MAXIMAL FAILURES OF SEQUENCE LOCALITY IN AEC SH932

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Dedicated to the memory of Andrzej Mostowski

ABSTRACT. We are interested in examples of AECs $\mathfrak{k}$ having some (extreme) behavior concerning types, preferably with amalgamation. Note, we deal with $\mathfrak{k}$ being sequence-local, i.e. local for increasing chains of length a regular cardinal (for types, equality of all restrictions implies equality, some call it *tame*). We construct an AEC $\mathfrak{k}$ and $\mathrm{LST}(\mathfrak{k}) = \theta, |\tau_{\mathfrak{K}}| = \theta$ such that the following class is maximal: $\{\kappa : \kappa$ is a regular cardinal and $\mathfrak{K}$ is not $(2^\kappa, \kappa)$-sequence-local$\}$. In fact, we have a direct characterization of this class of cardinals: the regular κ such that there is no uniform κ^+-complete ultrafilter (on any $\lambda > \kappa$). We also prove a similar result to "$(2^\kappa, \kappa)$-compact for types" for $\mathfrak{k}$.

Date: December 2, 2025.

1991 *Mathematics Subject Classification.* Primary: 03C48,03E55; Secondary: 03E05.

Key words and phrases. model theory, abstract elementary classes, types, locality, compactness of types.

The author thanks Alice Leonhardt for the beautiful typing up to 2019. First typed October 20, 2007. In later versions, the author thanks typing services generously funded Craig Falls, and we thank a typist for the careful and beautiful typing. We thank the ISF (Israel Science Foundation) by grant 1838(19) (2019-1023) and grant 2320/23 (2023-2027) and older grants, for partially supporting this research. References like [She, Th0.2=Ly5] mean the label of Th.0.2 is y5. The reader should note that the version on my website is usually more up-to-date than the one in the mathematical archive. Publication number 932 in the author list of publications.

0. Introduction

Recall AECs (abstract elementary classes); were introduced in [She87a]; and their (orbital) types defined in [She87b], see on them [She09b], [Bal09]. It has seemed to us obvious that even with $\mathfrak{k}$ having amalgamation, those types in general lack some good properties of the classical types in model theory. E.g. "(λ, κ)-sequence-locality where,

Definition 0.1. 1) We say that an AEC $\mathfrak{k}$ is a (λ, κ)-sequence-local (for types) <u>when</u> κ is regular and for every $\leq_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i \leq \kappa \rangle$ of models of cardinality λ and $p, q \in \mathscr{S}(M_\kappa)$ we have $(\forall i < \kappa)(p \restriction M_i = q \restriction M_i) \Rightarrow p = q$. We omit λ when we omit "$\|M_i\| = \lambda$".
2) We say an AEC $\mathfrak{k}$ is (λ, κ)-local when: $\kappa \geq$ LST$(\mathfrak{k})$ and if $M \in \mathfrak{k}_\lambda$ and $p_1, p_2 \in \mathscr{S}(M)$ and,

(∗) for every N we have, $N \leq_{\mathfrak{k}} M \wedge \|N\| \leq \kappa \Rightarrow p_1 \restriction N = p_2 \restriction N$ <u>then</u> $p_1 = p_2$.

3) We may replace λ by $\leq \lambda, < \lambda, [\mu, \lambda]$ with the obvious meaning (and allow λ to be infinity).

Of course, being sure is not a substitute for a proof; some examples of failures of being local were provided by Baldwin-Shelah [BS08, §2]. Also note our using: "Abelian groups without zero" is similar to e.g., the work [HS90]. Now [BS08] gives an example of the failure of (λ, κ)-sequence-locality for $\mathfrak{k}$-types in ZFC for some λ, κ, actually $\kappa = \aleph_0$. This was done by translating our problems to abelian group problems; using Abelian groups which are not Whitehead. While those problems seem reasonable by themselves, they may hide our real problem.

Here in §1 we get $\mathfrak{k}$, an AEC with the class $\{\kappa : (< \infty, \kappa)$-sequence-localness fail for $\mathfrak{k}\}$ being maximal, with amalgamation and the JEP. see Theorem 1.4; here the cardinality of its vocabulary as well as its LST number is θ for any given $\theta = \theta^{\aleph_0}$, we intend to deal with the other θ later. Also we deal with "compactness of types", getting classes with amalgamation; in [BS08], in some cases this was done there only in some universes of set theory; see §2.

We rely on a criterion from [BS08] to prove that $\mathfrak{k}$ has the JEP and amalgamation.

Question 0.2. Can $\{\kappa : \mathfrak{k}$ is $(< \kappa^+, \kappa)$-local$\}$ be "wild"? E.g. can it be all odd regular alephs? etc?

Similarly for $(< \infty, \kappa)$- sequentially local.

In §2 we deal with sequence-compactness of types.

Mostowski [Mos57] initiated the quest to find strengthenings of first-order logic that still have a "good model theory". Usually, one may add generalized quantifiers (e.g., $(\exists^{\geq \aleph_1} x)$) and/or allow certain infinitary operations (e.g., $\bigwedge_{\alpha < \lambda} \psi_\alpha$). There is much to be said on this topic; see the collection [Bar85] and, later, Väänänen's book [Vï1].

In particular, Lindström proved that one cannot expect too much: either the downward Löwenheim–Skolem property to $\aleph_0$ fails, or $\aleph_0$-compactness fails.

Now, *abstract elementary classes* (AECs) continue this, trying to deal directly at the model theory. E.g. concerning "a theory T in logic $\mathbb{L}(\exists^{\geq \aleph_1})$", we define the AEC $\mathfrak{k} = \mathfrak{k}_T$, by:

- $K_{\mathfrak{k}}$ is the model of T,

- $M \leq_{\mathfrak{k}} N$ iff in addition to $M \prec_{\mathbb{L}(\exists^{\aleph_1})} N$, which is naturally defined, we demand that, if $M \models (\exists^{\leq \aleph_0} x)\psi(x, \bar{a})$ then not only $N \models (\exists^{\leq \aleph_0})\varphi(x, \bar{a})$ but $N \models \varphi[b, a] \Rightarrow b \in M$.

Similarly, for e.g. the logic $\mathbb{L}_{\lambda^+, \aleph_0}$.

This work is part of the attempt to sort out which properties of first-order logic hold for AECs, particularly when $\mathfrak{k}$ is an AEC with amalgamation, in §2 the amalgamation property was added lately. This work was submitted to Jouko Väänänen in October 2009 for a volume in honor of Andrzej Mostowski, and deposited in the arXiv. Later and independently, Boney [Bon14] investigated such things mainly for compact cardinals, in particular, has results close to 1.8 (and 2.8).

We are grateful to the referees for their helpful comments and to Will Boney for pointing out a correction, and to Santiago Pinzón for corrections.

It is my pleasure to dedicate this to the memory of Andrzej Mostowski, who contributed so much to mathematical logic and particularly to starting other logics and generalized quantifiers in [Mos57].

1. An AEC with maximal failure of being local

Claim 1.1. *Assume*

$\circledast_1$ (a) $\kappa = \mathrm{cf}(\kappa) > \sigma \geq \aleph_0$ *and* $\sigma = \sigma^{\aleph_0}$,

(b) *there is no uniform* σ^+*-complete ultra-filter* D *on* κ

(c) τ_θ *is the vocabulary*

$\{E_n, E'_n : n \in [2, \omega)\} \cup \{F_c : c \in [\sigma]^{<\aleph_0}\} \cup \{R_e : e \in {}^\sigma\sigma\} \cup \{R_0, R_1\}$

where each R_e *is two-place predicate, each* F_c *is an unary function symbol,* R_0, R_1 *are two-place and unary predicates respectively and* E'_n, E_n *are* $(2n)$*-place predicates for* $n \geq 2$,

(d) τ^*_θ *is* $\tau_\theta \setminus \{R_1\}$.

Then

$\boxplus$ *there are* $I_\alpha, \pi_\alpha, M_{\ell,\alpha}$ *(for* $\ell = 1, 2$ *and* $\alpha \leq \kappa$*), and* g_α *(for* $\alpha < \kappa$*) satisfying:*

(a) I_α*, a set of cardinality* σ^κ*, is* $\subseteq$*-increasing continuous with* α.

(b) $M_{\ell,\alpha}$*, a* τ_θ*-model of cardinality* $\leq \theta^\kappa$*, is increasing continuous with* α

(c) π_α *is a function from* $M_{\ell,\alpha}$ *onto* I_α*, increasing continuous with* α,

(d) $|\pi_\alpha^{-1}\{t\}| \leq \sigma$ *for* $t \in I_\alpha, \alpha \leq \kappa$ *and* $\ell = 1, 2$,

(e) *if* $t \in I_{\alpha+1} \backslash I_\alpha$ *then* $\pi_\alpha^{-1}\{t\} \subseteq M_{\ell,\alpha+1} \backslash M_{\ell,\alpha}$,

(f) *for* $\alpha < \kappa, g_\alpha$ *is an isomorphism from* $M_{1,\alpha}$ *onto* $M_{2,\alpha}$ *respecting* π_α *which means* $a \in M_{1,\alpha} \Rightarrow \pi_\alpha(a) = \pi_\alpha(g_\alpha(a))$,

(g) *for* $\alpha = \kappa$ *there is no isomorphism from* $M_{1,\alpha}$ *onto* $M_{2,\alpha}$ *respecting* π_α.

Proof. Follows from 1.2 which is just a fuller version explicating the unary function F_c for $c \in G$; anyhow we shall use only 1.2. $\square_{1.1}$

Claim 1.2. *Assuming* $\circledast_1$ *of 1.1 we have:*

⊞ *there are* $I_\alpha, A_\alpha, \pi_\alpha, M_{\ell,\alpha}$ *(for* $\ell = 1, 2, \alpha \le \kappa$*) and* g_α *(for* $\alpha < \kappa$*) and* G *such that:*

(a) G *is an additive (so abelian) group of cardinality* σ*,*

(b) I_α *is a set, increasing continuous with* $\alpha, |I_\alpha| = \sigma^\kappa$*,*

(c) $M_{\ell,\alpha}$ *is a* τ_θ*-model, increasing continuous with* α*, of cardinality* θ^κ *with universe* A_α*,*

(d) π_α *is a function from* $M_{\ell,\alpha}$ *onto* I_α*, increasing continuous with* α*,*

(e) $F_c^{M_{\ell,\alpha}}(c \in G)$ *is a permutation of* $M_{\ell,\alpha}$*, increasing continuous with* α*,*

(f) $\pi_\alpha(a) = \pi_\alpha(F_c^{M_{\ell,\alpha}}(a))$ *for* $a \in M_{\ell,\alpha}$*,*

(g) $F_{c_1}^{M_{\ell,\alpha}}(F_{c_2}^{M_{\ell,\alpha}}(a)) = F_{c_1+c_1}^{M_{\ell,\alpha}}(a)$*,*

(h) $\pi_\alpha(a) = \pi_\alpha(b) \Leftrightarrow \bigvee_{c \in G} F_c^{M_{\ell,\alpha}}(a) = b$*,*

(i) *for* $\alpha < \kappa, g_\alpha$ *is an isomorphism from* $M_{1,\alpha}$ *onto* $M_{2,\alpha}$ *which respects* π_α *which means* $a \in M_{1,\alpha} \Rightarrow \pi_\alpha(a) = \pi_\alpha(g_\alpha(a))$*,*

(j) *there is no isomorphism from* $M_{1,\kappa} \restriction \tau_\theta$ *onto* $M_{2,\kappa} \restriction \tau_\theta$ *respecting* π_κ*,*

(k) $M_{1,\alpha} \restriction \tau^*_\theta = M_{2,\alpha} \restriction \tau^*_\theta$*, for* $\alpha \le \kappa$*.*

Discussion 1.3. We shall try to shed some light on 1.2 on how we intend to use [?] in the proof of [?], see Discussion 1.9. Note that, the models $M_{1,\alpha}$, $M_{2,\alpha}$ are almost the same.

