

\aleph_1 -FREE ABELIAN NON-ARCHIMEDEAN POLISH GROUPS

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ABSTRACT. An uncountable \aleph_1 -free group cannot admit a Polish group topology but an uncountable \aleph_1 -free abelian group can, as witnessed, for example, by the Baer-Specker group \mathbb{Z}^ω ; more strongly, \mathbb{Z}^ω is separable. In this paper we investigate \aleph_1 -free abelian non-Archimedean Polish groups. We prove two main results. The first is that there are continuum many separable (and so torsionless, and so \aleph_1 -free) abelian non-Archimedean Polish groups which are pairwise not topologically isomorphic. The second is that the following four properties are complete co-analytic subsets of the space of closed abelian subgroups of S_∞ : separability, torsionlessness, \aleph_1 -freeness and \mathbb{Z} -homogeneity.

1. INTRODUCTION

Recall that a topological group G is said to be Polish if its group topology is Polish, i.e., separable and completely metrizable, and it is said to be non-Archimedean Polish if, in addition, it admits a countable neighborhood basis at the identity consisting of open subgroups. In a meeting in Durham in 1997, D. Evans asked if a non-Archimedean Polish group can be an uncountable free group. Around the same time H. Becker and A. Kechris asked the same question without the assumption of non-Archimedeanity (cf. [1]). In [14] the second author settled the first question, and in the subsequent paper [15] he settled also the second question. It was later discovered that some of the impossibility results from [15] followed already from an old important result of Dudley [5]. All these results were later vastly generalized in [12] by both authors of the present paper in the context of graph products of groups. In another direction, Khelif proved in [10] that no uncountable \aleph_1 -free groups can be Polish, where we recall that a group G is said to be \aleph_1 -free (or \aleph_1 -free abelian) if every countable subgroup of G is free (or free abelian, respectively). Furthermore, an uncountable Polish group cannot be free abelian either, by [15]. In contrast, as well-known, the Baer-Specker group \mathbb{Z}^ω when topologized with the product topology is non-Archimedean Polish and \aleph_1 -free abelian (but not free abelian). This interesting state of affairs motivated us to investigate the structure of the class of \aleph_1 -free abelian non-Archimedean Polish groups, wondering in particular: how complicated is the class of these topological groups?

Recently, there has been significant interest in abelian (non-Archimedean) Polish groups. On one hand, with respect to the study of the Borel complexity of their actions, see e.g. [4], and on the other hand, in the context of homological algebra,

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see e.g. [2, 3]. Our study, which is algebraic in nature, complements these existing lines of inquiry by providing new structural algebraic insights into these groups.

We recall the following definitions as they will play a crucial role in our study:

Definition 1.1. Let A be a torsion-free abelian group.

- (1) We say that A is *separable* if every finite subset of A is contained in a free direct summand of A .
- (2) We say that A is *torsionless* if for every $0 \neq a \in A$ there is $f \in \text{Hom}(A, \mathbb{Z})$ such that $f(a) \neq 0$.
- (3) We say that A is \mathbb{Z} -*homogenous* if every element has type $\mathbf{0}$ (cf. 3.6).

It is well-known, see e.g. [6, Corollary 2.9], that we have the following:

$$\text{separable} \Rightarrow \text{torsionless} \Rightarrow \aleph_1\text{-free} \Rightarrow \mathbb{Z}\text{-homogenous}.$$

Now, it so happens that \mathbb{Z}^ω is not only \aleph_1 -free but actually separable. Thus, the motivating question of the paper mentioned above (on the existence of \aleph_1 -free abelian non-Archimedean Polish groups) can be strengthened to torsionless, or even separable, abelian groups. We will deal with all these properties in our paper.

The first result concerns the number of non-isomorphic groups realizing the four properties above, and it is of the strongest possible form. We recall that a completely decomposable abelian group is a group that is a direct sum of groups of rank 1.

