

DIAMOND ON KUREPA TREES

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ABSTRACT. We introduce a new weak variation of diamond that is meant to guess only the branches of a Kurepa tree. We demonstrate that this variation is considerably weaker than diamond by proving it is compatible with Martin's axiom. We then prove that this principle is nontrivial by showing it may consistently fail.

1. INTRODUCTION

Kurepa [Kur35] initiated the systematic study of uncountable trees through his attempt to solve Souslin's problem [Sou20] that asks whether the real line is the unique dense complete linear order with no endpoints satisfying that every pairwise disjoint collection of open intervals is countable. Kurepa showed that a counterexample is equivalent to the existence of what he called a *Souslin tree*. After sharing his findings with Aronszajn, the latter was able to construct a poor man's version of a Souslin tree that Kurepa then named an *Aronszajn tree*. A third type of an uncountable tree is now known as a *Kurepa tree*. These trees and their higher analogs were central in the development of set theory, where quite a few key concepts and techniques in forcing theory and large cardinals were discovered through solving problems concerning these objects. A milestone work here is Jensen's study [Jen72] of the fine structure of Gödel's inner model L [God40]. First, Jensen proved that a Souslin tree exists in L [Jen68]. Soon after, Solovay proved that a Kurepa tree exists in L and Silver [Sil71] extended it to outer universes under an optimal anti-large-cardinal assumption. Jensen (with a touch by Kunen) formulated the axiom \diamond as a *guessing principle* that holds in L but is sufficient for the construction of a Souslin tree in any universe of set theory [Jen72], and then Jensen, Kunen and Silver devised the stronger variation \diamond^+ as an axiom sufficient for the construction of a Kurepa tree [JK69].

Uncountable trees and diamond principles found many applications in infinite Ramsey theory, topology, measure theory and algebra. To give just one example, we mention that in [She74], the third author proved that \diamond^+ is sufficient to imply that all Whitehead groups of size \aleph_1 are free. And there are in addition some indirect applications, e.g., the Higman-Stone

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construction of an uncountable inverse system of rings whose limit is empty [HS54] which involves the rediscovery of an Aronszajn tree.

By now, a good understanding of diamond principles has been reached. For instance, the connection to other postulates such as the continuum hypothesis and forcing axioms has been determined [ST71, DJ74, DS78, SK80, Bau84, She10, GR12] and pump-up lemmas were discovered showing that weaker diamonds can be made stronger by various combinatorial maneuvers [Dev78, Dev79, Kun80, Mat87, She05].

Motivated by the current state of this understanding and by the fact that \diamond^+ is actually equivalent to the existence of a particular sort of a Kurepa tree [Kun80, Exercise VI.9], in this paper we revisit the very basics by proposing a generalization of diamond that becomes most interesting when tested against Kurepa trees and prove that all the familiar features of diamonds break down in the new context. Unlike the usual diamond, our new diamond does not imply the continuum hypothesis and it is compatible with forcing axioms. Furthermore, it is indestructible under various notions of forcing and can be made indestructible under all *proper* forcings. Three additional features that the classical diamond possesses and our new diamond lacks are: guessing uncountably often is equivalent to guessing stationarily often, guessing correctly modulo a countable error is equivalent to guessing correctly with no errors, and the feature that diamond cannot be added by a ccc forcing.

Next we make things precise. For this, let us first recall the definition of the trees under discussion.

Definition 1. A *binary tree of height ω_1* is a set T satisfying the following three requirements:

- $T \subseteq {}^{<\omega_1}2$, i.e., T consists of binary sequences of countable length;
- T is downward-closed, i.e., for all $t \in T$ and $\alpha < \text{dom}(t)$, $t \upharpoonright \alpha \in T$;
- for every $\alpha < \omega_1$, the α^{th} level of T , $T_\alpha := \{t \in T \mid \text{dom}(t) = \alpha\}$, is nonempty.

We denote by $\mathcal{B}(T) := \{f \in {}^{\omega_1}2 \mid \forall \alpha < \omega_1 (f \upharpoonright \alpha \in T_\alpha)\}$ the collection of all *cofinal branches* through T .

We now arrive at the main definition of this paper.

Definition 2. For a binary tree T of height ω_1 , the axiom $\diamond(T)$ asserts the existence of a sequence $\langle t_\alpha \mid \alpha < \omega_1 \rangle$ such that:

- for every $\alpha < \omega_1$, t_α is a function from α to 2;
- for every $f \in \mathcal{B}(T)$, the set $\{\alpha < \omega_1 \mid f \upharpoonright \alpha = t_\alpha\}$ is stationary.¹

Modulo the identification of sets with their characteristic functions, the classical \diamond is by definition equivalent to $\diamond({}^{<\omega_1}2)$, where ${}^{<\omega_1}2$ is the complete binary tree of height ω_1 . In particular, \diamond implies $\diamond(T)$ for every T .

¹It follows that we may assume that $\langle t_\alpha \mid \alpha < \omega_1 \rangle$ is a transversal for $\prod_{\alpha < \omega_1} \{t \in T_\alpha \mid t \text{ is extended by some element of } \mathcal{B}(T)\}$.

A decomposition theorem of Ulam [Ula30] easily implies that $\diamond(T)$ holds for every T such that $|\mathcal{B}(T)| \leq \aleph_1$. In addition, if $\diamond(T)$ holds then so does $\diamond(T')$ for some tree T' of size \aleph_1 with $\mathcal{B}(T') = \mathcal{B}(T)$. When put together, this shows that the study of $\diamond(T)$ should focus on those T of size \aleph_1 such that $|\mathcal{B}(T)| > \aleph_1$. This is a well-known class of trees. Indeed, it is the first class of the following definition.

Definition 3. A binary tree T of height ω_1 is:

- (1) *weak Kurepa* iff $|T| = \aleph_1 < |\mathcal{B}(T)|$;²
- (2) *Kurepa* iff $|T_\alpha| < \aleph_1$ for all $\alpha < \omega_1$ and $|\mathcal{B}(T)| > \aleph_1$;
- (3) *Aronszajn* iff $|T_\alpha| < \aleph_1$ for all $\alpha < \omega_1$ and $\mathcal{B}(T) = \emptyset$;
- (4) *Souslin* iff it is Aronszajn and for every uncountable $X \subseteq T$ there are $x, y \in X$ with $x \subsetneq y$.

Note that by Cantor's theorem, ${}^{<\omega_1}2$ happens to be a weak Kurepa tree iff the continuum hypothesis holds. Also note that it follows from the previous discussion and the trivial fact that every Kurepa tree is a weak Kurepa tree that the failure of diamond over a Kurepa tree would constitute the ultimate failure of our new principle, hence also of the classical one.

1.1. Main results. We will establish that $\diamond \implies \clubsuit \implies \clubsuit_w \implies \diamond(T)$ for every T of size \aleph_1 , and show that our new principle is considerably weaker than the other ones. Moreover, it will be shown that Martin's axiom is compatible with $\diamond(T)$ holding over a binary Kurepa tree T . This is a corollary to any of the following two results:³

Theorem A. *If T and S are two binary Kurepa trees and $|\mathcal{B}(T)| < |\mathcal{B}(S)|$, then $\diamond(T)$ holds.*

Theorem B. *It is consistent that there exists a binary Kurepa tree T such that $\diamond(T)$ holds and cannot be killed by a proper forcing.*⁴

There are additional results that we already hinted upon earlier on, but the main result of this paper is the finding that this very weak variation of diamond is nevertheless nontrivial:

Theorem C. *It is consistent that $\diamond(T)$ fails for some binary Kurepa tree T . Furthermore, such a tree T can be chosen to be either rigid or homogeneous. Furthermore, the failure of $\diamond(T)$ can be arranged together with $\diamond(T')$ holding for some binary Kurepa tree T' having the same number of branches as that of T .*

The model witnessing the preceding is obtained by a countable support iteration of proper notions of forcing for adding a branch that evades a

²These trees were first considered by Mitchell [Mit73], and the naming comes from [Tod81]. Baumgartner and Davies [Dav79, Bau83] call them *Canadian trees*.

³For every regular uncountable $\kappa = \kappa^{<\kappa}$, Martin's axiom together with $2^{\aleph_0} = \kappa$ may be introduced by a ccc (in particular, proper) forcing (see [Kun80, Theorem VIII.6.3]).

⁴A proper forcing may kill the Kurepa-ness of T , but then $\diamond(T)$ will hold trivially.

potential diamond sequence using models as side conditions. Curiously, the said Kurepa tree T will start its life in the ground model as a particular Aronszajn tree obtained from \diamond^+ .

1.2. Organization of this paper. In Section 2, we provide some preliminaries on trees, justify our focus on Kurepa trees that are binary and compare the principle $\diamond(T)$ with other standard set-theoretic hypotheses. The proof of Theorem A will be found there.

In Section 3, we study the existence of Kurepa trees T such that $\diamond(T)$ holds in an indestructible way. The proof of Theorem B will be found there.

In Section 4, we present a notion of forcing $\mathbb{Q}(T, \vec{t})$ to add a branch through an \aleph_1 -tree T that evades a given potential diamond sequence $\vec{t} = \langle t_\alpha \mid \alpha < \omega_1 \rangle$. We give a sufficient condition for $\mathbb{Q}(T, \vec{t})$ to be proper, and then turn to iterate it. The proof of Theorem C will be found there.

