SUPER BLACK BOXES REVISITED
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Abstract. Let $\kappa, \theta < \lambda$ be cardinals, with λ and κ regular. Concentrating on a simple case, we say that the triple $(\lambda, \kappa, \theta)$ has a *Super Black Box* when the following holds.

For some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ and $\overline{C} = \langle C_\delta : \delta \in S \rangle$, where C_δ is a club of δ of order type κ, for every coloring $\overline{F} = \langle F_\delta : \delta \in S \rangle$ with $F_\delta : {}^{C_\delta}\lambda \to \theta$, there exists $\langle c_\delta : \delta \in S \rangle \in {}^S\theta$ such that for every $f : \lambda \to \theta$, for stationarily many $\delta \in S$, we have $F_\delta(f \restriction C_\delta) = c_\delta$.

In an earlier work, it was proved (along with much more) that for a class of cardinals λ this holds for many pairs (κ, θ). (E.g. $\kappa < \aleph_\omega$ is large enough and $\beth_\omega(\theta) < \lambda$.) However, the most interesting cases (at least with regards to Abelian groups) are $\kappa = \aleph_0, \aleph_1$, which have not been covered there.

Here we restrict ourselves to the case where $\overline{F}$ is a so-called *continuous coloring*, which includes the case where F_δ is computed from some

$$\langle F'_{\delta,\beta}(f \restriction (C_\delta \cap \beta)) : \beta \in C_\delta \rangle.$$

This covers the cases we have in mind. We mainly prove results without any other caveats: e.g.

- For every θ and regular κ there exists such a λ.

We also deal with having multiple $\overline{C}$-s, and the existence of quite free subsets of ${}^\kappa\mu$.

Annotated Content

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References like (e.g.) [Sh:950, Th.0.2$_{=\mathsf{Ly5}}$] mean that y5 is the internal label of Theorem 0.2 in the TeXfile of [Sh:950]. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number 1268 in Saharon Shelah's list.

E.g. if μ is strong limit singular and $\lambda \in (\mu, 2^\mu)$, then there is a μ^+-free set $\Lambda \subseteq {}^{\mathrm{cf}(\mu)}\mu$ of cardinality λ. Earlier, this result was known for *almost* all such μ-s.

§ 0. Introduction

We continue (but do not rely on) [She05] and [She13b],[1] while [She20] presents another direction we could pursue. Compared to [She05], we restrict ourselves to the so-called *continuous colorings*, but the conditions on κ are greatly weakened.

Recalling the BB Trichotomy Theorem from [She13b, $1.22_{=\mathsf{Lh}.7}$], Case (B) there will be expanded upon in §2 here, and §3 will examine cases (C) and (A).

For the Trivial Dual Conjecture on abelian groups, see [She20], [She07], and [She13b].

We believe:

Thesis 0.1. Proving theorems with assumptions on cardinal arithmetic is better than just giving consistency results (usually via forcing). Another candidate for such hypotheses in $\mathbf{V}$ is of an inner model close to it.

The most famous cardinal-arithmetic assumption is the GCH, as it resolves many questions and makes many theorems easy to prove. But we believe that assuming some failures of GCH at specific cardinals can be illuminating as well.

Results related to this have applications for constructing Abelian groups and modules. Hopefully we will be able to apply the present results in [AGS] and [S$^+$].

Definition 0.2. 1) Let Λ be a set of sequences of length κ. We say that Λ is *free* when there exists a function $f : \Lambda \to \kappa$ such that

$$\big\langle \{\eta(i) : f(\eta) \leq i < \kappa\} : \eta \in \Lambda \big\rangle$$

is a sequence of pairwise disjoint sets.

(This definition is easily adaptable to (e.g.) $\Lambda \subseteq [\mu]^\kappa$.)

2) We say Λ is *μ-free* if every subset of Λ of cardinality $< \mu$ is free.

We would like to construct (e.g.) suitable λ-free Abelian groups. For this we may use the following fact:

$\boxplus_1$ Suppose μ is strong limit singular and

$$\kappa := \mathrm{cf}(\mu) < \mu < \lambda = \mathrm{cf}(\lambda) < 2^\mu < 2^\lambda.$$

Then there exists a μ-free subset of ${}^\kappa\mu$ of cardinality λ.

[Why? If $\mathrm{cf}(\mu) > \aleph_0$ this is proved in [She94, Ch.II, §3]. Note that the proof there has been extended to many cardinals with $\mathrm{cf}(\mu) = \aleph_0$. In §4 we shall prove this for *all* strong limit cardinals of cofinality $\geq \aleph_0$.]

Also recall (e.g., from [She13b]):

$\boxplus_2$ Suppose μ and κ are as above, and there is $\chi < \chi^{<\kappa} = 2^\mu$. Then there exists a μ-free subset of ${}^\kappa\mu$ of cardinality 2^μ.

We have reasonable Black Boxes (see [She05], [She13b]) and more here. The proofs in [She05] cover many specific cases (e.g. the result mentioned in the abstract). Say, if $\lambda := \mathrm{cf}(2^\mu) > \beth_\omega$, then for every large enough regular $\kappa < \aleph_\omega$ we have a black box on some $\overline{C} = \langle C_\delta : \delta \in S^\lambda_\kappa \rangle$. We shall prove it here for *all* such κ. (See 0.4 below.)

[1] Well, except for quoting one result of [She13b] in §3.

Convention 0.3. 0) For $\varnothing \in D \subseteq \mathcal{P}(\lambda)$, let $D^+ := \{A \subseteq \lambda : \lambda \setminus A \notin D\}$.

Note that if J is an ideal on λ, then $J^+ = \mathcal{P}(\lambda) \setminus J$.

1) Let up(λ) be the set of non-empty upward closed $\mathcal{D} \subset \mathcal{P}(\lambda)$.

E.g. a filter on λ is an example of such a set.

2) $\mathsf{cof}(\kappa)$ is the class of ordinals $\{\delta : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$ and $\mathsf{cof}(<\kappa) := \bigcup_{\theta<\kappa} \mathsf{cof}(\theta)$. (Usually κ is a regular cardinal.)

3) For κ regular, $S^\lambda_\kappa := \lambda \cap \mathsf{cof}(\kappa) = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ and $S^\lambda_{<\kappa} := \lambda \cap \mathsf{cof}(<\kappa)$.

* * *

We shall aim to state our results concisely, rather than with maximum possible generality. A major one is the following.

Theorem 0.4. *'If (A) then (B),' where:*

(A) (a) $\mu \geq \theta = \theta^{2^\kappa}$, *$\kappa$ is regular, and* $\alpha < \mu \Rightarrow |\alpha|^\kappa \leq \mu$.
(b) $\lambda := \mathrm{cf}(2^\mu)$ *and* $S \subseteq S^\lambda_\kappa$ *is stationary.*
(c) *For each* $\delta \in S$, *we have* $C'_\delta \subseteq \delta = \sup(C'_\delta)$ *with* $\mathrm{otp}(C'_\delta) = \kappa$.
(d) *For all* $\beta < \lambda$ *we have* $C^\bullet_\beta \subseteq \beta$ *such that* $2^{|C^\bullet_\beta|} \leq 2^\mu$.

(B) *There exists* $\overline{C} = \langle C_\delta : \delta \in S\rangle$ *with* $C_\delta \subseteq C'_\delta$ *and* $\sup(C_\delta) = \delta$ *such that* <u>if</u> $F_\beta : {}^{C^\bullet_\beta}(2^\mu) \to \theta$ *for* $\beta < \lambda$ <u>then</u> *there exists* $\langle c_{\delta,\beta} : \delta \in S,\ \beta \in C'_\delta\rangle$ *with* $c_{\delta,\beta} < \theta$ *such that for every* $\eta \in {}^\lambda(2^\mu)$, *for stationarily many* $\delta \in S$, *we have*

$$\beta \in C_\delta \Rightarrow F_\beta(\eta \restriction C^\bullet_\beta) = c_{\delta,\beta}.$$

The proof can be found on page 24.

Theorem 0.5. 1) *In* 0.4(A)(a), *we can weaken the demand* $(\forall\alpha < \mu)\big[|\alpha|^\kappa \leq \mu\big]$ *to* "$\mathbf{U}_\kappa(\mu) = \mu > \theta$." *(See Definition* 1.4(1)*.)*

2) *We can replace clause* 0.4(A)(d) *via the use of* $\mathbf{p}$ *as in* 2.1(1).

An additional result is as follows. (See §3-4 for context — specifically, 3.1 and 3.2.)

Theorem 0.6. *Assume μ is strong limit singular,* $\kappa := \mathrm{cf}(\mu)$,

$$\kappa + \theta < \mu < \lambda = \mathrm{cf}(\lambda) < 2^\mu < 2^\lambda,$$

and S is a stationary subset of S^λ_κ.

<u>Then</u> *we can find* $\overline{C} = \langle C^\delta_\gamma : \delta \in S,\ \gamma < \lambda\rangle$ *such that:*

- $\bullet_1$ $C^\delta_\gamma \subseteq \delta = \sup(C^\delta_\gamma)$
- $\bullet_2$ $\mathrm{otp}(C^\delta_\gamma) = \kappa$
- $\bullet_3$ $\overline{C}$ *is a* μ^+-free sequence.

  By this we mean: if $u \subseteq S \times \lambda$ *is of cardinality* $< \mu^+$, <u>then</u> *there exists some sequence* $\langle \beta^\delta_\gamma : (\delta,\gamma) \in u\rangle$ *with* $\beta^\delta_\gamma \in C^\delta_\gamma$ *such that*

$$\langle C^\delta_\gamma \setminus \beta^\delta_\gamma : (\delta,\gamma) \in u\rangle$$

  is a sequence of pairwise disjoint sets.

$\bullet_4$ *If* $\mathbf{F}^\delta_\gamma : {}^{C^\delta_\gamma}(2^\mu) \to \theta$ *for* $(\delta,\gamma) \in S \times \lambda$, *then we can find a*

$$\bar{c}^\delta = \langle c^\delta_\gamma : \gamma < \lambda \rangle \in {}^\lambda\theta$$

such that for any $\delta \in S$ *and* $f : \delta \to 2^\mu$, *for some (even 'many')* $\gamma < \lambda$, *we have*

$$\mathbf{F}^\delta_\gamma(f \restriction C^\delta_\gamma) = c^\delta_\gamma.$$

We may rephrase Theorem 0.4 as follows:

Theorem 0.7. *Suppose the assumptions in* 0.4(A) *all hold. Then for some* $\bar{C} = \langle C_\delta : \delta \in S \rangle$ *with* $C_\delta \subseteq \delta = \sup(C_\delta)$ *and* $C_\delta \subseteq C'_\delta$, *we have*

$$\mathrm{BB}_*(\lambda, \bar{C}, \bar{C}^\bullet, 2^\mu, \theta, \kappa).$$

By this we mean (κ is regular, and) clauses (A)(b)-(d) and (B) of 0.4 *all hold. I.e.*

⊞ (a) *κ is regular.*
(b) $\lambda := \mathrm{cf}(2^\mu)$ *and* $S \subseteq S^\lambda_\kappa$ *is stationary.*
(c) *For each* $\delta \in S$, *we have* $C_\delta \subseteq \delta = \sup(C_\delta)$ *with* $\mathrm{otp}(C_\delta) = \kappa$.
(d) *For all* $\beta < \lambda$, *we have* $C^\bullet_\beta \subseteq \beta$ *such that* $2^{|C^\bullet_\beta|} \le 2^\mu$.
(Without loss of generality $\beta \in \lambda \setminus \bigcup_\delta C_\delta \Rightarrow C^\bullet_\beta = \varnothing$.*)*
(e) *If* $F_\beta : {}^{C^\bullet_\beta}(2^\mu) \to \theta$ *then there exists* $\langle c_{\delta,\beta} : \delta \in S,\ \beta \in C'_\delta \rangle$ *with* $c_{\delta,\beta} < \theta$ *such that for every* $\eta \in {}^\lambda(2^\mu)$, *for stationarily many* $\delta \in S$, *we have*

$$\beta \in C_\delta \Rightarrow F_\beta(\eta \restriction C^\bullet_\beta) = c_{\delta,\beta}.$$

We also prove an analogous result for Double Black Boxes.

Theorem 0.8. *We have '*$(A) \wedge (B) \Rightarrow (C)$,*' where*

(A) (a) $\theta + \kappa < \mu < \min\{\lambda, \lambda_*\} \le \lambda_* + \lambda \le 2^\mu$
(b) $\kappa \in \mathrm{Reg} \cap \lambda$
(c) $S \subseteq S^\lambda_\kappa$ *is stationary, and* $\delta \in S \Rightarrow \mu^2 \mid \delta$.
(d) $\bar{C} = \langle C_\delta : \delta \in S \rangle$ *with* $C_\delta \subseteq \delta = \sup(C_\delta)$, $\mathrm{otp}(C_\delta) = \kappa$, *and*

$$\alpha \in C_\delta \Rightarrow \mu \mid \alpha.$$

(e) D_δ *is a filter on* λ_*. *(The default is the club filter if* λ_* *is regular,* $\{A \subseteq \lambda_* : |\lambda_* \setminus A| < \lambda_*\}$ *if it is singular, and* $\{\lambda_*\}$ *if it is finite.)*

(B) (a) *μ is strong limit singular, and* $\mu_* \le \mu^+$.
(b) $\lambda_* := \min\{\partial : 2^\partial > 2^\mu\} \le \mathrm{cf}(\lambda) \le 2^\mu$
(c) $\lambda < 2^\mu \wedge \mu_* := \mu^+$, *or there is a* μ_**-free subset of* ${}^\kappa\mu$ *of cardinality* λ.

(C) *We can find* $\bar{C}^\bullet$ *such that*

$$\mathrm{DBB}_*(\lambda, \bar{C}, \bar{C}^\bullet, 2^\mu, \theta, \kappa)$$

holds. This means
(a) *Clauses (A)(a)-(d) are satisfied.*
(b) $\bar{C}^\bullet = \langle C^\delta_\gamma : \delta \in S,\ \gamma < \lambda_* \rangle$, *where* $C^\delta_\gamma \subseteq \bigcup_{\beta \in C_\delta} [\beta, \beta+\mu)$ *and*

$$\beta \in C_\delta \Rightarrow \left|C^\delta_\gamma \cap [\beta, \beta+\mu)\right| = 1.$$

(c) $\overline{C}^\bullet$ *is* μ_*-free *(as defined in* $0.6\bullet_3$*).*[2]

(d) *If* $F^\delta_\gamma : {}^{C^\delta_\gamma}(2^\mu) \to \theta$ *(for* $(\delta, \gamma) \in S \times \lambda_*$*)* <u>*then*</u> *for some* $\langle c^\delta_\gamma : \gamma < \lambda_*,\ \delta \in S\rangle$ *with* $c^\delta_\gamma < \theta$*, for every* $f : \lambda \to 2^\mu$ *and stationarily many* $\delta \in S$*, some*[3] $\gamma < \lambda_*$*, we have*
$$F^\delta_\gamma(f \restriction C^\delta_\gamma) = c^\delta_\gamma.$$

Claim 0.9. 1) *If* 0.8(A) *holds and* $\lambda_* := 1$*, then* DBB_* *is equivalent to* BB_**.*

2) *In* 0.8 *(as in* 0.6*) we may change clause* $(C)(d)$ *to use* $\mathbf{F}^\delta_\gamma : {}^{C^\bullet_\delta}(2^\mu) \to \theta$*, where* $C^\bullet_\delta := \bigcup_{\beta \in C_\delta} [\beta, \beta + \mu)$*.*

Definition 0.10. 1) For a regular uncountable cardinal λ, let
$$\check{I}[\lambda] := \big\{S \subseteq \lambda : \text{some pair } (E, \bar{a}) \text{ satisfies part (2) below}\big\}.$$

2) We say that $(E, \bar{a})$ is a witness for $S \in \check{I}[\lambda]$ <u>when</u> ($S \subseteq \lambda$ and):

(A) E is a club of the regular cardinal λ.

(B) $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$, $a_\alpha \subseteq \alpha$, and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$.

(C) For every $\delta \in E \cap S$, a_δ is an unbounded subset of δ of order-type $< \delta$.

3) For $\kappa < \lambda$, we define $\check{I}_\kappa[\lambda] := \big\{S \subseteq \lambda : S \cap S^\lambda_{\mathrm{cf}(\kappa)} \in \check{I}[\lambda]\big\}$.

By [She79], [She93] and [She]:

Claim 0.11. *Let* λ *be regular uncountable.*

1) *We have* $S \in \check{I}[\lambda]$ <u>*iff*</u> *we can find a witness* $(E, \bar{a})$ *for it which satisfies:*

(a) $\delta \in S \cap E \Rightarrow \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)$

(b) *If* $\alpha \notin S$ *then* $\mathrm{otp}(a_\alpha) < \mathrm{cf}(\delta)$ *for some* $\delta \in S \cap E$*.*

2) $S \in \check{I}[\lambda]$ <u>*iff*</u> *there is a pair* $(E, \overline{\mathscr{P}})$ *which is a* weak witness *for it. By this we mean:*

(a) *E is a club of the regular uncountable* λ*.*

(b) $\overline{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$*, where* $\mathscr{P}_\alpha \subseteq \mathcal{P}(\alpha)$ *has cardinality* $< \lambda$*.*

(c) *If* $\alpha < \beta < \lambda$ *and* $\alpha \in u \in \mathscr{P}_\beta$ <u>*then*</u> $u \cap \alpha \in \mathscr{P}_\alpha$*.*

(d) *If* $\delta \in E \cap S$ <u>*then*</u> *some* $u \in \mathscr{P}_\delta$ *is an unbounded subset of* δ *of order type* $\leq \mathrm{cf}(\delta)$*. (We may restrict ourselves to the case where* δ *is a limit ordinal.)*

3) *Suppose* $(S, E, \overline{\mathscr{P}})$ *are as in part* (2) *and* C *is another club of* λ*.* <u>*Then*</u> *the triple* $(S_*, E_*, \overline{\mathscr{P}}_*)$*, defined below, satisfies part* (2) *as well.*

- $S_* := S \cap C$

[2] I.e. for all $u \subseteq S \times \lambda_*$ of cardinality $< \mu_*$ there exist $\langle \beta^\delta_\gamma : (\delta, \gamma) \in u\rangle$ such that
$$\langle C^\delta_\gamma \setminus \beta^\delta_\gamma : (\delta, \gamma) \in u\rangle$$
is a sequence of pairwise disjoint sets.

