mathcal{U}}$.
[Why? Easy.]
($\star_6$) For $\mathcal{U} \subseteq \lambda$, let $t_{\mathcal{U}} = \mathrm{ortp}(y/G_\lambda; G^*_{\mathcal{U}})$.
($\star_7$) For $\mathcal{U} \neq \mathcal{V} \subseteq \lambda$, $t_{\mathcal{U}} \neq t_{\mathcal{V}}$.
($\star_8$) $(\mathbf{K}, \leqslant_{\mathrm{pure}})$ is not λ-stable, for every λ.
[Why? Follows from ($\star_5$) and ($\star_7$).] ■

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