2. WARM UP

2.1. Forcing background. We assume the reader is comfortable with the method of forcing [Kun80], and is familiar with proper forcing [She98, Abr10], its characterization in terms of preservation of generalized stationary sets, and the fact that this class is closed under 2-step iterations as well as under countable support iterations.

2.2. Abstract, Hausdorff and binary trees. A *tree* is a partially ordered set $\mathbf{T} = (T, <_T)$ such that, for every $x \in T$, the cone $x_\downarrow := \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$; its order type is denoted by $\text{ht}(x)$. For any ordinal α , the α^{th} -level of the tree is the collection $T_\alpha := \{x \in T \mid \text{ht}(x) = \alpha\}$. The *height* of the tree is the first ordinal α for which $T_\alpha = \emptyset$. The tree is *normal* iff for every $t \in T$ and every ordinal α in-between $\text{ht}(t)$ and the height of the tree, there exists some $y \in T_\alpha$ with $t <_T y$. The tree is *Hausdorff* iff for every limit ordinal α and all $x, y \in T_\alpha$, if $x_\downarrow = y_\downarrow$, then $x = y$. In particular, a (nonempty) Hausdorff tree has a unique root.

An \aleph_1 -*tree* is a tree \mathbf{T} of height ω_1 all of whose levels are countable. A *Kurepa tree* (resp. *Aronszajn tree*) is an \aleph_1 -tree \mathbf{T} satisfying that the set $\mathcal{B}(\mathbf{T})$ of all uncountable maximal chains in \mathbf{T} has size $\geq \aleph_2$ (resp. is empty). Definition 2 generalizes to abstract trees of height ω_1 following Footnote 1 and by interpreting $f \restriction \alpha$ (for $f \in \mathcal{B}(\mathbf{T})$ and $\alpha < \omega_1$) as the unique element of T_α that belongs to f .

Hereafter, whenever we talk about a *binary* \aleph_1 -*tree*, we mean a set T as in Definition 1 such that T_α is countable for every $\alpha < \omega_1$, and we shall freely identify it with the Hausdorff \aleph_1 -tree $\mathbf{T} := (T, \subsetneq)$.

Lemma 2.1. *For every Hausdorff \aleph_1 -tree $\mathbf{T} = (T, <_T)$, there exists a binary \aleph_1 -tree S that is club-isomorphic to \mathbf{T} , i.e., for some club $D \subseteq \omega_1$, $(\bigcup_{\alpha \in D} T_\alpha, <_T)$ and $(\bigcup_{\alpha \in D} S_\alpha, \subsetneq)$ are order-isomorphic.*

In particular, if \mathbf{T} is Kurepa, then S is Kurepa and $\diamond(S)$ iff $\diamond(\mathbf{T})$.

Proof. Let $\mathbf{T} = (T, <_T)$ be a given Hausdorff \aleph_1 -tree. For every $\alpha < \omega_1$, fix an injection $\varphi_\alpha : T_{\alpha+1} \rightarrow \omega$. Also fix an injective sequence $\langle r_m \mid m < \omega \rangle$ of functions from ω to 2. We shall define an injection $\psi : T \rightarrow {}^{<\omega_1}2$ satisfying the following two requirements:

- (1) for every $t \in T$, $\text{dom}(\psi(t)) = \omega \cdot \text{ht}(t)$;
- (2) for all $t', t \in T$, $t' <_T t$ iff $\psi(t') \subsetneq \psi(t)$.

The definition of ψ is by recursion on the heights of the nodes in \mathbf{T} . By Clause (1) we are obliged to send the unique root of \mathbf{T} to \emptyset and by Clauses (1) and (2), for every $\alpha \in \text{acc}(\omega_1)$ and every $t \in T_\alpha$,⁵ we are obliged to set $\psi(t) := \bigcup \{\psi(t') \mid t' <_T t\}$.⁶ Thus, the only freedom we have is at nodes of successor levels. Here, for every $\alpha < \omega_1$ such that $\psi \upharpoonright T_\alpha$ has already been defined, and for every $t \in T_{\alpha+1}$, let t^- denote the immediate predecessor of t , and set

$$\psi(t) := \psi(t^-) \hat{\ } r_{\varphi_\alpha(t)}.$$

Finally, consider $S := \{s \upharpoonright \alpha \mid s \in \text{Im}(\psi), \alpha \leq \text{dom}(s)\}$. It is clear that S is a downward-closed subfamily of ${}^{<\omega_1}2$. Thus, to see that S is a binary \aleph_1 -tree, it suffices to prove that all of its levels are countable. However, by Clauses (1) and (2), more is true, namely, for every $\alpha < \omega_1$,

$$S_\alpha = \{\psi(t) \upharpoonright \alpha \mid t \in T_\alpha\}.$$

Consider the club $D := \{\alpha < \omega_1 \mid \omega \cdot \alpha = \alpha\}$. Evidently, ψ witnesses that $(\bigcup_{\alpha \in D} T_\alpha, <_T)$ and $(\bigcup_{\alpha \in D} S_\alpha, \subsetneq)$ are order-isomorphic. \square

Any tree $\mathbf{T} = (T, <_T)$ can be cofinally-embedded in a Hausdorff tree of the form (S, \subsetneq) by letting S be the downward closure of the collection of all $s : \alpha \rightarrow T$ that are order-preserving maps from a *successor* ordinal (α, \in) onto some downward-closed subset of $(T, <_T)$. The map that sends each $t \in T$ to the unique $s \in S$ with $\max(\text{Im}(s), <_T) = t$ embeds \mathbf{T} to the successor levels of S . In case that \mathbf{T} is non-Hausdorff, there is no better embedding. Since non-Hausdorff trees lack genuine limit levels and since our definition of diamond on trees has to do with stationary sets, the study here will be focused on the theory of diamond on Hausdorff Kurepa trees. By Lemma 2.1, then, we may moreover focus on binary Kurepa trees.

2.3. Diamonds and clubs. As said before, $\diamond \implies \clubsuit \implies \clubsuit_w \implies \diamond(T)$ for those T of size \aleph_1 . The first two implications are well-known. We now give the details of the last one.

Proposition 2.2. *If \clubsuit_w holds, then so does $\diamond(T)$ for every binary tree T of height ω_1 and size \aleph_1 .*

Proof. Suppose that \clubsuit_w holds. By [FSS97, p. 61], this means that we may fix a sequence $\langle B_\alpha \mid \alpha \in \text{acc}(\omega_1) \rangle$ such that each B_α is a cofinal subset of

⁵For a set of ordinals A , we write $\text{acc}(A) := \{\alpha \in A \mid \sup(\alpha) = \alpha > 0\}$.

⁶The injectivity here follows from Hausdorff-ness.

α of order-type ω , and, for every uncountable $B \subseteq \omega_1$, the following set is stationary:

$$G(B) := \{\alpha \in \text{acc}(\omega_1) \mid B_\alpha \setminus B \text{ is finite}\}.$$

Now, suppose T is a binary tree of height ω_1 and size \aleph_1 . Fix a bijection $\pi : \omega_1 \leftrightarrow T$. For every $\alpha \in \text{acc}(\omega_1)$, if there are $t_\alpha \in T_\alpha$ and a finite $F_\alpha \subseteq B_\alpha$ such that

$$t_\alpha = \bigcup \{\pi(\beta) \mid \beta \in B_\alpha \setminus F_\alpha\},$$

then we keep this unique t_α ; otherwise (including the case $\alpha \notin \text{acc}(\omega_1)$), we let t_α be an arbitrary choice of an element of T_α . To see that $\langle t_\alpha \mid \alpha < \omega_1 \rangle$ witnesses $\diamond(T)$, let $f \in \mathcal{B}(T)$. Consider the following club:

$$D := \{\delta \in \text{acc}(\omega_1) \mid \forall \gamma < \delta [\pi^{-1}(f \upharpoonright \gamma) < \delta]\}.$$

Note that for every $\delta \in D$, $\delta \leq \pi^{-1}(f \upharpoonright \delta) < \min(D \setminus (\delta + 1))$. In particular, the following set is uncountable:

$$B := \{\pi^{-1}(f \upharpoonright \delta) \mid \delta \in D\}.$$

We claim that for every $\alpha \in G(B)$, it is the case that $f \upharpoonright \alpha = t_\alpha$. Indeed, the set $F_\alpha := B_\alpha \setminus B$ is finite, and for every $\beta \in B_\alpha \setminus F_\alpha$, there exists a unique $\delta \in D$ such that $\delta \leq \beta = \pi^{-1}(f \upharpoonright \delta) < \min(D \setminus (\delta + 1))$. So, since $\sup(B_\alpha \setminus F_\alpha) = \alpha$, it is the case that

$$t_\alpha = \bigcup \{\pi(\beta) \mid \beta \in B_\alpha \setminus F_\alpha\} = f \upharpoonright \alpha,$$

as sought. □

Since the classical \diamond coincides with $\diamond^{(<\omega_1 2)}$, the preceding yields the implication $\clubsuit \wedge \text{CH} \implies \diamond$. Using [FSS97, Lemma 3.7], it also implies that $\diamond(T)$ is compatible with $\neg \text{CH}$. An alternative reasoning goes through the next proposition. Namely, add \aleph_2 many Cohen reals to a model of \diamond^+ .