[3] As in 0.6, it may be possible to strengthen this to 'many $\gamma < \lambda$' for some definition of *many*.

- $E_* := \{\delta \in S : \delta = \sup(E \cap C \cap \delta)\}$
- $\overline{\mathscr{P}}_* = \langle \mathscr{P}^*_\alpha : \alpha < \lambda \rangle$, *where* $\mathscr{P}^*_\alpha := \{\{\sup(E \cap C \cap \beta) : \beta \in C\} : C \in \mathscr{P}_\alpha\}$

4) *If* λ *is regular then* $S^{\lambda^+}_{<\lambda} \in \check{I}[\lambda]$.

§ 1. The framework

We will open by quoting some definitions from [She05] (although that paper is not a prerequisite). We investigate the notion of $\mathrm{Sep}(-)$ and define some relatives which we will need.

In §2 we will use only Sep_3 (although Sep_2 would actually be sufficient for proving 0.4, 0.5).

Convention 1.1. What we call BB here will be denoted as BB^0 in later sections.

Definition 1.2. Assume $\lambda > \kappa$ are regular cardinals, and let $\chi \leq \lambda$. Let

$$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$$

be a stationary subset of λ.

1) We say $\mathbf{p} = \big\langle (C_\delta, C'_\delta) : \delta \in S \big\rangle$ is a (λ, κ, χ)-BB-*parameter* <u>when</u>:

- (A) $C_\delta \subseteq \delta$, with $\sup(C_\delta) = \delta$ and $|C_\delta| < \chi$, <u>or</u> just $\mathrm{otp}(C_\delta) \leq \chi$.
- (B) $C'_\delta \subseteq \delta$, $\sup(C'_\delta) = \delta$, and $\mathrm{otp}(C'_\delta) = \kappa$. (We do not require that C_δ *or* C'_δ be closed in δ.)
- (C) For all $\alpha < \lambda$, the set
$$\big\{C_\delta \cap \alpha : \delta \in S,\ C'_\delta \ni \alpha\big\}$$
has cardinality $< \lambda$.

1A) We say $\mathbf{p}$ is *good* <u>when</u> in addition,

- $(C)^+$ For all $\alpha < \lambda$ the set[4] $\big\{(C_\delta \cap \alpha, C'_\delta \cap \alpha) : \delta \in S,\ C'_\delta \ni \alpha\big\}$ has cardinality $< \lambda$.

1B) If $\mathbf{p}$ just satisfies (1)(A)-(B), we call it a *weak* (λ, κ, χ)-BB-parameter.

2A) We say that $\mathbf{p}$ *does D-guess clubs*, where D is a filter on λ, <u>when</u> for every club $E \subseteq \lambda$,

$$\{\delta \in S : C'_\delta \subseteq E\} \in D^+.$$

2B) For $\mathbf{p}$ as above,

- (a) $\bar\beta_\delta = \langle \beta_{\delta,i} : i < \kappa\rangle$ will list the elements of C'_δ in increasing order.
- (b) $\beta_{\delta,<i} = \beta(\delta, <i) := \bigcup\limits_{j<i} (\beta_{\delta,j} + 1)$.

2C) We may write $\lambda_{\mathbf{p}}$, $\kappa_{\mathbf{p}}$, $\beta^{\mathbf{p}}_{\delta,i}$, etc. whenever there are multiple BB-parameters under discussion, or the identity of $\mathbf{p}$ is otherwise unclear from context.

2D) If $(\forall \delta \in S)[C_\delta = C'_\delta]$, then we may write $\mathbf{p} = \langle C_\delta : \delta \in S\rangle$. We may omit χ when $\chi := \min\{\theta : \delta \in S \Rightarrow \mathrm{otp}(C_\delta) \leq \theta\}$.

3) We say that $\overline{F} = \langle F_\delta : \delta \in S\rangle$ is a $(\mathbf{p}, \partial, \theta)$-*coloring* if $\theta \geq 2$, $\partial \geq 2$, and $F_\delta : {}^{C_\delta}\partial \to \theta$.

4) Let $\overline{F}$ (and $\mathbf{p}, \partial, \theta$) be as above, and D be a filter on λ. (The default choice will be the club filter.)

[4] $C_\delta \cap \alpha$ will suffice.

We say $\bar{c} \in {}^S\theta$ (or $\in {}^\lambda\theta$) is a $\mathbf{p}$-*D*-$\overline{F}$-*BB-sequence* <u>if</u> for every $\eta \in {}^\lambda\partial$ the set $\{\delta \in S : F_\delta(\eta \restriction C_\delta) = c_\delta\}$ is a member of D^+ (and in the default case, a stationary subset of λ).

5) We may omit $\mathbf{p}$ if both $\overline{C}$ and $\overline{C}'$ are clear from the context.

6) We say $\overline{C} = \langle C_\delta : \delta \in S\rangle$ is (λ, κ)-*good* <u>when</u>

- (A) S is a stationary subset of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ and a member of $\check{I}_\kappa[\lambda]$.
- (B) $C_\delta \subseteq \delta = \sup(C_\delta)$
- (C) $\mathrm{otp}(C_\delta) = \kappa$
- (D) For every $\beta < \lambda$ the set $\{C_\delta \cap \beta : \beta \in C_\delta,\ \delta \in S\}$ has cardinality $< \lambda$.

Claim 1.3. *Assume $\lambda > \kappa$ are regular cardinals and $\chi \in [\kappa, \lambda]$.*

1) *If S is a stationary subset of $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ <u>then</u> there exists a weak (λ, κ, χ)-*BB*-parameter $\mathbf{p}$ with $S_{\mathbf{p}} = S$.*

1A) *If $\chi = \lambda$ <u>then</u> we may omit "weak," and set $C_{\mathbf{p},\delta} := \delta$ for all $\delta \in S$.*

1B) *If $\lambda := \kappa^+$ and $S \subseteq S^\lambda_{<\kappa}$ is stationary, <u>then</u> we can add "for some club $E \subseteq \lambda$, there exists a a good $\mathbf{p}$ with $\overline{C}_{\mathbf{p}} = \langle C_\delta : \delta \in S \cap E\rangle$ such that each C_δ is a club of δ."*

2) *If S is a stationary subset of S^λ_κ and a member of $\check{I}_\kappa[\lambda]$, <u>then</u> in part (1) we may also add "$\mathbf{p}$ is* good, *with $S_{\mathbf{p}} := S \cap E$ for some club E of λ."*

3) *If $\lambda > \kappa^+$ <u>then</u> there exists a good (λ, κ, χ)-BB$_*$-parameter.*

4) *Every good (λ, κ, χ)-BB-parameter is a (λ, κ, χ)-BB-parameter, and every (λ, κ, χ)-BB-parameter is a weak one.*

Proof. Easy.

E.g. for part (3), use [She93, §1]. Part (1B) follows by [She91, 4.4]. $\square_{1.3}$

Definition 1.4. Let $\kappa \leq \mu$.

1) We define $\mathbf{U}_\kappa(\mu)$ to be

$$\min\big\{|\mathcal{U}| : \mathcal{U} \subseteq [\mu]^\kappa,\ (\forall v \in [\mu]^\kappa)(\exists u \in \mathcal{U})\big[|u \cap v| = \kappa\big]\big\}.$$

2) Let $\mathbf{U}'_\kappa(\mu)$ mean

$$\min\big\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\mu \text{ and } (\forall g \in {}^\kappa\mu)(\exists f \in \mathcal{F})(\exists^\kappa i < \kappa)\big[f(i) = g(i)\big]\big\}.$$

3) If $\mathcal{D} \in \mathrm{up}(\kappa)$ then we let

$$\mathbf{U}_{\mathcal{D}}(\mu) := \min\big\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\mu \text{ and } (\forall g \in {}^\kappa\mu)(\exists f \in \mathcal{F})\big[\big\{i < \kappa : f(i) = g(i)\big\} \in \mathcal{D}\big]\big\}.$$

4) Accordingly, if J is an ideal on κ (or an ideal on some set X containing $[X]^{\aleph_0}$) then we may write $\mathbf{U}_J(\mu)$ as shorthand for $\mathbf{U}_{J^+}(\mu)$, to keep notation consistent with [She05] and others.

Obviously,

Observation 1.5. 1) *If* $\mu \geq 2^\kappa$ *then* $\mathbf{U}_\kappa(\mu) = \mathbf{U}'_\kappa(\mu)$.

2) *If* $\mu = \mu^\kappa$ *(or just* $\alpha < \mu \Rightarrow |\alpha|^\kappa \leq \mu$ *and* $\mathrm{cf}(\mu) \neq \kappa$*), then* $\mathbf{U}_\kappa(\mu) = \mathbf{U}'_\kappa(\mu) = \mu$.

Definition 1.6.

1) When we write $\mathrm{Sep}(\chi, \mu, \partial, \theta, \Upsilon)$, we mean that there exists $\bar{f} = \langle f_\varepsilon : \varepsilon < \chi \rangle$ such that:

(A) $f_\varepsilon : {}^\mu\partial \to \theta$

(B) For every $\varrho \in {}^\chi\theta$, the set[5] $\mathrm{Sol}_\varrho := \big\{\nu \in {}^\mu\partial : (\forall \varepsilon < \chi)\big[f_\varepsilon(\nu) \neq \varrho(\varepsilon)\big]\big\}$ has cardinality $< \Upsilon$.

(The reader may assume $\Upsilon \leq \partial^\mu$, as the condition is vacuous otherwise.)

Such a sequence $\bar{f}$ will be called a *witness* for $\mathrm{Sep}(\chi, \mu, \partial, \theta, \Upsilon)$.

1A) If $\partial := \theta$, we may omit it.

2) We write $\mathrm{Sep}(\mu, \partial, \theta)$ to mean that $\mathrm{Sep}(\mu, \mu, \partial, \theta, \Upsilon)$ holds for some regular $\Upsilon \leq 2^\mu$.

As in part (1A), $\mathrm{Sep}(\mu, \theta) := \mathrm{Sep}(\mu, \theta, \theta)$.

2A) $\mathrm{Sep}(<\mu, \theta)$ will mean that $\mathrm{Sep}(\sigma, \mu, \theta, \theta, \Upsilon)$ holds for some $\sigma < \mu$ and Υ as in part (2).

3) We may write Sep_1 instead of Sep, to distinguish it from Sep_2 in 1.8 and Sep_3 in 1.14.

In [She05, $1.11_{=\mathsf{Ld.7}}$] we showed $\mathrm{Sep}(\mu, \theta)$ holds for many values of μ and θ. The following is a generalization of that theorem.

Claim 1.7. *If at least one of the following holds, then we have* $\mathrm{Sep}(\mu, \mu, 2^\mu, \theta, \Upsilon)$*:*

(a) $\mu = \mu^\theta$ *and* $\Upsilon := \theta$.

(b) $\mathbf{U}_\theta(\mu) = \mu$ *and* $\Upsilon := (2^{<\theta})^+ \leq 2^\mu$.

(c) *We have* $\mathbf{U}_{[\sigma]^{<\theta}}(\mu) = \mu$ *and* $\Upsilon := (2^{<\sigma})^+ \leq 2^\mu$ *for some* $\sigma \geq \theta$ *such that* $\sigma^\theta \leq \mu$.

(d) $\theta = \mathrm{cf}(\theta) < \mu$*,* μ *is strong limit singular of cofinality* $\neq \theta$*, and (e.g.)* $\Upsilon := (2^{\theta + \mathrm{cf}(\mu)})^+$.

(e) $\Upsilon := \beth_\omega(\theta) \leq \mu$.

Proof. Let $\bar{\eta} = \langle \eta_\beta : \beta < 2^\mu \rangle$ list ${}^\mu(2^\mu)$ without repetition. Let $\chi := \big(2^{2^\mu}\big)^+$, and let $N \prec (\mathcal{H}(\chi), \in)$ be of cardinality μ such that $(\mu+1) \cup \{\bar{\eta}\} \subseteq N$. Let $\bar{f} = \langle f_\varepsilon : \varepsilon < \mu \rangle$ list all the functions from ${}^\mu(2^\mu)$ to θ which are members of N. It will suffice to prove

$\boxplus_1$ For every $\varrho \in {}^\mu(2^\mu)$, the set
$$\mathrm{Sol}_\varrho := \big\{\eta \in {}^\mu(2^\mu) : (\forall \varepsilon < \mu)\big[f_\varepsilon(\eta) \neq \varrho(\varepsilon)\big]\big\}$$
has cardinality $< \Upsilon$.

For *this* it will suffice to prove

[5] 'Sol' stands for *solution.*

$\boxplus_2$ For every $\Lambda \subseteq {}^{\mu}(2^{\mu})$ of cardinality Υ, for some $\varepsilon < \mu$, we have

$$\{f_\varepsilon(\eta) : \eta \in \Lambda\} = \theta.$$

(That is, $f_\varepsilon \restriction \Lambda$ is a surjection onto θ.)

So for some $u \in [2^{\mu}]^{\Upsilon}$ (with $\mathrm{otp}(u) = \Upsilon$, for simplicity), let $\{\eta_\alpha : \alpha \in u\}$ list the elements of Λ without repetition.

Case 1: $\mu = \mu^{\theta}$ and $\Upsilon := \theta$.

For $\alpha \neq \beta \in u$, choose $\zeta_{\alpha,\beta} \in \mu$ such that $\nu_\alpha(\zeta_{\alpha,\beta}) \neq \nu_\beta(\zeta_{\alpha,\beta})$. Let

$$v := \{\zeta_{\alpha,\beta} : \alpha \neq \beta \in u\}.$$

So clearly $v \in [\mu]^{\leq\theta}$ and $g : {}^{v}2 \to \theta$ both belong to N, where g is the function which maps $\eta_\alpha \restriction v \mapsto \mathrm{otp}(u \cap \alpha)$ for each $\alpha \in u$ and sends all other elements of ${}^{v}2$ to zero. Let $f : {}^{\mu}(2^{\mu}) \to \theta$ be defined by $\eta \mapsto g(\eta \restriction v)$; clearly this is a member of N as well. Hence $f = f_\varepsilon$ for some $\varepsilon < \mu$.

Now check.

Case 2: $\mathbf{U}_\theta(\mu) = \mu$ and $\Upsilon := (2^{\theta})^{+} < 2^{\mu}$.

This is simply a special case of Case 3.

Case 3: $2^{\theta} \leq \mu$, and $\mathbf{U}_{[\sigma]^{<\theta}}(\mu) = \mu$ and $\Upsilon := (2^{<\sigma})^{+}$ for some $\sigma \in [\theta, \mu]$.

We will try to choose $(\alpha_i, \beta_i, \zeta_i)$ by induction on $i < \sigma$ such that:

- $\alpha_i \neq \beta_i$ are members of u.
- $\alpha_i, \beta_i \in u \setminus \{\alpha_j, \beta_j : j < i\}$
- $\zeta_i < \mu$
- $\eta_{\alpha_i}(\zeta_i) = 0$ and $\eta_{\beta_i}(\zeta_i) = 1$.
- If $j < i$ then $\eta_{\alpha_i}(\zeta_j) = \eta_{\beta_i}(\zeta_j)$.

Subcase A: We succeed.

Let $W_0 := \{\zeta_i : i < \sigma\}$ (so $W_0 \in [\mu]^{\sigma}$). Now, using '$\mathbf{U}_{[\sigma]^{<\theta}}(\mu) = \mu$,' there exists $W \in [W_0]^{\theta}$ which belongs to N, and we continue as in Case 1.

Subcase B: We get stuck at stage i_*, for some $i_* < \sigma$.

As $|u| = \Upsilon > 2^{|i_*|}$, there are $\alpha \neq \beta$ from u such that

$$\eta_\alpha \restriction \{\zeta_j : j < i_*\} = \eta_\beta \restriction \{\zeta_j : j < i_*\}.$$

This is an easy contradiction.

(Note that we can actually use $\Upsilon := \sum_{i<\sigma} (2^{|i|})^{+}$ instead of $(2^{<\sigma})^{+}$.)

Case 4: $\theta = \mathrm{cf}(\theta) < \mu$, μ is strong limit singular of cofinality $\neq \theta$, and $\Upsilon := (2^{\theta+\mathrm{cf}(\mu)})^{+}$.

This is also a special case of Case 3.

Case 5: $\Upsilon := \beth_\omega(\theta) \leq \mu$.

By [She00], this is also covered by Case 3. $\square_{1.7}$

We introduce the following relative of Sep $=$ Sep_1 which will be used in this work.

Definition 1.8. 1) Let $\mathrm{Sep}_2(\chi, \mu, \partial, \theta, \kappa, \mathcal{D})$ mean that there exists a sequence $\bar{f} = \langle f_{\varepsilon,i} : \varepsilon < \chi,\ i < \kappa \rangle$ *witnessing it.* By this, we mean that the following clauses hold.

(A) $\mathcal{D} \in \mathrm{up}(\kappa)$
(B) $f_{\varepsilon,i} : {}^\mu\partial \to \theta$
(C) If $\mathscr{P}_i \subseteq {}^\mu\partial$ has[6] cardinality $< \partial^\mu$ (for $i < \kappa$), <u>then</u> we can find a sequence $\bar{\varrho} = \langle \varrho_i : i < \kappa \rangle$ such that:
  (a) $\varrho_i \in {}^\mu\theta$
  (b) If $\bar{\nu} = \langle \nu_i : i < \kappa \rangle \in \prod\limits_{i<\kappa} \mathscr{P}_i$ <u>then</u> there exist $\varepsilon < \mu$ and $u \in \mathcal{D}$ such that
$$i \in u \Rightarrow f_{\varepsilon,i}(\nu_i) = \varrho_i(\varepsilon).$$

2) If $\mathcal{D} := [\kappa]^\kappa$ we may omit it. $\mathrm{Sep}_2(\mu, \theta, \kappa)$ will mean $\mathrm{Sep}_2(\mu, \mu, \theta, \theta, \kappa)$.