Theorem 1.2. *There are continuum many separable (hence torsionless, hence \aleph_1 -free) abelian non-Archimedean Polish groups which are pairwise not topologically isomorphic. Furthermore, all these groups can be taken to be inverse limits of countable torsion-free completely decomposable groups.*

The second result that we present is formulated in the framework of invariant descriptive set theory (see [8] for an excellent introduction to this subject). In [9], Kechris, Nies and Tent proposed the following program: as any non-Archimedean Polish group is topologically isomorphic to a closed subgroup of the topological group S_∞ of bijections of \mathbb{N} onto \mathbb{N} , we can study classification problems on non-Archimedean Polish groups with respect to the Effros structure on the set of closed subgroups of S_∞ , which is a standard Borel space (see [9] for details).

At the heart this program there is the following two-fold task:

- for natural classes of closed subgroups of S_∞ , determine whether they are Borel;
- if a class \mathcal{C} is Borel, study the Borel complexity of topological isomorphism on \mathcal{C} .

In this respect, in our second theorem, we prove that the four properties mentioned above are not Borel and, more strongly, that they are complete co-analytic.

Theorem 1.3. *Determining if a non-Archimedean Polish group is in \mathcal{C} is a complete co-analytic problem in the space of closed subgroups of S_∞ for the following classes \mathcal{C} of abelian groups:*

- (1) \mathbb{Z} -homogeneous;
- (2) \aleph_1 -free;
- (3) torsionless;
- (4) separable.

We conclude the paper with a result of independent interest:

Theorem 1.4. *There exists an inverse limit of free abelian groups (and so an abelian non-Archimedean Polish group) which is \mathbb{Z} -homogeneous but not \aleph_1 -free.*

The structure of the paper is as follows. In Section 2 we collect the necessary preliminaries. In Section 3 we introduce the main group-theoretic construction at the heart of the paper. In Section 4 we prove Theorems 1.2 and 1.3. Finally, in Section 5 we establish some complementary results.

2. THE DESCRIPTIVE SET THEORY SETTING

From here on, as is customary in set theory, we denote the set of natural numbers by ω . The aim of this section is to fix the coding, i.e., to specify which spaces we refer to in our results, together with introducing the necessary background.

Fact 2.1 ([4, Proposition 2.1]). Let G be a topological group, then the following are equivalent:

- (1) G is a non-Archimedean Polish abelian group;
- (2) G is isomorphic (as a topological group) to a closed subgroup of $\prod_{n < \omega} H_n$ (with the product topology), with each H_n abelian, countable and discrete;
- (3) G is the inverse limit of an inverse system $(G_n, f_{(n,m)} : m \leq n < \omega)$, with each G_n abelian, countable, discrete, where on G we consider the product topology.

Notation 2.2. Let X be a Polish space. The Effros structure on X is the Borel space consisting of the family $\mathcal{F}(X)$ of closed subsets of X together with the σ -algebra generated by the following sets \mathcal{C}_U , where, for $U \subseteq X$ open, we let:

$$\mathcal{C}_U = \{D \in \mathcal{F}(X) : D \cap U \neq \emptyset\}.$$

Notation 2.3. We denote by S_∞ the topological group of bijections of ω onto ω , where the topology is the one induced by the following metric:

- (1) if $x = y$, then $d(x, y) = 0$;
- (2) if $x \neq y$, then $d(x, y) = 2^{-n}$, where n is least such that $x(n) \neq y(n)$.

Fact 2.4. The closed subgroups of S_∞ form a Borel subset of $\mathcal{F}(S_\infty)$ (see [9]), which we denote by $\text{Sgp}(S_\infty)$, together with the Borel structure inherited from $\mathcal{F}(S_\infty)$.

Definition 2.5. By a tree on ω we mean a non-empty subset of $\omega^{<\omega}$ closed under initial segments (so in particular the empty sequence \emptyset is the root of the tree).

Notation 2.6. Given a tree $T \subseteq \omega^{<\omega}$ we denote by $[T]$ the set of $\eta \in \omega^\omega$ such that $\eta \upharpoonright n \in T$ for all $n < \omega$, i.e., the set of infinite branches of T .

