FAILURE OF SINGULAR COMPACTNESS FOR Hom

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ABSTRACT. Assuming Gödel's axiom of constructibility $\mathbf{V} = \mathbf{L}$, we construct an almost-free abelian group G of singular cardinality with the property that for every nontrivial subgroup $G' \subseteq G$ of smaller cardinality, $\mathrm{Hom}(G', \mathbb{Z}) \neq 0$, while $\mathrm{Hom}(G, \mathbb{Z}) = 0$. This provides a consistent counterexample to the singular compactness of nontrivial duality with respect to the functor $\mathrm{Hom}(-, \mathbb{Z})$.

§ 0. INTRODUCTION

Hill [6] proved that if an abelian group G has cardinality a singular cardinal with cofinality at most ω_1, and every subgroup of G of smaller cardinality is free, then G itself is free. This result forms one of the foundations of Shelah's *Singular Compactness Theorem* [9], in which he introduced an abstract notion of freeness and removed the restriction on cofinality. In particular, Shelah showed that if an abelian group G is of size a singular cardinal λ, and if every subgroup of G of cardinality $< \lambda$ is free, then G must be free. General references on singular compactness may be found in [3, 4], while applications are discussed in [5].

Compactness principles, and the corresponding phenomena of incompactness, occupy a central place in modern set-theoretic algebra. Broadly, a compactness principle asserts that if every small subobject of a given object satisfies a property Pr, then the object itself satisfies Pr. At the ASL Annual Meeting in Irvine (2008), Shelah [10] announced several theorems concerning singular incompactness, although these results were not published. The aim of this paper is to provide a systematic analysis of incompactness for abelian groups with respect to the duality functor $\mathrm{Hom}(-, \mathbb{Z})$, focusing on singular cardinals. In particular, we verify a prediction from [10]. The property of interest is:

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Pr_λ: Let G be a group of cardinality λ. If every nontrivial subgroup $G' \subseteq G$ of cardinality $< \lambda$ satisfies $\mathrm{Hom}(G', \mathbb{Z}) \neq 0$, then $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$.

For $\mu \leq \lambda$ we recall that $S^\lambda_\mu = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \mu\}$ is stationary in λ. Given a stationary set $S \subseteq \lambda$, we write $\diamondsuit_S$ for Jensen's diamond on S (see Definition 1.6). When $\lambda > \aleph_0$ is regular and $\diamondsuit_S$ holds for some stationary non-reflecting $S \subseteq S^\lambda_{\aleph_0}$, it is known (see [3]) that one may construct a λ-free abelian group G of cardinality λ such that $\mathrm{Hom}(G, \mathbb{Z}) = 0$. Any subgroup G' of cardinality $< \lambda$ is then free, and hence $\mathrm{Hom}(G', \mathbb{Z}) \neq 0$. Thus Pr_λ fails for such λ. These assumptions entail that λ is not weakly compact, since weak compactness implies reflection of stationary sets, whereas we require a non-reflecting stationary subset. In fact, the non–weak compactness of λ is necessary for the property Pr_λ; see Fact 1.5. However, the argument above does not extend to singular cardinals.

Our main contribution is to show that the failure of Pr_λ at singular cardinals is consistent with Gödel's axiom of constructibility. Although the main theorem is established in a more general setting, the following special case suffices for the purposes of the introduction.

Theorem 0.1. *Assume* $\mathbf{V} = \mathbf{L}$ *and that:*

- *(a)* $\langle \lambda_i : i < \kappa \rangle$ *is an increasing sequence of regular cardinals with limit* λ*;*
- *(b) each* $\lambda_i = \mu_i^+$ *is a successor cardinal, where* μ_i *is regular and not weakly compact;*
- *(c)* $\aleph_0 < \kappa \leq \chi < \lambda_0$*, where* κ *and* χ *are regular.*

Then there exists a χ*-free abelian group* G *of cardinality* λ *which is a counterexample to singular compactness at* λ *for the property* $\mathrm{Hom}(-, \mathbb{Z}) \neq 0$.

In fact, the assumptions of Theorem 0.1 can be weakened as follows:

- (a) $\langle \lambda_i : i < \kappa \rangle$ is an increasing sequence of regular cardinals with limit λ, with each $\lambda_i = \mu_i^+$;
- (b) $\aleph_0 < \kappa \leq \chi < \lambda_0$, where κ and χ are regular;
- (c) for each $i < \kappa$, there exist stationary, non-reflecting sets $S_i \subseteq S^{\lambda_i}_{\aleph_0}$ and $T_i \subseteq S^{\mu_i}_{\aleph_0}$ such that $\diamondsuit_{S_i}$ and $\diamondsuit_{T_i}$ hold;
- (d) there is no measurable cardinal $\leq \lambda$.

Our investigation is closely related to the classical Whitehead problem, concerning the vanishing of $\mathrm{Ext}(G, \mathbb{Z}) := \mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z})$, a property of considerable interest but greater complexity. In our forthcoming work [1], we study singular compactness in the context of Ext. Under the additional assumption $\mathbf{V} = \mathbf{L}$, the situation becomes simpler: by [8], an abelian group G of cardinality $\lambda > \aleph_0$ is free if and only if $\mathrm{Ext}(G, \mathbb{Z}) = 0$. Hence, by Shelah's singular compactness theorem for free groups [9], singular compactness also holds for the property $\mathrm{Ext}(-, \mathbb{Z}) = 0$.

Throughout the paper, all groups are abelian. For standard terminology from set-theoretic algebra, we refer to Eklof–Mekler [3] and Göbel–Trlifaj [5]. Background from group theory may be found in Fuchs [4].

§ 1. Preliminaries

In this section we establish the notation and conventions used throughout the paper, and recall several basic facts from set-theoretic algebra. All groups are abelian and written additively.

For abelian groups G and H, we write $\mathrm{Hom}(G, H) := \mathrm{Hom}_{\mathbb{Z}}(G, H)$.

Notation 1.1. Let u be an index set, and define $\mathbb{Z}_{[u]} := \bigoplus_{\alpha\in u} \mathbb{Z}x_\alpha$, so that $\langle x_\alpha : \alpha \in u\rangle$ is a basis for $\mathbb{Z}_{[u]}$.

Definition 1.2. Let κ be an infinite cardinal and G be an abelian group.

(a) The group G is called κ-free if every subgroup of G of cardinality less than κ is free.

(b) The group G is said to be strongly κ-free if it is κ-free and in addition every subset of G of cardinality $< \kappa$ is contained in a subgroup H of G of cardinality $< \kappa$ such that G/H is κ-free.

Remark 1.3. We note that if κ is a singular cardinal, then G is κ-free if and only if G is λ-free for every regular cardinal $\lambda < \kappa$.

Recall that a ring R is said *left-perfect* if every flat left R-module is projective. Since $\mathbb{Q}$ is flat but not projective, it follows that $\mathbb{Z}$ is not left-perfect. Moreover, the following result shows that for a weakly compact cardinal κ, every κ-free group of cardinality κ is free.

Fact 1.4. (See [3, Theorems VI.3.2 and VII.1.4].) Let λ be a weakly compact cardinal. If M is a $\leq\lambda$-generated module which is λ-free, then M is free.

We now verify that Pr_λ holds whenever λ is weakly compact.

