

# A UNIQUE $Q$ -POINT AND INFINITELY MANY NEAR-COHERENCE CLASSES OF ULTRAFILTERS

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**Abstract.** We show that in the model obtained by iteratively pseudo-intersecting a Ramsey ultrafilter via a length- $\omega_2$  countable support iteration of restricted Mathias forcing over a ground model satisfying CH, there is a unique  $Q$ -point up to isomorphism. In particular, it is consistent that there is only one  $Q$ -point while there are  $2^c$ -many near-coherence classes of ultrafilters.

**key-words:**  $Q$ -point, Ramsey ultrafilter, Mathias forcing

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## 0 Introduction

Throughout this paper, read *ultrafilter* as *non-principal ultrafilter on  $\omega$* . For  $x \subseteq \omega$ , we denote by  $[x]^\omega$  the set of infinite subsets of  $x$  and by  $[x]^{<\omega}$  the set of finite subsets of  $x$ .

Recall that an ultrafilter  $E$  is a  *$Q$ -point* if and only if for every interval partition  $\{[k_i, k_{i+1}) : i \in \omega\}$  of  $\omega$ , there exists some  $x \in E$  such that  $\forall i \in \omega : |x \cap [k_i, k_{i+1})| \leq 1$ . Furthermore, an ultrafilter  $\mathcal{U}$  is a *Ramsey ultrafilter* if and only if the *Maiden* has no winning strategy in the *ultrafilter game for  $\mathcal{U}$* , played between the *Maiden* and *Death*:

**DEFINITION 0.1.** *Let  $\mathcal{U}$  be an ultrafilter. The ultrafilter game for  $\mathcal{U}$  proceeds as follows:*

*The Maiden opens the game and plays some  $y_0 \in \mathcal{U}$ . Death responds by playing some  $n_0 \in y_0$ . In the  $(k+1)$ -th move, the Maiden having played  $y_0 \supseteq y_1 \supseteq \dots \supseteq y_k$ , and Death*

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having played  $n_0 < n_1 < \dots < n_k$ , the Maiden plays some  $y_{k+1} \in [y_k]^\omega \cap \mathcal{U}$ , and Death responds by playing some  $n_{k+1} \in y_{k+1}$ ,  $n_{k+1} > n_k$ .

Death wins if and only if  $\{n_i : i \in \omega\} \in \mathcal{U}$ .

It is well-known that every Ramsey ultrafilter is a  $Q$ -point. Canjar [4] showed that the existence of  $2^c$ -many Ramsey ultrafilters follows from the assumption  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ . The weaker assumption  $\text{cov}(\mathcal{M}) = \mathfrak{d}$  implies the existence of  $2^c$   $Q$ -points, as was shown by Millán [12]. It is well-known that in the Mathias model – the model obtained by a length- $\omega_2$  countable support iteration of unrestricted Mathias forcing over a ground model satisfying CH – there are no  $Q$ -points (see [9, Proposition 26.23]). In fact, the Mathias model contains no *rapid* ultrafilters, where an ultrafilter  $E$  is rapid if for every  $f \in {}^\omega\omega$  there exists some  $x \in E$  such that  $\forall n \in \omega : |x \cap f(n)| \leq n$  (note that every  $Q$ -point is rapid). It follows that both the Mathias model and the model considered in this paper satisfy  $\text{cov}(\mathcal{M}) = \omega_1 < \mathfrak{d} = \mathfrak{c} = \omega_2$ .

In contrast to the Mathias model, our model contains  $2^c$ -many rapid ultrafilters: It follows from an observation of Millán [12, page 222] that the existence of a single rapid ultrafilter  $E$  implies the existence of  $2^c$  of them, by considering the products  $\mathcal{U} \times E$  for different ultrafilters  $\mathcal{U}$ .<sup>2</sup>

While the consistency of the non-existence of  $Q$ -points is a well-established fact with a variety of witnesses apart from the Mathias model<sup>3</sup>, the construction of models containing only ‘few’  $Q$ -points seems to have received less attention. However, such models do arise naturally as models containing only few near-coherence classes of ultrafilters<sup>4</sup>: Indeed, Mildenberger [11] has constructed models with exactly two and exactly three near-coherence classes, and it is easy to see that these contain exactly one and exactly two  $Q$ -points, respectively: In her model with exactly two near-coherence classes, one class contains a Ramsey ultrafilter, while the other class contains an ultrafilter that is  $\omega_1$ -generated. Hence, this latter class cannot contain a  $Q$ -point, since her models satisfy  $\mathfrak{d} = \mathfrak{c} = \omega_2$  and such a  $Q$ -point would thus have to be  $<\mathfrak{d}$ -generated, which is impossible. Analogously, in Mildenberger’s model with exactly three near-coherence classes, two classes are represented by Ramsey ultrafilters, while the third contains an  $\omega_1$ -generated ultrafilter – giving exactly two  $Q$ -points in total.