Recalling Definition 1.2,

Definition 1.9. 1) We say that $\mathbf{p}$ has the (D, ∂, θ)-$\overline{F}$-*BB-property*[7] <u>when</u> there exists a $\mathbf{p}$-D-$\overline{F}$-BB-sequence, where:

(A) $\mathbf{p}$ is a (λ, κ, χ)-BB-parameter.
(B) D is a filter on λ.
(C) $\overline{F}$ is a $(\mathbf{p}, \partial, \theta)$-coloring.

2) We say that $\mathbf{p}$ has the (D, ∂, θ)-BB-property <u>when</u> it has (D, ∂, θ)-$\overline{F}$-BB-property for every $(\mathbf{p}, \partial, \theta)$-coloring $\overline{F}$.

3) If D is the club filter on λ, we may omit it.

We now quote the main claim of the previous paper – [She05, 1.10=Ld.6] – but we will not use it here.

Claim 1.10. *Assume*

(*a*) $\lambda := \mathrm{cf}(2^\mu)$
(*b*) *D is a μ^+-complete filter on λ extending the club filter.*
(*c*) $\kappa = \mathrm{cf}(\kappa) < \chi \leq \lambda$
(*d*) $\mathbf{p} = \big\langle (C_\delta, C'_\delta) : \delta \in S \big\rangle$ *is a good* (λ, κ, χ)-BB-*parameter, where* $S \in D$.
(*e*) $2^{<\chi} \leq 2^\mu$ *and* $\theta \leq \mu$.
(*f*) $\alpha < 2^\mu \Rightarrow \mathrm{trp}^+_\kappa(|\alpha|) \leq 2^\mu$ *(By this we mean that every tree with $|\alpha|$-many nodes and κ levels has $< 2^\mu$-many κ-branches.)*
(*g*) $\mathrm{Sep}_1(\mu, \theta)$.

<u>Then</u> $\mathbf{p}$ *has the* $(D, 2^\mu, \theta)$-BB-*property. (Note that this means that possibly $\theta > 2$; i.e. we have more than two colors.)*

[6] We may let this sequence be constant in i — if so we may write $\mathscr{P}$ instead of $\overline{\mathscr{P}} = \langle \mathscr{P}_i : i < \kappa \rangle$.
[7] Later, we will write ‘BB^0-property.’

Remark 1.11. 1) If (e.g.) μ is strong limit singular, $\kappa := \mathrm{cf}(\mu)$, $\lambda := \mathrm{cf}(2^\mu)$, and $\kappa + \theta < \mu$, <u>then</u> the only assumption in 1.10 which does not follow is clause (f) (which does hold for many regular $\kappa < \mu$ by [She00]). For more, see [She06].

Our aim here is to cover more cases of κ, and construct relatives of this property which are easier to use and have more applications.

2) By [She93, §1], there are are many S as required (usually from $\check{I}_\kappa[\lambda]$). Still, this restricts our choices.

3) 'Good $\mathbf{p}$' is also a restriction, as the result covers fewer S-s. In fact, if $S \notin \check{I}_\kappa[\lambda]$ then there is no good $\mathbf{p}$ with $S_\mathbf{p} := S$ (as we cannot find $\langle C'_\delta : \delta \in S \rangle$). Another one of our goals is to eliminate this assumption.

4) But we would like to have parallel results using Sep_2 or Sep_3. (This will be done in §2.)

5) An earlier definition of $\mathrm{Sep}_1(\mu, \theta)$ used '$\Upsilon < 2^\mu \vee \Upsilon := 2^\mu \in \mathrm{Reg}$' instead of '$\Upsilon \leq 2^\mu \wedge \Upsilon \in \mathrm{Reg}$.'

This was a natural generalization, because the notation is tailor-made for proofs which rely on induction on $\Upsilon \leq 2^\mu$. As the Υ argument is an upper bound for the cardinality of some specific set, clearly $\mathrm{Sep}_1(\ldots, \Upsilon)$ implies $\mathrm{Sep}_1(\ldots, \Upsilon^+)$. Moreover, as every successor ordinal is regular, if $\Upsilon < 2^\mu$ <u>then</u> $\Upsilon^+ < 2^\mu \vee \Upsilon^+ \in \mathrm{Reg}$. However, we would have to rewrite existing proofs to match this new definition, and that would be more trouble than it's worth.

Claim 1.12. *Assume κ is regular, $\mu > \theta = \theta^{<\kappa}$, and $\partial \in [2, 2^\mu]$. If at least one of the following holds <u>then</u> we have* $\mathrm{Sep}_2(\mu, \mu, \partial, \theta, \kappa)$.

(a) $\kappa \neq \mathrm{cf}(\mu)$, $\alpha < \mu \Rightarrow |\alpha|^\kappa \leq \mu$, *and* $\mathrm{Sep}_1(\mu, \partial, \theta)$.
(b) $\mathbf{U}'_\kappa(\mu) = \mu \geq \beth_\omega(\theta + \kappa)$
(c) $\mathbf{U}'_\kappa(\mu) = \mu$ *and* $\mathrm{Sep}_1(\mu, \partial, \theta)$.
(d) *We have* $\mathbf{U}_{[\sigma]^{<\theta}}(\mu) = \mu$ *for some* $\sigma \geq \theta$ *with* $\sigma^\theta \leq \mu$ *and* $(2^\sigma)^+ < 2^\mu$.

Proof. **Case (a):**

Let $\bar{f}^\circ = \langle f^\circ_\varepsilon : \varepsilon < \mu \rangle$ witness $\mathrm{Sep}_1(\mu, \partial, \theta)$ (hence f°_ε is a function from ${}^\mu\partial$ to θ). Let

$$\mathcal{F} := \{\nu \in {}^\kappa\mu : \mathrm{rang}(\nu) \text{ is a bounded subset of } \mu\}.$$

Recalling Definition 1.6(2), let Υ be a regular cardinal $\leq 2^\mu$ such that $\mathrm{Sep}_1(\mu, \mu, \partial, \theta, \Upsilon)$ holds.

By the assumption '$\alpha < \mu \Rightarrow |\alpha|^\kappa \leq \mu$,' clearly $|\mathcal{F}| = \mu$. Let $\langle \nu_\varepsilon : \varepsilon < \mu \rangle$ list the members of $\mathcal{F}$, and we shall define

$\circledast_1$ $\bar{f} = \langle f_{\varepsilon,i} : \varepsilon < \mu,\ i < \kappa \rangle$, where $f_{\varepsilon,i} := f^\circ_{\nu_\varepsilon(i)}$.

It will suffice to prove that $\bar{f}$ witnesses $\mathrm{Sep}_2(\mu, \mu, \partial, \theta, \kappa)$. So let $\mathscr{P}_i \subseteq {}^\mu\partial$ be of cardinality $< \partial^\mu = 2^\mu$ for $i < \kappa$, and we need to construct $\bar{\varrho}$ as in 1.8(1)(C).

Fix $i < \kappa$, so by 1.6(1)(B), for every $\rho \in {}^\mu\theta$ the set

$$\mathrm{Sol}_\rho := \big\{\nu \in {}^\mu\theta : (\forall \varepsilon < \chi)\big[f^\circ_\varepsilon(\nu) \neq \rho(\varepsilon)\big]\big\}$$

has cardinality $< \Upsilon$. As $|\mathscr{P}_i| < 2^\mu$ and[8] $\Upsilon = \mathrm{cf}(\Upsilon) \leq 2^\mu$, the set $\Lambda_i := \bigcup_{\nu \in \mathscr{P}_i} \mathrm{Sol}_\nu$ has cardinality $< 2^\mu$, so we can choose $\varrho_i \in {}^\mu\partial \setminus \Lambda_i$.

It will suffice to prove that $\langle \varrho_i : i < \kappa \rangle$ is as promised. So let $\bar\nu = \langle \nu_i : i < \kappa \rangle \in \prod_{i<\kappa} \mathscr{P}_i$, and we have to find $\varepsilon < \mu$ and $u \in [\kappa]^\kappa$ as promised in 1.8(1)(C)(b).

For each $i < \kappa$, by our choice of ϱ_i we know $\varrho_i \notin \mathrm{Sol}_{\nu_i}$. This means

$\circledast_2$ There is $\varepsilon_i < \mu$ such that $f^\circ_{\varepsilon_i}(\nu_i) = \varrho_i(\varepsilon_i)$.

As $\mathrm{cf}(\mu) \neq \mathrm{cf}(\kappa)$, there exists $\zeta < \mu$ such that the set $u := \{i < \kappa : \varepsilon_i < \zeta\}$ has cardinality κ (and even order type κ).

$\circledast_3$ Let $\nu \in {}^\kappa\zeta$ be the sequence

$$\nu(i) := \begin{cases} \varepsilon_i & \text{if } i \in u \\ 0 & \text{otherwise,} \end{cases}$$

and let $\varepsilon < \mu$ be such that $\nu = \nu_\varepsilon$.

Now ε is as required.

Why? For every $i \in u$, we have

$\circledast_4$ $f_{\varepsilon,i}(\nu_i) = f^\circ_{\nu_\varepsilon(i)}(\nu_*) = f^\circ_{\varepsilon_i}(\nu_i) = \varrho_i(\varepsilon_i)$.

[The first equality is the definition from $\circledast_1$, the second holds by the choice of ν in $\circledast_3$, and the third by the choice of ε_i in $\circledast_2$.]

Case (b): Assume $\mathbf{U}_\kappa(\mu) = \mu \geq \chi := \beth_\omega(\theta + \kappa)$.

By Case (e) of 1.7 this implies $\mathrm{Sep}_1(\mu, \theta)$, so let $\bar f^\circ = \langle f^\circ_\varepsilon : \varepsilon < \mu \rangle$ be a witness. Let $\mathcal{F} \subseteq {}^\kappa\mu$ be of cardinality μ witnessing $\mathbf{U}'_\kappa(\mu) = \mu$.

The rest is as in the proof of Case (a), except that in the end we choose $u \in [\kappa]^\kappa$ and $\varepsilon < \mu$ together such that

$$(\forall i \in u)\big[\nu_\varepsilon(i) = \varepsilon_i\big]$$

(which is possible by our choice of $\mathcal{F}$).

Case (c): Like Case (b).

Case (d): Similarly, using 1.7(c). $\square_{1.12}$

Claim 1.13. 1) $\mu = \mu^{\kappa+\theta}$ *implies* $\mathrm{Sep}_2(\mu, \mu, 2^\mu, \theta, \kappa, \{\kappa\})$.

2) *Suppose* $\mathbf{U}_\kappa(\mu) = \kappa$ *and* $\mathbf{U}_{[\sigma]^{<\theta}}(\mu) = \mu$ *for some* $\sigma \geq \theta$ *with* $\sigma^\theta \leq \mu$ *and* $(2^\theta)^+ < 2^\mu$.

Then we have $\mathrm{Sep}_2(\mu, \mu, 2^\mu, \theta, \kappa, [\kappa]^\kappa)$.

Proof. 1) Like the proof of Case (a) of 1.12, using $\mathcal{F} := {}^\kappa\mu$.

2) Use 1.12 Case (d) and the proof of Case (a). $\square_{1.13}$

[8] Recall that we permit $\Upsilon := 2^\mu$ if it is regular.

Definition 1.14. Assume $\lambda, \chi, \mu, \partial, \theta, \kappa$ are cardinals (with ∂ and θ possibly finite), and $\mathcal{D} \in \mathrm{up}(\kappa)$.

1) Let $\mathrm{Sep}_3(\lambda; \chi, \mu, \partial, \theta, \kappa, \mathcal{D})$ mean that there exist

$$\bar{f} = \langle f_{\varepsilon,i} : \varepsilon < \chi,\ i < \kappa \rangle$$

and $\overline{\mathscr{P}} = \langle \mathscr{P}_\xi : \xi < \lambda \rangle$ such that the following all hold.

(A) $f_{\varepsilon,i} : {}^\mu\partial \to \theta$

(B) $\mathscr{P}_\xi \subseteq {}^\mu\partial$, and $\overline{\mathscr{P}}$ is strictly $\subset$-increasing in ξ with union ${}^\mu\partial$.

(C) If $\xi < \lambda$ then we can we can find a sequence $\bar{\varrho} = \langle \varrho_i : i < \kappa \rangle$ such that:
 (a) $\varrho_i \in {}^\mu\theta$
 (b) If $\bar{\nu} = \langle \nu_i : i < \kappa \rangle \in {}^\kappa(\mathscr{P}_\xi)$ then there exist $\varepsilon < \chi$ and $u \in \mathcal{D}$ such that

$$i \in u \Rightarrow f_{\varepsilon,i}(\nu_i) = \varrho_i(\varepsilon).$$

2) $\mathrm{Sep}_4(\lambda; \chi, \mu, \partial, \theta, \kappa, \mathcal{D})$ is defined similarly, except that $\overline{\mathscr{P}}$ is redefined as a covering of ${}^\kappa({}^\mu\partial)$ (so each $\mathscr{P}_\xi \subset {}^\kappa({}^\mu\partial)$). Clause (C) remains the same, except that the antecedent to (C)(b) becomes "If $\bar{\nu} = \langle \nu_i : i < \kappa \rangle \in \mathscr{P}_\xi$ . . .' for consistency.

3) Again, the default value of $\mathcal{D}$ is $[\kappa]^\kappa$. For $\iota = 3, 4$, $\mathrm{Sep}_\iota(\lambda; \mu, \theta, \kappa)$ will mean $\mathrm{Sep}_\iota(\lambda; \mu, \mu, \theta, \theta, \kappa)$. (That is, $\chi \coloneqq \mu$ and $\partial \coloneqq \theta$.)

Observation 1.15. 1) **[Monotonicity:]** *If $\chi_1 \le \chi_2$, $\mu_1 \le \mu_2$, $\partial_1 \ge \partial_2$, $\theta_1 \ge \theta_2$, and $\mathcal{D}_1 \supseteq \mathcal{D}_2$, then*

$$\mathrm{Sep}_3(\lambda; \chi_1, \mu_1, \partial_1, \theta_1, \kappa, \mathcal{D}_1) \Rightarrow \mathrm{Sep}_3(\lambda; \chi_2, \mu_2, \partial_2, \theta_2, \kappa, \mathcal{D}_2).$$

1A) *Similarly for* Sep_4.

1B) *Similarly for* Sep_1*; i.e. if $\chi_1 \le \chi_2$, $\mu_1 \le \mu_2$, $\partial_1 \ge \partial_2$, $\theta_1 \ge \theta_2$, and $\Upsilon_1 \le \Upsilon_2 \le 2^{\mu_1}$,*[9] *then*

$$\mathrm{Sep}_1(\chi_1, \mu_1, \partial_1, \theta_1, \Upsilon_1) \Rightarrow \mathrm{Sep}_1(\chi_2, \mu_2, \partial_2, \theta_2, \Upsilon_2).$$

2) **[Connection to Sep_2:]** *Assume $\chi, \mu, \partial, \theta, \mathcal{D}$ are as in 1.14, and $\mathrm{cf}(\partial^\mu) > \kappa$.*[10] *Then*

$$\mathrm{Sep}_2(\chi, \mu, \partial, \theta, \kappa, \mathcal{D}) \Rightarrow \mathrm{Sep}_3(\mathrm{cf}(\partial^\mu); \chi, \mu, \partial, \theta, \kappa, \mathcal{D}) \Rightarrow \mathrm{Sep}_4(\mathrm{cf}(\partial^\mu); \chi, \mu, \partial, \theta, \kappa, \mathcal{D})$$

Proof. 1) Read the definition. (1A) and (1B) are similar.

2) **The first implication:**

Let $\langle \eta_\alpha : \alpha < \partial^\mu \rangle$ list ${}^\mu\partial$, let $\langle \Upsilon_\zeta : \zeta < \lambda \rangle$ be an increasing sequence of ordinals with limit ∂^μ, and let $\mathscr{P}_\zeta \coloneqq \{\eta_\alpha : \alpha < \Upsilon_\zeta\}$.

Now check.

The second implication:

Let $\langle \mathscr{P}_\xi : \xi < \mathrm{cf}(\partial^\mu) \rangle$ exemplify

$$\mathrm{Sep}_3(\mathrm{cf}(\partial^\mu); \chi, \mu, \partial, \theta, \kappa, \mathcal{D})$$

(that is, it is as in Definition 1.14(1).) Now $\langle {}^\kappa(\mathscr{P}_\xi) : \xi < \mathrm{cf}(\partial^\mu) \rangle$ is as required in 1.14(2). (Note that here we need the assumption '$\mathrm{cf}(\partial^\mu) > \kappa$.') $\square_{1.15}$

[9] Again, with equality only if 2^{μ_1} is regular.
[10] This is needed in the second $\Rightarrow$.

Conclusion 1.16. *We have* $\mathrm{Sep}_3(\lambda;\mu,\mu,2^\mu,\theta,\kappa,\mathcal{D})$ *when*

(A) $\kappa = \mathrm{cf}(\kappa)$
(B) *The triple* $(\mu, 2^\mu, \theta)$ *satisfies at least one of the conditions in* 1.7.
(C) $\mathbf{U}_{\mathcal{D}}(\mu) = \mu$
(D) $\lambda := \mathrm{cf}(2^\mu)$.

Proof. By the conclusion of 1.7, we have $\mathrm{Sep}_1(\mu,\mu,2^\mu,\theta,\Upsilon)$ for some Υ. Using clause (C) and repeating the proof of 1.12(a), we have $\mathrm{Sep}_2(\mu,\mu,2^\mu,\theta,\kappa,\mathcal{D})$. With that and 1.15(2), we get our conclusion. $\square_{1.16}$

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Remark 1.17. 1) We may need to sort out when '$\chi \neq \mu$' is actually needed in Sep_ι.

2) We may also consider the definition below.

Definition 1.18. Let $\kappa \le \mu$ and $\mathbb{S} \subseteq \mathcal{P}(\mathcal{P}(\kappa))$.

