BAUMGARTNER'S AXIOM AND SMALL POSETS

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Abstract. We contribute to the study of $\aleph_1$-dense sets of reals, a mainstay in set theoretic research since Baumgartner's seminal work in the 70s. In particular, we show that it is consistent with MA that there exists an $\aleph_1$-dense set of reals A so that, in any cardinal-preserving generic extension by a forcing of size $\aleph_1$, A and A^* do not contain uncountable subsets which are order isomorphic. This strengthens a result of Avraham and the second author and yields a different proof of a theorem of Moore and Todorcevic.

§ 0. Introduction

One of the most important early results in forcing theory and its applications is Baumgartner's theorem from [Bau73] that consistently all $\aleph_1$-dense sets of reals are order isomorphic. This statement is now commonly known in the literature as *Baumgartner's Axiom* and denoted BA, see [Swi25] for more on the background. Here a set of reals is $\aleph_1$-*dense* if between any two points there are $\aleph_1$-many. Clearly each such is isomorphic to one which is moreover $\aleph_1$-dense in the reals - i.e. has intersection size $\aleph_1$ with every nonempty real open interval. The idea behind Baumgartner's proof is to assume CH and show that, given a pair of $\aleph_1$-dense sets of reals A and B, there is a ccc forcing notion of size $\aleph_1$ which generically adds an isomorphism between them. From the modern perspective it is then clear that a careful bookkeeping suffices to provide a model of $\mathsf{BA}+2^{\aleph_0}=\aleph_2$. Baumgartner also showed later in [Bau84] that BA also follows from PFA.

A natural question is whether Baumgartner's result actually relies on CH i.e. given two $\aleph_1$-dense linear orders A and B is there necessarily a ccc forcing notion which generically adds an isomorphism between them. This was asked by Baumgartner in [Bau73]. Similarly one can ask whether BA follows from MA already and not just PFA. The answer to both these questions is "no", due to a celebrated result of Avraham and the second author from [AS81].

Theorem 0.1 ([AS81, Theorem 2])**.** *For any regular $\kappa>\aleph_1$ it is consistent that $2^{\aleph_0}=\kappa$,* MA *holds and there is an $\aleph_1$-dense $A\subseteq\mathbb{R}$ which is not isomorphic to its reverse ordering.*

Note that were there a ccc forcing notion in this model adding an isomorphism between A and its reverse ordering then they would have to already be isomorphic

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by MA. For this reason, in this model not only does BA fail but it can never be forced by a ccc forcing. Given this, it is natural to ask the following two questions.

Question 0.2. Given A and B which are $\aleph_1$-dense is there always a *proper* forcing notion of size less than the continuum which adds an isomorphism?

Question 0.3. Does PFA($\aleph_1$-sized posets) suffice to prove BA? What about MM($\aleph_1$-sized posets)?

Note that Baumgartner's proof provides in ZFC a proper forcing of size $2^{\aleph_0}$ which generically adds isomorphisms. In full generality these questions remain open however in this paper we provide the following partial answer.

Main Theorem 0.1. *For any regular cardinal $\kappa > \aleph_1$ it is consistent that* MA + $2^{\aleph_0} = \kappa$ *holds and there is an $\aleph_1$-dense $A \subseteq \mathbb{R}$ so that any forcing notion of size $\aleph_1$ which generically adds an isomorphism between A and its reverse ordering must necessarily collapse $\aleph_1$.*

In particular we have the following immediate corollary.

Corollary 0.4. *It is consistent with* ZFC *that there are A and B which are $\aleph_1$-dense sets of reals so that no proper forcing of size less than continuum can add an isomorphism between them.*

We remark in relation to this that in [MT17] it was shown that forcing $\mathsf{MA}_{\aleph_2}$ in the natural way forces that there is an $\aleph_1$-dense linear order that cannot be made isomorphic to its reverse in any outer model of the universe with the same ω_2, which provides a partial answer to the questions above as well. The results above allow us to give therefore an alternative proof of the observation made concerning the proof of [MT17, Theorem 1.4] made in the introduction to that paper.

Corollary 0.5 (Moore-Todorcevic, See [MT17, Theorem 1.4] and related discussions)**.** MA + ¬CH *is consistent with an $\aleph_1$-dense linear order that cannot be made isomorphic to its reverse ordering by any $\aleph_1$-sized forcing notion which does not collapse ω_1.*

Our proofs actually establish quite a bit more. The definition of a *good* linear order is given below, see Definition 1.5. For the purposes of the introduction we note that any set of uncountably many mutually generic Cohen reals is good. We show that if there are good, $\aleph_1$-dense linear orders, then under MA they necessarily have this indestructibility property.

Main Theorem 0.2. *Assume $\mathsf{MA}_{\aleph_1}$. If $A \subseteq \mathbb{R}$ is good and $\aleph_1$-dense then it cannot be made isomorphic to its reverse order by any forcing notion of size $\aleph_1$ which does not collapse $\aleph_1$.*

Already in [AS81] it was shown by the second author and Avraham that MA + $2^{\aleph_0} = \kappa$ is consistent with a good $\aleph_1$-dense linear order for any regular κ. This fact alongside Main Theorem 0.2 implies Main Theorem 0.1.

