CONSISTENCY OF SQUARE BRACKET PARTITION RELATION

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ABSTRACT. Characteristic earlier results were of the form $\mathrm{CON}(2^{\aleph_0} \to [\lambda]^2_{n,2})$, with $2^{\aleph_0}$ an ex-large cardinal, in the best case the first weakly Mahlo cardinal.

Characteristic new results are $\mathrm{CON}\big((2^{\aleph_0} = \aleph_m) + \aleph_\ell \to [\aleph_k]^2_{n,2}\big)$, for suitable $k < \ell < m$. So we improve in three respects: the continuum may be small (e.g. not a weakly Mahlo), we use no large cardinal, and the cardinals λ involved are $< 2^{\aleph_0}$ after the forcing.

§ 0. Introduction

In their seminal list of problems [EH71], Erdös and Hajnal posed the question (15(a)): does $2^{\aleph_0} \not\to [\aleph_1]^2_3$? Recently, Komjáth [Kom25] provided a comprehensive update on this topic.

We continue here works which start with the problem above:[She88, §2], [She92], [She89], [She95] [She96], [She00] and the work with Rabus [RS00], but we try to be self-contained.

The simplest case of our result is (recall 0.3 below):

Theorem 0.1. *Assume* GCH *for transparency.* *Then* *for some ccc forcing notion of cardinality* $\aleph_6$ *in the universe* $\mathbf{V}^{\mathbb{P}}$, *we have* $2^{\aleph_0} = \aleph_6$ *and for any* $n \geq 3$, $\aleph_5 \to [\aleph_2]^2_{n,2}$.

Proof. Choose $(\mu, \theta, \partial, \lambda)$ as $(\aleph_6, \aleph_5, \aleph_2, \aleph_0)$ and apply Theorem 0.2 and Fact 1.12 with $\partial_0 = \aleph_1$. $\square_{0.1}$

For Hypothesis 1.1, the main case is:

Theorem 0.2. *Assume* $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$, $\partial = \partial^\lambda$ *and* $2^{\partial^{+\ell}} = \partial^{+\ell+1}$ *for* $\ell = 0, 1, 2$ *and* $\partial^{+4} \leq \theta$. *Then* *for some* λ^+*-cc,* $(<\lambda)$*-complete forcing notion* $\mathbb{P}$ *of cardinality* μ *(so the forcing does not collapse any cardinal and preserves cardinal arithmetic outside* $[\lambda, \mu)$*), in the universe* $\mathbf{V}^{\mathbb{P}}$ *we have,* $2^\lambda = \mu$ *and for every* $\sigma < \lambda$, $\theta \to [\partial]^2_{\sigma,2}$

Proof. All this paper is dedicated to proving this theorem. Pedantically, choose $\partial = \kappa^+$, notice that Hypothesis 1.1 holds (by Fact 1.12) so we can apply Conclusion 1.11. $\square_{1.11}$

We may weaken $\mu = \mu^\theta$ to $\mu = \mu^\partial$ and replace $\partial = \kappa$ by ∂ being a suitable limit cardinal.

Recall,

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Definition 0.3. For possibly finite cardinals θ, ∂, σ and κ, let $\theta \to [\partial]^2_{\sigma,\kappa}$ mean:

- if $\mathbf{c}$ is a function from $[\theta]^2 := \{u \subseteq \theta \colon |u| = 2\}$ into σ, then there exists some subset $\mathscr{U}$ of θ of cardinality ∂ such that $\{\mathbf{c}(u) \colon u \in [\mathscr{U}]^2\}$ has at most κ-many members.

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§ 0(A). **Preliminaries.**

Notation 0.4.
1) $\mathrm{cof}(\delta)$ is the class of ordinals of cofinality $\mathrm{cf}(\delta)$.
2) For a set x, let $\mathrm{trcl}(x)$ be the transitive closure of x, that is, the minimal set Y such that $x \in Y$ and $(\forall y)(y \in Y \Rightarrow y \subseteq Y)$.
3) Let $\mathscr{H}(\lambda) := \{x \colon |\mathrm{trcl}(x)| < \lambda\}$.
4) Let $\mathrm{trcl}_{\mathrm{Ord}}(x)$ be defined similarly to $\mathrm{trcl}(x)$ considering ordinals as atoms (= elements), equivalently, the minimal set Y such that $x \in Y$ and

$$(\forall y)[y \in Y \wedge (\text{if } y \text{ is not an ordinal, then } y \subseteq Y)].$$

5) Let $\mathscr{H}_{<\kappa}(x) = \{x \colon \mathrm{trcl}_{\mathrm{Ord}}(x) \subseteq \mathscr{H}(x)$ but has cardinality $< \kappa\}$.

Notation 0.5.
(1) $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ are forcing notions.
(2) p, q, r called *conditions* are members of a forcing notion.
(3) $\mathbf{q}$ is as in Definition 1.3, some kind of $(< \lambda)$-support iterated forcing with extra information.

Notation 0.6. We may write e.g. $N[\mathbf{q}, \beta, u]$ instead $N_{\mathbf{q},\beta,u}$ to help with sub-scripts (or super-script).

Definition 0.7. Let θ, ∂, κ and λ be infinite cardinals. We say that $\theta \to_{\mathrm{sq}} (\partial)^{\lambda,2}_{\kappa}$ when $\theta > \partial \geq \kappa \geq \lambda$ and:

$\boxplus$ If (a) then (b), where:
- (a) $\mathscr{B}$ is an expansion of $(\mathscr{H}(\chi), \in, <_*)$, where $<_*$ is a well-ordering of $\mathscr{H}(\chi)$, $\chi > \theta$, and its vocabulary $\tau_{\mathscr{B}}$ has cardinality $\leq \lambda$.
- (b) There is a tuple $\mathbf{s} = (\mathscr{U}, \bar{N}, \bar{\pi})$ solving $\mathbf{p} = (\mu, \theta, \partial, \kappa, \lambda, \mathscr{B})$, which means:

  $\boxplus_{\mathbf{p},\mathbf{s}}$ for $u, v \in [\mathscr{U}]^{\leq 2}$,
  - $\bullet_1$ $\bar{N} = \langle N_u \colon u \in [\mathscr{U}]^{\leq 2} \rangle$,
  - $\bullet_2$ $\mathscr{U} \subseteq \theta$ is such that $\mathrm{otp}(\mathscr{U}) = \partial$,
  - $\bullet_3$ $N_u \prec \mathscr{B}$, $[N_u]^{<\lambda} \subseteq N_u$,
  - $\bullet_4$ $\varepsilon[\mathbf{s}] := \min(\mathscr{U})$,
  - $\bullet_5$ $N_u \cap \mathscr{U} = u$,
  - $\bullet_6$ $\|N_u\| = \kappa$ and $\kappa + 1 \subseteq N_u$,
  - $\bullet_7$ $N_u \cap N_v \prec N_{u \cap v}$,
  - $\bullet_8$ $\bar{\pi} = \langle \pi_{u,v} \colon u, v \in [\mathscr{U}]^{\leq 2}$ and $|u| = |v| \rangle$ such that if $|u| = |v|$, then $\pi_{u,v}$ is an isomorphism from N_v onto N_u mapping v onto u,
  - $\bullet_9$ if $u_1 \subseteq u_2$ and $v_1 \subseteq v_2$ all from $[\mathscr{U}]^{\leq 2}$ and $|u_2| = |v_2|$, $\pi''_{u_2,v_2}(v_1) = u_1$ then π_{u_1,v_1}, π_{u_2,v_2} are compatible functions[1],
  - $\bullet_{10}$ for $\ell = 1, 2$, the sets $N_u \cap \partial$ for $u \in [\mathscr{U}]^{\ell}$ are pairwise equal [2] and included in $N_{\emptyset}$.

[1] So e.g. it follows that: if $\zeta_1, \zeta_2 \in \mathscr{U}$ then $\pi_{\{\zeta_1\},\{\zeta_2\}} \restriction (N_{\emptyset} \cap N_{\{\zeta_2\}})$ is the identity map.

[2] Note that ∂ has two distinct roles: the size of $\mathscr{U}$ and the restriction on $N_u \cap \partial$. We may separate.

Observation 0.8. *If $\bar{N} = \langle N_u : u \in [\mathscr{U}]^{\leq 2}\rangle$ satisfies 0.7(b)$\bullet_1 + \bullet_7$, then:*

(∗) *For every $x \in \cup\{N_u : u \in [\mathscr{U}]^{\leq 2}\}$ the set $\{u \in [\mathscr{U}]^{\leq 2} : x \in N_u\}$ has one of the following forms:*
- *(a) $\{u\}$ for some $u \in [\mathscr{U}]^2$,*
- *(b) $\{\zeta\}$ for some $\zeta \in \mathscr{U}$,*
- *(c) $\{\{\zeta\}\} \cup \{\{\varepsilon, \zeta\} : \varepsilon \in \mathscr{U} \cap \zeta\}$ for some $\zeta \in \mathscr{U}$,*
- *(d) $\{\{\zeta\}\} \cup \{\{\zeta, \xi\} : \xi \in \mathscr{U} \setminus (\zeta + 1)\}$ for some $\zeta \in \mathscr{U}$,*
- *(e) $\{\emptyset\}$,*
- *(f) $\{\emptyset\} \cup \{\{\zeta\} : \zeta \in \mathscr{U}\}$,*
- *(g) $\{\emptyset\} \cup \{\{\zeta\} : \zeta \in \mathscr{U}\} \cup \{\{\varepsilon, \zeta\} : \varepsilon < \zeta$ are from $\mathscr{U}\}$.*

§ 1. The forcing

Our aim here is to prove the consistency of the following configuration:

$$2 < \sigma < \lambda = \lambda^{<\lambda} < \partial = \partial^{<\lambda} < \theta < \mu = \mu^\theta = 2^\lambda,$$

and having $\theta \to [\partial]^2_{\sigma,2}$.

A continuation is in preparation [S+], aiming to further develop the directions explored here, particularly for the case of superscript $\mathbf{n} > 2$, as dealt within [She92]. We also show there that we can weaken the requirements on the cardinals and have more pairs.