1) We define $\mathbf{U}_{\mathbb{S}}(\mu)$ to be

$$\min\big\{|\mathcal{U}| : \mathcal{U} \subseteq [\mu]^\kappa,\ (\forall g \in {}^\kappa\mu)(\exists \mathcal{A} \in \mathbb{S})(\forall u \in \mathcal{A})\big[\mathrm{rang}(g \restriction u) \in \mathcal{U}\big]\big\}.$$

2) $\mathbf{U}'_{\mathbb{S}}(\mu)$ means

$$\min\big\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\mu \text{ and } (\forall g \in {}^\kappa\mu)(\exists \mathcal{A} \in \mathbb{S})(\forall u \in \mathcal{A})(\exists f \in \mathcal{F})[f \restriction u = g \restriction u]\big\}.$$

Question 1.19. Can we prove existence results for $\mathrm{Sep}_\iota(\lambda;\mu,\theta,\kappa)$ for ($\iota = 3,4$ and) $\lambda \in (\mu, 2^\mu] \setminus \{\mathrm{cf}(2^\mu)\}$ regular? Can we disprove them?

Well, as in many cases, we have independence results. E.g.

Claim 1.20. 1) *Assume* $\mu = \mu^{<\mu} < \chi = \chi^\mu$. *Let* $\mathbb{P} := \mathsf{Cohen}_{\mu,\chi}$ *(i.e. the forcing adding* χ*-many* μ*-Cohens).*

Then in $\mathbf{V}^{\mathbb{P}}$*, for ever regular* $\lambda \in [\mu^+, \chi] \setminus \{\mathrm{cf}(\chi)\}$*, we have* $\mathrm{Sep}_3(\lambda;\mu,\theta,\kappa)$.

2) *If* $\theta \le \mu < \lambda = \mathrm{cf}(\lambda) \le 2^\mu$ *and* $\bar\eta = \langle \eta_\xi : \xi < \lambda\rangle \subseteq {}^\mu\theta$ *is a* μ*-Luzin sequence, then* $\mathrm{Sep}_3(\lambda;\mu,\mu,2^\mu,\theta,\kappa,\{\kappa\})$.

Recall that '$\bar\eta$ *is a* μ*-Luzin sequence' means that for every meagre* μ*-Borel set* $\mathbf{B} \subseteq {}^\mu\theta$ *there exists* $\alpha_* < \ell g(\bar\eta)$ *such that*

$$\alpha_* < \alpha < \ell g(\bar\eta) \Rightarrow \eta_\alpha \notin \mathbf{B}.$$

Proof. Straightforward. $\square_{1.20}$

Claim 1.21. *Let* $\mu = \mu^{<\mu} < \lambda = \mathrm{cf}(\lambda) < \chi \le 2^\mu$ *and assume (e.g.) the forcing axiom* $\mathbf{Ax}_{\mu,\chi}((1)^+_c, (2)^\varepsilon_c)$ *from* [She22]. *Let* $S := \{\delta \in S^\lambda_\kappa : \mu^\kappa \mid \delta\}$.[11]

Then

(A) *For* $\kappa = \mathrm{cf}(\kappa) < \mu$*, there exists* $\overline{C} = \langle C_\delta : \delta \in S\rangle$ *such that:*
  (a) C_δ *is an unbounded subset of* δ *of order type* κ.
  (b) $\overline{C}$ *is* μ^+*-free (see* 0.2*).*

[11] Here we mean μ^κ as exponentiation of ordinals.

(I.e. if $u \in \left[S^\lambda_\kappa\right]^{<\mu}$ *then there exists* $f \in \prod_{\delta \in u} C_\delta$ *such that* $\langle C_\delta \setminus f(\delta) : \delta \in u\rangle$ *is a sequence of pairwise disjoint sets.)*

(B) *Any* $\overline{C}$ *as in part* (A) *has* μ-uniformization*.*

That is, if $\bar{f} = \langle f_\delta : \delta \in S^\lambda_\kappa\rangle$ *with* $f_\delta \in {}^{C_\delta}\mu$ *then there exists* $f \in {}^\lambda\mu$ *such that*

$$(\forall \delta \in S^\lambda_\kappa)\big[f_\delta \subseteq^* f\big].$$

(By this *we mean* $\big|\{\alpha \in C_\delta : f_\delta(\alpha) \neq f(\alpha)\}\big| < \kappa$*.)*

Proof. **Clause (A):**

Choose $\langle C^\circ_\delta : \delta \in S\rangle$ such that:

$(*)_\delta$ If $\delta \in S$ (hence $\mu^\kappa \,\big|\, \delta$) then C°_δ is an unbounded subset of δ of order type μ^κ.

Now we define a forcing notion $\mathbb{Q}$.

$(*)_\mathbb{Q}$ (a) $p \in \mathbb{Q}$ iff p is of the form $\langle C_\delta : \delta \in u\rangle$, where:
- $\bullet_1$ $u \in [S]^{<\mu}$
- $\bullet_2$ $C_\delta \subseteq C^\circ_\delta$ is also unbounded in δ.
- $\bullet_3$ $\mathrm{otp}(C_\delta) = \kappa$

(b) $\mathbb{Q} \models$ '$p \leq q$' iff $u_p \subseteq u_q$ and $C_{p,\delta} = C_{q,\delta}$ for all $\delta \in u_p$.

Clearly $\mathbb{Q}$ is strategically α-complete for all $\alpha < \mu$ (i.e. [She22, 0.3(A)(1)$_{c\text{=Lx2}}$]). Moreover, this still holds for $\alpha = \mu$, and we have version $(2)_b$ of the μ^+-cc (from [She22, 0.3(B)$_{\text{=Lx2}}$]).

Clause (B): Similarly. $\square_{1.21}$

The following observation will give us some bounds.

Definition 1.22. For $\theta \in \kappa \cap \mathrm{Reg}$, let

$$\mathrm{inv}_\sigma(\kappa) \coloneqq \min\big\{|\mathcal{A}| : \mathcal{A} \subseteq [\kappa]^\kappa \text{ and } (\forall h \in {}^\kappa\sigma)(\exists i < \sigma)(\exists A \in \mathcal{A})[\mathrm{rang}(h \restriction A) \subseteq i]\big\}.$$

Observation 1.23. 1) *If* $\sigma = \mathrm{cf}(\sigma) < \kappa = \kappa^{<\kappa} < \lambda = \lambda^\kappa$ *and* $\mathbb{Q}$ *is the forcing which adds* κ*-many* σ*-Cohens, then* $\Vdash_\mathbb{Q}$ "$\mathrm{inv}_\sigma(\kappa) = \lambda = 2^\kappa$"*.*

2)*If* $\sigma = \mathrm{cf}(\sigma) \leq \partial = \partial^\kappa$ *and* $\mathbb{Q}$ *is a* $(<\partial)$*-complete forcing notion with calibre* μ *which does not collapse cardinals, then in* $\mathbf{V}^\mathbb{Q}$ *we have* $\mathrm{inv}_\sigma(\kappa)^{\mathbf{V}^\mathbb{Q}} \leq (2^\kappa)^\mathbf{V}$*.*

§ 2. The Black Box property

Definition 2.1. 1) We say $\mathbf{p} = (\langle (C_\delta, C'_\delta, \bar{C}_\delta) : \delta \in S\rangle, \mathcal{D})$ is a (λ, κ, χ)-BB^1-*parameter* <u>when</u>

- (A) (a) $\lambda > \kappa$ are regular.
  - (b) S is a stationary subset of S^λ_κ.
- (B) $C_\delta \subseteq \delta$ with $\sup(C_\delta) = \delta$.
- (C) (a) $C'_\delta \subseteq \delta$, $\sup(C'_\delta) = \delta$, and $\mathrm{otp}(C'_\delta) = \kappa$. (We do not require that C_δ *or* C'_δ be closed in δ.)
  - (b) Let $\bar{\beta}_\delta = \langle \beta_{\delta,i} : i < \kappa\rangle$ list C'_δ in increasing order.
- (D) (a) $\bar{C}_\delta = \langle C_{\delta,i} : i < \kappa\rangle$
  - (b) $C_{\delta,i} \subseteq C_\delta \cap \beta_{\delta,i}$
  - (c) $|C_{\delta,i}| < \chi$
- (E) For all $\alpha < \lambda$, the set
$$\{C_{\delta,i} : \delta \in S,\ i < \kappa,\ \beta_{\delta,i} = \alpha\}$$
has cardinality $< \lambda$.
- (F) $\mathcal{D} \in \mathrm{up}(\kappa)$.

1A) Above, replacing BB^1 by BB^2 means strengthening clause (1)(D)(b) to

- (b′) $C_{\delta,i} := C_\delta \cap \beta_{\delta,i} \setminus \beta_{\delta,<i}$.

1B) Replacing BB^1 by BB^3 means strengthening clause (1)(D)(b) to

- (b″) $C_{\delta,i} := C_\delta \cap \beta_{\delta,i}$.

2) We say $\overline{F} = \langle F_{\delta,i} : \delta \in S,\ i < \kappa\rangle$ is a *continuous* $(\mathbf{p}, \partial, \theta)$-BB^1-*coloring* <u>when</u> ($S = S_{\mathbf{p}}$ and)

- (A) $F_{\delta,i} : {}^{C_{\delta,i}}\partial \to \theta$ for $\delta \in S$ and $i < \kappa$ (recalling $C_{\delta,i} \subseteq C_\delta \cap \beta_{\delta,i}$).
- (B) For $\beta < \lambda$, the set $\mathcal{F}_\beta := \{F_{\delta,i} : \delta \in S,\ i < \kappa,\ \beta_{\delta,i} = \beta\}$ has cardinality $< \lambda$.

2A) For $\overline{F}$, ∂, and θ as above and D a filter on λ, we say $\bar{c} = \langle c_{\delta,i} : \delta \in S,\ i < \kappa\rangle \in {}^{S\times\kappa}\theta$ is a $\mathbf{p}$-D-$\overline{F}$-BB^1-*sequence* <u>when</u> for every $\eta \in {}^\lambda\partial$ the set
$$\{\delta \in S : (\exists^{\mathcal{D}} i < \kappa)[F_{\delta,i}(\eta \restriction C_{\delta,i}) = c_{\delta,i}]\}$$
is a member of D^+ (and in the default case, a stationary subset of λ).[12]

3) We say that $\gamma, \delta \in S$ are $\mathbf{p}$-*similar* <u>when</u>:

- $\bullet_1$ $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\delta)$ (Recall that we only demanded $\mathrm{otp}(C'_\gamma) = \mathrm{otp}(C'_\delta) = \kappa$.)
- $\bullet_2$ $i < \kappa \Rightarrow \mathrm{otp}(C_{\gamma,i}) = \mathrm{otp}(C_{\delta,i})$.

4) We say $\overline{F} = \langle F_\delta : \delta \in S\rangle$ is a *uniform* $(\mathbf{p}, \partial, \theta)$-*coloring* <u>when</u> the implication '(A) $\Rightarrow$ (B)' holds, where:

- (A) (a) δ_1 and δ_2 are $\mathbf{p}$-similar.
  - (b) $f_\ell : C_{\delta_\ell} \to \theta$ for $\ell = 1, 2$.

[12] Naturally, $(\exists^{\mathcal{D}} i < \kappa)\varphi_i$ is shorthand for $(\exists u \in \mathcal{D})(\forall i \in u)\varphi_i$.

(c) If $\gamma_\ell \in C_{\delta_\ell}$ for $\ell = 1, 2$, then

$$\mathrm{otp}(\gamma_1 \cap C_{\delta_1}) = \mathrm{otp}(\gamma_2 \cap C_{\delta_2}) \Rightarrow f_1(\gamma_1) = f_2(\gamma_2).$$

(B) $F_{\delta_1}(f_1) = F_{\delta_2}(f_2)$.

5) '$\langle F_{\delta,i} : \delta \in S,\ i < \kappa \rangle$ is a *uniformly continuous* $(\mathbf{p}, \partial, \theta)$-coloring' is defined similarly, but we replace (4)(B) with the demand

(B)′ $i < \kappa \Rightarrow F_{\delta_1,i}(f_1 \restriction C_{\delta_1,i}) = F_{\delta_2,i}(f_2 \restriction C_{\delta_2,i})$.

6) When we write σ-uniform instead of uniform, this means that in clause (4)(A)(a) we replace '$\mathbf{p}$-similar' by '$\mathcal{E}$-equivalent' for some equivalence relation $\mathcal{E}$ on S with $\leq \sigma$ equivalence classes satisfying

$$\gamma\ \mathcal{E}\ \delta \Rightarrow [\gamma \text{ is } \mathbf{p}\text{-similar to } \delta].$$

Remark 2.2. 1) Regarding BB^2 — the idea is that C_δ and the $C_{\delta,i}$-s are defined in terms of C'_δ and $\beta_{\delta,i}$ (recalling $C'_\delta = \{\beta_{\delta,i} : i < \kappa\}$).

On the one hand, we can choose $C_\delta := \delta$, in which case $\chi = \lambda$ and we may choose $C_{\delta,i} := [\beta_{\delta,<i}, \beta_{\delta,i}]$.

On the other hand, we may choose $C'_\delta := C_\delta$ and

$$C_{\delta,i} := \begin{cases} \{\beta_{\delta,i-1}\} & \text{if } i \text{ is successor} \\ \varnothing & \text{otherwise.} \end{cases}$$

In both cases we get clause 2.1(1)(E); that is,

$$\beta < \lambda \Rightarrow \big|\{C_{\delta,i} : \delta \in S,\ i < \kappa,\ \beta_{\delta,i} = \beta\}\big| < \lambda.$$

This will be used in clause $(*)_4$(d) in the proof of 2.5.

2) Uniformity (defined in 2.1(4)-(6)) is used only in 2.12.

3) Note that in Lemma 2.5 we may choose $\mathcal{D} = \mathcal{D}_{\mathbf{p}} := \{\kappa\}$ — the best, the desired case.

But then the assumption on Sep_3 is stronger, so it is better to apply it to $\mathcal{D} := [\kappa]^\kappa$; then we shall be able to upgrade the conclusion in 2.10, 2.11.

4) For the BB^1 version, we may omit $2.1(3)\bullet_1$.

Definition 2.3. 1) We say that $\mathbf{p}$ has the *continuous* (D, ∂, θ)-BB^1*-property* <u>when</u> it has the (D, ∂, θ)-$\overline{F}$-BB^1-property (see Definition 2.1(2A)) for every continuous $(\mathbf{p}, \partial, \theta)$-coloring $\overline{F}$.

By this, we mean that the implication '(A) $\Rightarrow$ (B)' holds, where:

(A) (a) $\mathbf{p}$ is a (λ, κ, χ)-BB-parameter.
(b) D is a filter on λ.
(c) $\overline{F}$ is a *continuous* $(\mathbf{p}, \partial, \theta)$-coloring.

(B) There exists a $\mathbf{p}$-D-$\overline{F}$-BB^1-sequence.

2) Again, if D is the club filter on λ plus $S_{\mathbf{p}}$, then we may omit it.

Discussion 2.4. The next claim is related to 1.10. We restrict ourselves to continuous colorings, but we gain by omitting demand 1.10(f), which restricted κ (that is, the cofinality of members of S). Also, if $\lambda = \lambda^{<\lambda}$ (hence $\lambda = 2^\mu$) then we may choose $\chi := \lambda$.

Lemma 2.5. *We have '$(A) \Rightarrow (B)$', where*

- (A) (a) $\mu < \lambda = \mathrm{cf}(\lambda) = \mathrm{cf}(2^\mu)$
  - (b) *D is a μ^+-complete filter on λ extending the club filter.*
  - (c) $\kappa = \mathrm{cf}(\kappa) < \chi \le \lambda$
  - (d) $\mathbf{p} = (\langle (C_\delta, C'_\delta, \overline{C}_\delta) : \delta \in S\rangle, \mathcal{D})$ *is a* (λ, κ, χ)-BB^1*-parameter, where* $S \in D^+$.
  - (e) $\theta \le \mu$ *and* $\theta^{<\chi} \le 2^\mu$.
  - (f) $\mathrm{Sep}_3(\lambda; \mu, \mu, \partial, \theta, \kappa, \mathcal{D})$.
- (B) $\mathbf{p}$ *has the continuous* $(D, 2^\mu, \theta)$-BB^1*-property.*

Remark 2.6. 1) If $C_{\delta_1,\varepsilon_1} = C_{\delta_2,\varepsilon_2} \Rightarrow F_{\delta_1,\varepsilon_1} = F_{\delta_2,\varepsilon_2}$, then in the proof below, we can replace $\mathcal{F}_\beta$ by $\mathcal{C}_\beta := \{C_{\delta,i} : \delta \in S,\ i < \kappa,\ \beta_{\delta,i} = \beta\}$.

2) The main case is $\partial := 2^\mu$.

Proof. First,

$(*)_1$ Let $\overline{F} = \langle F_{\delta,i} : \delta \in S,\ i < \kappa\rangle$ be a continuous $(\mathbf{p}, 2^\mu, \theta)$-coloring. It will suffice to show that there is a $\mathbf{p}$-D-$\overline{F}$-sequence.

Now by assumption $(A)(f)$,

$(*)_2$ Let $\bar{f} = \langle f_{\varepsilon,i} : \varepsilon < \mu,\ i < \kappa\rangle$ and $\overline{\mathscr{P}} = \langle \mathscr{P}_\xi : \xi < \lambda\rangle$ exemplify

$$\mathrm{Sep}_3(\lambda; \mu, \mu, \partial, \theta, \kappa, \mathcal{D}).$$

$(*)_3$ For each $\delta \in S$, choose $\bar{\varrho}^\delta = \langle \varrho^\delta_i : i < \kappa\rangle \in {}^\kappa({}^\mu\theta)$ as in Definition 1.14(1)(C). (Recall that $\delta \in S \Rightarrow \delta < \lambda$, so the δ here corresponds to the subscript ξ in the definition.)

Now for every $\varepsilon < \mu$ we suggest a possible $\mathbf{p}$-D-$\overline{F}$-sequence $\bar{c}_\varepsilon$:

$(*)_4$ $\bar{c}_\varepsilon = \langle c_{\varepsilon,\delta,i} : \delta \in S,\ i < \kappa\rangle$, where $c_{\varepsilon,\delta,i} := f_{\varepsilon,i}(\varrho^\delta_i) \in \theta$.