Fact 2.7. Let $\mathcal{T}(S_\infty)$ denote the set of trees on ω with no leaves topologized defining the distance between two trees $T_1 \neq T_2$ as $\frac{1}{2^n}$, where n is least n such that $T_1 \cap \omega^n \neq T_2 \cap \omega^n$. For $C \in \text{Sgp}(S_\infty)$, define $T_C \in \mathcal{T}(S_\infty)$ as follows:

$$\mathbf{B} : C \mapsto T_C = \{g \upharpoonright n : g \in C \text{ and } n < \omega\} \cup \{g^{-1} \upharpoonright n : g \in C \text{ and } n < \omega\}.$$

Then $\mathbf{B} : \text{Sgp}(S_\infty) \rightarrow \mathcal{T}(S_\infty)$ is Borel, it is injective and it has closed range; furthermore, when restricted to its range, the map \mathbf{B} is a Borel isomorphism.

By Fact 2.7, we can work with $\mathbf{B}(\text{Sgp}(S_\infty)) \subseteq \mathcal{T}(S_\infty)$ instead of $\text{Sgp}(S_\infty)$.

Proviso 2.8. *From now on we only look at inverse systems with onto bonding maps.*

Notation 2.9. We can look at an inverse system $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ of groups as a first-order structure in a language $L_{\text{inv}} = \{P_n, \cdot_n, f_{(n,m)} : m \leq n < \omega\}$, where:

- (1) the P_n 's are disjoint predicates denoting the groups A_n 's;
- (2) \cdot_n is a binary function symbol, interpreted as the group operation on A_n ;
- (3) the $f_{(n,m)}$'s are function symbols interpreted as morphisms from A_n to A_m .

Notation 2.10. Of course with respect to the language considered in 2.9 saying that a structure of the form $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ is an inverse system is axiomatizable in first-order logic. Thus, we can consider the Borel space of countable Pro-Groups with domain (a subset of) ω . We also assume that our inverse systems $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ are such that for $n \neq m$, $\text{dom}(A_n) \cap \text{dom}(A_m) = \emptyset$. Finally, naturally when the groups A_n are abelian, then we move to additive notation and so we write $+_n$ instead of \cdot_n and 0_{A_n} instead of e_{A_n} .

Definition 2.11. Let $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ be an inverse system as in 2.10. For every $\bar{a} = (a_n : n < \omega) \in \varprojlim(A)$ we define a permutation $\pi_{\bar{a}}$ of ω as:

$$\pi_{\bar{a}}(k) = \begin{cases} k +_{A_n} a_n, & \text{if there is } n < \omega \text{ s.t. } k \in A_n, \\ k, & \text{if } k \notin \bigcup_{n < \omega} A_n. \end{cases}$$

Observation 2.12. The map $\bar{a} \mapsto \pi_{\bar{a}}$ from 2.11 is an embedding of $\varprojlim(A)$ into S_∞ .

We omit the details of the next proposition as its proof is standard (use e.g. 2.7).

Proposition 2.13. *The map $A \mapsto \varprojlim(A)$ from Pro-Groups (recalling 2.10) into $\text{Sgp}(S_\infty)$ (recalling 2.4 and the embedding described in 2.11) is Borel.*

3. A FEW CRUCIAL CLAIMS ON TFAB'S

3.1. The engine

The following construction will be used to prove Theorems 1.2 and 1.3.

Notation 3.1. \mathbb{P} denotes the set of prime numbers.

Definition 3.2. Let $T \subseteq \omega^{<\omega}$ be a tree with no leaves and $L : T \rightarrow \mathcal{P}(\mathbb{P})$ be such that:

- (a) if $\eta \trianglelefteq \nu$, then $L(\nu) \subseteq L(\eta)$;
- (b) if $\eta \in T$, then $L(\eta) = \bigcup \{L(\nu) : \nu \in T \text{ and } \nu \text{ is a successor of } \eta\}$.