Fact 1.5. Assume λ is a weakly compact cardinal and G is a group of size λ such that $\mathrm{Hom}(G', \mathbb{Z}) \neq 0$ for every nontrivial subgroup $G' \subseteq G$ of cardinality $< \lambda$. Then $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$. In particular, Pr_λ holds.

Proof. Let $\theta > \lambda$ be sufficiently large and regular, and let M be an elementary submodel of $(\mathcal{H}(\theta), \in)$ of cardinality λ such that

$$M^{<\lambda} \subseteq M \qquad \text{and} \qquad \lambda,\, G \in M.$$

Since λ is weakly compact, there exists a model N of cardinality λ with $N^{<\lambda} \subseteq N$ and an elementary embedding $j : M \to N$ such that

$$j,\, M \in N, \qquad j(\alpha) = \alpha \text{ for all } \alpha < \lambda, \qquad j(\lambda) > \lambda.$$

In N, the group $j(G)$ has cardinality $j(\lambda)$, and G is a subgroup of $j(G)$ of size $\lambda < j(\lambda)$. By elementarity,

$$N \models \text{“}\mathrm{Hom}(G, \mathbb{Z}) \neq 0\text{”}.$$

This implies $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$ in the ambient universe, and the claim follows. □

Definition 1.6. Suppose $\lambda > \mu \geq \aleph_0$ are regular cardinals, and let $S \subseteq \lambda$ be stationary.

(1) *Jensen's diamond* $\diamondsuit_\lambda(S)$ asserts the existence of a sequence $(S_\alpha \mid \alpha \in S)$ such that for every $X \subseteq \lambda$, the set

$$\{\alpha \in S \mid X \cap \alpha = S_\alpha\}$$

is stationary.

(2) A useful consequence of $\diamondsuit_\lambda(S)$ is the following. Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ and $B = \bigcup_{\alpha<\lambda} B_\alpha$ be two λ-filtrations with $|A_\alpha|, |B_\alpha| < \lambda$. Then there exists a sequence of functions $(g_\alpha : A_\alpha \to B_\alpha \mid \alpha < \lambda)$ such that, for any function $g : A \to B$, the set $\{\alpha \in S \mid g{\restriction}_{A_\alpha} = g_\alpha\}$ is stationary in λ.

(3) S is *non-reflecting* if for every limit ordinal $\delta < \lambda$ of uncountable cofinality, the set $S \cap \delta$ is non-stationary in δ.

(4) We set $S^\lambda_\mu := \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \mu\}$.

Definition 1.7. Let $\mathcal{K}$ be the class of objects $\mathbf{k} := (\mu_\mathbf{k}, \theta_\mathbf{k}, K_\mathbf{k})$ consisting of:

(a) $\mu_\mathbf{k}$ is a limit ordinal, and $\theta_\mathbf{k} < \mu_\mathbf{k}$,

(b) $K_\mathbf{k}$ is an R-module with the set of elements $\theta_\mathbf{k}$, and $0_{K_\mathbf{k}} = 0$,

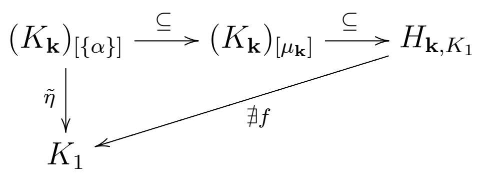
(c) if $K_1 \subseteq K_\mathbf{k}$ is an R-module, then we can find $(H_{\mathbf{k},K_1}, \eta_{\mathbf{k},K_1})$ such that:

  (α) $H_{\mathbf{k},K_1}$ is an R-module extending $(K_\mathbf{k})_{[\mu_\mathbf{k}]}$,

  (β) $H_{\mathbf{k},K_1}/(K_\mathbf{k})_{[\mu_\mathbf{k}]}$ is $(\mu_\mathbf{k}, K_\mathbf{k})$-free,

  (γ) $\eta_{\mathbf{k},K_1} \in {}^{\mu_\mathbf{k}}(K_1)$,

  (δ) there is no homomorphism $f : H_{\mathbf{k},K_1} \to K_1$ such that $f(x_\alpha) = \eta_{\mathbf{k},K_1}(\alpha)$ for $\alpha < \mu_\mathbf{k}$:



  where $\tilde\eta(x_\alpha) := \eta_{\mathbf{k},K_1}(\alpha)$.

In order to establish the existence of a framework $\mathbf{k} \in \mathbf{K}_1$, we first record the following result of Eklof–Mekler, see [2].

Fact 1.8. Let μ be an uncountable regular cardinal. Assume that $S \subseteq S^\mu_{\aleph_0}$ is stationary and non-reflecting, and that $\diamondsuit_S$ holds. Then there exists an indecomposable, strongly μ-free abelian group H of cardinality μ.

Lemma 1.9. *Assume $(R, K) = (\mathbb{Z}, \mathbb{Z})$. There exists $\mathbf{k} \in \mathbf{K}_1$ with $\mu_\mathbf{k} = \mu$ provided that:*

(a) $\mu = \mathrm{cf}(\mu) > \aleph_0$;

(b) $S \subseteq S^\mu_{\aleph_0}$ is stationary and non-reflecting;

(c) $\diamondsuit_S$ holds.

Proof. By Fact 1.8, there exists an indecomposable, strongly μ-free abelian group H of cardinality μ. We first note that $\mathrm{Hom}(H,\mathbb{Z})=0$. Indeed, if $0\neq f\in\mathrm{Hom}(H,\mathbb{Z})$, then $\mathrm{Im(f)}\cong\mathbb{Z}$ is free, so the exact sequence

$$0\longrightarrow \ker(f)\longrightarrow H\longrightarrow \mathrm{Im(f)}\longrightarrow 0$$

splits. Since H is indecomposable, this forces $\ker(f)=0$, hence H embeds into $\mathbb{Z}$, contradicting $|H|=\mu>\aleph_0$.

As H is strongly μ-free, there exists a subgroup $K\subseteq H$ of cardinality $<\mu$ such that $\mathbb{Z}\subseteq K$ and H/K is μ-free. Without loss of generality, write $K=\bigoplus_{i<\theta}\mathbb{Z}$ for some $\theta<\mu$. Fix a partition $\mu=\bigcup_{i<\mu}I_i$ such that the I_i are pairwise disjoint, $|I_i|=\theta$ for all $i<\mu$, and $I_i<I_j$ whenever $i<j$.

Define

$$G:=\bigoplus_{i<\mu}G_i,$$

where each $G_i\cong H$ and $G_i/\mathbb{Z}_{I_i}$ is μ-free. Since H is torsion-free, we have $\mathbb{Z}\subseteq H$, and hence

$$\mathbb{Z}_{[\mu]}\subseteq\bigoplus_{i<\mu}G_i=G.$$

Moreover,

$$\mathrm{Hom}(G,\mathbb{Z})\cong\mathrm{Hom}\Big(\bigoplus_{i<\mu}G_i,\mathbb{Z}\Big)\cong\prod_{i<\mu}\mathrm{Hom}(H,\mathbb{Z})=0.$$

We next verify that G is μ-free. Let $L\leq G$ with $|L|<\mu$. There exists $I\subseteq\mu$ with $|I|<\mu$ and subgroups $L_i\leq G_i$, each of size $<\mu$, such that $L\subseteq\bigoplus_{i\in I}L_i$. Since each L_i is free, so is $\bigoplus_{i\in I}L_i$, and hence L is free.