The rest of this paper is organized as follows. In the next section we introduce some combinatorial properties of $\aleph_1$-dense linear orders which will be used in the proof of Main Theorem 0.1. The following section proves that if MA holds alongside the existence of a linear order satisfying a certain combinatorial property, *slicewise coverable*, described in §1, then the conclusion of Main Theorem 0.1 holds. Finally in §3 we provide a model in which MA holds alongside such a linear order hence proving Main Theorem 0.1. A final section concludes with some further observations and open questions.

Throughout, our notation is standard, conforming to that of the monographs [Jec03] or [Kun11]. We fix some notation for linear orders which is slightly less

standard or well known. First, given a linear order $A \subseteq \mathbb{R}$ its reverse ordering is denoted $A^* = \{-r : r \in A\}$. Such an order is *reversible* if $A \cong A^*$. Clearly BA implies all $\aleph_1$-dense linear orders are reversible. Next we say that two linear orders $A, B \subseteq \mathbb{R}$ are *near* if there are uncountable subsets $A' \subseteq A$ and $B' \subseteq B$ so that $A' \cong B'$. If A and B are not near then we say they are *far*. Finally, let us say that a map $f : A \to B$ (for linear orders A and B) is *increasing* if $a < b$ implies $f(a) < f(b)$. A map $f : A \to B$ is *decreasing* or *order reversing* if $a < b$ implies $f(b) < f(a)$. A map $f : A \to B$ is *monotone* if it is either increasing or decreasing.

§ 1. Anti-reversibility conditions

The key steps in the proof of Main Theorem 0.1 require the construction of a linear order which is very far from being reversible. In [AS81] the following concept was used, though not given a name.

Definition 1.1. Let $A \subseteq \mathbb{R}$ be an $\aleph_1$-dense linear order. We say that A is *essentially increasing* if every partial, uncountable injection $f : A \rightharpoonup A$ is increasing on some uncountable subset.

Obviously, if A is essentially increasing then it is far from A^*. Moreover, the existence of an essentially increasing A implies that no ccc forcing notion can force A and A^* to be near, see [AS81, p. 106]. In [AS81] the following was proved.

Theorem 1.2 (Avraham-Shelah, [AS81])**.** *For any regular $\kappa > \aleph_1$ it is consistent that $2^{\aleph_0} = \kappa$, MA holds and there is an essentially increasing linear order.*

We will need a strengthening of being essentially increasing, which we call being slicewise coverable. Below we say that a *filtration* of an $\aleph_1$-dense linear order A is a $\subseteq$-increasing and continuous sequence $\vec{A} = \{A_\alpha : \alpha \in \omega_1\}$ so that the following are satisfied.

(A) $A_0 = \emptyset$
(B) For all $\alpha < \omega_1$ we have that $A_{\alpha+1} \setminus A_\alpha$ is a countable, dense subset of A.
(C) $A = \bigcup_{\alpha<\omega_1} A_\alpha$

Note that if $\vec{A}$ is a filtration of A then the set $\{A_{\alpha+1} \setminus A_\alpha : \alpha \in \omega_1\}$ forms a partition of A into ω_1-many countable, dense sets. In this set up let us refer to each $A_{\alpha+1} \setminus A_\alpha$ as a *slice* of the the filtration.

We now define slicewise coverability.

Definition 1.3. An $\aleph_1$-dense linear order is said to be *slicewise coverable* if given any filtration $\vec{A}$ of A there are countably many partial increasing functions $\{f_n\}_{n<\omega}$ with $f_n : A \rightharpoonup A$ so that $\bigcup_{n<\omega} f_n = \bigcup_{\alpha<\omega_1} (A_{\alpha+1} \setminus A_\alpha) \times (A_{\alpha+1} \setminus A_\alpha)$.

Note that the above says concretely that for every $n < \omega$ and every $\alpha \in \omega_1$ if $a \in \mathrm{dom}(f_n) \cap (A_{\alpha+1} \setminus A_\alpha)$ then $f_n(a) \in (A_{\alpha+1} \setminus A_\alpha)$ and, moreover every pair $(a, b) \in (A_{\alpha+1} \setminus A_\alpha)^2$ appears in this way for some $n < \omega$. In other words, each square of a slice is covered by the graphs of the f_n's.

Lemma 1.4. *Suppose $A \subseteq \mathbb{R}$ is $\aleph_1$-dense. If A is slicewise coverable, then it is essentially increasing.*

Proof. Assume A is a slicewise coverable, $\aleph_1$-dense linear order. Let $f : A \rightharpoonup A$ be a partial, uncountable function. Take an $\in$-increasing and continuous chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of some large H_θ with $f, A \in M_0$. Let $A_\alpha := M_\alpha \cap A$. This produces a filtration of A so that each slice if closed under f, by elementarity.