Proof. Let

$(*)_1$ (a) let $G = ([\sigma]^{<\aleph_0}, \Delta)$, i.e., the family of finite subsets of σ with the operation of symmetric difference. This is an abelian group satisfying $\forall x(x + x = 0)$, but we may identify $\varepsilon < \sigma$ with $\{\varepsilon\}$, so treating ordinals as atoms,

(b) let $\langle a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < \kappa, u \in G\rangle$ be a sequence without repetitions,

(c) for $\beta \le \kappa$ let $A_\beta = \{a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < 1 + \beta \text{ and } u \in G\}$,

(d) for $\beta \le \kappa$ let $I_\beta = ({}^\kappa\sigma) \times (1 + \beta)$,

(e) π_β be the function with domain A_β such that, we let $\pi_\beta(a_{f,\alpha,u}) = (f, \alpha)$ when $\alpha < 1 + \beta \le \kappa$,

(f) for each $\beta < \kappa$ we define a permutation g_β (of order 2) of A_β by $g_\beta(a_{f,\alpha,u}) = a_{f,\alpha,u+_G\{f(\beta)\}}$ hence $a \in A_\beta \Rightarrow \pi_\beta(g_\beta(a)) = \pi_\beta(a)$.

Note that

$(*)_2$ (a) $|G| = \sigma$,

(b) $\langle A_\beta : \beta \le \kappa\rangle$ is a $\subseteq$-increasing continuous, each A_β a set of cardinality σ^κ,

(c) $\langle I_\beta : \beta \le \kappa\rangle$ is $\subseteq$-increasing continuous, each I_β of cardinality σ^κ,

(d) π_β is a mapping from A_β onto I_β,

(e) if $t \in I_\alpha \subseteq I_\beta$ then $\pi_\beta^{-1}\{t\} = \pi_\alpha^{-1}\{t\}$ has cardinality $|G| = \sigma$,

(f) if $t \in I_{\alpha+1} \backslash I_\alpha$ then $\pi_{\alpha+1}^{-1}\{t\} \subseteq A_{\alpha+1} \backslash A_\alpha$,

(g) if $\alpha \le \beta \le \kappa$ then g_β maps A_α onto itself and $g_\beta \circ g_\beta$ is the identity.

For each $n \in [2, \omega)$ and $\beta \le \kappa$ we define equivalence relations $E'_{n,\beta}, E_{n,\beta}$ on ${}^n(A_\beta)$:

$(*)_3$ $\bar a E'_{n,\beta} \bar b$ iff $\pi_\beta(\bar a) = \pi_\beta(\bar b)$ where of course $\pi_\beta(\langle a_\ell : \ell < n\rangle) = \langle \pi_\beta(a_\ell) : \ell < n\rangle$.

$(*)_4$ $\bar a E_{n,\beta} \bar b$ iff $\bar a E'_{n,\beta} \bar b$ and there are $k < \omega$ and $\bar a_0, \dots, \bar a_k$ such that:
- (i) $\bar a_\ell \in {}^n(A_\beta)$,
- (ii) $\bar a = \bar a_0$,
- (iii) $\bar b = \bar a_k$,
- (iv) for each $\ell < k$ for some $\alpha_1, \alpha_2 < \kappa$ we have $g^{-1}_{\alpha_2}(g_{\alpha_1}(\bar a_\ell))$ is well defined and equal to $\bar a_{\ell+1}$ or $g_{\alpha_2}(g^{-1}_{\alpha_1}(\bar a_\ell))$ is well defined and equal to $\bar a_{\ell+1}$.

Note:

$(*)_{4.1}$
- (a) the two possibilities in $(*)_4(iv)$ are one as $g^{-1}_\alpha = g_\alpha$ so the first one is equal to the second,
- (b) g_α does not preserve $\bar a / E_{n,\beta}$!, in fact, $a, g_\alpha(a)$ are never $E_{n,\beta}$ equivalent,
- (c) clearly in $(*)_4$(iv) for $\ell < k$, the terms are well defined iff $\bar a_\ell \in {}^n(A_{\min\{\alpha_1,\alpha_2\}})$ because if $\alpha \le \beta$ then g_β maps A_α onto itself,
- (d) if $\alpha \le \beta, a \in A_\alpha$, then g_β maps $a/E_{n,\beta}$ onto itself
- (e) if $\alpha, \beta \le \kappa$, then g_α, g_β commute (on the intersection of their domains, $A_{\min\{\alpha,\beta\}}$).

[Why? Easy, e.g.

Clause b: Why? Let $a = a_{f_1,\gamma_1,u_1}$, $b = a_{f_2,\gamma_2,u_2}$. Now, on the one hand, if $g_\alpha(a) = b$ then $f_1 = f_2$, $\gamma_1 = \gamma_2$ and $u_1 +_G u_2 = u_1 \triangle u_2$ has cardinality 1, in fact is equal to $\{f_\alpha(\gamma)\}$. On the other hand, if $\langle a\rangle E_{n,\beta} \langle b\rangle$ then (by induction on k in the definition), we can prove $f_1 = f_2$, $\gamma_1 = \gamma_2$ and $u_1 +_G u_2 = u_1 \triangle u_2$ is a set of even cardinality.

Clause (e): Just recalling that G is a commutative group.]

Note,

$(*)_5$ For $n \in [2, \omega)$, we have:
- (a) $E'_{n,\beta}, E_{n,\beta}$ are indeed equivalence relations on ${}^n(A_\beta)$,
- (b) $E_{n,\beta}$ refine $E'_{n,\beta}$,
- (c) if $\bar a \in {}^n(A_\beta)$ then $\bar a / E'_{n,\beta}$ has exactly σ members,
- (d) if $\alpha < \beta \le \kappa$ then $E'_{n,\beta} \restriction {}^n(A_\alpha) = E'_{n,\alpha}$ and $E_{n,\beta} \restriction {}^n(A_\alpha) = E_{n,\alpha}$ (read $(*)_4(iv)$ carefully!),
- (e) if $\alpha < \beta \le \kappa, \bar a \in {}^n(A_\alpha)$ and $\bar b \in \bar a / E'_{n,\beta}$ then $\bar b \in {}^n(A_\alpha)$,
- (f) if $g_\alpha(\bar a_\ell) = \bar b_\ell$ for $\ell = 1, 2$ then: $\bar a_1 E'_{n,\beta} \bar a_2$ iff $\bar b_1 E'_{n,\beta} \bar b_2$.

Now we recall the vocabulary τ_θ of cardinality 2^σ from 1.1⊛(d) and for $\alpha \le \kappa$ we choose a τ_θ-model $M_{1,\alpha}$ such that:

$(*)_6$
- (a) $M_{1,\alpha}$ increasing with α and has universe A_α,
- (b) $\bullet_1$ let $R_0^{M_{1,\alpha}} = \{{}^2(A_\alpha)$: if $\bar a = \langle a_{f_\ell,\alpha_\ell,u_\ell} : \ell < 2\rangle$, then $u_1 = u_2\}$, and

$\bullet_2$ let $R_1^{M_{1,\alpha}} = \{^1(A_\alpha)$: if $\bar{a} = \langle a_{f,\alpha,uu_\ell} : \ell < 2\rangle$,
<u>then</u> u has even number of elements$\}$,

(c) $\tau_\theta = \tau(M_{1,\alpha})$ is defined in $\circledast_1$(c) from 1.1,

(d) for every function $e \in {}^\sigma\sigma$

$$R_e^{M_{1,\alpha}} = \{(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in A_\alpha \times A_\alpha : f_1 = e \circ f_2 \text{ and if } i < \sigma \text{ then } i \in u_1 \text{ iff } (|\{j \in u_2 : e(j) = i\}| \text{ is odd})\}$$

recalling $f_\ell \in {}^\kappa\sigma$,

(e) $\bullet_1$ $E_n^{M_{1,\alpha}} = E_{n,\alpha}$ for $n < \omega$,
$\bullet_2$ $(E'_{n_1})^{M_{1,\alpha}} = E'_{n,\alpha}$ for $n < \omega$
$\bullet_3$ $F_c^{M_{1,\alpha}} = F_c$ is defined by $F_c : A_\alpha \to A_\alpha$ satisfies $F_c(a_{f,\alpha,u}) := a_{f,\alpha,u+_G c}$ for $c \in G$.

(f) if $\alpha \le \beta_\ell < \kappa$ for $\ell = 1,2$ <u>then</u> $g_{\beta_2}^{-1} g_{\beta_1} \restriction A_\alpha$ is an automorphism of $M_{1,\alpha}$

(g) g_α is almost an automorphism of $M_{1,\alpha}$, it miss preserving R_1.

[Why is this possible? The universe of $M_{1,\alpha}$ is defined in clause (a), its vocabulary in clause (c), the interpretation of the predicates in clauses (b), (d), (e)$\bullet_1$, $\bullet_2$ and the interpretations of the functions symbols in clause (e)$\bullet_3$. So we are done with clauses (a)-(e), except concerning $M_{1,\alpha}$ being increasing with α, see (a). Also, we have to prove clauses (f) and (g). Of course, every g_α (hence $g_{\beta_2}^{-1} \circ g_{\beta_1}$) is a permutation of A_α, the universe of $M_{1,\alpha}$.

Now, to finish the proof of $(*)_5$(a), notice that there is no problem in proving the $M_{1,\alpha}$'s are increasing, e.g. by $(*)_5$(d), just check that.

We now prove clause (g) of $(*)_6$. Toward this, <u>first</u>, we shall show that for each $\alpha < \kappa, g_\alpha$ maps $R_e^{M_{1,\alpha}}$ onto itself.

Assume we are given a pair $(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2})$ from $A_\alpha \times A_\alpha$ so $\beta_1, \beta_2 < 1 + \alpha$. Clearly, $f_1 \ne e \circ f_2$ implies that

$$(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \notin R_e^{M_{1,\alpha}} \text{ and } (g_\alpha(a_{f_1,\beta_1,u_1}), g_\alpha(a_{f_2,\beta_2,u_2})) \notin R_e^{M_{1,\alpha}},$$

hence without loss of generality $f_1 = e \circ f_2$ so,

$(*)_{6.1}$ $(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in R_e^{M_{1,\alpha}}$ iff

$$u_1 = \{e(j) : j \in u_2 \text{ and } (\exists^{\text{odd}} \iota \in u_2)(e(\iota) = e(j))\}.$$

[Why? Read $(*)_6$(d) carefully, in particular note that if $i \notin \{e(j) : j \in u_2\}$ then $i \notin u_1$.]

$(*)_{6.2}$ $(g_\alpha(a_{f_1,\beta_1,u_1}), g_\alpha(a_{f_2,\beta_2,u_2})) \in R_e^{M_{1,\alpha}}$ iff
$(a_{f_1,\beta_1,u_1+\{f_1(\alpha)\}}, a_{f_2,\beta_2,u_2+_G\{f_2(\alpha)\}}) \in R_e^{M_{1,\alpha}}$ iff $u_1 +_G \{f_1(\alpha)\}$ is equal to the following set:

$$\big\{e(j) : j \in u_2 +_G \{f_2(\alpha)\} \text{ and } [\exists^{\text{odd}} \iota \in (u_2 +_G \{f_2(\alpha)\}][e(\iota) = e(j)]\big\}.$$

[Why? Straightforward but we shall elaborate, Inside $(*)_{6.2}$ the first "iff" holds by the definition of g_α, the second "iff" holds as in $(*)_{6.1}$.]

But $f_1 = e \circ f_2$ hence

$(*)_{6.3}$ $f_1(\alpha) = e(f_2(\alpha))$.

Next

$(*)_{6.4}$ letting $x = f_2(\alpha) < \sigma$ the following statements are equivalent:

(a) $u_1 = \{e(j) : j \in u_2 \text{ and } (\exists^{\rm odd}\iota \in u_2)(e(\iota) = e(j))\}$

(b) $u_1 +_G \{e(x)\} = \{e(j) : j \in u_2 +_G \{x\} \text{ and } \exists^{\rm odd}\iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}$.

[Why? We shall prove it for any $u_1, u_2 \in G$ and $x < \sigma$, By "G is of order two", it suffices to prove the "only if" so assume the equality in $(*)_{6.4}(a)$ and we have to prove the equality in $(*)_{6.4}(b)$.

