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UNIVERSAL GRAPHS BETWEEN A STRONG LIMIT SINGULAR AND ITS POWER

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ABSTRACT. The paper settles the problem of the consistency of the existence of a single universal graph between a strong limit singular and its power. Assuming that in a model of **GCH** κ is supercompact and the cardinals $\theta < \kappa$, $\lambda > \kappa$ are regular, as an application of a more general method, we obtain a forcing extension in which $\mathrm{cf}(\kappa) = \theta$, the Singular Cardinal Hypothesis fails at κ and there exists a universal graph at cardinality $\lambda \in (\kappa, 2^\kappa)$.

CONTENTS

INTRODUCTION

0(A). Background.

The existence of universal graphs at infinite cardinalities has received extensive investigation (where we mean that the graph G is universal at cardinality $|G|$ if every graph of the same cardinality is isomorphic to some induced subgraph of G). According to the classical result [Rad64], the so called countable random graph is a universal graph at $\aleph_0$ (which is also unique, up to isomorphism). A classical result (which now follows as a standard induction argument) establishes the existence of a κ^+-saturated graph on the set 2^κ [CK73]. Consequently, there exists a graph on 2^κ into which every graph on κ^+ embeds (and we can replace κ^+, 2^κ, κ^+-saturated with κ, $2^{<\kappa}$, κ-special). Therefore, assuming **GCH**, there exists a universal graph at every infinite cardinality. (However, concerning certain proper classes of graphs the situation is more intricate, even for the countable

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case, see [FK97], [Kom99], [KS95], [CS16], [KS19].) Regarding the problem of universal objects in more complex theories (i.e., beyond graphs) and the relevance of the present work in model theory, readers may consult the survey [She21] or earlier works such as [Dža05]. See also recent publications such as [She20] and [Sheb]. Another related question, the existence of universal Aronszajn trees has been extensively studied as well, see [Tod07], [DS21], and most recently [BNMV23].

However, without assuming **GCH**, it is generally much more challenging to construct universal objects. Furthermore, after adding κ^{++} Cohen subsets to a regular κ, there are no universal graphs on κ^+, as shown in [KS92].

Regarding positive results, for regular cardinals $\kappa < \lambda$, there consistently exists a universal graph of size λ, while $2^\kappa > \lambda$ [She90]. Moreover, the argument presented in [She90] also provides a universal ω-edge colored graph on ω_1 assuming $\neg$**CH**. Features of this method will be used in this paper. However, a recent study [SS21] proved that assuming $\neg$**CH**, the existence of a universal graph on ω_1 does not imply the existence of a universal ω-edge colored graph on ω_1. Furthermore, it should be noted that when considering specific classes of graphs, there are both negative [Koj98] and positive results [Mek90] for universal objects and weak universal families. (Given a class $\mathcal{K}_\lambda$ of models each of which is of cardinality λ, $\kappa < \lambda < 2^\kappa$, we say that the family $\mathcal{F} \subseteq \mathcal{K}_\lambda$ is a weak universal family for $\mathcal{K}_\lambda$ if every $G \in \mathcal{K}_\lambda$ embeds into some $G_* \in \mathcal{F}$, and $|\mathcal{F}| < 2^\kappa$.) It is also consistent that there exists a singular κ, $2^\kappa > \kappa^+$, and there is no universal graph on κ^+ [FT10, Theorem 3.3] (and it follows from their proof that κ is strong limit). For more consistency results in the absence of **GCH**, see [She93] and [DS04]. It is worth mentioning that dealing with the case $\lambda = \kappa^+$ was considerably easier in all the aforementioned cases.

In this paper, we investigate universal graphs in the interval between a strong limit singular cardinal and its power. The motivation for this question stems from the following observations. Recall that the cardinal exponentiation $2^{\aleph_0}$ can be quite large and at the same time relevant forcing axioms such as **MA** may hold. Similarly, for $\mu = \aleph_1 = 2^{\aleph_0}$, 2^μ can be large, or for $\mu = \mu^{<\mu}$, parallel results hold for forcing notions that are, for example, $< \mu$-complete and satisfy a strong form of μ^+-cc (the strong form is necessary, see [Shear]). On the other hand, much less is known for strong limit singular cardinals μ, and thus the existence of universals serves as a central test problem for examining the consistency of forcing axioms at μ.

In this paper, we continue the work of Džamonja-Shelah in [DS03], which demonstrated the consistency of the statement $(*)$ assuming the existence of a supercompact cardinal.

$(*)$ (a) μ is strong limit singular and $\mu^{++} < 2^\mu$,
(b) there is a graph G_* of cardinality μ^{++} which is universal for graphs of cardinality μ^+ (equivalently there is a sequence $\bar{G} = \langle G_\alpha : \alpha < \mu^{++} \rangle$ of graphs each of cardinality μ^+, universal for the family of such graphs).

for the case $\mathrm{cf}(\mu) = \aleph_0$, and later Cummings-Džamonja-Magidor-Morgan-Shelah proved this for arbitrary cofinality in [CDM+17]. Earlier, Mekler-Shelah [MS89] had proved such consistency results replacing (b) with uniformization principles; also starting naturally with a supercompact cardinal. Later, $(*)$ was proved to be consistent for small singular μ's too, see [CDM16], [Dav17].

Our goal is to address the naturally arising problem by replacing weak universal families (in the sense of $(*)$(b)) with single universal objects and by considering

λ in the range of $(\mu, 2^\mu)$ instead of restricting it to μ^+. Thus, we formulate the following assertions:

$(*)^+$ (a) μ is strong limit singular and $\mu^{++} < 2^\mu$,
(b) there is a universal graph G_* in μ^+, i.e. universal for graphs of cardinality μ^+, G_* itself is of cardinality μ^+,
(b)$^+$ as (b), but changing μ^+ for some cardinal in $(\mu, 2^\mu)$.

To initiate our proof, we consider a supercompact cardinal κ as our starting point. We demonstrate, as part of a more general axiomatic framework, that a stronger version of a universal on $\lambda > \kappa$ (e.g., $\lambda = \kappa^+$) is sufficient to guarantee the existence of a universal graph on λ even after forcing with a $\mathbb{P}$ that satisfies the axiomatic requirements. We first establish a general framework for the preparatory forcing, followed by the construction of a strong universal graph suited to the present framework, as in [She90]. (It is worth noting that certain large cardinal hypotheses are essential, as the failure of the Singular Cardinal Hypothesis itself implies the existence of an inner model with the Mitchell order $o(\kappa) = \kappa^{++}$ for a measurable cardinal κ; in fact, these are equiconsistent [Git91].)

The organization of the paper is as follows. In §1 we introduce the concept of $(\lambda, \kappa) - i$ $(i = 1, 2)$ systems, and in Claim 1.5 we prove that extending a ground model already admitting some strong version of universal using such a $(\lambda, \kappa) - i$ system results in a model with the desired universal object. In §2 we prove that Prikry forcing, Magidor forcing and Radin forcing give rise to a $(\lambda, \kappa) - 1$ system provided the relevant filters satisfy some reasonable directedness assumptions. In §3(A) we prepare the ground, in Claim 3.2 build the framework to force $(\lambda, \kappa) - 1$ systems using a supercompact cardinal. In §3(B) we construct a forcing for obtaining the strong universal, that fits in the framework in Claim 3.2.

In works in preparation we intend to replace graphs by more general classes; much of our work is not specific to graphs. Also for consistency of $(*)^+$ for a small singular μ, e.g. $\mu = \aleph_\omega = \beth_\omega$ [PS].

0(B). **Preliminaries.** We are interested in universal objects in the class of graphs, i.e. models of the first order language admitting no functions, only a single symmetric, non-reflexive binary relation. Under ordinals we always mean von Neumann ordinals, and for a set X the symbol $|X|$ always refers to the smallest ordinal with the same cardinality. If f is a mapping with $\mathrm{dom}(f) \supseteq X$, then $f``X = \{f(x) : x \in X\}$, i.e. the pointwise image of X. For a set X the symbol $\mathscr{P}(X)$ denotes the power set of X, while if κ is an ordinal we use the standard notation $[X]^\kappa$ for $\{Y \in \mathscr{P}(X) : |Y| = \kappa\}$, similarly for $[X]^{<\kappa}$, $[X]^{<\kappa}$, etc. By a sequence we mean a function on an ordinal, where for a sequence $\bar{s} = \langle s_\alpha : \alpha < \mathrm{dom}(\bar{s})\rangle$ the length of $\bar{s}$ (in symbols $\ell g(\bar{s})$) denotes $\mathrm{dom}(\bar{s})$. Moreover, for sequences $\bar{s}$, $\bar{t}$ let $\bar{s} \frown \bar{t}$ denote the natural concatenation (of length $\ell g(\bar{s}) + \ell g(\bar{t})$). For a set X, and ordinal α we use ${}^\alpha X = \{\bar{s} : \ell g(\bar{s}) = \alpha, \mathrm{ran}(\bar{s}) \subseteq X\}$, and for cardinals λ, κ we use the symbol $\lambda^\kappa = |{}^\kappa\lambda|$ (that is, the least ordinal equivalent to it).

We call a set $T \subseteq {}^{<\alpha}X$ a tree (where α is an ordinal), if T is downward closed, i.e. whenever $\bar{t} \in T$, $\gamma < \ell g(\bar{t})$, we have $\bar{t} \restriction \gamma \in T$. We call $\bar{t}$ a leaf, if there is no $\bar{s} \in T$ for which $t \subsetneq s$.

Regarding iterated forcing and quotient forcing we will mostly use the terminology of the survey [Bau76]. However we adhere to Conventions 0.1 and 0.2.

Convention 0.1. Regarding forcing we follow the convention that "$p \leq q$" means that q is stronger, i.e. giving more information.

Convention 0.2. A notion of forcing $\mathbb{P}$ is $<\mu$-directed closed ($<\mu$-closed, resp.), if for any directed (increasing, resp.) system $\{p_\alpha : \alpha < \nu < \mu\}$ there exists a common upper bound p_* in $\mathbb{P}$.

A filter $\mathcal{F} \subseteq \mathscr{P}(X)$ is κ-complete, if for each $\{F_\alpha : \alpha < \nu < \kappa\} \subseteq \mathcal{F}$ we have $\bigcap_{\alpha<\nu} F_\alpha \in \mathcal{F}$. A partial order P is $< \mu$-directed, if for each $\{p_\alpha : \alpha < \nu < \mu\} \subseteq P$, there exists a common upper bound $p_* \in P$. (For example, if $\mathcal{F} \subseteq \mathscr{P}(X)$ is a κ-complete filter on X, then $\mathcal{F}$ is $< \kappa$-directed with respect to the relation $\supseteq$.)

1. The framework and deducing the consistency results

1(A). **What we do.** In the present paper we introduce a general framework and apply it for the class of graphs.

We shall start with a large cardinal, such as a Laver indestructible supercompact, or with forcing a relative of it. We then have a two step forcing.

First, a forcing $\mathbb{P}$ with the following three properties:

(a) preserving the largeness of κ,
(b) moreover, in $\mathbf{V}^{\mathbb{P}}$ there is a normal κ-complete filter D on κ such that $(D, {}^*\supseteq)$ is λ^+-directed for a suitable cardinal $\lambda < 2^\kappa$,
(c) preparing the ground for the results we like to have on λ, e.g. has a strong version of "there is a universal graph in $\lambda, \lambda < 2^\kappa$".

Second, a forcing $\mathbb{Q}$ (in $\mathbf{V}^{\mathbb{P}}$) such that:

(d) $\mathbb{Q}$ makes κ singular,
(e) preserves κ is strong limit and 2^κ large.

Thirdly,

(f) to get the desired property of λ, we use $\mathbb{Q}$ that fits in the framework in Definition 1.2,
(g) then prove the existence of a universal object using the framework

In §1(B) Definition 1.2 defines the family of (λ, κ)-systems fitting (f), then we deduce the existence of universal graphs in λ (a case of (g)).

In §2 we shall prove that classical forcings for making κ singular fit our framework, i.e. satisfy (d)-(g).

In §3 we shall deal with finding $\mathbb{P}$ as in (a), (b), (c), so we have to combine the specific forcing (say forcing a universal graph in λ, i.e. clause (c)) and guaranteeing the existence of e.g. a normal ultrafilter of which is λ^+-complete in a suitable sense (i.e. clause (b)).

1(B). (λ, κ)**-systems.** The following is standard, but we have to include these definitions in order to avoid ambiguity, thus clarify what we mean under κ-Borel sets.

Definition 1.1. Assume that μ is a cardinal, Y is a set.

(1) We let $\mathscr{B}_\mu \subseteq \mathscr{P}({}^Y2)$ denote the set of μ-Borel subsets of ${}^Y2$, i.e. $\mathscr{B}_\mu({}^Y2) \subseteq \mathscr{P}({}^Y2)$ is the smallest family that satisfies
   - for each function $f : \mathrm{dom}(f) \to 2$ with $\mathrm{dom}(f) \in [Y]^{<\aleph_0}$ the basic open set (w.r.t. the product topology)

$$[f] := \{g \in {}^Y2 : g \supseteq f\} \in \mathscr{B}_\mu({}^Y2),$$

- whenever $\langle B_i : i \in \mu\rangle$ is a sequence with $(\forall i < \mu)\ B_i \in \mathscr{B}_\mu({}^Y 2)$, necessarily $\bigcup_{i\in\mu} B_i \in \mathscr{B}_\mu({}^Y 2)$,
- $\forall B \in \mathscr{B}_\mu({}^Y 2)$: $({}^Y 2 \setminus B) \in \mathscr{B}_\mu({}^Y 2)$.

(2) we say that the tree

$$T \subseteq {}^{<\omega}\{\cup, \neg, [f] :\ f : \mathrm{dom}(f) \to 2, \mathrm{dom}(f) \in\ [Y]^{<\aleph_0}\}$$

is a code for a set in $\mathscr{B}_\mu({}^Y 2)$ (in symbols, $T \in \mathrm{code}_\mu(Y)$), if
- $T \setminus \{\langle\rangle\}$ is non-empty, moreover, it has a stem $s \in T$ of length 1 (i.e. $\ell g(s) = 1$, and for each $t \in T$ with $\ell g(t) > 1$ $s \subseteq t$),
- T is well-founded, and
- for each $t \in T \setminus \{\langle\rangle\}$ we have that

$$t \text{ is a leaf of } T \iff t(\ell g(t) - 1) = [f] \text{ for a partial function } f \text{ above,}$$

- for each $t \in T \setminus \{\langle\rangle\}$, if $t(\ell g(t) - 1) = \neg$, then neither does T branch at t, nor is t a leaf (that is, $\exists! t' \in T$, $\ell g(t') = \ell g(t) + 1$, $t \subsetneq t'$), and
- for each $t \in T \setminus \{\langle\rangle\}$ with $t(\ell g(t) - 1) \neq \neg$, t has at most μ-many immediate successors, that is,

$$|\{s \in T :\ t \subsetneq s,\ \ell g(s) = \ell g(t) + 1\}| \leq \mu$$

(equivalently, $|T| \leq \mu$),

(3) we can define the evaluation B_T for $T \in \mathrm{code}_\mu(Y)$ in the obvious fashion, by induction on the rank of T. If $T = \{\langle [f]\rangle\}$, then we let $B_T = [f]$. Otherwise, T necessarily has a stem $s = \langle s(0)\rangle = \langle \cup\rangle$, or $s = \langle \neg\rangle$. For each $t \in T$, $\ell g(t) = 2$ we can naturally define the tree T_t below t, i.e.

$$T_t = \{u :\ \langle s(0)\rangle ^\frown u \in T,\ s(0)\rangle ^\frown u \supseteq t\}.$$

Now if $s(0)$ is the symbol $\cup$, then we let

$$B_T = \bigcup_{t\in T, \ell g(t)=2} B_{T_t}.$$

Otherwise, if $s(0) = \neg$, then there exists a unique $t \in T$, $\ell g(t) = 2$, and we let

$$B_T = {}^Y 2 \setminus B_{T_t}.$$

(4) Using the natural identification between ${}^Y 2$, and $\mathscr{P}(Y)$, we can talk about μ-Borel subsets of $\mathscr{P}(Y)$, $\mathscr{B}_\mu(\mathscr{P}(Y))$, and so about codes for μ-Borel subsets of $\mathscr{P}(Y)$.

Definition 1.2.

(1) We say $\mathbf{r}$ is a (λ, κ)−1-system <u>when</u> $\mathbf{r} = (\mathbb{R}, \underset{\sim}{X}, \leq_{\mathrm{pr}}, \mathscr{S}) = (\mathbb{R}_{\mathbf{r}}, \underset{\sim}{X}_{\mathbf{r}}, \leq_{\mathbf{r},\mathrm{pr}}, \mathscr{S}_{\mathbf{r}})$ satisfies the following

(a) κ is strongly inaccessible,
(b) $\lambda \in [\kappa^+, 2^\kappa)$,
(c) $\mathbb{R}$ is a forcing notion preserving "κ is strong limit",
(d) $\underset{\sim}{X}$ is an $\mathbb{R}$-name of a subset of κ,
(e) $\leq_{\mathrm{pr}} \subseteq \leq_{\mathbb{R}}$ is a quasi-order,
(f) for each $p \in \mathbb{R}$ we have $\mathscr{S}_p \subseteq \{\bar{q} \in {}^\kappa\mathbb{R} : p \leq_{\mathrm{pr}} q_\varepsilon \text{ for every } \varepsilon < \kappa\}$,

(g) whenever $p \in \mathbb{R}$, $\underset{\sim}{\tau}$ are such that $p \Vdash$ "$\underset{\sim}{\tau} \in \{0,1\}$" (a truth value), <u>then</u>:
   (∗) there are $\bar{q} \in \mathscr{S}_p, \bar{Y} = \langle Y_\varepsilon : \varepsilon < \kappa \rangle \in {}^\kappa V_\kappa$, $\bar{\mu} = \langle \mu_\varepsilon : \varepsilon < \kappa \rangle \in {}^\kappa\kappa$ and $\bar{T} = \langle T_\varepsilon : \varepsilon < \kappa \rangle$, where
      $\bullet_1$ each $T_\varepsilon \in \mathbf{V}$ is a code for a μ_ε-Borel set $B_\varepsilon \in \mathscr{B}_{\mu_\varepsilon}(\mathscr{P}(Y_\varepsilon))$ (in the sense of Definition 1.1 (2), (4)),
      $\bullet_2$ $q_\varepsilon \Vdash$ "$\underset{\sim}{\tau} = 1 \iff \underset{\sim}{X} \cap Y_\varepsilon \in B_{T_\varepsilon}$";

(h) for each $p \in \mathbb{R}$, and for each sequence $\langle \bar{q}_\alpha : \alpha < \lambda \rangle$ with $\forall \alpha < \lambda\ \bar{q}_\alpha \in \mathscr{S}_p$, there exists $q_* \in \mathbb{R}$ such that for every $\alpha < \lambda$ there exists $\varepsilon_\alpha < \kappa$ such that $q_{\alpha,\varepsilon_\alpha} \leq_{\mathbb{R}} q_*$.

(2) We say $\mathbf{r}$ is a $(\lambda,\kappa)-2$-system <u>when</u> above in clause (g) we restrict ourselves to $\underset{\sim}{\tau}$'s that are $\mathbb{R}_{\underset{\sim}{X}}$-names, where $\mathbb{R}_{\underset{\sim}{X}} \lessdot \mathbb{R}$ is the complete subforcing adding only $\underset{\sim}{X}[\mathbf{G}]$ (in other words, if $\mathbf{G} \subseteq \mathbb{R}$ is generic over $\mathbf{G}$, then letting $Z = \underset{\sim}{X}[\mathbf{G}]$, $\mathbf{V}[Z]$ is an $\mathbb{R}_{\underset{\sim}{X}}$-generic extension of $\mathbf{V}$);

(2A) We may omit the 1 in "1-system", so that "(λ,κ)-system" is always meant as "$(\lambda,\kappa) - 1$-system".

(3) We say $\mathbf{r}$ is nice <u>when</u> the forcing $\mathbb{R}_{\mathbf{r}}$ does not collapse any cardinal.

Discussion 1.3.

(1) Here we only deal with the question "when is there a universal graph in the cardinal λ?".

(2) Of course, in Definition 1.2, we are interested in the case $\Vdash_{\mathbb{R}_{\mathbf{r}}}$ "κ is singular".

(3) There are such $\mathbf{r}$'s: Prikry forcing, Magidor forcing, cases of Radin forcing, see Claim 2.1 and onwards. (In the specific case of Prikry forcing (g) can be simplified, as Y_ε will be an ordinal below κ, and the name $\underset{\sim}{\tau}$ will depend on the finite set in which the Prikry generic set meets the ordinal Y_ε.)

The following notion is necessary to phrase the framework for the main result (Claim 1.5).

Definition 1.4. Suppose that κ, λ are cardinals.

(1) We let K_κ denote the class of edge colored graphs with the set of colors indexed by κ, so formally it is defined as follows. The model M belongs to K_κ, <u>iff</u>
   (a) $M = (|M|, R^M_\varepsilon)_{\varepsilon<\kappa}$,
   (b) R^M_ε is a symmetric irreflexive two-place relation on $|M|$,
   (c) $\langle R^M_\varepsilon : \varepsilon < \kappa \rangle$ is a partition of $\{(a,b) : a \neq b \in |M|\}$.

(2) $(K_\kappa)_\lambda$ is the class of graphs in K_κ that have λ-many vertices, i.e. for $M \in K_\kappa$ we have
$$M \in (K_\kappa)_\lambda \iff \|M\| = \lambda.$$

Claim 1.5.

(1) Assume that
   (i) $\iota \in \{1,2\}$,
   (ii) κ, λ are fixed cardinals, $\kappa < \lambda < 2^\kappa$,
   (iii) $\mathbf{r} \in \mathbf{V}$ is a $(\lambda,\kappa) - \iota$-system, and let $\mathbf{V}_\iota = \mathbf{V}^{\mathbb{R}_{\mathbf{r}}}$ if $\iota = 1$; $\mathbf{V}_\iota = \mathbf{V}[X_{\mathbf{r}}]$ in case of $\iota = 2$.
   (iv) there is a universal member of $(K_\kappa)_\lambda$ (in $\mathbf{V}$),

   Then
$$\mathbf{V}_\iota \models \text{"there is a universal graph of cardinality } \lambda\text{".}$$

(2) Moreover, in general, if (i)–(iii) hold, and

(iv)$^\chi$ (in $\mathbf{V}$) there is a weak universal family of size χ in $(K_\kappa)_\lambda$, i.e. a system $\langle M_i : i < \chi\rangle$, for which for each $M \in (K_\kappa)_\lambda$ there exists $i_0 < \chi$ such that M can be embedded into M_{i_0} (in the sense of K_κ),

then

$$\mathbf{V}_\iota \models \exists\langle G_i : i < \chi\rangle : \quad \circledcirc_1\ (\forall i < \chi)\ G_i \text{ is a graph on } \lambda, \quad \circledcirc_2 \text{ and for every graph } G \text{ of size } \lambda \text{ there is } i_0 < \chi, \text{ s.t. } G \text{ can be embedded into } G_{i_0}. \tag{1.1}$$

Proof. (Claim 1.5) First note that it suffices to prove (2), as (1) is just a special case with χ being equal to 1.

$(*)_1$ Let (in $\mathbf{V}$) $\langle(U_\vartheta, \xi_\vartheta, Z_\vartheta) : \vartheta < \kappa\rangle$ list

$$\{(U, \xi, Z) : \ Z \in V_\kappa,\ \xi < \kappa \text{ is a cardinal}, \ U \text{ is a code for an } \xi-\text{Borel subset of } Z\}.$$

Assume that

$(*)_2$ there is a sequence $\bar{M} = \langle M_\delta : \delta < \chi\rangle$ in $(K_\kappa)_\lambda$ that forms a universal sequence for $(K_\kappa)_\lambda$ (in the universe $\mathbf{V}$, of course) i.e. $\bar{M}$ witnesses (iv)$^\chi$;

where $M_\delta = (\lambda, \ldots, R_\varepsilon^{M_\delta}, \ldots)_{\varepsilon<\kappa}$. It is enough to prove that $\mathbf{V}_\iota$ satisfies (1.1).

Now we define the sequence of $\mathbb{R}_{\mathbf{r}}$-names $\underset{\sim}{G}_\delta$ $(\delta < \chi)$ for graphs as follows.

$(*)_3$

(a) the set of nodes of $\underset{\sim}{G}_\delta$ is λ (and so $R^{\underset{\sim}{G}_\delta} \subseteq \lambda \times \lambda$),

(b) for $\alpha \neq \beta < \lambda$ let the truth value of "$(\alpha, \beta) \in R^{\underset{\sim}{G}_\delta}$" be defined as follows. For the unique $\vartheta < \kappa$ with $(\alpha, \beta) \in R_\vartheta^{M_\delta}$ we demand

$$\mathbf{V}_\iota \models (\alpha, \beta) \in R^{\underset{\sim}{G}_\delta} \iff \underset{\sim}{X} \cap Z_\vartheta \in B_{U_\vartheta}.$$

So clearly

$(*)_4$ for each $\delta < \chi$ $\underset{\sim}{G}_\delta$ is an $\mathbb{R}_{\mathbf{r}}$-name for a graph with set of nodes λ.

Hence it suffices to prove:

$(*)_5$ $\Vdash$ "$\mathbf{V}_\iota \models \langle\underset{\sim}{G}_\delta : \delta < \chi\rangle$ is a universal sequence in the class of graphs of size λ".

So why does $(*)_5$ hold? Assume

$(*)_{5.1}$ $p \Vdash$ "$\underset{\sim}{G}_* \in \mathbf{V}_\iota$ is a graph with set of nodes λ".

Let $\langle(\alpha_\gamma, \beta_\gamma) : \gamma < \lambda\rangle \in \mathbf{V}$ list the set of pairs (α, β) such that $\alpha < \beta < \lambda$. For each $\gamma < \lambda$ (considering the $\mathbb{R}_{\mathbf{r}}$-names $\underset{\sim}{\tau}_\gamma$ for the truth value of $(\alpha_\gamma, \beta_\gamma) \in R^{\underset{\sim}{G}_*}$) clause (g) of Definition 1.2(1) gives $\bar{q}_\gamma = \langle q_{\gamma,\varepsilon} : \varepsilon < \kappa\rangle \in \mathscr{S}_p$, $\bar{\zeta}_\gamma = \langle\zeta_{\gamma,\varepsilon} : \varepsilon < \kappa\rangle \in {}^\kappa\kappa$ and $\bar{T}_\gamma = \langle T_{\gamma,\varepsilon} : \varepsilon < \kappa\rangle$, $\langle Y_{\gamma,\varepsilon} : \varepsilon < \kappa\rangle$ such that for each $\gamma < \lambda$ and $\varepsilon < \kappa$

- $\bullet_1$ $T_{\gamma,\varepsilon}$ is a code for a $\zeta_{\gamma,\varepsilon}$-Borel subset of $\mathscr{P}(Y_{\gamma,\varepsilon})$ (in the sense of Definition 1.1(2)–(4)))
- $\bullet_2$ $q_{\gamma,\varepsilon} \Vdash_{\mathbb{R}} (\alpha_\gamma, \beta_\gamma) \in R^{\underset{\sim}{G}_*} \Leftrightarrow \underset{\sim}{X} \cap Y_{\gamma,\varepsilon} \in B_{T_{\gamma,\varepsilon}}$.

Now by clause (h) of Definition 1.2(1), there are $q_* \in \mathbb{R}$, $\langle\varepsilon_\gamma = \varepsilon(\gamma) : \gamma < \lambda\rangle \in {}^\lambda\kappa$ such that:

- $\bullet_3$ q_* is above $q_{\gamma,\varepsilon(\gamma)}$ for every $\gamma < \lambda$,

and recalling the enumeration from $(*)_1$, there exists $\langle\vartheta_\gamma = \vartheta(\gamma) : \gamma < \lambda\rangle \in {}^\lambda\kappa$ such that

- $\bullet_4$ $(T_{\gamma,\varepsilon(\gamma)}, \zeta_{\gamma,\varepsilon(\gamma)}, Z_{\gamma,\varepsilon(\gamma)}) = (U_{\vartheta(\gamma)}, \xi_{\vartheta(\gamma)}, Y_{\vartheta(\gamma)})$ holds for every $\gamma < \lambda$.

Now we define the model $M_* \in (K_\kappa)_\lambda \cap \mathbf{V}$ as follows:

$(*)_{5.3}$ (a) $M_* = (\lambda, (R_\alpha^{M_*})_{\alpha<\kappa})$, where
(b) for every $\vartheta \in \kappa$ we have

$$R_\vartheta^{M_*} = \{(\alpha_\gamma, \beta_\gamma) : \ (\gamma < \lambda) \ \wedge \ (\vartheta(\gamma) = \vartheta)\}.$$

Clearly

$(*)_{5.4}$ $M_* \in (K_\kappa)_\lambda$ (with the underlying set of nodes being λ), M_* belongs to $\mathbf{V}$.

Now choose a suitable $\delta < \chi$ and a function f so that:

$(*)_{5.5}$ $f : M^* \to M_\delta$ is an embedding, $f \in \mathbf{V}$

[which exists by $(*)_2$]. Finally it remains to check that

$(*)_{5.6}$ $q_* \Vdash$ "f is an embedding of $\underset{\sim}{G}_*$ into $\underset{\sim}{G}_\delta$".