Proposition 2.3. *Suppose that \mathbf{T} is a Kurepa tree such that $\diamond(\mathbf{T})$ holds, and that \mathbb{P} is a notion of forcing. If \mathbb{P} is σ -closed or if \mathbb{P}^2 satisfies the ccc, then $\diamond(\mathbf{T})$ remains to hold in $V^{\mathbb{P}}$.*

Proof. By [Sil71], a σ -closed forcing does not add new branches to \aleph_1 -trees. By [Ung13, Lemma 2.2], a forcing notion whose square satisfies the ccc does not add new branches to \aleph_1 -trees. In addition, stationary sets are preserved by σ -closed notions of forcing and by ccc notions of forcing. So, in both cases, the ground model witness to $\diamond(\mathbf{T})$ will survive as a witness in $V^{\mathbb{P}}$. □

Another simple argument shows that an instance of the parameterized diamond principle from [DHM04] is enough.

Proposition 2.4. *If $\Phi(\omega, =)$ holds, then so does $\diamond(T)$ for every binary Kurepa tree T .*

Proof. Suppose that $\Phi(\omega, =)$ holds. This means that for every function $F : {}^{<\omega_1}2 \rightarrow \omega$, there exists a function $g : \omega_1 \rightarrow \omega$ such that, for every function $f : \omega_1 \rightarrow 2$, the following set is stationary:

$$G(f) := \{\alpha < \omega_1 \mid F(f \upharpoonright \alpha) = g(\alpha)\}.$$

Now, suppose T is a binary Kurepa tree. For every $\alpha < \omega_1$, fix an enumeration $\langle t_\alpha^n \mid n < \omega \rangle$ of T_α . Fix a function $F : {}^{<\omega_1}2 \rightarrow \omega$ such that for all $\alpha < \omega_1$ and $t \in T_\alpha$,

$$F(t) = \min\{n < \omega \mid t = t_\alpha^n\},$$

and then let $g : \omega_1 \rightarrow \omega$ be the corresponding function given by $\Phi(\omega, =)$. A moment's reflection makes it clear that the transversal $\langle t_\alpha^{g(\alpha)} \mid \alpha < \omega_1 \rangle$ witnesses $\diamond(T)$. \square

Recall that a *Souslin tree* is an Aronszajn tree with no uncountable antichains. Every Souslin tree contains a Souslin subtree that is normal. As established in [DHM04], forcing with such trees gives an instance of $\Phi(\omega, =)$. A variant of that argument gives the following.

Proposition 2.5. *Forcing with a normal Souslin tree adds a $\diamond(T)$ -sequence for every ground model binary \aleph_1 -tree T .*

Proof. Working in V , suppose that $\mathbf{S} = (S, <_S)$ is a normal Souslin tree, and that T is a binary \aleph_1 -tree. As \mathbf{S} is normal and $\mathcal{B}(\mathbf{S}) = \emptyset$, for each $s \in S$, we may let $\langle s^n \mid n < \omega \rangle$ be a bijective enumeration of some infinite maximal antichain above s . For every $\alpha < \omega_1$, fix an enumeration $\langle t_\alpha^n \mid n < \omega \rangle$ of T_α . Given a generic G for $\mathbf{S}^* := (S, >_S)$, for every $\alpha < \omega_1$, denote by s_α the unique element of S_α that belongs to G , and then let $g(\alpha)$ denote the unique integer $n < \omega$ such that $(s_\alpha)^n$ belongs to G . We claim that the transversal $\langle t_\alpha^{g(\alpha)} \mid \alpha < \omega_1 \rangle$ witnesses $\diamond(T)$.

To see this, back in V , fix a condition $s \in S$, a name \dot{f} for a cofinal branch through T , and a club $C \subseteq \omega_1$; we shall find an $\alpha \in C$ and an extension of s forcing that $\dot{f} \upharpoonright \alpha$ coincides with $t_\alpha^{\dot{g}(\alpha)}$. Note that since \mathbf{S}^* is ccc, we can indeed restrict our attention to ground model clubs.

Fix a countable $M \prec H_{\omega_2}$, containing $\{\dot{f}, s, \mathbf{S}^*, T, C\}$. Write $\alpha := M \cap \omega_1$, and note that $\alpha \in C$, since $C \in M$. As \mathbf{S} is normal, fix $s' \in S_\alpha$ extending s . By a folklore fact, s' is an \mathbf{S}^* -generic branch over M , so in particular it determines $\dot{f} \upharpoonright \alpha$, and therefore there is an integer n such that

$$s' \Vdash \dot{f} \upharpoonright \alpha = t_\alpha^n.$$

Set $s'' := (s')^n$, so that $s'' \Vdash \dot{g}(\alpha) = n$. Then $s'' >_S s' >_S s$, and

$$s'' \Vdash \dot{f} \upharpoonright \alpha = t_\alpha^{\dot{g}(\alpha)},$$

as sought. \square

Corollary 2.6. *Suppose T is a normal binary Souslin tree. If T is almost-Kurepa, then in some ccc forcing extension, T is a binary Kurepa tree and $\diamond(T)$ holds.*

Proof. By definition, T is *almost-Kurepa* iff in the forcing extension by $\mathbb{P} := (T, \supseteq)$, T is a Kurepa tree. By Proposition 2.5, $\diamond(T)$ will hold in $V^{\mathbb{P}}$. \square

2.4. Weak variations. Devlin [Dev79, §2] showed that if there exists a sequence $\langle t_\alpha \mid \alpha < \omega_1 \rangle \in \prod_{\alpha < \omega_1} {}^\alpha 2$ such that, for every function $f : \omega_1 \rightarrow 2$, there is an infinite ordinal $\alpha < \omega_1$ with $f \upharpoonright \alpha = t_\alpha$, then \diamond holds. When combined with Theorem C, the next proposition shows that this does not generalize to our context.

Proposition 2.7. *Every \aleph_1 -tree \mathbf{T} admits a transversal $\langle t_\alpha \mid \alpha < \omega_1 \rangle$ such that $G(f) := \{\alpha < \omega_1 \mid f \upharpoonright \alpha = t_\alpha\}$ is uncountable for every $f \in \mathcal{B}(\mathbf{T})$.*

Proof. Suppose that $\mathbf{T} = (T, <_T)$ is an \aleph_1 -tree. Recall that for all $f \in \mathcal{B}(\mathbf{T})$ and $\alpha < \omega_1$, $f \upharpoonright \alpha$ denotes the unique element of T_α that belongs to f . Likewise, for every $t \in T$, and $\alpha \leq \text{ht}(t)$, we denote by $t \upharpoonright \alpha$ the unique element of T_α that is $<_T$ -comparable with t .

Now, for every limit ordinal $\beta < \omega_1$, fix a surjection $\phi_\beta : \omega \rightarrow T_{\beta+\omega}$, and then for every $n < \omega$, let $t_{\beta+n} := \phi_\beta(n) \upharpoonright (\beta+n)$. To see that $\langle t_\alpha \mid \alpha < \omega_1 \rangle$ is as sought, let $f \in \mathcal{B}(T)$. For every limit $\beta < \omega_1$, as $f \upharpoonright (\beta+\omega)$ is in T , we may find some $n < \omega$ such that $\phi_\beta(n) = f \upharpoonright (\beta+n)$. Consequently,

$$f \upharpoonright (\beta+n) = \phi_\beta(n) \upharpoonright (\beta+n) = t_{\beta+n}.$$

It follows that for some stationary $B \subseteq \omega_1$ and some $n' < \omega$, $G(f) \supseteq \{\beta+n' \mid \beta \in B\}$. In particular, $G(f)$ is uncountable. \square

In [Kun80, Theorem II.7.14], Kunen proved that \diamond^- is equivalent to \diamond . This means that if there exists a sequence $\langle T_\alpha \mid \alpha < \omega_1 \rangle \in \prod_{\alpha < \omega_1} [{}^\alpha 2]^{\leq \aleph_0}$ such that, for every function $f : \omega_1 \rightarrow 2$, the set $\{\alpha < \omega_1 \mid f \upharpoonright \alpha \in T_\alpha\}$ is stationary, then there exists a sequence $\langle t_\alpha \mid \alpha < \omega_1 \rangle \in \prod_{\alpha < \omega_1} {}^\alpha 2$ such that, for every function $f : \omega_1 \rightarrow 2$, the set $\{\alpha < \omega_1 \mid f \upharpoonright \alpha = t_\alpha\}$ is stationary.

In contrast, if \mathbf{T} is an \aleph_1 -tree, then for every $\alpha < \omega_1$, T_α is a countable subset of ${}^\alpha 2$ and for every $f \in \mathcal{B}(\mathbf{T})$, the set $\{\alpha < \omega_1 \mid f \upharpoonright \alpha \in T_\alpha\}$ is the whole of ω_1 . By Theorem C, this shows that in our context, guessing correctly modulo a countable error is not equivalent to guessing correctly.