The construction of models with exactly  $n$  near-coherence classes of ultrafilters for various finite  $n \geq 4$  would similarly yield the consistency of exactly  $m$   $Q$ -points for some  $m < n$ .<sup>5,6</sup>

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<sup>2</sup> $\mathcal{U} \times E$  is an ultrafilter on  $\omega \times \omega$  defined by  $\mathcal{U} \times E = \{x \subseteq \omega \times \omega : \{n \in \omega : \{m \in \omega : \langle n, m \rangle \in x\} \in E\} \in \mathcal{U}\}$ .

<sup>3</sup>such as the Laver and Miller models (see [13] and [3], respectively).

<sup>4</sup>Two ultrafilters  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are *nearly-coherent* if there is some finite-to-one  $f \in {}^\omega\omega$  such that  $f(\mathcal{U}_1) = f(\mathcal{U}_2)$ , where  $f(\mathcal{U}_i) := \{X \subseteq \omega : f^{-1}[X] \in \mathcal{U}_i\}$ . Note that two  $Q$ -points are nearly-coherent if and only if they are isomorphic.

<sup>5</sup>The inequality is strict since such a model must satisfy  $\mathfrak{u} < \mathfrak{d}$ , a result due to Banach and Blass [2]. Hence, one of the  $n$  near-coherence classes contains a  $<\mathfrak{d}$ -generated ultrafilter and thus no  $Q$ -point.

<sup>6</sup>See the note on the next page.

The model considered in this paper is of a different nature, however: It contains only one  $Q$ -point while its number of near-coherence classes is  $2^c$ , i.e., the model's lack of  $Q$ -points is not the consequence of a lack of near-coherence classes. This follows from the fact that dominating reals are added at each of the  $\omega_2$  stages of the iteration, which gives  $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2$  in the final extension. Since  $\mathfrak{b} \leq \mathfrak{u}$  (see Solomon [15]), we have  $\mathfrak{u} = \mathfrak{d} = \omega_2$  in our model, and hence there are  $2^c$ -many near-coherence classes of ultrafilters by Banach and Blass [2].

Incidentally, both our model as well as Mildner's models answer the two questions posed in a recent paper of Raghavan [5]. He asked whether it is consistent that there are  $Q$ -points, while (1) there is no Tukey maximal  $Q$ -point, and (2) the  $Q$ -points are not cofinal in the RK-ordering.<sup>7</sup> Note that the unique  $Q$ -point in our model is simultaneously a  $P$ -point, and these are never Tukey maximal (see [7, Corollary 19]).

**Note added in proof:** Based on the technique used below, in the forthcoming paper [10], Halbeisen, Horvath, and Özalp have established the consistency of the statement “There are exactly  $n$   $Q$ -points up to isomorphism and  $\mathfrak{u} = \mathfrak{d} = \mathfrak{c}$ ”, for any  $n \in \omega$ .

## 1 Definitions and Preliminaries

Before we state and prove our main result in the next section, let us give some definitions, relations, and related results.

**DEFINITION 1.1.** *Let  $\mathcal{U}$  be a Ramsey ultrafilter. Mathias forcing restricted to  $\mathcal{U}$ , written  $\mathbb{M}_{\mathcal{U}}$ , consists of conditions  $\langle s, x \rangle \in [\omega]^{<\omega} \times \mathcal{U}$  with  $\max s < \min x$ , ordered by*

$$\langle s, x \rangle \leq_{\mathbb{M}_{\mathcal{U}}} \langle t, y \rangle : \iff s \supseteq t \wedge x \subseteq y \wedge s \setminus t \subseteq y.$$

The forcing notion  $\mathbb{M}_{\mathcal{U}}$  clearly satisfies the c.c.c. and is therefore proper. We will need the following additional facts.

**FACT 1.2** (e.g., see [9, Theorem 26.3]). *Let  $\mathcal{U}$  be a Ramsey ultrafilter. The forcing notion  $\mathbb{M}_{\mathcal{U}}$  has the pure decision property, i.e., for any sentence  $\varphi$  in the forcing language and any  $\mathbb{M}_{\mathcal{U}}$ -condition  $\langle s, x \rangle$ , there exists  $y \in [x]^\omega \cap \mathcal{U}$  such that either  $\langle s, y \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \varphi$  or  $\langle s, y \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \neg\varphi$ .*

**DEFINITION 1.3.** *Recall that a forcing notion  $\mathbb{P}$  has the Laver property if for every  $\mathbb{P}$ -name  $\underline{g}$  for an element of  ${}^\omega\omega$  such that there exists  $f \in {}^\omega\omega \cap \mathbf{V}$  with*

$$\mathbb{P} \Vdash \forall n \in \omega : \underline{g}(n) \leq f(n),$$

*we have that  $\mathbb{P}$  forces that there exists  $c : \omega \rightarrow [\omega]^{<\omega}$  in  $\mathbf{V}$  with*

$$\forall n \in \omega : |c(n)| \leq 2^n \text{ and } \underline{g}(n) \in c(n).$$

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<sup>7</sup>See, for example, [6] and [7] on the Tukey- and RK-orderings of ultrafilters.