Recall that $q_* \geq q_{\gamma,\varepsilon(\gamma)}$ for each $\gamma < \lambda$ by $\bullet_3$. Fix $\gamma < \lambda$. Using $\bullet_2$ and $\bullet_4$ we get

$$q_{\gamma,\varepsilon(\gamma)} \leq q_* \Vdash \ (\alpha_\gamma, \beta_\gamma) \in R^{\underset{\sim}{G}_*} \ \Leftrightarrow \ \underset{\sim}{X} \cap Z_{\theta(\gamma)} \in B_{U_{\theta(\gamma)}}. \tag{1.2}$$

Also, note that by $(*)_{5.3}$ the color of the pair $(\alpha_\gamma, \beta_\gamma)$ in M_* is $\vartheta(\gamma)$, i.e. $(\alpha_\gamma, \beta_\gamma) \in R^{M_*}_{\vartheta(\gamma)}$, and as $f : M^* \to M_\delta$ is an embedding, clearly

$$(f(\alpha_\gamma), f(\beta_\gamma)) \in R^{M_\delta}_{\vartheta(\gamma)}.$$

Recalling $(*)_3$, we obtain

$$\Vdash \ [(f(\alpha_\gamma), f(\beta_\gamma)) \in R^{\underset{\sim}{G}_\delta} \iff \underset{\sim}{X} \cap Z_{\vartheta(\gamma)} \in B_{U_{\vartheta(\gamma)}}]. \tag{1.3}$$

Finally, combining (1.2) and(1.3) we obtain

$$q_* \Vdash \ [(\alpha_\gamma, \beta_\gamma) \in R^{\underset{\sim}{G}_*} \iff (f(\alpha_\gamma), f(\beta_\gamma)) \in R^{\underset{\sim}{G}_\delta}],$$

as desired. $\square_{\text{Claim 1.5}}$

Naturally we can ask:

Question 1.6.

(1) What can we say about universals in $(K_\kappa)_\lambda$?

(2) An old open problem concerns the case of the theory of triangle free graphs [Mek90], and similarly it is open for T_{feq} (equivalently T_{ceq}, see [Sheb]). On T_{feq} we refer the reader to [She93], or [DS04], and on consistent instances of non-existence of universals in case of T_{ceq} see [Sheb].

(3) Moreover, what can we say about $(\text{Mod}_T, \prec)$ for T simple? Or even NSOP_2? (Of cardinality $< \kappa$.) We have to be more careful because of, e.g. function symbols.

A work in preparation deals with 1.6(2), (3). Concerning 1.6(1) we have the following negative result (note that this does not reflect on Claim 1.5):

Claim 1.7. Assume κ is strong limit singular and $\kappa < \lambda < 2^\kappa$. Then in $(K_\kappa)_\lambda$ there is no universal member.

Proof. By [She06, Thm 1.13 and 1.14 (2) on RGCH]

$(*)_0$ there is a regular $\sigma \in (\text{cf}(\kappa), \kappa)$ such that $\lambda^{[\sigma,\kappa]} = \lambda$, i.e. there is $\mathscr{P}' \subseteq \{u \subseteq \lambda : |u| \leq \kappa\}$ of cardinality λ such that every $u \subseteq \lambda$ of cardinality $\leq \kappa$ is the union $< \sigma$ members of $\mathscr{P}'$.

Therefore, as $\sigma = \mathrm{cf}(\sigma) > \mathrm{cf}(\kappa)$, replacing each $u \in \mathscr{P}'$ with a collection $u_\alpha \in [u]^{<\kappa}$ $(\alpha < \mathrm{cf}(\kappa))$ satisfying $u = \bigcup_{\alpha<\mathrm{cf}(\kappa)} u_\alpha$ we obtain

$(*)_1$ there is $\mathscr{P} \subseteq \{u \subseteq \lambda : |u| < \kappa\}$ of cardinality λ such that every $u \subseteq \lambda$ of cardinality $\leq \kappa$ is the union $< \sigma$ members of $\mathscr{P}$.

Fix $M_* \in (K_\kappa)_\lambda$ and we shall prove that it is not universal; without loss of generality the universe of M_* is λ. Now for each $u \in \mathscr{P}$ and $\alpha < \lambda$ let

$$v(\alpha, u, M_*) = \{\varepsilon < \kappa : \text{ for some } \beta \in u \text{ we have } (\alpha, \beta) \in R_\varepsilon^{M_*}\},$$

so $v(\alpha, u, M_*) \subseteq \kappa$ has cardinality $< \kappa$. Let

$$\mathscr{P}_1 = \{w \in [v(\alpha, u, M_*)]^{\leq \mathrm{cf}(\kappa)} : \ u \in \mathscr{P}, \alpha \in \lambda\},$$

so

$(*)_2$ $\mathscr{P}_1 \subseteq [\kappa]^{\leq \mathrm{cf}(\kappa)}$.

Now

$(*)_3$ $|\mathscr{P}_1| \leq |\mathscr{P}| + 2^{<\kappa} \leq \lambda < 2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.

Hence

$(*)_4$ we can find $v \subseteq \kappa$ of cardinality $\mathrm{cf}(\kappa)$ which is not in $\mathscr{P}_1$, moreover, $u \in \mathscr{P}_1 \Rightarrow |u \cap v| < \mathrm{cf}(\kappa)$,

which is justified by the following argument: Let $\langle v_\gamma : \gamma < 2^\kappa \rangle$ be a sequence of members of $[\kappa]^{\mathrm{cf}(\kappa)}$ with any two having intersection of cardinality $< \mathrm{cf}(\kappa)$, hence for every $u \in \mathscr{P}_1, \{\gamma < 2^\kappa : |u \cap v_\gamma| = \mathrm{cf}(\kappa)\}$ has cardinality $\leq 2^{\mathrm{cf}(\kappa)} < \kappa$, so all but $\leq \lambda$ of the v_γ's are as required.

Now consider the following N:

$(*)_5$ (a) $N = (A \cup B, \ldots, R_\varepsilon^N, \ldots)_{\varepsilon<\kappa}$ belongs to $(K_\kappa)_{\sigma^{\mathrm{cf}(\kappa)}}$, where $|A| = \sigma$, $|B| = \sigma^{\mathrm{cf}(\kappa)}$, $A \cap B = \emptyset$,

(b) $R_\varepsilon^N \neq \emptyset$ iff $\varepsilon \in v$,

(c) letting $\langle \varepsilon_i : i < \mathrm{cf}(\kappa)\rangle$ list v (from $(*)_4$), for every sequence $\bar{\alpha} = \langle \alpha_i : i < \mathrm{cf}(\kappa)\rangle$ in A with no repetitions there is $\beta = \beta(\bar{\alpha}) \in B$ such that $(\alpha_i, \beta) \in R_{\varepsilon_i}^N$ for $i < \mathrm{cf}(\kappa)$.

Now if g embeds N into M_* then since $|\mathrm{Rang}(g \restriction A)| = \sigma < \kappa$, by $(*)_1$ it is the case that for some $\{u_\varepsilon : \varepsilon < \partial < \sigma\} \subseteq \mathscr{P}$, we have $\mathrm{Rang}(g \restriction A) = \cup\{u_\varepsilon : \varepsilon < \partial\}$. Now as $|A| = \sigma = \mathrm{cf}(\sigma)$ but $\partial < \sigma$, there is $\varepsilon < \partial$ such that $|u_\varepsilon \cap \mathrm{Rang}(g \restriction A)| \geq \sigma \geq \mathrm{cf}(\kappa)$ so we can choose pairwise distinct $\alpha_i \in A$ $(i < \mathrm{cf}(\kappa))$ such that $\{g(\alpha_i) : i < \mathrm{cf}(\kappa)\} \subseteq u_\varepsilon$. Let $\beta = \beta(\bar{\alpha}) \in B$ given by $(*)_5(c)$. So $g(\beta)$ is well-defined and we get an easy contradiction by $(*)_4$.

This shows that N cannot be embedded into M_*, hence we are done. $\square_{1.7}$

Remark 1.8. In fact, the argument above could be modified so that it works with weaker assumptions: the conditions $\beth_\omega(\mathrm{cf}(\kappa)) < \kappa$, and $(\alpha < \kappa \ \rightarrow \ |\alpha|^{\mathrm{cf}(\kappa)} < \kappa)$ together are sufficient.

2. Proving known forcings fit the framework

2(A). **Near a large singular.** Here we do not collapse cardinals, just change cofinalities.

Claim 2.1. There is a nice (λ, κ)-system $\mathbf{r}$ such that $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$ when the following hold:

(A) (a) $\kappa < \lambda < 2^\kappa$ are cardinals,

(b) D is a normal ultrafilter on κ,
(c) if $\mathscr{A} \subseteq D$ has cardinality $\leq \lambda$, then for some $B \in D$ we have $(\forall A \in \mathscr{A})(B \subseteq A \mod [\kappa]^{<\kappa})$ (e.g. D is generated by a $\subseteq^*_\kappa$-decreasing sequence of length of a regular cardinal $> \lambda$),
(d) $\mathbb{P}$ is the Prikry forcing for D (so $\mathbb{P}$ changes the cofinality of κ to $\aleph_0$ and adds no bounded subset of κ and satisfies the κ^+-c.c).

Proof. Recalling the definition of Prikry forcing for D:

$(*)_1$ (a) $p \in \mathbb{P}$ iff $p = (w, A) = (w_p, A_p)$, where $w_p \in [\kappa]^{<\aleph_0}$ and $A_p \in D$ and $[0, \max w_p] \cap A = \emptyset$,
(b) $p \leq_{\mathbb{P}} q$ iff $w_p \subseteq w_q \subseteq w_p \cup A_p$ and $A_p \supseteq A_q$.
We define the system $\mathbf{r}$ by letting:
$(*)_2$ (a) $\kappa_{\mathbf{r}} = \kappa$,
(b) $\lambda_{\mathbf{r}} = \lambda$,
(c) $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$,
(d) $\underaccent{\tilde}{X}_{\mathbf{r}} =$ the Prikry generic sequence $= \cup\{w_p : p \in \mathbf{G}_{\mathbb{P}}\}$,
(e) $\leq_{\text{pr}} = \leq_{\mathbf{r},\text{pr}}$ is defined by $p \leq_{\text{pr}} q$ iff $w_p = w_q \wedge A_p \supseteq A_q$ (and $p, q \in \mathbb{R}_{\mathbf{r}}$),
(f) for $p \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ let $\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \{\bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle$ and for some $B \in D$ we have $B \subseteq A_p$ and $\{A_{q_\varepsilon} : \varepsilon < \kappa\}$ list $\{A : A \subseteq A_p$ and $A \equiv B \mod [\kappa]^{<\kappa}\}$.

We still have to prove that $\mathbf{r}$ is as required, namely, that $\mathbf{r}$ satisfies conditions listed in Definition 1.2(1).

Now clauses (a)-(f) from Definition 1.2(1) hold trivially. For clause (g) fix p, $\underaccent{\tilde}{\tau}$, with $p \Vdash_{\mathbb{P}}$ "$\tau \in \{0, 1\}$". Recall the following well-known fact:
$(*)_3$ if $p \in \mathbb{P}$, $p \Vdash_{\mathbb{P}}$ "$\underaccent{\tilde}{\tau} \in \{0, 1\}$", then for some $A' \subseteq A_p$, $A' \in D$ we have:
if $\alpha \in \kappa$ and $u \subseteq A_p \cap \alpha$ is finite then $(w_p \cup u, A' \setminus \alpha)$ forces a value for $\underaccent{\tilde}{\tau}$.

[For the sake of completeness we prove $(*)_3$: by the Prikry-lemma, for each $s \in [A_p]^{<\aleph_0}$ there exists $A_s \subseteq A_p \setminus ((\max s) + 1)$, $A_s \in D$, such that $(w \cup s, A_s)$ decides the value of $\underaccent{\tilde}{\tau}$. Now let A' be the diagonal intersection of A_s's ($s \in [A_p]^{<\aleph_0}$), pedantically $\Delta_{\alpha<\kappa}(\bigcap_{s \in [\alpha+1]^{<\aleph_0}} A_s)$, it is straightforward to check that A' works.]

So given $p \in \mathbb{P}, \gamma$ and $\underaccent{\tilde}{\tau}$ as in clause (g) from Definition 1.2, let $A' \subseteq A_p$ be as in $(*)_3$ and let $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle$ be defined by: $q_\varepsilon \in \mathbb{P}, w_{q_\varepsilon} = w_p$ and $\{A_{q_\varepsilon} : \varepsilon < \kappa\}$ list $\{A \subseteq A_p : A \equiv A' \mod [\kappa]^{<\kappa}\}$.

We still have to choose the $Y_\varepsilon, T_\varepsilon$. For each ε choose $\zeta_\varepsilon \in A_{q_\varepsilon}$ such that $A_{q_\varepsilon} \setminus \zeta_\varepsilon = A' \setminus \zeta_\varepsilon$. Clause $(*)_3$ ensures that there is a function $f : [A_p \cap \zeta_\varepsilon]^{<\aleph_0} \to \{0, 1\}$ in $\mathbf{V}$ such that $q_\varepsilon \Vdash \underaccent{\tilde}{\tau} = f(\underaccent{\tilde}{X} \cap \zeta_\varepsilon)$. This means we can let $Y_\varepsilon = \gamma_\varepsilon$, and choose a γ_ε-Borel code T_ε such that whenever $w \in B_{T_\varepsilon}$ necessarily $w \in [\gamma_\varepsilon]^{<\aleph_0}$, and

$$q_\varepsilon \Vdash (\underaccent{\tilde}{\tau} = 1) \iff (\underaccent{\tilde}{X} \cap \zeta_\varepsilon) \in B_{T_\varepsilon}.$$

Lastly, for clause (h), assume $p \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ and $\bar{q} = \langle \bar{q}_\alpha : \alpha < \lambda\rangle$ satisfies $\bar{q}_\alpha \in \mathscr{S}_p$. So for each $\alpha < \lambda$ there exists $B_\alpha \subseteq A_p$ such that $\{A_{q_{\alpha,\varepsilon}} : \varepsilon < \kappa\}$ lists $\{A \in D : A \subseteq A_p,\ A \equiv B_\alpha \mod [\kappa]^{<\kappa}\}$, hence by clause (A)(c) of the assumption of the claim, there is $B \in D$, a subset of A_p such that $B \subseteq B_\alpha \mod [\kappa]^{<\kappa}$ for each $\alpha \in \lambda$ and let $q_* = (w_p, B)$ so clearly $p \leq_{\text{pr}} q_*$. Also for each $\alpha < \lambda$, for some $\zeta < \kappa$ we have $B \setminus \zeta \subseteq B_\alpha$. Finally, because $\bar{q}_\alpha \in \mathscr{S}_p$ we have that for some $\varepsilon < \kappa$ $A_{q_{\alpha,\varepsilon}} = (B_\alpha \setminus \zeta) \cup (A_p \cap \zeta) \supseteq B$ hence $q_{\alpha,\varepsilon} \leq q_*$.

We still have to prove that $\mathbf{r}$ is nice but as $\mathbb{P}$ satisfies the κ^+-c.c., and by the Prikry lemma this is obvious. □

Claim 2.2. There is a $(\lambda, \kappa) - 1$-system $\mathbb{R}_{\mathbf{r}}$ with $\mathbf{V}^{\mathbb{R}_{\mathbf{r}}} \models \mathrm{cf}(\kappa) = \theta$, when (B) holds:

(B) (a) $\theta = \mathrm{cf}(\theta) < \theta_* < \kappa < \lambda < 2^\kappa$,
(b) $\bar{D} = \langle D_i : i < \theta \rangle$ is a sequence of normal ultrafilters on κ, increasing in Mitchell order, i.e. $i < j \Rightarrow D_i \in \mathrm{Mos\,Col}({}^\kappa\mathbf{V}/D_j)$,
(c) each D_i $(i < \theta)$ is $< \lambda^+$-directed mod $[\kappa]^{<\kappa}$, i.e. satisfies the condition (A)(c) from Claim 2.1.

Moreover, the forcing $\mathbb{R}_{\mathbf{r}}$ changes the cofinality of κ to θ, preserves each cardinal and the function $\mu \mapsto 2^\mu$, satisfies the κ^+-c.c. Moreover, we can prescribe that in $\mathbf{V}^{\mathbb{P}}$ there is no new subset of θ_*.

Proof. Using [Kru07, Proposition 2.1], condition (b) implies the following.

Subclaim 2.3. If $\bar{D} = \langle D_i : i < \theta \rangle$ is an increasing (w.r.t. the Mitchell order) sequence of normal ultrafilters on κ, $\theta \leq \kappa$, then there exists a coherent sequence $\langle \bar{U}_\varepsilon : \varepsilon < \kappa + 1 \rangle$, $\bar{U}_\varepsilon = \langle U_\varepsilon(\alpha) : \alpha < o^U(\varepsilon) \rangle$ for some function $o^U : \kappa + 1 \to \kappa$ such that $\bar{D} = \bar{U}_\kappa$, which means:

$(\mathsf{T})_a$ for each $\varepsilon \leq \kappa$, $\alpha < o^U(\varepsilon)$ $U_\varepsilon(\alpha)$ is an ε-complete normal ultrafilter on ε,
$(\mathsf{T})_b$ moreover, for each $\varepsilon \leq \kappa$ and $\alpha < o^U(\varepsilon)$, letting $\mathbf{j}_{\varepsilon,\alpha} : \mathbf{V} \to \mathrm{Mos\,Col}({}^\varepsilon\mathbf{V}/U_{\varepsilon,\alpha})$ be the associated elementary embedding, we have

$$(\mathbf{j}_{\varepsilon,\alpha}(\bar{U} \restriction \varepsilon))_\varepsilon = \langle U_\varepsilon(\beta) : \beta < \alpha \rangle,$$

$(\mathsf{T})_c$ $\langle U_\kappa(\alpha) : \alpha < o^U(\kappa) \rangle = \langle D_\alpha : \alpha < \theta \rangle$.

Now we define the forcing $\mathbb{P}_{\overline{U}}$ to be the Magidor forcing associated with the sequence $\bar{D} = \bar{U}_\kappa = \langle U_\kappa(\alpha) : \alpha \leq \theta \rangle$, (see also [Mag78], or [Git10]), here we use the definition from [Git10, Definition 5.22].

Definition 2.4. Define $\mathbb{P}_{\overline{U}}$ to be the following (auxiliary) poset.

$(*_1)$ Let $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \kappa, A_\kappa \rangle \rangle \in \mathbb{P}_{\overline{U}}$, iff
(a) $A_\kappa \in \bigcap \bar{U}_\kappa = \bigcap_{\alpha < \theta} U_{\kappa,\alpha}$,
(b) each d_j $(j \leq n)$ is of the form
- either $\langle \varepsilon, A_\varepsilon \rangle$ for some $\varepsilon < \kappa$, where $o^U(\varepsilon) > 0$, moreover,

$$A_\varepsilon \in \bigcap \bar{U}_\varepsilon = \bigcap_{\gamma < o^U(\varepsilon)} U_{\varepsilon,\gamma}$$

(this case we define $\kappa(d_j) = \varepsilon$),
- or $d_j = \varepsilon$, when $o^U(\varepsilon) = 0$ (and we let $\kappa(d_j) = d_j = \varepsilon$).

(c) $\kappa(d_0) < \kappa(d_1) < \cdots < \kappa(d_n) < \kappa(d_{n+1}) = \kappa$,
(d) moreover, for each $j \leq n$, if d_{j+1} is a pair, then $\kappa(d_j) < \min A_{\kappa(d_{j+1})}$.

$(*_2)$ We define

$$p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \kappa, A_\kappa \rangle \rangle \leq q = \langle e_0, e_1, \ldots, e_m, e_{m+1} = \langle \kappa, B_\kappa \rangle \rangle,$$

if
(a) $m \geq n$, and
(b) there exists a sequence $0 \leq i_0 < i_1 < \cdots < i_n < j_{n+1} = m + 1$ such that for each $j \leq n + 1$ we have
- $\kappa(d_j) = \kappa(e_{i_j})$, and
- $B_{\kappa(d_j)} \subseteq A_{\kappa(d_j)}$,

(c) moreover, for each $k \leq m$ not of the form i_j $(j \leq n+1)$, if $i_+ = \min\{i_j : j \leq n+1,\ i_j > k\}$, then

$$B_{\kappa(e_k)} \cup \{\kappa(e_k)\} \subseteq A_{\kappa(d_{i_+})}.$$

$(*_3)$ Now if we define the pairwise disjoint sets Y_α $(\alpha < \theta)$ as

$$\delta \in Y_\alpha \iff o^U(\delta) = \alpha,$$

then

$$\{p \in \mathbb{P}_{\overline{U}} :\ p \geq \langle\langle \kappa, \bigcup_{\alpha<\theta} Y_\alpha\rangle\rangle\}$$

is the Magidor forcing changing the cofinality of κ to $\max\{\omega, \mathrm{cf}(\theta)\}$.

Definition 2.5. We define $p \leq_* q$ to be true iff $p \leq q$ and $\ell g(p) = \ell g(q)$.

We define the system $\mathbf{r}$ by letting:

$(*_4)$ (a) $\kappa_{\mathbf{r}} = \kappa$,
(b) $\lambda_{\mathbf{r}} = \lambda$,
(c) $\mathbb{R}_{\mathbf{r}} = \{p \in \mathbb{P}_{\overline{U}} :\ p \geq \langle\langle \kappa, \bigcup_{\alpha<\theta} Y_\alpha\rangle\rangle\}$,
(d) let $\underset{\sim}{X}_{\mathbf{r}}$ be the generic sequence, i.e.

$$\underset{\sim}{X}_{\mathbf{r}} = \cup\{\{\kappa(d_j) : j < \ell g(p)\} :\ p = \langle d_0, d_1, \dots, d_{\ell g(p)-1}\rangle \in \mathbf{G}_{\mathbb{P}}\} \setminus \{\kappa\},$$

(e) $\leq_{\mathrm{pr}} = \leq_{\mathbf{r},\mathrm{pr}}$ is defined by $p \leq_{\mathrm{pr}} q$ iff $p \leq_* q$,
(f) for $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$, let

$$\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \left\{ \begin{array}{l} \bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa\rangle, \text{ where} \\ (\bullet_1) q_\varepsilon = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{q_\varepsilon,\kappa}\rangle\rangle, \text{ and} \\ \text{for some } B \in \bigcap \bar{U}_\kappa \text{ we have} \\ (\bullet_2)\ B \subseteq A_{p,\kappa}, \text{ and} \\ (\bullet_3)\{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \wedge A_* \equiv B \mod [\kappa]^{<\kappa}\} \end{array} \right\}.$$

It is known that $\underset{\sim}{X}$ is a club of κ of order type θ, moreover, if condition $\langle\langle\beta\rangle, \langle\kappa, A\rangle\rangle$ is in the generic filter (for some $\beta < \kappa$, $o^U(\beta) = 0$, then the forcing adds no new subset to β. Therefore (it is not difficult to see that) by $(\mathsf{T})_b$ the set $\{\beta < \kappa : o^U(\beta) = 0\} \in U_{\kappa,0}$, and so we can limit ourselves to the subposet consisting of conditions above $\langle\langle\beta\rangle, \langle\kappa, \bigcup_{\alpha<\theta} Y_\theta\rangle\rangle$ for some $\beta \geq \theta_*$. In order to finish the proof of Claim 2.2 it suffices to verify that the forcing defined in $(*_3)$ is a $(\lambda, \kappa) - 1$-system.

Subclaim 2.6. If $\langle \bar{U}_\varepsilon :\ \varepsilon < \kappa + 1\rangle$ is a coherent sequence, where the ultrafilters $\{U_\kappa(\alpha) :\ \alpha < o^{\bar{U}}(\kappa)\}$ are $< \lambda^+$-directed mod $[\kappa]^{<\kappa}$, then the forcing $\mathbb{P}_{\bar{U}}$ from Definition 2.4 is a $(\lambda, \kappa) - 1$-system.

Proof. Now we have only to check the requirements of Definition 1.2(1). Recall the following properties of the Magidor forcing, see [Git10, Sec. 5.1 and 5.2].

Fact 2.7 (Prikry lemma)**.** For each $p \in \mathbb{P}_{\overline{U}}$ and each formula $\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ there exists $q \geq_* p$, $q \parallel \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$ (i.e. either $q \Vdash \sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$, or $q \Vdash \neg\sigma(\underset{\sim}{x}_0, \dots, \underset{\sim}{x}_m)$).

Notation 2.8. If $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{P}_{\overline{U}}$, and $i \leq n+1$, then $q \restriction (\kappa(d_i) + 1)$ refers to the condition $\langle d_0, d_1, \dots, d_i\rangle$.

Fact 2.9. Suppose that $\mathbf{G} \subseteq \mathbb{P}_{\overline{U}}$ is generic over $\mathbf{V}$, $p = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbf{G}$, $i \leq n+1$, $d_i = \langle \kappa(d_i), A_{\kappa(d_i)}\rangle$, then the filter $\mathbf{G} \restriction (\kappa(d_i)+1) := \{q \restriction (\kappa(d_i) + 1) :\ q \in \mathbf{G}\}$ is $\mathbf{V}$-generic over the Prikry forcing $\mathbb{P}_{\overline{U}\restriction(\kappa(d_i)+1)}$ associated with the coherent sequence $\langle \overline{U}_\delta = \langle U_\delta(\gamma) :\ \gamma < o^U(\delta)\rangle :\ \delta \leq \kappa(d_i)\rangle$.

The Prikry Lemma and the subforcing $\mathbb{P}_{\overline{U}\restriction(\delta+1)}$ together give the following.

Fact 2.10. For each $\delta < \kappa$, $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \delta, A_{p,\delta}\rangle\rangle \in \mathbb{P}_{\overline{U}\restriction(\delta+1)}$ and each formula $\sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$ there exists $q \in \mathbb{P}_{\overline{U}\restriction(\delta+1)}$,

$$q \geq_* p \text{ (in the sense of } \mathbb{P}_{\overline{U}\restriction(\delta+1)}),$$

such that for some $A \in \bigcap \overline{U}_\kappa$ we have

$$q \frown \langle \kappa, A\rangle \Vdash_{\mathbb{P}_{\overline{U}}} \sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m).$$

Lemma 2.11. *Suppose that $\sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$ is a formula, $\delta < \kappa$, $p = \langle d_0, d_1, \ldots, d_n\rangle \in \mathbf{G} \restriction (\kappa(d_n) + 1)$, $\delta = \kappa(d_n)$, the filter $\mathbf{G} \subseteq \mathbb{P}_{\overline{U}}$ is generic over $\mathbf{V}$ (so that $p = p' \restriction (\delta + 1)$ for some $p' \in \mathbf{G}$, and $p' \Vdash \delta \in \underset{\sim}{X}_{\mathfrak{r}}$).*

Then there exists $q \in \mathbf{G} \restriction (\delta+1)$, $\mathbb{P}_{\overline{U}\restriction(\delta+1)} \models q \geq p$, such that for some $A \in \bigcap \overline{U}_\kappa$ letting $q' = q \frown \langle \kappa, A\rangle$ we have $q' \in \mathbb{P}_{\overline{U}}$ and

$$q' = q \frown \langle \kappa, A\rangle \Vdash_{\mathbb{P}_{\overline{U}}} \sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m). \tag{2.1}$$

Proof. This is a standard density argument: First using Fact 2.9 $\mathbf{G} \restriction (\delta + 1) \subseteq \mathbb{P}_{\overline{U}\restriction(\delta+1)}$ is generic, and so by Fact 2.10 there exists $q \in \mathbf{G} \restriction (\delta + 1)$,

$$q \geq p \text{ (in the sense of } \mathbb{P}_{\overline{U}\restriction(\delta+1)}),$$

such that for some $A \in \bigcap \overline{U}_\kappa$ the condition $q \frown \langle \kappa, A\rangle \in \mathbb{P}_{\overline{U}}$ decides about σ. $\square_{\text{Lemma 2.11}}$

Similarly to the case of Prikry forcing, this has the following consequence.