Hypothesis 1.1. The parameter $\mathbf{p} = (\mu, \theta, \partial, \lambda, \lambda, \mathscr{B})$ consists of the following:

(a) $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$,

(b) $\theta \to_{\rm sq} (\partial)^{\lambda,2}_\lambda$ (see Definition 0.7, a variant of [She89, 2.1]); in our case using λ twice in intentional.

(c) σ will vary on the cardinal numbers from $(2, \lambda)$ and the "nice" μ-s are such that $\gamma < \mu \Rightarrow |\gamma|^\theta < \mu$.

(d)
- χ is e.g. $\beth_2(\mu)^+$,
- let $\mathscr{B}$ be an expansion of $(\mathscr{H}(\chi), \in, <^*_\chi)$ with vocabulary of cardinality λ such that for any finite set $u \subseteq \mathscr{H}(\chi)$, the Skolem hull of u $N_u :=$ Sk$(u, \mathfrak{C}_*)$ is of cardinality λ and $|N_u|^{<\lambda} \subseteq N$.

We intend to use $(< \lambda)$-support iterated forcing of quite a special kind but first, we define the iterand.

Definition 1.2.

(1) Let $\mathbf{A}$ be the set of objects $\mathbf{a}$ consisting of:

(a)
- $\gamma < \mu$ and $\sigma \in (2, \lambda)$,
- $\mathbb{P}$ is a forcing notion such that:

$$p \in \mathbb{P} \Rightarrow \mathrm{dom}(p) \in [\gamma]^{<\lambda} \wedge (\forall \alpha \in \mathrm{dom}(p))(p(\alpha) \in [\lambda \cup \gamma]^{<\lambda}),$$

- $\mathbb{P}$ is λ^+-cc and $(< \lambda)$-complete,
- the order $\leq_{\mathbb{P}}$ is: $p \leq_{\mathbb{P}} q$ iff:

$$\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \wedge (\forall \alpha \in \mathrm{dom}(p))[p(\alpha) \subseteq q(\alpha)],$$

(b)
- $\underset{\sim}{\mathbf{c}}$ is a $\mathbb{P}$-name of a function from $[\theta]^2$ to σ, (we may write $\underset{\sim}{\mathbf{c}}(\alpha, \beta)$ instead $\underset{\sim}{\mathbf{c}}(\{\alpha, \beta\})$ for $\alpha \neq \beta < \theta$).

(c) We have $(\mathscr{U}, \bar{N}, \bar{\pi})$ solving $\mathbf{p} = (\mu, \theta, \partial, \lambda, \lambda, \mathscr{B})$, (with $\mathscr{B}$ as in Definition 0.7⊞(b) and 1.1) such that $\mathbb{P}, \underset{\sim}{\mathbf{c}} \in N_u$ for every $u \in [\mathscr{U}]^{\leq 2}$.

(1A) In the context of Definition 1.2(1), $\mathbf{a} = (\gamma, \mathbb{P}, \underset{\sim}{\mathbf{c}}, \mathscr{U}, \bar{N}, \bar{\pi}) = (\gamma_{\mathbf{a}}, ...)$, so e.g. $N_{\mathbf{a},u} = N_u$.

(2) We say that the pair $(p, \bar{\iota})$ is a *solution* of $\mathbf{a} \in \mathbf{A}$, and write $(\mathbf{a}, p, \bar{\iota}) \in \mathbf{A}^+$, <u>when</u>:

(a) $\bar{\iota} = (\iota_1, \iota_2) \in \sigma \times \sigma$,

(b) $p \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$, recalling $\varepsilon(\mathbf{a}) = \min(\mathscr{U})$,

(c) if $p \le q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ and $\zeta_1 < \zeta_2$ are from $\mathscr{U}$ then there are q_1, q_2, r_1, r_2 such that for $\ell = 1, 2$, we have:

- $\bullet_0$ $q \le_{\mathbb{P}_{\mathbf{a}}} q_\ell$,
- $\bullet_1$ $q_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ and $q_1 \restriction (N_{\mathbf{a},\emptyset} \cap \gamma_{\mathbf{a}}) = q_2 \restriction (N_{\mathbf{a},\emptyset} \cap \gamma_{\mathbf{a}})$,
- $\bullet_2$ $r_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\zeta_1,\zeta_2\}}$,
- $\bullet_3$ $r_\ell \Vdash$ "$\underset{\sim}{\mathbf{c}}(\zeta_1, \zeta_2) = \iota_{\mathbf{a},\ell}$",
- $\bullet_4$ $r_\ell \restriction N_{\mathbf{a},\{\zeta_1\}}$ is $\le_{\mathbb{P}_{\mathbf{a}}}$-below $\pi^{\mathbf{a}}_{\{\zeta_1\},\{\varepsilon[\mathbf{a}]\}}(q_\ell)$,
- $\bullet_5$ $r_\ell \restriction N_{\mathbf{a},\{\zeta_2\}}$ is $\le_{\mathbb{P}_{\mathbf{a}}}$-below $\pi^{\mathbf{a}}_{\{\zeta_2\},\{\varepsilon[\mathbf{a}]\}}(q_{3-\ell})$.

(3) If $\mathbf{b} = (\mathbf{a}, p, \bar{\iota}) \in \mathbf{A}^+$ then let $\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}$ be the $\mathbb{P}$-name of the following forcing notion:

(∗) For $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$,

(a) the set of elements of $\mathbb{Q}_{\mathbf{b}} = \underset{\sim}{\mathbb{Q}}_{\mathbf{b}}[\mathbf{G}]$ is:

$$\Big\{u \in [\mathscr{U}]^{<\lambda} \colon \text{ if } \zeta_1 < \zeta_2 \text{ in } \mathscr{U}, \text{ then } \underset{\sim}{\mathbf{c}}\{\zeta_1, \zeta_2\}[\mathbf{G}] \in \{\iota_1, \iota_2\}, \text{ moreover}$$

$$\text{for some } q_1, q_2, r_1, r_2 \text{ as in Definition 1.2(1)(c)}(\bullet_1)\text{-}(\bullet_5), \text{ we have } r_1 \in \mathbf{G} \text{ or } r_2 \in \mathbf{G}\Big\},$$

(b) the order of $\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}[\mathbf{G}]$ is inclusion,

(c) the generic is $\underset{\sim}{\mathcal{V}}_{\mathbf{b}} = \bigcup \underset{\sim}{\mathbf{G}}_{\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}}$.

Definition 1.3.

(1) Let $\mathbf{Q} := \mathbf{Q}_{\mathbf{p}}$ be the class of $\mathbf{q}$ which consist of (below, $\alpha \le \lg(\mathbf{q})$ and $\beta < \lg(\mathbf{q})$ and e.g. $\mathbb{P}_\alpha = \mathbb{P}_{\mathbf{q},\alpha}$):

(a) $\lg(\mathbf{q})$ is an ordinal $\le \mu$,

(b) $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \colon \alpha \le \lg(\mathbf{q}), \beta < \lg(\mathbf{q})\rangle$ is a $(<\lambda)$-support iteration,

(c) $\mathbb{P}_\beta$ satisfies the λ^+-cc,

(d) $\underset{\sim}{\mathbb{Q}}_\beta$ is $\underset{\sim}{\mathbb{Q}}_{\mathbf{b}_\beta}$, where:

- $\bullet_1$ $\mathbf{b}_\beta := (\mathbf{a}_\beta, p^*_\beta, \bar{\iota}^*_\beta) \in \mathbf{A}^+$,
- $\bullet_2$ $\mathbf{a}_\beta := (\gamma_\beta, \mathbb{P}^\bullet_\beta, \underset{\sim}{\mathbf{c}}_\beta, \mathscr{U}_\beta, \bar{N}_\beta, \bar{\pi}_\beta) \in \mathbf{A}$,
- $\bullet_3$ $\mathbb{P}^\bullet_\beta$ is equal to $\mathbb{P}'_{\xi(\beta)}$ for some $\xi(\beta) = \xi_{\mathbf{q}}(\beta) \le \beta$ (on $\mathbb{P}'_\beta$, see below),
- $\bullet_4$ The sequence $\langle (\mathbb{P}_\gamma, \mathbb{P}'_\gamma, \mathbf{a}_\gamma, \mathbf{b}_\gamma, \xi(\gamma)) : \gamma < \beta\rangle$ belongs to $N_{\beta,u}$ for every $u \in [\mathscr{U}_\beta]^{\le 2}$.
- $\bullet_5$ Let $\mathscr{W}_\beta = \bigcup\{N_{\beta,u} \cap \beta \colon u \in [\mathscr{U}_\beta]^{\le 2}\}$,
- $\bullet_6$ we [3] have: for every $\gamma \in \mathscr{W}_\beta$ the set $\mathscr{W}_\beta \cap \mathscr{W}_\gamma$ has cardinality $\le \lambda$,
- $\bullet_7$ For every $\gamma \in \mathscr{W}_\beta$, there is $u = u_{\beta,\gamma} \in [\mathscr{U}_\beta]^{\le 2}$ such that $\mathscr{W}_\beta \cap \mathscr{W}_\gamma \subseteq N_{\beta,u}$ and without loss of generality u is minimal with this property.

(e) $\mathbb{P}'_\alpha$ is a dense subset of $\mathbb{P}_\alpha$, where,

- $\mathbb{P}'_\alpha$ is $\mathbb{P}_\alpha$ restricted to the set of conditions $p \in \mathbb{P}_\alpha$ such that: if $\beta \in \operatorname{dom}(p)$ then $p(\beta)$ is a member of $\mathbf{V}$ (not just a $\mathbb{P}_\alpha$-name) and if $\zeta_1 < \zeta_2$ are in $p(\beta) \subseteq \mathscr{U}_\beta$, then there are q_1, q_2, r_1, r_2 as in Definition 1.2(2)(c)($\bullet_1$)-($\bullet_5$) with $\mathbf{a}_\beta, \mathbf{b}_\beta$ here standing for $\mathbf{a}, \mathbf{b}$ there and
$$\bigvee_{\ell=1}^{2} (\forall \gamma \in \operatorname{dom}(r_\ell))[\gamma \in \operatorname{dom}(p) \wedge r_\ell(\gamma) \subseteq p(\gamma)].$$

(f) $\gamma_{\mathbf{q}} := \gamma(\mathbf{q}) := \sup\{\gamma_{\mathbf{q},\beta} \colon \beta < \lg(\mathbf{q})\}$, so $\mathbb{P}'_{\gamma(\mathbf{q})} \subseteq \mathscr{H}_{<\lambda}(\gamma_{\mathbf{q}})$; let $\mathbb{P}_{\mathbf{q}} := \mathbb{P}_{\lg(\mathbf{q})}$ and $\mathbb{P}'_{\mathbf{q}} := \mathbb{P}'_{\lg(\mathbf{q})}$.