If $e(x) \notin u_1$ and $x \notin u_2$ then and the equality is clear: we just add $e(x)$ to both sides of $(*)_{6.4}(a)$ to get $(*)_{6.4}(b)$.

Now assume $e(x) \notin u_1$ and $x \in u_2$ then the left side in clause $(*)_{6.4}(b)$ is the disjoint union of the left side of $(*)_{6.4}(a)$ (which is u_1) and $\{e(x)\}$. As we are assuming $e(x) \notin u_1$, necessarily $\exists^{\rm even}\iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}$, hence the right side of $(*)_{6.4}(b)$ is the disjoint union of the right side of $(*)_{6.4}(a)$ and $\{e(x)\}$. As we are assuming the equality in $(*)_{6.4}(a)$, the last two sentences implies the equality in $(*)_{6.4}(b)$.

Next assume $e(x) \in u_1$ and $x \notin u_2$. Then the left side in clause $(*)_{6.4}(b)$ is the result of subtracting $\{e(x)\}$ from the left side of $(*)_{6.4}(a)$ (which is u_1). As we are assuming $x \in u_2$ necessarily $\exists^{\rm odd}\iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}$, hence the right side of $(*)_{6.4}(b)$ is the result of subtracting $\{e(x)\}$ from the right side of $(*)_{6.4}(a)$. As we are assuming the equality in $(*)_{6.4}(a)$, the last two sentences implies the equality in $(*)_{6.4}(b)$.

Lastly assume $e(x) \in u_1$ and $x \in u_2$. Then the left side in clause $(*)_{6.4}(b)$ is the result of subtracting $\{e(x)\}$ from the left side of $(*)_{6.4}(a)$ (which is u_1). As we are assuming $e(x) \in u_1$, necessarily $\exists^{\rm odd}\iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}$, hence the right side of $(*)_{6.4}(b)$ is the result of subtracting $\{e(x)\}$ from the right side of $(*)_{6.4}(a)$ and $\{e(x)\}$. As we are assuming the equality in $(*)_{6.4}(a)$, the last two sentences implies the equality in $(*)_{6.4}(b)$.

So together we get equivalence, hence (for proving $(*)_6$(f)) the "first" holds. Second, we prove that g_α preserves "$\bar a, \bar b$ are $E_{n,\alpha}$-equivalent", "$\bar a, \bar b$ are $E'_{n,\alpha}$-equivalent" and their negations. That is, $\bar a, g_\alpha(\bar a)$ are not $E_{n,\alpha}$-equivalent, but as $(\forall\beta)(g_\beta = g^{-1}_\beta), \bar a, \bar b$ being $E_{n,\alpha}$-equivalent means that there is an even length pass from $\bar a$ to $\bar b$, in the graph $\{(\bar c, g_\beta(\bar c)) : \beta \in [\gamma, \kappa) \text{ and } \bar c \in {}^n(A_\gamma)\}$ where $\gamma = \min\{\gamma : \bar a, \bar b \in {}^n(A_\gamma)\}$. This proves another part of clause (g) of $(*)_6$.

Third, , g_α commutes with $F_c^{M_{1,\alpha}}$ for $c \in G$ because G is an Abelian group; thus completing the proof of $(*)_6$(g).

Fourth,it maps $R_0^{M_{1,\alpha}}$ to itself by the definition of g_α (in $(*)_2$(f)) and of $R_0^{M_{1,\alpha}}$ (in $(*)_6(b)\bullet_2$) and of G.

Lastly it does not preserve $R_1^{M'_{1,\alpha}}$ onto itself by the definition of $R_1^{M'_{1,\alpha}}$ in $(*)_6(b)\bullet_2$ and of g_α in $(*)(f)$.

This complete the proof of clause (g) of $(*)_6$.

Next, we should check clause $(*)_6(f)$. Now $g^{-1}_{\beta_2} g_{\beta_1}{\restriction}A_\alpha = (g_{\beta_2}{\restriction}A_\alpha)(g_{\beta_1}{\restriction}A_\alpha)$ by $(*)_2(g)$ and it has order 2 because G is of order 2.

By clause (g) of $(*)_2$ wet have to prove that $g^{-1}_{\beta_2} g_{\beta_1} \restriction A_\alpha$ maps $R_0^{M_{1,\alpha}}$ onto itself, which can b verified by definition of $M_{2,\alpha}$.

So we are done proving $(*)_6$.

$(*)_7$ for $\alpha < \kappa$ let $M_{2,\alpha}$ be the τ_θ-model with universe A_α such that g_α is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$.

Now we note

$(*)_8$ for $\alpha < \beta < \kappa, M_{2,\alpha} \subseteq M_{2,\beta}$, that is, $M_{2,\alpha}$ is a submodel of $M_{2,\beta}$.

[Why? By $(*)_6(b) \bullet_1 + (*)_7$ we have just to check that $R_1^{M_{2,\alpha}} = R_1^{M_{2,\beta}} \restriction M_{2,\alpha}$, which holds because for $a \in A_\alpha$ we have $a \in R_1^{M_{2,\alpha}}$ iff $g_\alpha^{-1}(a) \in R_1^{M_{1,\alpha}}$ iff $g_\alpha^{-1}(a) \in R_1^{M_{1,\beta}}$ iff $g_\beta(g_\alpha^{-1}(a))) \in R_1^{M_{1,\beta}}$. This clearly suffice.

$(*)_9$ let $M_{2,\kappa} := \cup\{M_{2,\alpha} : \alpha < \kappa\}$,

$(*)_{10}$ $M_{2,\kappa}$ well defined by $(*)_8$,

$(*)_{11}$ π_α is well defined by $(*)_1$(f),

$(*)_{12}$ except clause (j) the demands in the conclusion of $\boxplus$ of 1.2 were proved.

[Why? Just check.]

Note

$(*)_{13}$ if $(a_{f,\alpha,u_1}, a_{f,\alpha,u_2})$ is $E_{2,\alpha}$-equivalent to $(a_{f,\alpha,v_1}, a_{f,\alpha,v_2})$ then $G \models$ "$u_1 - u_2 = v_1 - v_2$".

[Why? By induction on the k from $(*)_4$.]

So, to finish, we assume toward contradiction:

$\boxtimes$ h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ which respects π_κ hence π_α for $\alpha < \kappa$, i.e. $h \restriction M_{1,\alpha}$ respect π_α, see clause $\boxplus$(i) of Claim 1.2.

So trivially

$\circledast_1$ if $\alpha < \kappa$, then $h(a_{f,\gamma,u}) \in \{a_{f,\gamma,v} : v \in G\}$ for $\gamma < 1 + \alpha$, and $\bar a \in {}^n(A_\alpha) \Rightarrow h(\bar a) \in \bar a / E'_{n,\alpha}$.

[Why? As $h \restriction M_{1,\alpha}$ respect π_α see $(*)_1(e)$ and $(*)_6(g) + (*)_7 + (*)_8 + (*)_9$ clearly $h(\bar a) \in \bar a / E'_{n,\alpha}$.]

Hence,

$\circledast_2$ for $f \in {}^\kappa\sigma$ and $\alpha < \kappa$ let $u_{f,\alpha} \in G$ be the $u \in G$ such that $h(a_{f,\alpha,\emptyset}) = a_{f,\alpha,u}$.

$\circledast_3$ for $f \in {}^\kappa\sigma, \alpha < \kappa$ and $v \in G$ we have $h(a_{f,\alpha,v}) = a_{f,\alpha,v+_G u_{f,\alpha}}$.

[Why? For $c \in G$, we know that h maps $F_c^{M_{1,\alpha}}$ onto $F_c^{M_{2,\alpha}}$ which is equal to $F_c^{M_{1,\alpha}}$. Apply this to $c = v$.]

$\circledast_4$ we define a partial order $\le$ on ${}^\kappa\sigma$ as follows:
$f_1 \le f_2$ <u>iff</u> there is a function $e \in {}^\sigma\sigma$ witnessing it; which means $f_1 = e \circ f_2$

$\circledast_5$ if $\alpha_1, \alpha_2 < \kappa$ and $f_1 \le f_2$ (are from ${}^\kappa\sigma$) then $|u_{f_1,\alpha_1}| \le |u_{f_2,\alpha_2}|$.

[Why? This follows from $\circledast_6$ below.]

$\circledast_6$ if $e \in {}^\sigma\sigma, f_2 \in {}^\kappa\sigma$ and $f_1 = e \circ f_2 \in {}^\kappa\sigma$ and $\alpha_1, \alpha_2 < \kappa$ then
$u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$.

[Why? Choose $\alpha < \kappa$ such that $\alpha > \alpha_1, \alpha > \alpha_2$ so $a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_1,\emptyset} \in M_{\ell,\alpha}$ for $\ell = 1,2$. Recall that h maps $R_e^{M_{1,\alpha}}$ onto $R_e^{M_{2,\alpha}}$ by $\boxtimes$ and $R_e^{M_{2,\alpha}} = R_e^{M_{1,\alpha}}$ because g_α maps $R_e^{M_{1,\alpha}}$ onto itself (by $(*)_6(g)$ <u>or</u> see the proof of $(*)_6$ above, the "first" in that proof). Now, let $x \in \sigma \setminus u_{f_2,\alpha_1}$, so see $(*)_6(d)$, i.e. the definition of $R_e^{M_{1,\alpha}}$, obviously $(a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_2,\{x\}}) \in R_e^{M_{1,\alpha}}$ so as h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ we have $(h(a_{f_1,\alpha_1,\emptyset}), h(a_{f_2,\alpha_2,\{x\}})) \in R_e^{M_{2,\alpha}}$ so by the previous sentence and by $\circledast_3$ and the definitions of $u_{f_\ell,\alpha_\ell}(\ell = 1,2)$ in $\circledast_2$ letting $v_1 = u_{f_1,\alpha_1}$, and $v_2 = v_{f_2,\alpha_2} +_G \{x\}$ we have $(a_{f_1,\alpha_1,v_1}, a_{f_2,\alpha_2,v_2}) \in R_e^{M_{1,\alpha}}$ which by the definitions of $R_e^{M_{1,\alpha}}$ in $(*)_6(d)$ implies $u_{f_1,\alpha_1} \subseteq \{e(i) \colon i \in u_{f_2,\alpha_2}\} \cup \{x\}$, which by the choice of x implies that $u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$ as promised.]

$\circledast_7$ (a) $|u_{f,\alpha_1}| = |u_{f,\alpha_2}|$ for $\alpha_1, \alpha_2 < \kappa, f \in {}^\kappa\sigma$,
(b) $\mathbf{n}(f) = |u_{f,\alpha}|$ is well defined for $\alpha < \kappa$,
(c) if $f_1 \le f_2$ then $\mathbf{n}(f_1) \le \mathbf{n}(f_2)$.

[Why? For clause (a) use $\circledast_6$ twice for the function $e = \mathrm{id}_\sigma$ and $f_1 = f_2 = f$. Clause (b) follows. Clause (c) holds by $\circledast_6$ equivalently by $\circledast_5$.]

$\circledast_8$ there are $f_* \in {}^\kappa\sigma$ and $\alpha_* < \kappa$ such that:
(i) if $f_* \le f \in {}^\kappa\sigma$ and $\alpha < \kappa$ then $|u_{f_*,\alpha_*}| = |u_{f,\alpha}|$
(ii) moreover if $f_* = e \circ f$ where $e \in {}^\sigma\sigma$ and $f \in {}^\kappa\sigma, \alpha < \kappa$ $u_{f_*,\alpha}$ so $\mathbf{n}(f_*) = \mathbf{n}(f)$
(iii) if $\alpha < \kappa, f_1 = e \circ f_2, f_* = e_1 \circ f_1, f_* = e_2 \circ f_2$ so $e, e_1, e_2 \in {}^\sigma\sigma$, <u>then</u> $e{\restriction}u_{f_2,\alpha}$ is one-to-one onto $u_{f_1,\alpha}$.