Claim 2.12. For each $p = \langle d_0, d_1, \ldots, d_n, \langle \kappa, A_{p,\kappa}\rangle\rangle \in \mathbb{P}_{\overline{U}}$ and $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0, 1\}$) there exists a set $A' \in \bigcap \bar{U}_\kappa$, $A' \subseteq A_{p,\kappa}$, such that the condition $p' = \langle d_0, d_1, \ldots, d_n, \langle \kappa, A'\rangle\rangle$ satisfies the following:

Whenever $\alpha \in A_{p,\kappa}$, $q = \langle e_0, e_1, \ldots, e_m, \langle \kappa, A_{q,\kappa}\rangle\rangle \geq p' = \langle d_0, d_1, \ldots, d_n, \langle \kappa, A'\rangle\rangle$ are given with $\kappa(e_m) \leq \alpha$, and q forces a value to $\underset{\sim}{\tau}$, then so does

$$q' = \langle e_0, e_1, \ldots, e_m, \langle \kappa, A' \cap (\alpha, \kappa)\rangle\rangle,$$

i.e.

$$q' \Vdash_{\mathbb{P}_{\overline{U}}} \text{“}\underset{\sim}{\tau} = 1\text{”}.$$

Proof. For each $\alpha \in A_{p,\kappa}$ define $B_\alpha \subseteq A_{p,\kappa}$ so that whenever

$$q = \langle e_0, e_1, \ldots, e_m, e_{m+1} = \langle \kappa, A_{q,\kappa}\rangle\rangle \geq p$$

(with $\kappa(e_0), \kappa(e_1), \ldots, \kappa(e_m) \leq \alpha$) decides the value of $\underset{\sim}{\tau}$, then so does

$$q' = \langle e_0, e_1, \ldots, e_{m+1} = \langle \kappa, B_\alpha\rangle\rangle.$$

This can be done easily: first for each possible $e_0, e_1, \ldots, e_m$ choose a set $B_{e_0,e_1,\ldots,e_m} \subseteq (\alpha, \kappa)$ with

$$\langle e_0, e_1, \ldots, e_m, \langle \kappa, B_{e_0,e_1,\ldots,e_m}\rangle\rangle \text{ deciding the value of } \underset{\sim}{\tau},$$

if such a $B_{e_0,e_1,\ldots,e_m}$ exists, otherwise just let $B_{e_0,e_1,\ldots,e_m} = A_{p,\kappa} \cap (\alpha, \kappa)$. Second, let $B_\alpha = \bigcap_{e_0,e_1,\ldots,e_m} B_{e_0,e_1,\ldots,e_m}$. Now it is easy to check that the diagonal intersection $A' = \Delta_{\alpha \in A_{p,\kappa}} B_\alpha \in \bigcap \bar{U}_\kappa$ works (note that the intersection of normal measures is a normal filter). $\square_{\text{Claim 2.12}}$

Claim 2.13. For every $p \in \mathbb{P}_{\overline{U}}$ and $\underset{\sim}{\tau}$, if $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$, then we can choose $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa \rangle \in \mathscr{P}_p$, $\langle \gamma_\varepsilon : \varepsilon < \kappa \rangle$, $\langle T_\varepsilon : \varepsilon < \kappa \rangle$, $\langle Y_\varepsilon : \varepsilon < \kappa \rangle$, where each T_ε is a code for a γ_ε-Borel subset of $\mathscr{P}(Y_\varepsilon)$ such that

$$q_\varepsilon \Vdash \underset{\sim}{\tau} = 1 \iff (\underset{\sim}{X} \cap \gamma_\varepsilon) \in B_{T_\varepsilon}.$$

Proof. First if $p = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{p,\kappa} \rangle \rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ are in the Lemma, let $A' = A'(p, \underset{\sim}{\tau}) \subseteq A_{p,\kappa}$ be given by Claim 2.12 and

($*_5$) let $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa \rangle \in \mathscr{S}_p$ be defined by: $q_\varepsilon \in \mathbb{P}, q_\varepsilon = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{q_\varepsilon,\kappa} \rangle \rangle$ where $\{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\}$ lists $\{A_* \subseteq A_{p,\kappa} : A_* \equiv A' \mod [\kappa]^{<\kappa}\}$.

We still have to choose $\gamma_\varepsilon, T_\varepsilon, Y_\varepsilon$. For each ε choose $\zeta_\varepsilon \in A_{q_\varepsilon,\kappa} \setminus \kappa(d_n)$ such that

$$A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) = A' \setminus (\zeta_\varepsilon + 1). \tag{2.2}$$

Now we claim that q_ε forces that $\underset{\sim}{\tau}$ only depends on $\mathbf{G} \restriction (\zeta_\varepsilon + 1)$ in the following sense:

Subclaim 2.14. If $q_\varepsilon \in \mathbf{G}$, then for some $q^* \in \mathbf{G}$ with $q^* \geq q_\varepsilon$ and $\delta \leq \zeta_\varepsilon$,

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) \rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{''}.$$

Proof. First observe that if $q_\varepsilon \in \mathbf{G}$, then by genericity there is some $\delta \leq \zeta_\varepsilon$, and $q' \geq q_\varepsilon$, $q' \in \mathbf{G}$, such that

$$q' \Vdash \max(\underset{\sim}{X} \cap (\zeta_\varepsilon + 1) = \delta), \tag{2.3}$$

i.e.

$$q' = \langle e_0, e_1, \dots, e_m, e_{m+1} = \langle \kappa, A_{q'} \rangle \rangle, \tag{2.4}$$

and for some $k \leq m$ we have

$$[\kappa(e_k) = \delta)] \quad \wedge \quad [A_{q',\kappa(e_{k+1})} \cap (\zeta_\varepsilon + 1) = \emptyset]. \tag{2.5}$$

Now by Lemma 2.11 there is some $q^* \in \mathbf{G}$, $A^* \in \bigcap \overline{U}_\kappa$ with

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A^* \rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{''}, \tag{2.6}$$

w.l.o.g. $q^* \geq q' \geq q_\varepsilon$. But then by the construction of $A' = A(p, \underset{\sim}{\tau})$ we have

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A' \setminus (\delta + 1) \rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{''}. \tag{2.7}$$

Therefore, as $A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) = A' \setminus (\zeta_\varepsilon + 1)$ by (2.2) (and $\delta \leq \zeta_\varepsilon$ by (2.3)),

$$A' \setminus (\delta + 1) \subseteq A' \setminus (\zeta_\varepsilon + 1) = A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1),$$

thus

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A' \setminus (\delta + 1) \rangle \leq q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) \rangle. \tag{2.8}$$

This means that by (2.7)

$$q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) \rangle \parallel \text{“}\underset{\sim}{\tau} = 1\text{”},$$

so recalling that $q_\varepsilon \leq q^*$, and $q^* \in \mathbf{G}$, we are done. $\square_{\text{Subclaim 2.14}}$

Now we claim that

$$q^* \geq q^* \restriction (\delta + 1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1) \rangle. \tag{2.9}$$

To this end first recall that

$$q^* = \langle d_0^*, d_1^*, \dots, d_\ell^*, d_{\ell+1}^* = \langle \kappa, A_{q^*} \rangle \rangle \geq q' \geq q_\varepsilon = \langle d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{q_\varepsilon,\kappa} \rangle \rangle,$$

where $\kappa(d_n) \leq \zeta_\varepsilon$ (by the choice of ζ_ε), and q' is from (2.10). Moreover, (2.5) implies that

$$q' = q' \restriction (\delta+1) \frown \langle e_{k+1}, e_{k+2}, \dots, e_m, e_{m+1} = \langle \kappa, A_{q'} \rangle \rangle,$$

where

$$A_{q',\kappa(e_{k+1})} \cap (\zeta_\varepsilon + 1) = \emptyset.$$

Now by $q' \leq q^*$ necessarily (for some $j \leq \ell$) $\kappa(d_j^*) = \delta$, and

$$q^* = q^* \restriction (\delta+1) \frown \langle d^*_{j+1}, d^*_{j+2}, \dots, d^*_\ell, d^*_{\ell+1} = \langle \kappa, A_{q'} \rangle \rangle, \tag{2.10}$$

and

$$A_{q^*,\kappa(d^*_{j+1})} \cap (\zeta_\varepsilon + 1) = \emptyset. \tag{2.11}$$

Then one the one hand,

$$A^{**} := \bigcup_{i \in (j,\ell+1]} (A_{q^*,\kappa(e_i)} \cup \{\kappa(e_i)\}) \cap (\zeta_\varepsilon + 1) = \emptyset,$$

and on the other hand,

$$A^{**} \subseteq A_{q_\varepsilon,\kappa},$$

since $q^* \geq q_\varepsilon$, so $A^{**} \subseteq A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon + 1)$, and recalling (2.10) we can conclude that (2.9) holds, indeed.

By Subclaim 2.14 $q_\varepsilon \in \mathbf{G}$ implies that there is always a $q^* \in \mathbf{G}$ and $\delta \leq \zeta_\varepsilon$ such that $q^* \restriction (\delta+1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle$ decides the value of $\underset{\sim}{\tau}$, and by (2.9)

$$q^* \restriction (\delta+1) \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle \in \mathbf{G}.$$

It is not difficult to check (using the definition of the partial order) that for every $q^{**} = \langle e_0, e_1, \dots, e_m \rangle \in \bigcup_{\delta \leq \zeta_\varepsilon} \mathbb{P}_{\overline{U} \restriction (\delta+1)}$,

$$\begin{aligned} q^{**} \in \mathbf{G} \iff & (\{\kappa(e_i) : i \leq m\} \\ & \subseteq \underset{\sim}{X} \cap (\zeta_\varepsilon+1) \subseteq \{\kappa(e_i) :\ i \leq m\} \cup (\cup \{A_{q^{**},\kappa(e_i)} :\ i \leq m\})\Big). \end{aligned}$$

Therefore, for any $q^{**} \geq q_\varepsilon$ with

$$q^{**} \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle \parallel \ \underset{\sim}{\tau} = 1,$$

fix the forced value $j_{q^{**}} \in \{0,1\}$:

$$q^{**} \frown \langle \kappa, A_{q_\varepsilon,\kappa} \setminus (\zeta_\varepsilon+1) \rangle \Vdash \ \underset{\sim}{\tau} = j_{q^{**}},$$

and fix the code $T_{q^{**}}$ for the 2^{ζ_ε}-Borel subset of $\mathscr{P}(\zeta_\varepsilon)$ with

$$q^{**} \in \mathbf{G} \iff \underset{\sim}{X} \cap (\zeta_\varepsilon+1) \in B_{T_{q^{**}}}.$$

Finally, let T_ε be the code for the 2^{ζ_ε}-Borel subset of $\mathscr{P}(\zeta_\varepsilon)$ defined as

$$B_{T_\varepsilon} = \cup \{B_{T_{q^{**}}} :\ q^{**} \geq q_\varepsilon,\ j_{q^{**}} = 1\}.$$

Then

$$q_\varepsilon \Vdash \ (\underset{\sim}{\tau} = 1) \iff ((X \cap \zeta_\varepsilon) \in B_{T_\varepsilon}),$$

and choosing $\gamma_\varepsilon = 2^{\zeta_\varepsilon}$, $Y_\varepsilon = \zeta_\varepsilon$ works, which completes the proof of Claim 2.13. $\square_{\text{Claim 2.13}}$

$\square_{\text{Subclaim 2.6}}$

Finally it remains to verify clause (h) from Definition 1.2. Fix $p \in \mathbb{P}$ and $\bar{q}_\alpha = \langle q_{\alpha,\varepsilon} : \varepsilon < \kappa \rangle \in \mathscr{S}_p$ $(\alpha < \lambda)$. Now recall $(*_4)$(f), and let $A'_\alpha \in \bigcap \bar{U}_\kappa = \bigcap_{\beta<\theta} U_{\kappa,\beta}$ the set corresponding to the sequence $\bar{q}_\alpha$, i.e. (if $d_0, d_1, \dots, d_n, d_{n+1} = \langle \kappa, A_{p,\kappa} \rangle$ denote the components of p)

$$\text{(2.12)} \quad \begin{array}{l} \bar{q}_\alpha = \langle q_{\alpha,\varepsilon} : \varepsilon < \kappa \rangle \text{ where } q_{\alpha,\varepsilon} = \langle d_0, d_1, \dots, d_n, \langle \kappa, A_{q_{\alpha,\varepsilon},\kappa} \rangle \rangle \text{ and} \\ \{A_{q_{\alpha,\varepsilon},\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \text{ and } A_* = A'_\alpha \mod [\kappa]^{<\kappa}\}. \end{array}$$

Then for each fixed $\beta < \theta$ as $A'_\alpha \in U_{\kappa,\beta}$ $(\forall \alpha < \lambda)$, using (B)(c) there is a pseudointersection in $U_{\kappa,\beta}$, i.e. a set $B_\beta \in U_{\kappa,\beta}$ such that $B_\beta \subseteq A_{p,\kappa}$, and

$(*_5)$ for each $\alpha < \lambda$ $|B_\beta \setminus A'_\alpha| < \kappa$.

Now taking the union of these pseudointersections, clearly

$(*_6)$ $B_* = \bigcup_{\beta<\theta} B_\beta \in \bigcap \bar{U}_\kappa$.

Therefore $(*_5)$ implies (recalling $\theta < \kappa$)

$(*_7)$ for each $\alpha < \lambda$: $|B_* \setminus A'_\alpha| < \kappa$, and we can infer that for some $\zeta_\alpha < \kappa$:

$$B_* \cap (\zeta_\alpha, \kappa) \subseteq A'_\alpha.$$

At this point we are ready to define q_*. We let $q_* = \langle d_0, d_1, \dots, d_n, \langle \kappa, B_* \rangle \rangle$, clearly $p \leq q_*$ as $B_* \subseteq A_{p,\kappa}$. Moreover, for any fixed $\alpha < \lambda$ by (2.12) there exists some $\varepsilon < \kappa$ with the property that

$(*_1)$ $A_{q_{\alpha,\varepsilon},\kappa} \cap (\zeta_\alpha, \kappa) = A'_\alpha \cap (\zeta_\alpha, \kappa) \supseteq B_* \cap (\zeta_\alpha, \kappa)$, and
$(*_2)$ $A_{q_{\alpha,\varepsilon},\kappa} \cap (\zeta_\alpha + 1) = B_* \cap (\zeta_\alpha + 1)$,

so $B_* \subseteq A_{q_{\alpha,\varepsilon},\kappa}$, thus concluding $q_{\alpha,\varepsilon} \leq_* q_*$. $\square_{2.2}$

Next we will give another example of a (λ, κ)-system, the Radin forcing, provided the measure sequence satisfies a similar $< \lambda^+$-directedness condition.

Definition 2.15. In order to state the following claim we need to prepare and introduce the notions below.

(i) Let κ be a cardinal, $\mathbf{j} : \mathbf{V} \to \mathbf{M}$ be an elementary embedding (into a transitive inner model $\mathbf{M}$) with $\mathrm{crit}(\mathbf{j}) = \kappa$. We call the sequence $\overline{F} = \langle F(\alpha) : \alpha < \mathrm{dom}(\overline{F}) \rangle$ a $\mathbf{j}$-sequence of ultrafilters, if
  (a) $F(0) = \kappa$,
  (b) $F(\alpha) \subseteq \mathscr{P}(\mathbf{V}_\kappa)$ for every $\alpha < \mathrm{dom}(\overline{F})$,
  (c) and for each $0 < \alpha < \mathrm{dom}(\overline{F})$, $\forall X \subseteq \mathbf{V}_\kappa$: $[X \in F(\alpha)$ iff $(\overline{F} \restriction \alpha) \in \mathbf{j}(X)]$.

(ii) for each ultrafilter sequence $\overline{F}$ that is a $\mathbf{j}$-sequence witnessed by some suitable $\mathbf{j}$ we let $\kappa(\overline{F})$ denote the critical point of the witnessing $\mathbf{j}$, thus the F_α's are concentrated on $\mathbf{V}_{\kappa(\overline{F})}$. For each ordinal α we mean $\kappa(\alpha) = \alpha$.

(iii) for an ultrafilter sequence $\overline{F}$ that is a $\mathbf{j}$-sequence witnessed by some suitable $\mathbf{j}$ we reserve the notation $\bigcap \overline{F}$ for the intersection of all $F(\alpha)$'s but $F(0)$, i.e.:

$$\cap \overline{F} := \bigcap_{0<\alpha<\mathrm{dom}(\overline{F})} F_*(\alpha).$$

Therefore, for each $\alpha < \mathrm{dom}(\overline{F})$ $F(\alpha)$ is a κ-complete normal ultrafilter on $\mathbf{V}_\kappa$, where under normality we mean that for each sequence $\langle X_\beta : \beta < \kappa \rangle$ in $F(\alpha)$ the diagonal intersection

$$\Delta_{\beta<\kappa} X_\beta = \{\overline{f} : \forall \gamma < \kappa(\overline{f}) : \overline{f} \in X_\gamma\} \in F(\alpha).$$

We will work with ultrafilter sequences $\overline{F}_*$ according to that almost every element of $V_{\kappa(\overline{F}_*)}$ is itself an ultrafilter sequence, i.e. the $F_*(\alpha)'$s are concentrated on the following classes:

(iv) Let $A^{(n)}$ $(n \in \omega)$ be the following sequence of classes

$$A^{(0)} = \{\overline{F} : \ \overline{F} \text{ is a } \mathbf{j}\text{-sequence of ultrafilters for some } \mathbf{j} : \mathbf{V} \to \mathbf{M}\},$$

and

$$A^{(n+1)} = \{\overline{F} \in A^{(n)} : \forall \alpha \in \mathrm{dom}(\overline{F}) \setminus \{0\} \ V_{\kappa(\overline{F})} \cap A^{(n)} \in F(\alpha)\}.$$

Finally let

$$\mathbf{A} = \bigcap_{n \in \omega} A^{(n)}.$$

(v) For any set $X \subseteq A^{(0)}$ and a set I of ordinals let

$$X \restriction I = \{\overline{F} \in X : \ \kappa(\overline{F}) \in I\}.$$

Claim 2.16. There is a (λ, κ)-system such that $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$ <u>when</u> the following hold:

(C) (a) $\theta_* < \kappa < \lambda < 2^\kappa$,

(b) $\overline{F}_*$ is an ultrafilter sequence consisting of κ-complete ultrafilters on $\mathbf{V}_\kappa$, $\overline{F}_* \in \mathbf{A}$.

(c) there exists $f : \kappa \to \kappa$ such that

$$\{\overline{F} : \ \mathrm{dom}(\overline{F}) < f(\kappa(\overline{F}))\} \in \bigcap \overline{F}_* = \bigcap_{0 < \alpha < \mathrm{dom}(\overline{F}_*)} F_*(\alpha)$$

(i.e. when for a witnessing $\mathbf{j}$ for $\overline{F}_*$ the inequality $\mathbf{j}(f)(\kappa) \geq \mathrm{dom}(\overline{F}_*)$ holds, for instance this holds if $\mathrm{dom}(\overline{F}_*) \leq (2^{2^\kappa})^{\mathbf{M}}$),

(d) $\bigcap \overline{F}_* = \bigcap_{0 < \alpha < \mathrm{dom}(\overline{F}_*)} F_*(\alpha)$ is $<\lambda^+$-directed in the following sense. For every sequence $\langle X_\alpha : \alpha < \lambda \rangle$ in $\bigcap \overline{F}_*$ there exists $X_* \in \bigcap \overline{F}_*$ such that

$$\forall \alpha < \lambda \ \exists \beta < \kappa : \ X_* \restriction (\beta, \kappa) \subseteq X_\alpha.$$

(e) $\mathbb{P} = \mathbb{P}_{\overline{F}_*}$ is the Radin forcing for $\overline{F}_*$ (see Definition 2.17), so preserves the function $\mu \mapsto 2^\mu$, moreover, we can prescribe that in $\mathbf{V}^{\mathbb{P}}$ there is no new subset of θ_*, and $\mathbb{P}$ satisfies the κ^+-c.c.

Proof. We will use the definition of the Radin forcing from [Git10, Definition 5.2]. Observe that the definition only depends on $\bigcap \overline{F}_*$.

Definition 2.17 ([Git10, Definition 5.2]). For an ultrafilter sequence $\overline{F}_* \in \mathbf{A}$ we define the Radin forcing $\mathbb{P}$ to be the collection of finite sequences of the form $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle$, where

($*_1$) (a) $A_{p,\kappa} \in \bigcap \overline{F}_* = \bigcap_{0 < \alpha < \mathrm{dom}(\overline{F}_*)} F_*(\alpha)$, $A_{p,\kappa} \in \mathbf{A}$,

(b) each d_j $(j \leq n)$ is either of the form

- $\langle \overline{F}_{d_j}, A_{d_j} \rangle$ where $\overline{F}_{d_j} \in \mathbf{A}$, $A_{d_j} \subseteq \mathbf{A}$, moreover,

$$A_{d_j} \in \bigcap \overline{F}_{d_j} = \bigcap_{0 < \gamma < \mathrm{dom}(\overline{F}_{d_j})} F_{d_j}(\gamma).$$

If $\varepsilon = \kappa(\overline{F}_{d_j})$ we may refer to $\langle \overline{F}_{d_j}, A_{d_j} \rangle$ as $\langle \overline{F}_{p,\varepsilon}, A_{p,\varepsilon} \rangle$, and we also define $\kappa(d_j) = \kappa(\overline{F}_{d_j})$.

- or $d_j = \varepsilon$ for some $\varepsilon < \kappa$ (when we let $\kappa(d_j) = \varepsilon$).

(c) $\kappa(d_0) < \kappa(d_1) < \cdots < \kappa(d_n) < \kappa(d_{n+1}) = \kappa$,
(d) moreover, for each $j \leq n$ if d_{j+1} is a triplet, then $A_{p,\kappa(d_{j+1})} \cap V_{\kappa(d_j)} = \emptyset$.

$(*_2)$ For the sequences

$$p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle,$$

$$q = \langle e_0, e_1, \ldots, e_n, e_{m+1} = \langle \overline{F}_*, A_{q,\kappa} \rangle \rangle,$$

we let $p \leq q$, if
(a) $m \geq n$, and
(b) there exists a sequence $0 \leq i_0 < i_1 < \cdots < i_n < j_{n+1} = m$ such that for each $j \leq n+1$ we have
- $\kappa(d_j) = \kappa(e_{i_j})$,
- and

either $\overline{F}_{p,\kappa(d_j)} = \overline{F}_{q,\kappa(e_{i_j})}$ and $A_{q,\kappa(e_{i_j})} \subseteq A_{p,\kappa(d_j)}$,

or $d_j = e_{i_j} = \kappa(d_j) = \kappa(e_{i_j})$,

(c) moreover, for each $l \leq m$ not of the form i_j $(j \leq n+1)$, if $i_l = \min\{i_j : j \leq n+1,\ i_j > l\}$, then

$$A_{q,\kappa(e_k)} \cup \{\overline{F}_{q,\kappa(e_k)}\} \subseteq A_{p,\kappa(d_l)}.$$

Definition 2.18. We define $p \leq_* q$ to be true iff $p \leq q$ and $\ell g(p) = \ell g(q)$.

We define the system $\mathbf{r}$ by letting:

(8) (a) $\kappa_{\mathbf{r}} = \kappa$,
(b) $\lambda_{\mathbf{r}} = \lambda$,
(c) $\mathbb{R}_{\mathbf{r}} = \mathbb{P}$,
(d) let $\underset{\sim}{X}_{\mathbf{r}}$ be the generic sequence, i.e.

$$\underset{\sim}{X}_{\mathbf{r}} = \cup \{ \{ \kappa(d_j), \overline{F}_{p,\kappa(d_j)} : j < \ell g(p) \} :\ p = \langle d_0, d_1, \ldots, d_{\ell g(p)-1} \rangle \in \mathbf{G}_{\mathbb{P}} \} \setminus \{\kappa\},$$

(e) $\leq_{\text{pr}} = \leq_{\mathbf{r},\text{pr}}$ is defined by $p \leq_{\text{pr}} q$ iff $p \leq_* q$,
(f) for $p = \langle d_0, d_1, \ldots, d_n, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle \in \mathbb{R}_{\mathbf{r}} = \mathbb{P}$ let

$$\mathscr{S}_p = \mathscr{S}_{\mathbf{r},p} := \left\{ \begin{array}{l} \bar{q} : \bar{q} = \langle q_\varepsilon : \varepsilon < \kappa \rangle, \text{ where} \\ (\bullet_1) q_\varepsilon = \langle d_0, d_1, \ldots, d_n, \langle \overline{F}_*, A_{q_\varepsilon,\kappa} \rangle \rangle, \text{ and} \\ \text{for some } B \in \bigcap \overline{F}_* \text{ we have} \\ (\bullet_2)\ B \subseteq A_{p,\kappa}, \text{ and} \\ (\bullet_3)\ \{A_{q_\varepsilon,\kappa} : \varepsilon < \kappa\} \text{ lists } \{A_* : A_* \subseteq A_{p,\kappa} \wedge A_* = B \mod [\kappa]^{<\kappa}\} \end{array} \right\}.$$

Now we check the requirements of Definition 1.2.

It is known that if a condition $\langle \langle \beta \rangle, \langle \overline{F}_*, A_\kappa \rangle \rangle$ is in the generic filter (for some $\beta < \kappa$) then the forcing adds no new subset of β. This implies that as $\bigcap \overline{F}_* \subseteq F_*(0)$, which is concentrated on the ordinals, i.e. on κ itself, w.l.o.g. we can assume that $\langle \beta, \langle \overline{F}_*, A \rangle \rangle \in \mathbf{G}$ for some $\beta \geq \theta_*$.

Now we have only to check the requirements of Definition 1.2. Recall the following properties of the Radin forcing, see [Git10, Sec. 5.1].

Fact 2.19 (Prikry lemma)**.** For each $p \in \mathbb{P}$ and each formula $\sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$ there exists $q \geq_* p$, $q \parallel \sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$ (i.e. either $q \Vdash \sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$, or $q \Vdash \neg\sigma(\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m)$).

The following claims, which complete the proof of Claim 2.16 have the same proofs as in the case of Magidor forcing. In Claim 2.20 condition (C)/(c) is essential for the argument.

Claim 2.20. For each $p = \langle d_0, d_1, \dots, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$) there exists a set $A' \in \bigcap \overline{F}_*$, $A' \subseteq A_{p,\kappa}$, such that whenever $q = \langle e_0, e_1, \dots, e_m, \langle \overline{F}_*, A_{q,\kappa} \rangle \rangle \geq p' = \langle d_0, d_1, \dots, d_n, \langle \overline{F}_*, A' \rangle \rangle$, $\alpha \geq \kappa(e_m)$ are given and q forces a value for $\underset{\sim}{\tau}$, then so does

$$q' = \big\langle e_0, e_1, \dots, e_m, \langle \overline{F}_*, A' \restriction (\alpha, \kappa) \rangle \big\rangle .$$

Claim 2.21. Suppose $p = \langle d_0, d_1, \dots, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle \in \mathbb{P}_{\overline{F}_*}$, $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$), and $\alpha \geq \kappa(d_n)$. If $p \in \mathbf{G}$, $\mathbf{G} \subseteq \mathbb{P}_{\overline{F}_*}$ is generic over $\mathbf{V}$, then there exists

$$q = \langle e_0, e_1, \dots, e_{m+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle \in \mathbb{P}_{\overline{F}_*},$$

$$q \in \mathbf{G},$$

where $\kappa(e_m) \leq \alpha$, $A_{q,\kappa} \cap \mathbf{V}_{\alpha+1} = \emptyset$, and there exists $A \subseteq A_{p,\kappa}$, $A \in \bigcap \overline{F}_*$, such that

$$q \restriction (\kappa(e_m)+1) \frown \langle \overline{F}_*, A \rangle \parallel \underset{\sim}{\tau} = 1.$$

Claims 2.20, 2.21 imply the following.

Claim 2.22. For each $p = \langle d_0, d_1, \dots, d_{n+1} = \langle \overline{F}_*, A_{p,\kappa} \rangle \rangle \in \mathbb{P}$, $\underset{\sim}{\tau}$ (with $p \Vdash \underset{\sim}{\tau} \in \{0,1\}$) there exists a set $A' \in \bigcap \overline{F}_*$, $A' \subseteq A_{p,\kappa}$, such that whenever $\alpha < \kappa$, and

$$p' = p \restriction (\alpha+1) \frown \langle \overline{F}_*, A_{p,\kappa} \restriction (\alpha+1) \cup A' \restriction (\alpha, \kappa) \rangle \rangle \in \mathbf{G},$$

$\mathbf{G} \subseteq \mathbb{P}_{\overline{F}_*}$ is a generic filter, then there exists $q \in \mathbf{G}$, q is of the form

$$q = q \restriction (\alpha+1) \frown \langle \overline{F}_*, A' \restriction (\alpha, \kappa) \rangle \rangle,$$

and

$$q \parallel \underset{\sim}{\tau} = 1.$$

Claim 2.23. Suppose that $p \in \mathbb{P}$ and $\underset{\sim}{\tau}$. If $p \Vdash \tau \in \{0,1\}$, then there exists $\bar{q} = \langle q_\varepsilon : \varepsilon < \kappa \rangle \in \mathscr{S}_p$, $\langle \gamma_\varepsilon : \varepsilon < \kappa \rangle \in {}^\kappa\kappa$, $\langle Y_\varepsilon : \varepsilon < \kappa \rangle \in {}^\kappa\mathbf{V}_\kappa$, $\langle T_\varepsilon : \varepsilon < \kappa \rangle$, such that each T_ε is a code for a γ_ε-Borel subset of $\mathscr{P}(Y_\varepsilon)$, and

$$q_\varepsilon \Vdash (\underset{\sim}{\tau} = 1) \iff (X \cap Y_\varepsilon) \in B_{T_\varepsilon}.$$

$\square_{\text{Claim 2.16}}$

3. The preparatory forcing

3(A). **The general framework.** This subsection is devoted to the preparatory forcing, in Claim 3.2 we provide a general framework to force a $(\lambda, \kappa) - 1$ system.

First we are going to define a variant of Mathias forcing, for which we need to recall the notations from Definition 2.15 (ii), (v), so if $I \subseteq \kappa$, $A \subseteq V_\kappa$, then

$$A \restriction I = \{x \in A : \kappa(x) \in I\},$$

where $\kappa(\alpha) = \alpha$ if α is an ordinal, $\kappa(\overline{F}) = \mathrm{crit}(\mathbf{j})$ for the elementary embedding $\mathbf{j}$ if $\overline{F}$ is a $\mathbf{j}$-sequence (and for every other x, we can let $\kappa(x) = -1$). Using this convention we will have Mathias forcing for filters in the context of Radin forcing, too, not only filters concentrated on κ.

Definition 3.1. For D a $<\kappa$-centered system (i.e. generating a κ-complete filter D^*) on $\cup D \subseteq \mathbf{V}_\kappa$ (so $D^* \subseteq \mathscr{P}(\cup D)$) we let $\mathbb{Q} = \mathbb{Q}_D$ be the following forcing notion:

(A) $p \in \mathbb{Q}$ iff
  (a) $p = (w, A) = (w_p, A_p)$, and for some $\sigma_p < \kappa$ we have
  (b) $w_p \subseteq \mathbf{V}_\kappa$, $w_p = w_p \restriction [0, \sigma_p)$ (so $w_p \in \mathbf{V}_\kappa$ holds, too)
  (c) $A_p \subseteq \cup D$, $A_p \in D^*$ and $A_p = A_p \restriction [\sigma_p, \kappa)$.