By assumption there are countably many $f_n : A \rightharpoonup A$, each of which is a partial increasing function so that $\bigcup_{n<\omega} f_n = \bigcup_{\alpha<\omega_1} [(A_{\alpha+1} \setminus A_\alpha) \times (A_{\alpha+1} \setminus A_\alpha)]$ and

therefore in particular we can cover the graph of f by these functions since each $A_{\alpha+1} \setminus A_\alpha$ was closed under f. Since f is uncountable however that means some $n < \omega$ we have that $f_n \cap f$ is uncountable, which completes the proof. □

We will need a separate combinatorial property of linear orders which will be useful in constructing slicewise coverable linear orders.

Definition 1.5. Let A be an $\aleph_1$-dense set of reals. We say that A is *good* if there is an injective enumeration $A = \{r_\xi : \xi < \omega_1\}$ so that the following holds: for every $n < \omega$ and every $\langle a_\alpha : \alpha < \omega_1 \rangle$ with $a_\alpha \in [\omega_1]^n$, there exist $\alpha < \beta$ such that $r_{a_\alpha(i)} \leq r_{a_\beta(i)}$ for every $i < n$.

In what follows, when we write "$A = \{r_\xi : \xi < \omega_1\}$ is good", we mean that $\langle r_\xi : \xi < \omega_1 \rangle$ is an (injective) enumeration witnessing the goodness of A.

The following is shown on [AS81, p. 164].

Lemma 1.6. *Let $A = \{r_\alpha : \alpha < \omega_1\}$ be a set of mutually generic Cohen reals over a fixed ground model V. Then A is good in $V[A]$.*

Being good can be preserved by finite support iterations. This was explained on [AS81, p. 166] but we provide a proof to make this presentation more self contained. First we need to define the class of posets preserving goodness.

Definition 1.7. Let $A = \{r_\xi : \xi < \omega_1\}$ be good. A forcing poset $\mathbb{P}$ is said to be *appropriate (for A)* iff for all $n \in \omega$ and $\langle (p_\alpha, a_\alpha) : \alpha < \omega_1 \rangle$ where $p_\alpha \in \mathbb{P}$ and $a_\alpha \in [\omega_1]^n$, there exist $\alpha < \beta$ such that $p_\alpha \parallel p_\beta$ and $r_{a_\alpha(i)} \leq r_{a_\beta(i)}$ for all $i < n$.

Observe that being appropriate implies ccc by ignoring the a_α part. Also, by [AS81, Lemma 12], if A is good and $\mathbb{P}$ is appropriate then $\mathbb{P}$ forces that A is still good. In fact this is an equivalent characterization.

Lemma 1.8. *Let $\mathbb{P}$ be a partial order and let A be good. The following are equivalent.*

(A) $\mathbb{P}$ is appropriate for A.
(B) $\mathbb{P}$ is ccc and preserves the goodness of A.

Proof. (A) $\implies$ (B): See [AS81, Lemma 12].

(B) $\implies$ (A): Fix a natural number $n < \omega$ and let $\{(p_\alpha, a_\alpha) : \alpha \in \omega_1\}$ be a set of pairs with $p_\alpha \in \mathbb{P}$ and $a_\alpha \in [\omega_1]^n$ as in the definition of goodness. Let $\dot{Z}$ be a $\mathbb{P}$-name for the set of $\alpha < \omega_1$ so that $p_\alpha \in \dot{G}$.

Claim 1.9. *There is a $p \in \mathbb{P}$ which forces that $\dot{Z}$ is uncountable.*

Proof of Claim. Otherwise $\Vdash_{\mathbb{P}}$"$\dot{Z}$ is countable". By the ccc, there is some countable ordinal γ so that $\Vdash_{\mathbb{P}} \dot{Z} \subseteq \check{\gamma}$. But this is not possible since for any p_α with $\alpha > \gamma$ we have that $p_\alpha \Vdash \check{\alpha} \in \dot{Z}$. □

Fix now a condition $p \in \mathbb{P}$ forcing that $\dot{Z}$ is uncountable. Let $G \ni p$ be generic over V and work in $V[G]$. By assumption, A is still good. Let $Z = \dot{Z}^G$. Consider now the set $\{(p_\alpha, a_\alpha) : \alpha \in Z\}$. Since A is good (in $V[G]$), there are p_α and p_β in G so that $r_{a_\alpha(i)} \leq r_{a_\beta(i)}$ for all $i < n$. Also, since p_α and p_β are both in G, they are compatible. But now, back in V we have that α and β witness compatibility. □

We now sketch the aforementioned preservation result.

Lemma 1.10. *Finite support iterations of appropriate posets are appropriate.*

Proof. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ be a finite support iteration of some ordinal length δ so that for each $\alpha < \delta$ we have that $\Vdash_\alpha$"$\dot{\mathbb{Q}}_\alpha$ is appropriate". We want to show by

induction on δ that $\mathbb{P}_\delta$ is appropriate. Note that since appropriate forcing notions are ccc we know that $\mathbb{P}_\delta$ is ccc.