[Why? First note that clause (ii), (iii) follows from clause (i)+$\circledast_6$. Second, if claus <u>then</u> $e \restriction u_{f,\alpha}$ is one-to-one from $u_{f,\alpha}$ ontoe (i) fails, then we can find a sequence $\langle (f_n, \alpha_n, e_n) : n < \omega \rangle$ such that

(α) $\alpha_n < \kappa, f_n \in {}^\kappa\sigma$ for $n < \omega$
(β) $f_n \le f_{n+1}$ say $f_n = e_n \circ f_{n+1}$ and $e_n \in {}^\sigma\sigma$
(γ) $(e_n, f_{n+1}, \alpha_{n+1})$ witness that (f_n, α_n) does not satisfy the demand (i) on (f_*, α_*) hence $\mathbf{n}(f_n) < \mathbf{n}(f_{n+1})$.

Recalling we assume $\sigma = \sigma^{\aleph_0}$, there are functions pr_ω and $e^n \in {}^\kappa\sigma$ for $n < \omega$ such that $\mathrm{pr}_\omega \colon {}^\omega\sigma \to \sigma$ is 1-to-1, onto and $\mathrm{pr}_\omega(\bar\alpha) = \beta \wedge n < \omega \Rightarrow e^n(\beta) = \alpha_n$. Now, define $f \in {}^\kappa\sigma$ by $f(\beta) = \mathrm{pr}_\omega(\langle f_n(\beta) \colon n < \omega \rangle)$, clearly $f \in {}^\kappa\sigma$ and $f_n = e^n \circ f$

So there is a sequence $\langle e^n : n < \omega \rangle$ satisfying $e^n \in {}^\sigma\sigma$ and $f \in {}^\kappa\sigma$ such that $f_n = e^n \circ f$ for each $n < \omega$. So $n < \omega \Rightarrow f_n \le f$ which by $\circledast_7(c)$ implies $\mathbf{n}(f_n) \le \mathbf{n}(f)$. As $\langle \mathbf{n}(f_n) : n < \omega \rangle$ is increasing, easily we get a contradiction.]

$\circledast_9$ $\mathbf{n}(f_*) > 0$, i.e. $\alpha < \kappa \Rightarrow u_{f_*,\alpha} \ne \emptyset$.

[Why? If $(\forall f \in {}^\kappa\sigma)(\forall \alpha < \kappa)(u_{f,\alpha} = \emptyset)$ <u>then</u> (by $\circledast_3$) we deduce h is the identity, a contradiction because $R_1^{M_{1,\alpha}} \ne R_1^{M_{2,\alpha}}$. Otherwise assume $u_{f,\alpha} \ne \emptyset$ hence as in the proof of $\circledast_8$ there is f' such that $f_* \le f' \wedge f \le f'$ so by $\circledast_5$ and $\circledast_8$ we have $0 < |u_{f,\alpha}| \le |u_{f',\alpha}| = |u_{f_*,\alpha_*}|$.]

$\circledast_{10}$ if $f \in {}^\kappa\sigma$ and $\alpha, \beta < \kappa$, then $u_{f,\alpha} = u_{f,\beta}$.

[Why? Recall $(*)_6(b)\bullet_1$, hence $(a_{f,\alpha,\emptyset}, a_{f,\beta,\emptyset}) \in R_0^{M_{1,\alpha}}$, so as h maps $R_0^{M_{1,\alpha}}$ onto $R_0^{M_{1,\alpha}}$ we have $(a_{f,u_{\ell,\alpha}}, a_{f,\alpha,u_{\ell,\beta}}) \in R_0^{M_{2,\alpha}}$, hence $u_{f,\alpha} = u_{f,\beta}$ is as promised.]

$\circledast_{11}$ Now fix f_*, α_* as in $\circledast_8$ for the rest of the proof, without loss of generality f_* is onto σ.

[Why? Clearly, Range(f_*) is a non-empty subset of σ, and as $\kappa > \sigma$ is regular, there is $i < \sigma$ such that $f_*^{-1}(\{i\})$ has cardinality $> \sigma$, (see 1.1. $\circledast_1$(a). Let $f \in {}^\kappa\sigma$ be such that $\alpha < \kappa \wedge f_*(\alpha) \neq i \Rightarrow f(\alpha) = f_*(\alpha)$ and $f{\restriction}(f_*^{-1}(\{i\}))$ is onto σ. Let $e \in {}^\sigma\sigma$ be such that $j \in \text{Rang}(f_*) \Rightarrow \sigma(j) = j$ and $j \notin \text{Range}(f_*) \Rightarrow \sigma(j) = i$. Easily, $f_* = e \circ f$, hence $f_* \leq f$ and f is onto σ, and so we can replace f_* by f, so indeed without loss of generality $\oplus_{11}$ holds.]

Let $u_{f_*,\alpha_*} = \{i_\ell^* : \ell < \ell(*)\}$ with $\langle i_\ell^* : \ell < \ell(*)\rangle$ increasing for simplicity. Now for every $f \in {}^\kappa\sigma$ such that $f_* \leq f$ and $\alpha < \kappa$ by $\circledast_8(ii),(iii)$ we know that if $e \in {}^\sigma\sigma \wedge f \in {}^\kappa\sigma \wedge f_* = e \circ f$ then $e{\restriction}u_{f,\alpha}$ is a one-to-one mapping from $u_{f,\alpha}$ onto u_{f_*,α_*}; but so $e \restriction u_{f,\alpha}$ is uniquely determined by $(f_*, \alpha_*, f, \alpha)$ so let $i_{f,\alpha,\ell} \in u_{f,\alpha}$ be the unique $i \in u_{f,\alpha}$ such that $e(i) = i_\ell^*$ (equivalently $(\exists\alpha)(f(\alpha) = i \wedge f_*(\alpha) = i_\ell^*)$).

Let

$$\mathcal{A} = \{A \subseteq \kappa : \text{ for some } f, f_* \leq f \text{ and } \alpha < \kappa \text{ we have } f^{-1}\{i_{f,0}\} \backslash \alpha \subseteq A\}$$

$\boxdot_1$ $\mathcal{A} \subseteq \mathcal{P}(\kappa) \backslash [\kappa]^{<\kappa} = [\kappa]^\kappa$.

[Why? As κ is regular, this means $A \in \mathcal{A} \Rightarrow A \subseteq \kappa \wedge \sup(A) = \kappa$ which holds by $\circledast_{10}$.]

$\boxdot_2$ $\kappa \in \mathcal{A}$.

[Why? By the definition of $\mathcal{A}$.]

$\boxdot_3$ if $A \in \mathcal{A}$ and $A \subseteq B \subseteq \kappa$ then $B \in \kappa$.

[Why? By the definition of $\mathcal{A}$.]

$\boxdot_4$ if $A_1, A_2 \in \mathcal{A}$ then $A =: A_1 \cap A_2$ belongs to $\mathcal{A}$.

[Why? Let $(f_\ell, e_\ell, \alpha_\ell)$ be such that $f_* = e_\ell \circ f_\ell$ and $f_\ell \in {}^\kappa\sigma, \alpha_\ell < \kappa$ and $f_\ell^{-1}\{i_{f_\ell,0}\} \backslash \alpha_\ell \subseteq A_\ell$ for $\ell = 1, 2$ and let $\alpha_0 = \max\{\alpha_1, \alpha_2\}$. Let pr:$\sigma \times \sigma \to \sigma$ be one-to-one and onto and define $f \in {}^\kappa\sigma$ by $f(\alpha) = \text{pr}(f_1(\alpha), f_2(\alpha))$. Clearly $f_\ell \leq f$ for $\ell = 1, 2$ hence $i_{f,0}$ is well defined and $i_{f,0} = \text{pr}(i_{f_1,0}, i_{f_2,0})$. Now for every $\alpha \in \kappa \setminus \alpha_0, f(\alpha) = i_{f_2,0} \Rightarrow f_1(\alpha) = i_{f_1,0} \wedge f_2(\alpha) = i_{f_{2,0}} \Rightarrow \alpha \in A_1 \wedge \alpha \in A_2 \Rightarrow \alpha \in A_1 \cap A_2 \Rightarrow \alpha \in A$ so $f^{-1}\{i_{f,0}\} \subseteq A$ hence $A \in \mathcal{A}$.]

$\boxdot_5$ if $A \subseteq \kappa$ then $A \in \mathcal{A}$ or $\kappa \backslash A \in \mathcal{A}$.

[Why? Define $f \in {}^\kappa\sigma$:

$$f(\alpha) = \begin{cases} 2f_*(\alpha) & \text{if } \alpha \in A \\ 2f_*(\alpha) + 1 & \text{if } \alpha \in \kappa \backslash A. \end{cases}$$

-

Let $e \in {}^\sigma\sigma$ be defined by $e(2i) = e(2i+1) = i$ so $f_*(i) = e(f(i))$ for $i < \sigma$; so $f_* \leq f$. Let $i = i_{f,0}$ so by the definition of $\mathcal{A}$ we have $f^{-1}\{i\} = f^{-1}\{i_{f,0}\} \in \mathcal{A}$.

But if i is even then $f^{-1}\{i\} \subseteq A$ and i is odd then $f^{-1}\{i\} \subseteq \kappa \backslash A$ so by $\boxdot_3$ we are done.]

$\boxdot_6$ $\mathcal{A}$ is a uniform ultrafilter on κ.

[Why? By $\boxdot_1 - \boxdot_5$.]

$\boxdot_7$ $\mathcal{A}$ is σ^+-complete.

[Why? Assume $B_\varepsilon \in \mathcal{A}$ for $\varepsilon < \sigma$ and let $B = \cap\{B_\varepsilon : \varepsilon < \sigma\}$.

Define $A_\varepsilon \subseteq \kappa$ for $\varepsilon < \sigma$ as follows:

- $A_\varepsilon = \bigcap\limits_{\zeta<\xi} B_\zeta \backslash B_\xi$ if $\varepsilon = 1 + \xi \geq 2$,
- $A_\varepsilon = \kappa \setminus B_0$ if $\varepsilon = 1$, and
- $A_\varepsilon = B$ if $\varepsilon = 0$.

Clearly $\langle A_\varepsilon : \varepsilon < \sigma\rangle$ is a partition of κ, let $f \in {}^\kappa\sigma$ be such that $f \restriction A_\varepsilon$ is constantly ε. Let $f' \in {}^\kappa\theta$ be such that $f \leq f' \wedge f_* \leq f'$. Now $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$ is included in some A_ε. If $\varepsilon = 0$ this exemplifies $\bigcap\limits_{\varepsilon<\sigma} B_\varepsilon \in \mathcal{A}$ as required. If $\varepsilon = 1 + \zeta < \sigma$, then $(f')^{-1}\{i_{f',0}\} \subseteq A_\zeta \subseteq \kappa \backslash B_\zeta$, contradiction to $\boxdot_6$ because $B_\varepsilon \in \mathcal{A}$ and $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$.]

So by the assumptions of 1.2, that is, $\circledast_1(b)$ of 1.1 we get a contradiction, coming from the assumption "toward contradiction, clause (j) of $\boxplus$ of 1.2 fails", so it holds, and the other clauses were proved so we are done. $\square_{1.2}$

Theorem 1.4. *For every θ there is an $\mathfrak{k} = \mathfrak{k}^*_\theta$ such that*

$\otimes$ (a) *$\mathfrak{k}$ is an AEC with* $\mathrm{LST}(\mathfrak{k}) = \theta, |\tau_\mathfrak{k}| = \theta$

(b) *$\mathfrak{k}$ admit intersections,*

(c) *$\mathfrak{k}$ has amalgamtion,*

(d) *if κ is a regular cardinal and there is no uniform θ^+-complete ultrafilter on κ, <u>then</u>: $\mathfrak{k}$ is not $(\leq 2^\kappa, \kappa)$-sequence-local for types, i.e., we can find an $\leq_\mathfrak{k}$-increasing continuous sequence $\langle M_i : i \leq \kappa\rangle$ of models and $p \neq q \in \mathscr{S}_\mathfrak{k}(M_\kappa)$ such that $i < \kappa \Rightarrow p \restriction M_i = q \restriction M_i$ and M_κ is of cardinality $\leq 2^\kappa$.*

We shall prove 1.4 below. As in [BS08, 1.2,§4] the aim of the definition of "admit intersections" is to ensure types behave reasonably.