Finally, note that $G/\mathbb{Z}_{[\mu]}\cong\bigoplus_{i<\mu}G_i/\mathbb{Z}_{I_i}$, and each summand on the right-hand side is μ-free. Thus $G/\mathbb{Z}_{[\mu]}$ is μ-free. Therefore $\mathbf{k}=(\mu,\aleph_0,\mathbb{Z})\in\mathbf{K}_1$. For any $K_1\subseteq\mathbb{Z}$, set $(H_{\mathbf{k},K_1},\eta_{\mathbf{k},K_1})=(G,\eta)$, where $\eta\colon\mu\to K_1$ is arbitrary. If $K_1\neq 0$, then $K_1\cong\mathbb{Z}$, and hence $\mathrm{Hom}(G,K_1)=0$, which verifies (δ) from Definition 1.7(c). □

We also need the following well-known result of Kurepa.

Fact 1.10. Assume $\mathrm{cf}(\lambda)>\aleph_0$ and let $\mathcal{T}$ be a tree of height λ whose levels are all finite. Then $\mathcal{T}$ has a cofinal branch.

§ 2. Controlling $\mathrm{Hom}(G,\mathbb{Z})$

In this section we prove our main result (see Theorem 2.6).

Discussion 2.1. Recall that a cardinal κ is *measurable* if it is uncountable and there exists a non-principal κ-complete ultrafilter $\mathcal{D}$ on κ, meaning that for every subset $S\subseteq\mathcal{D}$ of cardinality less than κ, the intersection $\bigcap S$ belongs to $\mathcal{D}$. It is a classical result that the existence of measurable cardinals cannot be established within ZFC.

Definition 2.2. Let G be an abelian group. The *dual* of G is the abelian group $\mathrm{Hom}(G, \mathbb{Z})$, denoted by G^*. For $g \in G$, define $\psi_g : G^* \to \mathbb{Z}$ by evaluation

$$\psi_g(f) := f(g), \quad f \in G^*.$$

The assignment $g \mapsto \psi_g$ defines a canonical map $\psi : G \to G^{**}$. We say that G is *reflexive* if ψ is an isomorphism.

Fact 2.3. (Lös–Eda, Shelah; see [3, Corollary III.1.5] and [11]). Let $\mu = \mu_{\text{first}}$ be the first measurable cardinal. The following hold:

(a) For any $\theta < \mu$, the group $\mathbb{Z}^{(\theta)}$ is reflexive; in fact, its dual is $\mathbb{Z}^\theta$.
(b) For any $\lambda \geq \mu$, the group $\mathbb{Z}^{(\lambda)}$ is not reflexive.
(c) There exists a reflexive group $G \subset \mathbb{Z}^\mu$ of cardinality μ.

Let Pr be any property of abelian groups, and let λ be a cardinal. Recall that *compactness for* (λ, Pr) means that if G is a group of cardinality λ and

$$\text{“for all } G' \subseteq G \text{ with } |G'| < \lambda,\ G' \text{ has Pr”},$$

then G itself has Pr. In this paper, we focus on the following specific property of abelian groups:

Notation 2.4. For a cardinal λ, let Pr_λ denote the property: If G is a group of size λ, and if for any nontrivial subgroup $G' \subseteq G$ of size less than λ, $\mathrm{Hom}(G', \mathbb{Z}) \neq 0$, then $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$.

We now turn to the primary framework for our construction.

Definition 2.5. Let θ be a cardinal.

(1) Let $\mathbf{M}_{1,\theta}$ be the class of objects

$$\mathbf{m} = \big(\lambda_{\mathbf{m}}, \langle G^{\mathbf{m}}_\alpha : \alpha \leq \alpha_{\mathbf{m}}\rangle, S_{\mathbf{m}}, \langle f_{\mathbf{m},s} : s \in S_{\mathbf{m}}\rangle\big)$$

consisting of:

(*a*) (*α*) $\lambda_{\mathbf{m}} = \mathrm{cf}(\lambda_{\mathbf{m}}) > \aleph_0$,
(*β*) $\lambda_{\mathbf{m}} \geq \alpha_{\mathbf{m}} := \ell g(\mathbf{m})$, where $\ell g(\mathbf{m})$ denotes the length of $\mathbf{m}$.

(*b*) (*α*) $\langle G^{\mathbf{m}}_\alpha : \alpha \leq \alpha_{\mathbf{m}}\rangle$ is an increasing and continuous sequence of abelian groups,
(*β*) $|G^{\mathbf{m}}_\alpha| < \lambda_{\mathbf{m}}$ for $\alpha < \alpha_{\mathbf{m}}$.

(*c*) $G^{\mathbf{m}}_\alpha / G^{\mathbf{m}}_0$ is free.

(*d*) The set

$$\{\beta < \alpha_{\mathbf{m}} : G^{\mathbf{m}}_{\beta+1} / G^{\mathbf{m}}_\beta \text{ is not free}\}$$

is non-reflecting and stationary.

(*e*) (*α*) $S_{\mathbf{m}}$ is a set of cardinality $\leq \theta$,
(*β*) $f_{\mathbf{m},s} \in \mathrm{Hom}(G^{\mathbf{m}}_{\alpha_{\mathbf{m}}}, \mathbb{Z})$ for each $s \in S_{\mathbf{m}}$.

(*f*) The family $\langle f_{\mathbf{m},s} : s \in S_{\mathbf{m}}\rangle$ is a free basis of a subgroup of $\mathrm{Hom}(G^{\mathbf{m}}_{\alpha_{\mathbf{m}}}, \mathbb{Z})$.

(2) The class $\mathbf{M}_{2,\theta}$ is defined as above, with the modification that in item (a)(β) we set $\alpha_{\mathbf{m}} = \lambda_{\mathbf{m}}$, and we further require:
   (g) For every $f \in \mathrm{Hom}(G^{\mathbf{m}}_{\alpha_{\mathbf{m}}}, \mathbb{Z})$, there exists $h \in \mathrm{Hom}(\prod_{S_{\mathbf{m}}} \mathbb{Z}, \mathbb{Z})$ such that
$$f(x) = h(\langle f_{\mathbf{m},s}(x) : s \in S_{\mathbf{m}}\rangle) \quad \text{for all } x \in G^{\mathbf{m}}_{\alpha_{\mathbf{m}}}.$$
   Namely, h is a factorization of f through
$$\pi \in \mathrm{Hom}(G^{\mathbf{m}}_{\alpha_{\mathbf{m}}}, \mathbb{Z}^{S_{\mathbf{m}}}),$$
   defined by $\pi(x) := \big(f_{\mathbf{m},s}(x) \mid s \in S_{\mathbf{m}}\big)$.
   (h) The homomorphism from $G^{\mathbf{m}}_{\alpha_{\mathbf{m}}}$ into ${}^{S_{\mathbf{m}}}\mathbb{Z}$, defined by $x \mapsto \langle f_{\mathbf{m},s}(x) : s \in S_{\mathbf{m}}\rangle$ is surjective.
   (i) For any nonzero subgroup $G' \subseteq G^{\mathbf{m}}_{\alpha}$ with $\alpha < \alpha_{\mathbf{m}}$, we have $\mathrm{Hom}(G', \mathbb{Z}) \neq 0$.

