Comparing Generalised Powers in ZF

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1 Introduction

The investigation of relations between infinite cardinals in ZF has a long tradition. The first such relation which was discovered is surely Cantor's Theorem, which states that for each infinite cardinal $\mathfrak{m}$ we have $\mathfrak{m} < 2^{\mathfrak{m}}$. Other results in ZF for infinite cardinals $\mathfrak{m}$ are that $2^{\mathfrak{m}} \not\leq \mathfrak{m}\times\mathfrak{m}$ (see Specker [5]) and that $\text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$ (see Halbeisen [1] or Halbeisen and Shelah [2]). Related to Specker's result, and Peng, Shen, Wu [4] proved that $2^{\mathfrak{m}} \not\leq^* \mathfrak{m}\times\mathfrak{m}$ (*i.e.*, there is no surjection from $\mathfrak{m}\times\mathfrak{m}$ onto $2^{\mathfrak{m}}$), and Tarski [6] proved that AC is equivalent to the statement that for every infinite cardinal $\mathfrak{m}$ we have $\mathfrak{m}^2 = \mathfrak{m}$. Another ZF result is that for any infinite cardinal $\mathfrak{m}$, $\text{seq}^{\text{1-1}}(\mathfrak{m}) \neq 2^{\mathfrak{m}}$, where $\text{seq}^{\text{1-1}}(\mathfrak{m})$ denotes the set of all finite injective sequences we can build with elements of some set of cardinality $\mathfrak{m}$.

Besides ZF results, we can also ask for consistency results. In this direction, among other results it was shown by Halbeisen, Plati, and Shelah [3], that the existence of infinite cardinals $\mathfrak{m}$ and $\mathfrak{n}$ such that $\text{fin}(\mathfrak{m}) < \mathfrak{m}^2$ and $\text{seq}^{\text{1-1}}(\mathfrak{n}) < \text{fin}(\mathfrak{n})$ is consistent with ZF.

One could remark that finite sets and finite sequences are just two of the many types of structures one can impose on a finite set. For instance, it would be easy to formalize the natural idea of a cycle and, given some cardinal $\mathfrak{n}$, to consider the cardinality $\text{cyc}(\mathfrak{n})$ of the set of finite cycles with elements in some set of cardinality $\mathfrak{n}$.

In this paper we present a generalization of the concept of finite subsets and finite injective sequences which entails all the possible finite *injective* structures one can think of, and

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then proceed to refine the techniques developed in [3] in order to simultaneously deal with any number of instances of that generalization. In general, we will obtain the best possible result if one restricts their attention to structures of some fixed finite size.

2 Preliminaries and main result

For a positive integer $n \in \omega$, let S_n be the group of all permutations of $n = \{0, \dots, n-1\}$. Let $H \leq S_n$ be a subgroup of S_n and let A be an infinite set. Furthermore, for sets X and Y, ${}^XY$ denotes the set of all functions from X to Y. For any $f \in {}^nA$, define

$$f/H := \big\{g \in {}^nA : g \text{ is one-to-one and } \exists\sigma \in H\, \forall m \in n\big(g(m) = f(\sigma m)\big)\big\}$$

and subsequently

$$[A]^H := \big\{f/H : f \in {}^nA \text{ and } f \text{ is one-to-one}\big\}$$

This is a strict generalization of the concept of finite subsets and finite injective sequences. Indeed, if $H = S_n$, then $[A]^H = [A]^n$ (the set of n-element subsets of A), whereas if $H = \{\iota\}$, where ι is the identity, then $[A]^H$ is the set of injective sequences of length n with elements in A. More broadly, as an instance of the fact that finite *injective* structures can in general be re-conducted to their group of automorphisms, we observe and briefly justify the following remark.

FACT 2.1. *For some positive integer $n \in \omega$, let $\mathcal{G} = (n, E)$ be a graph with vertex set $\{0, \dots, n-1\}$, and let $G = \mathrm{Aut}(\mathcal{G}) \leq S_n$ be the group of automorphisms of $\mathcal{G}$. Furthermore, let A be an infinite set, let $\mathcal{K}_A$ be the complete graph on A, and let $A_\mathcal{G}$ be the set of subgraphs $\mathcal{H}$ of $\mathcal{K}_A$ such that $\mathcal{H} \cong \mathcal{G}$. Then* ZF *proves that $|[A]^G| = |A_\mathcal{G}|$.*