2.5. Additional ways to get diamond. Kunen proved that \diamond cannot be introduced by a ccc forcing [Kun80, Exercise VII.H9], whereas Proposition 2.5 demonstrates that this is not the case with $\diamond(T)$. Another small ccc forcing that adds a diamond sequence for every ground model Kurepa tree is Cohen's forcing $\text{Add}(\omega, 1)$. The core of this fact can be restated combinatorially, as follows.

Proposition 2.8. *Suppose that \mathbf{T} is a Kurepa tree. If $\text{cov}(\mathcal{M}) > |\mathcal{B}(\mathbf{T})|$,⁷ then $\diamond(\mathbf{T})$ holds.*

⁷That is, if the real line cannot be covered by $|\mathcal{B}(\mathbf{T})|$ -many meager sets.

Proof. For every $\alpha < \omega_1$, fix an enumeration $\langle t_\alpha^n \mid n < \omega \rangle$ of T_α . Fix a partition $\langle S_i \mid i < \omega \rangle$ of ω_1 into stationary sets, and let $\pi : \omega_1 \rightarrow \omega$ be such that $\pi[S_i] = \{i\}$ for every $i < \omega$. For each $f \in \mathcal{B}(\mathbf{T})$, define a map $r_f : \omega \rightarrow \omega$ via:

$$r_f(i) := \min\{n < \omega \mid \{\alpha \in S_i \mid f \upharpoonright \alpha = t_\alpha^n\} \text{ is stationary}\}.$$

Finally, assuming $\text{cov}(\mathcal{M}) > |\mathcal{B}(\mathbf{T})|$, by [BJ95, Lemma 2.4.2], we may fix a function $g : \omega \rightarrow \omega$ such that $r_f \cap g \neq \emptyset$ for every $f \in \mathcal{B}(\mathbf{T})$. Then the transversal $\langle t_\alpha^{g(\pi(\alpha))} \mid \alpha < \omega_1 \rangle$ witnesses $\diamond(\mathbf{T})$. \square

Remark 2.9. \diamond^+ implies that $\diamond(\mathbf{T})$ holds for some Kurepa tree \mathbf{T} with $\text{cov}(\mathcal{M}) < |\mathcal{B}(\mathbf{T})|$. In Corollary 3.5 below, we get a model in which $\diamond(\mathbf{T})$ holds for some Kurepa tree \mathbf{T} with $\text{cov}(\mathcal{M}) = |\mathcal{B}(\mathbf{T})|$.

Remark 2.10. A proof similar to that of Proposition 2.8 shows that $\diamond(\mathbf{T})$ holds for every Kurepa tree \mathbf{T} with $\mathfrak{d}_{\omega_1} > |\mathcal{B}(\mathbf{T})|$.

Our next task is proving Theorem A. For this, let us recall the following definition.

Definition 2.11. A tree $\mathbf{T} = (T, <_T)$ of height ω_1 is:

- a *weak Kurepa tree* iff $|T| = \aleph_1 < |\mathcal{B}(\mathbf{T})|$;
- a *Jech–Kunen tree* iff $|T| = \aleph_1 < |\mathcal{B}(\mathbf{T})| < 2^{\aleph_1}$;
- *thick* iff $|\mathcal{B}(\mathbf{T})| = 2^{\aleph_1}$.

As mentioned earlier, CH implies that $(^{<\omega_1}2, \subsetneq)$ is a thick weak Kurepa tree.

Proposition 2.12. *Suppose that \mathbf{T} is a Kurepa tree, \mathbf{S} is a weak Kurepa tree, and $|\mathcal{B}(\mathbf{T})| < |\mathcal{B}(\mathbf{S})|$. Then $\diamond(\mathbf{T})$ holds.*

Proof. By [IR24, Corollary 7.18], if κ is a regular uncountable cardinal and there is a Kurepa tree with (at least) κ -many branches, then $\text{onto}(\{\aleph_1\}, J^{\text{bd}}[\kappa], \aleph_0)$ holds. The same proof works equally well against weak Kurepa trees. Therefore, $\kappa := |\mathcal{B}(\mathbf{T})|^+$ is a regular uncountable cardinal such that $\text{onto}(\{\aleph_1\}, J^{\text{bd}}[\kappa], \aleph_0)$ holds. This means that we may fix a map $c : \omega_1 \times \kappa \rightarrow \omega$ such that for every $B \in [\kappa]^\kappa$, there exists $i < \omega_1$ such that $c[\{i\} \times B] = \omega$. For each $\beta < \kappa$, define $g_\beta : \omega_1 \rightarrow \omega$ via $g_\beta(i) := c(i, \beta)$. It is easy to check that for every $\mathcal{R} \in [{}^{\omega_1}\omega]^{<\kappa}$, there exists some $\beta < \kappa$ such that $r \cap g_\beta \neq \emptyset$ for every $r \in \mathcal{R}$. From this point on, the proof continues the same way as that of Proposition 2.8, where the only change is that we fix a partition $\langle S_i \mid i < \omega_1 \rangle$ into \aleph_1 -many stationary sets and so each of the r_f 's is now a function from ω_1 to ω . \square

Corollary 2.13. CH implies $\diamond(\mathbf{T})$ for every non-thick Kurepa tree \mathbf{T} . \square

3. INDESTRUCTIBLE DIAMONDS

We would like to show that if the universe is close to being constructible, then there exists a Kurepa tree on which \diamond holds. The construction of such a tree relies on the concept of a *sealed* Kurepa tree:

Definition 3.1 (Hayut-Müller, [HM23]). A Kurepa tree \mathbf{T} is *sealed* if for every notion of forcing \mathbb{P} that preserves both ω_1 and ω_2 , \mathbb{P} does not add a new branch through \mathbf{T} .

Fact 3.2 (Poór-Shelah, [PS21, §4]). *If $\omega_1 = \omega_1^{\mathbb{L}[A]}$ and $\omega_2 = \omega_2^{\mathbb{L}[A]}$ for some $A \subseteq \omega_1$, then there exists a Kurepa tree \mathbf{T} such that $\mathcal{B}(\mathbf{T}) \subseteq \mathbb{L}[A]$.*⁸

Corollary 3.3 (Giron-Hayut, [GH23]). *If $\omega_1 = \omega_1^{\mathbb{L}[A]}$ and $\omega_2 = \omega_2^{\mathbb{L}[A]}$ for some $A \subseteq \omega_1$, then there exists a sealed Kurepa tree.* \square

Corollary 3.4. *Suppose that $V = \mathbb{L}[A]$ for some $A \subseteq \omega_1$. Then there exists a binary Kurepa tree T such that $\diamond(T)$ holds in any forcing extension preserving ω_1, ω_2 , and the stationary subsets of ω_1 .*

Proof. Let \mathbf{T} be a tree as in Fact 3.2. It can be verified that \mathbf{T} is Hausdorff, but, regardless, as described right after Lemma 2.1, there is a Hausdorff tree \mathbf{S} and an order-preserving injection π from \mathbf{T} to \mathbf{S} such that $\pi[T_\alpha] = S_{\alpha+1}$ for every $\alpha < \omega_1$. In particular, π induces a bijective correspondence between $\mathcal{B}(\mathbf{T})$ and $\mathcal{B}(\mathbf{S})$, meaning that we may as well assume that \mathbf{T} is Hausdorff. Next, as $A \subseteq \omega_1$, it is the case that \diamond holds in $\mathbb{L}[A]$, so $\diamond(\mathbf{T})$ holds as well. Suppose \mathbb{P} is a notion of forcing preserving ω_1 and ω_2 . The construction of \mathbf{T} takes place inside $\mathbb{L}[A]$, so

$$V^{\mathbb{P}} \models \mathcal{B}(\mathbf{T}) \subseteq \mathbb{L}[A].$$

As the definition of $\mathbb{L}[A]$ is absolute, all the branches through \mathbf{T} from $V^{\mathbb{P}}$ are already in V . Now, if \mathbb{P} also preserves stationary subsets of ω_1 , then the witness to $\diamond(\mathbf{T})$ in V will still witness $\diamond(\mathbf{T})$ in $V^{\mathbb{P}}$. Finally, by running the translation procedure of Lemma 2.1, we obtain a binary Kurepa tree with the same key features as the Hausdorff tree \mathbf{T} . \square

Corollary 3.5. *Each of the following is compatible with $\diamond(T)$ holding for some binary Kurepa tree T :*

- (1) *Martin's axiom and $2^{\aleph_0} = \aleph_2$;*
- (2) *There are no Souslin trees (in particular, \diamond fails), and GCH holds.*

Proof. Work in \mathbb{L} . By Corollary 3.4, we may fix a binary Kurepa tree T such that $\diamond(T)$ holds in any forcing extension preserving ω_1, ω_2 , and the stationary subsets of ω_1 .

(1) Let \mathbb{P} be some ccc notion of forcing such that in $\mathbb{L}^{\mathbb{P}}$, Martin's axiom holds and $2^{\aleph_0} = \aleph_2$. As \mathbb{P} preserves ω_1, ω_2 , and the stationary subsets of ω_1 , $\diamond(T)$ holds in $\mathbb{L}^{\mathbb{P}}$.