(1A) We may write either $\mathbb{P}_{\mathbf{q},\alpha}$ or $\mathbb{P}_\alpha$ whenever $\mathbf{q}$ is clear and $(\iota_{\mathbf{q},\beta,1}, \iota_{\mathbf{q},\beta,2})$ is $\bar{\iota}_{\mathbf{b}_\beta}$.

[3] Why? By 0.7(b)$\bullet_{10}$.

(2) Let $\leq_{\mathbf{p}}$ be the following two-place relation on $\mathbf{Q_p}$:

$$\mathbf{q}_1 \leq_{\mathbf{p}} \mathbf{q}_2 \ \underline{\text{iff}}\ \mathbf{q}_1 = \mathbf{q}_2 \restriction \lg(\mathbf{q}_1), \text{ see below.}$$

(3) For $\mathbf{q}_2 \in \mathbf{Q_p}$ and $\alpha_* \leq \lg(\mathbf{q}_2)$, we define $\mathbf{q}_1 := \mathbf{q}_2 \restriction \alpha_*$ by:

(a) $\lg(\mathbf{q}_1) = \alpha_*$,

(b) $(\mathbb{P}_{\mathbf{q}_1,\alpha}, \mathbb{P}'_{\mathbf{q}_1,\alpha}) = (\mathbb{P}_{\mathbf{q}_2,\alpha}, \mathbb{P}'_{\mathbf{q}_2,\alpha})$ for $\alpha \leq \alpha_*$,

(c) $(\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_1,\beta}, \mathbf{b}_{\mathbf{q}_1,\beta}, \xi_{\mathbf{q}_1}(\beta)) = (\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_2,\beta}, \mathbf{b}_{\mathbf{q}_2,\beta}, \xi_{\mathbf{q}_2}(\beta))$ for $\beta < \alpha_*$.

(4) We say that two conditions $p, q \in \mathbb{P}'_\alpha$ are *isomorphic*, <u>when</u>:

(a) $\mathrm{otp}(\mathrm{dom}(p)) = \mathrm{otp}(\mathrm{dom}(q))$, and

(b) if $\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ then:

- $\bullet_1$ $\mathrm{otp}(p(\beta)) = \mathrm{otp}(q(\beta))$,
- $\bullet_2$ if $\varepsilon \in p(\beta) \cap q(\beta)$ then $\mathrm{otp}(\varepsilon \cap p(\beta)) = \mathrm{otp}(\varepsilon \cap q(\beta))$,
- $\bullet_3$ if $\varepsilon \in p(\beta), \zeta \in q(\beta)$ and $\mathrm{otp}(\varepsilon \cap p(\beta)) = \mathrm{otp}(\zeta \cap q(\beta))$ then:

$$\pi_{\beta,\{\zeta\},\{\varepsilon\}}(p \restriction N_{\beta,\{\varepsilon\}}) = q \restriction N_{\beta,\{\zeta\}}.$$

- $\bullet_4$ if $\varepsilon < \varepsilon_1$ belong to $p(\beta), \zeta < \zeta_1$ belong to $q(\beta)$, $\mathrm{otp}(\varepsilon \cap p(\beta)) = \mathrm{otp}(\zeta \cap q(\beta))$ and $\mathrm{otp}(\varepsilon_1 \cap p(\beta)) = \mathrm{otp}(\zeta_1 \cap q(\beta))$ then:

$$\pi_{\beta,\{\zeta,\zeta_1\},\{\varepsilon,\varepsilon_1\}}(p \restriction N_{\beta,\{\varepsilon,\varepsilon_1\}}) = q \restriction N_{\beta,\{\zeta,\zeta_1\}}.$$

Remark 1.4. If we prefer in clause (d) ($\bullet_3$) of Definition 1.3 (1) to have $\xi(\beta) = \beta$, i.e., $\mathbb{P}^\bullet_\beta = \mathbb{P}'_\beta$, we need to add, e.g. "$\mu$ is regular and e.g. use a preliminary forcing $(\{\mathbf{q} \in \mathbf{Q_p} \colon \lg(\mathbf{q}) < \mu\}, \lhd)$".

Claim 1.5.

(0) For $\mathbf{q} \in \mathbf{Q_p}$*, we have:* $\mathbb{P}'_{\mathbf{q}} \models$ *"*$p \leq q$*" iff* $\{p, q\} \subseteq \mathbb{P}'_{\mathbf{q}}$, $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$*, and* $\beta \in \mathrm{dom}(p) \Rightarrow p(\beta) \subseteq q(\beta)$.

(1) For $\mathbf{q} \in \mathbf{Q_p}$*, any increasing sequence of members of length* $< \lambda$ *of* $\mathbb{P}'_{\mathbf{q}}$ *has a lub, in fact, if* $\delta < \lambda$, $\bar{p} = \langle p_i \colon i < \delta \rangle \in {}^\delta(\mathbb{P}'_{\mathbf{q}})$ *is increasing,* <u>*then*</u> *the following* $p \in \mathbb{P}'_{\mathbf{q}}$ *is a lub of* $\bar{p}$*; defined by:* $\mathrm{dom}(p) = \bigcup\{\mathrm{dom}(p_i) \colon i < \delta\}$*, and if* $\beta \in \mathrm{dom}(p)$ *then*

$$p(\beta) = \bigcup \{p_i(\beta) \colon i < \delta \text{ and } \beta \in \mathrm{dom}(p_i)\}.$$

We denote this p *by* $\lim(\bar{p})$*.*

(2) For $\mathbf{q} \in \mathbf{Q_p}$*, we have:*

- $p \in \mathbb{P}'_{\mathbf{q}}$ <u>*iff*</u>*:*
  - *(a)* p *is a function with domain* $\in [\lg(\mathbf{q})]^{<\lambda}$*,*
  - *(b) if* $\beta \in \mathrm{dom}(p)$ *then* $p(\beta)$ *belongs to* $[\mathscr{U}_\beta]^{<\lambda}$*.*
  - *(c) If* $\beta \in \mathrm{dom}(p)$ *and* $(\iota_1, \iota_2) = (\iota_{\mathbf{q},\beta,1}, \iota_{\mathbf{q},\beta,2})$ *then for every* $\zeta_1 < \zeta_2$ *from* $p(\beta)$, $(p \restriction \beta) \restriction N_{\mathbf{q},\beta,\{\zeta_1,\zeta_2\}} \Vdash_{\mathbb{P}_{\mathbf{q},\beta}}$ *"*$\underset{\sim}{\mathbf{c}}\{\zeta_1, \zeta_2\} \in \{\iota_1, \iota_2\}$*". Moreover, there are* q_1*,* q_2*,* r_1*,* r_2 *as in Definition 1.2(2)(c)(*$\bullet_1$*)-(*$\bullet_5$*) and*

$$\bigvee_{\ell=1}^{2} (\forall \gamma \in \mathrm{dom}(r_\ell))[\gamma \in \mathrm{dom}(p) \cap \beta \wedge r_\ell(\gamma) \subseteq p(\gamma)].$$

(3) If $\mathbf{q} \in \mathbf{Q_p}$ *and* $\alpha \leq \lg(\mathbf{q})$ *then* $\mathbf{q} \restriction \alpha \in \mathbf{Q_p}$*.*

(4) $\leq_{\mathbf{p}}$ *is a partial order on* $\mathbf{Q_p}$*.*

(5) If $\bar{\mathbf{q}} = \langle \mathbf{q}_j : j < \delta \rangle$ *is* $\leq_{\mathbf{p}}$*-increasing* <u>*then*</u> *it has a* $\leq_{\mathbf{p}}$*-lub,* $\lim(\bar{\mathbf{q}})$*, of length* $\cup\{\lg(\mathbf{q}_j) : j < \delta\}$*.*

(6) If $\beta < \lg(\mathbf{q})$, $\mathbf{a} = \mathbf{a}_{\mathbf{q},\beta}$, $u \in [\mathscr{U}_{\mathbf{a},\beta}]^{\leq 2}$ *and* $N_u = N_{\mathbf{a},u}$*,* <u>*then*</u>*:*

(∗) *if* $p \in \mathbb{P}'_{\mathbf{q}}$ *then* $q = p \restriction N_{\mathbf{q},\beta,u}$ *satisfies* $q \in N_u$ *and* $q \leq_{\mathbb{P}_{\mathbf{q}}} p$ <u>*where*</u> q *is defined by:*

- $\bullet_1$ $\mathrm{dom}(q) = \mathrm{dom}(p) \cap N_u \cap \beta$
- $\bullet_2$ *If* $\gamma \in \mathrm{dom}(q)$ *then* $q(\gamma) = p(\gamma) \cap N_u$*.*

(7) If (A) then (B), where:

(A) *(a)* $i_* < \lambda$,
(b) $p_i \in \mathbb{P}'_{\mathbf{q}}$ *for* $i < i_*$,
(c) *if* $i < j < i_*$, *then* p_i *and* p_j *are essentially comparable, i.e.:*
- *if* $\beta \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j)$ *then* $p_i(\beta) \subseteq p_j(\beta)$ *or* $p_j(\beta) \subseteq p_i(\beta)$.

(d) $\bar{p} = \langle p_i : i < i_* \rangle$.

(B) $\bar{p}$ *has a lub* p *called* $\lim(\bar{p})$ *or* $\lim(\{p_i : i < i_*\})$ *defined by:*
- $\mathrm{dom}(p) = \bigcup\{\mathrm{dom}(p_i) : i < i_*\}$,
- *if* $\beta \in \mathrm{dom}(p)$, *then*

$$p(\beta) = \bigcup\{p_i(\beta) : i < i_* \text{ satisfying } \beta \in \mathrm{dom}(p_i)\}.$$

Proof. Part (2) is crucial but easy to verify. Parts (0), (1), (3), and (4) are also easy.