Proof. We say that two embeddings ε and ε' of $\mathcal{G}$ into $\mathcal{K}_A$ are equivalent iff $\varepsilon[\mathcal{G}] = \varepsilon'[\mathcal{G}]$, i.e., ε and ε' map $\mathcal{G}$ to the same subgraph $\mathcal{H}$ of $\mathcal{G}$. Let $\mathscr{E}$ be the set of equivalence classes $\{[\varepsilon] : \varepsilon \text{ is an embedding of } \mathcal{G} \text{ into } \mathcal{K}_A\}$. Clearly $\mathscr{E}$ is in bijection with $A_\mathcal{G}$. On the other hand, each ε is simply an element of ${}^nA$, and one can easily see that $[\varepsilon] = [\varepsilon']$ exactly when some element $g \in G$ exists such that for each $i \in n$ we have $\varepsilon(i) = \varepsilon'(gi)$, implying that the natural bijection between $\mathscr{E}$ and $[A]^G$ is indeed well-defined. ⊣

For example, if $\mathcal{G}$ is the linear graph on n (for some $n \geq 2$), then $\mathrm{Aut}(\mathcal{G})$ is isomorphic to the cyclic group C_2 of order 2, and if $\mathcal{G}$ is an n-cycle, then $\mathrm{Aut}(\mathcal{G})$ is isomorphic to the dihedral group D_n of order $2n$.

We now state the main result of this work:

THEOREM 2.2. *Fix two natural numbers $l, n \in \omega$ and $l+1$ subgroups $G_0, \dots, G_l \leq S_n$ which are pairwise not conjugate to each other. It is consistent with* ZF *that there is an infinite set A such that $|[A]^{G_0}| < \dots < |[A]^{G_l}|$.*

For example, since for $n > 2$, $C_n \neq S_n$, Theorem 2.2 states that there is a model of ZF with a complete graph $\mathcal{K}$ on an infinite set, such that $\mathcal{K}$ contains strictly more complete graphs on n vertices than it contains linear graphs on n vertices.

The following Lemma implies that Theorem 2.2 is the best possible result if we restrict our attention to structures of some fixed finite size n.

LEMMA 2.3. *Let A be a set. Consider a natural number $n \in \omega$ and two subgroups $G, H \leq S_n$ which are conjugate to each other. Then,* ZF *proves that $|[A]^G| = |[A]^H|$.*

Proof. The claim is obvious whenever A is finite because in this case the cardinality of $[A]^G$ solely depends on the cardinality of G. Otherwise, we need to explicitely define a bijection. Let $\pi \in S_n$ so that $\pi H \pi^{-1} = G$. For any $f \in {}^nA$ and hence $f/G \in [A]^G$, define

$$\pi^*\colon f/G \mapsto f\pi/H$$

Let us show that this function is indeed well-defined, in the sense that the image does not depend on the choice of representative. Given a fixed $g \in G$, we have that $fg\pi/H = f\pi$ iff there exists some $h \in H$ such that $fg\pi = f\pi h$, or equivalently such that $fg = f\pi h\pi^{-1}$, which is true given the assumption on π. The fact that π^* is surjective is obvious, as given any $f \in {}^nA$ we have that $\pi^*\colon f\pi^{-1}/G \mapsto f/H$. For injectivity we rely once more on the definition of π: if $f\pi/H = t\pi/H$ then for some $h \in H$ we have $f\pi h = t\pi$ and hence $t = f\pi h \pi^{-1}$, but $\pi h \pi^{-1} \in G$, which gives $f/G = t/G$. ⊣

3 Proof of the Main Result

In this section, we first recall the construction, given in [3], of $\aleph_1$-homogeneous and $\aleph_1$-universal models. Then we show how to use these models in order to obtain the consistency result claimed in Theorem 2.2. Compared to [3], there's a difference in the construction of a *plain extension*, which will be essential for the consistency proof. Namely, if $l \in \omega$ is the number of subgroups, then here there's a dependency on the parameter $t \in l$. Instead, Proposition 3.2 does not change from its first appearance in [3], and is here included for completeness.