The successor stage follow from Lemma 1.8 above since clearly being good is preserved by two-step iterations. Let us therefore assume that δ is a limit ordinal and for all $\alpha < \delta$ we have $\mathbb{P}_\alpha$ is appropriate. Let $\{(p_\alpha, a_\alpha) : \alpha \in \omega_1\}$ be a sequence of pairs consisting of a condition p_α and a sequence a_α of countable ordinals of some fixed size $n < \omega$. By thinning out, we can find an uncountable set $X \in [\omega_1]^{\omega_1}$ so that the set $\{\mathrm{supp}(p_\alpha) : \alpha \in X\}$ forms a Δ-system. Let $\gamma < \delta$ be an ordinal above the maximum of the root. Consider now the set $\{(p_\alpha{\restriction}\gamma, a_\alpha)$. Since $\mathbb{P}_\gamma$ is assumed to be appropriate we can find $\alpha < \beta \in X$ so that $p_\alpha{\restriction}\gamma \parallel p_\beta{\restriction}\gamma$ and $r_{a_\alpha(i)} \leq r_{a_\beta(i)}$ for all $i < n$. Now however it follows that p_α and p_β are compatible which completes the proof of this case. □

§ 2. A Sufficient Condition

In this section we prove a sufficient condition for the existence of a linear order satisfying the conclusion of Main Theorem 0.1.

Theorem 2.1. *Suppose that there exists $A \subseteq \mathbb{R}$ such that the following hold:*

(A) A is dense in $\mathbb{R}$ and $\aleph_1$-dense and;
(B) A is slicewise coverable.

Let $\mathbb{Q}$ be an $\aleph_1$-preserving forcing of size $\aleph_1$. Then $\mathbb{Q}$ forces that A and A^* are far.

Proof. Fix A as in the statement of the theorem and enumerate its elements as $\{a_\alpha : \alpha \in \omega_1\}$. Suppose towards a contradiction that $\mathbb{Q}$ is a forcing notion of size $\aleph_1$ forcing that A and A^* are near. Let $\langle q_\xi : \xi < \omega_1\rangle$ be an injective enumeration of $\mathbb{Q}$. Let $\dot\pi$ be a $\mathbb{Q}$-name so that $\Vdash_{\mathbb{Q}}$" $\dot\pi$ is an order isomorphism between uncountable subsets of A and A^*". Without loss of generality we can assume that the lower cone of every element of $\mathbb{Q}$ is uncountable.

Let $\dot\pi_*$ denote the name for the uncountable, partial function from ω_1 to ω_1 defined in the extension by $\dot\pi_*(\alpha) = \beta$ if and only if $\dot\pi(a_\alpha) = a_\beta$. For each $p \in \mathbb{Q}$ and $\alpha \in \omega_1$ we can choose a triple $(q_{p,\alpha}, \beta_{p,\alpha}, \gamma_{p,\alpha})$ so that the following are satisfied:

(A) $q_{p,\alpha} \leq p$ and, if $\xi < \omega_1$ is such that $q_{p,\alpha} = p_\xi$, then $\xi > \alpha$;
(B) $\beta_{p,\alpha}, \gamma_{p,\alpha} \in (\alpha, \omega_1)$ (that is they are countable ordinals above α);
(C) $q_{p,\alpha} \Vdash \dot\pi_*(\check\beta_{p,\alpha}) = \check\gamma_{p,\alpha}$

Note that these exist since $\dot\pi$ is assumed to be uncountable (so there are $\beta_{p,\alpha}, \gamma_{p,\alpha} > \alpha$) and every lower cone is uncountable so we can always strengthen to such a q should the original index of a condition forcing the above statement be too small.

By considering a chain of elementary submodels, we can find a club $E \subseteq \omega_1$ with the following properties.

(A) If $\delta \in E$ then for every $\eta, \xi < \delta$ we have that $\beta_{p_\xi,\eta}, \gamma_{p_\xi,\eta} < \delta$ and, if $q_{p_\xi,\eta} = p_\zeta$, then $\zeta < \delta$ as well.
(B) For every $\delta \in E$, if $\delta' := \min(E \setminus (\delta + 1))$, then $\{a_i : \delta \leq i < \delta'\}$ is dense in A.
(C) $0 \in E$.

Fix now such a club E and enumerate its elements as $E = \{\delta_\alpha : \alpha < \omega_1\}$ in order. Observe that condition (B) ensures that $A_\alpha := \{a_i : i < \delta_\alpha\}$ forms a filtration. Choose functions f_n which are increasing and witness slicewise coverability of A with respect to this filtration.