We are now in a position to state and prove our main result:

Theorem 2.6. *Assume that:*
(a) $\langle \lambda_i : i < \kappa\rangle$ is an increasing sequence of regular cardinals with limit λ, and $\lambda_i = \mu_i^+$, for some regular cardinal μ_i,
(b) $\aleph_0 < \kappa \leq \chi < \lambda_0$ and κ, χ are regular,
(c) $S_i \subseteq S^{\lambda_i}_{\aleph_0}$ is stationary and non-reflecting, and $\diamondsuit_{S_i}$ holds,
(d) $T_i \subseteq S^{\mu_i}_{\aleph_0}$ is stationary and non-reflecting, and $\diamondsuit_{T_i}$ holds,
(e) there is no measurable cardinal $\leq \lambda$.

Then there is a χ-free abelian group G of cardinality λ which is counterexample to singular compactness in λ for Pr_λ.

Proof. We construct a χ-free abelian group G of cardinality λ endowed with the following property: for any nontrivial subgroup $G' \subseteq G$ of smaller cardinality, we have
$$\mathrm{Hom}(G', \mathbb{Z}) \neq 0,$$
while $\mathrm{Hom}(G, \mathbb{Z}) = 0$.

The idea of the construction is to equip the cardinals λ_i with an underlying tree structure $\mathcal{T}_i$, which allows precise control of homomorphisms from the building blocks G_i of the group
$$G := \bigcup_{i<\kappa} G_i.$$
The proof is organized in four stages. Stage (A) defines the tree structure $\mathcal{T}_i$. Stage (B), which is more involved, constructs by induction a system of χ-free groups together with homomorphisms from them to $\mathbb{Z}$. Stage (C) verifies that subgroups of smaller cardinality have nontrivial duals, and Stage (D) shows that there are no non-zero homomorphisms from G to $\mathbb{Z}$.

Stage (A): We define a tree $\mathcal{T}$ of height κ, whose i-th level $\mathcal{T}_i$ is defined as follows:
$(*)^i_A$: $\mathcal{T}_i$ is the set of all sequences η satisfying:
   (a) η has length $i+1$,

(b) for each $j \leq i$, $\eta(j) = (\eta(j,1), \eta(j,2))$,
(c) for each $j \leq i$, $\eta(j,1) < \lambda_j$ and $\eta(j,2) < \kappa$,
(d) if $j_1 < j_2 \leq i$, then $\eta(j_1,1) \leq \eta(j_2,1)$ and $\eta(j_1,2) \leq \eta(j_2,2)$,
(e) the range of η is finite,
(f) if $j_1 < j_2 \leq i$ and the sequence $\langle \eta(j,1) : j \in [j_1, j_2] \rangle$ is constant, then $j_2 < \eta(j_1, 2)$.

Set $\mathcal{T} := \bigcup_{i<\kappa} \mathcal{T}_i$, ordered by the end-extension relation $\triangleleft$. Then $(\mathcal{T}, \triangleleft)$ is a tree of height κ, whose i-th level is $\mathcal{T}_i$. Moreover, if $\eta \in \mathcal{T}_i$ and $i < j < \kappa$, there exists $\nu \in \mathcal{T}_j$ such that $\eta \triangleleft \nu$; that is, $\eta = \nu \restriction (i+1)$.

For truncated trees, we define, for $\alpha \leq \lambda_i$,

$$\mathcal{T}_{i,\alpha} := \{\eta \in \mathcal{T}_i : \eta(i,1) \leq \alpha\}.$$

In particular, $\mathcal{T}_i = \mathcal{T}_{i,\lambda_i}$.

Claim 2.7. *The tree $(\mathcal{T}, \triangleleft)$ has no branches of length κ.*

Proof. Assume, towards a contradiction, that there exists a branch

$$b := \langle \eta_i : i < \kappa \rangle$$

of $\mathcal{T}$. Then $\langle \eta_i : i < \kappa \rangle$ is $\triangleleft$-increasing. It follows that the sequence

$$\langle \eta_i(i,1) : i < \kappa \rangle$$

is non-decreasing in the ordinals. By clause $(*)^i_A(e)$, every initial segment takes only finitely many values, and since $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, the sequence must eventually stabilize. That is, there exists $i_* < \kappa$ such that

$$\eta_i(i,1) = \eta_{i_*}(i_*,1) \quad \text{for all } i \in [i_*, \kappa).$$

On the one hand, by $(*)^i_A(f)$, we have

$$\eta(i_*,2) > i \quad \text{for all } i < \kappa,$$

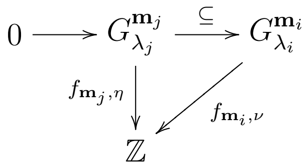
while on the other hand, $\eta(i_*,2) < \kappa$. This is a contradiction. $\square_{2.7}$

Stage (B): We define $\mathbf{m}_i$ by induction on $i < \kappa$ such that:

$(*)^i_B$: The sequence $\mathbf{m}_i$ satisfies:
(a) $\mathbf{m}_i = \big(\lambda_{\mathbf{m}_i}, \langle G^{\mathbf{m}_i}_\alpha : \alpha \leq \alpha_{\mathbf{m}_i} \rangle, S_{\mathbf{m}_i}, \langle f_{\mathbf{m}_i,s} : s \in S_{\mathbf{m}_i} \rangle\big) \in \mathbf{M}_{1,\lambda_i}$,
(b) $\lambda_{\mathbf{m}_i} = \alpha_{\mathbf{m}_i} = \lambda_i$, and the set of elements of $G^{\mathbf{m}_i}_{\lambda_i}$ has cardinality λ_i,
(c) $G_{<i} := \bigcup\{G^{\mathbf{m}_j}_{\lambda_j} : j < i\} \cup \{0\}$,
(d) $G^{\mathbf{m}_i}_0 := G_{<i}$,
(e) $S_{\mathbf{m}_i} := \mathcal{T}_i = \mathcal{T}_{i,\lambda_i}$,
(f) if $j < i$, then $\mathbf{m}_j \leq \mathbf{m}_i$, i.e.,

$$\eta \in \mathcal{T}_j \wedge \nu \in \mathcal{T}_i \wedge \eta \triangleleft \nu \implies f_{\mathbf{m}_j,\eta} \subseteq f_{\mathbf{m}_i,\nu},$$

which can be depicted as:



(g) $\langle f_{\mathbf{m}_i,\eta} : \eta \in \mathcal{T}_i \rangle$ is an independent subset of $\operatorname{Hom}(G^{\mathbf{m}_i}_{\lambda_i}, \mathbb{Z})$,

(h) $\bigcap\{\operatorname{Ker}(f_{\mathbf{m}_i,\eta}) : \eta \in \mathcal{T}_i\} = \{0\}$,

(i) for any $f \in \operatorname{Hom}(G^{\mathbf{m}_i}_{\lambda_i}, \mathbb{Z})$, there exist $\alpha < \lambda_i$ and $h \in \operatorname{Hom}\big(\prod_{\eta \in \mathcal{T}_{i,\alpha}} \mathbb{Z}, \mathbb{Z}\big)$ such that h is a factorization of f through $\pi_i \in \operatorname{Hom}(G^{\mathbf{m}_i}_{\alpha_{\mathbf{m}_i}}, \mathbb{Z}^{S_{\mathbf{m}_i}})$, defined by $\pi_i(x) = \big(f_{\mathbf{m}_i,s}(x) \mid s \in S_{\mathbf{m}_i}\big)$.