3.1 On the Construction of $\aleph_1$-homogeneous and $\aleph_1$-universal models

Let K be the class of all the pairs $(A, \{f_i\}_{i\in l}) \in K$ such that A is a (possibly empty) set and, for all $i \in l$, f_i is an injection $f_i\colon [A]^{G_i} \to [A]^{G_{i+1}}$. We will also refer to the elements of K as models. We define a partial ordering $\leq$ on K by stipulating

$$(A, \{f_i\}_{i\in l}) \leq (B, \{g_i\}_{i\in l}) \iff \begin{array}{l} A \subseteq B \\ \wedge\ \forall i \in l\ f_i \subseteq g_i \\ \wedge\ \forall i \in l\ \big(\operatorname{ran}(g_i \restriction ([B]^{G_i} \setminus [A]^{G_i})) \subseteq [B]^{G_{i+1}} \setminus [A]^{G_{i+1}}\big). \end{array}$$

When the functions involved are clear from the context, with a slight abuse of notation we will just write $A \leq B$ instead of $(A, \{f_i\}_{i\in l}) \leq (B, \{g_i\}_{i\in l})$ and $A \in K$ instead of

$(A, \{f_i\}_{i\in l}) \in K$. Moreover, given i and some element $a \in [A]^{G_i}$, we will write $\mathrm{ran}(a)$ to indicate the range of any of the elements in a.

We give two preliminary definitions. Given a model $(M, \{f_i\}_{i\in l})$ and a countable subset $A \subseteq M$, we define the *closure* $\mathrm{cl}(A, M)$ as the smallest superset of A that is closed under all f_i and pre-images with respect to the same functions. Constructively, we can characterize $\mathrm{cl}(A, M)$ as the following countable union: define $\mathrm{cl}_0 = \mathrm{cl}_0(A, M) := A$ and, for all $k \in \omega$,

$$\mathrm{cl}_{k+1} = \mathrm{cl}_k \cup \bigcup_{i\in l} \bigcup_{\substack{p\in[\mathrm{cl}_k]^{G_i}\\ p\neq\emptyset}} \mathrm{ran}\big(f_i(p)\big) \cup \bigcup_{i\in l} \bigcup_{q\in[\mathrm{cl}_k]^{G_{i+1}}\cap\mathrm{ran}(f_i)} \mathrm{ran}\big(f_i^{-1}(q)\big).$$

in order to finally define $\mathrm{cl}(A, M) := \bigcup_{i\in\omega} \mathrm{cl}_i$. Furthermore, for each $t \in l$, we set a standardized way to extend a *partial* model $(A, \{f'_i\}_{i\in l})$, where each f'_i is only a partial function, to an element of K: fix a $t \in l$ and consider $(A, \{f'_i\}_{i\in l})$, where A is a set and each f'_i is an injection with $\mathrm{dom}(f'_i) \subseteq [A]^G_i$ and $\mathrm{ran}(f'_i) \subseteq [A]^{G_{i+1}}$. Let $(M_0, \{f^0_i\}_{i\in l}) = (A, \{f'_i\}_{i\in l})$ and, for $j \in \omega$, define inductively $(M_{j+1}, \{f^{j+1}_i\}_{i\in l})$ as follows: M_{j+1} is the fully disjoint union

$$M_j \sqcup \bigsqcup_{i\in l} \bigsqcup_{\substack{P\in[M_j]^{G_i}\\ P\notin\mathrm{dom}(f^j_i)}} \{a^P_1, \dots, a^P_n\} \sqcup \bigsqcup_{t\in i\in l} \bigsqcup_{\substack{P\in[M_j]^{G_{i+1}}\\ P\notin\mathrm{ran}(f^j_i)}} \{a^P_1, \dots, a^P_n\}.$$

For what concerns each injection f^{j+1}_i, we naturally require the inclusion $f^j_i \subseteq f^{j+1}_i$, as well as the equality $\mathrm{dom}(f^{j+1}_i) = [M_j]^{G_i}$, where for $P \in [M_j]^{G_i} \setminus \mathrm{dom}(f^j_i)$, we define $f^{j+1}_i(P) := [\{\langle i, a^P_i\rangle : i \in n\}]$. Finally, for each i so that $t \in i \in l$ and for each $P \in [M_j]^{G_{i+1}} \setminus \mathrm{ran}(f^j_i)$, consider the class $x := [\{\langle i, a^P_i\rangle : i \in n\}] \in [M_{j+1}]^{G_i}$ and set $f^{j+1}_i(x) = P$. We are now in the position of defining the *plain extension* of $(A, \{f'_i\}_{i\in l})$ as

$$(M, \{f_i\}_{i\in l}) := \left(\bigcup_{j\in\omega} M_j,\ \Big\{\bigcup_{j\in\omega} f^j_i\Big\}_{i\in l}\right).$$

Given the previous definitions, we remark that given a model $M \in K$ and a countable subset $A \subseteq M$, we have that $\mathrm{cl}(A, M) \le M$, which proves the following:

Fact 3.1. *For every countable subset A of a model $M \in K$, there is a countable model N such that $A \subseteq N \le M$.*

Proposition 3.2 (CH). *There is a model M_* of cardinality $\mathfrak{c}$ in K such that:*

- *M_* is $\aleph_1$-universal, i.e., if $N \in K$ is countable then N is isomorphic to some $N_* \le M_*$.*
- *M_* is $\aleph_1$-homogeneous, i.e., if $N_1, N_2 \le M_*$ are countable and $\pi\colon N_1 \to N_2$ is an isomorphism then there exists an automorphism π_* of M_* such that $\pi \subseteq \pi_*$.*

- *If $N \leq M_*$ and $A \subseteq M_*$ are countable, then there is an automorphism π of M_* that fixes N pointwise, such that $\pi(A) \setminus N$ is disjoint from A.*

Proof. We construct the model M_* by induction on ω_1, where we assume that $\omega_1 = \mathfrak{c}$. Let $M_0 = \emptyset$. When M_α is already defined for some $\alpha \in \omega_1$, we can define

$$C_\alpha := \big\{N \leq M_\alpha : N \in K \text{ and } N \text{ is countable}\big\}.$$

The construction of $M_{\alpha+1}$, starting from M_α, consists of a disjoint union of two differently built sets of models. First, for each element $N \in C_\alpha$, let S_N be a system of representatives for the *strong* isomorphism classes of all the models $M \in K$ such that $N \leq M$ with M countable. Here, by *strong* we mean that, for two models M_1 and M_2 with $N \leq M_1, M_2$, it is not enough to be isomorphic in order to belong to the same class, but we require that there exists an isomorphism between M_1 and M_2 that fixes N pointwise, which we can express by saying that M_1 *is isomorphic to* M_2 *over* N. We first extend M_α by the set

$$M'_\alpha = \bigsqcup_{N \in C_\alpha} \bigsqcup_{M \in S_N} M \setminus N,$$

where "$\bigsqcup$" indicates that we have a *disjoint union*, and now we define $M_{\alpha+1}$ as the plain extension of $M_\alpha \sqcup M'_\alpha$. Finally, for non-empty limit ordinals δ define $M_\delta = \cup_{\alpha\in\delta} M_\alpha$, and let

$$M_* = \bigcup_{\alpha \in \omega_1} M_\alpha.$$

It remains to show that the model M_* has the required properties: First we notice that M_* has cardinality $|M_*| = \mathfrak{c}$, as required, and since, by construction, M_1 is $\aleph_1$-universal, M_* is also $\aleph_1$-universal. In order to show that M_* is $\aleph_1$-homogeneous, we make use of a back-and-forth argument. Let $N_1, N_2 \leq M_*$ be countable models and $\pi\colon N_1 \to N_2$ an isomorphism. Let $\{x_\alpha : \alpha \in \omega_1\}$ be an enumeration of the elements of M_* and let $I_0 := N_1$. If x_{δ_1} is the first element (with respect to this enumeration) in $M_* \setminus I_0$, then, by Fact 3.1, there exists a countable model $I'_1 \leq M_*$ such that $I_0 \leq I'_1$ and $x_{\delta_1} \in I'_1$. Similarly, there is a countable model J'_1 with $N_2 \leq J'_1 \leq M_*$ such that there exists an isomorphism $\pi'_1\colon I'_1 \to J'_1$ with $\pi \subseteq \pi'_1$. Now, let x_{γ_1} be the first element in $M_* \setminus J'_1$: for the same reason as above we can find countable models J_1, I_1 such that $I'_1 \leq I_1 \leq M_*$ and $J'_1 \leq J_1 \leq M_*$, together with $x_{\gamma_1} \in J_1$ and the fact that there exists an isomorphism $\pi_1\colon I_1 \to J_1$ with $\pi'_1 \subseteq \pi_1$. Proceed inductively with $x_{\delta_{\alpha+1}}$ being the first element in $M_* \setminus I_\alpha$ and find countable models $I_\alpha \leq I'_{\alpha+1} \leq M_*$, $J_\alpha \leq J'_{\alpha+1} \leq M_*$ and an isomorphism $\pi'_{\alpha+1}\colon I'_{\alpha+1} \to J'_{\alpha+1}$ with $x_{\delta_{\alpha+1}} \in I'_{\alpha+1}$ and $\pi_\alpha \subseteq \pi'_{\alpha+1}$. As in the second part of the base step, let $x_{\gamma_{\alpha+1}}$ be the first element in $M_* \setminus J'_{\alpha+1}$ and find countable models $I'_{\alpha+1} \leq I_{\alpha+1} \leq M_*$, $J'_{\alpha+1} \leq J_{\alpha+1} \leq M_*$, with an isomorphism $\pi_{\alpha+1}\colon I_{\alpha+1} \to J_{\alpha+1}$ such that $x_{\gamma_{\alpha+1}} \in J_{\alpha+1}$ and $\pi'_{\alpha+1} \subseteq \pi_{\alpha+1}$. We naturally take the union at limit stages and finally obtain $\pi_* = \cup_{\alpha\in\omega_1} \pi_\alpha$, which is the required automorphism of M_*.