$(*)_5$ Let $\mathcal{F}_\beta := \{F_{\delta,i} : \delta \in S,\ i < \kappa,\ \beta_{\delta,i} = \beta\}$, so
- $\mathrm{dom}(F_{\delta,i}) = {}^{(C_{\delta,i})}({}^\mu\partial)$ (That is, the set of functions $f : C_{\delta,i} \to {}^\mu\partial$.)
- $\mathrm{rang}(F_{\delta,i}) \subseteq \theta$
- If $F = F_{\delta,i} \in \mathcal{F}_\beta$, then we may write C_F as a well-defined shorthand for $C_{\delta,i}$.

$(*)_6$ Let $\mathcal{C}' := \bigcup_{\delta \in S} C'_\delta$.

If for some $\varepsilon < \mu$ the sequence $\bar{c}_\varepsilon$ is as required (i.e. it is a $\mathbf{p}$-D-$\overline{F}$-sequence) then we are done. So toward contradiction, assume

$\boxplus_{6.1}$ For all $\varepsilon < \mu$ we can choose $\eta_\varepsilon \in {}^\lambda({}^\mu\partial)$ and $E_\varepsilon \in D$ such that

$$\delta \in E_\varepsilon \Rightarrow \big\{i < \kappa : c_{\varepsilon,\delta,i} = F_{\delta,i}(\eta \restriction C_{\delta,i})\big\} \notin \mathcal{D}.$$

$(*)_7$ For every $\beta \in \mathcal{C}'$ and $F \in \mathcal{F}_\beta$, let

- $\rho_{\beta,F} := \langle F(\eta_\varepsilon \restriction C_F) : \varepsilon < \mu \rangle$ (so $\rho_{\beta,F} \in {}^\mu\theta$).
- $j_{\beta,F} := \min\{\zeta < \lambda : \rho_{\beta,F} \in \mathscr{P}_\zeta\}$
- $\zeta_\beta := \sup\{j_{\beta,F} : F \in \mathcal{F}_\beta\} < \lambda$

[Why can we do this? Because $|\mathcal{F}_\beta| < \lambda = \mathrm{cf}(\lambda)$ by 2.1(2)(B).]

$(*)_8$ Let $E_* := \{\delta < \lambda : \delta \text{ limit, and } \beta < \delta \Rightarrow \zeta_\beta < \delta\}$.
(a) E_* is a club of λ.
(b) $E := E_* \cap \bigcap\limits_{\varepsilon<\mu} E_\varepsilon \in D$

[Why is this in the filter? We declared each $E_\varepsilon \in D$ by $\boxplus_{6.1}$, and clause (a) holds by our construction. By assumption $(A)(b)$, D contains all clubs and is μ^+-complete.]

$(*)_9$ For every $\delta \in E$ there exists $\varepsilon = \varepsilon_\delta < \mu$ such that

$$\{i < \kappa : F_{\delta,i}(\eta_\varepsilon \restriction C_{\delta,i}) = c_{\varepsilon,\delta,i}\} \in \mathcal{D}.$$

[Why? By the choice of $\bar{\varrho}^\delta$ — that is, by the definition of Sep_3 — because $\varepsilon < \mu \Rightarrow \eta_\varepsilon \in \mathscr{P}_\delta$.]

$(*)_{10}$ For some $\varepsilon < \mu$,

$$A_\varepsilon := \{\delta \in E \cap S : \varepsilon_\delta = \varepsilon\} \in D^+.$$

[Why? By $(*)_9$ and the fact that D is μ^+-complete.]

But this is a contradiction. $\square_{2.5}$

Remark 2.7. It is nice to successfully predict the values of $\langle F_{\delta,i}(\eta) \restriction C_{\delta,i} : i \in u\rangle$ on some $u \in [\kappa]^\kappa$, but it would be better to succeed for $u = \kappa$.

One possibility: what if we just assume $\theta = \theta^{<\kappa}$, and for each $u \subseteq \kappa$ we define $\mathbf{p}_{[u]}$ by $(S_{\mathbf{p}_{[u]}}, \bar{C}_{\mathbf{p}_{[u]}}) := (S_\mathbf{p}, \bar{C}_\mathbf{p})$, but

$$C'_{\mathbf{p}_{[u]},\delta} := \{\alpha \in C'_{\mathbf{p},\delta} : \mathrm{otp}(C'_{\mathbf{p},\delta} \cap \alpha) \in u\}?$$

Or use a regressive function $h : u \to \kappa$? Something close is done below.

Definition 2.8. Let $\mathbf{p}$ be a (λ,κ,χ)-BB^1-parameter.

For $A \in [\kappa]^\kappa$, we define a (λ,κ,χ)-BB^1-parameter

$$\mathbf{p}_A = \mathbf{p}[A] = (\langle C_{A,\delta}, C'_{A,\delta}, \bar{C}_{A,\delta} : \delta \in S\rangle, \mathcal{D}_A)$$

by

- $C_{A,\delta} := C_\delta$
- $C'_{A,\delta} := \{\beta \in C'_\delta : \mathrm{otp}(C'_\delta \cap \beta) \in A\}$.
- $\bar{C}_{A,\delta} = \langle C_{A,\delta,i} : i < \kappa\rangle$ is defined by $C_{A,\delta,i} := C_{\delta,h_A^{-1}(i)}$, where $h_A : A \to \kappa$ is the function $i \mapsto \mathrm{otp}(A \cap i)$.[13]
- $\mathcal{D}_A := \{\kappa\}$

Observation 2.9. 1) $\mathbf{p}_A$*, as defined above, is indeed a* (λ,κ,χ)-BB^1*-parameter.*

2) *If* $\mathbf{p}$ *is good then so is* $\mathbf{p}_A$*.*

[13] Note that this function is invertible as it is strictly increasing.

Claim 2.10. *Assume* $\mathbf{p}$ *is a* (λ,κ,χ)-BB^1*-parameter,* $\mathcal{D}_{\mathbf{p}} := [\kappa]^\kappa$*,* D *is a* $(2^\kappa)^+$*-complete filter on* λ*,* $\partial = \partial^{2^\kappa}$*, and* $\theta = \theta^{2^\kappa}$*.*

Then $\mathbf{p}$ *has the continuous* (D,∂,θ)-BB^1*-property* *iff* $\mathbf{p}_A$ *has the continuous* (D,∂,θ)-BB^1*-property for some* $A \in [\kappa]^\kappa$*.*

Proof. The $\Leftarrow$ implication is obvious, so we concentrate on $\Rightarrow$. Let $\mathbf{p}$ be a (λ,κ,χ)-BB^1-parameter and D be as above.

$\circledast_1$ Toward contradiction, assume that $\mathbf{p}_A$ fails the continuous (D,∂,θ)-BB^1-property for all $A \in [\kappa]^\kappa$.

$\circledast_2$ (a) So for $A \in [\kappa]^\kappa$, let
$$\overline{F}^1_A = \langle F^{A,1}_{\delta,i} : \delta \in S,\ i < \kappa\rangle$$
be a continuous $(\mathbf{p}_A,\partial,\theta)$-coloring witnessing this failure. (I.e. there is no $\mathbf{p}_A$-D-$\overline{F}_A$-BB^1-sequence $\bar{c} \in {}^{S\times\kappa}\theta$.)

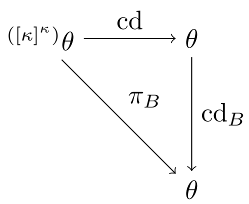
(b) Now $F^A_{\delta,i} : {}^{C_{\delta,i}}\partial \to \theta$ is defined as follows:

- $\bullet_1$ If $i \in A$ then $F^A_{\delta,i} := F^{A,1}_{\delta,\mathrm{otp}(i\cap A)}$.
- $\bullet_2$ If $i \in \kappa \setminus A$ then $F^A_{\delta,i}$ is the constantly zero.

Naturally, we choose

$\circledast_3$ (a) $\mathrm{cd} : {}^{([\kappa]^\kappa)}\theta \to \theta$ and $\mathrm{cd}^* : {}^{([\kappa]^\kappa)}\partial \to \partial$, both bijections.

(b) For $B \in [\kappa]^\kappa$, let $\mathrm{cd}_B : \theta \to \theta$ be defined so that the following diagram commutes:



where π_B is the function which sends $\langle \zeta_A : A \in [\kappa]^\kappa\rangle \mapsto \zeta_B$.

(c) $\mathrm{cd}^*_B : \partial \to \partial$ will be defined analogously.

Next,

$\circledast_4$ Choose $\overline{F} = \langle F_{\delta,i} : \delta \in S,\ i < \kappa\rangle$ as follows: $\mathrm{dom}(F_{\delta,i}) := {}^{C_{\delta,i}}\partial$, and for η in the domain we define
$$F_{\delta,i}(\eta) := \mathrm{cd}\big(\langle F^A_{\delta,i}(\eta) : A \in [\kappa]^\kappa\rangle\big).$$

By our assumption,

$\circledast_5$ There exists a $\mathbf{p}$-D-$\overline{F}$-BB^1-sequence $\bar{c} = \langle c_{\delta,i} : \delta \in S,\ i < \kappa\rangle \in {}^{S\times\kappa}\theta$.

Next,

$\circledast_6$ For every $A \in [\kappa]^\kappa$, we choose $\bar{c}_A := \big\langle \mathrm{cd}_A(c_{\delta,i}) : \delta \in S,\ i < \kappa\big\rangle$.

If $\bar{c}_A$ is a $\mathbf{p}_A$-D-$\overline{F}_A$-BB^1-sequence for some A, then we get our contradiction.

Therefore, assume:

$\circledast_7$ For each $A \in [\kappa]^\kappa$ there exist $\eta_A \in {}^\lambda\partial$ and $E_A \in D$ such that

$$(\forall\delta \in S \cap E_A)(\exists i < \kappa)\big[F^{A,1}_{\delta,h_A^{-1}(i)}(\eta_A \restriction C_{\delta,h_A^{-1}(i)}) \neq c^A_{\delta,h_A^{-1}(i)}\big].$$

(Equivalently, $(\forall\delta \in S \cap E_A)(\exists i \in A)\big[F^A_{\delta,i}(\eta_A \restriction C_{\delta,i}) \neq c^A_{\delta,i}\big]$.)

Now,

$\circledast_8$ $E := \bigcap\limits_{A\in[\kappa]^\kappa} E_A \in D.$

[Why? Because we assumed D is $(2^\kappa)^+$-complete.]

Next,

$\circledast_9$ Define $\eta \in {}^{([\kappa]^\kappa)}\partial$ as the function

$$\alpha \mapsto \mathrm{cd}^*\big(\langle \eta_A(\alpha) : A \in [\kappa]^\kappa \rangle\big).$$

Now we can finish as in the proof of 2.5. $\square_{2.10}$

Conclusion 2.11. *Assume clause (A) of Theorem 2.5. Also suppose $\mathcal{D}_{\mathbf{p}} = [\kappa]^\kappa$, $2^\kappa \le \mu$, and $\theta^{2^\kappa} = \theta$.*

For some $A \in [\kappa]^\kappa$, $\mathbf{p}_A$ has the $(D, 2^\mu, \theta)$-BB^1-property.

Proof. By 2.5 we know

$(*)_1$ $\mathbf{p}$ has the continuous $(D, 2^\mu, \theta)$-BB^1-property.

Let $\partial := 2^\mu$. We would like to apply 2.10, so let us check its assumptions. First, $\mathbf{p}$ is a (λ, κ, χ)-BB^1-parameter by 2.5(A)(d), one of our assumptions.

We also assumed $\mathcal{D}_{\mathbf{p}} = [\kappa]^\kappa$ and $\theta^{2^\kappa} = \theta$, so we don't have to worry about those.

'D is a μ^+-complete filter' was assumed in 2.5(A)(b) and we added $2^\kappa \le \mu$, so D is also $(2^\kappa)^+$-complete.

Lastly, $\partial^{2^\kappa} = \partial$ holds because we defined $\partial := 2^\mu$; hence $(2^\mu)^\mu = 2^\mu$, and again $2^\kappa \le \mu$.

Therefore the conclusion in 2.10 holds, giving us our desired conclusion. $\square_{2.11}$

Conclusion 2.12. 1) *We can add the following to the conclusion of* 2.5.

If $\lambda_* = \mathrm{cf}(\lambda_*) \ge \lambda$, <u>then</u> there exists a $\mathbf{p}^*$ such that

(a) $\mathbf{p}^*$ *is a good $(\lambda_*^+, \kappa, \chi)$-$\mathrm{BB}_*$-parameter.*
(b) $\mathbf{p}^*$ *has the continuous λ-uniform $(D, 2^\mu, \theta)$-BB^0-property.*

2) *If $\mathbf{p}$ is a (λ, κ, χ)-BB^0-parameter with the $(D, 2^\mu, \theta)$-BB^0-property, <u>then</u> $(\mathbf{p}, \{\kappa\})$ is a (λ, κ, χ)-BB^1-parameter with the $(D, 2^\mu, \theta)$-BB^1-property.*

Proof. 1) By [She91, §4], as in [She05, §2].

2) Easy, by the definitions. $\square_{2.12}$

Remark 2.13. We can say more in 2.12, replacing λ_*^+ by λ' weakly inaccessible or successor of singular: see [She05].

* * *

We are now ready to prove Theorem 0.4 as promised in the introduction.

> The reader would be well-advised to keep the statements of Theorem 0.4 on page 4, Definition 2.1(1) on page 18, and Lemma 2.5 on page 20 close at hand while reading this proof.

Proof of 0.4: Let us define a BB^1-parameter $\mathbf{p}$ (as in Definition 2.1) as follows:

- $\bullet_1$ $(\lambda_{\mathbf{p}}, \kappa_{\mathbf{p}}) := (\lambda, \kappa)$ from 0.4(A)(a)-(b) (so $\lambda > \kappa$ are regular, and so 2.1(1)(A)(a) holds).
- $\bullet_2$ $\chi := \min\{\sigma : 2^\sigma > \mu\}$ (so $\chi \leq \mu$).
- $\bullet_3$ $S_{\mathbf{p}} := S$, the stationary subset of S^λ_κ from 0.4(A)(b) (so 2.1(1)(A)(b) holds).
- $\bullet_4$ $C'_{\mathbf{p},\delta} := C'_\delta$ from 0.4(A)(c). They satisfy 2.1(1)(C)(a), and so we let $\bar\beta_\delta = \langle \beta_{\delta,i} : i < \kappa\rangle$ list its elements as in 2.1(1)(C)(b).
- $\bullet_5$ Let $C_{\mathbf{p},\delta} := \bigcup\limits_{i<\kappa} C^\bullet_{\beta_{\delta,i}}$, where $\langle C^\bullet_\beta : \beta < \lambda\rangle$ was given in 0.4(A)(d).
- $\bullet_6$ If $\beta \in C'_\delta \wedge \mathrm{otp}(C'_\delta \cap \beta) = i$ <u>then</u> we define $C_{\delta,i} := C^\bullet_\beta$.[14]
  - (a) Clearly each $C_{\delta,i} \subseteq C_{\mathbf{p},\delta} \cap \beta_{\delta,i}$, giving us 2.1(1)(D)(b).
  - (b) Also, $|C_{\delta,i}| < \chi$ as $2^{|C_{\delta,i}|} \leq 2^\mu$ by 0.4(A)(d), hence 2.1(1)(D)(c) holds.
- $\bullet_7$ $\mathcal{D}_{\mathbf{p}} := [\kappa]^\kappa$.

Now,

$\boxplus_1$ $\mathbf{p}$ is indeed a BB^1-parameter.

[Why? The only demand we have not already checked off our list is clause 2.1(1)(E). For a given $\alpha < \lambda$, the set defined there is a singleton $\{C^\bullet_{\beta_{\delta,i}}\}$, and so this easily holds.]

Let D be the club filter on λ.

$\boxplus_2$ All the assumptions in Lemma 2.5 have been satisfied.

Why? 2.5(A)(a) holds by 0.4(A). (We defined $\lambda := \mathrm{cf}(2^\mu)$ in 0.4(A)(b).)

Clause (A)(b) holds by our choice of D above.

For (A)(c), $\chi \leq \mu$ by $\bullet_2$. Recalling $\theta = \theta^{2^\kappa}$ and $\theta < \mu$, clearly $\kappa < \chi$. As $\lambda := \mathrm{cf}(2^\mu)$ $(> \mu)$, we have $\lambda \geq \chi$.

Clause (A)(d) demands that $\mathbf{p}$ is a BB^1-parameter, which we have just proved.

For (A)(e), note that $\theta \leq \mu$ by 0.4(A)(a). Let $\sigma < \chi$; necessarily, $2^\sigma \leq \mu$ by our choice of χ, so as $\theta < \mu$ we have $\theta^\sigma \leq 2^\mu$. Therefore $\theta^{<\chi} \leq 2^\mu$ as required.

Lastly, we have to prove (A)(f):

$$\mathrm{Sep}_3(\lambda; \mu, \mu, \partial, \theta, \kappa, \mathcal{D}).$$

[Why is this true? We shall prove it using 1.16, recalling $\mathcal{D} := [\kappa]^\kappa$. Now we need to verify that demands 1.16(A)-(D) hold. Clauses (A) and (D) were among our

[14] Note that this is well-defined, as the sequence $\langle \mathrm{otp}(\beta \cap C) : \beta \in C\rangle$ is strictly increasing for any set C of ordinals.

assumptions, and '$\mathbf{U}_{\mathcal{D}}(\mu) = \mu$' holds because this is equivalent to $\mathbf{U}_\kappa(\mu) = \kappa$ (as $\mathcal{D} := [\kappa]^\kappa$), which holds because $\alpha < \mu \Rightarrow |\alpha|^\kappa \leq \mu$.

The last of the four clauses is 1.16(B); if we have this, then we will be able to conclude $\mathrm{Sep}_3(\lambda; \mu, \mu, \partial, \theta, \kappa, \mathcal{D})$.