Definition 1.5. We say an AEC $\mathfrak{k}$ admits intersections <u>when</u> there is a function $c\ell_\mathfrak{k}$ such that:

(a) $c\ell_\mathfrak{k}(A, M)$ is well defined iff $M \in K_\mathfrak{k}$ and $A \subseteq M$

(b) $c\ell_\mathfrak{k}(A, M)$ is preserved under isomorphisms and $\leq_\mathfrak{k}$-extensions; that is:
- h is an isomorphisms from $M_1 \in K_\mathfrak{k}$ onto M_2 and $A_1 \subseteq M_1$ then $c\ell_\mathfrak{k}(\{h(a) : a \in A_1\}, M_2) = \{h(b) : b \in c\ell_\mathfrak{k}(A, M_1)\}$, and
- if $A \subseteq M_1 \leq_\mathfrak{k} M_2$ then $c\ell_\mathfrak{k}(A, M_1) = c\ell_\mathfrak{k}(A, M_2)$.

(c) for every $M \in K_\mathfrak{k}$ and non-empty $A \subseteq M$ the set $B = c\ell_\mathfrak{k}(A, M)$ satisfies: $M \restriction B \in K_\mathfrak{k}, M \restriction B \leq_\mathfrak{k} M$; noting that for every M_1, N we have $A \subseteq M_1 \leq_\mathfrak{k} N \wedge M \leq_\mathfrak{k} N \Rightarrow B \subseteq M_1$;

(d) we may use $cl_\mathfrak{k}(A, M)$ for $M \restriction c\ell_\mathfrak{k}(A, M)$.

Claim 1.6. *Assume $\mathfrak{k}$ is an* AEC *admitting intersections.* *Then* $\mathrm{ortp}_{\mathfrak{k}}(a_1, M, N_1) = \mathrm{ortp}_{\mathfrak{k}}(a_2, M, N_2)$ *iff letting $M_\ell = N_\ell \restriction c\ell_{\mathfrak{k}}(M \cup \{a_\ell\})$, there is an isomorphism from M_1 onto M_2 over M mapping a_1 to a_2.*

Proof. It should be clear from the definition. $\square_{1.6}$

Remark 1.7. In Theorem 1.4 we can many times demand $\|M_\kappa\| = \kappa$, e.g., if $(\exists\lambda)(\kappa = 2^\lambda)$.

Note we now show that 1.4 is the best possible.

Claim 1.8. *1) If $\mathfrak{k}$ satisfies clause (a) of 1.4, (i.e. $\mathfrak{k}$ is an AEC with* LST*-number $\le \theta$ and $|\tau_{\mathfrak{k}}| \le \theta$) and κ is regular and fails the assumption of clause (d) of 1.4, that is, $\kappa > \theta$ and there is a uniform θ^+-complete ultrafilter on κ,* *then* *the conclusion of clause (d) of 1.4 fails, that is $\mathfrak{k}$ is κ-sequence local for types.*
2) If D is a θ^+-complete ultrafilter on κ and $\mathfrak{k}$ is an AEC with $\mathrm{LST}(\mathfrak{k}) \le \theta$ *then* *ultraproducts by D preserve "$M \in \mathfrak{k}$", "$M \le_{\mathfrak{k}} N$", i.e.*

$\boxtimes$ *if $M_i, N_i (i < \kappa)$ are $\tau(\mathfrak{K})$-models and $M = \prod_{i<\kappa} M_i/D$ and $N = \prod_{i<\kappa} N_i/D$* *then*:
- (a) $M \in K$ *if* $\{i < \kappa : M_i \in \mathfrak{k}\} \in D$
- (b) $M \le_{\mathfrak{k}} N$ *if* $\{i : M_i \le_{\mathfrak{k}} N_i\} \in D$.

Proof. Recall that if D is θ^+-complete, then it is σ^+-complete where $\sigma = \theta^{\aleph_0}$ (and much more, it is θ'-complete for the first measurable $\theta' > \theta$).
1) So assume

$\boxplus$
- (a) $\langle M_i : i \le \kappa \rangle$ is $\le_{\mathfrak{k}}$-increasing
- (b) $M_\kappa = N_0 \le_{\mathfrak{k}} N_\ell$ for $\ell = 1, 2$
- (c) $p_\ell = \mathrm{ortp}_{\mathfrak{k}}(a_\ell, N_0, N_\ell)$ for $\ell = 1, 2$
- (d) $i < \kappa \Rightarrow p_1 \restriction M_i = p_2 \restriction M_i$.

We shall show $p_1 = p_2$, this is enough.
Without loss of generality

$(*)_1$
- (a) $a_1 = a_2$ call it a,
- (b) $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$.

By (d) of $\boxplus$ we have:

$(d)^+$ for each $i < \kappa$ there are even $m_i < \omega$ and $\langle N_{i,n} : n \le m_i \rangle$ such that:
- (α) $N_{i,0} = N_1$,
- (β) $N_{i,m_i} = N_2$ or just h_i is an isomorphism from N_{i,m_i} onto N_2 such that $h_i \restriction (M_i \cup \{a\})$ is the identity,
- (γ) $a \in N_{i,\ell}$ and $M_i \le_{\mathfrak{k}} N_{i,\ell}$,
- (δ) if $2m + 2 \le m_i$ then $N_{i,2m+1} \le_{\mathfrak{k}} N_{i,2m}, N_{i,2m+2}$.

As $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ without loss of generality $i < \kappa \Rightarrow m_i = n_*$. Let χ be such that $\langle M_i : i \le \kappa \rangle, \langle \langle N_{i,n} : n \le n_* \rangle : i < \kappa \rangle$ and $\mathfrak{k}_{\mathrm{LST}(\mathfrak{k})}$ all belongs to $\mathcal{H}(\chi)$; concerning $\mathfrak{k}_{\mathrm{LST}(\mathfrak{k})}$ this means $\tau_{\mathfrak{k}}$ and $\mathrm{LST}(\mathfrak{k})$ belongs to $\mathcal{H}(\chi)$ and as usual without loss of generality$\tau_{\mathfrak{k}}$ has cardinality at most 2^σ hence $\{M \in K_{\mathfrak{k}} : M \in \mathcal{H}(\mathrm{LST}^+_{\mathfrak{k}})\}$ and $\le_{\mathfrak{k}} \restriction \mathcal{H}(\mathrm{LST}^+_{\mathfrak{k}})$ belongs to $\mathcal{H}(\chi)$; those hold by $(*)_1(b)$. Let $\mathfrak{B}$ be the ultrapower

$(\mathcal{H}(\chi),\in)^\kappa/D$ and $\mathbf{j}_0$ the canonical embedding of $(\mathcal{H}(\chi),\in)$ into $\mathfrak{B}$ and let $\mathbf{j}_1$ be the Mostowski-Collapse of $\mathfrak{B}$ to a transitive set $\mathcal{H}$ and let $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$. So $\mathbf{j}$ is an elementary embedding of $(\mathcal{H}(\chi),\in)$ into $(\mathcal{H},\in)$ and even an $\mathbb{L}_{\theta^+,\theta^+}$-elementary one. Recall that by $(*)_1(b)$ we are assuming without loss of generality $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$ hence $\mathbf{j}(\tau_{\mathfrak{k}}) = \tau_{\mathfrak{k}}$ hence by part (2), $\mathbf{j}$ preserves "$N \in K_{\mathfrak{k}}$", "$N^1 \leq_{\mathfrak{k}} N^2$" and "$h$ is an isomorphism from N' onto N''.

So $\mathbf{j}(\langle M_i : i \leq \kappa\rangle)$ has the form $\langle M^*_i : i \leq \mathbf{j}(\kappa)\rangle$ but $\mathbf{j}(\kappa) > \kappa_* := \bigcup\limits_{i<\kappa} \mathbf{j}(i)$ by the uniformity of D and let $\mathbf{j}(\langle\langle N_{i,n} : n \leq n_*\rangle : i < \kappa\rangle = \big\langle\langle N^*_{i,n} : n \leq n^*\rangle : i < \mathbf{j}(\kappa)\big\rangle$ and $\mathbf{j}(\langle h_i : i < \kappa\rangle) = \langle h^*_i : i < \mathbf{j}(\kappa)\rangle$.

So

(a) $\mathbf{j}{\restriction} M_\kappa$ is a $\leq_{\mathfrak{k}}$-embedding of M_κ into $M^*_{\mathbf{j}(\kappa)}$ hence even into $M^*_{\kappa_*}$,

(b) $M^*_{\kappa_*} \leq_{\mathfrak{k}} N^*_{i,n}$ and $\mathbf{j}(a) \in N^*_{i,n}$ for $i < \kappa, n \leq n_*$,

(c) $N^*_{i,0} = \mathbf{j}(N_1)$,

(d) h_{κ_*} is an isomorphism from N_{κ_*,n_*} onto $\mathbf{j}(N_2)$,

(e) $N^*_{\kappa_*,2m+1} \leq_{\mathfrak{k}} N^*_{\kappa_*,2m}, N^*_{\kappa_*,2m+2}$ for $2m+1 < n_*$,

(f) $\mathbf{j}(a) \in N_{\kappa_*,m}$ for $m \leq n_*$,

(g) h_{κ_*} is an isomorphism from N_{κ_*,n_*} into $\mathbf{j}(N_2)$ over $M^*_{\kappa_*} \cup \{\mathbf{j}(a)\}$. Hence,

(h) $M^*_{\kappa_*} \leq_{\mathfrak{k}} \mathbf{j}(N_2)$,

(i) $\mathrm{ortp}(\mathbf{j}(a)),\ M^*_{\kappa_*},\ \mathbf{j}(N_1) = \mathrm{ortp}(\mathbf{j}(a)),\ M^*_{\kappa_*},\ \mathbf{j}(N_2)$,

Also,

(j) letting $M^\bullet_\kappa = M^*_{\kappa_*}{\restriction}\{\mathbf{j}(a)\colon a \in M_\kappa\}$, we have $M^\bullet_\kappa \leq_{\mathfrak{k}} M_{\kappa_*}$ and $\mathbf{j}{\restriction} M_\kappa$ is an isomorphism from M_κ onto $M^\bullet_\kappa$,

(k) For $\ell = 1,2$, let $N^\bullet_\ell = \mathbf{j}(N_\ell){\restriction}\{\mathbf{j}(a)\colon a \in N_\ell\}$, we have $M^\bullet_\kappa \leq_{\mathfrak{k}} N^\bullet_\ell \leq_{\mathfrak{k}} \mathbf{j}(N_\ell)$.

Hence (by (i) and monotonicity),

(l) $\mathrm{ortp}(\mathbf{j}(a), M^\bullet_\kappa, N^\bullet_1) = \mathrm{ortp}(\mathbf{j}(a), M^\bullet_\kappa, N^\bullet_2)$.

By preservation under isomorphisms,

(m) $\mathrm{ortp}(a, M_\kappa, N_1) = \mathrm{ortp}(a, M_\kappa, N_2)$.

Together, we are done.

2) By the representation theorem of AEC, see [She09a, §1]. $\square_{1.8}$

Discussion 1.9. We try to help the reader by pointing out some things in the proof of Theorem 1.4.

(1) If the reader do not mind having $\tau_{\mathfrak{k}}$ to be of cardinality $2^{(\theta^{\aleph_0})}$ then we can replace R_2 essentially by the R_e $(e \in {}^\sigma\sigma)$ and omit S and $\{d_i\colon i < \theta\}$. This simplifies somewhat, so in 1.9 we follow it.

(2) We rely on the conclusion in 1.2. There we have two increasing continuous sequences of models $\bar{M}_\ell = \langle M_{\ell,\alpha} : \alpha \leq \kappa\rangle$ for $\ell = 1,2$.

Now $M_{1,\alpha}, M_{2,\alpha}$ are very similar:

$(*)$ For $\alpha < \kappa$, they are not just isomorphic but have the same universe, and the difference is only in the interpretation of R_1.