To show the last property of the theorem, let $N \leq M_*$ and $A \subseteq M_*$ be both countable. Since the cofinality of ω_1 is greater than ω, we can find by construction both a countable model M satisfying the properties $A \subseteq M$, $N \leq M \leq M_*$ and a further countable model M' with $N \leq M' \leq M_*$ such that $M' \cap (A \setminus N) = \emptyset$, and such that there exists an isomorphism $i\colon M \to M'$ with i fixing N pointwise. Now, by M_* being $\aleph_1$-homogeneous we obtain an automorphism i_* extending i, as required. ⊣

3.2 Consistency Proof

We are now ready to prove the consistency result claimed in Theorem 2.2. Let $l, n \in \omega$ and let $G_0, \ldots, G_l \leq S_n$ be pairwise non-conjugate subgroups. In order to show that it is consistent with ZF that there is an infinite set A with $|[A]^{G_0}| < \cdots < |[A]^{G_l}|$, we have to find a model of ZF in which this statement holds, which is — by the Jech-Sochor Theorem — provided by finding a permutation model of ZFA in which this statement holds.

We consider the permutation model $\boldsymbol{\mathcal{V}}$ that arises naturally by considering the elements of the $\aleph_1$-universal and $\aleph_1$-homogeneous model M_* as the set of atoms and the automorphisms $\mathrm{Aut}(M_*)$ as the group of permutations. In particular, each permutation in the group preserves all the injections $f_i\colon [M]^{G_i} \to [M]^{G_{i+1}}$ that the model $(M_*, \{f_i\}_{i \in l})$ comes with. The normal ideal of supports is given by the countable subsets. If A is the set of atoms, clearly in $\boldsymbol{\mathcal{V}}$ we have, for all $i \in l$, an injection from $[A]^{G_i}$ into $[A]^{G_{i+1}}$. Assume now towards a contradiction that there is in $\boldsymbol{\mathcal{V}}$, for some fixed $t \in l$, an injection $h\colon [A]^{G_{t+1}} \hookrightarrow [A]^{G_t}$. Let S be a support of h and let M be the closure of S with respect to M_*. Consider some n-element set E which is disjoint from M and call N the plain extension of $M \cup E$ with respect to the index t. Let ε be an element of $[E]^{G_{t+1}}$ and consider $h(\varepsilon)$.

The first step of the proof is to realize that, because of the third property in Prop. 3.2, we can assume that $E \subseteq \mathrm{cl}(M \cup \mathrm{ran}(h(\varepsilon)))$. This inclusion requires $\mathrm{ran}(h(\varepsilon))$ to be, for some $k \in l+1$ and some $P \in [N]^{G_k}$, a single whole block $\{a_1^P, \ldots, a_n^P\}$ in the construction of the plain extension of $M \cup E$. Next, from the same inclusion we deduce that the following holds: consider the undirected graph whose verteces are given by all the blocks added in the construction of the plain extension of $M \cup E$ and by E itself, and whose edges connect, when the range of the considered P is a whole block, the block $\mathrm{ran}(P)$ with $\{a_1^P, \ldots, a_n^P\}$. Then, $\mathrm{ran}(h(\varepsilon))$ is in the same connected component as E. Let us now proceed with a few remarks and definitions regarding this graph. Notice that each vertex v distinct from E is in the form $\{a_1^P, \ldots, a_n^P\}$, and hence naturally comes with a unique index i, namely that $i \in l+1$ such that $P \in [N]^{G_i}$. We will call this index the *pure interpretation* of v. Similarly, every vertex v has an interpretation for each other edge involving v. More explicitely, each time some $k \in l+1$ and $Q \in [v]^{G_k}$ are considered in the construction of the plain extension, we get a new vertex $w = \{a_1^Q, \ldots, a_n^Q\}$ and an edge between v and w, to which we associate k as the relative interpretation of v. Finllay, this graph is acyclic, reason for which there is a unique path between E and $\mathrm{ran}(h(\varepsilon))$. Given this unique path $\langle s_0, \ldots, s_m \rangle$, each intermediate node v has two interpretations, one for each edge of the path involving v, which we will intuitively call left $L(v)$ and right $R(v)$. We now need the following:

Lemma 3.3 (Algebraic Claim). *Let G and H be subgroups of S_n which are not conjugate to each other and let E be a set with $|E| \geq n$. A pair $(g, h) \in [E]^G \times [E]^H$ is said to be concording if the range of g equals the range of h. Let $(g, h) \in [E]^G \times [E]^H$ be a concording pair and let f be an element of g. Enumerate the range $E = \{r_i : r_i = f(i)\}$ and let G^* be the group that acts on E in the same way as how G acts on n. Then, modulo swapping G with H, there exists some permutation $\sigma \in G^*$ such that $\sigma(g) = g$ and $\sigma(h) \neq h$.*

Proof. Choose a representative $\tilde{f} \in h$ and, for all $i \in n$, define k_i so that $\tilde{f}\colon i \mapsto r_{k_i}$. Let

τ be the permutation of n sending $i \mapsto k_i$. Pick a permutation $\sigma^* \in G^*$ and recall that $\sigma^* \tilde{f} \in h$ iff there exists some $\pi \in H$ such that, for all $i \in n$, we have $\tilde{f}(\pi i) = \sigma^* \tilde{f}(i)$. That is, such that $\sigma^* \tilde{f}(i) = \sigma^* r_{k_i} = r_{\sigma k_i} = r_{\sigma\tau i}$ equals $\tilde{f}(\pi i)$, i.e. $\tau\pi i = \sigma\tau i$. From this last equality it is clear that all the $\sigma^* \tilde{f}$ belong to h if and only if $G \subseteq \tau H \tau^{-1}$. Similarly, each analogous $\pi^*(f)$ belongs to g if and only if $\tau^{-1} H \tau \subseteq G$. $\dashv$

Our goal is to find a point on the path, hence a block, on which to apply the algebraic claim in order to destroy the pair $(\varepsilon, h(\varepsilon))$ while fixing pointwise the support S of h. Start from $s_0 = E$: if $R(E)$ is not $t+1$, then we can apply the algebraic claim in order to either move e and pointwise fix $\langle s_1, \ldots, s_m \rangle$ or fix e and move $\langle s_1, \ldots, s_m \rangle$ to some pointwise disjoint sequence of blocks, hence we assume that $R(E) = t+1$, which implies $L(s_1) = t+2$. Assume now that $L(s_1) \neq R(s_1)$. Then we can reach our goal by applying the algebraic claim at this point, but it is important to remark how we can apply the claim thanks to how we constructed the plain extension. Indeed, the algebraic claim concludes that at least one among two options holds. If we are in the case in which we can preserve the relevant element in $[s_1]^{G_{L(s_1)}}$ while moving the one in $[s_1]^{G_{R(s_1)}}$, then the fact that we can apply the claim would also be true even if we did not add any pre-image for injections with indeces i with $t \in i \in l$. On the other hand, the fact that the plain extension is symmetric enough to accept a permutation which preserves the relevant class in $[s_1]^{G_{R(s_1)}}$ and destroys the one in $[s_1]^{G_{L(s_1)}}$ is due to the fact that we added those pre-images, since we need to move E. To further clarify, we say that adding pre-images only for those indeces i with $t \in i \in l$ is sufficient thanks to the fact that we could reduce our proof to the case in which the right interpretation of s_0 is $t+1$. Proceeding this way, we obtain that the left interpretation of s_m is greater or equal to $t+1$, and in particular $L(s_m) \neq t$, which is what we will now use to conclude. Since $h(\varepsilon) \in [s_m]^{G_t}$, we can finally apply the algebraic claim in order to either move $h(\varepsilon)$ while preserving the relevant element of $[s_m]^{G_{L(s_m)}}$ and fixing pointwise each block in $\langle s_0, \ldots, s_{m-1} \rangle$, or while fixing $h(\varepsilon)$ and moving $\langle s_0, \ldots, s_{m-1} \rangle$ to some pointwise disjoint sequence of blocks, hence concluding the proof.

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