(5) For this, define $\mathbf{q} := \lim(\bar{\mathbf{q}})$ naturally, but we elaborate.

$(*)$ (a) $\lg(\mathbf{q}) = \bigcup\{\lg(\mathbf{q}_i) : i < \delta\}$,
(b) if $i < \delta$ and $\alpha \leq \lg(\mathbf{q}_i)$, then $(\mathbb{P}_{\mathbf{q},\alpha}, \mathbb{P}'_{\mathbf{q},\alpha}) = (\mathbb{P}_{\mathbf{q}_i,\alpha}, \mathbb{P}'_{\mathbf{q}_i,\alpha})$,
(c) if $i < \delta$ and $\beta < \lg(\mathbf{q}_i)$, then $(\mathbb{Q}_{\mathbf{q},\beta}, \mathbf{a}_{\mathbf{q},\beta}, \mathbf{b}_{\mathbf{q},\beta}) = (\mathbb{Q}_{\mathbf{q}_i,\beta}, \mathbf{a}_{\mathbf{q}_i,\beta}, \mathbf{b}_{\mathbf{q}_i,\beta})$,
(d) $(\mathbb{P}_{\mathbf{q},\lg(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})})$ is $(\bigcup\{\mathbb{P}_{\mathbf{q}_i} : i < \delta\}, \bigcup\{\mathbb{P}'_{\mathbf{q}_i} : i < \delta\})$ when $\mathrm{cf}(\delta) \geq \lambda$,
(e) if $\mathrm{cf}(\delta) < \lambda$, then $(\mathbb{P}_{\mathbf{q},\lg(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})})$ are defined as inverse limit. Then,
- $\mathbb{P}'_{\mathbf{q}} := \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})}$ is dense in $\mathbb{P}_{\mathbf{q}}$ because by Definition 1.2(3), for each $\beta < \lg(\mathbf{q}_j)$ with $j < \delta$, $\mathbb{Q}_{\mathbf{b}[\beta,\mathbf{q}_j]}$ is closed under increasing unions of length $< \lambda$.

Recalling that in Definition 1.3(1)(c), we use β and not α, "$\mathbb{P}_{\mathbf{q}}$ satisfies the λ^+-cc" is not required for proving 1.5 (5), only "if $\beta < \lg(\mathbf{q})$ then $\mathbb{P}_{\mathbf{q},\beta}$ satisfies the λ^+-cc", which is clear. Note that even though we formally do not need it here, the chain condition of $\mathbb{P}_{\mathbf{q}}$ will be proved in claim 1.6.

(6) Note that:

(a) If $\gamma \in \mathrm{dom}(q)$ then $\gamma \in N_u$ and $q(\gamma) \subseteq N_u$,
(b) As $\mathrm{dom}(q)$ and $q(\gamma)$ for $\gamma \in \mathrm{dom}(q)$ has cardinality $< \lambda$ and $[N_u]^{<\lambda} \subseteq N_u$ so recalling clause (a) obviously $q \in N_u$.
(c) To prove q is in $\mathbb{P}'_{\mathbf{q}}$ we need, for $\gamma \in \mathrm{dom}(q)$ and $\zeta_1 < \zeta_2$ from $q(\gamma) \subseteq \mathscr{U}_\gamma$ to verify the condition in 1.5(2)(c).
(d) But as $\gamma \in N_u$ hence $\mathbf{q} \restriction (\gamma+1)$ and ζ_1, ζ_2 belong to N_u, also $N_{\mathbf{q},\gamma,\{\zeta_1\}}$, $N_{\mathbf{q},\gamma,\{\zeta_2\}}$, $N_{\mathbf{q},\gamma,\{\zeta_1,\zeta_2\}}$ belong to N_u hence are included in it so we can finish easily.

(7) Follows by our definitions. $\square_{1.5}$

We now arrive to the

Crucial Claim 1.6. *If* $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$ *then* $\mathbb{P}_{\mathbf{q}}$ *satisfies* λ^+*-cc. Moreover* $\mathbb{P}_{\mathbf{q}}$ *is* λ^+*-Knaster.*

Proof. It suffices, by 1.3(1)(e), to prove that $\mathbb{P}'_{\mathbf{q}} = \mathbb{P}'_{\mathbf{q},\lg(\mathbf{q})}$ satisfies the λ^+-cc, so assume:

$(*)_1$ (a) Let $\bar{p} = \langle p_\xi : \xi < \lambda^+ \rangle$, where $p_\xi \in \mathbb{P}'_{\mathbf{q}}$,
(b) it suffices to prove that for some $\zeta < \xi < \lambda^+$, p_ζ and p_ξ are compatible.

[Why? By the definitions.]

$(*)_2$ For some stationary set $S \subseteq \mathrm{cof}(\lambda) \cap \lambda^+$, we have:
- $\bullet_1$ $\langle \mathrm{dom}(p_\xi) : \xi \in S \rangle$ is a Δ-system with heart $w_* \in [\lg(\mathbf{q})]^{<\lambda}$, and
- $\bullet_2$ if $\beta \in w_*$ then $\langle p_\xi(\beta) : \xi \in S \rangle$ is a Δ-system.

[Why? By the Delta system lemma, the proof using Fodor's lemma recalling $\lambda = \lambda^{<\lambda}$.]

$(*)_3$ Without loss of generality, $\langle p_\xi : \xi \in S\rangle$ are pairwise isomorphic (see Definition 1.3(4)).

[Why? Easy because for every $\mathbf{a}, u$ the model $N_{\mathbf{a},u}$ has cardinality λ.]

$(*)_4$ For $\gamma < \beta$ from w_*, we have:

- $\bullet_1$ Let $\mathscr{W}_\beta, u_{\beta,\gamma}$ be as in 1.3(1)(d)$\bullet_5$.
- $\bullet_2$ Without loss of generality, $u_{\beta,\gamma}$ is disjoint to $N_{\mathbf{q},\beta,\{\zeta\}} \setminus N_{\mathbf{q},\beta,\emptyset} \cap \mu$ for every $\zeta \in \mathscr{U}_\beta$ and is disjoint to $N_{\mathbf{q},\beta,\{\varepsilon,\zeta\}} \setminus N_{\mathbf{q},\beta,\emptyset} \cap \mu$ for every $\varepsilon < \zeta$ from $\mathscr{U}_\beta$.

[Why? As for any $\gamma < \beta$ from w_* we have to omit from $\mathscr{U}_\beta$ at most two elements and w_* has cardinality $< \lambda$.]

$(*)_5$ We fix $\xi(1) \neq \xi(2)$ from S and we shall prove that $p_{\xi(1)}$ and $p_{\xi(2)}$ have a common upper bound; this suffices for proving the Crucial Claim 1.6.

$(*)_6$ For $\beta \in w_*$:

(a) for $\ell \in \{1,2\}$, consider the sequence $\langle \alpha^\beta_{\xi(\ell),\varepsilon} : \varepsilon < \varepsilon_\beta\rangle$ listing the set $p_{\xi(\ell)}(\beta)$ in increasing order

(b) Why ε_β and not $\varepsilon_{\beta,\ell}$? as the two sequences have the same length because $p_{\xi(1)}, p_{\xi(2)}$ are isomorphic, see Definition 1.3(4) $\bullet_1$.

(c) Let $\mathscr{S}_\beta := \{\varepsilon < \varepsilon_\beta : \alpha^\beta_{\xi(1),\varepsilon} \neq \alpha^\beta_{\xi(2),\varepsilon}\}$,

(d) so by Definition 1.3 (4) $\bullet_2$ the sets $\{\alpha^\beta_{\xi(1),\varepsilon} : \varepsilon \in \mathscr{S}_\beta\}$, $\{\alpha^\beta_{\xi(2),\varepsilon} : \varepsilon \in \mathscr{S}_\beta\}$ are disjoint and disjoint to $\{\alpha^\beta_{\xi(1),\varepsilon} : \varepsilon \in \varepsilon_\beta \setminus \mathscr{S}_\beta\} = \{\alpha^\beta_{\xi(2),\varepsilon} : \varepsilon \in \varepsilon_\beta \setminus \mathscr{S}_\beta\}$.

Let $\bar\beta = \langle \beta_i : i \le i_*\rangle$ list the closure of $\{\alpha, \alpha+1 : \alpha \in w_*\} \cup \{0, \lg(\mathbf{q})\}$ in increasing order, so necessarily $i_* < \lambda$ and clearly it suffices:

$(*)_7$ To choose $q_i \in \mathbb{P}'_{\mathbf{q},\beta_i}$ a common upper bound of $\{p_{\xi(1)} \restriction \beta_i, p_{\xi(2)} \restriction \beta_i\}$ increasing with $i \le i_*$ by induction on $i \le i_*$ such that:

$(*)$ If $\beta \in w_* \setminus \{\beta_j : j < i\}$ and $\zeta(1), \zeta(2)$ are from $\mathscr{S}_\beta$ then:

- $\bullet_1$ $\mathrm{dom}(q_j) \cap N_{\beta,\{\alpha_{\xi(1),\zeta(1)},\alpha_{\xi(2),\zeta(2)}\}}$ is a subset of
$$N_{\beta,\{\alpha_{\xi(1),\zeta(1)}\}} \cup N_{\beta,\{\alpha_{\xi(2),\zeta(2)}\}} \cup N_{\beta,\emptyset},$$
- $\bullet_2$ if $\ell = 1,2$ and $\gamma \in \mathrm{dom}(q_j) \cap N_{\beta,\{\alpha_{\xi(\ell),\zeta(\ell)}\}}$ then $q_i(\gamma) = p_{\xi(\ell)}(\gamma)$ or $\gamma \in N_{\beta,\emptyset}$

Let us carry the induction.

<u>Case 1</u>: $i = 0$. Clearly, this case is trivial, letting $q_0 = \emptyset$.

<u>Case 2</u>: i is a limit ordinal.

In this case, let $q_i := \lim\langle q_j : j < i\rangle$, so by Claim 1.5(1), q_i is well-defined and is as required by the definition of the order and satisfies $(*)_7$.

<u>Case 3</u>: $i = j+1$ and $\beta_j \notin w_*$.