FACT 1.4 (e.g., see [9, Corollary 26.8]). *Let  $\mathcal{U}$  be a Ramsey ultrafilter. The forcing notion  $\mathbb{M}_{\mathcal{U}}$  has the Laver property.*

FACT 1.5 (e.g., see [14, Ch. VI, 2.10D]). *The Laver property is preserved under countable support iterations of proper forcing notions.*

## 2 Result

Now, we are ready to state the

MAIN THEOREM. *It is consistent that there is a unique  $Q$ -point while there are  $2^c$ -many near-coherence classes of ultrafilters.*

The proof is given by the following construction and the subsequent results: Assume that the ground model  $\mathbf{V}$  satisfies CH. By induction, we define:

- (i) A countable support iteration  $\mathbb{P}_{\omega_2} := \langle \mathbb{P}_{\xi}, \mathcal{Q}_{\xi} : \xi \in \omega_2 \rangle$  of c.c.c. forcing notions,
- (ii) A sequence  $\langle \mathcal{U}_{\xi} : \xi \in \omega_2 \rangle$ , such that

$$\forall \xi \in \omega_2 : \mathbb{P}_{\xi} \Vdash \text{“}\underline{\mathcal{U}}_{\xi} \text{ is a Ramsey ultrafilter extending } \bigcup_{\iota \in \xi} \mathcal{U}_{\iota}\text{”}$$

and  $\mathcal{Q}_{\xi}$  is a  $\mathbb{P}_{\xi}$ -name for Mathias forcing restricted to  $\mathcal{U}_{\xi}$ ,

Assume that we are in step  $\xi \in \omega_2$ . Let  $G_{\xi}$  be  $\mathbb{P}_{\xi}$ -generic over  $\mathbf{V}$  and work in  $\mathbf{V}[G_{\xi}]$ . Note that since  $\mathbb{P}_{\xi}$  is a countable support iteration of proper forcing notions that are forced to be of size  $\leq \omega_1$ , we have  $\mathbf{V}[G_{\xi}] \models \text{CH}$  (e.g., see [1, Theorem 2.12]). For each  $\iota \in \xi$ , let  $\eta_{\iota}$  be the Mathias real added at stage  $\iota$ .

If  $\xi = \xi' + 1$ ,  $\eta_{\xi'}$  pseudo-intersects  $\mathcal{U}_{\xi'}[G_{\xi}]$  and we may construct a Ramsey ultrafilter on  $\eta_{\xi'}$  using CH (and extend it to  $\omega$  to obtain  $\mathcal{U}_{\xi}$ ). Similarly, if  $\xi$  is a limit ordinal and  $\text{cf}(\xi) = \omega$ , we can build  $\mathcal{U}_{\xi}$  on a pseudo-intersection of the tower  $\langle \eta_{\iota} : \iota \in \xi \rangle$ . Finally, if  $\text{cf}(\xi) = \omega_1$ , then  $\bigcup_{\iota \in \xi} \mathcal{U}_{\iota}[G_{\xi}]$  is already a Ramsey ultrafilter, since no new reals are added at stage  $\xi$ . For the same reason we also have that  $\mathcal{U}_{\omega_2} := \bigcup_{\xi \in \omega_2} \mathcal{U}_{\xi}[G]$  is a Ramsey ultrafilter in  $\mathbf{V}[G]$ , where  $G$  is  $\mathbb{P}_{\omega_2}$ -generic over  $\mathbf{V}$ .

FACT 2.1 (see, e.g., [1, Theorem 2.10]).  *$\mathbb{P}_{\omega_2}$  is proper and satisfies  $\omega_2$ -c.c..*

We need to show that  $\mathcal{U}_{\omega_2}$  is the only  $Q$ -point in  $\mathbf{V}[G]$ . To see this, assume by contradiction that  $\mathbf{V}[G] \models \text{“}E \text{ is a } Q\text{-point and not isomorphic to } \mathcal{U}_{\omega_2}\text{”}$ .

LEMMA 2.2. *There exists  $\delta \in \omega_2$  such that  $E \cap \mathbf{V}[G_{\delta}] \in \mathbf{V}[G_{\delta}]$  and  $\mathbf{V}[G_{\delta}] \models \text{“}E \cap \mathbf{V}[G_{\delta}] \text{ is a } Q\text{-point and not isomorphic to } \mathcal{U}_{\delta}\text{”}$ .*

*Proof.* Fix  $\xi \in \omega_2$  and consider names  $\underline{e}_\xi, \underline{i}_\xi, \underline{s}_\xi, \underline{b}_\xi$  and  $\underline{f}_\xi$  such that  $\mathbb{P}_{\omega_2}$  forces that