Here we define $M'_{0,\alpha}$ by Restricting ourselves to I_α, Q_α, S where S cde $G, {}^\sigma\sigma$ and Q_α code ${}^\kappa\sigma$ and for each $t \in J$ attached is something like a copy of $M_{\ell,\alpha}$,

Now, for $\ell = 1,2$ we shall define $M'_{\ell,\alpha}$ adding a new element t^*_ℓ, which code the R_1, i.e.

$$R_1^{M_{\ell,\alpha}} = \{\bar{a}{}^\frown\langle t^*_\ell\rangle\colon \bar{a} \in R_1^{M_{\ell,i}}\}.$$

So this translate "$M_{1,\alpha} \cong M_{2,\alpha} \iff \alpha < \kappa$" to

$$\mathrm{otp}(t^*_1, M'_{0,\alpha}, M'_{1,\alpha}) = \mathrm{otp}(t^*_2, M'_{0,\alpha}, M'_{2,\alpha}) \iff \alpha < \kappa$$

where $M'_{0,\alpha}$ is obtained from $M'_{\ell,\alpha}$ by omitting Q, J

Proof. Proof of 1.4

Recall $\sigma = \theta^{\aleph_0}$. Let $G = ([\sigma]^{<\aleph_0}, \Delta)$ and let $\langle c_i : i < \sigma\rangle$ list the members of G. We define $\tau = \tau_{\mathfrak{k}}$, by:

$\boxplus_1$ $\tau = \tau^\bullet_\sigma \cup \{S, S_0, S_1, S_2, J, I, A, \pi, Q, R_1, R_2, R_3,\} \cup \{H_n : n \le \omega\} \cup \{F_1, F_2\} \cup \{d_i : i < \sigma\}$, so of cardinality σ, where:
- (a) $\tau^\bullet_\theta = \{E_n, E'_n : n < \omega\} \cup \{R_0\}$,
- (b) R_0, R_3 are binary predicates, E_n, E'_n are $(2n)$-place predicates for $n \in [2, \omega)$,
- (c) $S, S_0, S_1, S_2, Q, J, I, A$ are unary predicates,
- (d) π is an unary function symbol,
- (e) R_1, R_2 are two place and three place predicates respectively,
- (f) F_2 H_n is an unary function symbol,
- (g) d_i is an individual constant for $i < \sigma$.
- (h) H is a two place function symbol.
- (i) $F_1, +$ are two-place function symbols.

We define K as a class of τ-models by (note that the function symbols are interpreted as partial functions):

$\boxtimes_2$ $M \in K$ iff (up to isomorphism):
- (a) $\langle S^M_0, S^M_1, S^M_2, Q^M, I^M, J^M, A\rangle$ is a partition of $|M|$, (recall that they are unary),
- (b) $S^M = S^M_0 \cup S^M_1 \cup S^M_2$.
- (c) $(E'_n)^M \subseteq {}^{2n}(A^M)$ is an equivalence relation on ${}^n(A^M)$, and E^M_n an equivalence relation on ${}^n(A^M)$ refining it, for $n \in [2, \omega)$,
- (d) R_0 is a binary relation $\subseteq A^M \times A^M$,
- (e) R^M_1 is a binary relation $\subseteq A^M \times J^M$,
- (f) π^M is a function from A^M into I^M,
- (g) $R^{M_0}_2 \subseteq A^M \times A^M \times S^M_2$ (play the role of R_e's in 1.2).
- (h) $\{d^M_i : i < \sigma\}$ are pairwise distinct elements of S^M_0,
- (i)
  - $\bullet_1$ $+^M$ is a two place function on S^M_1,
  - $\bullet_2$ $H^M : S^M_0 \to S^M_1$ is one-to-one
  - $\bullet_3$ $G = (S^M_1 .+^M)$ is an Abelian group satisfying $(\forall x \in G)(x + x = 0_G)$ and $\mathrm{rng}(H^M)$ is a basis of G.
- (j)
  - (α) $R^M_3 \subseteq A^M \times J$ and for $t \in J$, define: $R^M_3[t] = \{a \in A_0 : M \models R_3[a, t]\}$,
  - (β)
    - $(E'_n)^M \subseteq \bigcup\{B \times B : B \in \mathcal{B}^M\}$, where $\mathcal{B}^M = \{R^M_3[t] : t \in J\}$ is a partition of A^M,
    - $E^M_n \subseteq (E'_n)^M$,
    - the domain of F^M_1 is $A^M \times S^M_1$ and it maps $B \times S^M_1$ onto B for $B \in \mathcal{B}$; also if $a \in A^M$ and $c_1, c_2 \in S^M_1$ then $F^M_1(F^M_1(a, c_1), c_2) = F^M_1(a, c_1 + c_2)$; (it play the role of the F_c-s in 1.2),
    - if $\bar a = \langle a_i : i < n\rangle \in {}^n(A^M)$ then $\bar a/(E'_n)^M$ is equal to $\{\langle F^M_1(a_i, c) : c \in S_1\}$,

- F_2^M is a function from A^M onto Q^M,
- if $a \in A^M$ then $\{\langle b,b\rangle : \langle b,b\rangle (E'_n)^M \langle a,a\rangle\}$ is equal to $\{\langle b,b\rangle \in A^M \times A^M : F_2(b) = F_2^M(a)\}$ and to $\{(b,b)\}\colon b = F_1^M(a,c)$ for some $c \in S_1^M$.

$\boxtimes_3$ We define $\le_{\mathfrak{k}}$ as being a submodel, in particular $M \le_{\mathfrak{k}} N \Rightarrow \pi^N{\restriction}M = \pi^M$, $F^N{\restriction}M = F^M$, etc.

Easily

$\boxtimes_4$ $\mathfrak{k} = (K, \le_{\mathfrak{k}})$ is an AEC where $\le_{\mathfrak{k}}$ is the two place relation on K of being a sub-model.

$\boxtimes_5$ We define the closure operation $c\ell = c\ell_{\mathfrak{k}}$ with domain $\{(A,M) : A \subseteq M \in K\}$ such that for every (A,M) from the domain of $c\ell(A,M)$, is the minimal set B such that:

- $\bullet_1$ $B \subseteq M$,
- $\bullet_2$ $A \subseteq B$,
- $\bullet_3$ B is closed under the functions $\pi^M, F_1^M, +^M$,
- $\bullet_4$ $d_i^M \in B$ for $i < \sigma$, hence $S_1^M \subseteq B$,
- $\bullet_5$ for $c \in S_1^M$ we have if $F_1^M(a,c) = b$ then all of them $a \in B$ iff $b \in b$.

$\boxtimes_6$ Indeed $c\ell$ witnesses $\mathfrak{k}$ admit intersections.

[Why? Just check.]

$\boxtimes_7$ $\mathfrak{k}$ has disjoint amalgamation.

[Why? So assume $M_0 \subseteq M_\ell$ are from K and without loss of generality$M_1 \cap M_2 = M_0$ and we shall find a member M of K extending all of them. We define Mas follows:

- $\bullet_1$ $a \in M$ if $a \in M_1$ or $a \in M_2$,
- $\bullet_2$ if P is a predicate from τ not equal to E'_n or E_n for some $n \in [2,n)$ then $P^M = P^{M_1} \cup P^{M_2}$,
- $\bullet_3$ if F is a function symbol from $\mathfrak{k}$ then $F^M = F^{M_1} \cup F^{M_2}$,
- $\bullet_4$ if $n \in [2,\omega)$ the $(E'_n)^M$ is the closure of $(E'_n)^{M_1} \cup (E'_n)^{M_2}$ to an equivalence relation; similarly for E_n.

]

Obviously $M_\ell \subseteq M$ for $\ell = 0,1,2$ so we just have to check that $M \in K$. This is straightforward. So $\boxtimes_7$ holds indeed.

Assume κ is as in clause (d) of 1.4, we use the $M_{\ell,\alpha} (\ell = 1,2, \alpha \le \kappa)$ as well as I_α, π_α constructed inside the proof of 1.2 (the main relevant properties are stated in 1.2). They are not in the right vocabulary and universe, so let $M'_{\ell,\alpha}$ be the following τ-model:

$\boxtimes_8$ (a) <u>elements</u>: The universe of the model $M'_{\ell,\alpha}$ is the disjoint union of the following sets (and we have $S^{M'_{\ell,\alpha}} = S_0^{M'_{\ell,\alpha}} \cup S_1^{M'_{\ell,\alpha}} \cup S_2^{M'_{\ell,\alpha}}$):

- $S_0^{M'_{\ell,\alpha}} = \sigma$
- $S_1^{M'_{\ell,\alpha}} = [\sigma]^{<\aleph_0}$;
- $S_2^{M_{\ell,\alpha}} = {}^\sigma\sigma$,

- $I^{M'_{\ell,\alpha}} = I_\alpha$,
- $J^{M'_{\ell,\alpha}} = \{t^*_\ell\}$ where t^*_ℓ is just a new element,
- $A^{M'_{\ell,\alpha}} = \{\ell\} \times |M_{\ell,\alpha}| = \{\iota\} \times A_\alpha$,

(we assume disjointness)

(b) for $\ell = 1,2$ the mapping $a \mapsto (\iota, a)$ for $a \in M_{\ell,\alpha}$ is an isomorphism from $(M_{\ell,\alpha}{\restriction}A^*_\alpha){\restriction}\tau^\bullet_\theta$ onto $(M'_{\ell,\alpha}{\restriction}A_\iota^{M'_{\ell,\alpha}}){\restriction}\tau^\bullet$ so $(E'_n)^{M_{\ell,\alpha}}, E_n^{M_{\ell,\alpha}}, R_1^{M_{\ell,\alpha}}$ are defined.

(c) $I^{M'_{\ell,\alpha}} = I_\alpha$,

(d) $\pi^{M'_{\ell\alpha}}$ is defined by $\pi((\ell,x)) = \pi(x)$

(e) (i) $R_1^{M'_{\ell,\alpha}}$ is the set of $\langle(\ell,\alpha_k),t^*_\ell\rangle$ such that $\langle a_k\rangle \in R_1^{M_{\ell,\alpha}}$,
(ii) $R_3^{M_{\ell,\alpha}} = \{(a,t^*_\ell) : a \in A^{M_{\ell,\alpha}}\}$,

(f) $d_i^{M'_{\ell,\alpha}} = i$ for $i < \theta$,

(g) $R_2^{M'_{\ell,\alpha}} = \{((\ell,a),(\ell,b),e) \colon e \in {}^\sigma\sigma,$ and $(a,b) \in R_e^{M_{\ell,\alpha}}\}$,

(h) H^M is the function mapping d_i to $\{d_i\}$,

(i) $F_1^{M'_{\ell,\alpha}}$ maps $(a,c) \in A^{M'_{\ell,\alpha}} \times S_1^{M'_{\ell,\alpha}}$ to $F_c^{M_{\ell,\alpha}}(a)$

(k) $F_2^{M'_{\ell,\alpha}} : A^{M'_{\ell,\alpha}} \to Q^{M'_{\ell,\alpha}}$ maps $(a_{f,\alpha,u},\alpha,u)$ to $a_{f,\alpha}$.

Let $M'_{0,\alpha} = M'_{\ell,\alpha} \restriction \left(S^{M'_{\ell,\alpha}} \cup I^{M'_{\ell,\alpha}} \cup Q^{M_{\ell,\alpha}} \cup J^{M'_{\ell,\alpha}}\right)$ for $\ell = 1,2$ and $\alpha \le \kappa$ (we get the same result for $\ell = 1,2$).

Note easily

$\boxtimes_9$ $M'_{0,\alpha} = M'_{1,\alpha} \cap M'_{2,\alpha}$

$\boxtimes_{10}$ $\langle M'_{\ell,\alpha} : \alpha \le \kappa\rangle$ is $\le_{\mathfrak{k}}$-increasing and continuous for $\ell = 0,1,2$,

[Why? Easy to check.]

$\boxtimes_{11}$ $\mathrm{ortp}_{\mathfrak{k}}(t^*_1, M'_{0,\alpha}, M'_{1,\alpha}) = \mathrm{ortp}_{\mathfrak{k}}(t^*_2, M'_{0,\alpha}, M'_{2,\alpha})$ for $\alpha < \kappa$.