Remark 2.8. For each $\alpha < \lambda_i$, the cardinality of $\mathcal{T}_{i,\alpha}$ is less than λ_i. This fact will be useful to show that

$$|G^{\mathbf{m}_j}_{\alpha}| < \lambda_{\mathbf{m}_j} \quad \text{for } \alpha < \lambda_{\mathbf{m}_j},$$

see the subsequent discussion in ($\sharp$) below.

Assume that $i < \kappa$ and that the sequence $\langle \mathbf{m}_j : j < i \rangle$ has been defined. Fix a diamond sequence

$$\langle F_{i,\delta} : \delta \in S_i \rangle, \quad F_{i,\delta} : \delta \to \mathbb{Z}.$$

Notation 2.9. Let $\langle \beta_i(\gamma) : \gamma < \lambda_i \rangle$ be an increasing, continuous sequence of ordinals cofinal in λ_i, with $\beta_i(0) = 0$.

We proceed by setting $G_{<i} := \bigcup_{j<i} G^{\mathbf{m}_j}_{\lambda_j} \cup \{0\}$. Also, for $\eta \in \mathcal{T}_i$, define $f_{<i,\eta} : G_{<i} \to \mathbb{Z}$ by

$$f_{<i,\eta} := \bigcup_{j<i} f_{\mathbf{m}_j, \eta\restriction(j+1)}.$$

In particular, $G_{<0} = \{0\}$ and $f_{<0,\eta} : G_{<0} \to \mathbb{Z}$ is the zero map. We shall choose $\mathbf{m}_{i,\gamma}$ by induction on $\gamma < \lambda_i$ such that:

$(*)^{\gamma}_{C}$: (a) $\mathbf{m}_{i,0}$ is defined as

- (α) $\ell g(\mathbf{m}_{i,0}) = 0$,
- (β) $\lambda_{\mathbf{m}_{i,0}} := \chi + \sup_{j<i} \lambda_{\mathbf{m}_j}$ with the convention that $\sup_{j<0} \lambda_{\mathbf{m}_j} = 0$
- (γ) $G^{\mathbf{m}_{i,0}}_0 := G_{<i}$,
- (δ) $S_{\mathbf{m}_{i,0}} := \mathcal{T}_{i,\beta_i(0)}$,
- (ϵ) for $\eta \in \mathcal{T}_{i,\beta_i(0)}$, $f_{\mathbf{m}_{i,0},\eta} := f_{<i,\eta}$.

(b) $\langle \mathbf{m}_{i,\gamma} : \gamma < \lambda_i \rangle$ is an increasing and continuous sequence from $\mathbf{M}_{1,\lambda_i}$ with $S_{\mathbf{m}_{i,\gamma}} = \mathcal{T}_{i,\beta_i(\gamma)}$ and $\ell g(\mathbf{m}_{i,\gamma}) = \alpha_{i,\gamma} < \lambda_i$, which means:

- (α) if $\rho < \gamma$, then $\mathbf{m}_{i,\rho} \le \mathbf{m}_{i,\gamma}$,
- (β) if γ is a limit ordinal, then $\mathbf{m}_{i,\gamma} := \bigcup_{\rho<\gamma} \mathbf{m}_{i,\rho}$. In particular, we have the following equalities:
  - (β_1) $\alpha_{i,\gamma} = \sup_{\rho<\gamma} \alpha_{i,\rho}$,
  - (β_2) $G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}} = \bigcup_{\rho<\gamma} G^{\mathbf{m}_{i,\rho}}_{\alpha_{i,\rho}}$,

(β_3) $S_{\mathbf{m}_{i,\gamma}} = \mathcal{T}_{i,\beta_i(\gamma)}$,

(β_4) $f_{\mathbf{m}_{i,\gamma},\eta} = f_{i,<\eta} \cup \bigcup_{\rho<\gamma} f_{\mathbf{m}_{i,\rho},\eta\restriction\rho+1}$ for any $\eta \in \mathcal{T}_{i,\beta_i(\gamma)}$.

(c) If $\rho < \gamma$, and $\rho \notin S_i$, then $G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}}/G^{\mathbf{m}_{i,\rho}}_{\alpha_{i,\rho}}$ is free.

(d) $\bigcap\{\mathrm{Ker}(f_{\mathbf{m}_{i,\gamma},\eta}) : \eta \in \mathcal{T}_{i,\beta_i(\gamma)}\} = \{0\}$.

(e) $G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}}$ has set of elements an ordinal $\delta_i(\gamma) < \lambda_i$.

(f) Recall that $\langle F_{i,\delta} : \delta \in S_i\rangle$ denotes the diamond sequence. We now collect the following notations and assumptions:

(α) $\gamma = \alpha_{i,\gamma} \in S_i$,

(β) The underlying set of $G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}}$ is γ,

(γ) $\mathrm{Im}(\mathrm{F_{i,\gamma}}) \subseteq \mathbb{Z}$ is nonzero; in particular, $\mathrm{Im}(\mathrm{F_{i,\gamma}}) = \mathrm{n}\mathbb{Z} \cong \mathbb{Z}$ for some nonzero $n \in \mathbb{Z}$,

(δ) $F_{i,\gamma}$ is a homomorphism from $G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}}$ onto $\mathrm{Im}(\mathrm{F_{i,\gamma}})$,

(ϵ) $F_{i,\gamma} \notin \langle f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)}\rangle$, where $\langle f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)}\rangle$ denotes the subgroup of $\mathrm{Hom}(G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}}, \mathbb{Z})$ generated by $\{f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)}\}$.

We then define $\mathbf{m}_{i,\gamma+1}$ so that $F_{i,\gamma}$ has no extension to a homomorphism from $G^{\mathbf{m}_{i,\gamma+1}}_{\alpha_{i,\gamma+1}}$ into K_1. Namely, we have the following commutative diagram:

$$\begin{array}{ccc} G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}} & \xrightarrow{\subseteq} & G^{\mathbf{m}_{i,\gamma+1}}_{\alpha_{i,\gamma+1}} \\ {\scriptstyle F_{i,\gamma}}\downarrow & & \downarrow{\scriptstyle \nexists} \\ \mathrm{Im}(\mathrm{F_{i,\gamma}}) & \xrightarrow{=} & \mathrm{Im}(\mathrm{F_{i,\gamma}}), \end{array}$$

where the right vertical arrow does not exist. To simplify notation, we let $G_{i,\gamma} := G^{\mathbf{m}i,\gamma}\alpha_{i,\gamma}$. We then define $G_{i,\gamma,\rho} := G_{i,\gamma}$ for any $\rho \leq \alpha_{i,\gamma}$. Also, we abbreviate the function $f_{\mathbf{m}i,\gamma,\eta}$ as $f_{i,\gamma,\eta}$ for any index $\eta \in \mathcal{T}_{i,\beta_i(\gamma)}$.

The starting case $\gamma = 0$ is trivial, since it can be defined as in $(*)^\gamma_C$(a). By the induction hypothesis and the definition of $f_{i,0,\eta}$, we have:

- The sequence $\langle f_{i,0,\eta} : \eta \in \mathcal{T}_{i,\beta_i(0)}\rangle$ is an independent subset of $\mathrm{Hom}(G_{i,0}, \mathbb{Z})$,
- Each $f_{i,0,\eta}$ extends $f_{<i,\eta}$,
- $\bigcap\{\mathrm{Ker}(f_{i,0,\eta}) : \eta \in \mathcal{T}_{i,\beta_i(0)}\} = \{0\}$.