Flipping back to page 16, it says "The triple $(\mu, 2^\mu, \theta)$ satisfies at least one of the conditions in 1.7." Of the options, 1.7(b) holds by 0.4(A), and so we are done.]

We obtain the conclusion of 1.16, giving us 2.5(A)(f), and $\boxplus_2$ has been proven.

Now by the conclusion of Lemma 2.5, $\mathbf{p}$ has the continuous $(D, 2^\mu, \theta)$-BB^1-property, and so by 2.10, we know

- $\boxplus_3$ For some $A \in [\kappa]^\kappa$, $\mathbf{p}_A$ (as defined in 2.8) has the continuous $(D, 2^\mu, \theta)$-BB^1-property.

Now to finish the proof we need to provide a $\overline{C} = \langle C_\delta : \delta \in S \rangle$ satisfying 0.4(B). Let us choose $\langle C^A_\delta : \delta \in S \rangle$, where

$$C^A_\delta := \{\beta \in C_{\mathbf{p},\delta} : \mathrm{otp}(C_{\mathbf{p},\delta} \cap \beta) \in A\}.$$

Now check. $\square_{0.4}$

Proof of 0.5. Similar. $\square_{0.5}$

§ 3. The DBB Property

The following result relies on [She13b, 2.2$_{=\text{Ld.6}}$].

Theorem 3.1. *We have '(A) ⇒ (B)', where*

- (*A*) (*a*) $\lambda := \min\{\partial : 2^\partial > 2^\mu\}$ *(so $\lambda > \mu$ is regular).*
  - (*b*) *Let D be a μ^+-complete filter on λ extending the co-bounded filter.*
  - (*c*) $\bar{C} = \langle C_\gamma : \gamma < \lambda\rangle$, *where* $C_\gamma \subseteq \mu$.
  - (*d*) $\theta \in [2, \mu]$
  - (*e*) $\mathrm{Sep}_1(\mu, \mu, \theta, \theta, \Upsilon)$ *for some* $\Upsilon < 2^\mu$ *(or possibly* $\Upsilon := 2^\mu \in \mathrm{Reg}$ *as before).*
- (*B*) *If* $\mathbf{F}_\gamma : {}^{C_\gamma}(2^\mu) \to \theta$ *for* $\gamma < \lambda$ *then we can find a*

$$\bar{c} = \langle c_\gamma : \gamma < \lambda\rangle \in {}^\lambda\theta$$

such that for any $f : \mu \to 2^\mu$, *for* D^+*-many* $\gamma < \lambda$, *we have*

$$\mathbf{F}_\gamma(f \restriction C_\gamma) = c_\gamma.$$

Proof. By [She13b, 2.2(β)$_{=\text{Ld.6}}$]. $\square_{3.1}$

Definition 3.2. Suppose $\lambda = \mathrm{cf}(\lambda) > \mu \geq \kappa = \mathrm{cf}(\kappa)$ and $\mu_* \leq \mu^+$.

1) We say that $\mathbf{p}$ is a $(\lambda, \lambda_*, \mu, \mu_*, \kappa)$-*DBB-parameter*[15] when:

- (A) (a) $\lambda \geq \lambda_* > \kappa$ are regular cardinals.
  - (b) $S \subseteq S^\lambda_\kappa$ is a stationary subset of λ.
- (B) $\mathbf{p}$ consists of $\bar{C} = \bar{C}_0 = \langle C_\delta : \delta \in S\rangle$ and $\bar{C}_1 = \langle C^\delta_\gamma : \delta \in S,\ \gamma < \lambda_*\rangle$ such that
  - (a) $\langle C_\delta : \delta \in S\rangle$ is as usual (that is, $C_\delta \subseteq \delta = \sup(C_\delta)$ and $\mathrm{otp}(C_\delta) = \kappa$), but we add the demand

$$\alpha \in C_\delta \Rightarrow \alpha > \mu \wedge \mu \,\big|\, \alpha.$$

  - (b) $C^\delta_\gamma \subseteq \bigcup\limits_{\alpha \in C_\delta} [\alpha, \alpha + \mu)$ such that

$$\big|C^\delta_\gamma \cap [\alpha, \alpha + \mu)\big| = 1$$

  for all $\delta \in S$, $\gamma < \lambda_*$, and $\alpha \in C_\delta$. (So $\mathrm{otp}(C^\delta_\gamma) = \kappa$.)
  - (c) $\bar{C}_1$ is a μ_**-free sequence.*
  By this we mean: if $u \subseteq S \times \lambda_*$ is of cardinality $\leq \mu_*$, then there exists some sequence $\bar{\beta} = \langle \beta^\delta_\gamma : (\delta, \gamma) \in u\rangle$ with $\beta^\delta_\gamma \in C^\delta_\gamma$ such that

$$\langle C^\delta_\gamma \setminus \beta^\delta_\gamma : (\delta, \gamma) \in u\rangle$$

  is a sequence of pairwise disjoint sets.

2) We say that $\mathbf{p}$ has the $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$-*DBB-property* when in addition to the above,

- (C) If $\mathbf{F}^\delta_\gamma : {}^{C^\delta_\gamma}(2^\mu) \to \theta$ for $\gamma < \lambda_*$ and $\delta \in S$, then we can find sequences

$$\bar{c}^\delta = \langle c^\delta_\gamma : \gamma < \lambda_*\rangle \in {}^{\lambda_*}\theta$$

such that for any $\delta \in S$ and $f : \delta \to 2^\mu$, for some $\gamma < \lambda_*$, we have

$$\mathbf{F}^\delta_\gamma(f \restriction C^\delta_\gamma) = c^\delta_\gamma.$$

[15] DBB stands for *Double Black Box.*

3) If we say $\mathbf{p}$ *guesses clubs*, we mean $\overline{C}_{\mathbf{p}}$ does.

If $\lambda_* := \lambda$ then we may omit it. Similarly if $\mu_* := \mu^+$.

Remark 3.3. 1) In Definition 3.2, we can make the following changes:

- $\bullet_1$ In clause (1)(B), we add
  (B)(d) $\overline{D} = \langle D_\delta : \delta \in S\rangle$, with each D_δ a filter on λ_*.
- $\bullet_2$ Then in clause (2)(C), we replace "for some $\gamma < \lambda_*$" by 'for D_δ-many $\gamma < \lambda_*$.'

2) Adopting this change, we would add an additional clause to Claim 3.4(1)$\boxplus$:

$\boxplus(f)$ $\bullet_1$ $\overline{D} = \langle D_\delta : \delta \in S\rangle$, with each D_δ a filter on λ_*.
$\bullet_2$ $\lambda_\bullet \in D_\delta$
$\bullet_3$ $D_\delta \restriction \lambda_\bullet$ is a μ^+-complete filter on $\lambda_\bullet$ extending the co-bounded filter.

3) The proof of 3.4 would not change.

Claim 3.4. 1) *If $\boxplus$ below holds, <u>then</u> there exists a $\mathbf{p}$ with the $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$-DBB-property.*

$\boxplus$ (a) $\kappa = \mathrm{cf}(\mu) < \mu$
(b) $\lambda_\bullet \leq \lambda_* \leq \lambda$, *where λ is regular and* $\lambda_\bullet := \min\{\partial : 2^\partial > 2^\mu\}$.
(c) $\bullet_1$ $\mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu) > \lambda$ *and* $\mu_* := \mu^+$,
<u>or</u>
$\bullet_2$ $\mu_* \leq \mu$ *and there exists a μ_*-free subset of ${}^\kappa\mu$ of cardinality λ (see* §4*).*
(d) $\theta \in [2, \mu]$
(e) $\mathrm{Sep}_1(\mu, \theta)$.

1A) *If* (1)$\boxplus$ *holds and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa$ and $\mu^2 \mid \delta\}$ is stationary in λ, <u>then</u> there exists a $\mathbf{p}$ with the $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$-DBB-property and $S_{\mathbf{p}} = S$.*

1B) *If we assume $S \in \check{I}_\kappa[\lambda]$ then*[16] *we can add "$\overline{C}_{\mathbf{p}}$ is (λ, κ)-good" (see* 1.2(6)*) and "$S_{\mathbf{p}} := S$."*

1C) *Moreover, in* (1B) *we could replace '$S_{\mathbf{p}} := S$' by '$S_{\mathbf{p}} := S \cap E$ for some club $E \subseteq \lambda$' and add 'if $\beta \in C_{\mathbf{p},\delta_1} \cap C_{\mathbf{p},\delta_2}$ then $C_{\mathbf{p},\delta_1} \cap \beta = C_{\mathbf{p},\delta_2} \cap \beta$.'*

1D) *In both* (1A) *and* (1B)*, we may add "$\overline{C}_{\mathbf{p}}$ guesses clubs."*

1E) *Note that if μ is (e.g.) strong limit singular, then* (1) $\boxplus(c)$ *holds even if* $\mathrm{cf}(\mu) = \aleph_0$.

2) *In part* (1)*, we may replace clause* $\boxplus\,(c)$ *by*

$(c)'$ $\bullet_1$ $\mathrm{pp}_J(\mu) > \lambda$ *for some ideal* $J \supseteq [\kappa]^{<\kappa}$.
$\bullet_2$ $\mu = \mu^{<\kappa}$

[16] Recall that such an S exists because λ and κ are regular with $\lambda > \kappa^+$ (as $\lambda > \mu > \kappa$), and so we can apply [She93, §1].

2A) *We can adopt* $\bullet_1$ *above if we weaken clause* 3.2(1)(B)(c) *to* "(μ_*, J)*-free.*"

3) *Alternatively,*

(c)″ $\bullet_1$ *As above.*
$\bullet_2$ $2^{\mu^{<\kappa}} < 2^\mu$.

Remark 3.5. 1) Concerning the Double Black Box property in Definition 3.2(2), we may allow $\mathbf{F}^\delta_\gamma$ to have domain ${}^{\mathcal{C}^\delta_\gamma}(2^\mu)$, where $\mathcal{C}^\delta_\gamma := \bigcup_{\beta \in C^\delta_\gamma} [\beta, \beta + \mu)$, and range $\subseteq \theta$. (Alternatively, we may use $\mathcal{C}^\delta_\gamma := \big\{\beta : (\exists \beta' \in C^\delta_\gamma)[\beta + \mu = \beta' + \mu]\big\}$.)

[Why? Because $\big|{}^\mu(2^\mu)\big| = 2^\mu$.]

2) Assume $\lambda := 2^\mu \notin$ Reg. Then the proof still works for a weaker version of the DBB property, where 'λ is regular' (in 3.2(1)(A)(a)) is replaced by '$\mathrm{cf}(\lambda) > \kappa$.' (Of course, we still demand that λ_* and κ are regular.)

3) By §4, if μ is strong limit singular above then we can have $\mu_* := \mu^+$.

Proof. 1) First, choose a stationary $S \subseteq S^\lambda_\kappa$ such that

$$\delta \in S \Rightarrow \mu^2 \,\big|\, \delta.$$

Next, choose $\overline{C}$ as in 3.2(1)(B)(a); this is possible by our choice of S. Third, by $\boxplus$, choose a μ_*^+-free sequence

$$\langle \rho_\gamma : \gamma < \lambda \rangle \subseteq {}^\kappa\mu.$$

[Why can we do this? In subclause $\boxplus(c)\bullet_1$, this follows from [She94, Ch.II, §3]; if $\bullet_2$ holds then this is obvious.]

Without loss of generality $\gamma < \lambda \Rightarrow \rho_\gamma(i) = i \mod \kappa$. Let $\langle \rho^*_{\delta,\gamma} : \delta \in S,\ \gamma < \lambda \rangle$ list $\langle \rho_\gamma : \gamma < \lambda \rangle$ without repetition: we can do this because $|S \times \lambda| = \lambda$.

Let $\langle \beta^\delta_i : i < \kappa \rangle$ list C_δ in increasing order, and let

- $\rho_{\delta,\gamma} := \langle \beta^\delta_i + \rho^*_{\delta,\gamma}(i) : i < \kappa \rangle$
- $C^\delta_\gamma := \mathrm{rang}(\rho_{\delta,\gamma})$.

Let D be the club filter on λ. So

$$\mathbf{p} := (\lambda, \kappa, S, D, \langle C_\delta : \delta \in S \rangle, \langle C^\delta_\gamma : \delta \in S,\ \gamma < \lambda \rangle)$$

is well-defined.

Next we have to check that $\mathbf{p}$ is indeed a $(\lambda, \lambda_*, \mu, \mu_*, \theta, \kappa)$-DBB-parameter: that is, all clauses of 3.2(1).

First, the demands on the cardinals in the beginning of the definition hold, as does clause (A).

Clause 3.2(B)(a): Holds by the choice of $\langle C_\delta : \delta \in S \rangle$.

Clause (B)(b): Holds by our choice of the C^δ_γ-s.

Clause (B)(c):

Let $u \in [S \times \lambda]^{<\mu_*}$. By the choice of $\langle \rho_\gamma : \gamma < \lambda \rangle$, we can find a function $h : u \to \kappa$ such that

$(*)_1$ If $(\delta_1,\gamma_1) \neq (\delta_2,\gamma_2)$ are from u and $i \geq \max\big(h(\delta_1,\gamma_1), h(\delta_2,\gamma_2)\big)$, <u>then</u> $\rho^*_{\delta_1,\gamma_1}(i) \neq \rho^*_{\delta_2,\gamma_2}(i)$.

Hence

$(*)_2$ If $(\delta_1,\gamma_1) \neq (\delta_2,\gamma_2)$ are from u and $i_\ell \in \big[h(\delta_\ell,\gamma_\ell),\kappa\big)$ for $\ell = 1,2$, <u>then</u>
$$\rho^*_{\delta_1,\gamma_1}(i_1) \neq \rho^*_{\delta_2,\gamma_2}(i_2).$$

[Why? If $i_1 \neq i_2$ then recall $\gamma < \lambda \Rightarrow \rho_\gamma(i) \equiv i \mod \kappa$. The $i_1 = i_2$ case is just $(*)_1$.]

So clause (B)(d) does indeed hold. Together we have proved that $\mathbf{p}$ is a $(\lambda,\lambda_*,\mu,\mu_*,\theta,\kappa)$-DBB-parameter.

Lastly, why does it have the $(\lambda,\lambda_*,\mu,\mu_*,\theta,\kappa)$-DBB-property?

Let $\langle \mathbf{F}^\delta_\gamma : \gamma < \lambda_*,\ \delta \in S\rangle$ be given, as in 3.2(2)(C). To finish we need to produce $\bar{c}_\delta = \langle c^\delta_\gamma : \gamma < \lambda_*\rangle$ as required there.

Fix $\delta \in S$, and let $h_\delta : \mu \to \delta$ be such that
$$\varepsilon < \mu \wedge \varepsilon \equiv i \bmod \kappa \Rightarrow h_\delta(\varepsilon) := \beta_{\delta,i} + \varepsilon.$$
As $\langle \beta_{\delta,i} : i < \kappa\rangle$ is increasing and $\mu \,\big|\, \beta_{\delta,i}$ for all $i < \kappa$, we know h_δ is well-defined and one-to-one.

Let $\overline{C}'_\delta = \langle C'_{\delta,\gamma} : \gamma < \lambda_\bullet\rangle$ be defined as
$$C'_{\delta,\gamma} := \big\{\varepsilon < \mu : h_\delta(\varepsilon) \in C^\delta_\gamma\big\}.$$

For $\gamma < \lambda_\bullet$, define $F_{\delta,\gamma} : {}^{C'_{\delta,\gamma}}(2^\mu) \to \theta$ as the function
$$f \mapsto \mathbf{F}^\delta_\gamma(f \circ h_\delta).$$
By 3.1 applied to $\overline{C}'_\delta$ and $\langle F_{\delta,\gamma} : \gamma < \lambda_\bullet\rangle$, we get a sequence of ordinals $\langle c^\gamma_\delta : \gamma < \lambda_\bullet\rangle$ as guaranteed there.

As λ_* may be any cardinal in the interval $[\lambda_\bullet,\lambda]$, we need to pad out this sequence with extra terms. Simply let $c^\gamma_\delta := 0$ (and $F_{\delta,\gamma}$ be identically zero) for $\gamma \in [\lambda_\bullet,\lambda_*)$, and we reader may check that the resulting sequence is as desired.

1A) Similarly.

1B-C) By the definition of $\check{I}_\kappa[\lambda]$ and Claim 0.11.

1D) Use [She94, Ch.III, §1] and 0.11(3).

1E) By §4.

2) Similarly, but when choosing $\bar{\rho} = \langle \rho_\gamma : \gamma < \lambda\rangle$ we only require that it is (μ^+, J)-free.

Then we let $\mathrm{cd} : {}^{\kappa>}\lambda \to \lambda$ be a bijection.

3) Similarly as well. $\square_{3.4}$

Discussion 3.6. Let μ be strong limit singular of cofinality $\kappa < \mu$, and $\lambda = \lambda_* := \min\{\partial : 2^\partial > 2^\mu\}$.

(A) (a) If $\lambda < 2^\mu$ and $\kappa > \aleph_0$, then 3.4 $\boxplus$ holds (see [She94]).

(b) What about $\lambda < 2^\mu$ and $\kappa := \aleph_0$? Still, we knew 3.4$\boxplus$ held in many cases (e.g. for a club of $\mu < \mu_*$, where $\mu_* := \beth_\delta > \kappa := \mathrm{cf}(\mu_*) > \aleph_0$). In §4 we shall prove that it *always* holds.

(c) 3.4 would seem to be helpful for constructing (e.g.) μ^+-free Abelian groups.

(B) But what about the $\lambda = \lambda_* = 2^\mu$ case? In this case we have $\lambda = \lambda^{<\lambda}$, a condition which is again helpful in constructions. Can we construct an entangled linear order of cardinality λ^+? Recall that by [She00] or [She06] we have $(D\ell)^*_\Lambda$. Can we use several pairwise disjoint subsets of λ?

Alternatively, find a subset of ${}^\lambda\theta$ for some regular θ (e.g. $\mathrm{cf}(2^{\aleph_0})$)?