We define an inverse system $\text{inv}(T, L) = (G_n, f_{(m,n)} : m \leq n < \omega)$ as follows:

- (i) $Z_n := \sum \{x_\eta : \eta \in T \cap \omega^n\} \leq G_n \leq \sum \{\mathbb{Q}x_\eta : \eta \in T \cap \omega^n\} := Q_n$;
- (ii) $G_n = \langle \frac{1}{p}x_\eta : \eta \in T \cap \omega^n \text{ and } p \in L(\eta) \rangle_{Q_n}$;
- (iii) for $m \leq n < \omega$, $f_{(n,m)}$ is the unique homomorphism from G_n into G_m s.t.:

$$\eta \in T \cap \omega^n \Rightarrow x_\eta \mapsto x_{\eta \upharpoonright m}.$$

Notice that the inverse system $\text{inv}(T, L) = (G_n, f_{(n,m)} : m \leq n < \omega)$ is well-defined because of conditions (a) and (b) above. It being well-defined, we do the following:

$$(T, L) \mapsto \text{inv}(T, L) \mapsto \varprojlim(\text{inv}(T, L)) := G(T, L).$$

So the group $G(T, L)$ is a well-defined Polish non-Archimedean abelian group, when equipped with the topology inherited from the product topology on $\prod_{n < \omega} G_n$.

Remark 3.3. Notice that the inverse systems $\text{inv}(T, L) = (G_n, f_{(n,m)} : m \leq n < \omega)$ defined in 3.2 have bonding maps $f_{(n,m)}$ which are onto; this is because we ask that the tree T has no leaves and because we ask that condition (b) from there holds. Recall that in 3.2 we only consider trees with no leaves and so, for T 's as in 3.2, the set $[T]$ is always non-empty (cf. 2.6 for a definition of $[T]$).

Remark 3.4. Notice also that for $\text{inv}(T, L) = (G_n, f_{(n,m)} : m \leq n < \omega)$ as in 3.2 we have that the groups G_n are completely decomposable, i.e., they are direct sums of groups of rank 1, in fact it follows immediately from the definitions that:

$$G_n = \sum_{\eta \in \omega^n \cap T} \langle x_\eta \rangle_{G_n}^*,$$

where $\langle X \rangle_{G_n}^*$ denotes pure closure (cf. [7, pg. 151]).

For background on the notions that we now introduce see e.g. [7, Chapter 13]. We will often abbreviate ‘‘torsion-free abelian group’’ with TFAB.

Definition 3.5. Let $A \in \text{TFAB}$ and let $(p_i : i < \omega)$ be the list of the prime numbers in increasing order. For $a \in A$, we define the characteristic of a in A , denoted as $\chi_A(a) = \chi(a)$, as follows: $\chi(a) = (h_{p_i}(a) : i < \omega)$, where $h_{p_i}(a)$ is the supremum of the $k < \omega$ such that $p_i^k \mid a$, where the supremum is taken in $\omega \cup \{\infty\}$ (so the value ∞ is allowed).

Definition 3.6. (1) Two characteristics $(k_i : i < \omega)$ and $(\ell_i : i < \omega)$ are said to be equivalent if $k_i = \ell_i$ for almost all $i < \omega$ and both k_i and ℓ_i are finite whenever $k_i \neq \ell_i$. The equivalence classes of characteristics are called *types*.
(2) Given $A \in \text{TFAB}$ we denote by $\mathbf{t}_A(a)$ the type of a in A ; we may simply write $\mathbf{t}(a)$ when A is clear from the context.
(3) We denote by $\mathbf{0}$ the type of the characteristic $(0, 0, \dots)$.
(4) For $A \in \text{TFAB}$ and $a, b \in A$ we say that $\mathbf{t}(a) \leq \mathbf{t}(b)$ if there are characteristics $\chi = (k_i : i < \omega) \in \mathbf{t}(a)$ and $\nu = (\ell_i : i < \omega) \in \mathbf{t}(b)$ such that $(k_i : i < \omega) \leq (\ell_i : i < \omega)$, where the order is the pointwise order, i.e., $k_i \leq \ell_i$, for all $i < \omega$.

We will need the following well-known facts.

Fact 3.7 ([7, (C) on pg. 411]). Let G and H be torsion-free abelian and suppose that there is a homomorphism $f : G \rightarrow H$, then, for every $g \in G$, $\mathbf{t}(g) \leq \mathbf{t}(f(g))$.