In this case, $\mathrm{dom}(p_{\xi(1)}) \cap \mathrm{dom}(p_{\xi(2)}) \cap \beta_i \subseteq \beta_j$, hence the condition
$$q_i := q_j \cup \big(p_{\xi(1)} \restriction [\beta_j, \beta_i]\big) \cup \big(p_{\xi(2)} \restriction [\beta_j, \beta_i]\big)\big)$$
is as promised.

<u>Case 4</u>: $i = j+1$ and $\beta_j \in w_*$.

By the choice of $\bar\beta$, clearly $\beta_i = \beta_j + 1$ and let $\mathscr{S} = \mathscr{S}_{\beta_j}$.

Recalling 1.3(1)(d) and 0.7(b)($\bullet_8$), we have:

$(*)_8$ $\mathbf{a}_{\beta_j} = \mathbf{a}_{\mathbf{q},\beta_j}$ determine:

(a) $\bar\pi_{\beta_j} = \langle \pi_{u,v} : u, v \in [\mathscr{U}_{\beta_j}]^{\le 2}$ and $|u| = |v|\rangle$,

(b) $\bar N_{\beta_j} = \langle N_u : u \in [\mathscr{U}_{\beta_j}]^{\le 2}\rangle$,

(c) for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$, let:

- $v[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1),\varepsilon(1)}, \alpha_{\xi(1),\varepsilon(2)}\}$, and
- $u[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1),\varepsilon(1)}, \alpha_{\xi(2),\varepsilon(2)}\}$.

(d) for $\varepsilon \in \mathscr{S}$, let $v[\varepsilon] = \{\alpha_{\xi(1),\varepsilon}\}$ and $u[\varepsilon] = \{\alpha_{\xi(2),\varepsilon}\}$,
(e) $\bar\iota = \bar\iota^*_{\beta_j}$, see 1.3 (1) (d) $\bullet_1$.
(f) $\gamma_j = \xi_{\mathbf{q}}(\beta_j)$; see 1.3(1)(d) $\bullet_3$.

We shall now define $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$ such that:

$(*)_9$ (a) $p_{\varepsilon(1),\varepsilon(2)} \in \mathbb{P}_{\gamma_j} \cap N_{u[\varepsilon(1),\varepsilon(2)]}$, hence $\mathrm{dom}(p_{\varepsilon(1),\varepsilon(2)}) \subseteq \gamma_j \cap N_{u[\varepsilon(1),\varepsilon(2)]}$,
(b) if $\varepsilon(1) = \varepsilon(2)$, then $p_{\varepsilon(1),\varepsilon(2)} \restriction (\gamma_j \cap N_{v[\varepsilon(1)]})$, $p_{\xi(1)} \restriction N_{v[\varepsilon(1)]}$ are essentially comparable; see 1.5(7)(A)(c), moreover the first is $\le_{\mathbb{P}_{\mathbf{q}}}$-above the second,
(c) if $\varepsilon(1) = \varepsilon(2)$, then $p_{\varepsilon(1),\varepsilon(2)} \restriction (\gamma_j \cap N_{u[\varepsilon(2)]})$, $p_{\xi(2)} \restriction N_{u[\varepsilon(2)]}$ are essentially comparable, moreover the first is $\le_{\mathbb{P}_{\mathbf{q}}}$-above the second,
(d) $p_{\varepsilon(1),\varepsilon(2)}$ satisfies 1.3(1)(e)$\bullet$ with $(\gamma_j, \varepsilon(1), \varepsilon(2))$ here standing for $(\beta, \zeta_1, \zeta_2)$ there,
(e) $\{q_j \restriction N_\emptyset\} \cup \{p_{\varepsilon(1),\varepsilon(2)} \restriction N_\emptyset \colon \varepsilon(1), \varepsilon(2) \in \mathscr{S}\}$ are pairwise essentially comparable,
(f) if $\varepsilon(1) \neq \varepsilon(2)$ then $p_{\varepsilon(1),\varepsilon(2)} \restriction N_{\{\alpha_{\varepsilon(\ell)}\}} \le p_{\xi(\ell)} \restriction N_{\{\alpha_{\varepsilon(\ell)}\}}$ for $\ell = 1, 2$.
(g) if $\mathscr{S}_* \subseteq \mathscr{S} \times \mathscr{S}$ then the lub $q_{\mathscr{S}_*}$ of $\{q_j[N_{u[\varepsilon(1),\varepsilon(2)]}] \colon \varepsilon(1), \varepsilon(2) \in \mathscr{S}_*\}$ satisfies the condition in $(*)_7$.

We have to show two things: $\boxplus_1$ and $\boxplus_2$. The first says we can choose them (the $p_{\varepsilon(1),\varepsilon(2)}$-s), the second that this is enough.

$\boxplus_1$ we can choose $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$ as required in $(*)_7$.

We consider two possible cases:

Case 4.1: $\varepsilon(1) \neq \varepsilon(2)$.

Let $p_{\varepsilon(1),\varepsilon(2)} = \pi(p_{\xi(1)} \restriction N_{v[\varepsilon(1),\varepsilon(2)]})$, where $\pi = \pi_{u[\varepsilon(1),\varepsilon(2)],v[\varepsilon(1),\varepsilon(2)]}$.

Why is $(*)_9$ preserved? Most clauses are obvious, but $(*)_9(g)$ deserve elaboration, recalling that we have to satisfy $(*)_7$.

So let $\beta \in \mathscr{W}_* \setminus \{\beta_\iota \colon \iota < i\}$, hence for some $j(*) < i_*$, we have $\beta = \beta_{j(*)}$, hence we have $\beta_{j(*)} \ge \beta_i$ hence $\beta_{j(*)} > \beta_j$ and we have $\mathscr{S}_* \subseteq \mathscr{S} \times \mathscr{S}$ and deal with $q_{\mathscr{S}_*}$.

For this, it is enough to consider the cases:

$\oplus_1$ $\mathscr{S}_* = \{\zeta(1), \zeta(2)\}$, where $\zeta(1) = \varepsilon(1)$ and $\zeta(2) = \varepsilon(2)$ hence from $\mathscr{S}$, so $\zeta(1) \neq \zeta(2)$,
$\oplus_2$ $\mathscr{S}_* = \{\zeta(1), \zeta(2)\}$ where $\zeta(1) \neq \zeta(2)$ are from $\mathscr{S}$ but $(\zeta(1), \zeta(2)) \neq (\varepsilon(1), \varepsilon(2))$.

Easy to check.

Case 4.2: $\varepsilon(1) = \varepsilon(2)$.

In this case, we pick some sequence $\langle p_{\varepsilon,\varepsilon} \colon \varepsilon \in \mathscr{S}\rangle$ by choosing $p_{\varepsilon,\varepsilon}$ by induction on $\varepsilon \in \mathscr{S}$. Now, $p_{\varepsilon,\varepsilon} \in \mathbb{P}'_{\beta_j} \cap N_{u[\varepsilon(1),\varepsilon(2)]}$ is such that:

$(*)$ (a) $p_{\varepsilon,\varepsilon}$ is $\le_{\mathbb{P}'_{\mathbf{q},\beta_j}}$-above $p_{\xi(1)} \restriction N_{v[\varepsilon]}$ and above the restriction $p_{\xi(2)} \restriction N_{u[\varepsilon]}$,
(b) $\langle p_{\zeta,\zeta} \restriction N_\emptyset \colon \zeta \in (\varepsilon + 1) \cap \mathscr{S}\rangle$ is $\le_{\mathbb{P}_{\beta[j]}}$-increasing, and
(c) there are q_1, q_2, r_1, r_2 as in Definition 1.3(2)(c) ($\bullet_1$)-($\bullet_5$) with $\mathbf{b}_{\mathbf{q},\beta_j}$ standing here for $(\mathbf{a}, p, \bar\iota)$ there such that:

$$\bigvee_{\ell=1}^{2} (\forall \gamma \in \mathrm{dom}(r_\ell)) \left[\gamma \in \mathrm{dom}(p_{\varepsilon,\varepsilon}) \wedge r_\ell(\gamma) \subseteq p_{\varepsilon,\varepsilon}(\gamma)\right].$$

We can choose $p_{\varepsilon,\varepsilon}$ by the properties of $\mathbf{b}_{\beta_j}$

Having defined all the $p_{\varepsilon(1),\varepsilon(2)}$-s we can proceed.

$\boxplus_2$ The following set of members of $\mathbb{P}_{\beta_i}$ has a common upper bound q_*:
- $p_{\xi(1)}, p_{\xi(2)}$, and
- $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathscr{S}$.

[Why? Recall Claim 1.5(2) and 1.2(1)(c)($\bullet_1$) by 1.5(7), clause (A) there holds, in particular sub-clause (A)(c). The main point is that:

(∗) $\langle N_{v[\varepsilon(1),\varepsilon(2)]} \cap \gamma_j \setminus (N_{v[\varepsilon(1)]} \cup N_{u[\varepsilon(1)]}) \colon \varepsilon(1), \varepsilon(2) \in \mathscr{S} \rangle$ is a sequence of pairwise disjoint sets.

Why? As "$N_u \cap N_v \subseteq N_{u \cap v}$ for $u, v \in [\mathscr{U}_{\beta_j}]^{<2}$ by 0.7$\bullet_7$.

So q_* from $\boxplus_2$ is a common upper bound of $p_{\xi(1)}, p_{\xi(2)}$, as promised. $\square_{1.6}$

Remark 1.7. 1) No need so far, but we may add in $(*)_4$ of the proof of Crucial Claim 1.6 the following item:

(d) if $\beta \in w_*$ and $\langle \alpha_{\zeta,\beta,i} \colon i < \iota_{\zeta,\beta} \rangle$ lists in increasing order the members of $p_\zeta(\beta)$ for $\zeta \in S$, then:
- $\langle \iota_{\zeta,\beta} \colon \zeta \in S \rangle$ is constant called i_β,
- for $i < i_\beta$, the sequence $\langle \alpha_{\zeta,\beta,i} \colon \zeta \in S \rangle$ is constant or increasing,
- if $i, j < i_\beta$ the sequence of truth values

$$\langle \text{Truth value}(\alpha_{\zeta,\beta,i} < \alpha_{\xi,\beta,j}) \colon \zeta < \xi \text{ are from } S \rangle$$

  is constant, and
- if $i, j < i_\beta$, $\zeta \neq \xi$ are from S and $\alpha_{\zeta,\beta,i} = \alpha_{\xi,\beta,j}$ then $i = j$.