(C) Again, if $\lambda = 2^\mu$ then we may try to use

$$\mathfrak{d} := \big\{\theta \in \mu \cap \mathrm{Reg} : \big(\exists \mu' \in (\mu, \lambda)\big)\big[\mathrm{cf}(\mu') = \theta \wedge \mathrm{pp}_{\theta\text{-comp}}(\mu') =^+ \lambda\big]\big\}$$

as in [She13b] whenever $\mathfrak{d} := \{\kappa\}$ does not work. The new proof is as in [She20], using [She13a].

(D) However, we can use BB_k in clause (C). We consider $\mu_0 < \ldots < \mu_{3n}$ as above (i.e. all strong limit of cofinality $\kappa < \mu_0$). For each ℓ we choose $\mathbf{p}_\ell$ as in 3.4, except that their free-ness (in the sense of [She20]) is such that their "product" is $\aleph_{n\cdot\kappa^+}$-free, and they have a Black Box as there.

Definition 3.7. 1) For $\Lambda_* \subseteq \Lambda_\bullet \subseteq {}^\kappa\mu$, we say that $\Lambda_\bullet$ is (θ_2, θ_1)-*free over*[17] Λ_* <u>when</u> $\theta_2 \geq \theta_1$ and for every $\Lambda \subseteq \Lambda_\bullet \setminus \Lambda_*$ of cardinality $< \theta_2$ there is a *witness* $(\bar\Lambda, h)$. By this we mean:

(A) $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ is a partition of Λ into γ_*-many sets, each of cardinality $< \theta_1$ (so γ_* is an ordinal $< \theta_2$).

(B) $h : \Lambda \to \kappa$.

(C) If $\gamma < \gamma_*$, $\eta \in \Lambda_\gamma$, and $i \in \big[h(\eta), \kappa\big)$, <u>then</u>

$$\eta(i) \notin \big\{\rho(j) : j < \kappa,\ \rho \in \bigcup_{\beta<\gamma} \Lambda_\beta \cup \Lambda_*\big\}.$$

2) For $\Omega \subseteq \big\{(\theta_2, \theta_1) \in \mathrm{Card} \times \mathrm{Card} : \theta_2 \geq \theta_1\big\}$, we say $\Lambda_\bullet$ is Ω-free over Λ_* <u>when</u> it is (θ_2, θ_1)-free over Λ_* for every $(\theta_2, \theta_1) \in \Omega$.

3) Suppose $\lambda = \mathrm{cf}(\lambda) > \mu \geq \kappa = \mathrm{cf}(\kappa)$ and Ω is as in part (2).

We say that $\mathbf{p}$ is a $(\lambda, \lambda_*, \mu, \Omega, \kappa)$-*DBB-parameter* <u>when</u> clauses (A) and (B)(a)-(b) of Definition 3.2 hold, and

(B)(c)′ $\overline{C}_1$ is Ω-free.

Observation 3.8. *Assume (for transparency) that $\Lambda_\bullet \subseteq {}^{\kappa>}\mu$ is tree-like. (That is, $\eta \neq \nu \in \Lambda_\bullet \wedge \eta(i) = \nu(j) \ \Rightarrow\ i = j \wedge \eta \restriction i = \nu \restriction i$.)*

If $\Lambda_\bullet$ is of cardinality $< \theta$ and (θ, κ^+)-free over $\varnothing$, <u>then</u> $\Lambda_\bullet$ is free.

Proof. See [She20, §1]. $\square_{3.8}$

Claim 3.9. 1) *If $\boxplus$ holds <u>then</u> there exists a $\mathbf{p}$ with the $(\lambda, \lambda, \mu, \Omega, \theta, \kappa)$-DBB-property, where*

[17] We may omit λ_* if it is empty.

$\boxplus$ (a) $\kappa = \mathrm{cf}(\mu) < \mu$

(b) $\lambda = \lambda^{<\lambda} = 2^\mu$

(c) $\mathrm{pp}^+_{J^{\mathrm{bd}}_\kappa}(\mu) > \lambda$

(d) $\theta \in [2, \mu]$

(e) $\mathrm{Sep}(\mu, \theta, \Upsilon)$ *for some* $\Upsilon \le \mu$.

(f) $\Omega := \{(\kappa^{+\kappa}, \kappa^{+4})\}$

2) *Like part* (1), *but replacing clause* $\boxplus(f)$ *by*

$(f)'$ $\Omega := \big\{(\theta^{+\kappa}, \theta^{+4}) : \theta \in [\kappa, \mu)\big\}$

3) *In parts* (1) *and* (2), *we my replace clause* $\boxplus(c)$ *by*

$(c)'$ $\bullet_1$ $\mathrm{pp}_J(\mu) \ge \lambda$ *for some ideal* $J \supseteq [\kappa]^{<\kappa}$.

$\bullet_2$ $\mu = \mu^{<\kappa}$

as in 3.4(2).

4) *If* S *is a stationary subset of* $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ *then we can demand* $S_{\mathbf{p}} := S$, *and we can add "*$\overline{C}_{\mathbf{p}} = \langle C_{\mathbf{p},\delta} : \delta \in S\rangle$ *guesses clubs." If* $S \in \check{I}_\kappa[\lambda]$ *then we can add "*$\overline{C}_{\mathbf{p}}$ *is* (λ, κ)*-good."*

5) *In part* (2), *we can replace* $\mu = \mu^{<\kappa}$ *by* $2^{\mu^{<\kappa}} < 2^\lambda$.

Remark 3.10. The $\lambda = \lambda^{<\lambda}$ is not necessary; just otherwise 3.4 gives us more.

Proof. 1) Like the proof of 3.4, but in the choice of $\bar{\rho}$ (at the beginning of the proof) we replace 'μ^+-free' by 'Ω-free.'

[Why is this possible? Use θ in the beginning of the proof of [She20, $1.26_{=\mathrm{La51}}$] (which relies on [She13a, $0.4\text{-}0.6_{=\mathrm{Ly19,y22,y40}}$]).]

2) As above.

3) Similarly to the proof of 2.5(2).

4-5) Clear. $\square_{3.9}$

§ 4. When do quite free subsets $\Lambda \subseteq {}^{\kappa}\mu$ exist?

We know that if $\kappa := \mathrm{cf}(\mu) < \mu < \lambda < \mathrm{pp}_J(\mu)$, where J is an ideal on κ, then there exists a $<_J$-increasing (μ^+, J)-free sequence $\bar{f} \in {}^{\lambda}({}^{\kappa}\mu)$. This implies that if μ is strong limit singular of uncountable cofinality and $\lambda \in (\mu, 2^\mu)$, then there is a μ^+-free subset $\Lambda \subseteq {}^{\kappa}\mu$ of cardinality λ. This also holds for many cardinals μ with $\mathrm{cf}(\mu) = \aleph_0$ (see [She94]). This has applications for (e.g.) the existence of μ^+-free Abelian groups with trivial dual (The TDC_μ).

We intend to prove this for all μ-s.

This is an example of the thesis "assuming negations of GCH may help in proving interesting results." Above, we considered singular cardinals. What about regulars? If $2^\theta = 2^{<\mu} < 2^\mu$ (hence $\theta < \mu$) then by an old result with Devlin [DS78] we have Weak Diamond on μ. For results from this century, [CS16] with Chernikov showed that by pcf considerations, for every μ there exist $\mu_0 := \mu < \mu_1 < \ldots < \mu_n := 2^\mu$ for $n \leq 6$ such that $\mathrm{trp}^+_{\kappa_\ell}(\mu_\ell) > \mu_{\ell+1}$ for $\ell < n$ (where $\kappa_\ell := \mathrm{cf}(\mu_\ell) < \mu_\ell$). (See 4.2 below.)

Can we get freeness results? Here we see if we can get "for every $\ell < n$ there exists a quite free subset $\Lambda_\ell \subseteq \lim\limits_{\kappa_\ell} \mathcal{T}$ of cardinality μ_ℓ."

A characteristic neat result (referenced in §0) is as follows.

Theorem 4.1. *If μ is strong limit singular, $\kappa := \mathrm{cf}(\mu)$, and $\lambda \in (\mu, 2^\mu)$, then there is a μ^+-free subset $\Lambda \subseteq {}^{\kappa}\mu$ of cardinality λ.*

Recall:

Definition 4.2. 1) For $\kappa = \mathrm{cf}(\kappa) \leq \mu$, let $\mathrm{trp}^+_\kappa(\mu)$ be the minimal λ such that there is no sub-tree $\mathcal{T} \subseteq {}^{\kappa>}\mu$ of cardinality μ with $\geq \lambda$-many κ-branches.

(trp stands for *tree power.*)

2) For J an ideal on $\kappa < \mu$ and $\Lambda \subseteq {}^{\kappa}\mu$, we say Λ is (λ, J)-*free* when for every $\Lambda' \in [\Lambda]^{<\lambda}$ there exists $f : \Lambda' \to J$ such that

$$\eta \neq \nu \in \Lambda' \wedge i \in \kappa \setminus \big(f(\eta) \cup f(\nu)\big) \Rightarrow \eta(i) \neq \nu(i).$$

3) For J an ideal on κ and $\bar{A} = \langle A_\varepsilon : \varepsilon < \kappa \rangle$, let $T^+_J(\bar{A})$ be the minimal λ such that there is no[18] $\Lambda \subseteq \prod\limits_{\varepsilon<\kappa} (A_\varepsilon \cup \{0\})$ of cardinality λ such that

$$\eta \neq \nu \in \Lambda \Rightarrow \big\{i < \kappa : 0 \neq \eta(i) \neq \nu(i) \neq 0\big\} \equiv \kappa \mod J.$$

3A) Let $\mathbf{T}^+_J(\bar{A})$ be the minimal λ such that there is no Λ satisfying $\boxplus_{\bar{A},\Lambda}$ below.

$\boxplus_{\bar{A},\Lambda}$ (a) $\Lambda \subseteq \bigcup\limits_{a \in J^+} \prod\limits_{i \in a} (A_i \cup \{0\})$

(b) $|\Lambda| \geq \lambda$

(c) $\eta \neq \nu \in \Lambda \Rightarrow \big\{i \in \mathrm{dom}(\eta) \cap \mathrm{dom}(\nu) : \eta(i) = \nu(i)\big\} \in J.$

[18] We add the $\{0\}$ just so we don't have to worry about $A_\varepsilon := \varnothing$.

3B) Above, if $\bar{A}$ is the constant sequence $\langle B : \varepsilon < \kappa\rangle$, then we may write $T^+_J(B)$ and $\mathbf{T}_J(B)$.

3C) The default value of J is $[\kappa]^{<\kappa}$. If we omit J, this is what we mean.

4) For $\chi \geq \lambda \geq \theta \geq \sigma$, $\mathrm{cov}(\chi,\lambda,\theta,\sigma)$ means that there exists a $\mathscr{P} \subseteq [\lambda]^{<\theta}$ of cardinality $\leq \chi$ such that every $u \in [\lambda]^{<\theta}$ is contained in the union of $< \sigma$-many members of $\mathscr{P}$.

Observation 4.3. 1) *Without loss of generality we may strengthen* 4.2(2) *to*

$$\eta \neq \nu \in \Lambda' \wedge \big[i \in \kappa \setminus f(\eta)\big] \wedge \big[j \in \kappa \setminus f(\nu)\big] \Rightarrow \eta(i) \neq \nu(j).$$

2) *Similarly in* 4.2(3).

3) *If* $\mu = \mu^{<\kappa} < \mu^\kappa$ *then* $\mathrm{trp}_\kappa(\mu) = \mu^\kappa$ *(and even* $\mathrm{trp}^+_\kappa(\mu) = (\mu^\kappa)^+$*). So if* μ *is strong limit of cofinality* κ *then* $\mathrm{trp}_\kappa(\mu) = 2^\mu$*.*

4) *If* $\chi \geq \lambda \geq \theta > \sigma$ *and* σ *is regular uncountable, then* $\mathrm{cov}(\chi,\lambda,\theta,\sigma)$ *holds* <u>*iff*</u>

$$\chi \geq \sup\big\{\mathrm{pp}_{\mathrm{cf}(\mu)\text{-comp}}(\mu) : \mu \in [\theta,\lambda],\ \mathrm{cf}(\mu) \in [\sigma,\theta)\big\}.$$

5) *Assume* $\bar{A} = \langle A_\varepsilon : \varepsilon < \kappa\rangle$ *with* $|A_\varepsilon| \geq 2^\kappa$ *and* J *an ideal on* κ *such that*

$$u \in J^+ \Rightarrow T^+_J(\bar{A}) = T^+_{J\restriction u}(\bar{A} \restriction u).$$

Then $\mathbf{T}^+_J(\bar{A}) = T^+_J(\bar{A})$ <u>*or*</u> $\mathbf{T}^+_J(\bar{A}) = \sigma$ *and* $T^+_J(\bar{A}) = \sigma^+$ *for some* σ *of cofinality* $\leq 2^\kappa$*.*

6) *In part* (5)*, if* $J := [\kappa]^{<\kappa}$ *then* $\mathbf{T}^+_J(\bar{A}) = T^+_J(\bar{A})$*.*

7) *In parts* (5) *and* (6)*, we may replace* 2^κ *by* $\mathrm{cf}(\mathcal{P}(\kappa) \setminus J, \subseteq)$*.*

Proof. 1-2) As $\mu \geq \kappa$, we can define the function $\eta' \in {}^\kappa\mu$ by $\eta'(i) := \mathrm{pr}(\eta(i), i)$ for $i < \kappa$ (where $\mathrm{pr} : \mu \times \mu \to \mu$ is some bijection), and then replace η by η'.

3) Classical.

4) By [She94].

5) Clearly $\mathbf{T}^+_J(\bar{A}) \geq T^+_J(\bar{A})$. To get the '$\leq$' direction, assume

$\circledast_1$ $\chi \in \big(2^\kappa, \mathbf{T}^+_J(\bar{A})\big)$ (or just $\big|\mathcal{P}(\kappa)/J\big| < \chi < \mathbf{T}^+_J(\bar{A})$).

For $\alpha < \chi$, let $\eta_\alpha \in \prod\limits_{i \in u_\alpha} A_i$ (where $u_\alpha \in J^+$) witness $\circledast_1$.

For each $u \in J^+$, define $W_u := \{\alpha < \chi : u_\alpha = u\}$. If for some u the set W_u has cardinality χ, then clearly $T^+_{J\restriction u}(\bar{A} \restriction u) > \chi$ and we are done.

So assume

$$u \in J^+ \Rightarrow |W_u| < \chi;$$

recalling $\chi > \big|\mathcal{P}(\kappa)/J\big|$, necessarily $\mathrm{cf}(\chi) \leq \big|\mathcal{P}(\kappa)/J\big|$ and $\chi = \sup\big\{|W_u| : u \in J^+\big\}$.

Now $\langle \eta_\alpha : \alpha \in W_u\rangle$ essentially witnesses '$T^+_{J\restriction u}(\bar{A} \restriction u) \geq |W_u|$,' and we can easily finish.

(Note that if $\mathbf{T}^+_J(\bar{A}) > \chi^+$ then we could have used χ^+ instead of χ.)

6) for $A \in J^+ = \mathcal{P}(\kappa) \setminus J$, let $h_A : \kappa \to A$ be the inverse of the function

$$i \mapsto \mathrm{otp}(A \cap i).$$

For each $A_i \in \bar{A}$ (that is, for each $i < \kappa$), let pr_i be a bijection between $(A_i \cup \{0\}) \times J^+$ and A_i.

Beginning as in the proof of part (3), for each $\alpha < \chi$ we define $\eta'_\alpha \in \prod\limits_{i<\kappa} A_i$ as follows.

$\circledast_\alpha$ For $\alpha < \chi$ and $i < \kappa$, we let $\eta'_\alpha(i) := \mathrm{pr}_i\big(h_{u_\alpha}(i), u_\alpha\big)$

Now $\langle \eta'_\alpha : \alpha < \chi \rangle$ witnesses '$\mathbf{T}^+_J(\bar{A}) > \chi^+$.'

7) Left to the reader. $\square_{4.3}$

Now comes the section's main results: 4.4, 4.5, 4.6. (The reader may concentrate on the case '$\kappa_* := \kappa$, $\mu_* := \mu$' and on part (1) of 4.4.)

Claim 4.4. 1) *If clauses (a)-(e) below hold, then there is a (μ_*^+, J)-free subset $\Lambda \subseteq {}^\kappa\mu$ of cardinality λ.*

(a) *$\kappa_* \le \kappa$ are regular, and J is a κ_*-complete ideal on κ.*
(b) $\kappa := \mathrm{cf}(\mu) < \mu_* \le \mu \le \lambda \le \chi$
(c) $\mu^\kappa \ge \chi^+$
(d) $\mathrm{cov}(\chi, \lambda, \mu_*, \kappa_*^+)$
(e) $\alpha < \mu_* \Rightarrow |\alpha|^\kappa < \mu$

2) *In part (1), we may replace clauses (c) and (e) by*

(c)′ $T^+_J(\mu) > \chi^+$
(e)′ $\alpha < \mu \Rightarrow \mathbf{T}^+_J(|\alpha|) \le \chi^+$.

3) *If we weaken '$T^+_J(\mu) > \chi^+$' in clause (c)′ to 'There exists a sequence of ordinals $\bar{\alpha}$ with $\mu = \bigcup\limits_{j<\kappa} \alpha_j$, each $\alpha_i < \mu$, and $\mathbf{T}^+_J(\bar{\alpha}) > \chi^+$,' then we may still conclude that there is a μ_*^+-free subset of $\bigcup\limits_{u \in J^+} \prod\limits_{i \in u} \alpha_i$.*

Proof. 1) We shall prove that part (1) follows from part (2). To that end, we need to prove that assumptions (1)(a)-(e) imply (2)(c)′,(e)′.