(B) $\mathbb{Q} \models p \leq q$ iff
  (a) $p, q \in \mathbb{Q}$,
  (b) $w_p \subseteq w_q \subseteq w_p \cup A_p$,
  (c) $A_p \supseteq A_q$,

(C) $\underset{\sim}{w} = \cup\{w_p : p \in \underset{\sim}{\mathbf{G}}\}$.

Claim 3.2. If (A) and (B) hold, then so does (C), where:

(A) $\mathbf{v} = (\mathbf{V}_0, \kappa, \mathbf{h}, \mathbf{p}, \mathbf{G}_\kappa, \mathbf{V}_1)$ satisfies:
  (a) $\mathbf{V}_0$ is a universe of set theory,
  (b) in $\mathbf{V}_0$ κ is supercompact and $\mathbf{h} : \kappa \to \mathscr{H}(\kappa)$ is a Laver diamond,
  (c) $\mathbf{p}$ is the Easton support iteration $\langle \mathbb{P}_{\mathbf{p},\alpha}, \underset{\sim}{\mathbb{Q}}_{\mathbf{p},\beta} : \alpha \leq \kappa, \beta < \kappa \rangle = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ built as specified in Definition 3.4$(\bullet)_I$–$(\bullet)_{II}$, and $(\bullet)_a$–$(\bullet)_b$ using $\mathbf{h}$ (essentially as in Laver [Lav78]) and let $\mathbb{P}_\mathbf{p} = \mathbb{P}_{\mathbf{p},\kappa}$ (hence for $\alpha < \kappa$ also $\mathbb{P}^0_\alpha \in V_\kappa^{\mathbf{V}_0}$),
  (d) $\mathbf{G}_\kappa = \mathbf{G}_{\mathbf{p},\kappa} \subseteq \mathbb{P}_\mathbf{p}$ is generic over $\mathbf{V}_0$ and $\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_\kappa]$.

(B) (a) $\kappa < \lambda < \chi = \chi^\lambda$ (in $\mathbf{V}_0$, of course),
  (b) $\mathbb{P}^1_\chi = \langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi \rangle \in \mathbf{V}_1$ is an iteration with $<\kappa$ support such that $\mathbb{P}^1_\chi$ is λ^+-c.c. and $<\kappa$-directed closed, preserving cardinals,
  (c) for each $\alpha < \chi$
$$\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi.$$
  (d) for the set $S^* \subseteq \chi$ there is a system $\langle \underset{\sim}{D}_\delta : \delta \in S^* \rangle \in \mathbf{V}_1$, $\underset{\sim}{D}_\delta$ is a $\mathbb{P}^1_\delta$-name of a subset of $\mathscr{P}^{\mathbf{V}_1^{\mathbb{P}^1_\delta}}(V_\kappa)$, and if

(3.1)
$$\mathbf{V}_1^{\mathbb{P}^1_\delta} \models \begin{array}{l} \underset{\sim}{D}_\delta \text{ generates a } \kappa\text{-complete filter, satisfying} \\ (\forall \alpha < \kappa)\ |(\cup D_\delta) \restriction \alpha| < \kappa, \end{array}$$

then the forcing $\underset{\sim}{\mathbb{Q}}^1_\delta$, $\delta \in S^*$ is of the form $Q_{\underset{\sim}{D}_\delta}$, the forcing from Definition 3.1. Moreover, we assume that each $D \in [\mathscr{P}^{\mathbf{V}_1^{\mathbb{P}^1_\chi}}(V_\kappa)]^{\leq \lambda}$ that satisfies (3.1) appears as a D_δ for some $\delta \in S^*$, i.e.

$$(\#)\ \mathbf{V}_1^{\mathbb{P}^1_\chi} \models \forall D \in [\mathscr{P}(V_\kappa)]^{\leq \lambda}:$$
$$[\text{if } D \text{ generates a } <\kappa\text{-complete filter, and}$$
$$\forall \alpha < \kappa:\ |(\cup D) \restriction \alpha| < \kappa,$$
$$\text{then } (D = D_\delta \text{ for some } \delta \in S^*).]$$

(C) in $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ we have 2^κ is χ, and the following.
  (a) There is a κ-complete normal ultrafilter U, which is $<\lambda^+$-directed mod $[\kappa]^{<\kappa}$.
  (b) (Setting for Magidor forcing:) There is a sequence $\bar{U} = \langle U_i : i < \kappa \rangle$ of normal ultrafilters on κ, strictly increasing in the Mitchell order, i.e. $i < j \Rightarrow U_i \in \operatorname{Mos}\operatorname{Col}({}^\kappa(\mathbf{V}^{\mathbb{P}^1_\chi})/U_j)$, such that each U_i is $<\lambda^+$-directed mod $[\kappa]^{<\kappa}$.

(c) (Setting for Radin forcing:) For any $\Upsilon \geq \kappa$ and η there is a κ-complete fine normal ultrafilter W on $[\Upsilon]^{<\kappa}$ such that for the elementary embedding $\mathbf{j}_W$ of $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ with critical point κ we have (letting $\overline{U}$ denote the measure sequence associated to $\mathbf{j}_W$):

($\star$) for every $\sigma \leq \min(\mathrm{dom}(\overline{U}, \eta))$ if the filter $\bigcap(\overline{U} \restriction \sigma) = \bigcap_{\gamma<\sigma} U_\gamma$ concentrates on a set $X \subseteq V_\kappa$ with $(\forall \alpha < \kappa)\ |X \restriction \alpha| < \kappa$, then $\bigcap(\overline{U} \restriction \sigma)$ is $< \lambda^+$-directed in the following sense: Whenever $\langle A_i : i < \lambda \rangle$ $(\forall i < \lambda\ A_i \in \bigcap(\overline{U} \restriction \sigma))$ is given, there exists $A_* \in \bigcap(\overline{U} \restriction \sigma)$ such that

$$\forall i \in \lambda\ \exists \delta_i < \kappa :\ A_* \restriction [\delta_i, \kappa) \subseteq A_i. \tag{3.2}$$

In particular κ is supercompact.

Remark 3.3. This continues Džamonja-Shelah [DS03].

Proof. First we have to construct the iteration $\mathbb{P}^0$ using the Laver function $\mathbf{h} : \kappa \to \mathscr{H}(\kappa) \in \mathbf{V}_0$. The construction $\mathbb{P}^0 = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta :\ \alpha \leq \kappa, \beta < \kappa \rangle$ goes by induction, we follow [Lav78], only with a slight technical modification which we will need in the proof of (C)(b).

Let $\mathbf{h}$ be as in [Lav78] (i.e.

$(\bullet)_1$ for each $\lambda \geq \kappa$, $x \in \mathscr{H}(\lambda^+)$ there exists a κ-complete fine normal ultrafilter U on $[\lambda]^{<\kappa}$ such that for the associated elementary embedding $\mathbf{j}_U$

$$\mathbf{j}_U(\mathbf{h})(\kappa) = x).$$

Definition 3.4. We define $\mathbb{P}^0 = \langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta :\ \alpha \leq \kappa, \beta < \kappa \rangle$ and $\langle \mu_\alpha :\ \alpha < \kappa \rangle$ by induction. If $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta :\ \alpha < \gamma, \beta < \gamma \rangle$ are already defined, then

$(\bullet)_I$ if γ is strongly inaccessible then $\mathbb{P}^0_\gamma$ is the direct limit (i.e. we use bounded support),

$(\bullet)_{II}$ otherwise let $\mathbb{P}^0_\gamma$ be the inverse limit of $\mathbb{P}^0_\beta$'s $(\beta < \gamma)$ (i.e. for a function p with $\mathrm{dom}(f) = \gamma$ $p \in \mathbb{P}^0_\gamma$ iff $(\forall \beta < \gamma)\ p \restriction \beta \in \mathbb{P}^0_\beta$).

Second,

$(\bullet)_a$ if $\sup\{\mu_\alpha :\ \alpha < \gamma\} \leq \gamma$, and γ is strongly inaccessible, moreover, $\mathbf{h}(\gamma)$ happens to be of the form $\langle \underset{\sim}{Q}_*, \mu_*, \underset{\sim}{U} \rangle$, where $\underset{\sim}{Q}_*$ is a $\mathbb{P}^0_\gamma$-name for a $<\gamma$-directed closed notion of forcing, μ_* is an ordinal, $\underset{\sim}{U}$ is a (possibly trivial) $\mathbb{P}^0_\gamma$-name, then let

$$\underset{\sim}{\mathbb{Q}}^0_\gamma = \underset{\sim}{\mathbb{Q}}_*, \quad \mu_\gamma = \mu_*.$$

$(\bullet)_b$ In the remaining case let $\underset{\sim}{\mathbb{Q}}^0_\gamma$ be the trivial forcing, $\mu_\gamma = \gamma$.

Recall $\mathbf{G}^0_\kappa \subseteq \mathbb{P}^0_\kappa$ is generic over $\mathbf{V}_0$ so that $\mathbf{V}_0[\mathbf{G}^0_\kappa] = \mathbf{V}_1$, and let $\mathbf{G}^1_\chi \subseteq \mathbb{P}^1_\chi$ be generic over $\mathbf{V}_1$, let $\mathbf{V}_2 = \mathbf{V}_1[\mathbf{G}^1_\chi] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$. Note that as $|\mathbb{P}^0_\kappa| = \kappa$ and $\kappa < \lambda$, (B)(a) implies that

$(\bowtie)_1$ $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}^0_\kappa] \models\ \chi^\lambda = \chi^{\lambda\cdot\kappa} = \chi$, thus $\mathrm{cf}(\chi) > \lambda$ is preserved, too. Since κ is strongly inaccessible, and $\mathbb{P}^0$ is an Easton support iteration, where $\underset{\sim}{\mathbb{Q}}^0_\beta$ is $< \alpha$-closed for $\alpha < \beta$, and for stationarily many α's $|\mathbb{P}^0_\alpha| = \alpha$ (actually for each strongly inaccessible cardinal α), by standard arguments

$(\bowtie)_2$ $\mathbb{P}^0_\kappa$ has the κ-cc (so forcing with it preserves the regularity of κ), moreover

$(\bowtie)_3$ $\mathbb{P}^0_\kappa$ preserves κ to be strongly inaccessible. Also note that as $\mathbb{P}^1_\chi$ is $< \kappa$-closed

$(\bowtie)_4$ $V_\kappa^{\mathbf{V}_2} = V_\kappa^{\mathbf{V}_1}$, and $\mathbf{V}_2 \models$ "κ is still strongly inaccessible."

First observe that because of our cardinal arithmetic assumptions $\chi^\kappa \leq \chi^\lambda = \chi$ in (B)(a), and as $|\mathbb{P}^0_\kappa| = \kappa$, not only do we have $(\bowtie)_1$ $(\chi^\lambda)^{\mathbf{V}_1} = \chi^{\lambda \cdot \kappa} = \chi$, but by an easy induction (and by the λ^+-cc) $|\mathbb{P}^1_\chi|^{\mathbf{V}_1} = \chi$, so

$(\bowtie)_5$ $|\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi| = \chi$ up to equivalence (and so obviously χ^+-cc). Recalling $\chi^\lambda = \chi$ again, clearly

$(\bowtie)_6$ $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models 2^\chi = (2^\chi)^{\mathbf{V}_0}$,

$(\bowtie)_7$ $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models 2^\kappa = \chi$.

Definition 3.5. We have to introduce the following objects.

- $(\bullet)_2$ Let $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$ be an elementary embedding with critical point κ such that $(\mathbf{j}(\mathbf{h}))(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \check{\emptyset} \rangle$ ($\check{\emptyset} = \emptyset$ is the canonical name for the empty set) and $\mathbf{j}(\kappa) > \chi$, ${}^\chi\mathbf{M} \subseteq \mathbf{M}$,
- $(\bullet)_3$ Let $\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \mathbf{j}(\kappa), \beta < \mathbf{j}(\kappa) \rangle = \mathbf{j}(\langle \mathbb{P}^0_\alpha, \underset{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa \rangle)$ so $\underset{\sim}{\mathbb{Q}}^0_\kappa = \underset{\sim}{\mathbb{P}}^1_\chi$, and
- $(\bullet)_4$ let $\underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)} = \mathbf{j}(\underset{\sim}{\mathbb{P}}^1_\chi)$, i.e.

  $$(\text{a } \mathbb{P}^0_{\mathbf{j}(\kappa)}\text{-name for a } < \mathbf{j}(\kappa)\text{-directed closed notion of forcing})^{\mathbf{M}}.$$

  (Recall that $\underset{\sim}{\mathbb{P}}^1_\chi$ is a $\mathbb{P}^0_\kappa$-name for the iteration $\langle \underset{\sim}{\mathbb{P}}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi \rangle \in \mathbf{V}_0^{\mathbb{P}^0_\kappa}$.)

Similarly to $(\bowtie)_4$, recalling ${}^\chi M \subseteq M$,

$(\bowtie)_8$ $V_\kappa^{\mathbf{M}[\mathbf{G}^0_{\kappa+1}]} = V_\kappa^{\mathbf{M}[\mathbf{G}^0_\kappa]} = V_\kappa^{\mathbf{V}_2}$, and $(\kappa \text{ is strongly inaccessible})^{\mathbf{M}[\mathbf{G}^0_{\kappa+1}]}$.

From now on we will identify $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ with the $(\kappa+1)$-step iteration $\mathbb{P}^0_{\kappa+1}$, and also

$(\bowtie)_9$ $\mathbf{G}^0_{\kappa+1} = \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$ is a generic subset of $\mathbb{P}^0_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ (over $\mathbf{V}_0$).

Remark 3.6. Having completed the requirements of Claim 3.2 we remark that given a scheme for an iteration fitting all our assumptions except perhaps (B)(d), it is easy to adapt it to have (#) using $\chi^\lambda = \chi$ $(\bowtie)_1$.

Now we can prove the statements in 3.2(C).

Case 1. First we verify 3.2(C)(a).

We would like to find an appropriate κ-complete ultrafilter in $\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$, for which we will use the basic trick: using the elementary embedding $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$, then extending $\mathbf{V}_0$ with $\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$, and extending $\mathbf{M}$ with $\mathbf{G}^0_{\kappa+1} (= \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi)$, and finding a single condition in $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)} / \mathbf{G}^0_{\kappa+1}$ compatible with $\{\mathbf{j}(p \restriction \{\kappa\}) = \mathbf{j}(p) \restriction \{\mathbf{j}(\kappa)\} : p \in \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi\}$ giving us sufficient information (just as if there existed some lifting $\tilde{\mathbf{j}} : \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \to \mathbf{M}[\mathbf{H}^0_{\mathbf{j}(\kappa)} * \mathbf{H}'_{\mathbf{j}(\chi)}])$ of $\mathbf{j}$ extending it). (Here the quotient $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)} / \mathbf{G}^0_{\kappa+1}$ is formally

$$\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)} / \mathbf{G}^0_{\kappa+1} = \{(p \restriction (\kappa, \mathbf{j}(\kappa)), \underset{\sim}{q}) : (p, \underset{\sim}{q}) \in \mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}\},$$

and

$$\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)} / \mathbf{G}^0_{\kappa+1} \models (p \restriction (\kappa, \mathbf{j}(\kappa)), q) \leq (p' \restriction (\kappa, \mathbf{j}(\kappa)), q'),$$

if there exists $p_* \in \mathbf{G}^0_{\kappa+1}$ such that

$$\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)} \models (p_* \frown p\restriction(\kappa, \mathbf{j}(\kappa)), \underset{\sim}{q}) \leq (p_* \frown p'\restriction(\kappa, \mathbf{j}(\kappa)), \underset{\sim}{q}').)$$

We will need the following facts.

Fact 3.7. The filter $\mathbf{G}^0_{\kappa+1}$ is generic over $\mathbf{M}$ as well, and the forcing notions $\mathbb{P}^0_{j(\kappa)}/\mathbf{G}^0_{\kappa+1}$ and $(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ $(\gamma \leq \mathbf{j}(\chi))$ are well-defined and $< \chi^+$-directed closed in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.

Proof. Note that $\mathbf{G}^0_{\kappa+1}$ is generic, as $\mathbb{P}^0_{\kappa+1} \subseteq \mathbf{M} \subseteq \mathbf{V}_0$.

For the second assertion we first recall that a pair $(p, \underset{\sim}{q}) \in (\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}}(\chi))/\mathbf{G}^0_{\kappa+1}$ iff $p = p_0 \restriction (\kappa, \mathbf{j}(\kappa))$ for some $p_0 \in \mathbb{P}^0_{\mathbf{j}(\kappa)}$, and $(\Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)}} \underset{\sim}{q} \in \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)})^{\mathbf{M}}$. We only have to refer to the construction of the iteration Definition 3.4, i.e. recall that

(i) $\Vdash_{\mathbb{P}^0_\kappa}$ "$\underset{\sim}{\mathbb{P}}^1_\chi$ is a $< \kappa$-support iteration of $< \kappa$-directed closed forcing notions", and

(ii) for each $\alpha \leq \beta < \kappa$ we have that $\Vdash_{\mathbb{P}^0_\beta}$ "$\underset{\sim}{\mathbb{Q}}^0_\beta$ is $< \beta$-directed closed", and is the trivial forcing if $\beta < \sup\{\mu_\varrho : \varrho < \beta\}$ (in particular, if $\beta < \sup\{\mu_\varrho : \varrho < \alpha\}$),

(iii) for each $\alpha < \beta < \kappa$, where β is limit and $\mathrm{cf}(\beta) < \mu_\alpha$ the iteration $\mathbb{P}^0_\beta$ is the inverse limit of $\mathbb{P}^0_\delta$'s $(\delta < \beta)$.

So using [Bau78, Thm. 5.5], for each $\alpha < \beta < \kappa$ the quotient $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi)/\mathbf{G}^0_\alpha$ (of the $\kappa + 1$-long iteration $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi = \mathbb{P}^0_{\kappa+1}$) is $< \beta$-directed closed in $\mathbf{V}_0[\mathbf{G}^0_\alpha]$ provided $\beta \leq \sup\{\mu_\varrho : \varrho < \alpha\}$, and $\mathbb{P}^0_\alpha$ has the β-cc. (In typical applications $\underset{\sim}{\mathbb{Q}}^0_\alpha$ is the trivial forcing.) Thus by elementarity (letting $\alpha = \kappa + 1$, $\beta = \chi^+ = \mu_\kappa$, recalling $\mathbb{P}^0_{\kappa+1}$ has the χ^+-cc by $(\bowtie)_5$, and $(\chi^+)^{\mathbf{M}} = \chi^+$ by ${}^\chi\mathbf{M} \subseteq \mathbf{M}$):

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \text{“}(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}}(\chi))/\mathbf{G}^0_{\kappa+1} \text{ is } < \chi^+\text{-directed closed.”}$$

$\square_{\text{Fact 3.7}}$

Fact 3.8. $\mathbf{V}_1 \models$ “$(\mathbb{P}^0_{j(\kappa)} * \underset{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ is $< \chi^+$-directed closed.”

Fact 3.8 follows from Fact 3.9.

Fact 3.9. $\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models {}^\chi\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.

Proof. For, pick a name $\underset{\sim}{f}$ for a function $\underset{\sim}{f} : \chi \to \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$, and observe that w.l.o.g. we can assume that $\underset{\sim}{f} : \chi \to \mathrm{ORD}$, i.e. for each $\alpha < \chi$, $f(\alpha)$ is an ordinal, in particular $\mathrm{ran}(f) \subseteq \mathbf{M}$. Now for each α there exists a maximal antichain $A_\alpha = \{a^\alpha_i : i < |A_\alpha|\} \subseteq \mathbb{P}^0_{\kappa+1}$, and $\{x^\alpha_i : i < |A_\alpha|\} \subseteq \mathbf{M}$, s.t. $a^\alpha_i \Vdash \underset{\sim}{f}(\alpha) = x^\alpha_i$. As $\mathbb{P}^0_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$ is of power χ, we have $|A_\alpha| \leq \chi$ trivially, therefore as $\mathbf{M}$ is closed under sequences of length χ ($(\bullet)_2$, Definition 3.5) $\langle (x^\alpha_i, a^\alpha_i) : \alpha < \chi, i < |A_\alpha| \rangle \in M$, which means that there is indeed a name $\underset{\sim}{g} \in \mathbf{M}$, such that $\Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi} \underset{\sim}{f} = \underset{\sim}{g}$. $\square_{\text{Fact 3.9}}$

Definition 3.10. (In $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$) for $\zeta \in S^*$ we let

(1) $\varepsilon_\zeta \in \mathbf{V}[\mathbf{G}^0_\kappa * \underset{\sim}{\mathbf{G}}^1_{\zeta+1}]$ denote the generic subset of $V^{\mathbf{V}_1}_\kappa$ (or just κ) given by $\mathbb{Q}^1_\zeta$, i.e.

$$\Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1}} \underset{\sim}{\varepsilon}_\zeta = \cup\{\underset{\sim}{\varepsilon} : \exists \underset{\sim}{A} : (\underset{\sim}{\varepsilon}, \underset{\sim}{A}) \in \mathbf{G}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\}$$

(after identifying $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1} = \mathbb{P}^0_\kappa * (\underset{\sim}{\mathbb{P}}^1_\zeta * \underset{\sim}{\mathbb{Q}}^1_\zeta)$ with $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta) * \underset{\sim}{\mathbb{Q}}^1_\zeta$).

(2) Define $\mathscr{N}_\zeta$ to be a set of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-names of subsets of V_κ containing exactly one name from each equivalence class, i.e. no $\underset{\sim}{A} \neq \underset{\sim}{B} \in \mathscr{N}_\zeta$ satisfy $\Vdash_{\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta} \underset{\sim}{A} = \underset{\sim}{B}$, but each set in the extension is represented.

Observe that (as $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta) * \underset{\sim}{\mathbb{Q}}^1_\zeta, \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta \in \mathbf{M}$) we can assume that

$(\bowtie)_{10}$ $\mathscr{N}_\zeta \subseteq \mathbf{M}$,

and as $|V_\kappa^{\mathbf{V}_2}| = \kappa$, and by the λ^+-cc (B)(b)

$(\bowtie)_{11}$ $|\mathscr{N}_\zeta| \leq |\mathbb{P}^0_\kappa * \mathbb{P}^1_\zeta|^\lambda = \chi$,

so by ${}^\chi\mathbf{M} \subseteq \mathbf{M}$:

$(\bowtie)_{12}$ $\mathscr{N}_\zeta \in \mathbf{M}$, and $\mathbf{j} \restriction \mathscr{N}_\zeta \in \mathbf{M}$.

(3) Using the notation

$$\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} = \{\underset{\sim}{A} \in \mathscr{N}_\zeta : \ (\underset{\sim}{\varepsilon}, \underset{\sim}{A}) \in \mathbf{G}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} \text{ for some } \underset{\sim}{\varepsilon}\},$$

note that $\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} \in \mathbf{M}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\zeta+1}]$ (so $\underset{\sim}{\mathscr{A}}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}$ is a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{\zeta+1}$-name for a set of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-names). Now similarly

$$\mathbf{j}``\mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta} = \{\mathbf{j}(\underset{\sim}{A}) : \ \underset{\sim}{A} \in \mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\} \in \mathbf{M}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\zeta+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$$

is a set of $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$-names, and each of which collection corresponds to a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta/\mathbf{G}^0_{\kappa+1}$-name, we can define the $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}/\mathbf{G}^0_{\kappa+1}$-name $\underset{\sim}{A}'_{\mathbf{j}(\zeta)} \in \mathbf{M}$ for a subset of $V_{\mathbf{j}(\kappa)}$ so that

$$(\text{in } \mathbf{M}[\mathbf{G}^0_{\kappa+1}] :) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}/\mathbf{G}^0_{\kappa+1}} \ \underset{\sim}{A}'_{\mathbf{j}(\zeta)} = \cap\{\mathbf{j}(\underset{\sim}{A}) : \ \underset{\sim}{A} \in \mathscr{A}_{\underset{\sim}{\mathbb{Q}}^1_\zeta}\}.$$

Claim 3.11. There is a sequence $\langle q_\zeta : \ \zeta \leq \chi\rangle \in \mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$ such that:

$(*)_{1.1}$ (a) $q_\zeta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$, and if $\varepsilon < \zeta \leq \chi$, then $q_\varepsilon \leq q_\zeta$,

(b) $q_\zeta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\zeta)})/\mathbf{G}^0_{\kappa+1}$ (i.e. $q_\zeta \restriction \mathbf{j}(\kappa) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)}} q_\zeta(\mathbf{j}(\kappa)) \in \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$),

(c) whenever $p \in \mathbf{G}^0_{\kappa+1} \cap (\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta)$ then

$$(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1} \models \mathbf{j}(p) \leq q_\zeta$$

(i.e. $\mathbf{j}(p) \leq q_\zeta$ in the order of the quotient forcing $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$),

(d) whenever $\underset{\sim}{A}$ is a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\zeta$-name of a subset of κ (so $\mathbf{j}(\underset{\sim}{A})$ is a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)}$-name for a subset of $\mathbf{j}(\kappa)$) then for $\kappa \in$ORD

$$q_\zeta \parallel_{(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\zeta)})/\mathbf{G}^0_{\kappa+1}} \ \kappa \in \mathbf{j}(\underset{\sim}{A}).$$

(e) if $\zeta \in S^*$ (from $(\#)$ of (d)) then we have the following: If $D_\zeta := \underset{\sim}{D}_\zeta[\mathbf{G}^1_\zeta]$ generates a κ-complete filter on V_κ (in $\mathbf{V}_1[\mathbf{G}^1_\zeta] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\zeta]$) <u>then</u> (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$ in the poset $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}/\mathbf{G}^0_{\kappa+1}$)

$$(q_{\zeta+1}(\mathbf{j}(\kappa)))\,(\mathbf{j}(\zeta)) \geq \left(\varepsilon_\zeta \cup \left(\underset{\sim}{A}'_{\mathbf{j}(\zeta)} \restriction \{\kappa\}\right), \underset{\sim}{A}'_{\mathbf{j}(\zeta)} \restriction (\kappa+1, \mathbf{j}(\kappa))\right). \tag{3.3}$$

(In this generality this will be relevant for the proof (c). For D_ζ's for which $D_\zeta \subseteq \mathscr{P}(\kappa)$ it is enough to ensure that if for each $A \in D_\zeta$ we have $\kappa \in \mathbf{j}(A)$ (forced by q_ζ), then $q_{\zeta+1} \Vdash \ \kappa \in \tilde{\mathbf{j}}(\varepsilon_\zeta)$.)

Proof. Working in $\mathbf{V}_2 = \mathbf{V}_0[\mathbf{G}_{\kappa+1}]$ we can define the q_η's ($\eta \le \chi$, $q_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$) by induction on η. Assume that q_ξ's ($\xi < \eta$) are chosen and (a)–(e) hold. First we choose q'_ξ satisfying (a), (c), (e) which we will then further strengthen to get $q_\xi \ge q'_\xi$.

Recalling Fact 3.8, let $q'_0 \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(0)})/\mathbf{G}^0_{\kappa+1} = \mathbb{P}^0_{\mathbf{j}(\kappa)}/\mathbf{G}^0_{\kappa+1}$ be the empty condition.

For η limit we choose $q'_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}``\eta})/\mathbf{G}^0_{\kappa+1}$ to be an upper bound of the increasing sequence $\langle q_\xi : \xi < \eta\rangle$ satisfying (c). Now it follows from standard arguments that q'_η satisfies (c), even if $\mathbb{P}^0_\kappa * \mathbb{P}^1_\eta$ is bigger than the direct limit of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$'s ($\xi < \eta$) (in the case $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\eta = \bigcup_{\xi<\eta} \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$ it is automatic), but for completeness we elaborate:

If $p \in \mathbf{G}^0_{\kappa+1}$ is fixed, $p \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$, then for each $\xi < \eta$ let $p_\xi \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi \subseteq \mathbb{P}^0_{\kappa+1}$ be such that $p\restriction\kappa \Vdash_{\mathbb{P}^0_\kappa} p(\kappa)\restriction\xi = p_\xi(\kappa)$. Now if $\mathrm{cf}(\eta) < \kappa$, then $\sup\{\mathbf{j}(\xi) : \xi < \eta\} = \mathbf{j}(\eta)$, and so $\mathbf{j}(p)(\mathbf{j}(\kappa))$ is the least upper bound for the system $\{\mathbf{j}(p_\xi(\kappa)) = \mathbf{j}(p_\xi)(\mathbf{j}(\kappa)) : \xi < \eta\}$, and

$$(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1} \models \mathbf{j}(p_\xi) \le q_\xi$$

by our hypothesis. If $\mathrm{cf}(\eta) \ge \kappa$, then by the κ-cc of $\mathbb{P}^0_\kappa$ $(\bowtie)_2$ there exists a $\xi < \eta$ such that

$$\Vdash_{\mathbb{P}^0_\kappa} p(\kappa) = p(\kappa)\restriction\xi,$$

and so $p \in \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi$ (remember, $\mathbb{P}^1_\chi$ is a $<\kappa$ support iteration). This in turn implies

$$(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1} \models \mathbf{j}(p) \le q_\xi \le q'_\eta.$$

If $\eta = \xi + 1$ is a successor and

- if $\xi \notin S^*$,

then using simply the $<(2^\chi)^+$-directed closedness of $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}$ (by Fact 3.8) define $q'_\eta \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi+1)})/\mathbf{G}^0_{\kappa+1}$ to be an upper bound of $q_\xi \in \mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi)}$ and the set $\{\mathbf{j}(p) : p \in (\mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1}) \cap \mathbf{G}^0_{\kappa+1}\}$.