(2) As GCH, \diamond and \square_{\aleph_1} all hold, we may let \mathbb{P} be Jensen's poset for forcing GCH and the Souslin hypothesis (see [DJ74]). \mathbb{P} preserves ω_1, ω_2 , and the stationary subsets of ω_1 , hence $\diamond(T)$ holds in $\mathbb{L}^{\mathbb{P}}$. \square

⁸This was generalized by Hayut and Müller [HM23, Lemma 15] to any successor of a regular uncountable cardinal κ such that $\kappa^+ = (\kappa^+)^{\mathbb{L}}$.

A Kurepa tree that is sealed for all proper forcings can also be added by forcing, and in fact such a notion of forcing was already devised by D. H. Stewart in his 1966 Master's thesis.

Definition 3.6 (Stewart [Ste66], see [Jec71]). For a cardinal $\kappa > \aleph_1$, the forcing \mathbb{S}_κ consists of all triples (T_p, ϵ_p, b_p) such that:

- (1) $\epsilon_p \in \text{acc}(\omega_1)$;
- (2) T_p is a countable downward-closed subfamily of ${}^{\leq \epsilon_p} 2$ such that (T_p, \varnothing) is a normal tree of height $\epsilon_p + 1$;
- (3) b_p is an injection from a countable subset of κ to the top level of T_p ,

and the ordering is given by $q \leq p$ iff

- $\epsilon_q \geq \epsilon_p$;
- $T_q \supseteq T_p$ and $T_q \upharpoonright (\epsilon_p + 1) = T_p$;
- $\text{dom}(b_q) \supseteq \text{dom}(b_p)$;
- for every $\xi \in \text{dom}(b_p)$, $b_q(\xi) \supseteq b_p(\xi)$.

Note that \mathbb{S}_κ is σ -closed, and that, assuming CH, it has the \aleph_2 -cc. The following establishes Theorem B.

Proposition 3.7. *Suppose $\kappa > \aleph_1$ is some cardinal. In the forcing extension by \mathbb{S}_κ , there exists a binary Kurepa tree T such that $\diamond(T)$ holds and no proper forcing adds a new branch through T , let alone kills diamond over it.*

Proof. Suppose G is \mathbb{S}_κ -generic. Let $T := \bigcup \{T_p \mid p \in G\}$ denote the generic binary tree added by \mathbb{S}_κ . For each $\xi < \kappa$, let

$$f_\xi := \{t \in T \mid \exists p \in G (\xi \in \text{dom}(b_p) \wedge t \subseteq b_p(\xi))\}.$$

A density argument shows that T is an \aleph_1 -tree and that $\langle f_\xi \mid \xi < \kappa \rangle$ is an injective sequence of branches through it. By a theorem of Baumgartner, any σ -closed forcing that adds a new subset of ω_1 forces \diamond to hold (see [Sak06, Theorem 3.1]), hence $\diamond(T)$ hold in $V[G]$.

In [Jec71, p. 10], Jech proved that $\{f_\xi \mid \xi < \kappa\}$ enumerates *all* cofinal branches through T . It is a folklore fact that this remains the case in any extension by a proper notion of forcing. Hints of this fact may be found in [Tod84, Lemma 8.14] and [Kos05, Proposition 37], but for completeness, we give the details here. Following [Kos05, Definition 1], we consider the following set:

$$\mathcal{S} := \{X \in [\kappa]^{\aleph_0} \mid X \cap \omega_1 \in \text{acc}(\omega_1), T_{X \cap \omega_1} = \{f_\xi \upharpoonright (X \cap \omega_1) \mid \xi \in X\}\}.$$

Claim 3.7.1. \mathcal{S} is stationary.

Proof. Back in V , pick a name \dot{C} for a club in $[\kappa]^{\aleph_0}$, and a condition p . Recursively construct a sequence $\langle (p_n, X_n) \mid n < \omega \rangle$ such that $(p_0, X_0) := (p, \omega)$ and, for every $n < \omega$:

- $p_{n+1} \leq p_n$;
- $\epsilon_{p_{n+1}} > \epsilon_{p_n}$;
- $\text{dom}(b_{p_{n+1}}) \supseteq X_n$;

- every element of T_{p_n} is extended by some element of $\text{Im}(b_{p_{n+1}})$;
- $X_{n+1} \in [\kappa]^{\aleph_0}$ with $X_{n+1} \supseteq X_n \cup \text{dom}(b_{p_n}) \cup (\sup(X_n \cap \omega_1) + 1)$;
- $p_{n+1} \Vdash X_{n+1} \in \dot{C}$.

Set $X := \bigcup_{n < \omega} X_n$ and note that $X \cap \omega_1 \in \text{acc}(\omega_1)$. Define a condition q by letting:

- (1) $\text{dom}(b_q) := \bigcup_{n < \omega} X_n$;
- (2) for all $n < \omega$ and $\xi \in X_n$, $b_q(\xi) := \bigcup_{n < m < \omega} b_{p_m}(\xi)$;
- (3) $\epsilon_q := \sup_{n < \omega} \epsilon_{p_n}$;
- (4) $T_q := \bigcup_{n < \omega} T_{p_n} \cup \text{Im}(b_q)$.

It is clear that $q \leq p_n$ for all $n < \omega$, and hence $q \Vdash X \in \dot{C}$. In addition, Clause (4) implies that $q \Vdash X \in \dot{S}$. \square

Now, let \mathbb{Q} be any proper forcing in $V[G]$, and work in $V[G][H]$, where H is \mathbb{Q} -generic. Towards a contradiction, suppose that f is a branch through T distinct from any of the f_ξ 's. As \mathbb{Q} is proper, \mathcal{S} remains stationary, so we may fix an elementary submodel $M \prec H_\theta$ (for a large enough regular cardinal θ) containing $\{T, f, \langle f_\xi \mid \xi < \kappa \rangle\}$ such that $X := M \cap \kappa$ is in \mathcal{S} . By elementarity,

$$M \models \forall \xi < \kappa (f \neq f_\xi).$$

Denote $\alpha := X \cap \omega_1$. As f is a branch through T , we get that $f \upharpoonright \alpha \in T_\alpha$. As $X \in \mathcal{S}$, we may find a $\xi \in X$ such that $f \upharpoonright \alpha = f_\xi \upharpoonright \alpha$. But $\xi \in X = M \cap \kappa$, and then, by elementarity,

$$M \models f = f_\xi.$$

This is a contradiction. \square

Despite the fact that the generic tree added by \mathbb{S}_κ is sealed for proper forcings, we can still add new branches to it without collapsing cardinals or adding reals using the quotient forcing $\mathbb{S}_{\kappa^+}/\mathbb{S}_\kappa$. More generally, consider the countable support iteration $(\langle \mathbb{P}_\xi \mid \xi \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_\xi \mid \xi < \kappa \rangle)$, for an uncountable cardinal κ , where \mathbb{Q}_0 is the Jech partial order for adding a Souslin tree T , and, for every nonzero $\xi < \kappa$,

$$\mathbb{P}_\xi \Vdash \dot{\mathbb{Q}}_\xi = T.$$

By Proposition 3.7 and the upcoming proposition, \mathbb{P}_κ is proper (even σ -strategically closed) for any choice of κ , and yet the quotient forcings of the form $\mathbb{P}_\mu/\mathbb{P}_\lambda$ are not proper, whenever $\aleph_2 \leq \lambda < \mu$.

Proposition 3.8. \mathbb{P}_κ has a dense subset that is isomorphic to \mathbb{S}_κ .

Proof. Let R denote the collection of all *rectangular* conditions in \mathbb{P}_κ , i.e., the collection of all conditions q for which there exists some $\delta < \omega_1$ such that:

- $q(0)$ is a tree of height $\delta + 1$,
- for every $\xi \in \kappa \setminus \{0\}$, $q(\xi)$ is either trivial or a node at the δ^{th} -level of $q(0)$,

- all maximal nodes of $q(0)$ which are of the form $q(\xi)$, for $\xi \in \kappa \setminus \{0\}$, are pairwise distinct.

To see that R is isomorphic to \mathbb{S}_κ , note that we can map a condition $q \in R$ to a condition $\phi(q)$ in \mathbb{S}_κ , by keeping the tree coordinate, and using the function $b_{\phi(q)}$ to record all maximal nodes of $q(0)$ that are of the form $q(\xi)$. More precisely, we put:

- $T_{\phi(q)}(0) := q(0)$,
- $\epsilon_{\phi(q)} := \text{ht}(q(0)) - 1$,
- $b_{\phi(q)}(\xi) := q(\xi)$, whenever $q(\xi)$ belongs to the top level of $q(0)$.

This mapping is clearly order-preserving, with the image

$$\{p \in \mathbb{S}_\kappa \mid 0 \notin \text{dom}(b_p)\},$$

which is obviously isomorphic to \mathbb{S}_κ . Moreover, it is straightforward to define an inverse of ϕ .