If γ is a limit ordinal, set $\alpha_{i,\gamma} = \sup_{\rho<\gamma} \alpha_{i,\rho}$ and define $\mathbf{m}_{i,\gamma}$ as in $(*)^\gamma_C$(b)(β).

Suppose that $\mathbf{m}_{i,\gamma}$ has already been defined. We now proceed to define $\mathbf{m}_{i,\gamma+1}$. First, assume that one of the following cases occurs:

(h_1) $\gamma \notin S_i$ or at least one of the hypotheses $(*)^\gamma_C$(f)(α)-(δ) fails, or

(h_2) $\gamma \in S_i$, the hypotheses $(*)^\gamma_C$(i)(α)-(δ) hold, but either
- $G_{i,\gamma}$ does not have domain γ, or
- $F_{i,\gamma} \notin \mathrm{Hom}(G_{i,\gamma}, \mathbb{Z})$, or
- $F_{i,\gamma} \in \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)}\rangle$.

In this case, define $\mathbf{m}_{i,\gamma+1}$ as follows:

(1) $\alpha_{i,\gamma+1} := \alpha_{i,\gamma} + 1$,

(2) $\mathbf{m}_{i,\gamma} \leq \mathbf{m}_{i,\gamma+1}$,

(3) $S_{\mathbf{m}_{i,\gamma+1}} := \mathcal{T}_{i,\beta_i(\gamma+1)}$,
(4) $G_{i,\gamma+1} := G_{i,\gamma+1,\alpha_{i,\gamma}+1} := G_{i,\gamma} \oplus \mathbb{Z}_{[u_{i,\gamma}]}$, where $u_{i,\gamma} := \mathcal{T}_{i,\beta_i(\gamma+1)}$,
(5) For $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$, set $f_{i,\gamma+1,\eta} := f_{i,\gamma,\eta} \oplus \pi_\eta$, where $\pi_\eta : \mathbb{Z}_{[u_{i,\gamma}]} \to \mathbb{Z}x_\eta$ is the projection, and for $\eta \notin \mathcal{T}_{i,\beta_i(\gamma)}$, $f_{i,\gamma,\eta}$ is the zero map.

Then suppose that $\mathbf{m}_{i,\gamma}$ is defined and the following holds:

(h_3) $\gamma \in S_i$, $G_{i,\gamma}$ has domain γ, hypotheses $(*)^\gamma_C$(f)(α)-(δ) hold, $F_{i,\gamma} \in \mathrm{Hom}(G_{i,\gamma}, \mathbb{Z})$, and $F_{i,\gamma} \notin \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)}\rangle$.

In this case, we define $\mathbf{m}_{i,\gamma+1}$ so as to satisfy $(*)^\gamma_B$(i). Let $\alpha_{i,\gamma+1} := \alpha_{i,\gamma} + 1$ and set

$$G^{\mathbf{m}_{i,\gamma+1}}_\rho := G^{\mathbf{m}_{i,\gamma}}_\rho = G_{i,\gamma,\rho}, \quad \forall \rho \leq \alpha_{i,\gamma}.$$

It remains to define

$$G_{i,\gamma+1} = G^{\mathbf{m}_{i,\gamma+1}}_{\alpha_{i,\gamma}+1} \quad \text{and} \quad f_{i,\gamma+1,\eta} = f_{\mathbf{m}_{i,\gamma+1},\eta} : G_{i,\gamma+1} \to \mathbb{Z}, \quad \eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}.$$

For each $\beta < \lambda_i$, define

$$G^{[\beta]}_{i,\gamma} := \{x \in G_{i,\gamma} : \eta \in \mathcal{T}_i \wedge \eta(i,1) < \beta \implies f_{i,\gamma,\eta}(x) = 0\}.$$

Then $\langle G^{[\beta]}_{i,\gamma} : \beta < \lambda_i\rangle$ is increasing, and since $|G_{i,\gamma}| < \lambda_i$, there exists $\beta_{i,\gamma} < \lambda_i$ such that

$$G^{[\beta]}_{i,\gamma} = G^{[\beta_{i,\gamma}]}_{i,\gamma}, \quad \forall \beta \in (\beta_{i,\gamma}, \lambda_i).$$

Set

$$K_{i,\gamma} := \mathrm{Im}\big(\mathrm{F_{i,\gamma}} \restriction \mathrm{G^{[\beta_{i,\gamma}]}_{i,\gamma}}\big),$$

and note that

$$\chi + \sum_{j<i} \lambda_j + \big|\mathcal{T}_{i,\beta_i(\gamma+1)}\big| + \aleph_0 < \lambda_i.$$

Consequently,

$$\chi + \sum_{j<i} \lambda_j + \big|\mathcal{T}_{i,\beta_i(\gamma+1)}\big| + \aleph_0 \leq \mu_i.$$

We set $\mu_{i,\gamma} := \mu_i$ and

$$\mathbf{k}_{i,\gamma} := (\mu_i, \omega, \mathbb{Z}).$$

Combining Lemma 1.9 with assumption 2.6(d), we obtain $\mathbf{k}_{i,\gamma} \in \mathcal{K}$. Using Notation 1.1, define

$$(K_{i,\gamma})_{[u]} := \bigoplus_{\alpha\in u} K_{i,\gamma} x_\alpha \quad \text{for any index set } u.$$

Finally, set

$$(H_*, \phi_*) := \big(H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}, \phi_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}\big),$$

so that

- H_* is a free abelian group of size $\mu_{i,\gamma}$ extending $(K_{i,\gamma})_{[u_{i,\gamma}]}$,
- $\phi_* : u_{i,\gamma} \to K_{i,\gamma} \setminus \{0\}$,
- $H_*/(K_{i,\gamma})_{[\mu_{i,\gamma}]}$ is $\mu_{i,\gamma}$-free,
- there is no homomorphism $f : H_* \to K_{i,\gamma}$ such that $f(x_\eta) = \phi_*(\eta)$ for all $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$.

Finally, for each β with $\beta_{i,\gamma} \leq \beta < \lambda_i$ and $b \in K_{i,\gamma} = \mathrm{Im}(\mathrm{F}_{\mathrm{i},\gamma} \restriction \mathrm{G}^{[\beta_{\mathrm{i},\gamma}]}_{\mathrm{i},\gamma})$, there exists $y_{b,\beta} \in G^{[\beta_{i,\gamma}]}_{i,\gamma} \subseteq G_{i,\gamma}$ such that

$(*)_{\beta,b}$ (a) If $\eta \in \mathcal{T}_i$ and $\eta(i,1) < \beta$, then $f_{i,\gamma,\eta}(y_{b,\beta}) = 0$,
(b) $F_{i,\gamma}(y_{b,\beta}) = b$.

Since $|G_{i,\gamma}| < \lambda_i$, for each $b \in K_{i,\gamma}$ as above, there exists some fixed $y_b \in G_{i,\gamma}$ such that the set

$$X_b := \{\beta < \lambda_i : \beta_{i,\gamma} \leq \beta \text{ and } y_{b,\beta} = y_b\}$$

is stationary in λ_i.

The assignment $x_\eta \mapsto y_{\phi_*(\eta)}$ induces a morphism

$$g_{i,\gamma} : (K_{i,\gamma})_{[u_{i,\gamma}]} \longrightarrow G_{i,\gamma}.$$

Recall that $u_{i,\gamma} = \mathcal{T}_{i,\beta_i(\gamma+1)}$ and that

$$\mathrm{id} : (K_{i,\gamma})_{[u_{i,\gamma}]} \longrightarrow H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$$

denotes the natural inclusion.