- (i) “ $\underline{e}_\xi$  is an enumeration (in  $\omega_1$ ) of  $\underline{E} \cap \mathbf{V}[G_\xi]$ ”. For each  $\alpha \in \omega_1$  and  $n \in \omega$  let  $\mathcal{E}_{\xi, \alpha, n} \subseteq \mathbb{P}_{\omega_2}$  be a maximal antichain deciding “ $n \in \underline{e}_\xi(\alpha)$ ”.
- (ii) “ $\underline{i}_\xi$  is an enumeration (in  $\omega_1$ ) of the set of interval partitions of  $\omega$  in  $\mathbf{V}[G_\xi]$ ”. Note that we may assume that  $\underline{i}_\xi$  is a  $\mathbb{P}_\xi$ -name.
- (iii) “For all  $\alpha \in \omega_1$ ,  $\underline{s}_\xi(\alpha)$  is an element of  $\underline{E}$  that intersects each interval in the interval partition  $\underline{i}_\xi(\alpha)$  in at most one point”. Let  $\mathcal{S}_{\xi, \alpha, n} \subseteq \mathbb{P}_{\omega_2}$  be a maximal antichain deciding “ $n \in \underline{s}_\xi(\alpha)$ ”.
- (iv) “ $\underline{b}_\xi$  is an enumeration (in  $\omega_1$ ) of all permutations of  $\omega$  in  $\mathbf{V}[G_\xi]$ ”. We may again assume that  $\underline{b}_\xi$  is a  $\mathbb{P}_\xi$ -name.
- (v) “For all  $\alpha \in \omega_1$ ,  $\underline{f}_\xi(\alpha)$  is a pair  $\langle \underline{x}_\alpha, \underline{y}_\alpha \rangle$  such that  $\underline{x}_\alpha$  is in  $\underline{E}$ ,  $\underline{y}_\alpha$  is in  $\mathcal{U}_{\omega_2}$  and  $\underline{b}_\xi(\alpha)[\underline{x}_\alpha]$  is disjoint from  $\underline{y}_\alpha$ ”. Let  $\mathcal{X}_{\xi, \alpha, n} \subseteq \mathbb{P}_{\omega_2}$  be a maximal antichain deciding “ $n \in \underline{x}_\alpha$ ”, and define  $\mathcal{Y}_{\xi, \alpha, n}$  analogously.

Since  $\mathbb{P}_{\omega_2}$  satisfies  $\omega_2$ -c.c., there exists for each  $\xi \in \omega_2$  some  $\gamma_\xi \in \omega_2$  greater than  $\xi$  such that all the above antichains consist of  $\mathbb{P}_{\gamma_\xi}$ -conditions. Recursively define  $\lambda(0) = 0$ ,  $\lambda(\xi + 1) = \gamma_{\lambda(\xi)}$  and for limit ordinals  $\xi$  :  $\lambda(\xi) = \bigcup_{\iota \in \xi} \lambda(\iota)$ , for  $\xi \leq \omega_1$ . Set  $\delta := \lambda(\omega_1)$  and consider the extension  $\mathbf{V}[G_\delta]$ . Since  $\text{cf}(\delta) = \omega_1$ , we have that  $E \cap \mathbf{V}[G_\delta] = \bigcup_{\iota \in \omega_1} E \cap \mathbf{V}[G_{\lambda(\iota)}]$ , and since each  $E \cap \mathbf{V}[G_{\lambda(\iota)}]$  is an element of  $\mathbf{V}[G_\delta]$  by (i),  $E \cap \mathbf{V}[G_\delta]$  is an element of  $\mathbf{V}[G_\delta]$  (and an ultrafilter). Furthermore, any interval partition of  $\omega$  in  $\mathbf{V}[G_\delta]$  already appears in some  $\mathbf{V}[G_{\lambda(\iota)}]$ ,  $\iota \in \omega_1$ , where it equals  $\underline{i}_{\lambda(\iota)}[G_{\lambda(\iota)}](\alpha)$  for some  $\alpha \in \omega_1$ . Since  $\underline{s}_{\lambda(\iota)}[G_\delta](\alpha) \in E \cap \mathbf{V}[G_\delta]$ , we obtain that  $E \cap \mathbf{V}[G_\delta]$  is a  $Q$ -point. Finally and analogously, any permutation of  $\omega$  in  $\mathbf{V}[G_\delta]$  already appears in  $\mathbf{V}[G_{\lambda(\iota)}]$  for some  $\iota \in \omega_1$  and hence there are witnesses  $\underline{x}_\alpha[G_\delta] \in E \cap \mathbf{V}[G_\delta]$  and  $\underline{y}_\alpha[G_\delta] \in \mathcal{U}_{\omega_2} \cap \mathbf{V}[G_\delta] = \mathcal{U}_\delta$  witnessing that  $E \cap \mathbf{V}[G_\delta]$  and  $\mathcal{U}_\delta$  are not isomorphic.  $\dashv$

We now designate  $\mathbf{V}[G_\delta]$  as the new ground model and rename the  $Q$ -point  $E \cap \mathbf{V}[G_\delta]$  to  $E$  and the Ramsey ultrafilter  $\mathcal{U}_\delta$  to  $\mathcal{U}$ . Note that by the Factor-Lemma (e.g., see [8, Theorem 4.6]), the quotient  $\mathbb{P}_{\omega_2}/G_\delta$  is again isomorphic to a countable support iteration of restricted Mathias forcings. In particular, by Facts 1.4 and 1.5,  $\mathbb{P}_{\omega_2}/G_\delta$  is isomorphic to the two-step iteration  $\mathbb{M}_{\mathcal{U}} * \underline{R}$ , where  $\mathbb{M}_{\mathcal{U}} \Vdash$  “ $\underline{R}$  has the Laver property”.