Concerning clause (c)′: as $\mathrm{cf}(\mu) = \kappa$, there exists an increasing sequence $\langle \mu_i : i < \kappa \rangle$ with limit μ. By clause (e), without loss of generality $\mu_i = (\mu_i)^\kappa > 2^\kappa$. By clause (1)(c) and basic cardinal arithmetic,

$$\Big|\prod_i \mu_i\Big| = \Big|\sum_i \mu_i\Big|^\kappa = \mu^\kappa.$$

Hence there is a sequence $\langle \eta_\alpha : \alpha < \chi^+ \rangle$ of members of $\prod\limits_{i<\kappa} \mu_i$ without repetition, and a one-to-one function $\mathrm{cd}_i : {}^i(\mu_i) \to \mu_i$ for each $i < \kappa$. For $\alpha < \lambda$, we define $\nu_\alpha \in {}^\kappa\mu$ by

$$i < \kappa \Rightarrow \nu_\alpha(i) := \mathrm{cd}_i(\eta_\alpha \restriction (i+1)).$$

Easily, $\alpha < \beta < \chi^+ \Rightarrow \big|\{i < \kappa : \nu_\alpha(i) = \nu_\beta(i)\}\big| < \kappa$. So by the assumption on J, the sequence $\langle \nu_\alpha : \alpha < \chi^+ \rangle$ witnesses $T^+_J(\mu) > \chi^+$.

As for clause (2)$(e)'$, it follows immediately from (1)(e).

2) First,

$(*)_1$ Let $\mathscr{P} \subseteq [\lambda]^{<\mu_*}$ be the covering guaranteed by Definition 4.2(4), recalling that we assumed $\mathrm{cov}(\chi, \lambda, \mu_*, \kappa_*^+)$ in clause (1)(d). (So in particular, $|\mathscr{P}| \le \chi$.)

$(*)_2$ Let $\Lambda^* \subseteq {}^\kappa\mu$ witness $T_J^+(\mu) > \chi^+$ (as assumed in clause (2)$(c)'$). Note this implies $|\Lambda^*| \ge \chi^+$.

We shall try to choose η_α and $\bar{a}_\alpha$ by induction on $\alpha < \lambda$ such that

$(*)_3^\alpha$
(a) $\eta_\alpha \in \Lambda^*$
(b) $\bar{a}_\alpha = \langle a_u^\alpha : u \in \mathscr{P}\rangle$
(c) $a_u^\alpha \in J$
(d) if $u \in \mathscr{P}$, $\beta \in u \cap \alpha$, and $i \in \kappa \setminus \big(a_u^\alpha \cup a_u^\beta\big)$, <u>then</u> $\eta_\alpha(i) \neq \eta_\beta(i)$.

Assuming we have succeeded up to Stage α, for each $u \in \mathscr{P}$ let

$$S_{\alpha,u} := \big\{\eta_\beta(i) : \beta \in u \cap \alpha,\ i \in \kappa \setminus a_u^\beta\big\}.$$

So $S_{\alpha,u} \in [\mu]^{<|u|+\kappa}$; and as $|u| + \kappa < \mu_*$, we may say $S_{\alpha,u} \in [\mu]^{<\mu_*}$.

For $\eta \in \Lambda^*$, let

$$W_{\alpha,u,\eta} := \eta^{-1}(S_{\alpha,u})$$

and

$$\Lambda_{\alpha,u} := \big\{\eta \in \Lambda^* : W_{\alpha,u,\eta} \notin J\big\}.$$

Now

- $\bullet_{3.1}$ $|\Lambda_{\alpha,u}| < \mathbf{T}_J^+(S_{\alpha,u})$.

[Why? Because $\big|\{\eta \restriction W_{\alpha,u,\eta} : \eta \in \Lambda_{\alpha,u}\}\big| < \mathbf{T}_J^+(S_{\alpha,u})$ by its definition.]

As $|S_{\alpha,u}| \le |u| + \kappa$, we may conclude $|\Lambda_{\alpha,u}| < \mathbf{T}_J^+(|u| + \kappa)$, and therefore

- $\bullet_{3.2}$ $|\Lambda_{\alpha,u}| \le \chi$.

[Why? By clause $(e)'$ of our assumptions.]

Therefore (recalling $(*)_1$)

$(*)_4$ $\Lambda_\alpha := \bigcup\limits_{u \in \mathscr{P}} \Lambda_{\alpha,u}$ has cardinality $\le |\mathscr{P}| + \chi = \chi < |\Lambda^*|$.

So we can choose $\eta_\alpha \in \Lambda^* \setminus \Lambda_\alpha$, and now:

$(*)_5$ For each $u \in \mathscr{P}$ with $u \ni \alpha$, we can choose a_u^α as required in $(*)_3^\alpha$(d).

[Why? Because $\eta_\alpha \notin \Lambda_{\alpha,u}$.]

Lastly, let $\Lambda := \{\eta_\alpha : \alpha < \lambda\}$.

$(*)_6$ $\Lambda \in [\Lambda^*]^\lambda$ is μ^+-free.

Why? First, $\bigcup \mathscr{P} = \lambda$ by the definition of $\mathscr{P}$ in $(*)_1$. Hence if $\alpha < \beta < \lambda$ then for some $u \in \mathscr{P}$ we have $\beta \in u$. But $\eta_\beta \notin \Lambda_{\alpha,u}$ by our choices, and hence $\eta_\alpha \neq \eta_\beta$. Moreover, $\eta_\alpha \neq_J \eta_\beta$ (that is, $\{i < \kappa : \eta_\alpha(i) = \eta_\beta(i)\} \in J$). Hence $|\Lambda| = \lambda$.

Second, let $u^* \in [\lambda]^{<\mu_*}$. Then by the choice of $\mathscr{P}$ (in $(*)_1$) there is a sequence

$$\langle u_i : i < \kappa_* \rangle \subseteq \mathscr{P}$$

such that $u^* \subseteq \bigcup_i u_i$. For $\alpha \in u^*$, let $i_\alpha = i(\alpha) := \min\{i < \kappa_* : \alpha \in u_i\}$ and let $a^*_\alpha := a^\alpha_{u_{i(\alpha)}}$. Let

$$W := \big\{(\alpha,\beta) \in u^* \times u^* : \big(\exists j \in \kappa \setminus (a^*_\alpha \cup a^*_\beta)\big)\big[\eta_\alpha(j) = \eta_\beta(j)\big]\big\}.$$

Easily, $\alpha \in u^* \Rightarrow \big|\{\beta : (\alpha,\beta) \in W\}\big| \leq \kappa_*$. Hence (noting that W is a symmetric relation on u^*) we can find an equivalence relation E on u such that $(\alpha,\beta) \in W \Rightarrow \alpha \mathrel{E} \beta$ and every equivalence class is of cardinality $\leq \kappa_*$.

For S an equivalence class of E, let $\langle \alpha^S_j : j < j_S \leq \kappa \rangle$ list S without repetition. For $\alpha := \alpha^S_j \in S$, let

$$a_\alpha := a^*_\alpha \cup \big\{i \in \kappa_* : (\exists \varepsilon < j)\big[\eta_{\alpha^S_\varepsilon}(i) = \eta_\alpha(i)\big]\big\}.$$

This is a member of J because $j < \kappa_*$ and J is κ_*-complete (by assumption (1)(a)), and by our choice of Λ.

So $\langle a_\alpha : \alpha \in u \rangle$ witnesses the freeness demand in $(*)_6$.

3) Similarly, but in $(*)_2$ we choose Λ^* to witness clause $(c)''$. $\square_{4.4}$

Claim 4.5. *If clauses* (a)-(f) *below hold, then there is a* (μ^+, J)*-free subset* $\Lambda \subseteq {}^\kappa\mu$ *of cardinality* λ.

- (a) *J is a κ-complete ideal on κ.*
- (b) $\kappa := \mathrm{cf}(\mu) < \mu < \mu_\bullet \leq \lambda \leq \chi$
- (c) $\mathbf{T}^+_J(\mu) > \chi^+$
- (d) $\kappa_\bullet := \mathrm{cf}(\mu_\bullet) \in [\kappa^+, \mu)$ *and*

$$\partial \in (\mu, \mu_\bullet) \wedge \mathrm{cf}(\partial) \in [\kappa^+, \mu) \Rightarrow \mathrm{pp}_{\mathrm{cf}(\partial)}(\partial) < \mu_\bullet.$$

- (e) $\lambda^+ < \mathrm{pp}^+_{J_\bullet}(\mu_\bullet)$, *where $J_\bullet$ is a κ^+-complete ideal on $\kappa_\bullet$.*[19]
- (f) $\alpha < \mu \Rightarrow |\alpha|^{\kappa_\bullet} < \mu$.

Proof. First,

$(*)_1$ There exists $\bar\alpha^\bullet = \langle \alpha^\bullet_i : i < \kappa \rangle \in {}^\kappa\mu$ such that $T^+_J(\bar\alpha^\bullet) > \chi^+$.

[Why? by assumption (c) there exists $\Lambda \subseteq \bigcup_{b \in J^+} {}^b\mu$ witnessing $\mathbf{T}^+_J(\mu) > \chi^+$.

Now let $\langle \mu_i : i < \kappa \rangle$ be increasing with limit μ; so for every $\eta \in \Lambda$ there is an increasing function $h_\eta : \kappa \to \kappa$ such that $i \in \mathrm{dom}(\eta) \Rightarrow \eta(i) < \mu_{h_\eta(i)}$. As $\big|\{h_\eta : \eta \in \Lambda\}\big| \leq 2^\kappa \leq \chi$, clearly for some $h \in {}^\kappa\kappa$ the set $\Lambda_h := \{\eta \in \Lambda : h_\eta = h\}$ has cardinality $\geq \chi^+$.

So let $\alpha^\bullet_i := \mu_{h(i)}$ witness $(*)_1$.

Second,

$(*)_2$ We can apply 4.4(2) for each $\lambda' \in (\mu, \mu_\bullet)$, with $\kappa, \kappa, J, \mu, \mu, \lambda', \lambda'$ here standing in for $\kappa_*, \kappa, J, \mu_*, \mu, \lambda, \chi$ there.

[19] Alternatively, maybe $J_\bullet$ is just $\big(\kappa + \mathrm{cf}(J, \subseteq)\big)^+$-complete.

We have to check that all the assumptions hold.

Clause (a): Holds by clause (a) of our assumptions.

Clause (b): Holds by clause (b) of our assumptions, recalling $\mu < \lambda' < \mu_\bullet$.

Clause (c)′: Holds by $(*)_1$.

Clause (d): Holds by the second phrase in assumption (d) and 4.3(4).

Clause (e),(e)′: Directly implied by assumption (f).

So we get

$(*)_3$ There is a (J, μ^+)-free $\Lambda \subseteq \prod_{i<\kappa} d_i$ of cardinality $\mu_\bullet$.

[Why? Recalling $\kappa_\bullet := \mathrm{cf}(\mu_\bullet) < \mu < \mu_\bullet$, there exists an increasing sequence of cardinals $\langle \lambda_\varepsilon : \varepsilon < \kappa_\bullet \rangle$ with $\lambda_0 > \mu$ and limit $\mu_\bullet$.

Applying the conclusion of 4.4(1)+(2) which we just obtained in $(*)_2$, for each $\varepsilon < \kappa_\bullet$ there exists a μ-free subset $\Lambda_\varepsilon \subseteq {}^\kappa\mu$ of cardinality λ_ε. For each $\eta \in \Lambda_\varepsilon$, define $\nu^\varepsilon_\eta := \langle \kappa_\bullet \cdot \eta(i) + \varepsilon : i < \kappa \rangle$. Now $\Lambda = \{\nu^\varepsilon_\eta : \varepsilon < \kappa_\bullet,\ \eta \in \Lambda_\varepsilon\}$ as promised.]

$(*)_4$ Let $\langle \eta^*_\alpha : \alpha < \mu_\bullet \rangle$ list the members of Λ without repetition.

Third, applying Definition 4.2(3) and clause (e) of the assumption, recalling [She94, Ch.II, 3.1],

$(*)_5$ There is a $<_{J_\bullet}$-increasing sequence $\bar f = \langle f_\gamma : \gamma < \lambda^+ \rangle \subseteq {}^{\kappa_\bullet}\mu_\bullet$ (hence all functions in the sequence are pairwise $\neq_{J_\bullet}$).

By [She94],

$(*)_6$ Without loss of generality, $\bar f \restriction \lambda = \langle f_\gamma : \gamma < \lambda \rangle$ is a $J_\bullet$-free sequence and $\gamma < \lambda \Rightarrow f_\gamma < f_\lambda$.

$(*)_7$ For each $i < \kappa$, we choose a one-to-one function g_i such that
 (a) $\mathrm{dom}(g_i) := \bigcup_{b \in J^+_\bullet} {}^b(\alpha^\bullet_i)$
 (b) $\mathrm{rang}(g_i) \subseteq \mu$

[Why does such a g_i exist?

The set $W_i := \bigcup_{b \in J^+_\bullet} {}^b(\alpha^\bullet_i)$ is a subset of $\bigcup_{u \subseteq \kappa_\bullet} {}^u(\alpha^\bullet_i)$, which has cardinality $|\alpha^\bullet_i|^{\kappa_\bullet}$; this is $< \mu$ by clause (f) of our assumptions. So clearly there is an injection from W_i into μ; this satisfies $(*)_7$, and so it will be our g_i.]

$(*)_8$ For each $\gamma < \lambda$ we choose $\eta_\gamma \in {}^\kappa\mu$ as follows:
For each $i < \kappa$ let $\nu_{\gamma,i} := \big\langle \eta^*_{f_\gamma(\xi)}(i) : \xi < \kappa_\bullet \big\rangle \in {}^{\kappa_\bullet}(\alpha^\bullet_i)$. Now let

$$\eta_\gamma := \big\langle g_i(\nu_{\gamma,i}) : i < \kappa \big\rangle.$$

Lastly,

$(*)_9$ $\bar\eta = \langle \eta_\gamma : \gamma < \lambda \rangle$ is as promised.

Why? Clearly $\bar\eta$ is a sequence of members of ${}^\kappa\mu$, without repetition. Let $u \in [\lambda]^{\leq \mu}$, and we shall prove that $\bar\eta \restriction u$ is free.

$(*)_{9.1}$ Without loss of generality $|u| < \mu$.

[Why? By [She19].]

$(*)_{9.2}$ We can find $\langle b^*_\gamma : \gamma \in u\rangle \subseteq J_\bullet$ such that <u>if</u> $\gamma \neq \varepsilon \in u$ and $j \in \kappa_\bullet \setminus \left(b^*_\gamma \cup b^*_\varepsilon\right)$ <u>then</u> $f_\gamma(j) \neq f_\varepsilon(j)$.

[Why? By $(*)_6$.]

$(*)_{9.3}$ Let $W := \left\{f_\gamma(j) : \gamma \in u,\ j < \kappa_\bullet\right\}$ (so $W \in [\mu_\bullet]^{<\mu}$). Hence there exists a sequence $\langle a_\beta : \beta \in W\rangle \in {}^W J$ witnessing that $\bar{f} \restriction W$ is free.

$(*)_{9.4}$ We can choose $\langle a_\gamma : \gamma \in u\rangle$ such that
- $a_\gamma \in J$
- $\{j < \kappa_\bullet : a_{f_\gamma(j)} \subseteq a_\gamma\} \in J^+_\bullet$.

[Why? As $b_\gamma \in J^+_\bullet$, $b^*_\gamma \in J_\bullet$, and $J_\bullet$ is κ^+-complete.]

$(*)_{9.5}$ If $\gamma \neq \varepsilon \in u$ and $j \in \kappa \setminus \left(a_\gamma \cup a_\varepsilon\right)$ <u>then</u> $\eta_\gamma(j) \neq \eta_\varepsilon(j)$.

[Why? Put clauses $(*)_{9.1}$-$(*)_{9.4}$ together.]

Now we are done. $\square_{4.5}$

Claim 4.6. *If μ is strong limit, $\kappa := \mathrm{cf}(\mu) < \mu$, and $\lambda \in (\mu, 2^\mu)$, <u>then</u> there is a μ^+-free $\lambda \subseteq {}^\kappa\mu$ of cardinality λ.*

Proof. Let $\Theta_{\mu,\kappa} := \left\{\chi \in (\mu, \lambda] : \mathrm{cf}(\chi) \in [\kappa^+, \mu) \text{ and } \mathrm{pp}_{\mathrm{cf}(\chi)\text{-complete}}(\chi) \geq \lambda^+\right\}$.

If $\Theta_{\mu,\kappa}$ is empty then we can apply 4.4.

[Why? Choose $\kappa_* := \kappa$, $\mu_* := \mu$, $J := J^{\mathrm{bd}}_\kappa$, and we have to verify the assumptions of 4.4. Clauses (a), (b), and (c) are obvious. Clause (e) says '$\alpha < \mu \Rightarrow |\alpha|^\kappa < \mu$,' and this follows from μ being strong limit. Lastly, clause (d) holds by 4.2(4).]

So assume $\Theta_{\mu,\kappa} \neq \varnothing$. Let $\mu_\bullet := \min \Theta_{\mu,\kappa}$, and apply 4.5.

[Why can we do this? We should check assumptions 4.5(a)-(f). Choose $\chi := \lambda$, $\kappa_\bullet := \mathrm{cf}(\mu_\bullet)$, and $J_\bullet$ a $\kappa_\bullet$-complete ideal on $\kappa_\bullet$ satisfying $\mathrm{pp}_{J_\bullet}(\mu_\bullet) \geq \lambda^+$.

Clauses (a) and (b) are obvious.

Clause (c) says '$T^+_J(\mu) > \chi^+$', which holds because $J := J^{\mathrm{bd}}_\kappa$ and μ is strong limit of cofinality κ.

In clause (d), the first statement ($\mathrm{cf}(\mu_\bullet) \in \left[\kappa^+, \mu\right)$) holds by the definition of $\Theta_{\mu,\kappa}$ and choice of $\mu_\bullet$. The second statement holds by the same reasoning.

Clause (e) holds as $\mu_\bullet \in \Theta_{\mu,\kappa}$ and the choice of $J_\bullet$.

Clause (f) holds as $\mu > \kappa_\bullet$ is strong limit.] $\square_{4.6}$

Discussion 4.7. 1) In 4.5, there is no harm in adding "$J_\bullet$ is $\kappa_\bullet$-complete."

2) Furthermore, if we add '$\kappa_\bullet > 2^\kappa$' then we can weaken clause 4.5(c) to '$\mathbf{T}^+_J(\mu) > \chi^+$.'

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