Why? By the isomorphism g_α from $M_{1,\alpha}$ onto $M_{2,\alpha}$ respecting π_α in 1.1. That is we define h such that it is the isomorphism for $M'_{1,\alpha}$ onto $M^\bullet_{2,\alpha}$ over $M'_{0,\alpha}$ mapping t^*_1 to t^*_2 and mapping $(1,x) \in A^{M'_{1,\alpha}}$ to $(2, g_\alpha(x))$.

Now, check.

$\boxtimes_{12}$ $\mathrm{ortp}_{\mathfrak{k}}(t^*_1, M'_{0,\kappa}, M'_{1,\kappa}) \ne \mathrm{ortp}_{\mathfrak{k}}(t^*_2, M'_{0,\kappa}, M'_{2,\kappa})$.

[Why? By the non-isomorphism in 1.1; extension will not help.]

$\square_{1.4}$

2. Compactness of types in AEC

Baldwin [Bal09] asks "Can we in ZFC prove that some AEC has amalgamation and JEP but fails compactness of types?". The background is that in [BS08] we construct one using diamonds.

To me, the question is to show that this class can be very large (in ZFC).

Here we omit amalgamation and accomplish both by direct translations of problems of existence of models for theories in $\mathbb{L}_{\kappa^+,\kappa^+}$, first in the propositional logic. So whereas in [BS08] we have an original group G^M, here instead we have a set P^M of propositional "variables" and Γ^M, set of such sentences (and relations and functions explicating this; so really we use coding but are a little sloppy in stating this obvious translation).

In [BS08] we have I^M, set of indexes, 0 and H, set of Whitehead cases, H_t for $t \in I^M$, here we have I^M, each $t \in I^N$ representing a theory $P^M_t \subseteq P^M$ and in J^M we give each $t \in I^M$ some models $\mathcal{M}^M_s : P^M \to \{\text{true,false}\}$. This is set up so that amalgamation holds.

Notation 2.1. In this section types are denoted by $\mathbf{p}, \mathbf{q}$ because p, q are used for propositional variables.

Definition 2.2. 1) We say that an AEC $\mathfrak{k}$ has $(\leq \lambda, \kappa)$-sequence-compactness (for types) <u>when</u>: if $\langle M_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and $i < \kappa \Rightarrow \|M_i\| \leq \lambda$ and $\mathbf{p}_i \in \mathscr{S}^n(M_i)$ for $i < \kappa$ satisfying $i < j < \kappa \Rightarrow \mathbf{p}_i = \mathbf{p}_j \restriction M_i$ <u>then</u> there is $\mathbf{p}_\kappa \in \mathscr{S}^n(M_\kappa)$ such that $i < \kappa \Rightarrow \mathbf{p}_\kappa \restriction M_i = \mathbf{p}_i$.
2) We define "$(= \lambda, \kappa)$-sequence-compactness" similarly. Let (λ, θ)-sequence-compactness mean $(\leq \lambda, \kappa)$-compactness.

Question 2.3. Can we find an AEC $\mathfrak{k}$ with amalgamation and JEP such that $\{\kappa : \mathfrak{k}$ have (λ, κ)-compactness of types for every $\lambda\}$ is complicated, say:

- (a) not an end segment of the class of cardinals but with "arbitrarily large" members
- (b) any $\{\kappa : \kappa \text{ satisfies } \psi\}, \psi \in \mathbb{L}_{\theta^+,\theta^+}$ (second order) when $\text{LST}_{\mathfrak{k}} \leq \theta$.

Definition 2.4. Let $\theta \geq \aleph_0$, we define $\mathfrak{k} = \mathfrak{k}_\theta = \mathfrak{k}(\theta)$ as follows:

- (A) the vocabulary $\tau_{\mathfrak{k}}$ consist of $F_i (i \leq \theta), R_\ell (\ell = 1, 2), P, \Gamma, I, J, c_i\ (i < \theta), F_i (i \leq \theta)$, (pedantically see later),
- (B) the universe of $M \in K_{\mathfrak{k}}$ is the disjoint union of P^M, Γ^M, I^M, J^M so P, Γ, I, J are unary predicates,
- (C) (a) P^M a set of propositional variables (i.e. this is how we treat them)
  - (b) $c - i$ an individual constant such that $c^M_i \in P^M$ are pairwise distinct for $i < \theta$, Γ^M is a set of sentences of one of the forms $\varphi = (p), \varphi = (r \equiv p \wedge q), \varphi = (q \equiv \neg p), \varphi = (q \equiv \bigwedge_{i<\kappa} p_i)$, so $p, q, r, p_i \in P^M$,
  - (c) the function $F^M_i : \Gamma^M \to P^M$ for $i < \theta$ are such that for every $i < \theta$ and $\varphi \in \Gamma^M$ we have (below $F_0(\varphi)$ tell us how φ is composed, F_{i+1} give from what):
    - (α) if $\varphi = (p)$ and $i < \theta$ then $F_{1+i}(\varphi) = p, F_0(\varphi) = c_0$
    - (β) if $\varphi = (r \equiv p \wedge q)$ <u>then</u> $F_i(\varphi)$ is c_1 if $i = 0$, is p if $i = 1$, is q if $i = 2$, and is r if $i \geq 3$
    - (γ) if $\varphi = (q \equiv \neg p)$ then $F_i(\varphi)$ is c_2 if $i = 0$, is p if $i = 1$, and is q if $i \geq 2$
    - (δ) if $\varphi = (q \equiv \bigwedge_{j<\theta} p_j)$ then $F_i(\varphi)$ is c_3 if $i = 0$, is q if $i = 1$ and is c_{2+j} if $i = 2 + j$
  - (d) I a set of theories, i.e. $R^M_1 \subseteq \Gamma \times I$ and for $t \in I$ let $\Gamma^M_t = \{\psi \in \Gamma^M : \psi R^M_1 t\} \subseteq \Gamma^M$

(e) J is a set of models, i.e. $R_2^M \subseteq (\Gamma \cup P) \times J$ and for $s \in J$ we have $\mathcal{M}_s^M$ is the model, i.e. function giving truth values to (some) $p \in P^M$, i.e.

(α) $\mathcal{M}_s^M(p)$ is true if $pR_2^M s$; is false if $\neg(pR_2^M s)$

(β) $(\varphi, s) \in R_2^M$ iff computing the truth value of φ in $\mathcal{M}_s^M$ we get truth

(f) $F_\theta^M : J^M \to I^M$ such that $s \in J^M \Rightarrow \Gamma^M_{F_\kappa^M(s)} = \{\varphi \in \Gamma^M : (\varphi, s) \in R_2^M\}$, i.e. the set of sentences from Γ^M which $\mathcal{M}_s^M$ satisfies,

(g) $(\forall t \in I^M)(\exists s \in J^M)(F_\theta^M(s) = t)$

(D) $M \leq_{\mathfrak{k}} N$ iff $M \subseteq N$ are $\tau_{\mathfrak{k}}$-models from $K_{\mathfrak{k}}$.

Claim 2.5. *$\mathfrak{k}$ is an AEC and LST$(\mathfrak{k}) = \kappa$.*

Proof. Obvious. $\square_{2.5}$

Claim 2.6. *1) $\mathfrak{k}$ has the JEP.*

2) $\mathfrak{k}$ has the amalgamation property.

Proof. 1) Just like disjoint unions (also of the relations and functions) except for the individual constants c_i (for $i < \theta$), or see the proof of part (2).

2) So assume that $M_0 \subseteq M_\ell$ for $\ell = 0, 1$ are in $K_{\mathfrak{k}}$, without loss of generality $M_1 \cap M_2 = M_0$ and we shall find $M \in K_{\mathfrak{k}}$ extending all of them.

(a) the universe of M is $|M_1| \cup |M_2|$.

(b) Similarly for the predicates.

(c) As the functions are all unary, the situation is similar:

- for $i < \theta$ and $\ell \leq 2$, $F_i^{M_\ell} : \Gamma^{M_\ell} \to \Gamma^{M_\ell}$ and $F_i^{M_\ell} {\restriction} \Gamma^{M_0} = F_i^{M_0}$. Clearly $F_i^M = F_i^{M_1} \cup F_i^{M_2}$ and we have:

  ⊕ F_i^M is a well-defined function from Γ^M into P^M extending $F_i^{M_\ell}$ for $i < \theta$

(b) Similarly for $i = \theta$.

It is easy to check then that $\ell = 2 \Rightarrow M_\ell \subseteq M$ and that all clauses of the definition of "$M \in K_{\mathfrak{k}}$" hold. $\square_{2.6}$

Claim 2.7. *Assume $M_0 \leq_{\mathfrak{k}} M_\ell$ for $\ell = 0, 1$ and $|M_0| = P^{M_0} \cup \Gamma^{M_0} = P^{M_\ell} \cup \Gamma^{M_\ell}$ for $\ell = 1, 2$ and $a_\ell \in I^{M_\ell}$ for $\ell = 1, 2$. Then* $\text{ortp}_{\mathfrak{k}}(a_1, M_0, M_1) = \text{ortp}_{\mathfrak{k}}(a_2, M_0, M_2)$ *iff $\Gamma^{M_1}_{a_1} = \Gamma^{M_2}_{a_2}$.*

Proof. The if direction, $\Leftarrow$:

Let h be a one to one mapping with domain M_1 such that $h {\restriction} M_0 =$ the identity, $h(a_1) = a_2$ and $h(M_1) \cap M_2 = M_0 \cup \{a_2\}$. Renaming without loss of generality h is the identity. Now define M_3 as $M_1 \cup M_2$, as in 2.6, now $a_1 = a_2$ does not cause trouble because $P^{M_0} = P^{M_\ell}, \Gamma^{M_0} = \Gamma^{M_\ell}$ for $\ell = 1, 2$.

The "only if" direction, $\Rightarrow$:

Obvious. $\square_{2.7}$

Claim 2.8. *Assume λ, θ, κ are such that:*

(a) $\theta \leq \kappa$ are regular $\leq \lambda$

(b) $\langle \Gamma_i : i \leq \kappa \rangle$ is $\subseteq$-increasing continuous sequence of sets of propositional sentences in $\mathbb{L}_{\theta^+, \aleph_0}$ such that $[\Gamma_i$ has a model $\Leftrightarrow i < \kappa]$

(c) $|\Gamma_\kappa| \le \lambda$.

Then $\mathfrak{k}$ *fail* (λ,θ)*-sequence-compactness (for types).*

Remark 2.9. We may wonder but: for $\theta = \aleph_0$, compactness holds? Yes.

Proof. Without loss of generality $|\Gamma_\kappa| = \lambda$. Without loss of generality $\langle p^*_\varepsilon : \varepsilon < \lambda\rangle$ are pairwise distinct propositions variables appearing in Γ_κ (but not necessarily $\in \Gamma_0$) and each $\psi \in \Gamma_i$ is of the form (p) or $r \equiv p \wedge q$ or $r \equiv \neg p$ or $r \equiv \bigwedge\limits_{i<\theta} p_i$.

Let P_i be the set of propositional variables appearing in Γ_i; without loss of generality $|P_i| = \lambda$.

We choose a $\tau(\mathfrak{k}_\theta)$-model M_i for $i \le \kappa$ such that:

$\boxplus_1$ (a) $|M_i| = P_i \cup \Gamma_i$, and $\tau(M_i) = \tau_{\mathfrak{k}}$,
(b) $P^{M_i} = P_i$ and $\Gamma^{M_i} = \Gamma_i$,
(c) $F^{M_i}_\varepsilon$ (for $\varepsilon < \kappa$) are defined naturally),
(d) $I^{M_i} = \emptyset = J^{M_i}$, hence $R^M_1 = R^M_2 = \emptyset = F^M_\theta$.

Clearly

$\boxplus_2$ (a) $M_i \in K_{\mathfrak{k}}$,
(b) $\langle M_i : i \le \kappa\rangle$ is $\le_{\mathfrak{k}}$-increasing and continuous.

Let $\mathcal{M}_i : P_i \to \{\text{true, false}\}$ be a model of Γ_i.