It remains to show the following

**PROPOSITION 2.3.** *Let  $E$  be a  $Q$ -point and  $\mathcal{U}$  a Ramsey ultrafilter such that  $E$  and  $\mathcal{U}$  are not isomorphic. Let  $\mathbb{M}_{\mathcal{U}}$  be Mathias forcing restricted to  $\mathcal{U}$  and let  $\underline{R}$  be a  $\mathbb{M}_{\mathcal{U}}$ -name such that  $\mathbb{M}_{\mathcal{U}} \Vdash$  “ $\underline{R}$  has the Laver property”. Then  $\mathbb{M}_{\mathcal{U}} * \underline{R} \Vdash$  “ $E$  cannot be extended to a  $Q$ -point”.*

*Proof.* It suffices to show that if  $\langle p, \underline{q} \rangle \in \mathbb{M}_{\mathcal{U}} * \underline{R}$  and a  $\mathbb{M}_{\mathcal{U}} * \underline{R}$ -name  $\underline{a}$  for a strictly increasing element of  ${}^\omega\omega$  are such that

$$\langle p, \underline{q} \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \forall n \in \omega : \underline{a}(n) \in (\eta(n-1), \eta(n)],$$

then there exists some  $v \in E$  and some  $\langle \bar{p}, \bar{q} \rangle \leq_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \langle p, \underline{q} \rangle$  such that

$$\langle \bar{p}, \bar{q} \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underline{R}} |\text{range}(\underline{a}) \cap v| < \omega.$$

Recall that  $\eta$  is the canonical  $\mathbb{M}_{\mathcal{U}}$ -name for the Mathias real (assume  $\mathbb{M}_{\mathcal{U}} \Vdash \eta(-1) = -\infty$ ).

Note that  $\underline{a}$  is forced by  $\mathbb{M}_{\mathcal{U}}$  to be dominated by  $\eta$ . Hence, by the Laver property of  $\underline{R}$ , there exists a  $\mathbb{M}_{\mathcal{U}}$ -name  $\underline{c}$  for a function from  $\omega$  to  $[\omega]^{<\omega}$  and some  $\langle p', \underline{q}' \rangle \leq_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \langle p, \underline{q} \rangle$  such that

$$\langle p', \underline{q}' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \forall n \in \omega : \underline{a}(n) \in \underline{c}(n) \text{ and } |\underline{c}(n)| \leq 2^n.$$

We may assume without loss of generality that  $p' \Vdash_{\mathbb{M}_{\mathcal{U}}} \forall n \in \omega : \underline{c}(n) \subseteq (\eta(n-1), \eta(n)]$ . Let  $\underline{C}$  be a  $\mathbb{M}_{\mathcal{U}}$ -name for an element of  $[\omega]^\omega$  such that  $p' \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} = \bigcup \text{range}(\underline{c})$ . Hence, we have

$$\langle p', \underline{q}' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underline{R}} \forall n \in \omega : \underline{a}(n) \in \underline{C} \cap (\eta(n-1), \eta(n)] \text{ and } |\underline{C} \cap (\eta(n-1), \eta(n)]| \leq 2^n.$$

**LEMMA 2.4.** *Write  $p' = \langle s, x_0 \rangle$ . There exists  $x_1 \in [x_0]^\omega \cap \mathcal{U}$  such that the  $\mathbb{M}_{\mathcal{U}}$ -condition  $\langle s, x_1 \rangle \leq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_0 \rangle$  has the following property:*

*For every  $t \in [x_1]^{<\omega}$ , there exists  $C_t \in [\omega]^{<\omega}$  such that*

$$\langle s \cup t, x_1 \setminus (\max t)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} \cap (\max t)^+ = C_t.$$

*Proof.* We define a strategy for the Maiden in the ultrafilter game for  $\mathcal{U}$ , which will not be a winning strategy since  $\mathcal{U}$  is a Ramsey ultrafilter.

Since  $\mathbb{M}_{\mathcal{U}}$  has pure decision, there exists  $C_\emptyset \subseteq (\max s)^+$  and  $y_0 \in [x_0]^\omega \cap \mathcal{U}$  such that  $\langle s, y_0 \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} \cap (\max s)^+ = C_\emptyset$ . The Maiden starts by playing  $y_0$ .

Assume  $y_0 \supseteq y_1 \supseteq \dots \supseteq y_k$  and  $n_0 < n_1 < \dots < n_k$  have been played, where  $\forall i \leq k : y_i \in \mathcal{U}$  and  $n_i \in y_i$ . Again by pure decision, for each  $t \subseteq \{n_0, n_1, \dots, n_k\}$  with  $\max t = n_k$ , there exists  $z_t \in [y_k \setminus n_k^+]^\omega \cap \mathcal{U}$  and  $C_t \subseteq n_k^+$  such that  $\langle s \cup t, z_t \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underline{C} \cap (n_k)^+ = C_t$ . The Maiden plays

$$y_{k+1} := \bigcap_{\substack{t \subseteq \{n_i : i \leq k\} \\ \max t = n_k}} z_t.$$

Since Death wins, we have that  $x_1 := \{n_i : i \in \omega\} \in \mathcal{U}$ . It is easy to check that this  $x_1$  satisfies the lemma.  $\dashv$

The following lemma strengthens the previous one.