Otherwise,

- if $\xi \in S^*$,

(where $\eta = \xi + 1$) then recall that by the definition of $\underset{\sim}{\mathbb{Q}}^1_{\xi+1}$ each $p \in (\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_{(\xi+1)})$, the coordinate $(p(\kappa))(\xi+1)$ is a $(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\xi)$-name for a pair (ε, A) with $\varepsilon = \varepsilon \restriction (0,\gamma)$ for some $\gamma < \kappa$, and where $A \subseteq V_\kappa^{\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\xi]}$, $A = A \restriction [\gamma, \kappa)$. Note that $\underset{\sim}{D}_\xi$ generates a $<\kappa$-closed filter on V_κ, therefore $\mathbf{j}(\underset{\sim}{D}_\xi)$ generates a $<\mathbf{j}(\kappa)$-closed filter on $V_{\mathbf{j}(\kappa)}$. We claim that

(3.4)

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi+1)}/\mathbf{G}^0_{\kappa+1} \models q'_\xi \frown \left(\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}\right) \ge \mathbf{j}(p) \text{ whenever } p \in \mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1} \cap \mathbf{G}^0_{\kappa+1},$$

where $q'_\xi \frown \left(\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}\right)$ denotes the condition that agrees with q'_ξ on coordinates below $\mathbf{j}(\xi)$, and $(\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)})$ at $\mathbf{j}(\xi)$. Note that by our hypothesis it suffices to check that

$$\forall p \in \mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1} \cap \mathbf{G}^0_{\kappa+1} : \quad (\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}) \ge \mathbf{j}(p)(\mathbf{j}(\xi)).$$

But a contradiction may only arise if for some $x = \mathbf{j}(x) \in V_\kappa^{\mathbf{V}_1}$ it were the case that

$$\mathbf{j}(p) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi)+1}} \mathbf{j}(x) \in \mathbf{j}(\varepsilon_\xi) (= \mathbf{j}(\varepsilon)_{\mathbf{j}(\xi)}),$$

equivalently,

$$p \Vdash_{\mathbb{P}^0_\kappa * \mathbb{P}^1_{\xi+1}} x \in \varepsilon_\xi,$$

while

$$q'_\xi \frown (\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)}) \Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\xi+1)}/\mathbf{G}^0_{\kappa+1}} x = \mathbf{j}(x) \notin \mathbf{j}(\varepsilon)_{\mathbf{j}(\xi)}$$

(or the other way around). However, this is impossible as clearly $x \in \varepsilon_\xi$ by $p \in \mathbf{G}^0_{\kappa+1}$ and the very definition of ε_ξ, and by the fact that $q'_\xi \frown (\varepsilon_\xi, \underset{\sim}{A}'_{\mathbf{j}(\xi)})$ forces $\mathbf{j}(\varepsilon)_{\mathbf{j}(\xi)} \restriction \kappa = \varepsilon_\xi$.

Having the claim established we can choose $q'_{\xi+1}$ so that $q'_{\xi+1}(\mathbf{j}(\kappa))(\mathbf{j}(\xi))$ satisfies (3.3) (with $\zeta = \xi$), hence (e) as well.

Finally, for (d), first note that we can assume $\underset{\sim}{A} \in \mathscr{N}_\eta$, so there are at most χ-many such names. Now choosing an increasing sequence of conditions $\langle q''_\gamma : \gamma < \chi \rangle$ in $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$ with $q''_0 = q'_\eta$, we can decide for each name $\underset{\sim}{X}$ the statement $\kappa \in \mathbf{j}(\underset{\sim}{X})$. So using the $< \chi^+$-directed closedness of $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\eta)})/\mathbf{G}^0_{\kappa+1}$ in $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$ (Fact 3.8), we can choose q_η to be an upper bound of the sequence $\langle q''_\gamma : \gamma < \chi \rangle$, yielding (d) as desired.

Finally, q_χ is defined to be an upper bound of the q_η's ($\eta < \chi$).

$\square_{\text{Claim 3.11}}$

Fact 3.12. By the definition of $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi$, and the way q_χ was constructed, we have:

$(\bowtie)_{13}$ For each $\delta \in S^*$, if D_δ generates a κ-complete ultrafilter on V_κ, then

$$\Vdash_{\mathbb{P}^0_{\kappa+1}} \forall \underset{\sim}{A} \in \underset{\sim}{D}_\delta \ \exists \alpha < \kappa \text{ s.t. } (\underset{\sim}{\varepsilon}_\delta \restriction (\alpha, \kappa) \subseteq \underset{\sim}{A}),$$

$(\bowtie)_{14}$ moreover, (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$) by (e)

$$q_\chi \Vdash_{(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \underset{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}} \forall d \left(\kappa(d) = \kappa \ \wedge \ d \in \bigcap_{\underset{\sim}{A} \in \underset{\sim}{D}_\delta} \mathbf{j}(\underset{\sim}{A}) \right) \to \big(d \in \mathbf{j}(\underset{\sim}{\varepsilon}_\delta)\big)$$

(where $\kappa(d)$ is defined in Definition 2.15(ii)).

$(\bowtie)_{15}$ If $\delta \in S^*$, then ε_δ is a pseudointersection of D_δ.

$\mathbf{j}$ and q_χ define the normal ultrafilter

$(\bullet)_5$ $D^\bullet = \{\underset{\sim}{A}[\mathbf{G}^0_{\kappa+1}] : \Vdash_{\mathbb{P}^0_{\kappa+1}} \underset{\sim}{A} \subseteq \kappa, \ q_\chi \Vdash \text{“}\kappa \in \mathbf{j}(\underset{\sim}{A})\text{''}\} \subseteq \mathscr{P}(\kappa)$,

$(\bowtie)_{16}$ and if $D_\delta \subseteq D^\bullet$, then $\varepsilon_\delta \in D^\bullet$.

This together with (#) completes the proof of (C)(a).

Case 2. For 3.2(C)(b) we proceed as follows.

In $\mathbf{V}_1^{\mathbb{P}^1_\chi}$ we have to find a sequence $\bar{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of normal measures on κ increasing in the Mitchell order, such that each U_α satisfies our closedness properties, namely, whenever $\langle X_\nu : \nu < \lambda \rangle$ is a sequence in U_α, there exists $X' \in U_\alpha$, $|X' \setminus X_\nu| < \kappa$ for each $\nu < \lambda$. Let U_0 be the normal ultrafilter provided by appealing to (C)(a) which we have already proved.

Working in $\mathbf{V}_1[\mathbf{G}^1_\chi] = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$ we will construct the sequence by induction, so fixing $\alpha < \kappa$, we assume that U_β's are already defined for $\beta < \alpha$. So we

$(\bullet)_6$ let $\overline{\underset{\sim}{U}}$ be a $\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi = \mathbb{P}^0_{\kappa+1}$-name for $\langle U_\beta : \beta < \alpha \rangle \in \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$, where $1_{\mathbb{P}^0_{\kappa+1}}$ forces that $\overline{\underset{\sim}{U}} = \langle \underset{\sim}{U}_\beta : \beta < \alpha \rangle$ is an increasing sequence of κ-complete normal ultrafilters w.r.t. the Mitchell-order of length α, each $\underset{\sim}{U}_\beta$ is $<\lambda^+$-directed modulo $[\kappa]^{<\kappa}$.

And fix an elementary embedding $\mathbf{j}_*: \mathbf{V}_0 \to \mathbf{M}_*$ with critical point κ, ${}^\chi\mathbf{M}_* \subseteq \mathbf{M}_*$ with

$$\mathbf{j}_*(\mathbf{h})(\kappa) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \underset{\sim}{U} \rangle \tag{3.5}$$

(recall the definition of $\mathbf{h}$ $(\bullet)_1$, this is possible). We are going to define a normal ultrafilter U_α associated with $\mathbf{j}_*$, above the U_β's w.r.t. the Mitchell-order.

Defining $\underset{\sim}{\mathbb{P}}'_* = \mathbf{j}_*(\underset{\sim}{\mathbb{P}}^1)$, and letting $(\mathbb{P}^0_*)_{\mathbf{j}_*(\kappa)} = \mathbf{j}_*(\mathbb{P}^0_\kappa)$, observe that by the definition of $\mathbb{P}^0_\kappa$ (Definition 3.4)

$$\mathbf{j}_*(\mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi) = (\mathbb{P}^0_*)_{j_*(\kappa)} * (\underset{\sim}{\mathbb{P}}'_*)_{\mathbf{j}_*(\chi)},$$

and

$$(\mathbb{P}^0_*)_{\kappa+1} = \mathbb{P}^0_\kappa * \underset{\sim}{\mathbb{P}}^1_\chi.$$

Now our fixed $\mathbf{G}^0_{\kappa+1} \subseteq \mathbb{P}^0_{\kappa+1}$ is generic over $\mathbf{V}_0$ and also over $\mathbf{M}_*$.

With a slight abuse of notation (in the proof of Case 2 from now on, in order to avoid notational awkwardness) we will refer to $(\mathbb{P}^0_*)_{\mathbf{j}_*(\kappa)}$ as $\mathbb{P}^0_{\mathbf{j}_*(\kappa)}$, and to $(\underset{\sim}{\mathbb{P}}'_*)_{\mathbf{j}_*(\chi)}$ as $\mathbb{P}'_{\mathbf{j}_*(\chi)}$; moreover, observe that all the preceding facts and claims hold in this setting, we only used that $\mathbf{j}(\mathbf{h}(\kappa)) = \langle \underset{\sim}{\mathbb{P}}^1_\chi, \chi^+, \underset{\sim}{x} \rangle$ for some name $\underset{\sim}{x}$, which obviously holds for $\mathbf{j}_*$ as well (where $\underset{\sim}{x}$ is not arbitrary anymore). In this new setting we appeal to Claim 3.11, obtaining the condition $q^*_\chi \in \mathbb{P}^0_{\mathbf{j}_*(\kappa)+1}/\mathbf{G}^0_{\kappa+1}$, and the κ-complete normal ultrafilter

$$D^\bullet_* = \{\underset{\sim}{A}[\mathbf{G}^0_{\kappa+1}] :\ \mathbf{M}_*[\mathbf{G}^0_{\kappa+1}] \models \text{“} q^*_\chi \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1}} \kappa \in \mathbf{j}_*(\underset{\sim}{A})\text{”}\} \tag{3.6}$$

(which is a κ-complete normal ultrafilter over $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$, belonging to $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$) and $< \lambda^+$-directed w.r.t. $\supseteq^*$. We only need to prove Claim 3.13, implying that the filter $D^\bullet_*$ dominates $\{U_\beta : \beta < \alpha\}$ w.r.t. the Mitchell order:

Claim 3.13. For each $\beta < \alpha$ there exists a sequence $\langle W_\gamma :\ \gamma < \kappa \rangle \in \mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$, where

- for $D^\bullet_*$-many $\gamma < \kappa$ the set W_γ is an ultrafilter over γ,
- for each $X \in \mathscr{P}(\kappa) \cap \mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$

$$X \in U_\beta \iff \{\gamma < \kappa :\ (X \cap \gamma) \in W_\gamma\} \in D^\bullet_*.$$

Proof. Using (reinterpreting) (3.5)

$$\left\{ \begin{array}{ll} \gamma < \kappa : & \mathbf{h}(\gamma) = \langle \underset{\sim}{x}_\gamma, \mu_\gamma, \underset{\sim}{y}_\gamma \rangle,\ \text{where } \underset{\sim}{y}_\alpha \text{ is a } \mathbb{P}^0_{\gamma+1}\text{-name} \\ & \text{for a sequence of subsets of } \mathscr{P}(\gamma) \text{ of length } \alpha), \\ & \underset{\sim}{x}_\alpha = \underset{\sim}{\mathbb{Q}}^0_\alpha, \end{array} \right\} \in D^\bullet_* \cap \mathbf{V}_0.$$

Now suppose that $\beta < \alpha$ is fixed. Since $\underset{\sim}{x}_\gamma$ is name for a sequence of length α, we can easily get a name for its β'th coordinate. This way, we can fix $Y \in D^\bullet_* \cap \mathbf{V}_0$, and the sequence $\langle \underset{\sim}{W}_\gamma :\ \gamma < \kappa \rangle$ such that

($\blacktriangle_1$) for each $\gamma \in Y$, $\underset{\sim}{W}_\gamma$ is a $\mathbb{P}^0_{\gamma+1}$-name for a subset of $\mathscr{P}(\gamma)$ (the β'th coordinate of x_γ), and

($\blacktriangle_2$) $\mathbf{j}_*(\langle \underset{\sim}{W}_\gamma :\ \gamma < \kappa \rangle)(\kappa) = \underset{\sim}{U}_\beta$.

In what follows, we will prove that the natural candidate $W_\gamma = \underset{\sim}{W}_\gamma[\mathbf{G}^0_{\kappa+1}]$ $(\gamma < \kappa)$ works (utilizing standard arguments, so a reader familiar with this kind of proofs can jump to Case 3).

For a fixed $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underaccent{\sim}{X} \in \mathbf{V}_0$ (for a subset of κ) define the $\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi$-name $\underaccent{\sim}{Z}_X \in \mathbf{V}_0$ as follows.

$$1_{\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi} \Vdash \underaccent{\sim}{Z}_X = \{\gamma < \kappa : \ \underaccent{\sim}{X} \restriction \gamma \in \underaccent{\sim}{W}_\gamma\}. \tag{3.7}$$

We only have to verify that

$$\underaccent{\sim}{X}[\mathbf{G}^0_{\kappa+1}] \in \underaccent{\sim}{U}_\beta[\mathbf{G}^0_{\kappa+1}] \ \text{ iff } \ \underaccent{\sim}{Z}_X[\mathbf{G}^0_{\kappa+1}] \in D^\bullet_*. \tag{3.8}$$

But the latter is defined (by (3.6)) as

$$\begin{gathered}\underaccent{\sim}{Z}_X[\mathbf{G}^0_{\kappa+1}] \in D^\bullet_*,\\ \Updownarrow \\ (\text{in } \mathbf{M}_*[\mathbf{G}^0_{\kappa+1}]) \ q^*_\chi \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1}} \kappa \in \mathbf{j}_*(\underaccent{\sim}{Z}_X).\end{gathered}$$

Therefore, as $\mathbf{j}_*(\underaccent{\sim}{W})(\kappa) = \underaccent{\sim}{U}_\beta$ by (3.7), and

$$\mathbf{M}_*[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models (q^*_\chi \Vdash \kappa \in \mathbf{j}_*(\underaccent{\sim}{Z}_X) \iff q^*_\chi \Vdash \mathbf{j}_*(\underaccent{\sim}{X}) \restriction \kappa \in \underaccent{\sim}{U}_\beta)$$

(since $\mathbf{j}_*(\langle \underaccent{\sim}{W}_\gamma : \ \gamma < \kappa\rangle)(\kappa) = \underaccent{\sim}{U}_\beta$), we observe that in order to get (3.8) it suffices to show the following

$$\underaccent{\sim}{X}[\mathbf{G}^0_{\kappa+1}] \in \underaccent{\sim}{U}_\beta[\mathbf{G}^0_{\kappa+1}] \ \text{ iff } \ q^*_\chi \Vdash \mathbf{j}_*(\underaccent{\sim}{X}) \restriction \kappa \in \underaccent{\sim}{U}_\beta. \tag{3.9}$$

But then by the elementarity of $\mathbf{j}_*$ (and $\text{crit}(\mathbf{j}_*) = \kappa$)

$$\forall \alpha < \kappa, \forall p \in \mathbb{P}^0_\kappa * \underaccent{\sim}{\mathbb{P}}^1_\chi : \quad p \Vdash_{\mathbb{P}^0_\kappa * \underaccent{\sim}{\mathbb{P}}^1_\chi} \check{\alpha} \in \underaccent{\sim}{X} \iff \mathbf{j}_*(p) \Vdash_{\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \underaccent{\sim}{\mathbb{P}}'_{\mathbf{j}_*(\chi)}} \check{\alpha} \in \mathbf{j}_*(\underaccent{\sim}{X}),$$

and recalling $p \in \mathbf{G}^0_{\kappa+1}$ implies $q^*_\chi \geq \mathbf{j}_*(p)$ in the quotient forcing $\mathbb{P}^0_{\mathbf{j}_*(\kappa)} * \mathbb{P}'_{\mathbf{j}_*(\chi)}/\mathbf{G}^0_{\kappa+1})$ we get that

$(*)_1$ (in $\mathbf{M}_*[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]$) the condition q^*_χ forces $\mathbf{j}_*(\underaccent{\sim}{X}) \restriction \kappa$ to be equal to $\underaccent{\sim}{X}[\mathbf{G}^0_{\kappa+1}]$.

This yields (3.9), completing the proof of Claim 3.13 and Case 2. $\square_{\text{Claim 3.13}}$

Case 3. For 3.2(C)(c). We fix $\Upsilon > \kappa$, and η, and we would like to define the κ-complete fine normal ultrafilter W on $[\Upsilon]^{<\kappa}$ that satisfies $(\star)$ from (c). First we redefine the elementary embedding $\mathbf{j}$ from Definition 3.5 (as well as $\mathbb{P}^0_{\mathbf{j}(\kappa)}$, $\underaccent{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)}$):

Definition 3.14.

- $(\bullet)_1$ Let $\rho = |2^{(\Upsilon \cdot \chi)^\kappa} + \eta|$, and
- $(\bullet)_2$ define $\mathbf{j} : \mathbf{V}_0 \to \mathbf{M}$ to be an elementary embedding with critical point κ such that $(\mathbf{j}(\mathbf{h}))(\kappa) = \langle \underaccent{\sim}{\mathbb{P}}^1_\chi, \rho^+, \check{\emptyset}\rangle$ ($\check{\emptyset} = \emptyset$ is the canonical name for the empty set) and $\mathbf{j}(\kappa) > \rho$, ${}^\rho\mathbf{M} \subseteq \mathbf{M}$,
- $(\bullet)_3$ Let $\langle \mathbb{P}^0_\alpha, \underaccent{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \mathbf{j}(\kappa), \beta < \mathbf{j}(\kappa)\rangle = \mathbf{j}(\langle \mathbb{P}^0_\alpha, \underaccent{\sim}{\mathbb{Q}}^0_\beta : \alpha \leq \kappa, \beta < \kappa\rangle)$ so $\underaccent{\sim}{\mathbb{Q}}^0_\kappa = \underaccent{\sim}{\mathbb{P}}^1_\chi$, and let $\underaccent{\sim}{\mathbb{P}}'_{\mathbf{j}(\chi)} = \mathbf{j}(\underaccent{\sim}{\mathbb{P}}^1_\chi)$.

Similarly as in Facts 3.7, 3.8, 3.9 we can get the following.

Fact 3.15. The filter $\mathbf{G}^0_{\kappa+1}$ is generic over $\mathbf{M}$ as well, and the forcing notions $\mathbb{P}^0_{j(\kappa)}/\mathbf{G}^0_{\kappa+1}$ and $(\mathbb{P}^0_{j(\kappa)} * \underaccent{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ ($\gamma \leq \mathbf{j}(\chi)$) are well-defined and $< |2^\Upsilon + \eta|^+$-directed closed in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.

Fact 3.16. $\mathbf{V}_1 \models$ "$(\mathbb{P}^0_{j(\kappa)} * \underaccent{\sim}{\mathbb{P}}'_\gamma)/\mathbf{G}^0_{\kappa+1}$ is $< |2^\Upsilon + \eta|^+$-directed closed."

Fact 3.8 follows from Fact 3.17.

Fact 3.17. $\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models {}^{2^\Upsilon + \eta}\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \subseteq \mathbf{M}[\mathbf{G}^0_{\kappa+1}]$.

Using this new $\mathbf{j}$, we will extract the ultrafilter $W \subseteq \mathscr{P}([\Upsilon]^{<\kappa})$ (in the sense of $\mathbf{V}_0[\mathbf{G}^0_{\kappa+1}]$), and the sequence of ultrafilters $\overline{U}$ as well from the information provided by $\mathbf{G}^0_{\kappa+1} = \mathbf{G}^0_\kappa * \mathbf{G}^1_\chi$, and $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$ (given by Claim 3.11), and then we will prove that it is indeed a measure sequence corresponding to the elementary embedding $\mathbf{j}_W$. Obviously,

($\circledcirc_1$) $\mathbf{j}(\kappa) > \chi$, ${}^\chi M \subseteq M$.

Observe that Claim 3.11 is true in this setting as well, and let the master condition $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\chi)})/\mathbf{G}^0_{\kappa+1}$ be given by it. First we claim that by possibly extending q_χ, we can assume that

($\circledcirc_2$) For each $A \in \mathscr{P}([\Upsilon]^{<\kappa}) \cap \mathbf{V}_2$ the condition $q_\chi \in (\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)})/\mathbf{G}^0_{\kappa+1}$ decides about (the truth value of) "($\mathbf{j}$"$\Upsilon \in \mathbf{j}(A)$)" (in $\mathbf{M}[\mathbf{G}^0_{\kappa+1}]$).

To this end we first count the possible A's. Recall that $\mathbb{P}^1_\chi$ is $<\kappa$-closed (B)/(b)

$$[\chi]^{<\kappa} \cap \mathbf{V}_2 = [\chi]^{<\kappa} \cap \mathbf{V}_1 = [\chi]^{<\kappa} \cap \mathbf{V}_0[\mathbf{G}^0_\kappa],$$

and as $|\mathbb{P}^0_\kappa| = \kappa$,

$$|[\Upsilon]^{<\kappa} \cap \mathbf{V}_2| \leq (\Upsilon \cdot \chi)^\kappa. \tag{3.10}$$

Second, as $|\mathbb{P}^0_\kappa * \mathbb{P}^1_\chi| = \chi$, we have

$$\mathbf{V}_2 = \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi] \models \mathscr{P}([\chi]^{<\kappa})| \leq (2^{(\chi\cdot\Upsilon)^\kappa})^{\mathbf{V}_0} \leq \rho. \tag{3.11}$$

Now using Fact 3.8 we can extend q_χ to another condition q_* in (at most) ρ-many steps (in $(\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{j(\zeta)})/\mathbf{G}^0_{\kappa+1}$) so that

($\circledcirc_3$) for each name $\underaccent{\sim}{A}$ for a subset of $[\chi]^{<\kappa}$

$$\mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \ q_* \parallel \ \mathbf{j}\text{``}\Upsilon \in \underaccent{\sim}{A},$$

and so (by possibly replacing q_χ by q_*) ($\circledcirc_2$) holds, indeed. Now we can define the κ-complete, fine, normal ultrafilter

$$W = \{\underaccent{\sim}{A}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\kappa] \in [\Upsilon]^{<\kappa} : \ q_\chi \Vdash \mathbf{j}\text{``}\Upsilon \in \mathbf{j}(\underaccent{\sim}{A})\} \in \mathbf{V}_2. \tag{3.12}$$

Now let $\mathbf{j}_W : \mathbf{V}_2 \to \mathbf{M}_W = \mathrm{Mos}({}^{[\Upsilon]^{<\kappa}}\mathbf{V}_2/W)$ be the corresponding elementary embedding, and let $\overline{U} = \langle U_\alpha : \ \alpha < \mathrm{dom}(\overline{U})\rangle$ be the ultrafilter sequence of maximal length associated to $\mathbf{j}_W$, that is, the following hold in $\mathbf{V}_2$.

($\boxminus_1$) $U_0 = \kappa$, and for each $\alpha \in \mathrm{dom}(\overline{U})$, $\alpha > 0$ the set $U_\alpha \subseteq \mathscr{P}(V_\kappa)$ is a κ-complete normal ultrafilter satisfying

$$\forall A \subseteq V_\kappa : \ A \in U_\alpha \iff U \restriction \alpha \in \mathbf{j}_W(A)$$

(therefore for each $\alpha < \mathrm{dom}(\overline{U})$, $U \restriction \alpha \in \mathbf{M}_W$),

($\boxminus_2$) $\overline{U} \notin \mathbf{M}_W$.

Claims 3.18 and 3.19 complete the proof of 3.2(C)(c) as we study the ultrafilter sequence $\overline{U}\restriction(\min(\mathrm{dom}(\overline{U}), \eta)$.

Claim 3.18. For every ultrafilter sequence $\overline{F} \in \mathbf{M}_W$ with $\kappa(\overline{F}) = \kappa$ there exists a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-name $\underaccent{\sim}{\overline{F}}' \in \mathbf{M}$ for an ultrafilter sequence with $\kappa(\overline{F}') = \kappa$ such that for each name $\underaccent{\sim}{A}$ for a subset of $V_\kappa^{\mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]}$ we have

$$\overline{F} \in \mathbf{j}_W(\underaccent{\sim}{A}[\mathbf{G}^0_\kappa * \mathbf{G}^1_\chi]) \iff \mathbf{M}[\mathbf{G}^0_{\kappa+1}] \models \ q_* \Vdash \underaccent{\sim}{\overline{F}}' \in \mathbf{j}(\underaccent{\sim}{A}).$$

Claim 3.19. Suppose that $\sigma \leq \min(\mathrm{dom}(\overline{U}, \eta))$, and assume $\{\overline{\underset{\sim}{F}_i} : i < \sigma\} \subseteq \mathbf{M}$ is a set of ($\mathbb{P}_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-names for) ultrafilter sequences with $\kappa(\underset{\sim}{F}_i) = \kappa$ $(i < \sigma)$.

If the filter

$$F_* = \bigcap_{i<\sigma} \{A \subseteq V_\kappa^{\mathbf{V}_2} :\ q_\chi \Vdash \overline{\underset{\sim}{F}_i} \in \mathbf{j}(A)\}$$

satisfies $(\forall \alpha < \kappa) : |\cup F_* \restriction \alpha| < \kappa$, then F_* is $< \lambda^+$-directed in the sense that for any system $\langle X_\alpha : \alpha < \lambda \rangle$ in F_* there is a set $X' \in F_*$ s.t. for each $\alpha < \lambda$ there exists $\delta < \kappa$ with $X' \restriction [\delta, \kappa) \subseteq X_\alpha$.

Proof. (Claim 3.18) Instead of factoring through our elementary embeddings (after forcing) we provide a direct calculation. Fix the ultrafilter sequence $\overline{F} \in \mathbf{M}_W$, and pick a function $f \in \mathbf{V}_2$, $\mathrm{dom}(f) = [\Upsilon]^{<\kappa}$, $\mathbf{j}_W(f)(\mathbf{j}_W``\Upsilon) = \overline{F}$, where we can assume that

$$\forall x \in \mathrm{dom}(f)\ f(x) \text{ is an u.f. sequence with } \kappa(f(x)) = \mathrm{otp}(\kappa \cap x). \tag{3.13}$$

Now we can fix a $\mathbb{P}^0_\kappa * \mathbb{P}^1_{\underset{\sim}{\chi}}$-name $\underset{\sim}{f} \in \mathbf{V}_0$ of f, such that $1_{\mathbb{P}^0_\kappa * \mathbb{P}^1_{\underset{\sim}{\chi}}}$ forces (3.13). Now as $\underset{\sim}{f} \in \mathbf{V}_0$ is a $\mathbb{P}^0_\kappa * \mathbb{P}^1_{\underset{\sim}{\chi}}$-name for a function with $\mathrm{dom}(f) = [\Upsilon]^{<\kappa}$, by elementarity $\mathbf{j}(\underset{\sim}{f})$ is a $\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}$-name for a function with domain $[\mathbf{j}(\Upsilon)]^{<\mathbf{j}(\kappa)}$. Now, as $\mathbf{j}``\Upsilon \in \mathbf{M} \cap [\mathbf{j}(\Upsilon)]^{\leq \rho}$, there is a name $\underset{\sim}{F}' \in \mathbf{M}$ such that

$$\mathbf{M} \models\ \Vdash_{\mathbb{P}^1_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}(\chi)}} \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) = \underset{\sim}{F}'. \tag{3.14}$$

It only remains to check that for each $X \subseteq V_\kappa^{\mathbf{V}_2}$ the conditions "$F \in \mathbf{j}_W(X)$" and "$q_* \Vdash \underset{\sim}{F}' \in \mathbf{j}(X)$" are equivalent. More precisely, we prove the following.