It remains to show that R is dense. Let p be an arbitrary condition in \mathbb{P}_κ . Fix a countable $M \prec H_\theta$, for a sufficiently large regular cardinal θ , containing all relevant objects. Let $\delta := \omega_1 \cap M$, and let $G \subseteq P \cap M$ be a \mathbb{P}_κ -generic filter over M , containing p . For $\xi \in \kappa \cap M$, let us denote by $G(\xi)$ the projection of G to the coordinate ξ . Therefore $\bigcup G(0)$ is a tree of height δ , with countable levels, and for each $\xi \in \kappa \cap M \setminus \{0\}$, $G(\xi)$ determines a cofinal branch f_ξ through $G(0)$, given by the formula

$$f_\xi := \bigcup \{t \in {}^{<\omega_1}2 \mid \exists r \in G (r \upharpoonright \xi \Vdash r(\xi) = t)\}.$$

The latter follows from the observation that for any $\epsilon < \delta$, the set

$$D_{\xi, \epsilon} := \{r \in \mathbb{P}_\kappa \mid \exists t \in {}^{<\omega_1}2 (r \upharpoonright \xi \Vdash r(\xi) = t \wedge \text{dom}(t) \geq \epsilon)\},$$

is dense in \mathbb{P}_κ , and belongs to M .

Finally, we define a condition q by the following considerations:

- $q(0) := G(0) \cup \{f_\xi \mid \xi \in M \cap \kappa\}$,
- $q(\xi) := f_\xi$ for every $\xi \in M \cap \kappa \setminus \{0\}$,
- $q(\xi) := \emptyset$ for every $\xi \in \kappa \setminus M$.

Then q is a rectangular condition extending p , as sought. \square

4. THE FAILURE OF DIAMOND ON A KUREPA TREE

The main result of this section is a consistent example of a binary Kurepa tree on which diamond fails. At the end of this section, we shall derive a few additional corollaries.

As a first step, we present a notion of forcing with side conditions for adding a branch through a given normal binary \aleph_1 -tree T . The poset takes a transversal \vec{t} of the complete binary tree (${}^{<\omega_1}2, \subseteq$) as a second parameter, and ensures that the generic branch will disagree on a club with this transversal.

Definition 4.1. Let T be a normal binary \aleph_1 -tree, and let $\vec{t} \in \prod_{\alpha < \omega_1} {}^\alpha 2$.

The forcing $\mathbb{Q}(T, \vec{t})$ consists of all triples $p = (x_p, \mathcal{M}_p, f_p)$ that satisfy all of the following:

- (1) x_p is a node in T ;
- (2) \mathcal{M}_p is a finite, \in -increasing chain of countable elementary submodels of H_{ω_1} ;
- (3) f_p is a partial function from \mathcal{M}_p to ω_1 ;
- (4) for every $M \in \mathcal{M}_p$:
 - $\text{dom}(x_p) \geq M \cap \omega_1$,
 - $x_p \upharpoonright (M \cap \omega_1) \neq \vec{t}(M \cap \omega_1)$;
 - $f_p \upharpoonright M \in M$.

The ordering is defined by letting $q \leq p$ iff

- (1) $x_q \supseteq x_p$;
- (2) $\mathcal{M}_q \supseteq \mathcal{M}_p$;
- (3) for every $M \in \text{dom}(f_p)$, $M \in \text{dom}(f_q)$ and $f_q(M) \geq f_p(M)$.

Remark 4.2. $\mathbb{Q}(T, \vec{t}) \subseteq H_{\omega_1}$, so that $\mathbb{Q}(T, \vec{t}) \in H_{\mathfrak{c}^+}$.

Lemma 4.3. *Suppose that T is a normal binary \aleph_1 -Souslin tree, and $\vec{t} \in \prod_{\alpha < \omega_1} {}^\alpha 2$. Let $p \in \mathbb{Q}(T, \vec{t})$. Then for every countable $M^* \prec H_{\omega_2}$ with $\mathbb{Q}(T, \vec{t}) \in M^*$ such that $M^* \cap H_{\omega_1} \in \mathcal{M}_p$, p is M^* -generic.*

Proof. Fix a model M^* as above, and let $D \in M^*$ be a dense open subset of $\mathbb{Q}(T, \vec{t})$; we need to find an $r \in D \cap M^*$ compatible with p .

Fix a large enough $\gamma \in M^* \cap \omega_1$ such that $N \cap \omega_1 \subseteq \gamma$ for every $N \in \mathcal{M}_p \cap M^*$. Define a condition

$$\bar{p} := (x_p \upharpoonright \gamma, \mathcal{M}_p \cap M^*, f_p \upharpoonright M^*),$$

and note that $\bar{p} \in M^*$.

Claim 4.3.1. *The set $D' := \{x_q \mid q \in D, q \leq \bar{p}\}$ is dense in (T, \supseteq) below $x_{\bar{p}}$.*

Proof. Let y be any extension of $x_{\bar{p}}$. Evidently, $(y, \mathcal{M}_{\bar{p}}, f_{\bar{p}})$ is a legitimate condition. Pick $q \in D$ such that $q \leq (y, \mathcal{M}_{\bar{p}}, f_{\bar{p}})$. Then x_q is an extension of y that lies in D' , as sought. \square

Denote $\delta := M^* \cap \omega_1$. Since T is a Souslin tree lying in M^* , any node in T_δ is (T, \supseteq) -generic over M^* . In particular, $x_p \upharpoonright \delta$ is (T, \supseteq) -generic over M^* , and it follows that there is an $x \in D' \cap M^*$ such that

$$x \subseteq x_p \upharpoonright \delta.$$

Now, pick any q witnessing $x \in D'$, so that $x = x_q \subseteq x_p$. By elementarity, we can assume that $q \in M^*$.

Define a condition r by letting $x_r := x_p$, $\mathcal{M}_r := \mathcal{M}_p \cup \mathcal{M}_q$, and f_r be such that $\text{dom}(f_r) = \text{dom}(f_p) \cup \text{dom}(f_q)$ and

$$f_r(\alpha) := \max(\{f_p(\alpha) \mid \alpha \in \text{dom}(f_p)\} \cup \{f_q(\alpha) \mid \alpha \in \text{dom}(f_q)\}).$$

It is not hard to see that r is a legitimate condition extending both p and q . \square

Lemma 4.4. *Suppose that T is a normal binary \aleph_1 -Souslin tree, and $\vec{t} \in \prod_{\alpha < \omega_1} {}^\alpha 2$. Then:*

- (1) $\mathbb{Q}(T, \vec{t})$ is proper;
- (2) $\mathbb{Q}(T, \vec{t})$ adds a branch through T that evades \vec{t} on a club.

Proof. (1) To verify properness, fix $p \in \mathbb{Q}(T, \vec{t})$, and a countable $M^* \prec H_{\omega_2}$, containing everything relevant including p . Denote $\delta := M^* \cap \omega_1$. As T is normal and Souslin, we may be able to extend x_p to some node $x \in T_\delta$ distinct from $\vec{t}(\delta)$.

The triple $(x, \mathcal{M}_p \cup \{M^* \cap H_{\omega_1}\}, f_p)$ is a condition in $\mathbb{Q}(T, \vec{t})$, and by Lemma 4.3, it is an M^* -generic condition.

- (2) Let G be a generic filter. It suffices to verify that the following uncountable set is moreover closed:

$$D := \{M \cap \omega_1 \mid \exists p \in G (M \in \mathcal{M}_p)\},$$

that is, every $\gamma \in \kappa \setminus D$ is not an accumulation point of D .

Let \dot{D} be a name for the above set, and suppose $p \Vdash \dot{\gamma} \in \kappa \setminus \dot{D}$. By extending p , we can assume that it forces that $\dot{\gamma} = \gamma$, and γ lies between two consecutive elements $N_0 \in N_1$ of \mathcal{M}_p . By extending p further, we can assume that $f_p(N_0) \geq \gamma$. Since γ is forced to not be the height of any element of \mathcal{M}_p , it follows that $N_0 \cap \omega_1 < \gamma$, and

$$p \Vdash \dot{D} \cap (N_0 \cap \omega_1, \gamma) = \emptyset,$$

as sought. \square

Definition 4.5. Given a sequence of binary \aleph_1 -trees T^0, \dots, T^n , the tree product $T^0 \otimes \dots \otimes T^n$ is the collection of all $(x_0, \dots, x_n) \in T^0 \times \dots \times T^n$ such that $\text{dom}(x_0) = \dots = \text{dom}(x_n)$, and the ordering is such that a node (x_0, \dots, x_n) is below a node (y_0, \dots, y_n) iff $x_i \subsetneq y_i$ for all $i \leq n$.

In our context, it will be useful to know of the following fact which follows from [BRY25, Corollary 5.7] as in the proof of [BRY25, Corollary 5.3].

Fact 4.6 ([BRY25, §5]). *If \diamond^+ holds, then there is a binary \aleph_1 -Aronszajn tree T and a sequence $\vec{T} = \langle T^\eta \mid \eta < \omega_2 \rangle$ of normal binary \aleph_1 -subtrees of T such that, for every nonempty $a \in [\omega_2]^{<\omega}$, the tree product $\bigotimes_{\eta \in a} T^\eta$ is an \aleph_1 -Souslin tree.*

It is well-known that if the product $T \otimes S$ of two normal binary \aleph_1 -trees is Souslin, then

$$(T, \supseteq) \Vdash \text{“}S \text{ is Souslin”}.$$

Thus, it is natural to expect that furthermore

$$\mathbb{Q}(T, \vec{t}) \Vdash \text{“}S \text{ is Souslin”}.$$

The next lemma shows that this is indeed the case.