LEMMA 2.5. Assume  $\langle s, x_1 \rangle$  is as in the conclusion of the previous lemma. There exists  $x_2 \in [x_1]^\omega \cap \mathcal{U}$  such that  $\langle s, x_2 \rangle$  has the following property:

For every  $t \in [x_2]^{<\omega}$ , every  $m \in x_2 \setminus \max t$  and all  $n, n' \in x_2 \setminus m^+$ , it holds that  $C_{t \cup \{n\}} \cap m^+ = C_{t \cup \{n'\}} \cap m^+$ .

*Proof.* We again prove this by playing the ultrafilter game for  $\mathcal{U}$ . Assume  $y_0 := x_1 \supseteq y_1 \supseteq \dots \supseteq y_k$  and  $n_0 < n_1 < \dots < n_k$  have been played. For every  $t \subseteq \{n_0, n_1, \dots, n_k\}$  and every  $d \subseteq n_k^+$  consider the set

$$P_{t,d} := \{n \in y_k \setminus n_k^+ : C_{t \cup \{n\}} \cap n_k^+ = d\}.$$

Note that for every  $t \subseteq \{n_0, n_1, \dots, n_k\}$ , the set  $\{P_{t,d} : d \subseteq n_k^+\}$  is a partition of  $y_k \setminus n_k^+$  into finitely many pieces. Hence, there exists one  $d_t \subseteq n_k^+$  such that  $P_{t,d_t} \in \mathcal{U}$ .

The Maiden plays

$$y_{k+1} := \bigcap_{t \subseteq \{n_i : i \leq k\}} P_{t,d_t}.$$

Death will win and hence  $x_2 := \{n_i : i \in \omega\} \in \mathcal{U}$ . It is again not hard to check that  $x_2$  satisfies the lemma.  $\dashv$

The following fact will be needed later.

FACT 2.6. Without loss of generality, we may assume that for all  $n \in \{\max s\} \cup x_2$ , if  $n$  is the  $j$ 'th element of  $s \cup x_2$  in increasing order, then  $n > 2^{j+1}$ .

*Proof.* Note that the conclusion of Lemmata 2.4 and 2.5 also holds for each  $\langle s', x' \rangle \leq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_2 \rangle$ . Hence, we simply trim  $x_2$  such that the enumeration of  $s \cup x_2$  dominates  $2^{j+1}$  above  $|s|$  and replace  $s$  with  $s \cup \{\min x_2\}$  and  $x_2$  with  $x_2 \setminus \{\min x_2\}$ .  $\dashv$

Next, let  $N$  be a countable elementary submodel of some large enough  $\mathcal{H}_\chi$  such that  $\{\mathcal{U}, \mathbb{M}_{\mathcal{U}}, \mathcal{C}, \langle s, x_2 \rangle\} \in N$ . By induction, construct a sequence  $N_0 \subseteq N_1 \subseteq \dots$  of finite subsets of  $N$  such that

- (i)  $\{\mathcal{U}, \mathbb{M}_{\mathcal{U}}, \mathcal{C}, \langle s, x_2 \rangle, s, x_2\} \subseteq N_0$ ,
- (ii)  $\bigcup_{i \in \omega} N_i = N$ ,
- (iii)  $\forall i \in \omega : k_i := N_i \cap \omega \in \omega$ .
- (iv)  $\forall i \in \omega : \forall t \in [\omega]^{<\omega} : t \in N_i \iff t \subseteq N_i$ ,
- (v) If  $\langle m, l, D \rangle \in (\omega \times \omega \times [\omega]^{<\omega}) \cap N_i$ , then  $m, l, D \in N_i$  (and hence  $D \subseteq N_i$  by the previous condition).

- (vi)  $\forall i \in \omega$  : If  $\varphi(x, a_0, \dots, a_l)$  is a formula of length less than 2025 with  $a_0, \dots, a_l \in N_i$  and  $N \models \exists x \varphi(x, a_0, \dots, a_l)$ , then there exists  $b \in N_{i+1}$  such that  $N \models \varphi(b, a_0, \dots, a_l)$ .

LEMMA 2.7.  $\langle s, x_2 \rangle$  forces that

$$\forall i \in \omega \setminus \{0, 1\} : \mathcal{C} \setminus (\max s)^+ \cap [k_{i-1}, k_i] \neq \emptyset \implies \begin{cases} \text{range}(\eta) \cap [k_{i-2}, k_{i-1}] \neq \emptyset, \text{ or} \\ \text{range}(\eta) \cap [k_{i-1}, k_i] \neq \emptyset, \text{ or} \\ \text{range}(\eta) \cap [k_i, k_{i+1}] \neq \emptyset. \end{cases}$$

*Proof.* Assume  $\langle s \cup t, x' \rangle \leq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_2 \rangle$ ,  $a \in \omega \setminus (\max s)^+$  and  $i \in \omega \setminus \{0, 1\}$  are such that

$$\langle s \cup t, x' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} a \in \mathcal{C} \setminus (\max s)^+ \cap [k_{i-1}, k_i].$$

We show that  $\langle s \cup t, x' \rangle$  forces one of the three possible conclusions in the statement of the lemma.