Fact 3.8. [6, Theorem 2.3] Let $A \in \text{TFAB}$. Then A is \aleph_1 -free if and only if, for every finite $S \subseteq A$, the pure closure of S in A (denoted $\langle S \rangle_A^*$) is free abelian.

3.2. Nice pairs

We now present a technical definition necessary for our approach to work, the fact that this or similar technical adjustments are necessary is justified by Section 5.2.

Definition 3.9. We call (T, L) a nice pair when for some S and \bar{p} we have that:

- (a) S is a subtree of T and T is a subtree of $\omega^{<\omega}$ with no leaves;
- (b) $\eta \in S$ implies that $0 \notin \text{ran}(\eta)$;
- (c) $\bar{p} = (p_\eta : \eta \in S)$ is a sequence of primes without repetitions;
- (d) $S^+ = \{\eta \frown \nu : \eta \in S, \nu \in \{0\}^{<\omega}\} \subseteq T$;
- (e) $p_{\eta \frown \nu} = p_\eta$, when $\eta \in S$ and $\nu \in \{0\}^{<\omega}$;
- (f) L is the function with domain T defined as follows:

$$L(\eta) = \{p_\nu : (\nu \triangleleft \eta \text{ or } \eta \triangleleft \nu) \text{ and } \nu \in S^+\}.$$

Observation 3.10. If (T, L) a nice pair, then (T, L) is as in 3.2.

Convention 3.11. For the rest of this subsection, we fix (T, L) , S , S^+ and \bar{p} as in 3.9 and we let $G = G(T, L)$. For $\eta \in [T]$, we let $L(\eta) = \bigcap \{L(\eta \upharpoonright n) : n < \omega\}$. Also, for $n < \omega$, we let $f_{(\omega, n)}$ be the obvious projection of G onto G_n . For $\nu \in [T]$, we let $x_\nu = (x_{\nu \upharpoonright n} : n < \omega) \in G$. Recall also that $f_{(n, m)}$ denote the bonding map from G_n onto G_m .

Claim 3.12. If S has an ω -branch (i.e., there is $\eta \in [T]$ such that, for all $n < \omega$, $\eta \upharpoonright n \in S$), then G is not \mathbb{Z} -homogeneous, and so G is not \aleph_1 -free.

Proof. In this case $L(\eta) = \bigcap \{L(\eta \upharpoonright n) : n < \omega\} = \{p_{\eta \upharpoonright n} : n < \omega\}$ is infinite, and so in G we have that x_η is divisible by p for every $p \in L(\eta)$. \blacksquare

Definition 3.13. For $\eta \in T \cup [T]$, let K_η be the additive subgroup of \mathbb{Q} generated by $\{\frac{1}{p} : p \in L(\eta)\} \subseteq \mathbb{Q}$; explicitly, K_η is the subgroup of \mathbb{Q} consisting of:

$$\left\{ \frac{b}{\prod_{\ell < k} p_\ell} : b \in \mathbb{Z}, k < \omega, p_\ell \in L(\eta) \right\}.$$

Main Claim 3.14. If S has no ω -branch, then $G = G(T, L)$ is separable (and so G is torsionless, and so G is \aleph_1 -free).

Proof. We show that G is separable. To this end, let $X \subseteq G$ be finite.

(*₁) We claim that it suffices to prove that there are $n < \omega$ and finite $Y \subseteq \omega^n \cap T$ such that letting $\pi_{(n, Y)}$ be the natural projection of G_n onto $G_{(n, Y)} := \langle Y \rangle_{G_n}^*$ we have:

- (a) for every $\nu \in Y$, $L(\nu)$ is finite;
- (b) $g = \pi_{(n, Y)} \circ f_{(\omega, n)}$;
- (c) $g \in \text{Hom}(G, G_{(n, Y)})$;
- (d) $g_0 := g \upharpoonright \langle X \rangle_G^*$ is an isomorphism from $\langle X \rangle_G^*$ onto $G_{(n, Y)} = \langle Y \rangle_{G_n}^*$.