Lemma 4.7. *Suppose T and S are normal binary \aleph_1 -trees the product of which is Souslin, and let $\vec{t} \in \prod_{\alpha < \omega_1} {}^\alpha 2$. Then*

$$\mathbb{Q}(T, \vec{t}) \Vdash \text{“}S \text{ is Souslin”}.$$

Proof. Let $q \in \mathbb{Q}(T, \vec{t})$, and fix a $\mathbb{Q}(T, \vec{t})$ -name \dot{A} for a maximal antichain in S . Fix a countable $M^* \prec H_{\omega_2}$ containing $\{\mathbb{Q}(T, \vec{t}), q, \dot{A}\}$. Denote $\delta := M^* \cap \omega_1$, and pick a node $x \in T_\delta \setminus \{\vec{t}(\delta)\}$ extending x_q . We define an extension $q' \leq q$ by letting:

$$q' := (x, \mathcal{M}_q \cup \{M^* \cap H_{\omega_1}\}, f_q).$$

It is clear that q' is indeed a condition and $q' \leq q$. We claim that

$$q' \Vdash \text{“}\dot{A} \cap (S \upharpoonright \delta) \text{ is a maximal antichain in } S\text{”}.$$

To see this, pick a condition $p \leq q'$ and a node $y \in S_\delta$; our aim is to find a condition $r \leq p$ that forces y to extend an element of \dot{A} .

Fix a large enough $\gamma < \delta$ such that $N \cap \omega_1 \subseteq \gamma$ for every $N \in \mathcal{M}_p \cap M^*$. Let $\bar{x} := x_p \upharpoonright \gamma$ and $\bar{y} := y \upharpoonright \gamma$. Define a condition

$$\bar{p} := (\bar{x}, \mathcal{M}_p \cap M^*, f_p \upharpoonright M^*).$$

Now, we turn to run a recursion producing a family

$$\mathcal{E} = \{(q_\alpha, y_\alpha) \mid \alpha < \theta\},$$

satisfying that for each α :

- (1) $q_\alpha \in \mathbb{Q}(T, \vec{t})$ with $q_\alpha \leq \bar{p}$;
- (2) $y_\alpha \in S$ with $\bar{y} \subseteq y_\alpha$;
- (3) $q_\alpha \Vdash \exists a \in \dot{A} (a \subseteq y_\alpha)$;
- (4) $\text{dom}(x_{q_\alpha}) = \text{dom}(y_\alpha)$;
- (5) for all $\beta < \alpha$, $(x_{q_\beta}, y_\beta) \perp (x_{q_\alpha}, y_\alpha)$ in $T \otimes S$.

We continue the construction until it is not possible to choose the next element. Because $T \otimes S$ is Souslin, Requirement (5) ensures that the construction terminates after countably many steps, so the ordinal θ ends up being countable.

Claim 4.7.1. *The set $\{(x_{q_\alpha}, y_\alpha) \mid \alpha < \theta\}$ is a maximal antichain above (\bar{x}, \bar{y}) in $T \otimes S$.*

Proof. Suppose not. Then we may pick $(x', y') \in T \otimes S$ extending (\bar{x}, \bar{y}) that is incompatible with (x_{q_α}, y_α) for every $\alpha < \theta$. By possibly extending both x' and y' , we may assume that $x' = x_{q'}$ for some condition $q' \leq \bar{p}$ which moreover satisfies:

$$q' \Vdash \exists a \in \dot{A} (a \subseteq y').$$

But now we can add (q', y') to \mathcal{E} , contradicting its maximality. \square

Since $q \in M^*$, by making canonical choices in the recursive construction we may secure that the countable family \mathcal{E} be a subset of M^* . So, by Claim 4.7.1, we may find an $\alpha < \theta$ such that $(x_{q_\alpha} \subseteq x$ and $y_\alpha \subseteq y)$. Since $\mathcal{E} \subseteq M^*$, we infer that $q_\alpha \in M^*$. Now, as in the proof of Lemma 4.3,

we see that there exists a condition r extending both p and q_α . Recalling Requirement (3) in the choice of q_α , we get that

$$r \Vdash \exists a \in \dot{A} (a \subseteq y),$$

as sought. \square

We are now in conditions to prove the core of Theorem C.

Theorem 4.8. *It is consistent that there exists a binary Kurepa tree T such that $\diamond(T)$ fails.*

Proof. We start with a model of GCH and \diamond^+ . Using Fact 4.6, fix a binary \aleph_1 -tree T and a sequence $\vec{T} = \langle T^\xi \mid \xi < \omega_2 \rangle$ of normal binary \aleph_1 -subtrees of T such that $\bigotimes_{\xi \in a} T^\xi$ is an \aleph_1 -Souslin tree for every nonempty $a \in [\omega_2]^{<\omega}$.⁹ Note that for all $\xi \neq \eta$, the trees T^ξ and T^η have a countable intersection.

Let \mathcal{P} denote the collection of all pairs (\mathbb{R}, τ) such that \mathbb{R} is a notion of forcing lying in H_{ω_2} , and τ is a nice \mathbb{R} -name for an element of $\prod_{\alpha < \omega_1} T_\alpha$. Using GCH, we may fix a (repetitive) enumeration $\langle (\mathbb{R}_\xi, \tau_\xi) \mid \xi < \omega_2 \rangle$ of \mathcal{P} in such a way that every pair is listed cofinally often.

Finally, we force with \mathbb{P}_{ω_2} , where $(\langle \mathbb{P}_\xi \mid \xi \leq \omega_2 \rangle, \langle \dot{Q}_\xi \mid \xi < \omega_2 \rangle)$ is the countable support iteration satisfying that for every $\xi < \omega_2$:

- (i) $\mathbb{P}_\xi \Vdash \dot{Q}_\xi = \dot{Q}(T^\xi, \sigma_\xi)$, and
- (ii) σ_ξ is a nice \mathbb{P}_ξ -name for an element of $\prod_{\alpha < \omega_1} T_\alpha$ such that if $\mathbb{R}_\xi = \mathbb{P}_\eta$ for some $\eta \leq \xi$, then σ_ξ is the lift of τ_η (from a \mathbb{P}_η -name to a \mathbb{P}_ξ -name).

As CH holds, and recalling Remark 4.2, we infer that $(\mathbb{P}_\xi, \sigma_\xi) \in \mathcal{P}$ for all $\xi < \omega_2$.

Claim 4.8.1. *For every $\xi < \omega_2$, \mathbb{P}_ξ forces that $\dot{Q}(T^\xi, \sigma_\xi)$ is proper.*

Proof. By Lemma 4.4(1), it suffices to prove that for every $\xi < \omega_2$,

$$\mathbb{P}_\xi \Vdash \text{“}T^\xi \text{ is Souslin”}.$$

We shall prove by induction on $\xi < \omega_2$ a stronger claim, namely that for every finite tuple $\xi < \eta_0 < \dots < \eta_n < \omega_2$,

$$\mathbb{P}_\xi \Vdash \text{“}T^\xi \otimes T^{\eta_0} \otimes \dots \otimes T^{\eta_n} \text{ is Souslin”}.$$

There are three cases to consider:

- $\xi = 0$. This is since the sequence \vec{T} was given by Fact 4.6.
- $\xi + 1$. Recall that

$$\mathbb{P}_{\xi+1} \simeq \mathbb{P}_\xi * \dot{Q}(T^\xi, \sigma_\xi).$$

Let $\xi + 1 < \eta_0 < \dots < \eta_n < \omega_2$ be a finite tuple. By the induction hypothesis,

$$\mathbb{P}_\xi \Vdash \text{“}T^\xi \otimes T^{\xi+1} \otimes T^{\eta_0} \otimes \dots \otimes T^{\eta_n} \text{ is Souslin”}.$$

⁹The tree T given by the fact is moreover Aronszajn, but this feature is not necessary for our application here.

By invoking Lemma 4.7 in $V^{\mathbb{P}^\xi}$, we infer that

$$\mathbb{P}_\xi * \dot{\mathbb{Q}}(T^\xi, \sigma_\xi) \Vdash \text{“}T^{\xi+1} \otimes T^{\eta_0} \otimes \dots \otimes T^{\eta_n} \text{ is Souslin”},$$

as sought.

- $\xi \in \text{acc}(\omega_2)$. We apply a result from [AS93, §3] or [Miy93] stating that for any \aleph_1 -tree S , the property “ S is Souslin” is preserved at a limit stage of a countable support iteration of proper forcings. \square

A standard name counting argument shows that CH is preserved in every intermediate stage, so \mathbb{P}_{ω_2} satisfies the \aleph_2 -cc.¹⁰ In addition, \mathbb{P}_{ω_2} is proper, so our forcing preserves all cardinals.

Claim 4.8.2. \mathbb{P}_{ω_2} forces that T is Kurepa.

Proof. For every $\xi < \omega_2$, \mathbb{P}_{ω_2} projects to a forcing of the form $\mathbb{Q}(T^\xi, \vec{t})$, and then Lemma 4.4(2) implies that in $V^{\mathbb{P}_{\omega_2}}$, there exists a branch f_ξ through T^ξ , and in particular, through T . As the elements of $\langle T^\xi \mid \xi < \omega_2 \rangle$ have a pairwise countable intersection, in $V^{\mathbb{P}_{\omega_2}}$, $\langle f_\xi \mid \xi < \omega_2 \rangle$ is injective. \square

Claim 4.8.3. \mathbb{P}_{ω_2} forces that $\diamond(T)$ fails.