Claim 2.2. *For each $p \in \mathbb{Q}$ and each $\alpha < \omega_1$, there exists $n \in \omega$ such that $f_n(a_{\beta_{p,\alpha}}) = a_{\gamma_{p,\alpha}}$.*

Proof. Fix $p \in \mathbb{Q}$ and $\alpha < \omega_1$. There is a unique $i < \omega_1$ such that $\delta_i \leq \alpha < \delta_{i+1}$. By (A) in the definition of E, $\beta_{p,\alpha}, \gamma_{p,\alpha} < \delta_{i+1}$. On the other hand, by (B) in the definition of $(q_{p,\alpha}, \beta_{p,\alpha}, \gamma_{p,\alpha})$, we have that $\beta_{p,\alpha} > \alpha \geq \delta_i$ and similarly for $\gamma_{p,\alpha}$. Therefore, $(a_{\beta_{p,\alpha}}, a_{\gamma_{p,\alpha}}) \in (A_{i+1} \setminus A_i) \times (A_{i+1} \setminus A_i)$, and so, by the definition of the f_m's, there is some $n \in \omega$ with $f_n(a_{\beta_{p,\alpha}}) = a_{\gamma_{p,\alpha}}$. □

Consider the set

$$S_n := \left\{ q \in \mathbb{Q} : \exists \alpha < \omega_1 \exists p \in \mathbb{Q}[q = q_{p,\alpha} \wedge f_n(a_{\beta_{p,\alpha}}) = a_{\gamma_{p,\alpha}}] \right\}.$$

Claim 2.3. *For every $n < \omega$, the set S_n is an antichain.*

Proof of Claim. Fix $n < \omega$. If $q_{p_0,\alpha_0}, q_{p_1,\alpha_1} \in S_n$ then we have that $f_n(a_{\beta_{p_i,\alpha_i}}) = a_{\gamma_{p_i,\alpha_i}}$ for $i < 2$ and hence $\{(a_{\beta_{p_0,\alpha_0}}, a_{\gamma_{p_0,\alpha_0}}), (a_{\beta_{p_1,\alpha_1}}, a_{\gamma_{p_1,\alpha_1}})\}$ is an increasing function so q_{p_0,α_0} and q_{p_1,α_1} cannot be in the same generic as $\dot{\pi}$ was forced to be order reversing. □

For each $n < \omega$ extend S_n to some maximal antichain I_n. Let $\dot{\xi}_n$ denote the unique countable ordinal so that $p_{\dot{\xi}_n} \in \dot{G} \cap \check{I}_n$. Let $\dot{\xi}$ be a name for $\sup_{n\in\omega} \dot{xi}_n$. Since ω_1 is not collapsed, it follows that $\Vdash \dot{\xi} < \omega_1^V$. Pick $\varepsilon < \omega_1$ and $\xi < \omega_1$ so that $p = p_\varepsilon \Vdash \dot{\xi} = \check{\xi}$. Let $\alpha \in E$ be larger than ε and ξ and finally consider the triple $(q_{p,\alpha}, \beta_{p,\alpha}, \gamma_{p,\alpha})$. If $\zeta < \omega_1$ is such that $q_{p,\alpha} = p_\zeta$, then $\zeta > \alpha > \xi$ by (B) in the definition of $q_{p,\alpha}$. Moreover, there is an $n < \omega$ so that $f_n(a_{\beta_{p,\alpha}}) = a_{\gamma_{p,\alpha}}$ and hence $q_{p,\alpha} \in S_n$. Thus, $q_{p,\alpha}$ forces that $\dot{\xi}_n = \zeta$, contradicting that p, which was a weaker condition, forced that $\dot{\xi}_n < \dot{\xi} < \zeta$. □

§ 3. Constructing a Model of MA with a slicewise coverable linear order

We now complete the proof of Main Theorem 0.1 by constructing a model of $\mathsf{MA} + 2^{\aleph_0} = \kappa$ in which there is a slicewise coverable, $\aleph_1$-dense linear order. By Theorem 2.1 this suffices.

Theorem 3.1. *Assume* GCH *and let κ be a regular cardinal greater than $\aleph_1$. There is a ccc forcing notion $\mathbb{P}$ so that in any generic extension by $\mathbb{P}$ there is an $\aleph_1$-dense $A \subseteq \mathbb{R}$ so that the following hold.*

(A) $2^{\aleph_0} = \kappa$ and MA *holds and;*
(B) A is slicewise coverable.

Towards proving this theorem we need a forcing notion which will allow us to cover the slices of a given filtration.

Definition 3.2. Let $A = \{r_\xi : \xi < \omega_1\}$ be $\aleph_1$-dense and good. Given a filtration $\vec{A}$ of A, let $\mathbb{Q}_{\vec{A}}$ be the set of finite, order-preserving partial functions $q : A \rightharpoonup A$ such that

$$\forall \xi, \eta < \omega_1 [q(r_\xi) = r_\eta \to \exists \alpha < \omega_1 (r_\xi, r_\eta \in A_{\alpha+1} \setminus A_\alpha)].$$

Order $\mathbb{Q}_{\vec{A}}$ by (reverse) inclusion.

Lemma 3.3. *Let A be $\aleph_1$-dense and good, and let $\vec{A} = \langle A_\alpha : \alpha \in \omega_1 \rangle$ be a filtration. The following hold.*

(A) $\mathbb{Q}_{\vec{A}}$ is ccc.
(B) $\mathbb{Q}_{\vec{A}}$ generically adds a function $f : A \to A$ which is defined on an uncountable set, is increasing and has the property that for every $\alpha < \omega_1$ we have $a \in A_{\alpha+1} \setminus A_\alpha$ if and only if $f(a) \in A_{\alpha+1} \setminus A_\alpha$ for every $a \in \mathrm{dom}(p)$.
(C) $\mathbb{Q}_{\vec{A}}$ is appropriate.