($\sharp$) Define

$$G_{i,\gamma+1} := \frac{G_{i,\gamma} \times H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}}{\big\langle (g_{i,\gamma}(k), -\mathrm{id}(k)) : k \in (K_{i,\gamma})_{[u_{i,\gamma}]} \big\rangle}.$$

Recall that $H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ is of size $\mu_{i,\gamma} < \lambda_i$. Combining this with an inductive argument along with $(*)$, we conclude that $|G_{i,\gamma+1}| < \lambda_i$.

Notation 2.10. For $g \in G_{i,\gamma}$ and $h \in H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$, let $[(g,h)] \in G_{i,\gamma+1}$ denote the equivalence class of (g,h).

By the definition of the push-out construction, this gives us two maps

$$h_{i,\gamma} : G_{i,\gamma} \longrightarrow G_{i,\gamma+1}, \qquad k_{i,\gamma} : H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} \longrightarrow G_{i,\gamma+1},$$

such that $h_{i,\gamma} \circ g_{i,\gamma} = k_{i,\gamma}$. In fact $h_{i,\gamma}$ is defined by $h_{i,\gamma}(x) := [(x,0)]$. The situation is depicted in the following commutative diagram:

$$\begin{array}{ccc} H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} & \xrightarrow{k_{i,\gamma}} & G_{i,\gamma+1} \\ \mathrm{id}\uparrow & & \uparrow h_{i,\gamma} \\ (K_{i,\gamma})_{[u_{i,\gamma}]} & \xrightarrow{g_{i,\gamma}} & G_{i,\gamma} \end{array}$$

Claim 2.11. *Let $g \in G_{i,\gamma}$ and $h \in (K_{i,\gamma})_{[u_{i,\gamma}]}$. Then there exists an element $\tilde{g} \in G_{i,\gamma}$ such that $[(g,h)] = [(\tilde{g},0)]$ in $G_{i,\gamma+1}$.*

Proof. Set $\tilde{g} := g + g_{i,\gamma}(h) \in G_{i,\gamma}$. Then

$$(g,h) - (\tilde{g},0) = (g - \tilde{g},\, h) = (-g_{i,\gamma}(h),\, h) \in \Big\langle \big(g_{i,\gamma}(k), -\mathrm{id}(k)\big) : k \in (K_{i,\gamma})_{[u_{i,\gamma}]} \Big\rangle.$$

Hence $[(g,h)] = [(\tilde{g},0)]$ in $G_{i,\gamma+1}$. $\square_{2.11}$

Clearly, the map $h_{i,\gamma} : G_{i,\gamma} \to G_{i,\gamma+1}$ is an embedding, and we may identify $G_{i,\gamma} \subseteq G_{i,\gamma+1}$ via $h_{i,\gamma}$. Let $g \in G_{i,\gamma}$ and $h \in H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$, and define

$$\psi : G_{i,\gamma+1} \longrightarrow H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}/(K_{i,\gamma})_{[u_{i,\gamma}]}, \quad \psi([(g,h)]) := h + (K_{i,\gamma})_{[u_{i,\gamma}]}.$$

As consequences of well-known general facts, ψ is well-defined, surjective, and also

$$\ker(\psi) = \{[(g,h)] : g \in G_{i,\gamma},\ h \in (K_{i,\gamma})_{[u_{i,\gamma}]}\} \overset{(2.11)}{=} \{[(\tilde{g},0)] : g \in G_{i,\gamma}\} \cong G_{i,\gamma}.$$

So, $G_{i,\gamma+1}/G_{i,\gamma} \cong H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}/(K_{i,\gamma})_{[u_{i,\gamma}]}$. Since the latter quotient is $\mu_{i,\gamma}$-free by assumption, the same holds for $G_{i,\gamma+1}/G_{i,\gamma}$. The next key observation is the non-extendability property of $F_{i,\gamma}$. Suppose, toward a contradiction, that $F : G_{i,\gamma+1} \to K_{i,\gamma}$ extends $F_{i,\gamma}$. Then the map

$$f := F \circ k_{i,\gamma} : H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} \longrightarrow K_{i,\gamma}$$

satisfies

$$f(x_\eta) = F \circ k_{i,\gamma}(x_\eta) = F_{i,\gamma} \circ g_{i,\gamma}(x_\eta) = \phi_*(\eta)$$

for all $\eta \in u_{i,\gamma}$, contradicting the choice of $(H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}, \phi_*)$.

Next, we define the map $f_{i,\gamma+1,\eta}$. Let $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$. For any $h \in H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ and $g \in G_{i,\gamma}$, set

$$f_{i,\gamma+1,\eta}([(h,g)]) := f_{i,\gamma,\eta}(g).$$

This defines a homomorphism

$$f_{i,\gamma+1,\eta} = f_{\mathbf{m}_{i,\gamma+1,\eta}} : G_{i,\gamma+1} \longrightarrow \mathbb{Z}.$$

To verify that $f_{i,\gamma+1,\eta}$ is well-defined, it suffices to check that $f_{i,\gamma,\eta} \circ g_{i,\gamma} = 0$. Indeed, for a given $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$, choose $\beta \in X_{\phi_*(\eta)}$ with $\eta(i,1) < \beta$. Then by $(*)_{\beta,\phi_*(\eta)}$(a),

$$f_{i,\gamma,\eta} \circ g_{i,\gamma}(x_\eta) = f_{i,\gamma,\eta}(y_{\phi_*(\eta)}) = f_{i,\gamma,\eta}(y_{\phi_*(\eta),\beta}) = 0.$$

Also, the family $\{f_{i,\gamma+1,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}\}$ is independent, and

$$\bigcap_{\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}} \operatorname{Ker}(f_{i,\gamma+1,\eta}) = \{0\}.$$

Finally, for $\eta \in \mathcal{T}_i$, define

$$f_{i,\eta} := \bigcup_{\gamma<\lambda_i} f_{i,\gamma,\eta}.$$

We are now prepared to state the following claim.

Claim 2.12. *The set* $\{f_{i,\eta} : \eta \in \mathcal{T}_i\}$ *generates* $\operatorname{Hom}(G^{\mathbf{m}_i}_{\lambda_i}, \mathbb{Z})$.