2) We can make our choice of q_1, q_2, r_1, r_2 canonical, that is:

(A) In 1.2(2) we replace $(\mathbf{a}, p, \bar{\iota})$ by $(\mathbf{a}, p, \bar{\iota}, \mathbb{F})$, where:
- $\bullet_1$ $\mathbb{F}_{\zeta_1,\zeta_2}(q) = (q_1, q_2, r_1, r_2) = \langle \mathbb{F}_{\zeta_1,\zeta_2,\ell}(q) : \ell = 1, 2, 3, 4 \rangle$
- $\bullet_2$ if also $\zeta_3 < \zeta_4$ are from $\mathscr{U}$ then $\pi^{\mathbf{a}}_{\zeta_3,\zeta_4,\zeta_1,\zeta_2} \mathbb{F}_{\zeta_1,\zeta_2,\ell} = \mathbb{F}_{\zeta_3,\zeta_4,\ell}$, where if $p \leq q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ and $\zeta_1 < \zeta_2$ are from $\mathscr{U}$, then $\langle \mathbb{F}_{\zeta_1,\zeta_1,\ell}(p,q) \colon \ell < \mu \rangle$ is the quadruple (q_1, q_2, r_1, r_2) as in 1.2(1)(c)($\bullet_1$)-($\bullet_5$).

(B) In 1.2(3) similarly and in 1.3(1)(d)

(C) In 1.5(1)(d) use $\mathbb{F}_\beta$,

(D) In the proof of 1.6, in $(*)_7 \boxplus_1$, case $4.2(*)_{4.2}$ we use $\mathbb{F}_{\beta_j}$,

(E) Update the proof of 1.8 accordingly.

Claim 1.8. *If (A) then (B), where:*

(A) *(a)* $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$,
(b) $2 < \sigma < \lambda$,
(c) $\underset{\sim}{\mathbf{c}}$ *is a* $\mathbb{P}_{\mathbf{q}}$*-name of a function from* $[\theta]^2$ *into* σ.
(d) $p_* \in \mathbb{P}_{\mathbf{q}}$.

(B) *There is some* $\mathbf{b} \in \mathbf{A}^+$ *such that* $\mathbb{P}_{\mathbf{b}} = \mathbb{P}'_{\mathbf{q}}$ *and* $\underset{\sim}{\mathbf{c}}_{\mathbf{b}} = \underset{\sim}{\mathbf{c}}$ *and* $p_* \leq_{\mathbb{P}_{\mathbf{q}}} p_{\mathbf{b}}$.

Proof. Recalling Hypothesis 1.1(b), on the one hand, it is clear how to choose $\mathbf{a} \in \mathbf{A}$ such that $\mathbb{P}_{\mathbf{a}} = \mathbb{P}'_{\mathbf{q}}$ and $\underset{\sim}{\mathbf{c}}_{\mathbf{a}} = \underset{\sim}{\mathbf{c}}$. On the other hand, the choice of $p_{\mathbf{b}}$ and $\bar{\iota}_{\mathbf{b}}$ is similar to the proof of [She88, 2.1]. We now elaborate.

First, we can find $\mathbf{a}$ such that:

$(*)^1_{\mathbf{a}}$ (a) $\mathbf{a} \in \mathbf{A}$,
(b) $\mathbb{P}_{\mathbf{a}} = \mathbb{P}'_{\mathbf{q}}$,
(c) $\gamma = \lg(\mathbf{q})$,
(d) $\underset{\sim}{\mathbf{c}}_{\mathbf{a}} = \underset{\sim}{\mathbf{c}}$.

Why can we find? Because we have chosen $\mathbb{P}_{\mathbf{a}}$ as in $(*)^1_{\mathbf{a}}$(b), it is λ^+-cc by Claim 1.6; also $\gamma, \underset{\sim}{\mathbf{c}}_{\mathbf{a}}$ are as is required in Definition 1.2. Lastly we can choose $(\mathscr{U}_{\mathbf{a}}, \bar{N})$ as required because $\theta \to_{\text{sq}} (\partial)^{\lambda,2}_{\lambda}$ holds by Hypothesis 1.1 clause (b) and 0.7 in particular clause (b)$\bullet_{10}$.

We are left with choosing some appropriate $(p, \bar{\iota})$ and then let $\mathbf{b} = (\mathbf{a}, p, \bar{\iota})$. Let

$$Y := \{(q_1, q_2) \colon q_1, q_2 \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}} \text{ are above } p_* \text{ and, } q_1 \restriction (N_{\mathbf{a},\emptyset} \cap \lg(\mathbf{q})) = q_2 \restriction (N_{\mathbf{a},\emptyset} \cap \lg(\mathbf{q}))\},$$

and let $\leq_Y$ be the following two place relation on Y:

$(*)_2$ $(p_1, p_2) \leq_Y (q_1, q_2)$ <u>iff</u>:

(a) $(p_1, p_2) \in Y$ and $(q_1, q_2) \in Y$,
(b) $p_1 \leq_{\mathbb{P}'_{\mathbf{q}}} q_1$ and $p_2 \leq_{\mathbb{P}'_{\mathbf{q}}} q_2$.

Clearly,

$(*)_3$ $(Y, \leq_Y)$ is a $(< \lambda)$-complete partial order.

[Why? Recalling 1.5(1).]

$(*)_4$ For $(p_1, p_2) \in Y$, let
(a) $\mathrm{solv}(p_1, p_2)$ be the set of pairs (ι_0, ι_1) such that for any $\zeta_1 < \zeta_2$ from $\mathscr{U}_{\mathbf{a}}$, there are r_1, r_2 such that for $\ell = 1, 2$ clauses $\bullet_2$-$\bullet_5$ of Definition 1.2(2)(c) hold.
(b) $\mathrm{solv}^+(p_1, p_2) := \bigcap \{\mathrm{solv}(q_1, q_2) \colon (p_1, p_2) \leq_Y (q_1, q_2) \in Y\}$.

$(*)_5$ (a) if $(p_1, p_2) \leq_Y (q_1, q_2)$ then:
$$\mathrm{solv}(p_1, p_2) \supseteq \mathrm{solv}(q_1, q_2) \supseteq \mathrm{solv}^+(q_1, q_2) \supseteq \mathrm{solv}^+(p_1, p_2),$$
(b) if $(p_1, p_2) \in Y$ then $\mathrm{solv}(p_1, p_2) \neq \emptyset$.

[Why? The first inclusion in Clause (a) holds because $\leq_{\mathbb{P}_{\mathbf{q}}}$ is transitive. The other inclusions are clear, and Clause (b) is easy too.]

$(*)_6$ If $(p_1, p_2) \in Y$ then for some (q_1, q_2) and $\bar{\iota}$, we have:
(a) $(p_1, p_2) \leq_Y (q_1, q_2) \in Y$,
(b) if $(q_1, q_2) \leq_Y (q'_1, q'_2)$ <u>then</u> $\bar{\iota} \in \mathrm{solv}(q'_1, q'_2)$, moreover, $\mathrm{solv}(q_1, q_2) = \mathrm{solv}(q'_2, q'_2) = \mathrm{solv}^+(q'_1, q'_2) = \mathrm{solv}^+(q_1, q_2)$.

[Why? Recalling $\sigma < \lambda$, hence $|\sigma \times \sigma| < \lambda$ and $(Y, \leq_Y)$ is λ-complete by $(*)_3$.]

$(*)_7$ For $p \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$, let $\mathrm{solv}(p)$ be the set of $\bar{\iota} \in \sigma \times \sigma$ such that there is (q_1, q_2) such that:
$\bullet_1$ $p \leq_{\mathbb{P}_{\mathbf{q}}} q_1$, $p \leq_{\mathbb{P}_{\mathbf{q}}} q_2$ and
$\bullet_2$ $(q_1, q_2) \in Y$,
$\bullet_3$ $\bar{\iota} \in \mathrm{solv}^+(q_1, q_2)$,
$\bullet_4$ $\mathrm{solv}(q_1, q_2) = \mathrm{solv}^+(q_1, q_2)$.

$(*)_8$ (a) if $p \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then $\mathrm{solv}(p) \neq \emptyset$,
(b) if $p \leq_{\mathbb{P}'_{\mathbf{a}}} q$ are from $\mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then $\mathrm{solv}(p) \supseteq \mathrm{solv}(q)$,
(c) if $p \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then for some q and $\bar{\iota}$, for every q', we have $q \leq_{\mathbb{P}'_{\mathbf{q}}} q' \wedge q' \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}} \Rightarrow \bar{\iota} \in \mathrm{solv}(q')$.

[Why? Clause (a) follows by $(*)_6$, Clause (b) by the definitions, and Clause (c) holds as $\mathbb{P}'_{\mathbf{a}}$ and even $\mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ is λ-complete and $|\sigma \times \sigma| < \lambda$.]

Now, applying $(*)_8$(c) to p_* finish the proof of 1.8. $\square_{1.8}$

Claim 1.9. *If (A) then (B), where:*

(A) *(a)* $\mathbf{q} \in \mathbb{Q}_{\mathbf{p}}$ *and* $\mathbf{q}_0 <_{\mathbf{p}} \mathbf{q}$,
(b) $\gamma(\mathbf{q}) < \mu$, *so* $\lg(\mathbf{q}) < \mu$,
(c) $\mathbf{b} \in \mathbf{A}_{\mathbf{p}}$ *and* $\mathbb{P}_{\mathbf{b}} = \mathbb{P}_{\mathbf{q}_0}$.

(B) *There exists some* $\mathbf{q}_1$ *such that:*
(a) $\mathbf{q} \leq_{\mathbf{p}} \mathbf{q}_1$,
(b) $\lg(\mathbf{q}_1) = \lg(\mathbf{q}) + 1$,
(c) $\mathbf{b}_{\lg(\mathbf{q})}[\mathbf{q}_1] = \mathbf{b}$.

Proof. Easy. $\square_{1.9}$

Lastly, before arriving at the main conclusion, we have to prove the following.