Proof. To establish (A), we argue by contradiction. We can then let n be the smallest positive integer so that there exists an uncountable antichain all of whose elements have size n. Let W be an uncountable antichain of $\mathbb{Q}_{\vec{A}}$ with $|q| = n$ for all $q \in W$.

Claim 3.4. *We may assume, shrinking W if necessary, that for all $p, q \in W$ with $p \neq q$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \emptyset$ and $\mathrm{ran}(p) \cap \mathrm{ran}(q) = \emptyset$.*

Proof. If $n = 1$, then the elements of W are all singletons, and the conclusion is easy. Suppose therefore that $n > 1$ and let $W' \in [W]^{\aleph_1}$ be such that $\{\mathrm{dom}(q) : q \in W'\}$ forms a Δ-system with root R. For each $r \in R$, $q(r)$ can take values in a countable set (by the definition of $\mathbb{Q}_{\vec{A}}$), so, by the finiteness of R, we may find $W'' \in [W']^{\aleph_1}$ so that, if $p, q \in W'$, then $p \cup q$ is a function. Now note that $\{q{\restriction}(\mathrm{dom}(q) \setminus R) : q \in W''\}$ is a family of conditions of size $n - |R|$, which moreover forms an antichain. By the minimality of n, $|R| = 0$, so the first half of the conclusion follows upon replacing W by W''.

To get pairwise disjoint ranges, simply apply the previous paragraph to the family $\{q^{-1} : q \in W\}$. □

Set, for every $q \in W$,

$$u_q := \{\xi < \omega_1 : r_\xi \in \mathrm{dom}(q)\}$$

and

$$v_q := \{\xi < \omega_1 : r_\xi \in \mathrm{ran}(q)\}.$$

We identify u_q with its increasing enumeration, so that, for example, $u_q(i)$ denotes the ith element (in the ordinal ordering) of $\mathrm{dom}(q)$ whenever $i < n$.

For each $q \in W$ and $i, j < n$ with $i \neq j$, fix rational numbers d^q_{ij} and e^q_{ij} so that the following hold:

- d^q_{ij} is between $r_{u_q(i)}$ and $r_{u_q(j)}$;
- e^q_{ij} is between $r_{v_q(i)}$ and $r_{v_q(j)}$;

Also, fix for every $q \in W$ a permutation σ_q in[1] S_n so that, for every $i < n$, $q(r_{u_q(i)}) = r_{v(\sigma_q(i))}$.

Shrinking W if necessary, we may assume that, for each $i \neq j$, the map $q \mapsto (d^q_{ij}, e^q_{ij})$ is constant, say with value (d_{ij}, e_{ij}). Similarly, we may assume that the map $q \mapsto \sigma_q$ is constant, say with value σ.

We now apply the goodness of A to the family of $2n$-tuples $\langle (u_q, v_q) : q \in W\rangle$ to find $p, q \in W$ with $p \neq q$ and

$$\forall i < n \left[r_{u_p(i)} \leq r_{u_q(i)} \wedge r_{v_p(i)} \leq r_{v_q(i)}\right].$$

We will show that p and q are compatible, which will contradict that W forms an antichain. To see this, fix $x \in \mathrm{dom}(p)$ and $y \in \mathrm{dom}(q)$ with $x < y$ we need to see that $p \cup q$ is order-preserving on $\{x, y\}$. Fix $i, j < n$ so that $x = r_{u_p(i)}$ and $y_{u_q(j)}$. We now distinguish two cases. Suppose first that $i \neq j$. Suppose that $r_{u_p(i)} < r_{u_p(j)}$, the other case being symmetric. We have that $x = r_{u_p(i)} < d_{ij} < r_{u_p(j)}$ by the choice of d_{ij}, and therefore $r_{u_q(i)} < d_{ij} < r_{u_q(j)} = y$ by the same reason. In particular, $x < y$. On the other hand, since p is order-preserving and $r_{u_p(i)} < r_{u_p(j)}$, we infer that $p(r_{u_p(i)}) < p(r_{u_p(j)})$, hence

$$p(x) = r_{v_p(\sigma(i))} < e_{\sigma(i)\sigma(j)} < r_{v_p(\sigma(j))}$$

and so

$$r_{v_q(\sigma(i))} < e_{\sigma(i)\sigma(j)} < r_{v_q(\sigma(j))} = q(y).$$

Putting everything together, $p(x) < q(y)$, as desired.

[1] S_n is the symmetric group with n-elements

We are now left with the case $i = j$. We have that $p(x) = r_{v_p(\sigma(i))}$ and $q(y) = r_{v_q(\sigma(i))}$. By the choice of p and q, $r_{u_p(i)} \leq r_{u_q(i)}$ and $r_{v_p(\sigma(i))} \leq r_{v_q(\sigma(i))}$, which immediately yields that $x < y$ and $p(x) < q(y)$.