Proof. Suppose, toward a contradiction, that there exists

$$f \in \operatorname{Hom}(G^{\mathbf{m}_i}_{\lambda_i}, \mathbb{Z}) \setminus \langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle.$$

Choose $\gamma \in S$ such that $G_{i,\gamma}$ has domain γ, $f \restriction \gamma = F_{i,\gamma}$, and

$$f \restriction \gamma \notin \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_i \rangle.$$

By construction, the map $f \restriction \gamma$ cannot be extended to a homomorphism from $G_{i,\gamma+1}$ to $\mathbb{Z}$, yielding a contradiction. $\square_{2.12}$

We also note that clause $(*)^{\gamma}_{B}$(i) holds by Claim 2.12. Indeed, given any $f \in \mathrm{Hom}(G^{\mathbf{m}_i}_{\lambda_i}, \mathbb{Z})$, there exist $\eta_0, \dots, \eta_{n-1} \in \mathcal{T}_i$ and coefficients $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{Z}$ such that $f = \sum_{k=0}^{n-1} \alpha_k\, f_{i,\eta_k}$. Pick $\alpha < \lambda_i$ large enough so that $\eta_k(i,1) \le \alpha$ for all $k < n$. By Fact 2.3(a), for each $i < \kappa$ we have

$$\mathrm{Hom}\left(\prod_{\mathcal{T}_{i,\alpha}} \mathbb{Z},\, \mathbb{Z}\right) \cong \bigoplus_{\eta \in \mathcal{T}_{i,\alpha}} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{\eta \in \mathcal{T}_{i,\alpha}} \mathbb{Z}x_\eta.$$

In particular, there exists $h \in \mathrm{Hom}(\prod_{\mathcal{T}_{i,\alpha}} \mathbb{Z}, \mathbb{Z})$ such that

$$h\big(\langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle\big) = \sum_{k<n} \alpha_k f_{i,\eta_k}.$$

Clause $(*)^{\gamma}_{C}$(b) follows from the fact that S_i is non-reflecting. Hence there exists a club $C \subseteq \gamma$ with $\min(C) = \rho$ such that $C \cap S_i = \emptyset$. Then $G_{i,\gamma}/G_{i,\rho}$ is the union of the increasing and continuous sequence $\langle G_{i,\tau}/G_{i,\rho} : \tau \in C \rangle$. By the induction hypothesis, each $G_{i,\tau}/G_{i,\mu}$ is free for all $\mu < \tau$ in C, and therefore $G_{i,\gamma}/G_{i,\rho}$ is free as well.

Having defined the sequence $\langle \mathbf{m}_{i,\gamma} : \gamma < \lambda_i \rangle$ with properties $(*)^{\gamma}_{C}$, we set

$$\mathbf{m}_i := \bigcup_{\gamma < \lambda_i} \mathbf{m}_{i,\gamma}.$$

This completes the inductive construction of $\langle \mathbf{m}_i : i < \kappa \rangle$ as required by $(*)^{i}_{B}$.

Stage (C): In this step, we show that for each i, $\mathbf{m}_i \in \mathbf{M}_{2,\lambda_i}$ (see Definition 2.5(2)). Items (a)–(e) of Definition 2.5(1) and the equalities $\alpha_{\mathbf{m}_i} = \lambda_i = \lambda_{\mathbf{m}_i}$ are immediate.

For clause (i), let $\alpha < \lambda_i$ and $0 \neq G' \subseteq G^{\mathbf{m}_i}_{\alpha}$. Then there exists $\gamma < \lambda_i$ such that $G' \subseteq G^{\mathbf{m}_{i,\gamma}}_{\alpha_{i,\gamma}}$. Fix $0 \neq x \in G'$. By property $(*)^{\gamma}_{C}$(d),

$$\bigcap \{\mathrm{Ker}(f_{i,s}) : s \in \mathcal{T}_i\} = \{0\}.$$

Hence there exists $s \in \mathcal{T}_i$ with $f_{i,s}(x) \neq 0$. In particular, the restriction $f_{i,s} \restriction G' \in \mathrm{Hom}(G', \mathbb{Z})$ is non-zero, as required.

Stage (D): In this stage we conclude the proof of Theorem 2.6. For each $i < \kappa$, set

$$G_i := G_{\mathbf{m}_i, \lambda_i}, \qquad G_{<i} := \bigcup_{j<i} G_j.$$

Then $\langle G_i : i < \kappa \rangle$ is increasing and continuous, and for each i the quotient $G_i/G_{<i}$ is $(\chi + \sum_{j<i} \lambda_j)$-free. Indeed, for all $\gamma < \lambda_i$ we have $\mu_{i,\gamma} \ge \chi + \sum_{j<i} \lambda_j$, and each $G_{i,\gamma+1}/G_{i,\gamma}$ is $\mu_{i,\gamma}$-free.

Define

$$G := \bigcup_{i<\kappa} G_i.$$

Then G is an abelian group of size λ. We first show that G is χ-free. To this end, let $H \leq G$ be a subgroup of size $< \chi$. Then the sequence

$$\langle H \cap G_i : i < \kappa \rangle$$

is increasing and continuous. For each $i < \kappa$,

$$(H \cap G_i)/(H \cap G_{<i}) \cong ((H \cap G_i) + G_{<i})/G_{<i},$$

and this group is free since $G_i/G_{<i}$ is $(\chi + \sum_{j<i} \lambda_j)$-free, and therefore χ-free. Hence $H = \bigcup_{i<\kappa}(H \cap G_i)$ is free.

Next, let $H \leq G$ be a nonzero subgroup of size $< \lambda$. Choose $i < \kappa$ such that $H \cap G_{<i} \neq \{0\}$ and $|H| < \lambda_i$. By Definition 2.5(2)(i),

$$\mathrm{Hom}(H \cap G_i, \mathbb{Z}) \neq 0.$$

Moreover,

$$(H \cap G_i)/(H \cap G_{<i}) \cong ((H \cap G_i) + G_{<i})/G_{<i}$$

is free, and therefore $\mathrm{Hom}(H, \mathbb{Z}) \neq 0$.

Finally, we show that $\mathrm{Hom}(G, \mathbb{Z}) = 0$. Suppose, towards a contradiction, that $0 \neq f \in \mathrm{Hom}(G, \mathbb{Z})$. By $(*)^i_B$(i), for each $i < \kappa$ there exist $\alpha_i < \lambda_i$ and

$$h_i \in \mathrm{Hom}\Big(\prod_{\mathcal{T}_{i,\alpha_i}} \mathbb{Z}, \mathbb{Z}\Big)$$

such that

$$f(x) = h_i\big(\langle f_{\mathbf{m}_i,\eta}(x) : \eta \in \mathcal{T}_{i,\alpha_i}\rangle\big), \qquad x \in G_i.$$

By Fact 2.3(a),

$$\mathrm{Hom}\Big(\prod_{\mathcal{T}_{i,\alpha_i}} \mathbb{Z}, \mathbb{Z}\Big) \cong \bigoplus_{\eta \in \mathcal{T}_{i,\alpha_i}} \mathbb{Z}x_\eta.$$

By [3, Corollary III.3.3], for each i there exists a finite set $u_i \subseteq \mathcal{T}_{i,\alpha_i}$ such that

$$f_{\mathbf{m}_i,\eta}(x) = 0 \text{ for all } \eta \in u_i \implies f(x) = 0, \qquad x \in G_i.$$

Since $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, there exists n_* such that the set

$$\mathcal{V}_1 := \{i < \kappa : |u_i| = n_*\}$$

is unbounded in κ.

For $i < j < \kappa$, define

$$\mathrm{pr}_{i,j} : \mathcal{T}_j \to \mathcal{T}_i, \qquad \mathrm{pr}_{i,j}(\eta) = \eta \restriction (i+1).$$

If $\eta \in u_j$, then $\eta \restriction (i+1) \in u_i$, since $G_i \subseteq G_j$ and $f_{\mathbf{m}_i,\eta\restriction(i+1)} \subseteq f_{\mathbf{m}_j,\eta}$. Thus $\mathrm{pr}_{i,j}$ maps u_j onto u_i.

Let $\mathcal{T}$ be the tree of height κ whose i-th level is u_i. This is a well-defined tree of uncountable height with finite levels. By Fact 1.10, $\mathcal{T}$ carries a cofinal branch, contradicting Claim 2.7. $\square$

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