($\circ$) For every fixed $\mathbb{P}^0_\kappa * \mathbb{P}^1_{\underset{\sim}{\chi}}$-name $\underset{\sim}{X}$ for a subset of $V_\kappa^{\mathbf{V}_2}$

$$F \in \mathbf{j}_W(\underset{\sim}{X}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\underset{\sim}{\chi}}]) \iff q_* \Vdash \underset{\sim}{F}' \in \mathbf{j}(\underset{\sim}{X}).$$

As $F = \mathbf{j}_W(f)(\mathbf{j}_W``\Upsilon)$ we can reformulate the lhs. of the statement as

$$\mathbf{V}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\underset{\sim}{\chi}}] \models\ \{y \in [\Upsilon]^{<\kappa} :\ f(y) \in X\} \in W,$$

i.e. for some $p \in \mathbf{V}_0[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\underset{\sim}{\chi}}]$

$$p \Vdash_{\mathbb{P}^0_\kappa * \mathbb{P}^1_{\underset{\sim}{\chi}}} \{y \in [\Upsilon]^{<\kappa} :\ \underset{\sim}{f}(y) \in \underset{\sim}{X}\} \in \underset{\sim}{W}.$$

Now for the $\mathbb{P}^0_\kappa * \mathbb{P}^1_{\underset{\sim}{\chi}}$-name $\underset{\sim}{C} := \{y \in [\Upsilon]^{<\kappa} :\ \underset{\sim}{f}(y) \in \underset{\sim}{X}\}$ we have (by ($\circledcirc_2$) and (3.12))

$$\underset{\sim}{C}[\mathbf{G}^0_\kappa * \mathbf{G}^1_{\underset{\sim}{\chi}}] \in W \iff q_* \Vdash \mathbf{j}``\Upsilon \in \mathbf{j}(\underset{\sim}{C}). \tag{3.15}$$

(Recall that q_* decides this by ($\circledcirc_3$) as $\underset{\sim}{C}$ is a name for a subset of $[\Upsilon]^{<\kappa}$.) Now

$$\begin{aligned}(\Vdash_{\mathbb{P}^0_{\mathbf{j}(\kappa)} * \mathbb{P}'_{\mathbf{j}\chi}})\ \mathbf{j}(\underset{\sim}{C}) &= \mathbf{j}(\{y \in [\Upsilon]^{<\kappa} :\ \underset{\sim}{f}(y) \in \underset{\sim}{X}\}) \\ &= (\{y \in [\mathbf{j}(\Upsilon)]^{<\mathbf{j}(\kappa)} :\ \mathbf{j}(\underset{\sim}{f})(y) \in \mathbf{j}(\underset{\sim}{X})\}),\end{aligned}$$

so the rhs. of (3.15) is equivalent to

$$q_* \Vdash \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) \in \mathbf{j}(\underset{\sim}{X}), \tag{3.16}$$

so recalling the definition of $\underset{\sim}{F}'$, $\Vdash \mathbf{j}(\underset{\sim}{f})(\mathbf{j}``\Upsilon) = \underset{\sim}{F}'$ by (3.14) (3.16) is clearly equivalent to $q_* \Vdash \underset{\sim}{F}' \in \mathbf{j}(\underset{\sim}{X})$, therefore ($\circ$) holds, as desired. $\square_{\text{Claim 3.18}}$

Proof. (Claim 3.19) Fix $\langle \underset{\sim}{F}'_i : i < \sigma \rangle$ as in Claim 3.18.

We only have to recall how we constructed q_χ, which ensures the existence of the desired pseudointersection. Fix a sequence $\langle X_\alpha : \alpha < \lambda \rangle$ in the filter F_*. Now let $D' = \{X_\alpha : \alpha < \lambda\}$, which is equal to D_ζ for some $\zeta < \chi$ by (#) from our assumptions (B)/(d). Now by our assumptions

$$(\forall i < \sigma)\ (\forall X \in D_\zeta)\ q_\chi \Vdash \underset{\sim}{F}_i \in \mathbf{j}(X),$$

so since $A'_{\mathbf{j}(\zeta)}$ is the name for the intersection of the $\mathbf{j}(\underset{\sim}{A})$'s, where $\underset{\sim}{A}$ runs over the $< \kappa$-complete filter generated by D_ζ (Definition 3.10) $A'_{\mathbf{j}(\zeta)}$

$$(\forall i < \sigma)\ q_\chi \Vdash \underset{\sim}{F}_i \in A'_{\mathbf{j}(\zeta)}.$$

Finally, recalling Definition 3.10 and (3.3) from Claim 3.11 we get that for the generic sequence ε_ζ (which is a pseudointersection of the $D' = D_\zeta$)

$$q_{\zeta+1} \Vdash \mathbf{j}(\varepsilon_\zeta) \restriction (\kappa+1) = \varepsilon_\zeta \cup (A'_{\mathbf{j}(\zeta)} \restriction [\kappa, \kappa+1)),$$

which means

$$(\forall i < \sigma)\ \ q_\chi \Vdash \overline{\underset{\sim}{F}_i} \in \mathbf{j}(\varepsilon_\zeta),$$

and we are done.

$\square_{\text{Claim 3.19}}$

$\square_{\text{Claim 3.2}}$

3(B). **The preliminary forcing for obtaining $(\kappa, \lambda) - 1$ systems together with a universal in $(K_\kappa)_\lambda$.** This subsection deals with the application of Claim 3.2, we show that it is possible to force a universal object in $(K_\kappa)_\lambda$ with a notion of forcing satisfying requirements from Claim 3.2.

Conclusion 3.20. Assume

- κ is supercompact,
- $\kappa < \lambda < \chi = \chi^\lambda$,
- λ is regular,
- $(\forall \theta)(\theta \in \mathrm{Card} \wedge \kappa \leq \theta < \lambda \Rightarrow 2^\theta = \theta^+)$, and
- $\sigma = \mathrm{cf}(\sigma) < \kappa$.

Then for some forcing extension $\mathbf{V}^{\mathbb{P}}$ preserving cardinals $\geq \kappa$ and cofinalities $> \kappa$ and $\leq \sigma$, we have that in $\mathbf{V}^{\mathbb{P}}$:

(1) $2^\kappa = \chi$,
(2) κ is a strong limit singular of cofinality σ,
(3) and there is a universal graph in cardinality λ.

Proof. We shall use 1.2, but we have to justify it. That is, we need a forcing fitting in the scheme in Claim 3.2 with $\mathbf{V}_0 = \mathbf{V}$, specifying the $(< \kappa)$-directed-complete iteration $\mathbb{P}^1_\chi = \langle \mathbb{P}^1_\alpha, \mathbb{Q}^1_\beta :\ \alpha \leq \chi, \beta < \chi \rangle \in \mathbf{V}_1 = \mathbf{V}^{\mathbb{P}^0_\kappa}$ in which we are free to choose $\underset{\sim}{\mathbb{Q}}_\beta$'s on β's outside $S^* \subseteq \chi$. (And then conclusion (C)/(a) or (b) with Claim 1.5 together with Claim 2.1 or 2.2 will give the desired consistency result.) Our task is to construct (in $\mathbf{V}_1$) a suitable iteration $\mathbb{P}^1_\chi$, and to check that $\mathbb{P}^1_\chi$

$(\mathsf{T})_1$ is $<\kappa$-directed closed,
$(\mathsf{T})_2$ is of cardinality χ (up to equivalence),
$(\mathsf{T})_3$ has the λ^+-c.c.,
$(\mathsf{T})_4$ does not collapse any cardinals, and
$(\mathsf{T})_5$ $\mathbf{V}_1 \models\ \Vdash_{\mathbb{P}^1_\chi}$ "there is a universal graph in $(K_\kappa)_\lambda$",

$(\mathsf{T})_6$ and we can choose $S^* \in [\chi \setminus \{0,1\}]^\chi$, $S^* \in \mathbf{V}_1$, $|\chi \setminus S^*| = \chi$, and the $\mathbb{P}^1_\delta$-names $\underset{\sim}{D}_\delta$ $(\delta \in S^*)$ satisfying (B)(d) from Claim 3.2.

We will do the same as in [She90], we define (in $\mathbf{V}_1$)

(1) $\mathbb{Q}^1_0$ to be the forcing of χ-many stationary sets of λ, any two intersecting in a set of size smaller than κ,
(2) $\underset{\sim}{\mathbb{Q}}^1_\beta$ for $\beta \in \chi \setminus (S^* \cup \{0\})$ the main iteration from [She90] just with κ-many colors (i.e. in the class K_κ instead of simple graphs, which is just equivalent to K_2): forcing a generic random graph, and the embeddings into it with $< \kappa$-support partial functions.

We need to check that the iteration is indeed λ^+-cc, which will be ensured by showing that (in $\mathbf{V}_1$) $\mathbb{Q}^1_0$ is λ^+-cc, and in $(\mathbf{V}_1)^{\mathbb{Q}^1_0}$ the iteration of $\mathbb{Q}^1_\alpha$'s $(0 < \alpha < \chi)$, i.e. $\mathbb{P}^1_\chi / \mathbf{G}^1_1$ has the κ^+-cc .

First for future reference we have to remark that by the construction of $\mathbb{P}^0_\kappa$

$(*)_1$ in $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{P}^0_\kappa}$ κ is still strongly inaccessible (as we noted in $(\bowtie)_3$). As $|\mathbb{P}^0_\kappa| = \kappa$ our cardinal arithmetic assumptions above κ are also preserved.

Working in $\mathbf{V}_1$, Lemma 3.21 concerns the first step $\mathbb{Q}^1_0$ which we can define to be $Q(\lambda, \chi, \kappa)$ as in [Bau76, Sec. 6.], see (b) in Definition 3.22.

Lemma 3.21. *In $\mathbf{V}_1$ there exists a forcing poset $\mathbb{Q}^1_0$ that is $<\kappa$-directed closed, of power χ, having the λ^+-cc, preserving cardinals from $(\kappa, \lambda]$, and*

$$\mathbf{V}_1^{\mathbb{Q}^1_0} \models \exists \{S_\alpha : \alpha < \chi\} \subseteq \mathscr{P}(\lambda),$$
$$\text{a system of stationary sets s.t. } \forall \alpha < \beta < \chi : |S_\alpha \cap S_\beta| < \kappa.$$

Proof.

Definition 3.22. First we define the following auxiliary posets.

(a) For a regular cardinal μ we let $Q'(\lambda, \chi, \mu)$ be the set of functions f satisfying
  (i) $\mathrm{dom}(f) \in [\chi]^{<\mu}$,
  (ii) for each $\alpha \in \mathrm{dom}(f)$ $f(\alpha) \in [\lambda]^{<\mu}$,
  with $f \leq g$, iff
  (iii) $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$,
  (iv) $\forall \alpha \in \mathrm{dom}(f)$: $f(\alpha) \subseteq g(\alpha)$,
  (v) for each $\alpha \neq \beta \in \mathrm{dom}(f)$: $f(\alpha) \cap f(\beta) = g(\alpha) \cap g(\beta)$.
(b) Let $Q(\lambda, \chi, \kappa) \subseteq \prod_{\mu \in \mathrm{Reg} \cap [\kappa, \lambda]} Q'(\lambda, \chi, \mu)$ be the collection of the following functions f
  (i) $(\forall \mu < \nu \in \mathrm{Reg} \cap [\kappa, \lambda])$, $(\forall \alpha \in \mathrm{dom}(f_\mu))$: $f_\mu(\alpha) \subseteq f_\nu(\alpha)$
  with the pointwise ordering inherited from the full product

$$\prod_{\mu \in \mathrm{Reg} \cap [\kappa, \lambda]} Q'(\lambda, \chi, \mu).$$

Definition 3.23. We let $\mathbb{Q}^1_0 = Q(\lambda, \chi, \kappa) \in \mathbf{V}_1$.

For later reference we note the following. Recall that $\chi^\lambda = \chi$ holds by our assumptions.

Observation 3.24. For each $\mu \in \mathrm{Reg} \cap [\kappa, \lambda]$ $|Q'(\lambda, \chi, \mu)| \leq \chi^{<\mu} \cdot \lambda^{<\mu} = \chi$. Therefore $|\mathbb{Q}^1_0| = \chi$.

By [Bau76, Lemma 6.3], recalling $(\sigma \in \mathrm{Card} \cap [\kappa, \lambda)) \to (2^\sigma = \sigma^+)$ by our premises, so $\lambda^{<\lambda} = \lambda$ we have the following.

Claim 3.25. $Q(\lambda, \chi, \kappa)$ is λ^+-cc, $<\kappa$-directed closed, preserving cofinalities and cardinals.

Clearly

$(\ddagger)_1$ every directed subset of power less than κ in $\mathbb{Q}^1_0 = Q(\lambda, \chi, \kappa)$ has a least upper bound.

Now obviously, in $\mathbf{V}_1^{\mathbb{Q}^1_0}$

$(\ddagger)_2$ the generic subsets S_α $(\alpha < \chi)$ defined by $\Vdash_{\mathbb{Q}^1_0} \underset{\sim}{S}_\alpha = \cup\{f_\kappa(\alpha) : f \in \underset{\sim}{\mathbf{G}}\}$ form a κ-almost disjoint system, i.e. if $\alpha < \beta$, then $\Vdash |\underset{\sim}{S}_\alpha \cap \underset{\sim}{S}_\beta| < \kappa$,

we only need to verify that

$(\ddagger)_3$ for each $\alpha < \chi$ the subset

$$S_\alpha \ \text{ is a stationary subset of } \lambda,$$

which is a standard argument, but for the sake of completeness we elaborate. (In fact, recalling [Bau76, Lemmas 6.3-6.5.] with the aid of the following it is easy to argue that $(S_\alpha \cap E^\lambda_{\geq\kappa})$ i.e. restricting S_α to points of cofinality at least κ is stationary.)

Claim 3.26 ([Bau76, Lemmas 6.3-6.5.]). The notion of forcing $Q(\lambda, \chi, \kappa)$ is equivalent to the two-step iteration $Q(\lambda, \chi, \kappa^+) * \underset{\sim}{q}'(\lambda, \chi, \kappa, \underset{\sim}{F})$ where

$$\mathbf{V}_1^{Q(\lambda,\chi,\kappa^+)} \models \begin{array}{l} \bullet\ F_\alpha\ (\alpha \in \chi) \text{ is the generic sequence in } [\lambda]^\lambda (\text{given by } Q(\lambda, \chi, \kappa^+)), \\ \bullet\ Q'(\lambda, \chi, \kappa, F) \subseteq Q'(\lambda, \chi, \kappa) \text{ defined by} \\ \quad [f \in Q'(\lambda, \chi, \kappa, \underset{\sim}{F}) \iff \forall \alpha \in \mathrm{dom}(f)\ f(\alpha) \subseteq F_\alpha]. \end{array}$$

Moreover, $Q(\lambda, \chi, \kappa^+)$ is $<\kappa^+$-closed, (in $\mathbf{V}_1^{Q(\lambda,\chi,\kappa^+)}$), and $Q'(\lambda, \chi, \kappa, F)$ has the $\kappa^+ - cc$).

Looking at the definition of the forcing $Q(\lambda, \chi, \kappa)$, if we are given a condition p, and a $Q(\lambda, \chi, \kappa)$-name $\underset{\sim}{C}_*$ for a club set in λ, first recall that $Q(\lambda, \chi, \kappa)$ is $<\kappa$-closed (Claim 3.26), in particular $< \omega_1$-closed, as κ is strongly inaccessible. We can define an increasing sequence p^j $(j < \omega)$ in $Q(\lambda, \chi, \kappa)$ with $p^0 = p$, and an increasing sequence of ordinals ϱ_j $(j < \kappa)$ satisfying $p^j \Vdash \varrho_j \in \underset{\sim}{C}_*$, and if $j < k$, then $\sup \cup\{p^j_\lambda(\beta) : \beta \in \mathrm{dom}(p^j_\lambda)\} < \varrho_k$. This is possible, as $|\mathrm{dom}(p_j)| < \lambda$, as well as $|p^j_\lambda(\beta)| < \lambda$, and λ is regular. Then clearly any upper bound of the p^j's forces $\varrho_\omega := \sup\{\varrho_j : j < \omega\} \in \underset{\sim}{C}_*$, but as the least upper bound does not say anything about the statements $\varrho_\omega \in \underset{\sim}{S}_\beta$ $(\beta < \chi)$ we can extend it to a condition p' with $\varrho_\omega \in p'_\mu(\alpha)$ for each $\mu \in \mathrm{Reg} \cap [\kappa, \lambda]$ (thus $p' \Vdash \varrho_\omega \in \underset{\sim}{S}_\alpha \cap \underset{\sim}{C}_*$). This completes the proof of Lemma 3.21. $\square_{\text{Lemma 3.21}}$

As $\mathbb{Q}^1_0$ as already defined in Definition 3.23 we can define the iteration $\langle \mathbb{P}^1_\alpha, \underset{\sim}{\mathbb{Q}}^1_\beta : \alpha \leq \chi, \beta < \chi \rangle$ for which we have to choose a suitable S^*.

Definition 3.27. We let $0, 1 \notin S^* \subseteq \chi$ be such that $|S^*| = \chi$, $|\chi \setminus S^*| = \chi$.

Definition 3.28. We let $\langle \mathbb{P}^1_\alpha,\ \mathbb{Q}^1_\beta :\ \alpha \le \chi, \beta < \chi\rangle$ be the following $<\kappa$-support iteration. The definition of the $\mathbb{P}^1_\beta$-name $\underset{\sim}{\mathbb{Q}}^1_\beta$ goes by induction on β as follows, distinguishing three cases. But first

- ⊛ we have to remark that in steps with $\beta \in S^*$ we will only assume that $\underset{\sim}{D}_\beta$ is a $\mathbb{P}^1_\beta$-name for a system of subsets if $V_\kappa^{\mathbf{V}_1}$, where
$$\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{D}_\beta \in [\mathscr{P}(V_\kappa^{\mathbf{V}_1})]^{\le\lambda},$$
first we will deduce some properties of $\mathbb{P}^1_\chi$ based on only this weak assumption up until the end of the proof of Lemmas 3.35 and 3.34 and then we will verify that the $\underset{\sim}{D}_\beta$'s ($\beta \in S^*$) can be suitably chosen (during the inductive process of defining the iteration $\mathbb{P}^1_\chi$) so that the iteration fulfills all our remaining demands from $(\mathsf{T})_1$–$(\mathsf{T})_6$. This will be a standard bookkeeping argument.
- ⊛ Similarly, for steps in $\chi \setminus S^* \setminus \{0,1\}$ up until the end of the proof of Lemmas 3.35 and 3.34 we only assume that $\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{M}_\beta \in (K_\kappa)_\lambda$, i.e. is a $\mathbb{P}^1_\beta$-name for a κ-colored graph on λ.
- For every $M = \langle |M|, R^M_\alpha :\ \alpha < \kappa\rangle \in (K_\kappa)_\lambda$ we will use the notation $c_M : [\lambda]^2 \to \kappa$ denoting the color of the edge between i and j, i.e.
$$c_M(i,j) = \alpha \iff (i,j) \in R^M_\alpha.$$

Case 1 ($\beta = 1$). Let $\mathbb{Q}^1_1 \in \mathbf{V}_1^{\mathbb{Q}^1_0}$ be the forcing for obtaining a random κ-colored graph on λ with conditions of power $<\kappa$, i.e. $q \in \mathbb{Q}^1_1$ iff

(i) $q \subseteq \{[i\ R_\gamma\ j] :\ i \neq j < \lambda,\ \gamma < \kappa\}$,
(ii) $\forall i \neq j < \lambda$ we have
$$([i\ R_\gamma\ j], [i\ R_{\gamma'}\ j] \in q) \longrightarrow (\gamma = \gamma'),$$
(iii) $|q| < \kappa$,

with the usual ordering. Then

$(\diamond)_1$ the generic object $\underset{\sim}{M}_* = \langle \lambda, \underset{\sim}{R}^{M_*}_\alpha :\ \alpha < \kappa\rangle$ satisfies
$$\Vdash_{\mathbb{P}^1_2} \langle \underset{\sim}{R}^{M_*}_\alpha :\ \alpha < \kappa\rangle \text{ is a partition of } [\lambda]^2.$$

Case 2 ($\beta \in \chi \setminus S^* \setminus \{0,1\}$). In order to define $\mathbb{Q}^1_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ (formally a $\mathbb{P}^1_\beta$-name $\underset{\sim}{\mathbb{Q}}^1_\beta \in \mathbf{V}_1$) we first need to work in $\mathbf{V}_1' = \mathbf{V}_1^{\mathbb{P}^1_1} (= \mathbf{V}_1^{\mathbb{Q}^1_0})$ as preparation. Let Υ be a large enough regular cardinal, and define the continuous increasing chain $\overline{N}_\beta = \langle N_{\beta,\gamma} :\ \gamma < \lambda\rangle \in \mathbf{V}_1'$ so that

- β, $\mathbb{P}^1_\beta$, $\langle \overline{N}_\gamma :\ \gamma \in \beta \setminus S^* \setminus \{0,1\}\rangle$, $\mathbf{G}^1_1 \in N_{\beta,0}$ (the generic filter over $\mathbb{P}^1_1$),
- $\kappa + 1 \subseteq N_{\beta,0}$,
- for each $\gamma < \lambda$:
  $(\bullet)_a$ $N_{\beta,\gamma} \prec \left(\mathscr{H}(\Upsilon)^{\mathbf{V}_1'}, \in\right)$,
  $(\bullet)_b$ $|N_{\beta,\gamma}| < \lambda$,
  $(\bullet)_c$ $N_{\beta,\gamma} \cap \lambda$ is an initial segment of λ
  $(\bullet)_d$ $N_{\beta,\gamma} \cap \lambda < N_{\beta,\gamma+1} \cap \lambda$,
  $(\bullet)_e$ for $\varepsilon < \lambda$ limit $N_{\beta,\varepsilon} = \bigcup_{\gamma<\varepsilon} N_{\beta,\gamma}$,

and

$(\diamond)_2$ let $\xi_\beta(\gamma) = N_{\beta,\gamma} \cap \lambda$ ($\gamma < \lambda$).

So the set $\{\xi_\beta(\gamma): \gamma < \lambda\}$ is a club subset of λ, and as S_β is stationary (Lemma 3.21) the set $C_\beta = \mathrm{cl}(S_\beta \cap \{\xi_\beta(\gamma): \gamma < \lambda\})$ (i.e. the smallest closed set containing $S_\beta \cap \{\xi_\beta(\gamma): \gamma < \lambda\}$) is a club. Therefore the system $\langle N_{\beta,\gamma}: \gamma < \lambda \wedge \xi_\beta(\gamma) \in C_\beta\rangle$ (after reparametrizing) clearly satisfies $(\bullet)_a$–$(\bullet)_e$, hence we can assume that

$(\diamond)_3$ $\{\xi_\beta(\gamma+1): \gamma \in \lambda\} \subseteq S_\beta$,

and we let

$(\diamond)_4$ $N^*_\beta = \{\xi_\beta(\gamma): \gamma \in \lambda\}$.

For later reference we remark that the κ-almost disjointness of the S_α's and $(\diamond)_3$ together implies

$(\diamond)_5$ if $\beta \neq \alpha < \chi$ then $|\{\xi_\beta(\delta+1): \delta \in \lambda\} \cap \{\xi_\alpha(\delta+1): \delta \in \lambda\}| < \kappa$.

Now the forcing $\mathbb{Q}^1_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ is defined so that it shall give an embedding f_β of the κ-colored graph $M_\beta \in \mathbf{V}_1^{\mathbb{P}^1_\beta}$ into M_*, formally defined by

$(\diamond)_6$ $q \in \mathbb{Q}^1_\beta$, iff

(i) q is a set of elementary conditions of the following form
- $[f_\beta(i) = j]$, where $j \in \{\xi_\beta(\nu+1): \kappa i \leq \nu < \kappa(i+1)\}$ (so necessarily $i < j$),
- $[j \notin \mathrm{ran}(f_\beta)]$ for some $j < \lambda$,

(this is necessary for the κ-cc),

(ii) the collection q corresponds to a partial injection, and free of any explicitly contradictory subset of terms, by which we mean that
  (a) there are no $i, j \in \lambda$ s.t. $[f_\beta(i) = j]$, $[j \notin \mathrm{dom}(f_\beta)] \in q$,
  (b) there are no $i, j_0 \neq j_1 \in \lambda$ s.t. $[f_\beta(i) = j_0]$, $[f_\beta(i) = j_1] \in q$,
  (c) there are no $[f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in q$ s.t. $c_{M_\beta}(i_0, i_1) \neq c_{M_*}(j_0, j_1)$.

Note that f_β's are automatically injective by (i).

(iii) $|q| < \kappa$.

Case 3 ($\beta \in S^*$). As $\underset{\sim}{D}_\beta$ is a $\mathbb{P}^1_\beta$-name for a system of subsets of $V_\kappa^{\mathbf{V}_1}$, if additionally for each $\alpha < \kappa$ $|(\cup \underset{\sim}{D}_\beta) \restriction \alpha| < \kappa$ holds (and if $\underset{\sim}{D}_\beta$ generates a proper κ-complete filter), then we define $\mathbb{Q}^1_\beta$ to be the Mathias forcing $\mathbb{Q}_{D_\beta}$ from Definition 3.1, otherwise we can let $\mathbb{Q}^1_\beta$ to be the trivial forcing. Note that this requirement ensures that

$(\diamond)_7$ if $(w, A) \in \mathbb{Q}^1_\beta$, then $|w| < \kappa$.

This completes Definition 3.28.

Now it is straightforward to check that each $\mathbb{Q}^1_\alpha$ is (forced to be) $< \kappa$-directed closed, so $\mathbb{P}^1_\chi$ is a $< \kappa$-support iteration of $< \kappa$-directed closed posets, $\mathbb{P}^1_\chi$ itself is $< \kappa$-directed closed by [Bau78, Thm 2.7]. (In particular it does not add any new sequence of length $< \kappa$.) Since forcing M_* goes by $< \kappa$-approximations ($\Vdash_{\mathbb{P}^1_1} (\underset{\sim}{q} \in \underset{\sim}{\mathbb{Q}}^1_1) \to (|\underset{\sim}{q}| < \kappa)$, we have:

Observation 3.29. For each $\beta \in \chi \setminus S^* \setminus \{0, 1\}$ forcing with $\mathbb{Q}^1_\beta$ over $\mathbf{V}_1^{\mathbb{P}^1_\beta}$ adds an embedding $f_\beta: M_\beta \to M_*$.

We already saw that $\mathbb{P}^1_1 = \mathbb{Q}^1_0$ is λ^+-cc (Lemma 3.21), now we prove that in $\mathbf{V}_1^{\mathbb{P}^1_1}$ the quotient forcing $\mathbb{P}^1_\chi / \mathbf{G}^1_1$ has the κ^+-cc (no matter how we choose the $\mathbb{P}^1_\beta$-name

$\underset{\sim}{D}_\beta$, or $\underset{\sim}{M}_\beta$, at first only assumed to satisfy ⊛ for $2 \leq \beta < \chi$), after which not only will the λ^+-ccness of $\mathbb{P}^1_1 * (\mathbb{P}^1_\chi/\mathbf{G}^1_1)$ follow (and of $\mathbb{P}^1_\chi$, too), but some easy calculation will be sufficient for ensuring $(\mathsf{T})_2$–$(\mathsf{T})_6$. In order to prove the antichain condition we will need some technical preparation, the same way as in [She90]. Recalling that each $\mathbb{P}^1_\alpha$ is $<\kappa$-closed (and $(\diamond)_7$) is straightforward to prove (by induction on α) that

$(*)_2$ The set

$$D^\bullet_\alpha = \{p \in \mathbb{P}^1_\alpha : \ \forall \gamma \in \mathrm{dom}(p) \quad \begin{array}{l} (\beta \in S^*) \ \rightarrow \ [\, \exists w_\gamma \in \mathbf{V}_1 \text{ s.t. } \ \Vdash_{\mathbb{P}^1_\gamma} p(\gamma) = (\check{w}_\gamma, \underset{\sim}{A}_\gamma)\,] \\ (\beta \notin S^*) \ \rightarrow \ [\, \exists s_\gamma \in \mathbf{V}_1 \text{ s.t. } \ \Vdash_{\mathbb{P}^1_\gamma} p(\gamma) = \check{s}_\gamma \}\,] \end{array}$$

is a dense subset of $\mathbb{P}^1_\alpha$.

$(*)_3$ Therefore, in the quotient forcing $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$ (as defined in [Bau78], or see below) the set

$$D^0_\alpha = \{p \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 : \ \exists q_0 \in \mathbf{G}^1_1 : \ \langle q_0 \rangle \cup p \in D^\bullet_\alpha\} \ \in \mathbf{V}'_1$$

is dense (where $\mathbb{P}^1_\alpha/\mathbf{G}^1_1 = \{p \restriction (\mathrm{dom}(p) \setminus \{0\}) : \ p \in \mathbb{P}^1_\alpha\} \in \mathbf{V}'_1$, and $p \leq_{\mathbb{P}^1_\alpha/\mathbf{G}^1_1} q$, iff for some $r_0 \in \mathbf{G}^1_1 \subseteq \mathbb{P}^1_1$: $\langle r_0 \rangle \cup p \leq_{\mathbb{P}^1_\alpha} \langle r_0 \rangle \cup q$).

$(*)_4$ With a slight abuse of notation (in order to avoid further notational awkwardness) we will identify each condition $p \in D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$ with the function on the same domain, but for each $\gamma \in \mathrm{dom}(p)$
- if $\beta \in S^*$ then writing $p(\beta) = (w, \underset{\sim}{A})$ (instead of some $\mathbb{P}^1_\beta$-name satisfying $\langle q_0 \rangle \cup p \restriction \beta \Vdash_{\mathbb{P}^1_\beta} p(\beta) = (\check{w}, \underset{\sim}{A})$ for some $q_0 \in \mathbf{G}^1_1$),
- if $\beta \notin S^*$, $\beta > 0$, then writing $p(\beta) = s$, where s is a set of symbols as in Case 1, 2 in Definition 3.28 (instead of $\langle q_0 \rangle \cup p \restriction \beta \Vdash_{\mathbb{P}^1_\beta} p(\beta) = \check{s}$ for some $q_0 \in \mathbf{G}^1_1$).