Claim 1.10.
(1) Assume $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, $\alpha < \lg(\mathbf{q})$ *and* $\mathbf{b} = \mathbf{b}_{\mathbf{q},\alpha} = (\mathbf{a}_\alpha, p_\alpha, \bar{\iota}_\alpha) = (\mathbf{a}, p, \bar{\iota})$, *then:*

- $\Vdash_{\mathbb{P}_{\mathbf{q},\alpha+1}}$ "$\mathcal{V}_{\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}} \in [\mathscr{U}_{\mathbf{a}_\alpha}]^\partial$ *and for every* $\alpha \neq \beta \in \mathcal{V}_{\underset{\sim}{\mathbb{Q}}_{\mathbf{b}}}$, $\underset{\sim}{\mathbf{c}}_{\mathbf{a}_\alpha}\{\alpha, \beta\} \in \{\iota_1, \iota_2\}$".

(2) If $\mathbf{b} = (\mathbf{a}, p, \bar\iota) \in \mathbf{A}^+$, $\mathrm{cf}(\partial) > \lambda$, *and in* $\mathbf{V}^{\mathbb{P}_\mathbf{a}}$, $\underset{\sim}{\mathbb{Q}}_\mathbf{b}$ *satisfies the* λ^+*-cc,* *then* *for some* $p \in \underset{\sim}{\mathbb{Q}}_\mathbf{b} \cap \mathbb{P}_\mathbf{a} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ *we have*[4] $p \Vdash_{\underset{\sim}{\mathbb{Q}}_\mathbf{b}}$ *"*$\underset{\sim}{\mathcal{V}}_{\mathbb{Q}_\mathbf{b}} \in [\mathscr{U}_\mathbf{a}]^\partial$ *and for every* $\alpha \neq \beta \in \underset{\sim}{\mathcal{V}}_{\mathbb{Q}_\mathbf{b}}$, $\mathbf{c}_\mathbf{a}\{\alpha,\beta\} \in \{\iota_1, \iota_2\}$*".*

Proof. (1) The second phrase in both conclusion holds by the definitions of $\mathbb{Q}_\mathbf{b}$.

By the proof of "$\mathbb{P}_\mathbf{q}$ satisfies the λ^+-cc", we can show for $\varepsilon < \partial$, the density of the set

$$\mathscr{I}_\varepsilon := \{p \in \mathbb{P}'_\mathbf{q} \colon \alpha \in \mathrm{dom}(p) \text{ and there is } \beta \in p(\alpha) \text{ such that } \varepsilon < \mathrm{otp}(\mathscr{U}_{\mathbf{a}_\alpha} \cap \beta)\}.$$

(2) Easily, for every $\beta \in \mathscr{U}_\mathbf{a}$ we can choose $p^0_\beta = \{\beta\}$, $q_\beta = \{(p, p^0_\beta)\}$. Clearly, $q_\beta \in \mathbb{P}_\alpha * \mathbb{Q}_\mathbf{b}$ for $\beta \in \mathscr{U}_\mathbf{a}$. So by the λ^+-cc for some $\beta \in \mathscr{U}_\mathbf{a}$, $q_\beta \Vdash$ "$\{\varepsilon \in \mathscr{U}_\mathbf{a} \colon q_\varepsilon \in \underset{\sim}{\mathbb{Q}}_\mathbf{b}\} \in [\mathscr{U}_\mathbf{a}]^\partial$; well assuming $\mathrm{cf}(\partial) > \lambda$. $\square_{1.10}$

Conclusion 1.11. *There exists a forcing notion* $\mathbb{P}$ *satisfying the following conditions:*

- *(a)* $\mathbb{P}$ *is* λ^+*-cc of cardinality* μ.
- *(b)* $\mathbb{P}$ *is* $(<\lambda)$*-complete; hence, it collapses no cardinals, changes no cofinalities, and preserves cardinal arithmetic outside the interval* $[\lambda, \mu)$.
- *(c)* $\Vdash_\mathbb{P}$ *"*$2^\lambda = \mu$*".*
- *(d)* $\Vdash_\mathbb{P}$ *"*$\theta \to [\partial]^2_{\sigma,2}$*" for every* $\sigma \in (2, \lambda)$.

Proof. Choose a $\le_\mathbf{p}$-increasing continuous sequence $\langle \mathbf{q}_\alpha \colon \alpha < \mu\rangle \in {}^\mu(\mathbf{Q}_\mathbf{p})$ such that $\lg(\mathbf{q}_\alpha) = \alpha$, $\mathbb{P}_{\mathbf{q}_\alpha}$ has cardinality $\le (|\alpha| + \lambda)^{<\lambda}$ and,

- if $\alpha < \mu$ and $\Vdash_{\mathbb{P}_{\mathbf{q}_\alpha}}$ "$\underset{\sim}{\mathbf{c}} \colon [\theta]^2 \to \sigma$", then for unboundedly many $\beta \in [\alpha, \mu)$, $\underset{\sim}{\mathbf{c}}_{\mathbf{q}_{\beta+1,\beta}} = \underset{\sim}{\mathbf{c}}$.

The existence of $\mathbf{b}_\beta[\mathbf{q}_{\beta+1}]$ with $\underset{\sim}{\mathbf{c}}[\mathbf{b}_\beta[\mathbf{q}_{\beta+1}]] = \underset{\sim}{\mathbf{c}}$ as required holds by Claim 1.8 and Claim 1.9 below.

Clearly $\bigcup\{\mathbb{P}_{\mathbf{q}_\beta} \colon \beta < \mu\}$ is a forcing notion as is required. $\square_{1.11}$

Conclusion 1.11 is meaningful because:

Fact 1.12. Assume that $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$, and $[\alpha < \mu \Rightarrow |\alpha|^\lambda < \mu]$, $\theta > \beth_2(\kappa)$ and $\partial = \kappa^+$, $\kappa = \kappa^\lambda$. Then the demands in Hypothesis 1.1 hold.

Remark 1.13. To justify the assumption, notice that:

- (A) Omitting $\kappa = \kappa^\lambda$ does not help.
- (B) $\theta \to_{\mathrm{sq}} (\partial)^{2 \le \lambda}_\partial$ implies $\theta \to (\partial)^2_{2^\partial}$, hence necessarily $\theta > 2^{2^\partial}$.

With stronger lower bound on θ, see [She89].

The main point is proving $\theta \to_{\mathrm{sq}} (\partial)^{\le\lambda,2}_\partial$. For this, see [She89], $\theta = \beth_m(\partial)$ for some small m suffice, on this the bounds in 1.11 depends; we intend to return to this in [S+]. Anyhow just $\theta < \partial^{+\omega}$ and GCH in $[\partial, \partial^{+\omega}]$ would suffice for me.

Proof. The point is to prove $\theta \to_{\mathrm{sq}} (\partial)^{\lambda,2}_\partial$. Let $\mathscr{B}$ be as in 0.7(a), $\partial_1 = 2^\kappa$, $\partial_2 = 2^{\partial_1}$, and $\theta > \partial_2$.

Let $\chi > 2^\mu$, and $\mathfrak{C}_*$ be an expansion of $(\mathscr{H}(\chi), \in, <^*_\chi, \mathscr{B})$ with vocabulary of cardinality λ such that for any finite set $u \subseteq \mathscr{H}(\chi)$, the Skolem hull of u, $N_u := \mathrm{Sk}(u, \mathfrak{C}_*)$ is of cardinality λ and $|N_u|^{<\lambda} \subseteq N_u$.

Let $\mathfrak{C}_2 \prec_{\mathbb{L}_{\partial(1)^+,\partial(1)^+}} \mathfrak{C}_*$ be of cardinality ∂_2 such that $\partial_2 + 1 \subseteq \mathfrak{C}_2$. Let $\beta_1 := \min(\theta \setminus \mathfrak{C}_2)$. Similarly, choose $\mathfrak{C}_1 \prec_{\mathbb{L}_{\partial,\partial}} \mathfrak{C}_*$ of cardinality ∂_1 such that $\partial_1 + 1 \subseteq \mathfrak{C}_1$ and $\{\mathfrak{C}_2, \beta_0\} \subseteq \mathfrak{C}_1$.

Let $\mathfrak{C}_0 = \mathfrak{C}_1 \cap \mathfrak{C}_2$ and choose $\beta_0 \in \beta_1 \cap \mathfrak{C}_2 \subseteq \theta \cap \mathfrak{C}_2$ realizing the $\mathbb{L}_{\partial,\partial}$-type which β_1 realizes over $\mathfrak{C}_0$.

Now,

[4] We may omit p but it does not matter.

$(*)_1$ choose $\alpha_\varepsilon \in \mathfrak{C}_0 \cap \theta$ by induction on $\varepsilon < \partial$, such that:

- α_ε, β_1 realize the same first-order type in $\mathfrak{C}_*$ over the set $\{\beta_2\} \cup (A_\varepsilon \cap \mathfrak{C}_0)$, where:

$$A_\varepsilon = \mathrm{Sk}_{\mathfrak{C}}(\{\alpha_\zeta : \zeta < \varepsilon\} \cup \{\beta_1, \beta_0\}).$$

$(*)_2$ Let $N^\bullet_\emptyset = N_{\{\beta_0,\beta_1\}} \cap \mathfrak{C}_0$.

Note,

$(*)_3$ for $\varepsilon < \zeta < \partial$, the following pairs realize the same type over N^*_0 in $\mathfrak{C}_*$:

$\bullet_1$ $(\alpha_\varepsilon, \alpha_\zeta)$,
$\bullet_2$ $(\alpha_\varepsilon, \beta_0)$,
$\bullet_3$ $(\alpha_\varepsilon, \beta_1)$,
$\bullet_4$ (β_0, β_1).

[Why? For the equality of $\bullet_1$ and $\bullet_2$ note the choice of α_ε.
For the equality of $\bullet_2$ and $\bullet_3$, note the choice of β_0.
For the equality of $\bullet_3$ and $\bullet_4$ note the choice of α_ε.]