By possibly extending  $t$ , we may assume that  $t$  contains at least one element that is greater than  $a$ . Let  $l_0 := \max(t \cap a)$  and  $l^* := \min(t \setminus a)$ . Furthermore, let  $m^* := \max(x_2 \cap l^*)$ . Hence,  $l_0$  and  $l^*$  are consecutive elements of  $t$  and  $l_0 \leq m^* < l^*$  and  $l_0 < a \leq l^*$ . We distinguish between two cases:

**Case I.** Assume  $l_0 \leq m^* \leq a \leq l^*$ .

If  $l^* \in [k_{i-1}, k_i)$ , we are done, since this means that  $\langle s \cup t, x' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} l^* \in \text{range}(\eta) \cap [k_{i-1}, k_i)$ . Hence, assume  $l^* \notin [k_{i-1}, k_i)$ , i.e.,  $l^* \notin N_i$ . Note that  $l^*$  witnesses that

$$N \models \exists l : l = \min(x_2 \setminus a).$$

Hence, by (vi), we have that  $l^* \in N_{i+1}$  and thus  $l^* \in [k_i, k_{i+1})$ .

**Case II.** Assume  $l_0 < a < m^* < l^*$ .

Let  $t' := t \cap a$ , i.e.,  $l_0 := \max t'$ , and let  $i^* \in \omega \setminus \{0\}$  be such that  $l_0 \in [k_{i^*-1}, k_{i^*})$ , i.e.,  $l_0$  first appears in  $N_{i^*}$ . If  $i^* = i$ , we are again done, hence assume that  $a \notin N_{i^*}$ . We will show that  $i^* = i - 1$ .

Let  $j \in \omega$  be such that  $l^*$  is the  $j$ 'th elements of  $s \cup t$  in increasing order. By Lemmata 2.4 and 2.5, there is  $C_{t' \cup \{l^*\}} \subseteq (l^*)^+$  such that

$$\langle s \cup t' \cup \{l^*\}, x_2 \setminus (l^*)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \mathcal{C} \cap (l^*)^+ = C_{t' \cup \{l^*\}}.$$

Set  $D^* := C_{t' \cup \{l^*\}} \cap (l_0, m^*)$ . Since

$$\langle s \cup t' \cup \{l^*\}, x_2 \setminus (l^*)^+ \rangle \geq_{\mathbb{M}_{\mathcal{U}}} \langle s \cup t, x' \rangle,$$

and since  $l_0 < a < m^*$  by assumption, we must have  $a \in D^*$ . Furthermore, note that  $D^* \subseteq C_{t' \cup \{l^*\}} \cap (l_0, l^*]$  and thus  $|D^*| =: \gamma \leq 2^j$ .

Now,  $m^*$ ,  $l^*$  and  $D^*$  witness that

$$N \models \exists \langle m, l, D \rangle : \begin{cases} m, l \in x_2 \setminus l_0^+, m < l, \text{ and} \\ D \subseteq (l_0, m), \text{ and} \\ |D| = \gamma, \text{ and} \\ \langle s \cup t' \cup \{l\}, x_2 \setminus l^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \mathcal{C} \cap (l_0, m) = D. \end{cases}$$

Since  $l_0$  is the  $(j-1)$ 'th element of  $s \cup t'$ , we have  $l_0 > 2^j$  by Fact 2.6.<sup>8</sup> Hence, since  $l_0 \in N_{i^*}$ , it follows that  $\gamma \in N_{i^*}$ . Thus, all the parameters in the above formula lie in  $N_{i^*}$ , which implies that there exists  $\langle m^\dagger, l^\dagger, D^\dagger \rangle \in N_{i^*+1}$  satisfying the formula.

CLAIM.  $l^\dagger \geq a$

Note that the proof of this claim will finish the proof of the Lemma, since  $l^\dagger \in N_{i^*+1}$  by (v) and thus  $a \in N_{i^*+1} \setminus N_{i^*}$ .

*Proof of Claim.* Assume by contradiction that  $l^\dagger < a$ , i.e.,

$$l_0 < m^\dagger < l^\dagger < a < m^* < l^*.$$

By Lemma 2.5, we have that

$$C_{t' \cup \{l^\dagger\}} \cap (m^\dagger) = C_{t' \cup \{l^*\}} \cap (m^\dagger).$$

Since  $\langle s \cup t' \cup \{l^\dagger\}, x_2 \setminus (l^\dagger)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \mathcal{C} \cap (l_0, m^\dagger) = D^\dagger$ , it follows that  $C_{t' \cup \{l^*\}} \cap (m^\dagger) = D^\dagger$  and hence  $D^\dagger = D^* \cap (l_0, m^\dagger)$ . However, both  $D^\dagger$  and  $D^*$  have size  $\gamma$  and thus  $D^* \subseteq (l_0, m^\dagger)$ , which is a contradiction to the fact that  $a \in D^*$  and  $a > m^\dagger$ . This completes the proof of the Claim and Lemma 2.7 as well.  $\dashv$

We now only need one final lemma to finish the proof of the proposition and thus of the MAIN THEOREM.