Item (B) is clear from the way the poset is constructed.

The proof of (C) is a straightforward modification of that of (A), so we point out the only difference and leave the details to the interested reader. Given $\{(p_\alpha, a_\alpha) : \alpha < \omega_1\}$ with $|p_\alpha| = n$ and $|a_\alpha| = m$ for every $\alpha < \omega_1$, apply goodness to the sequence of $2n + m$ tuples $(u_{p_\alpha}, v_{p_\alpha}, a_\alpha)$ rather than just $(u_{p_\alpha}, v_{p_\alpha})$. The argument is then as before, *mutatis mutandis*. □

Lemma 3.5. *Let A be good and $\vec{A}$ be a filtration. The finite-support product $\prod^{fin}_{n<\omega} \mathbb{Q}_{\vec{A}}$ is ccc, appropriate and generically adds a family $\{f_n\}_{n<\omega}$ of partial, strictly increasing functions $A \to A$ which respect the slices of the filtration and $\bigcup_{n<\omega} f_n = \bigcup_{\alpha<\omega_1} (A_{\alpha+1} \setminus A_\alpha) \times (A_{\alpha+1} \setminus A_\alpha)$*

Proof. Since $\mathbb{Q}_{\vec{A}}$ is ccc when A is good it follows that $\mathbb{Q}_{\vec{A}}$ forces itself to remain ccc by the fact that it is appropriate. It follows therefore that the finite-support product is itself ccc and in fact appropriate since finite support iterations of ccc, appropriate forcing notions are themselves appropriate. By a density argument, the union of the generic functions cover $\bigcup_{\alpha<\omega_1} (A_{\alpha+1} \setminus A_\alpha) \times (A_{\alpha+1} \setminus A_\alpha)$, and we are done. □

Let us now finish the proof of Theorem 3.1. This is similar to the proof of [AS81, Theorem 2].

Proof of Theorem 3.1. Assume GCH and first add $\aleph_1$-many Cohen reals. Call this set A. Again by [AS81], this is a good set. By a finite support iteration of length κ we can, using some bookkeeping, force with every ccc, appropriate partial order of size $<\kappa$. By standard arguments we get that in the final model $2^{\aleph_0} = \kappa$ and Martin's axiom holds for appropriate posets. By Lemma 1.10 A is still good. Also, as previously remarked (and observed in [AS81]), in fact there are no ccc inappropriate posets so full MA holds. Moreover, we can now apply the forcing notion $\prod^{\mathsf{fin}}_{n<\omega} \mathbb{Q}_{\vec{A}}$ to any filtration of $\vec{A}$ to obtain a sequence of functions witnessing slicewise-coverability for that filtration. This completes the proof. □

§ 4. Discussion and Open Questions

We conclude the paper with some final observations and questions for further research. We first note that proof of Theorem 3.1 actually shows the following.

Theorem 4.1. *Let $A \subseteq \mathbb{R}$ be good and $\aleph_1$-dense. If $\mathsf{MA}_{\aleph_1}$ holds then A is slicewise coverable and hence cannot be made isomorphic to its reverse by any forcing notion of size $\aleph_1$ which preserves $\aleph_1$.*

In fact, it is easy to see that the above results have nothing to do with $\aleph_1$. Therefore we get the following, where the notions of "good" and "slicewise coverable" are generalized above $\aleph_1$ in the obvious way. Note that if A is a κ-dense set of mutually generic Cohen reals it will good.

Theorem 4.2. *Let κ be a cardinal and let $A \subseteq \mathbb{R}$ be good and κ-dense. If MA_κ holds then A is slicewise coverable.*

As mentioned in the introduction, a corollary of Theorem 4.1 is an alternative proof of a related (though different) fact shown in [MT17]. This theorem can also be understood in the context of *rigidity of* $\aleph_1$-sized structures under MA. Under MA, often structures of size $\aleph_1$ retain some strong "incompactness" property by forcing notions preserving $\aleph_1$ or at least ccc forcing. For example this is true for

Aronszajn trees, all of which are special (and hence indestructibly Aronszajn) under MA but not in general. If A is $\aleph_1$-dense, then for every countable and dense (in itself) subset $B \subseteq A$, we have that $B \cong B^*$ by Cantor's Theorem, but A need not be isomorphic (or even near) to its reverse.

We note that the implication "good $\rightarrow$ slicewise coverable" can fail in the absence of MA:

Lemma 4.3. *It is consistent for any cardinal κ that $\kappa < 2^{\aleph_0}$ and there is a good, κ-dense $A \subseteq \mathbb{R}$ which is not slicewise coverable.*

We remark that if $\kappa = \aleph_1$ then adding $\aleph_1$-many Cohen reals over a model of CH will witness the lemma as CH will hold again in $V[A]$ and hence by the main result of [Bau73], there will be a ccc forcing notion of size $\aleph_1$ which adds an isomorphism between A and A^*.