$(*)_4$ $\bullet_1$ $N^*_{\{\varepsilon,\zeta\}} = N_{\{\alpha_\varepsilon,\alpha_\zeta\}}$, so for $\varepsilon < \zeta < \partial$,
$\bullet_2$ $N^*_{\{\varepsilon,\zeta\}} \prec \mathfrak{C}_0$,
$\bullet_3$ $N^\bullet_\emptyset \prec N^*_{\{\varepsilon,\zeta\}}$.

$(*)_5$ for $\varepsilon < \zeta < \partial$, let $f_{\{\varepsilon,\zeta\}}$ be the isomorphism from $N^*_{\{\varepsilon,\zeta\}}$ onto $N_{\{\beta_0,\beta_1\}}$.

[Why does it exist? by $(*)_3$.]

$(*)_6$ $f_{\{\varepsilon,\zeta\}}$ is the identity on $N^\bullet_\emptyset$ (and $N^\bullet_\emptyset \prec N_{\{\varepsilon,\zeta\}}$).

[Why? By $(*)_2$.]

$(*)_7$ if $\varepsilon(0) < \zeta(0) < \partial$, $\varepsilon(1) < \zeta(1) < \partial$ and $\{\varepsilon(0), \zeta(0)\} \cap \{\varepsilon(1), \zeta(1)\} = \emptyset$, then

$$N^*_{\{\varepsilon(0),\zeta(0)\}} \cap N^*_{\{\varepsilon(1),\zeta(1)\}} = N_{\{\beta_0,\beta_1\}} \cap \mathfrak{C}_0 = N^\bullet_\emptyset.$$

[Why? The second equality holds by $(*)_2$; without loss of generality $\zeta(0) < \zeta(1)$. Now,

- $\bullet_1$ $N^*_{\{\varepsilon(0),\zeta(0)\}} \cap N^*_{\{\varepsilon(1),\zeta(1)\}} = N^*_{\{\varepsilon(0),\zeta(0)\}} \cap N_{\{\alpha_{\varepsilon(1)},\beta_1\}}$ by the choice of $\zeta(1)$.
- $\bullet_2$ if $\zeta(0) < \varepsilon(1)$ then
 $N^*_{\{\varepsilon(0),\zeta(0)\}} \cap N_{\{\alpha_{\varepsilon(1)},\beta_1\}} = N^*_{\{\varepsilon(0),\zeta(0)\}\}} \cap N_{\{\beta_0,\beta_1\}} = N_{\{\varepsilon(0),\zeta(0)\}} \cap N^\bullet_\emptyset = N^\bullet_\emptyset$ because the first equality follows by the choice of $\alpha_{\varepsilon(1)}$ second equality by $(*)_4\bullet_2$ and $(*)_2$; the third equality by $(*)_3$.
- $\bullet_3$ if $\varepsilon(0) < \varepsilon(1) < \zeta(0)$, then:

$$\begin{aligned} N^*_{\{\varepsilon(0),\zeta(0)\}} \cap N_{\{\alpha_{\varepsilon(1)},\beta_1\}} &= N_{\{\alpha_{\varepsilon(0)},\beta_0\}} \cap N_{\{\alpha_{\varepsilon(1)},\beta_1\}} \\ &= \left(N^*_{\{\alpha_{\varepsilon(0)},\beta_0\}} \cap \mathfrak{C}_0\right) \cap \left(N_{\{\alpha_{\varepsilon(0)},\beta_1\}} \cap \mathfrak{C}_0\right) \\ &= \left(N_{\{\alpha_{\zeta(0)},\beta_1\}} \cap \mathfrak{C}_0\right) \cap \left(N_{\{\alpha_{\zeta(0)},\beta_1\}} \cap \mathfrak{C}_0\right) \\ &= \left(N_{\{\alpha_{\zeta(0)},\beta_1\}} \cap \mathfrak{C}_0\right) \cap \left(N_{\{\beta_0,\beta_1\}} \cap \mathfrak{C}_0\right) \\ &= \left(N_{\{\alpha_{\varepsilon(0)},\beta_1\}} \cap \mathfrak{C}_0\right) \cap N^\bullet_\emptyset = N^\bullet_\emptyset. \end{aligned}$$

[Why? The first equality holds by the choice of β_0. The second equality as $N_{\{\varepsilon(0),\zeta(0)\}} \subseteq \mathfrak{C}_0$ and the first equality. The third equality holds by the choice of β_0. The fourth equality holds by the choice of $\alpha_{\zeta(0)}$. The fifth equality holds by the choice of $N^\bullet_\emptyset$ i.e., $(*)_2$. Finally, the sixth equality holds as $N_{\{\alpha_{\varepsilon(1)},\beta_1\}} \supseteq N^\bullet_\emptyset$ by the choice of $\alpha_{\varepsilon(2)}$.]

$\bullet_4$ If $\varepsilon(1) < \varepsilon(0)$, then:

$$\begin{aligned} N^*_{\{\varepsilon(0),\zeta(0)\}} \cap N_{\{\alpha_{\varepsilon(1)},\beta_1\}} &= N_{\{\alpha_{\varepsilon(0)},\beta_0\}} \cap N_{\{\alpha_{\varepsilon(1)},\beta_1\}} \\ &= \Big(N^*_{\{\alpha_{\varepsilon(0)},\beta_0\}} \cap \mathfrak{C}_0\Big) \cap \Big(N^*_{\{\alpha_{\varepsilon(1)},\beta_1\}} \cap \mathfrak{C}_0\Big) \\ &= \Big(N_{\{\alpha_{\varepsilon(0)},\beta_1\}} \cap \mathfrak{C}_0\Big) \cap \Big(N_{\{\alpha_{\varepsilon(1)},\beta_1\}} \cap \mathfrak{C}_0\Big) \\ &= \big(N_{\{\beta_0,\beta_1\}} \cap \mathfrak{C}\big) \cap \Big(N_{\{\alpha_{\varepsilon(1)},\beta_1\}} \cap \mathfrak{C}\Big) \\ &= N^*_\emptyset \cap \Big(N_{\{\alpha_{\varepsilon(1)},\beta_1\}} \cap \mathfrak{C}_0\Big) = N^\bullet_\emptyset. \end{aligned}$$

[Why? The first equality holds by the choice of β_0. The second one holds as $N_{\{\varepsilon(0),\zeta(0)\}} \subseteq \mathfrak{C}_0$ and the first equality. The third equality holds by the choice of β_0. The fourth equality holds by the choice of $\alpha_{\varepsilon(0)}$. The fifth equality holds by the choice of $N^\bullet_\emptyset$, i.e., by $(*)_2$. Finally, the sixth equality holds as $N_{\{\alpha_{\varepsilon(1)},\beta_1\}} \supseteq N^\bullet_\emptyset$ and by the choice of $\alpha_{\varepsilon(0)}$.

Recalling $\bullet_1$ and the division to cases in $\bullet_2, \bullet_3$ and $\bullet_4$, we are done proving $(*)_6$.]

$(*)_7$ if $\varepsilon < \zeta(1) < \zeta(2) < \partial$, then $N_{\{\varepsilon,\zeta(1)\}} \cap N_{\varepsilon,\zeta(2)} = N^\uparrow_{\{\varepsilon\}} := N_{\{\alpha_\varepsilon,\beta_1\}} \restriction \mathfrak{C}_0$.

[Why? By the choice of $\alpha_{\zeta(2)}$ and $\alpha_{\zeta(1)}$.]

$(*)_8$ if $\zeta_1 < \zeta_2 < \varepsilon < \partial$, then $N^*_{\{\zeta_1,\varepsilon\}} \cap N^*_{\{\zeta_2,\varepsilon\}} = N^\downarrow_{\{\varepsilon\}}$, where $N^\downarrow_{\{\varepsilon\}} := f^{-1}_{\varepsilon,\varepsilon+1}(N_{\{\beta_0,\beta_1\}})$.

[Why? For $\zeta < \partial$, $N_{\{\alpha_\zeta,\beta_0\}} \cap \mathfrak{C}_0 = N_{\{\alpha_{\zeta_3},\beta_1\}} \cap \mathfrak{C}_0$ by the choice of β_0, and $\alpha_\varepsilon, \beta_0$ realize the same type of $\mathfrak{C}_*$ over $\{\beta_1\} \cup (A_\varepsilon \cap \mathfrak{C}_0)$.]

$(*)_9$
- Let $N^*_{\{\varepsilon\}}$ be the $\mathrm{Sk}(N^\uparrow_{\{\varepsilon\}} \cup N^\downarrow_{\{\varepsilon\}}, \mathfrak{C}_*)$, and
- let $M^*_\varepsilon = \Big(\mathrm{Sk}(\bigcup_{\ell<5} N^*_{\{5\varepsilon+\ell\}} \cup \{N_{\{5\varepsilon+m,\delta\varepsilon+n\}} : m < n < 5\}^*)\}\Big)_{\ell<5}$,
- let M^+_ε be M^*_ε expanded by:
- $c^{M^+_\varepsilon}_\ell = \alpha_{5\varepsilon+\ell}$ for $\ell < 5$,
- $p^{M^+_\varepsilon}_\ell = |N^*_{\{5\varepsilon+\ell\}}|$ for $\ell < 5$,
- $P^{M^+_\varepsilon}_{m,n} = |N^*_{\{5\varepsilon+m,5\varepsilon+n\colon m<n<5\}}|$.

$(*)_{10}$ There is some $\mathscr{U}_1 \in [\partial]^\partial$ such that:
- $\langle M^*_\varepsilon : \varepsilon \in \mathscr{U}\rangle$ is a Δ-system with heart $N^*_\emptyset$,
- the M_ε are pairwise isomorphic.

[Why? Because $\partial = \partial_0$ and $\partial_0 = (\partial_0)^+$ by the Δ-system lemma.]

$(*)_{11}$ $\langle N^*_u : u \in \mathscr{U}_2\rangle$ is a required when $\mathscr{U}_2 = \{5\varepsilon+2 : \varepsilon \in \mathscr{U}_1\}$ and $N^*_{\{5\varepsilon+2\}} = M^*_\varepsilon$.

Pedantically, $\mathscr{U}_3 = \{\alpha_\zeta : \zeta \in \mathscr{U}_2\}$ and $N^*_{\{\alpha_\zeta\,:\,\zeta\in u\}} = N^*_u$ for $u \in [\mathscr{U}_3]^{\le 2}$. $\square_{1.12}$

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