LEMMA 2.8. *Let  $I := \{[k_i, k_{i+1}) : i \in \omega\}$  be any interval partition of  $\omega$  and  $E$  and  $\mathcal{U}$  non-isomorphic  $Q$ -points. Then there exist  $v \in E$  and  $u \in \mathcal{U}$  such that*

$$\forall i \in \omega \setminus \{0\} : v \cap [k_i, k_{i+1}) \neq \emptyset \implies \begin{cases} u \cap [k_{i-1}, k_i) = \emptyset, \text{ and} \\ u \cap [k_i, k_{i+1}) = \emptyset, \text{ and} \\ u \cap [k_{i+1}, k_{i+2}) = \emptyset. \end{cases}$$

*Proof.* Say that a  $Q$ -point element *selects* from an interval partition if it intersects each interval in exactly one point. Let  $v_0 \in E$  and  $u_0 \in \mathcal{U}$  be such that they select from  $I$ . Let  $f$  be an order-preserving bijection from  $v_0$  to  $u_0$ , extended to a permutation of  $\omega$ . Thus, for each  $i \in \omega$ ,  $f$  sends the element selected by  $v_0$  in  $[k_i, k_{i+1})$  to the element

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<sup>8</sup>Note that the additional requirement in Fact 2.6 that  $\max s$  is already larger than  $2^{|s|}$  is needed here, since  $l_0$  could be  $\max s$ .

selected by  $u_0$  in  $[k_i, k_{i+1})$ . Since  $E$  and  $\mathcal{U}$  are non-isomorphic, there exist  $v_1 \in [v_0]^\omega \cap E$  and  $u_1 \in [u_0]^\omega \cap \mathcal{U}$  such that  $u_1 \cap f[v_1] = \emptyset$ . Hence, for all  $i \in \omega \setminus \{0\}$ :

$$v_1 \cap [k_i, k_{i+1}) \neq \emptyset \implies u_1 \cap [k_i, k_{i+1}) = \emptyset.$$

Both  $E$  and  $\mathcal{U}$  contain the set

$$y_\varepsilon := \bigcup_{\substack{i \in \omega \\ i \equiv \varepsilon \pmod{3}}} [k_i, k_{i+1}),$$

each for exactly one  $\varepsilon = \varepsilon(E), \varepsilon(\mathcal{U}) \in 3$ . Let  $v_2 := v_1 \cap y_{\varepsilon(E)} \in E$  and  $u_2 := u_1 \cap y_{\varepsilon(\mathcal{U})} \in \mathcal{U}$ . If  $\varepsilon(E) = \varepsilon(\mathcal{U})$  then  $v_2$  and  $u_2$  satisfy the lemma, hence assume without loss of generality that  $\varepsilon(E) = 0$  and  $\varepsilon(\mathcal{U}) = 1$ .

Let  $\bar{v}_0 \in E$  and  $\bar{u}_0 \in \mathcal{U}$  be elements that select from the interval partition

$$\{[k_i, k_{i+2}) : i \in \omega, i \equiv 0 \pmod{3}\} \cup \{[k_i, k_{i+1}) : i \in \omega, i \equiv 2 \pmod{3}\}.$$

Again, by considering a permutation of  $\omega$  that maps the element selected by  $\bar{v}_0$  in any interval to the element selected by  $\bar{u}_0$  in the same interval, we find  $\bar{v}_1 \in [\bar{v}_0]^\omega \cap E$  and  $\bar{u}_1 \in [\bar{u}_0]^\omega \cap \mathcal{U}$  such that  $\bar{v}_1$  and  $\bar{u}_1$  never select from the same interval. Now, clearly,  $v_1 \cap \bar{v}_1 \in E$  and  $u_1 \cap \bar{u}_1 \in \mathcal{U}$  work.  $\dashv$

We can now finish the proof of the proposition and hence of the main theorem: Let  $v \in E$ ,  $u \in \mathcal{U}$  be given by the previous lemma for the interval partition  $\{[k_i, k_{i+1}) : i \in \omega\} \cup \{[0, k_0)\}$  constructed in the proof of Lemma 2.7. Let  $G * H$  be any  $\mathbb{M}_{\mathcal{U}} * \mathbb{R}$ -generic filter containing  $\langle\langle s, x_2 \rangle, \underline{q}' \rangle$ . By Lemma 2.7, we have that in  $\mathbf{V}[G * H]$ , whenever  $\text{range}(\underline{a}[G * H]) \setminus (\max s)^+$  intersects one of the intervals  $[k_i, k_{i+1})$ , then the Mathias real  $\eta$  intersects  $[k_i, k_{i+1})$  or one of the adjacent intervals  $[k_{i-1}, k_i)$  or  $[k_{i+1}, k_{i+2})$ . Since  $\text{range}(\eta)$  is almost contained in  $u$ , the same is true for  $u$  in place of  $\eta$  above some  $n \geq (\max s)^+$ . Hence,  $\text{range}(\underline{a}[G * H]) \setminus n$  is disjoint from  $v$ .  $\dashv$

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