Proof. Fix a cardinal κ and let A be a κ-dense set of mutually generic Cohen reals. Work in $V[A]$, where we know that A is good. We note the following two facts. First, as is well known, see e.g. [Bla, p. 473], in $V[A]$ the set of Cohen reals is not meager. However observe that any slicewise coverable set $B \subseteq \mathbb{R}$ must in fact be meager. To see this, fix any filtration $\vec{B} = \{B_\alpha : \alpha \in \kappa\}$ and any family of countably many increasing functions $f_n : B \rightarrow B$ as in the definition of slicewise coverability. Note that every $x \in B$ there will be an $n < \omega$ so that $f_n(x) = x$. Now, note that for any given $n < \omega$ we have that the graph of f_n is nowhere dense in $\mathbb{R}^2$. The result now follows. $\square$

In fact there is another way as well to see this which gives more information.

Lemma 4.4. MA(σ-*linked*) *is consistent with a good linear order which is not slicewise coverable.*

Proof. As one can force MA(σ-linked) by a σ-linked forcing over a model of CH, it suffices to show that σ-linked forcing cannot add a countable family of increasing functions covering a filtration that was not already covered by such a family of functions. Therefore let $A \subseteq \mathbb{R}$ be a set which is not slicewise coverable and let $\vec{A}$ be such a filtration with no family of functions. Suppose towards a contradiction that $\mathbb{P}$ is σ-linked and let $\{\dot{f}_n : n < \omega\}$ be forced to be a family of increasing functions covering $\vec{A}$ as in the definition of slicewise coverability. Let $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$ be the partition into countably many linked pieces. If $n, m \in \omega$ then let $\hat{f}_{n,m} : A \rightarrow A$ be the partial, increasing function defined by $\hat{f}_{n,m}(x) = y$ if and only if some $p \in \mathbb{P}_m$ forces that $\dot{f}_n(\check{x}) = \check{y}$. It is easy to verify that by linked-ness each $\hat{f}_{n,m}$ is a monotone increasing function and collectively they will cover the square of the filtration. This is a contradiction however, which completes the proof. $\square$

Note that the above implies that the forcing notions of the form $\mathbb{Q}_{\vec{A}}$ are not in general σ-linked. It is not clear what the relationship is between good, essentially increasing and slicewise coverable in ZFC.

Question 4.5. What is the ZFC-provable behavior of the classes of good, essentially increasing and slicewise coverable $\aleph_1$-dense suborders of $\mathbb{R}$ in ZFC?

It would be particularly interesting to know whether examples like this are at the heart of the failure of BA under MA. One way to phrase this is the following.

Question 4.6. Does MA imply that every $\aleph_1$-dense linear order is either reversible or good?

As we mentioned in the introduction, the original motivation for this work was to determine whether PFA for $\aleph_1$-sized posets (or even MM for $\aleph_1$-sized posets) suffices to prove BA. While the results presented above hint at a negative answer, we cannot yet rule this out completely.

Question 4.7. Do either PFA or MM for $\aleph_1$-sized posets prove BA?

By what we have shown it would be enough to answer the following in the affirmative, though this is not clear.

Question 4.8. If $A \subseteq \mathbb{R}$ is a good, $\aleph_1$-dense linear order, is the property that A is good preserved by countable support iterations of proper forcing notions?

Observe finally that by the proof PFA implies BA in [Bau84], there is always a proper poset $\mathbb{P}$ of size continuum adding an isomorphism from between two given $\aleph_1$-dense A and B. By the results above there may not always be one of size $\aleph_1$. This suggests a cardinal characteristic which might give some interesting information.

Question 4.9. Suppose $A, B \subseteq \mathbb{R}$ are $\aleph_1$-dense. What can be said about the least size of a $\mathbb{P}$ which is proper and adds an isomorphism between them? Can it be strictly between $\aleph_1$ and $2^{\aleph_0}$?

References

[AS81] Uri Abraham and Saharon Shelah, *Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic*, Israel J. Math. **38** (1981), no. 1-2, 161–176. MR 599485

[Bau73] James E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. **79** (1973), no. 2, 101–106. MR 317934

[Bau84] ———, *Applications of the proper forcing axiom*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 913–959. MR 776640

[Bla] Andreas Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), vol. 1, Springer, pp. 395–490.

[Jec03] Thomas Jech, *Set theory*, Springer monographs in mathematics, Springer-Verlag, Berlin, 2003. MR 1940513

[Kun11] Kenneth Kunen, *Set theory*, Studies in Logic (London), vol. 34, College Publications, London, 2011. MR 2905394

[MT17] Justin Tatch Moore and Stevo Todorcevic, *Baumgartner's isomorphism problem for $\aleph_2$-dense suborders of $\mathbb{R}$*, Arch. Math. Logic **56** (2017), no. 7-8, 1105–1114. MR 3696077

[Swi25] Corey Bacal Switzer, *Weak Baumgartner axioms and universal spaces*, Topology Appl. **373** (2025), Paper No. 109530, 18. MR 4939512

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