We define a model $N_i \in K_{\mathfrak{k}}$ for $i < \kappa$ (but not for $i = \kappa$!)

$\boxtimes$ (a) $M_i \le_{\mathfrak{k}} N_i$,
(b) $P^{N_i} = P^{M_i}$,
(c) $\Gamma^{N_i} = \Gamma^{M_i}$,
(d) $I^M = \{t_j : j < i\}$,
(e) $J^M = \{s_j : j < i\}$,
(f) $F^{N_i}_\kappa(s_j) = t_j$,
(g) $R^{N_i}_1 = \bigcup\{\Gamma_j \times \{t_j\} : j < i\}$,
(h) $R^{N_i}_2$ is chosen such that $\mathcal{M}^{N_j}_{s_j}$ is $\mathcal{M}_j$.

Now

$(*)_1$ $\mathbf{p}_i = \text{ortp}_{\mathfrak{k}}(t_i, M_i, N_i) \in \mathscr{S}^1(M_i)$.

[Why? Trivial.]

$(*)_2$ $i < j < \kappa \to \mathbf{p}_i = \mathbf{p}_j{\restriction}M_j$.

[Why? Let $N_{i,j} = N_j{\restriction}(M_j \cup \{s_j, t_j\})$.

Easily $\text{ortp}(t_j, M_i, N_{i,j}) \le \mathbf{p}_j$ and $\text{ortp}(t_j, M_i, N_{i,j}) = \mathbf{p}_j$ by the claim 2.7 above.]

$(*)_3$ there is no $\mathbf{p} \in \mathscr{S}^1(M_\theta)$ such that $i < \kappa \Rightarrow \mathbf{p}_j{\restriction}M_i = \mathbf{p}_i$.

Why? We prove more:

$(*)_4$ there is no (N, t) such that
(a) $M_\kappa \le_{\mathfrak{k}} N$,

(b) $t \in I^N$,

(c) $(\forall \varphi \in \Gamma^{M_\kappa})[\varphi R_1^N t]$.

[Why? As then $\Gamma_\kappa = \Gamma^M$ has a model contradiction to an assumption.] $\square_{2.8}$

So e.g.

Conclusion 2.10. If $\kappa > \theta$ are regular with no θ^+-complete uniform ultrafilter on κ and $\lambda = 2^\kappa$, then $\mathfrak{k} = \mathfrak{k}_\theta$ is not (λ, κ)-sequence-compact.

Remark 2.11. Recall if D is an ultrafilter on θ then $\min\{\sigma' : D \text{ is not } \sigma'\text{-complete}\}$ is $\aleph_0$ or a measurable cardinal.

Proof. Not novel but we elaborate.

$(*)_1$ Let M be the model with the following characteristics:

- $\bullet_1$ its universe is $\mathscr{H}_{\le\kappa}(\lambda)$, where $\mathscr{H}_{\le\theta}(\lambda)$ considering ordinals as atoms, and for any μ

$$\mathscr{H}_{\le\theta}(\mu) = \{x \colon \text{trcl}(x) \text{ and has cardinality } \le \theta \text{ and every ordinal from it is } < \mu\},$$

  where $\text{trcl}(x)$ is the transitive closure of x,
- $\bullet_2$ $P_0^M = \theta$, $c_i^M = i$ for $i < \theta$ and $c_\theta = \kappa$,
- $\bullet_3$ $R^M = \in \restriction \mathcal{H}_{\le\theta}(\lambda)$,
- $\bullet_4$ $<_*^M$ is a well-ordering of $\mathcal{H}_{\le\theta}(\lambda)$,
- $\bullet_5$ the vocabulary of M has cardinality κ and has elimination of quantifiers and Skolem functions.

$(*)_2$ Let $M_\bullet$ be an expansion of M such that for some $\chi > 2^\lambda$, and $N \prec (\mathcal{H}(\chi), \in)$ of cardinality θ, $|N| = \theta$, $\theta + 1 \subseteq N$ and $M \in N$ and M_* is gotten from M by:

for every $\varphi(\bar{c}) \in \mathbb{L}_{\theta^+,\aleph_0}(\tau_M) \cap N$ so $\lg(\bar{x})$ finite we add $P_\varphi^M = \{\bar{a} \in {}^{\lg(\bar{x})}M \colon M \models \varphi[\bar{a}]\}$ so P^M in a new $\lg(\bar{x})$-place predicate.

$(*)_3$ (a) Let $\tau_* = \tau(M_\bullet) \cup \{c_b \colon b \in M\} \cup \{c\}$,

(b) Let M^+ be the expansion of $M_\bullet$ to a τ_*-model with $c_b^{M^+} = b$ for $b \in M$.

$(*)_4$ For $i \le \kappa$, let Γ_i be the union of he following sets:

(a) Γ_i^1 is the set of quantifiers free formulas which M^+ satisfies will we consider only those generated from the atomic by $\neg\varphi$, $\varphi \wedge \psi$, and $\varphi \equiv \bigwedge_{i<\theta} \varphi_i$,

(b) $\Gamma_i^2 = \{P_0(c) \wedge b_j Rc \colon j < i\}$,

(c) $\Gamma_i^3 = \{\sigma(\bar{d}) = c_\varepsilon \equiv \neg p_{\sigma(\bar{d})=c_\varepsilon} : \bar{d} \in {}^{\omega>}(|M| \cup; \{c\}, \sigma(-,-) \text{ a term for } \tau(M)\}$

(d) $\Gamma_i^4 = \{q_{\sigma(d)\varepsilon} P_1(\sigma(\bar{d})) \wedge p_{\sigma(\bar{d}),\varepsilon} \colon \sigma(\bar{d}), \varepsilon \text{ as above }\}$,

(e) $\Gamma_i^5 = \{q_{sigma(d)} \equiv \bigwedge_{\varepsilon<\theta} v p_{(\sigma(\bar{d}),\varepsilon} \colon \wedge p_{\sigma(\bar{d}),\varepsilon} \colon \sigma(\bar{d}) \text{ as above }\}$,

$(*)_5$ If $i < \kappa$ then Γ_i has a model.

[Why? Expand M^+ by interpreting c as c_i.]

$(*)_6$ Γ_κ has no model.

[Why? If Γ_κ has a model N, then without loss of generality, $N \restriction \tau(M^+)$ extend M^+, and l et N' be the restriction of N to the closure of $\{c_b \colon b \in M^+\} \cup \{c^{N'}\}$ under the Skolem functions. Easily $N \prec N'$ and $D = \{b \subseteq \kappa \colon N' \models \text{“}c \in c_b''\}$. Now check.]

$(*)_7$ Now 2.8, applied to $\langle \Gamma_i \colon i \le \kappa \rangle$ gives the desired result.

$\square_{2.10}$

Conclusion 2.12. In Claim 2.8 if $\lambda = \lambda^\kappa$ then we can allow $\langle \Gamma_i : i \le \kappa \rangle$ to be a sequence of theories in $\mathbb{L}_{\theta^+,\theta^+}(\tau), \tau$ any vocabulary of cardinality $\le \lambda$.

Proof. Without loss of generality, we can add Skolem functions (each with $\le \kappa$ places) in particular. So Γ_i becomes universal, and adding propositional variables for each quantifier-free sentence and writing down the obvious sentences, we get a set of propositional sentences, and we get Γ_i as there. $\square_{2.12}$

Note that:

Conclusion 2.13. 1) Let $\mathbf{C}_\theta = \{\kappa : \kappa = \text{cf}(\kappa)$ and for every λ and AEC $\mathfrak{k}$ with $\text{LST}(\mathfrak{k}) \le \theta, |\tau_{\mathfrak{k}}| = \theta$ have (λ,κ)-sequence-compactness of types $\}$ is the class $\{\kappa : \kappa = \text{cf}(\kappa) > \theta$ and there is a uniform θ^+-complete ultrafilter on $\kappa\}$.
2) In $\mathbf{C}_\theta$ we can replace "every λ" by $\lambda = 2^\theta + \kappa$.

Proof. Put together 2.10 and **??**. $\square_{2.13}$

Of course, a complementary result (showing the main claim is best possible) is:

Claim 2.14. *If $\mathfrak{k}'$ is an AEC,* $\text{LST}(\mathfrak{k}') \le \theta$ *and on κ there is a uniform θ^+-complete ultrafilter on κ and θ is regular and λ any cardinality* <u>then</u> *$\mathfrak{k}'$ has (λ,κ)-compactness of types.*

Proof. We sha;; write down a set of sentences Γ_ζ from $\mathbb{L}_{\theta^+,\theta^+}(\tau^+_{\mathfrak{k}})$ for $\zeta \le \theta$ expressing the demands.

Let $\langle M_i : i \le \kappa \rangle$ be $<_{\mathfrak{k}}$-increasing continuous, $\|M_i\| \le \lambda, \mathbf{p}_i = \text{ortp}_{\mathfrak{k}}(a_i, M_i, N_i)$ so $M_i \le_{\mathfrak{k}} N_i$ such that $i < j < \kappa \Rightarrow \mathbf{p}_i = \mathbf{p}_j{\restriction}M_i$. Without loss of generality $\|N_i\| \le \lambda$.

Let $\langle N_{i,j,\ell} : \ell \le n_{i,j} \rangle, \pi_{i,1}$ witness $\mathbf{p}_i = \mathbf{p}_j{\restriction}M_i$ for $i < j < \kappa$ (i.e. $M_i \le_{\mathfrak{k}} N_{i,j,\ell}$ (without loss of generality $\|N_{i,j,\ell}\| \le \lambda$), $N_{i,j,0} = N_i, N_{i,j,n_{i,j}} = N_j, a_i \in N_{i,j,\ell}, \bigwedge\limits_{\ell < n_{i,j,\ell}} (N_{i,j,\ell} \le_{\mathfrak{k}} N_{i,j,\ell+1} \vee N_{i,j,\ell+1} \le_{\mathfrak{k}} N_{i,j,\ell})$ and $\pi_{i,j}$ be an isomorphism from N_j onto $N_{i,j,n_{i,j}}$ over M_i mapping a_j to a_i.

Let $\tau^+ = \tau \cup \{F_{\varepsilon,n} : \varepsilon < \theta, n < \omega\}$ be disjoint union, $\text{arity}(F_{\varepsilon,n}) = n$. Let $\langle M^+_i : i \le \kappa \rangle$ be $\subseteq$-increasing, M^+_i a τ^+-expansion of M_i such that $u \subseteq M^+_i \Rightarrow M_i{\restriction}c\ell_{M^+_i}(u) \le_{\mathfrak{k}} M_i$. Similarly for $i < k < \kappa$, we have $\langle N^{+,\varepsilon}_{i,j,\ell} : \ell \le n_{i,j,\ell} \rangle; \varepsilon \in \{1,\ell\}$ such that $N^{+,\varepsilon}_{i,j,\ell}$ is a τ^+-expansion of $N_{i,j,\varepsilon}$ as above such that $(\forall \ell < n_{i,j,\ell})(\exists \varepsilon \in \{1,2\})(N^{+,\varepsilon}_{i,j,\ell} \subset N^{+,\varepsilon}_{i,j,\ell+1} \vee N^{+,\varepsilon}_{i,j,\ell+1} \subseteq N^{+,\varepsilon}_{i,j,\ell})$.

Now write down a translation of the question, "is there $\mathbf{p}$ such that..."

Assume that D is a uniform θ^+-complete ultrafilter on κ.

For each $i < \kappa$ let $\mathcal{U}_i \in D$ be such that $i < j \in \mathcal{U}_i \Rightarrow n_{i,j} = n^*_i$. Let moreover n_* and $\mathscr{U} \in D$ be such that for every $i \in \mathscr{U}$ we have $n_i = n_*$. Let $N_{i,\kappa,\ell} = \prod\limits_{j \in \mathcal{U}_i} N_{i,j,\ell}/D$. So $\langle N_{i,\kappa,\ell} : \ell \le n^*_\ell \rangle$ are as above. Let $M = \prod\limits_{i<\kappa} M_i/D, \pi_{i,\theta} = \prod\limits_{j \in \mathcal{U}_i} \pi_{i,j}/D$, etc. $\square_{2.14}$

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