Proof. Otherwise, as $\prod_{\alpha < \omega_1} T_\alpha$ lies in the ground model, we may fix a nice \mathbb{P}_{ω_2} -name \vec{t} for a transversal $\vec{t} \in \prod_{\alpha < \omega_1} T_\alpha$ that witnesses $\diamond(T)$ in the extension. As \mathbb{P}_{ω_2} has the \aleph_2 -cc and \vec{t} is a nice name for an \aleph_1 -sized set, there is a large enough $\eta < \omega_2$ such that all nontrivial conditions appearing in \vec{t} belong to \mathbb{P}_η . It thus follows that \vec{t} admits a nice \mathbb{P}_η -name, say, τ . Clearly, $(\mathbb{P}_\eta, \tau) \in \mathcal{P}$, so we may find a large enough $\xi \in [\eta, \omega_2)$ such that $(\mathbb{R}_\xi, \tau_\xi) = (\mathbb{P}_\eta, \tau)$. Recalling Clauses (i) and (ii) in the definition of our iteration, it is the case that $\mathbb{P}_{\xi+1} \simeq \mathbb{P}_\xi * \dot{\mathbb{Q}}(T^\xi, \sigma_\xi)$ where σ_ξ is a \mathbb{P}_ξ -name for \vec{t} . By Lemma 4.4(2), then, $\mathbb{P}_{\xi+1}$ introduces a branch through T that evades \vec{t} on a club. So \vec{t} cannot witness $\diamond(T)$ in \mathbb{P}_{ω_2} . This is a contradiction. \square

This completes the proof. \square

4.1. Ramifications. Proposition 2.12 may suggest that the validity of $\diamond(T)$ for a Kurepa tree T depends only on the cardinal $|\mathcal{B}(T)|$. However, the next corollary shows that this is not the case:

Corollary 4.9. *It is consistent that there exist two binary Kurepa trees T' and T such that $|\mathcal{B}(T')| = |\mathcal{B}(T)| = 2^{\aleph_1}$, $\diamond(T')$ holds, but $\diamond(T)$ fails.*

Proof. Work in L . By Corollary 3.4, we may fix a binary Kurepa tree T' such that $\diamond(T')$ holds in any forcing extension preserving ω_1 , ω_2 , and the stationary subsets of ω_1 . As L satisfies GCH and \diamond^+ , the proof of Theorem 4.8 provides us with an \aleph_2 -cc proper notion of forcing \mathbb{P}_{ω_2} that introduces a binary Kurepa tree T on which \diamond fails. Altogether, $L^{\mathbb{P}_{\omega_2}}$ is a model satisfying the desired configuration. \square

¹⁰Recall [Bau83, Theorem 2.2].

Corollary 4.10. *It is consistent that there exists a binary Kurepa tree T such that $\diamond(T)$ fails and T is uniformly homogeneous.*¹¹

Proof. The proof is almost identical to that of Theorem 4.8. We start with a model of GCH and \diamond^+ . Using Fact 4.6, we fix a binary \aleph_1 -tree T and a sequence $\langle T^\xi \mid \xi < \omega_2 \rangle$ of normal binary \aleph_1 -subtrees of T such that $\bigotimes_{\xi \in a} T^\xi$ is an \aleph_1 -Souslin tree for every nonempty $a \in [\omega_2]^{<\omega}$. Now, let T' be the collection of all functions $t' \in {}^{<\omega_1}2$ for which there exists some $t \in T$ such that $\text{dom}(t') = \text{dom}(t)$ and $\{\alpha \in \text{dom}(t) \mid t(\alpha) \neq t'(\alpha)\}$ is finite. Then T' is a uniformly homogeneous \aleph_1 -tree having each of the T^ξ 's as a normal binary \aleph_1 -subtree. So, we can continue with the proof of Theorem 4.8 using T' instead of T . \square

Corollary 4.11. *It is consistent that there exists a binary Kurepa tree S such that $\diamond(S)$ fails and S is rigid.*

Proof. The proof is quite close to that of Theorem 4.8. We start with a model of GCH and \diamond^+ . In particular, there exists a binary Kurepa tree. Setting $(\kappa, \mathcal{U}) := (\aleph_1, \aleph_2)$, this means that there exists a κ -tree with \mathcal{U} -many cofinal branches. Instead of using Fact 4.6, we appeal to [BRY25, Corollary 5.7] with κ, \mathcal{U} and $\varsigma := \omega$ to obtain a downward-closed subfamily $T \subseteq {}^{<\omega_1}\omega$ such that (T, \subseteq) is an Aronszajn tree, every node in T admits infinitely many immediate successors, and there exists a sequence $\langle T^\xi \mid \xi < \omega_2 \rangle$ of normal downward-closed subtrees of T such that $\bigotimes_{\xi \in a} T^\xi$ is an \aleph_1 -Souslin tree for every nonempty $a \in [\omega_2]^{<\omega}$. We force with \mathbb{P}_{ω_2} , where $(\langle \mathbb{P}_\xi \mid \xi \leq \omega_2 \rangle, \langle \dot{Q}_\xi \mid \xi < \omega_2 \rangle)$ is the countable support iteration satisfying that for every $\xi < \omega_2$, $\mathbb{P}_\xi \Vdash \text{“}\dot{Q}_\xi = \dot{Q}(T^\xi, \sigma_\xi)\text{”}$, where σ_ξ is a nice \mathbb{P}_ξ -name for an element of $\prod_{\alpha < \omega_1} T_\alpha$ obtained from some bookkeeping sequence. This time, a typical transversal $\vec{t} \in \prod_{\alpha < \omega_1} T_\alpha$ is an element of $\prod_{\alpha < \omega_1} {}^\alpha\omega$ instead of $\prod_{\alpha < \omega_1} {}^\alpha 2$, but everything goes through and we end up in a generic extension in which T is a Kurepa tree on which diamond fails.

Next, let S be the binary \aleph_1 -tree produced by the proof of Lemma 2.1 when fed with the tree $\mathbf{T} := (T, \subseteq)$. Since \mathbf{T} is Kurepa on which diamond fails, S is a binary Kurepa tree and $\diamond(S)$ fails.

Recall that the proof of Lemma 2.1 made use of an injective sequence $\langle r_m \mid m < \omega \rangle$ of functions from ω to 2. For our purpose here, we shall moreover assume that for all $m \neq m'$, it is the case that $\Delta(r_m, r_{m'}) = \min\{m, m'\}$.¹²

To prove that S is rigid, we must establish the following.

Claim 4.11.1. *Suppose that $\pi : S \leftrightarrow S$ is an automorphism of (S, \subseteq) . Then π is the identity map.*

Proof. Suppose not, so that $\pi(s) \neq s$ for some $s \in S$. By the definition of S and as π is order-preserving, it follows that there exists a $t \in T$ such that $\pi(\psi(t)) \neq \psi(t)$. Let $\beta < \omega_1$ be the least for which there exists a $t \in T_\beta$

¹¹The definition of a *uniformly homogeneous* tree may be found in [BRY25, §4].

¹²This can easily be arranged by defining $r_m : \omega \rightarrow 2$ via $r_m(n) := 1$ iff $n < m$.

with $\pi(\psi(t)) \neq \psi(t)$. Clearly, β is a successor ordinal, say, $\beta = \alpha + 1$. Fix $t_0 \neq t_1$ in $T_{\alpha+1}$ such that $\pi(\psi(t_0)) = \psi(t_1)$ and note that the minimality of β implies that $t_0 \upharpoonright \alpha = t_1 \upharpoonright \alpha$, which we hereafter denote by t . Now, since every node in T admits infinitely many immediate successors, we may fix $t_2, t_3 \in T_{\alpha+1}$ such that:

- t_2, t_3 are immediate successors of t ,¹³
- $\pi(\psi(t_2)) = \psi(t_3)$, and
- $\min\{\varphi_\alpha(t_2), \varphi_\alpha(t_3)\} > \max\{\varphi_\alpha(t_0), \varphi_\alpha(t_1)\}$.

Recalling the proof of Lemma 2.1, for every $i < 4$, letting $m_i := \varphi_\alpha(t_i)$, it is the case that

$$\psi(t_i) = \psi(t) \hat{\ } r_{m_i}.$$

As $m_0 < m_2$, $\Delta(r_{m_0}, r_{m_2}) = m_0$ and

$$\Delta(\psi(t_0), \psi(t_2)) = \omega \cdot \alpha + m_0.$$

Likewise,

$$\Delta(\pi(\psi(t_0)), \pi(\psi(t_2))) = \Delta(\psi(t_1), \psi(t_3)) = \omega \cdot \alpha + m_1.$$

As $m_0 \neq m_1$, we infer that

$$\Delta(\psi(t_0), \psi(t_2)) \neq \Delta(\pi(\psi(t_0)), \pi(\psi(t_2))),$$

contradicting the fact that π is an automorphism of S . □

This completes the proof. □

Data availability. Data sharing not applicable to this article as datasets were neither generated nor analyzed.

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¹³Possibly, $t_2 = t_3$.

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