Note that (as $\mathbb{P}^1_1$ is $<\kappa$-closed and $D^0_\alpha \subseteq \mathbf{V}_1$)

$(*)_5$ for any $\alpha \leq \chi$, and increasing sequence $\overline{p} = \langle p_\zeta : \ \zeta < \varepsilon < \kappa \rangle$ in D^0_α if $\bar{p} \in \mathbf{V}'_1$, then $\bar{p}$ has a least upper bound in $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$, which we will denote by $\lim_{\zeta<\varepsilon} p_\zeta$, and this limit is in D^0_α. For the sake of completeness we include the formal definition of $\lim_{\zeta<\varepsilon} p_\zeta$. The limit of $\overline{p} = \langle p_\zeta : \ \zeta < \varepsilon < \kappa \rangle$ is the function p^*, where
(a) $\mathrm{dom}(p^*) = \bigcup_{\zeta<\varepsilon} \mathrm{dom}(p_\zeta)$,
(b) for $\beta \in S^* \cap \mathrm{dom}(p^*)$ $p^*(\beta) = (\bigcup_{\zeta<\varepsilon} w_{p_\zeta(\beta)}, \underset{\sim}{A}_\beta)$, where $p_\zeta(\beta) = (w_{p_\zeta(\beta)}, A_{p_\zeta(\beta)})$, and $\underset{\sim}{A}_\beta$ is the $\mathbb{P}^1_\beta$-name defined so that $\Vdash_{\mathbb{P}^1_\beta} \underset{\sim}{A}_\beta = \bigcap_{\zeta<\varepsilon} \underset{\sim}{A}_{p_\zeta(\beta)}$ holds,
(c) for $\beta \in \chi \setminus S^*$, $\beta > 0$, set $p^*(\beta) = \bigcup_{\zeta<\varepsilon} p_\zeta(\beta)$.

Definition 3.30. In $\mathbf{V}'_1$ for each $\alpha \leq \chi$, $\delta \leq \lambda$, for each condition $p \in D^0_\alpha$ we define $p^{[\delta]}$ to be the function with $\mathrm{dom}(p^{[\delta]}) = \mathrm{dom}(p)$, and
(a) if $1 \in \mathrm{dom}(p)$, then $p^{[\delta]}(1) = \{[i \ R_\gamma \ j] \in p(1) : \ i, j < \delta\}$,
(b) for $1 < \beta \in \mathrm{dom}(p) \cap S^*$ we let $p^{[\delta]}(\beta) = p(\beta)$,
(c) otherwise (for $1 < \beta \in \mathrm{dom}(p) \setminus S^*$) we let

$$p^{[\delta]}(\beta) = \begin{array}{c} \{[f_\beta(i) = j] \in p(\beta) : \ i, j < \max\{\xi_\beta(\gamma) : \ \gamma < \lambda, \ \xi_\beta(\gamma) \leq \delta\}\} \\ \cup \\ \{[j \notin \mathrm{ran}(f_\beta)] \in p(\beta) : \ j < \max\{\xi_\beta(\gamma) : \ \gamma < \lambda, \ \xi_\beta(\gamma) \leq \delta\}\}. \end{array}$$

Observe that, because of each p and each $p(\beta)$ ($\beta \in \mathrm{dom}(p)$) has support of size $<\kappa$, and $\lambda > \kappa$ is regular,

$(*)_6$ for each $\alpha \leq \chi$, $p \in D^0_\alpha \subseteq (\mathbb{P}^1_\alpha/\mathbf{G}^1_1)$ we have $p^{[\delta]} = p$ for every large enough δ, and

$(*)_7$ clearly $p^{[\delta]} \restriction \beta = (p \restriction \beta)^{[\delta]}$ (for $\beta < \chi$).

Note that for $p \in D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$ the reduced function $p^{[\delta]}$ is in $\mathbf{V}'_1$ (even in $\mathbf{V}_1$), but is not necessarily a condition in $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$. Nevertheless,

$(*)_8$ for $p \leq q \in D^0_\alpha$ with $p^{[\delta]}, q^{[\delta]} \in D^0_\alpha$ (i.e. if they are conditions in $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$) we obviously have $p^{[\delta]} \leq q^{[\delta]}$.

It is straightforward to check the following (by induction on α).

Observation 3.31. For each $\alpha \leq \chi$, $p \in D^0_\alpha$ and $\delta < \lambda$

(a) $p^{[\delta]}$ is an actual condition (i.e. belongs to $D^0_\alpha \subseteq \mathbb{P}^1_\alpha/\mathbf{G}^1_1$), iff for every $\beta \in \mathrm{dom}(p)$
- $p^{[\delta]} \restriction \beta \in \mathbb{P}^1_\beta$, and
- (letting $\delta^-_\beta = \max(N^*_\beta \cap (\delta+1))$

$$\begin{array}{l}(\forall i_0, j_0, i_1, j_1) \ \text{ if } [f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in p(\beta), \text{ then}: \\ j_0, j_1 < \delta^-_\beta \longrightarrow p^{[\delta]} \restriction \beta \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_0, i_1) = c_{M_*}(j_0, j_1).\end{array} \tag{3.17}$$

(b) In particular, for limit α

$$p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 \iff \Big[(\text{for cofinally many } \varepsilon < \alpha): \ p^{[\delta]} \restriction \varepsilon \in \mathbb{P}^1_\varepsilon)\Big],$$

(c) while for $\alpha = \beta + 1$

$$p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1 \iff p^{[\delta]} \restriction \beta \in \mathbb{P}^1_\beta/\mathbf{G}^1_1 \text{ and (3.17) holds.}$$

The following notion and lemma is of central importance.

Definition 3.32. In $\mathbf{V}^{\mathbb{P}^1_1}_1$ for $\alpha \leq \chi$ define

$$D^*_\alpha = \{p \in D^0_\alpha : \ (\forall \delta < \lambda) \ \ p^{[\delta]} \in \mathbb{P}^1_\alpha/\mathbf{G}^1_1\}.$$

Having Observation 3.31 in our mind it is easy to check the following.

$(*)_9$ Whenever $\langle p_\zeta : \zeta < \varepsilon < \kappa\rangle$ is an increasing sequence in D^*_α, then $\lim_{\zeta<\varepsilon} p_\zeta \in D^*_\alpha$.

This leads to the statements about how $p \in D^*_\alpha$ and $p \restriction \beta \in D^*_\beta$ ($\beta < \alpha$) relate to each other.

Observation 3.33. For each $\alpha \leq \chi$, $p \in D^0_\alpha$

(a) $p \in D^*_\alpha$, iff for every $\beta \in \mathrm{dom}(p)$ and for every $\delta < \lambda$

$$p \restriction \beta \in D^*_\beta,$$

and (letting $\delta^-_\beta = \max(N^*_\beta \cap (\delta+1))$

$$\begin{array}{l}(\forall i_0, j_0, i_1, j_1) \ \text{ if } [f_\beta(i_0) = j_0], [f_\beta(i_1) = j_1] \in p(\beta), \text{ then:} \\ j_0, j_1 < \delta^-_\beta \longrightarrow p^{[\delta]} \restriction \beta \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_0, i_1) = c_{M_*}(j_0, j_1).\end{array} \tag{3.18}$$

(b) In particular, for limit α

$$p \in D^*_\alpha \iff (\text{for cofinally many } \varepsilon < \alpha): \ p \restriction \varepsilon \in D^*_\varepsilon,$$

(c) while for $\alpha = \beta + 1$

$$p \in D^*_\alpha \iff [p \restriction \beta \in D^*_\beta] \text{ and [for each } \delta < \lambda \text{ (3.18) holds for } \beta.].$$

We are ready to state the two lemmas on which the correctness of the entire construction depends. Lemma 3.35 makes it possible to enumerate and embed all possible graphs on λ into M_*, which can be proved relying on Lemma 3.34.

Lemma 3.34. *For $\alpha \leq \chi$*

$(\blacksquare)^1_\alpha$

$$\mathbf{V}_1^{\mathbb{P}_1^1} \models D^*_\alpha \textit{ is dense in } \mathbb{P}^1_\alpha / \mathbf{G}^1_1.$$

Lemma 3.35. *For every $\alpha \leq \chi$*

$(\blacksquare)^2_\alpha$

$$\mathbf{V}_1^{\mathbb{P}_1^1} \models \mathbb{P}^1_\alpha / \mathbf{G}^1_1 \textit{ has the } \kappa^+\textit{-cc.}$$

Proof. We proceed by induction, and prove Lemmas 3.34 and 3.35 simultaneously: More exactly we prove Lemma 3.34 for α provided that both Lemmas hold for β's less than α, and we verify the κ^+-cc property for $\mathbb{P}^1_\alpha$ assuming that D^*_α is a dense subset of $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$. For $\alpha \leq 2$ (when $\mathbb{P}^1_2/\mathbf{G}^1_1$ is essentially the forcing $\mathbb{Q}^1_1$ of the random graph Case 1 of Definition 3.28) the statement $(\blacksquare)^1_\alpha$ clearly holds, moreover, $(\blacksquare)^2_\alpha$ holds recalling $\kappa^{<\kappa} = \kappa$.

Suppose we know that for each $\varepsilon < \alpha$ $(\blacksquare)^1_\alpha$ and $(\blacksquare)^2_\alpha$ hold. Assume first that α is limit. If $\mathrm{cf}(\alpha) \geq \kappa$, then $\mathbb{P}^1_\alpha = \bigcup_{\varepsilon<\alpha} \mathbb{P}^1_\varepsilon$, $D^*_\alpha = \bigcup_{\varepsilon<\alpha} D^*_\varepsilon$, so the latter is dense, we are done.

Second, if α is limit, but $\mathrm{cf}(\alpha) < \kappa$, then let $\langle \eta_\theta : \theta < \mathrm{cf}(\alpha)\rangle$ be a continuous increasing sequence with limit α, let $p_{-1} \in D^0_\alpha$ be arbitrary. We will choose the increasing sequence $\langle p_\theta : \theta < \mathrm{cf}(\alpha)\rangle$ in D^0_α with $p_0 \geq p_{-1}$, and $p_\theta \restriction \eta_\theta \in D^*_{\eta_\theta}$. This would suffice as for each $\theta < \mathrm{cf}(\kappa)$ the sequence $p_\varrho \restriction \eta_\theta$ ($\varrho < \mathrm{cf}(\alpha)$) is eventually in $D^*_{\eta_\theta}$, so for $p^* = \lim_{\varrho<\mathrm{cf}(\alpha)} p_\varrho$ using $(*)_9$ we have $p^* \restriction \eta_\theta \in D^*_{\eta_\theta}$, leading to

$$(\forall \theta < \mathrm{cf}(\alpha))\ p^* \restriction \eta_\theta \in D^*_{\eta_\theta},$$

so by (b) we are done. For the construction of the p_θ's, as D^0_α and $D^*_{\eta_\theta}$'s are $<\kappa$-closed we only have to ensure that $p_\theta \in D^0_\alpha$ can be chosen so that not only $p_\theta \geq p_\varrho$ ($\varrho < \theta$), but $p_\theta \restriction \eta_\theta \in D^*_{\eta_\theta}$. Now applying the induction hypothesis, we can find $p^*_\theta \in D^*_{\eta_\theta}$ such that it extends $(\lim_{\varrho<\theta} p_\varrho) \restriction \eta_\theta$ (in $\mathbb{P}^1_{\eta_\theta}/\mathbf{G}^1_1$). Finally, let p_θ be the least upper bound of p^*_θ and $(\lim_{\varrho<\theta} p_\varrho)$ (in fact for θ limit we did not even have to appeal to the induction hypothesis if $\bar{\eta}$ is continuous).

Third, if $\alpha = \beta + 1$, let $p_{-1} \in D^0_\alpha$ be arbitrary and we will extend $p_{-1} \restriction \beta$ to $p^* \in D^*_\beta$ (using $(\blacksquare)^1_\beta$) in such a way that the right hand side of Observation 3.33(c) holds for $p = p^* \cup \langle p_{-1}(\beta)\rangle$ (so that $p \in D^*_\alpha$, $p \geq p_{-1}$).

For this, let $\{j_\theta : \theta < \nu\}$ enumerate $\{j < \lambda : [f_\beta(i) = j] \in p_{-1}(\beta)$ for some $i < \lambda\}$ in increasing order, and we can fix the system $\{i_\theta : \theta < \nu\}$ so that

$(\odot)_1$ $\{i_\theta : \theta < \nu\}$ is such that for each θ $[f_\beta(i_\theta) = j_\theta] \in p_{-1}(\beta)$.

Note that by Definition 3.28/Case 2/(i)

$(\odot)_2$ for each θ: $i_\theta < j_\theta$,

and also we can choose γ_θ for each $\theta < \nu$ such that $\xi_\beta(\gamma_\theta) = j_\theta$, thus

$(\odot)_3$ we have

$$\{j < \lambda : \exists i < \lambda\ [f_\beta(i) = j] \in p_{-1}(\beta)\} = \{j_\theta : \theta < \nu\} = \{\xi_\beta(\gamma_\theta) : \theta < \nu\}.$$

Now we construct the increasing sequence $\langle p_\theta : \theta < \nu\rangle$ in D^*_β with the properties

(α) $p_{-1} \restriction \beta \leq p_0$, (β) for each $\theta < \nu$, for each $\varepsilon_0 < \varepsilon_1 < \theta$

$$p_\theta^{[\xi_\beta(\gamma_{\varepsilon_1}+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_{\varepsilon_0}, i_{\varepsilon_1}) = c_{M_*}(j_{\varepsilon_0}, j_{\varepsilon_1}).$$

This clearly suffices, as we can let $p^* = \lim_{\theta<\nu} p_\theta \in D^*_\beta$, and then $p = p^* \cup \langle p_{-1}(\beta)\rangle$ belongs to D^*_α, $(\blacksquare)^1_\alpha$ follows, indeed. (To see that the condition p belongs to D^*_α, recall $j_{\varepsilon_1} = \xi_\beta(\gamma_{\varepsilon_1})$ so $\xi_\beta(\gamma_{\varepsilon_1}+1)$ is the minimal $\delta < \lambda$ with $p^{[\xi_\beta(\gamma_{\varepsilon_1}+1)]}(\beta)$ containing the symbol $[f_\beta(i_{\varepsilon_1}) = j_{\varepsilon_1}]$, therefore by Observation 3.33(c) we are done.)

Appealing to the induction hypothesis, let $p_0 \in D^*_\beta$, $p_0 \geq p_{-1}$. Using the $< \kappa$-closedness of D^*_β $(*)_9$ it is enough to deal with the successor case, that is, for each θ choose $p_{\theta+1}$ such that $p_{\theta+1}^{[\xi_\beta(\gamma_\theta+1)]}$ forces that the partial function $i_\varepsilon \mapsto j_\varepsilon$ $(\varepsilon \leq \theta)$ is an embedding of $\underset{\sim}{M}_\beta \restriction \{i_\varepsilon : \varepsilon \leq \theta\}$ into $\underset{\sim}{M}_* \restriction \{j_\varepsilon : \varepsilon \leq \theta\}$. Using again $(*)_9$

$(\odot)_6$ it suffices to show that for each $\varepsilon < \theta$ and $q \geq p_{-1} \restriction \beta$, where $q \in D^*_\beta$, there exists $q' \in D^*_\beta$, $q' \geq q$

$$q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta).$$

We will see that this follows from the following (formally) more general lemma, stated here for later reference.

Lemma 3.36. *For every $\beta \leq \chi$, $q \in D^*_\beta$, $\delta < \lambda$, i', $i'' < \max(N^*_\beta \cap (\delta+1))$ there exists $q' \in D^*_\beta$, $q' \geq q$ such that*

$$q'^{[\delta]} \text{ forces a value to } c_{M_\beta}(i', i'').$$

Moreover, if q satisfies

$$(3.19)\qquad \begin{array}{l} (\forall \gamma \in \mathrm{dom}(q) \setminus S^*)(\forall i, j) : \\ \big[([f_\beta(i) = j)] \in q(\gamma) \setminus q^{[\delta]}(\gamma)) \longrightarrow (j = \max(N^*_\gamma \cap (\delta+1)) \ \wedge j < \delta)\big] \\ \ \textit{and } (q(1) = q^{[\delta]}(1)) \end{array}$$

(hence $\delta \notin N^*_\gamma$ for $\gamma \in \mathrm{dom}(q) \setminus S^*$), then there exists q' for which additionally:

$$(\forall \gamma \in \mathrm{dom}(q') \setminus S^*) : \quad q'(\gamma) \setminus q'^{[\delta]}(\gamma) = q(\gamma) \setminus q^{[\delta]}(\gamma) \ .$$

(Here we remark that lemma is for every β, and uses the κ^+-cc property of $\mathbb{P}^1_\beta/\mathbf{G}^1_1$, but we will only apply it to our fixed β, for proving $(\odot)_6$, that is, to complete the proof of $((\blacksquare)^1_\beta \wedge (\blacksquare)^1_\beta) \rightarrow (\blacksquare)^1_\alpha$.)

Proof. (Lemma 3.36) So fix $q \in D^*_\beta$, let ϱ be chosen so that $\xi_\beta(\varrho) = \max(N^*_\beta \cap (\delta+1))$, so $i', i'' < \xi_\beta(\varrho) \leq \delta$, and recall that for the model $N_{\beta,\varrho} \prec (\mathscr{H}^{\mathbf{V}'_1}(\Upsilon), \in)$ we know that $i', i'', \underset{\sim}{M}_\beta, \mathbb{P}^1_\beta, \mathbf{G}^1_1 \in N_{\beta,\varrho}$ (and thus $\mathbb{P}^1_\beta/\mathbf{G}^1_1 \in N_{\beta,\varrho}$). So we can find $A \in N_{\beta,\varrho}$ such that A is a maximal antichain in $D^*_\beta \subseteq \mathbb{P}^1_\beta/\mathbf{G}^1_1$, each $p \in A$ decides the value of $c_{M_\beta}(i', i'')$. But as $\mathbb{P}^1_\beta/\mathbf{G}^1_1$ has the κ^+-cc, and $\kappa + 1 \subseteq N_{\beta,\varrho}$ we have that $A \subseteq N_{\beta,\varrho}$.

So

$(\boxplus)_1$ let $q' \in D^*_\beta$ be a common upper bound of q and some $q'' \in A$.

We have to argue that not only $q' \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i', i'') = c_*$ (for some $c_* < \kappa$) but

$$(3.20)\qquad q'^{[\delta]} \Vdash_{\mathbb{P}^1_\beta/\mathbf{G}^1_1} c_{M_\beta}(i', i'') = c_*.$$

For (3.20) it is enough to prove that $q''^{[\delta]} = q''$, because then $q'^{[\delta]} \geq q''^{[\delta]} = q''$ (by $(*)_8$), which decides $\underset{\sim}{c}_{M_\beta}(i', i'')$, yielding (3.20), as we wanted. But as $q'' \in N_{\beta,\varrho}$, and $\lambda \cap N_{\beta,\varrho} = \xi_\beta(\varrho) \leq \delta$, we have $\mathrm{dom}(q'') \subseteq N_{\beta,\varrho}$. Now for each $\zeta \in \mathrm{dom}(q'') \setminus S^* \setminus \{0,1\}$ we have $\langle N_{\zeta,\iota} : \iota < \lambda \rangle \in N_{\beta,\varrho}$ (recall Case 2 from Definition 3.28), so $\xi_\beta(\varrho)$ is an accumulation point of the $\xi_\zeta(\iota)$'s. Hence we get that

$(\boxplus)_2$ for each $\zeta \in \mathrm{dom}(q'') \setminus S^* \setminus \{0,1\}$ $\xi_\beta(\varrho) = \xi_\zeta(\iota)$ for some $\iota < \lambda$ (in fact, for $\iota = \xi_\beta(\varrho)$),

so $q''^{[\xi_\beta(\varrho)]} = q''^{[\delta]} = q''$, we are done.

Finally, for the moreover part, if $\gamma \in \mathrm{dom}(q) \setminus S^*$, let $\delta^-_\gamma = \max(N_\gamma \cap (\delta + 1)))$, and define i^-_γ to be the unique ordinal s.t.

$$[f_\gamma(i^-_\gamma) = \delta^-_\gamma] \in q(\gamma) \tag{3.21}$$

(if there exists). Note that our conditions on q imply that if i^-_γ is defined, then $i^-_\gamma < \delta^-_\gamma$, and by our conditions (3.19)

$$\delta^-_\gamma < \delta.$$

Now by induction and by the first part define $q'' \geq q$ such that for every $\gamma \in \mathrm{dom}(q'') \setminus S^*$ with i^-_γ defined

$$([f_\gamma(i) = j] \in q''^{[\delta]}(\gamma)) \;\rightarrow\; q''^{[\delta]} \restriction \gamma \text{ decides the value } c_{\underset{\sim}{M}_\gamma}(i, i^-_\gamma),$$

and

$$([f_\gamma(i) = j] \in q''^{[\delta]}(\gamma)) \;\rightarrow\; q''^{[\delta]}(1) \text{ decides the value } c_{\underset{\sim}{M}_*}(j, \delta^-_\gamma)$$

(in fact this latter follows from $j, \delta^-_\gamma < \delta$ and (3.21)). Now clearly $q''^{[\delta]} \geq q^{[\delta]}$, and we can define the condition q' to be the least upper bound of $q''^{[\delta]}$ and q (which is just adding symbols $[f_\gamma(i^-_\gamma) = \delta^-_\gamma] \in q(\gamma)$): this is possible, as for every γ with i^-_γ defined we have that $q''^{[\delta]} \restriction \gamma$ forces that $q''^{[\delta]}(\gamma) \cup \{[f_\gamma(i^-_\gamma) = \delta^-_\gamma]\}$ is indeed a partial embedding. $\square_{\text{Lemma 3.36}}$

Turning back to the statement from $(\odot)_6$, as $j_\varepsilon < j_\theta = \xi_\beta(\gamma_\theta) < \xi_\beta(\gamma_\theta + 1)$ we also have $i_\varepsilon, i_\theta < \xi_\beta(\gamma_\theta)$ (thus obviously $i_\varepsilon, i_\theta < \xi_\beta(\gamma_\theta + 1)$). Apply the lemma with $\delta = \xi_\beta(\gamma_\theta + 1)$, $i' = i_\varepsilon$, $i'' = i_\theta$,

$(\odot)_7$ let $q' \in D^*_\beta$ be given by the lemma, so

$$q' \Vdash_{\mathbb{P}^1_\beta / \mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta) \tag{3.22}$$

(which is obvious, as

$(\odot)_8$ $q' \geq p_{-1} \restriction \beta$, and p_{-1} is a proper condition in D^0_α with $[f_\beta(i_\theta) = j_\theta]$, $[f_\beta(i_\varepsilon) = j_\varepsilon] \in p_{-1}(\beta)$, hence $q' \frown \langle p_{-1}(\beta) \rangle$, too).

It remains to argue that

$$q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta / \mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_{M_*}(j_\varepsilon, j_\theta). \tag{3.23}$$

But $q'^{[\xi_\beta(\gamma_\theta+1)]} \Vdash_{\mathbb{P}^1_\beta / \mathbf{G}^1_1} c_{M_\beta}(i_\varepsilon, i_\theta) = c_*$ (for some $c_* < \kappa$) and if $[j_\varepsilon \; R_{c_*} \; j_\theta] \notin q'^{[\xi_\beta(\gamma_\theta+1)]}(1)$ (so does not belong to $q'(1)$), then adding $[j_\varepsilon \; R_{c_*+1} \; j_\theta]$ to the first coordinate of q' would lead to a contradiction with (3.22). This verifies that assuming the induction hypotheses for β, the assertion $(\blacksquare)^1_\alpha$ holds, i.e. the set $D^*_{\beta+1} = D^*_\alpha$ is dense in $\mathbb{P}^1_\alpha / \mathbf{G}^1_1$.

Now assuming that D^*_α is dense we are ready to prove that $\mathbb{P}^1_\alpha/\mathbf{G}^1_1$ has the κ^+-cc. So let $\langle p_\gamma : \gamma < \kappa^+\rangle$ be an antichain in D^*_α. By extending each p_γ

$(\odot)_9$ we can assume that for each $\gamma < \kappa^+$

(i) for each $\beta' \in \mathrm{dom}(p_\gamma)$, for each $i_0, i_1, j_0 < j_1$ with $[f_{\beta'}(i_0) = j_0], [f_{\beta'}(i_1) = j_1] \in p_\gamma(\beta')$ the condition $p^{[j_1]} \restriction \beta'$ decides the value $c_{\underset{\sim}{M}_{\beta'}}(i_0, i_1)$,

(ii) for each $\gamma < \kappa^+$ the condition $p_\gamma(1)$ is a complete graph on some set L_γ with its edges colored, i.e.

$$L_\gamma = \{i < \lambda : \exists i' < \lambda \; \exists \varepsilon < \kappa \; [i \; R_\varepsilon \; i'] \in p_\gamma(1)\},$$
$$\text{so } (\forall i, j \in L_\gamma) \; (\exists \delta < \kappa) : \; [i \; R_\delta \; j] \in p_\gamma(1).$$

(iii) for each $\gamma < \kappa^+$ and $\beta' \neq \beta'' \in \mathrm{dom}(p_\gamma) \setminus S^* \setminus \{0,1\}$ we have

$$\{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\} \subseteq L_\gamma$$

(recall that $|\{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\}| < \kappa$ by $(\diamond)_5$),

(iv) for each $\gamma < \kappa^+$ and $\beta' \in \mathrm{dom}(p_\gamma) \setminus S^* \setminus \{0,1\}$, for each $j < \lambda$ if either $[j \notin \mathrm{ran}(f_{\beta'})] \in p_\gamma(\beta')$, or $[f_{\beta'}(i) = j] \in p_\gamma(\beta')$ (for some $i < \lambda$), then $j \in L_\gamma$,

(v) for each $\gamma < \kappa^+$, $\beta' \in \mathrm{dom}(p_\gamma)$ and $j < \lambda$, if $j \in L_\gamma$, then

$$(j \in \{\xi_{\beta'}(\rho+1) : \rho < \lambda\}) \Rightarrow \begin{cases} \text{either} & [j \notin \mathrm{ran}(f_{\beta'})] \in p_\gamma(\beta') \\ \text{or (for some } i) & [f_{\beta'}(i) = j] \in p_\gamma(\beta'), \end{cases}$$

(vi) the set $L_\gamma \subseteq \lambda$ is closed, of limit order type.

[This is possible, a simple induction using Lemma 3.36, and the fact

$$[f_\beta(i) = j] \in p_\gamma(\beta) \;\rightarrow\; j \in N^*_\beta$$

(and $(*)_9$) yield that there is $p'_\gamma \geq p_\gamma$ in D^*_α, with $(p'_\gamma \restriction \beta)^{[j_1]}$ determining the value $c_{\underset{\sim}{M}_\beta}(i_0, i_1)$ whenever $[f_{\beta_0}(i_0) = j_0] \in p_\gamma(\beta_0)$, $[f_{\beta_1}(i_1) = j_1] \in p_\gamma(\beta_1)$ (for some $j_0 < j_1$, or if either of the i's belongs to the universe of $p_\gamma(1)$). Now repeating this ω-many times we get a condition satisfying (i). Then we can obtain an even stronger condition satisfying (ii)–(vi) by only adding symbols of the form $[j \notin \mathrm{ran}(f_{\beta'})]$ at coordinates $1 < \beta' \in \chi \setminus S^*$ and extending also $p'_\gamma(1)$.]

As κ is strongly inaccessible in $\mathbf{V}_1$ (by $(*)_1$), and in $\mathbf{V}_1^{\mathbb{P}^1_1}$ (as $\mathbb{P}^1_1$ is $< \kappa$-closed), we can apply the delta system lemma, so w.l.o.g. $\langle \mathrm{dom}(p_\gamma) : \gamma < \kappa^+\rangle$ forms a delta system. By applying the delta system lemma again we can assume that for each $\beta' \in \cap\{\mathrm{dom}(p_\gamma) : \gamma < \kappa\} \setminus S^*$ each of the following systems of sets forms a delta system:

- L_γ $(\gamma < \kappa^+)$,
- $I_\gamma(\beta') = \left\{ \begin{matrix} i : & [f_{\beta'}(i) = j] \in p_\gamma(\beta') \vee \exists j \in [\xi_{\beta'}(\kappa i), \xi_{\beta'}(\kappa(i+1))) \\ & [j \notin \mathrm{ran}(f_\beta)] \in p_\gamma(\beta') \end{matrix} \right\} (\gamma < \kappa^+).$

Therefore (recalling that each $i < \lambda$ has κ-many possible images) there are $\xi \neq \zeta < \kappa^+$, such that p_ξ and p_ζ has no explicitly contradictory terms on the coordinates concerning the κ-colored graphs, and agreeing in the first part of the condition on the coordinates dedicated to Mathias forcing, under which we mean the following (w.l.o.g. we can assume that $\xi = 0$, $\zeta = 1$):

$(\odot)_{10}$ for each $i, j \in L_0(1) \cap L_1(1)$ there exists some $\varepsilon < \kappa$ s.t. $[i \; R_\varepsilon \; j] \in p_0(1) \cap p_1(1)$,

$(\odot)_{11}$ for $\beta' \in \chi \setminus S^* \setminus \{0,1\}$ (if $\beta' \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$) the set $p_0(\beta') \cup p_1(\beta')$ determines a partial injection from a subset of λ to a subset of λ, i.e. satisfies (ii)(a), (b) (from Definition 3.28 Case 2),

$(\odot)_{12}$ for $\beta \in S^* \cap \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$ $p_0(\beta) = (w_\beta, \underset{\sim}{A}_{0,\beta})$, $p_1(\beta) = (w_\beta, \underset{\sim}{A}_{1,\beta})$ for some $w_\beta \in [V_\kappa^{\mathbf{V}_1}]^{<\kappa}$, and $\mathbb{P}^1_\beta$-names $\underset{\sim}{A}_{0.\beta}$, $\underset{\sim}{A}_{1,\beta}$.

Now p_0 and p_1 appear as good candidates for a compatible pair in our supposed antichain, but we cannot take just the upper bound coordinate wise, as for coordinates $\beta' > 1$ outside S^* it will not necessarily force that $p_0(\beta') \cup p_1(\beta')$ is an embedding of $\underset{\sim}{M}_{\beta'}$ to $\underset{\sim}{M}_*$. Although it is not immediate, Claim 3.37 shows that we can construct a common upper bound, which will complete the proof of $(\blacksquare)^2_\alpha$ for α.

Claim 3.37. There exists a condition $q \in D^*_\alpha$ extending both p_0 and p_1.

Proof. $(\bullet)_1$ By adding symbols of the form $[j \notin \mathrm{ran}(f_\beta)]$ to $p_0(\beta)$, $p_1(\beta)$ we can assume the following (not harming $(\odot)_{11}$)

$(\bullet)_{1a}$ for $1 < \beta \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ if $[f_\beta(i) = j_\theta] \in p_0(\beta) \cup p_1(\beta)$ holds for no i then $[j_\theta \notin \mathrm{ran}(f_\beta)] \in p_0(\beta) \cap p_1(\beta)$,

$(\bullet)_{1b}$ whenever $\beta' \neq \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$, $j^* \in \{\xi_{\beta'}(\rho+1) : \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\} \cap j_\varrho$ and there is no i with $[f_{\beta'}(i) = j^*] \in p_0(\beta') \cup p_0(\beta')$ then $[j^* \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cap p_1(\beta')$,

$(\bullet)_2$ Let $\{j_\varepsilon : \varepsilon < \varrho\}$ be a continuous increasing sequence for which,

$(\bullet)_{2a}$ whenever $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$, and j is such that either $[j \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta')$, or $[f_{\beta'}(i) = j] \in p_0(\beta') \cup p_1(\beta')$ for some i, then $j = j_\theta$ for some $\theta < \varrho$. (Therefore, $L_0 \cup L_1 = \{j : [j\ R_\nu\ j'] \in p_0(1) \cup p_1(1)$ for some $j' < \lambda,\ \nu < \kappa\} \subseteq \{j_\theta : \theta < \varrho\}$.)

Let $j_\varrho = \sup\{j_\theta : \theta < \varrho\}$, let $j_{\varrho+1}$ be an ordinal which is bigger than $\min(N^*_{\beta'} \setminus j_\varrho)$ for any $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$.

$(\bullet)_3$ We construct the increasing sequence $\langle q_\varepsilon : \varepsilon < \varrho + 2\rangle$ in D^*_α satisfying

$$q_\varepsilon^{[j_\varepsilon]} \geq p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]},$$

$(\bullet)_4$ and also we require that for each $\varepsilon < \varrho$ the strict inequality $q_\varepsilon(\beta') \gneq q_\varepsilon^{[j_\varepsilon]}(\beta')$ is possible if and only if $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus \{1\}$ and $(\delta^{\beta'}_\varepsilon)^- = \max(N^*_{\beta'} \cap (j_\varepsilon + 1)) < j_\varepsilon$ hold, and then for each such β' the difference

$$q_\varepsilon(\beta') \setminus q_\varepsilon^{[j_\varepsilon]}(\beta') = \begin{cases} \{[f_{\beta'}(i) = (\delta^{\beta'}_\varepsilon)^-]\}, & \text{if } [f_{\beta'}(i) = (\delta^{\beta'}_\varepsilon)^-] \in p_0(\beta') \cup p_1(\beta'), \\ \{\left[(\delta^{\beta'}_\varepsilon)^- \notin \mathrm{ran}(f_{\beta'})\right]\}, & \text{if } \left[(\delta^{\beta'}_\varepsilon)^- \notin \mathrm{ran}(f_{\beta'})\right] \in p_0(\beta') \cup p_1(\beta'). \end{cases}$$

While otherwise, if neither $[(\delta^{\beta'}_\varepsilon)^- \notin \mathrm{ran}(f_{\beta'})]$ belongs to $p_0(\beta') \cup p_1(\beta')$ nor is there an i with $[f_{\beta'}(i) = (\delta^{\beta'}_\varepsilon)^-] \in p_0(\beta') \cup p_1(\beta')$, then $q_\varepsilon(\beta') = q_\varepsilon^{[j_\varepsilon]}(\beta')$. (Since for the generic embedding $f_{\beta'}$ $\mathrm{ran}(f_{\beta'}) \subseteq N^*_{\beta'}$ must hold, roughly speaking q_ε contains all the information from p_0 and p_1 strictly below j_ε.)

Now we claim that provided the sequence $\langle q_\varepsilon : \varepsilon < \varrho + 2\rangle$ exists there is a common upper bound of p_0 and p_1.

Observation 3.38. $q_{\varrho+1}$ is an upper bound of p_0 and p_1.

Claim 3.39. There exists a sequence $\langle q_\varepsilon : \varepsilon < \varrho + 2\rangle$ satisfying $(\bullet)_3$, $(\bullet)_4$.

Proof. We define q_0 to be the upper bound of $p_0^{[j_0]}$ and $p_1^{[j_0]}$ to satisfy $(\bullet)_{1a}$, $(\bullet)_{1b}$: For $\beta' \in S^*$ if $p_0(\beta') = (w_{\beta'}, \underset{\sim}{A}_{0,\beta'})$, $p_1(\beta') = (w_{\beta'}, \underset{\sim}{A}_{1,\beta'})$ then we let $s_0(\beta') = (w, \underset{\sim}{B}_{\beta'})$ (where $\underset{\sim}{B}_{\beta'}$ is the $\mathbb{P}^1_{\beta'}$-name satisfying $\Vdash_{\mathbb{P}^1_{\beta'}} \underset{\sim}{B}_{\beta'} = \underset{\sim}{A}_{0,\beta'} \cap \underset{\sim}{A}_{1,\beta'}$). Because of $q_0 = q_0^{[j_\varepsilon]}$ (by $(\bullet)_3$), and recalling $(\odot)_9$/(iv) for $\gamma = 0, 1$, $q_0(1)$ can only be the empty condition. Furthermore, for $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$, $\beta' > 1$ we let

$(\triangle)_1$ $q_0(\beta') = \{[j \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta') : \ j < j_0 \ \wedge \ j \leq \sup(N^*_{\beta'} \cap j_0)\}$.

So q_0, $q_0^+ \in D^0_\alpha$ in fact belong to D^*_α, and we obviously have $(\bullet)_3$, $(\bullet)_4$.

Now suppose that q_θ's are already defined for $\theta < \varepsilon$, and we shall construct q_ε, but we need to deal with limit and successor ε's differently.

Case A (ε is limit). Let $s_\varepsilon = \lim_{\theta<\varepsilon} q_\theta \in D^*_\alpha$, we argue that we can choose a suitable extension of s_ε to be q_ε. For q_ε we only extend s_ε on coordinates $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus (\{1\} \cup S^*)$. So fix such a β'. First, if $j_\varepsilon \notin N^*_{\beta'}$ (hence $N^*_{\beta'}$ is bounded in j_ε) then we let $q_\varepsilon(\beta') = s_\varepsilon(\beta')$. Second, if $j_\varepsilon \in N^*_{\beta'}$, and it is an accumulation point of $N^*_{\beta'}$, then again we do nothing, we just let $q_\varepsilon(\beta') = s_\varepsilon(\beta')$. But if j_ε is a successor of $(j_\varepsilon^{\beta'})^- = \max(N^*_{\beta'} \cap j_\varepsilon)$ in $N^*_{\beta'}$, then first note that

$(\triangle)_2$ $p_0^{[j_\varepsilon]}(\beta') \cup p_1^{[j_\varepsilon]}(\beta') \subseteq p_0^{[(j_\varepsilon^{\beta'})^-]}(\beta') \cup p_1^{[(j_\varepsilon^{\beta'})^-]}(\beta') \cup \{[j_\theta \notin \mathrm{ran}(f_{\beta'})] : \ j_\theta \geq (j_\varepsilon^{\beta'})^-\} \cup \{[f_{\beta'}(i) = (j_\varepsilon^{\beta'})^-] : \ i < (j_\varepsilon^{\beta'})^-\}$

(in fact j's between two consecutive elements of $N^*_{\beta'}$ are irrelevant in terms of the forcing and the embedding $f_{\beta'}$). Moreover, as ε is limit (and $\langle j_\theta : \ \theta < \varrho + 2\rangle$ is closed by $(\bullet)_2$) there is $\theta \in \varepsilon$ with $j_\theta \in ((j_\varepsilon^{\beta'})^-, j_\varepsilon)$, and by $(\bullet)_3$, $(\bullet)_4$ we have (for such θ)

$(\triangle)_3$ $q_\theta(\beta') \subseteq s_\varepsilon(\beta') \subseteq s_\varepsilon^{[(j_\varepsilon^{\beta'})^-]}(\beta') \cup \{[(j_\varepsilon^{\beta'})^- \notin \mathrm{ran}(f_{\beta'})], [f_{\beta'}(i) = (j_\varepsilon^{\beta'})^-] : \ i < (j_\varepsilon^{\beta'})^-\}$.

Again

$(\triangle)_4$ $s_\varepsilon(\beta') \supseteq p_0^{[(j_\varepsilon^{\beta'})^-]}(\beta') \cup p_1^{[(j_\varepsilon^{\beta'})^-]}(\beta')$, and

$(\triangle)_5$ $s_\varepsilon(\beta') \supseteq (p_0(\beta') \cup p_1(\beta')) \cap \Big\{[(j_\varepsilon^{\beta'})^- \notin \mathrm{ran}(f_{\beta'})], [f_{\beta'}(i) = (j_\varepsilon^{\beta'})^-] : \ i < (j_\varepsilon^{\beta'})^-\Big\}$.

So there is no problem adding $\{[j_\theta \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta') : \ (j_\varepsilon^{\beta'})^- < j_\theta < j_\varepsilon\}$ to $s_\varepsilon(\beta')$ obtaining $q_\varepsilon(\beta')$. In each of the cases it is also easy to check $(\bullet)_4$.

Case B ($\varepsilon = \theta + 1$). We summarize first which symbols the $q_\varepsilon(\beta')$'s ($\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$) would have to include in order for q_ε to satisfy $q_\varepsilon^{[j_\varepsilon]} \geq p_0^{[j_\varepsilon]}, p_1^{[j_\varepsilon]}$, and $(\bullet)_4$. Of course only the case $\beta' \notin S^*$ is relevant.

$(\triangle)_6$ for $\beta' = 1$ the set to cover is

$$p_0^{[j_\varepsilon]}(1) \cup p_1^{[j_\varepsilon]}(1) \setminus q_\theta(1) = \{[j_\theta \ R_\tau \ j] \in p_0(0) \cup p_1(0) : \ j < j_\theta, \ \tau < \kappa\}. \tag{3.24}$$

By $(\bullet)_{2a}$

$(\triangle)_7$ for $1 < \beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$ the set $q_\varepsilon(\beta')$ has to include the set

$$\{[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta') : \ i \in \lambda\} \tag{3.25}$$

(which is actually either a singleton, or the empty set) and

$$\{[j \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta') : \ j \in \Big((\delta_\theta^{\beta'})^-, \delta_\varepsilon^{\beta'})^-\Big] \cup \{j_\theta\} \setminus \{j_\varepsilon\} \tag{3.26}$$

(where $(\delta_\theta^{\beta'})^- = \sup(N^*_{\beta'} \cap (j_\theta+1))$, $(\delta_\varepsilon^{\beta'})^- = \sup(N^*_{\beta'} \cap (j_\varepsilon+1))$, possibly $(\delta_\theta^{\beta'})^- = (\delta_\varepsilon^{\beta'})^- \leq j_\theta$). Recall that if $[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ for some i, then necessarily $j_\theta \in N^*_{\beta'}$, hence $(\delta_\theta^{\beta'})^- = j_\theta$.

First we are going to extend q_θ to a condition q_θ^+ with $q_\theta^+(1)$ including the set in (3.24), and for $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \setminus S^*$ the condition $q_\theta^+(\beta')$ including the set in (3.25).

Subclaim 3.40. There exists $q_\theta^+ \geq q_\theta$ in D^*_α with

$(*)_a$ $q_\theta^+(1) \supseteq \{[j_\theta \ R_\tau \ j] \in p_0(0) \cup p_1(0) : \ j < j_\theta, \ \tau < \kappa\}$,
$(*)_b$ for each $0 < \beta' \notin S^*$

$$q_\theta^+(\beta') \ni [j_\theta \notin \mathrm{ran}(f_{\beta'})], \text{ if } [j_\theta \notin \mathrm{ran}(f_{\beta'})] \in p_0(\beta') \cup p_1(\beta'),$$

$$q_\theta^+(\beta') \supseteq \{[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta') : \ i < j_\theta\},$$

while

$(*)_c$ $q_\theta^+(1) \subseteq q_\theta^{+[j_\theta]}(1) \cup \{[j \ R_\nu \ j_\theta] : \ j < j_\theta, \ \nu < \kappa\}$,
$(*)_d$ and for each $1 < \beta' \notin S^*$

$$q_\theta^+(\beta') \subseteq q_\theta^{+[(j_\theta^{\beta'})^-]}(\beta') \cup \{[f_{\beta'}(i) = j_\theta] : \ i < j_\theta\} \cup \{[j_\theta \notin \mathrm{ran}(f_{\beta'})]\}.$$

Assuming Subclaim 3.40 (which guarantees that q_θ^+ satisfies $(\bullet)_4$) we only have to add symbols of the form $[j \notin \mathrm{ran}(f_{\beta'})]$ (sets in (3.26)) to the $q_\theta^+(\beta')$'s to obtain the condition $q_{\theta+1} = q_\varepsilon$ satisfying $(\bullet)_3$ and $(\bullet)_4$, therefore Subclaim 3.40 will finish the proof of Claim 3.39

Proof. (Subclaim 3.40)

$(\blacktriangle)_1$ For each fixed β' where $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ with $[f_{\beta'}(i) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ for some i let $i_\theta^{\beta'}$ denote this unique i.

Now observe that

$(\blacktriangle)_2$ for each β' with $i_\theta^{\beta'}$ defined, for each $j' < j_\theta$ with $[f_{\beta'}(i') = j'] \in q_\theta(\beta')$ for some i' note that $i' < j' \leq (\delta_\theta^{\beta'})^- = j_\theta$ and $i_\theta^{\beta'} < (\delta_\theta^{\beta'})^- = j_\theta$, so we can apply Lemma 3.36, and thus each condition q in D^*_α can be extended to $q' \in D^*_\alpha$ with $q'^{[j_\theta]}$ deciding the color $c_{\underset{\sim}{M}_{\beta'}}(i', i_\theta^{\beta'})$.

So enumerating all possible pairs (β', i') (that are as in $(\blacktriangle)_2$) and recalling $(*)_9$ we infer that

$(\blacktriangle)_3$ for some $q^* \geq q_\theta$ the condition $q^{*[j_\theta]} \restriction \beta' \in D^*_\alpha$ decides the color $c_{\underset{\sim}{M}_{\beta'}}(i', i_\theta^{\beta'})$ for all such pairs from $\{(\beta', i') : \ \beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1), \ \exists j \ [f_{\beta'}(i') = j] \in q_\theta\}$,
$(\blacktriangle)_4$ repeat this for pairs in $\{(\beta', i') : \ \exists j \ [f_{\beta'}(i') = j] \in q^{*[j_\theta]}\}$, and let $q^{**} \in D^*$ be the condition obtained after countable many such steps,

so

$(\blacktriangle)_5$ the condition $q^{**} \in D^*_\alpha$, $q^{**} \geq q_\theta$ with $q^{**[j_\theta]} \restriction \beta'$ deciding the color $c_{\underset{\sim}{M}_{\beta'}}(i', i_\theta^{\beta'})$ for all $(\beta', i') \in \{(\beta', i') : \ \beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1), \ \exists j \ [f_{\beta'}(i') = j] \in q^{**[j_\theta]}(\beta')$, Finally recall that by $(\bullet)_4$ $q_\theta(1) = q_\theta^{[j_\theta]}(1)$, and for each $\beta' \in \mathrm{dom}(q_\theta) \setminus S^*$ then $q_\theta(\beta') \setminus q_\theta^{[j_\theta]}(\beta')$ can only be non-empty if $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ (and if it is indeed non-empty then it is a singleton $[j_\theta \notin \mathrm{ran}(f_{\beta'})]$ or $[f_{\beta'}(i) = j_\theta]$, where $(\delta_\theta^{\beta'})^- < j_\theta$).

$(\blacktriangle)_6$ This means that after possibly replacing $q_\theta^{**}(\beta')$ by $q^{**[j_\theta]}(\beta') \cup q_\theta(\beta')$ using $(\blacktriangle)_5$ it is easy to see that we get a condition $q^{**} \in D_\alpha^*$ (which still satisfies both $(\bullet)_4$ and $(\blacktriangle)_5$).

Now we are at the position to construct the desired q_θ^+ as an extension of q^{**}. (In order to include the symbols listed in $(*)_a$ and $(*)_b$ for β''s with $(\delta_\theta^{\beta'})^- = j_\theta$, but constructing a proper condition in D_α^*), our task is to determine the color $\nu(j^*, j_\theta) = c_{M_*}(j^*, j_\theta)$ (i.e. add $[j^* \ R_{\nu(j^*,j_\theta)} \ j_\theta]$ to $q^{**}(1)$) for each j^* and β' such that

- $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$,
- and for some i^* $[f_{\beta'}(i^*) = j^*] \in q^{**[j_\theta]}(\beta')$,

so that $\nu(j^*, j_\theta) = c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'})$ (this latter value is the color forced by $q^{**[j_\theta]} \restriction \beta'$ by $(\blacktriangle)_5$). Then adding also the symbols $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ will work.

So fix a pair j^*, j_θ as above. Now we will make use of the preparations $(\odot)_9$ and $(\bullet)_1$ and show that there are no contradicting demands concerning the value of $\nu(j^*, j_\theta)$. We distinguish the following cases.

Case 1. For some $\nu^* < \kappa$ we have $[j^* \ R_{\nu^*} \ j_\theta] \in p_0(1) \cup p_1(1)$. Then necessarily $j^* = j_\eta$ for some $\eta < \theta$, and the only option is to

$$\text{put } [j_\eta \ R_{\nu^*} \ j_\theta] \in q_\varepsilon^+(1), \tag{3.27}$$

i.e. define $\nu(j_\eta, j_\theta) = \nu^*$. Note that this implies $j_\eta, j_\theta \in L_0$. Pick an arbitrary $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ satisfying $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ and for some i^* $[f_{\beta'}(i^*) = j_\eta] \in q^{**}(\beta')$.

If $\beta' \in \mathrm{dom}(p_0)$, then by $(\odot)_9/(\mathrm{v})$, which implies that both $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta], [f_{\beta'}(i^*) = j_\eta] \in p_0(\beta')$, so by $(\odot)_9/(\mathrm{i})$ $p_0^{[j_\theta]} \restriction \beta'$ forces a value to $c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'})$. Hence, $q^{**[j_\theta]} \restriction \beta' \geq q_\theta^{[j_\theta]} \restriction \beta' \geq p_0^{[j_\theta]} \restriction \beta'$ forces the same value for $c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'})$ (by our hypothesis on q_θ $(\bullet)_3$), which is ν^*.

Otherwise, assume that $\beta' \notin \mathrm{dom}(p_0)$ (so necessarily $\beta' \in \mathrm{dom}(p_1)$ and $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_1(\beta')$, and $j_\theta \in L_1$). Then again (by our construction and $(\bullet)_1/(\bullet)_{1a}$) the only way that $[f_{\beta'}(i^*) = j_\eta] \in q_\theta$ can happen for some i^* is when $[f_{\beta'}(i^*) = j_\eta] \in p_1(\beta')$, but then $(\odot)_9/(\mathrm{iv})$ implies that $j_\eta \in L_1$, so $[j_\eta \ R_{\nu^*} \ j_\theta] \in p_1(\beta')$ is a member of $p_1(\beta')$, too, and then we can proceed as in the case above (i.e. arguing that $p_1^{[j_\theta]} \restriction \beta' \Vdash c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'}) = \nu^*$).

Case 2. For no $\nu^* < \kappa$ do we have $[j^* \ R_{\nu^*} \ j_\varepsilon] \in p_0(1) \cup p_1(1)$.

Case 2A. $j^* = j_\eta$ for some $\eta < \theta$ (so by (ii) necessarily $|\{j_\eta, j_\theta\} \cap (L_0 \setminus L_1)| = |\{j_\eta, j_\theta\} \cap (L_1 \setminus L_0)| = 1$). We can assume that $j_\eta \in L_0 \setminus L_1$, $j_\theta \in L_1 \setminus L_0$. This means that

$(\blacktriangle)_7$ for no β' does there exist i such that $[f_{\beta'}(i) = j_\eta] \in p_1(\beta')$, and similarly, $[f_{\beta'}(i) = j_\theta] \in p_0(\beta')$ is impossible

by our assumption $(\odot)_9/(\mathrm{iv})$ on p_0 and p_1. So by $(\bullet)_1/(\bullet)_{1a}$ $[f_{\beta'}(i) = j_\eta] \in q_\theta(\beta')$ is only possible for any $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ if $[f_{\beta'}(i) = j_\eta] \in p_0(\beta') \cup p_1(\beta')$, so this case necessarily $[f_{\beta'}(i) = j_\eta] \in p_0(\beta')$. Summing up, for each β' with the prospective q_θ^+ forcing $j_\eta \in L_0 \setminus L_1$, $j_\theta \in L_1 \setminus L_0$ to be in the range of $f_{\beta'}$ the only possibility is that

$$[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_1(\beta'), \text{ and} \tag{3.28}$$

$$\text{for some } i^* \ [f_{\beta'}(i^*) = j_\eta] \in p_0(\beta'). \tag{3.29}$$

Now we argue that at most one such $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ may exist (then by $(\blacktriangle)_5$ we can put $[j^* \ R_{\nu^*} \ j_\varepsilon] \in q_\theta^+(\beta')$ with $\nu^* < \kappa$ defined by $q^{**[j_\theta]} \restriction \beta' \Vdash c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'}) = \nu^*$, and we are done).

So assume on the contrary, let $\beta' \neq \beta''$ be such that (3.28), (3.29) hold. Then clearly $\beta', \beta'' \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$, and $j_\theta, j_\eta \in \{\xi_{\beta'}(\rho+1) : \ \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \rho < \lambda\}$, then by our assumption (on all the p_γ's) $(\odot)_9$/(iii) contradicts $(\blacktriangle)_7$.

Case 2B. j^* is not of the form j_θ for any $\theta < \varepsilon$. This case we argue that at most one $\beta' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ could exist with $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$ satisfying that for some i^* $[f_{\beta'}(i^*) = j^*] \in q^{**}(\beta')$. (Then again by $(\blacktriangle)_5$ we can put $[j^* \ R_{\nu^*} \ j_\theta] \in q_\theta^+(\beta')$ with $\nu^* < \kappa$, $q^{**[j_\theta]} \restriction \beta' \Vdash c_{\underset{\sim}{M}_{\beta'}}(i^*, i_\theta^{\beta'}) = \nu^*$.)

So if there are $\beta' \neq \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ with

- $[f_{\beta'}(i^*) = j^*] \in q_\theta(\beta')$ for some i^*,
- $[f_{\beta''}(i^{**}) = j^*] \in q_\theta(\beta'')$ for some i^{**},
- $[f_{\beta'}(i_\theta^{\beta'}) = j_\theta] \in p_0(\beta') \cup p_1(\beta')$,
- $[f_{\beta''}(i_\theta^{\beta''}) = j_\theta] \in p_0(\beta'') \cup p_1(\beta'')$,

then again as in Case 2A we can get to an easy contradiction (i.e. $\beta', \beta'' \in \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$, and $j^* \in \{\xi_{\beta'}(\rho+1) : \ \rho < \lambda\} \cap \{\xi_{\beta''}(\rho+1) : \ \rho < \lambda\}$, hence $(\bullet)_1/(\bullet)_{1b}$ implies $[j^* \notin \mathrm{ran}(f_\beta)] \in p_0(\beta') \cap p_1(\beta')$, similarly for β''. Now recall $q^{**} \geq q_\theta$ and $(\bullet)_4$).

$\square_{\text{Subclaim 3.40}}$

$\square_{\text{Claim 3.39}}$

$\square_{\text{Claim 3.37}}$

$\square_{\text{Lemmas 3.34 and 3.35}}$

Having proven that $\mathbb{P}^1_\chi$ (and each $\mathbb{P}^1_\alpha$, $\alpha \leq \chi$) is the composition of a λ^+-cc and a κ^+-cc forcing, so itself λ^+-cc, we have $(\top)_3$. Moreover, recall Claim 3.25 and that $\mathbb{Q}^1_0 = Q(\lambda, \chi, \kappa)$, so $\mathbb{Q}^1_0$ does not collapse any cardinal, while $\mathbb{P}^1_\chi / \mathbf{G}^1_1$ is κ^+-cc, $< \kappa$-closed, so $\mathbb{P}^1_\chi$ being the composition of the forcings preserving cardinals itself does not collapse any cardinal, we get $(\top)_4$. An easy calculation yields the following.

Claim 3.41. For each $\alpha < \chi$ we have $\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi$. Therefore, up to equivalence $\mathbb{P}^1_\chi$ is of power χ.

Proof. For $\mathbb{P}^1_1 = \mathbb{Q}^1_0$ we already know $|\mathbb{Q}^1_1|$ by Observation 3.24. We have to prove the two statements simultaneously by induction on α. As $\mathbb{P}^1_\chi$ is a $< \kappa$-support iteration, and $\chi^{<\kappa} \leq \chi^\lambda = \chi$, by our premises it is enough to prove for the successor case. So for each $\alpha < \chi$ it is enough to show that $\mathbf{V}_1^{\mathbb{P}^1_\alpha} \models |\mathbb{Q}^1_\alpha| \leq \chi$. For $\alpha = 1$ as $\mathbb{Q}^1_1$ is a forcing of a κ-colored random graph on λ with conditions of size $< \kappa$ we get that $|\mathbb{Q}^1_1| = \lambda^{<\kappa} \leq \chi$ (in fact $|\mathbb{Q}^1_1| = \lambda$).

For α with $1 < \alpha \notin S^*$ (so Definition 3.28, Case 2). Again, each condition in $\mathbb{Q}^1_\alpha$ can be coded by a partial function of size $< \kappa$ on λ to $\lambda + 1$, so $|\mathbb{Q}^1_\alpha| = \lambda^{<\kappa} \leq \chi$.

Finally, for $\alpha \in S^*$ (Definition 3.28, Case 3), $\mathbb{Q}^1_\alpha = \mathbb{Q}_{D_\alpha}$ is the Mathias type forcing from Definition 3.1, where D_α is a system of subsets of $V_\kappa^{\mathbf{V}_1}$ generating a κ-complete filter, so $|\mathbb{Q}^1_\alpha| \leq (2^{|V_\kappa|})^{\mathbf{V}_1^{\mathbb{P}^1_\alpha}} = (2^\kappa)^{\mathbf{V}_1^{\mathbb{P}^1_\alpha}} \leq \chi$ (because $|\mathbb{P}^1_\alpha| = \chi$, $\mathbb{P}^1_\alpha$ is λ^+-cc, and we assumed $(\chi^\lambda)^{\mathbf{V}_1} = \chi$).

$\square_{\text{Lemma 3.41}}$

So now we are ready to complete the definition of $\mathbb{P}^1_\chi$ by prescribing the names $\underset{\sim}{D}_\delta$ ($\delta \in S^*$) and $\underset{\sim}{M}_\delta$ ($1 < \delta \notin S^*$), which are standard easy bookkeeping arguments (using $|\mathbb{P}^1_\chi| = \chi$ and the λ^+-cc), but for the sake of completeness we elaborate. This will prove $(\mathsf{T})_5$ and $(\mathsf{T})_6$, so complete the proof of Conclusion 3.20.

Claim 3.42. The system of $\underset{\sim}{D}_\delta$'s can be chosen so that for every $\mathbb{P}^1_\chi$-name $\underset{\sim}{D}$ with $\mathbf{V}_1 \Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{D} \in [\mathscr{P}(V_\kappa)]^{\leq\lambda}$ there exists a $\delta \in S^*$, such that for the $\mathbb{P}^1_\delta$-name $\underset{\sim}{D}_\delta$ we have $\Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{D} = \underset{\sim}{D}_\delta$.

Proof. It is obvious that by $\chi^\lambda = \chi$ (so $\mathrm{cf}(\chi) > \lambda$) and the λ^+-cc for every such $\underset{\sim}{D}$ there is a nice $\mathbb{P}^1_\delta$-name for some $\delta < \chi$. As forcing with the $< \kappa$-closed $\mathbb{P}^1_\chi$ does not add new elements to V_κ we get that for each δ there are $\chi^{\kappa\cdot\lambda} = \chi$-many such nice names. Also, as $|S^*| = \chi$ we can partition $S^* = \bigcup_{\alpha<\chi} S^*_\alpha$ with $S^*_\alpha \cap \alpha = \emptyset$, $|S^*_\alpha| = \chi$, we can let $\langle \underset{\sim}{D}_\delta : \ \delta \in S^*_\alpha \rangle$ list the nice names for subsets of $\mathscr{P}(V_\kappa)$. $\square_{\text{Claim 3.42}}$

A similar calculation yields the following.

Claim 3.43. The system of $\underset{\sim}{M}_\delta$'s can be chosen so that for every $\mathbb{P}^1_\chi$-name for a κ-colored graph $\underset{\sim}{M}$ on λ there exists a $1 < \delta \notin S^*$, such that for the $\mathbb{P}^1_\delta$-name $\underset{\sim}{M}_\delta$ we have $\Vdash_{\mathbb{P}^1_\chi} \underset{\sim}{M} = \underset{\sim}{M}_\delta$.

Proof. Easy. $\square_{\text{Claim 3.43}}$

$\square_{3.20}$

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