We show that (*₁) holds, i.e., that the conditions listed there suffice to conclude that G is separable. For this purpose, first of all, notice that $G_{(n, Y)}$ is a free direct summand of G , because $G_n = \bigoplus \{K_\nu x_\nu : \nu \in T \cap \omega^n\}$ and, for every $\nu \in Y$, $L(\nu)$ is finite and so $K_\nu \cong \mathbb{Z}$. Secondly, observe that then also $\langle X \rangle_G^*$ is free, since by (c), g_0 is an isomorphism. Observe now that, by clause (b), we have that $g_0^{-1} \circ g$ is an homomorphism from G into $\langle X \rangle_G^*$. Also, by clause (c), clearly g_0^{-1} is an isomorphism from $G_{(n, Y)}$ onto $\langle X \rangle_G^*$. Notice now that for $a \in \langle X \rangle_G^*$ we have that:

$$g_0^{-1} \circ g(a) = g_0^{-1}(g(a)) = g_0^{-1}(g_0^{-1}(a)) = a.$$

Hence, we have that $g_0^{-1} \circ g$ is a retraction of G onto the free subgroup $\langle X \rangle_G^*$ and so X is contained in a free direct summand of G , as desired. Hence, (*₁) holds indeed.

So now that we have showed that (*₁) suffices, for the rest of the proof we will focus on showing that the conditions from (*₁) can be fulfilled (cf. the final (*₁₉)).

(*₂) For every $n < \omega$, $f_{(n+1, n)}$ maps $f_{(\omega, n+1)}(\langle X \rangle_G^*)$ onto $f_{(\omega, n)}(\langle X \rangle_G^*)$, hence:

- (₁) $\text{rk}(f_{(\omega, n+1)}(\langle X \rangle_G^*)) \geq \text{rk}(f_{(\omega, n)}(\langle X \rangle_G^*))$;
- (₂) those ranks are $\leq |X|$ which is finite.

(*₃) For some $n_* < \omega$, $(\text{rk}(f_{(\omega, m)}(\langle X \rangle_G^*)) : m \geq n_*)$ is constant, call it k_* , and for every $n_2 > n_1 \geq n_*$, $f_{(n_2, n_1)} \upharpoonright f_{(\omega, n_2)}(\langle X \rangle_G^*)$ is an isomorphism from $f_{(\omega, n_2)}(\langle X \rangle_G^*)$ onto $f_{(\omega, n_1)}(\langle X \rangle_G^*)$.

[Why? As $n_2 > n_1 \geq n_*$ and so $\text{rk}(f_{(\omega, n_2)}(\langle X \rangle_G^*)) = \text{rk}(f_{(\omega, n_1)}(\langle X \rangle_G^*))$, recalling that $f_{(n_2, n_1)}$ maps $f_{(\omega, n_2)}(\langle X \rangle_G^*)$ onto $f_{(\omega, n_1)}(\langle X \rangle_G^*)$, we have that the map $f_{(n_2, n_1)} \upharpoonright f_{(\omega, n_2)}(\langle X \rangle_G^*)$ has trivial kernel and so it is 1-to-1, and trivially it is also onto.]

(*₄) If $n \geq n_*$, then $f_{(\omega, n)} \upharpoonright \langle X \rangle_G^*$ is an isomorphism from $\langle X \rangle_G^*$ onto $f_{(\omega, n)}(\langle X \rangle_G^*)$.

Why (*₄) holds? Let $n_* \leq n < \omega$ and suppose the negation of the thesis. Then there is $y \in \langle X \rangle_G^* \setminus \{0\}$ such that $f_{(\omega, n)}(y) = 0$. Let $y = (y_m : m < \omega)$. Then we can find $k \geq n$ such that $f_{(\omega, k)}(y) = y_k \neq 0$. But then $y_k \in G_k \setminus \{0\}$ and $f_{(k, n)}(y_k) = 0$, and so $f_{(k, n)}$ has non-trivial kernel, contradicting the choice of n_* (recall that by hypothesis $n \geq n_*$ and that $f_{(k, n)} : G_k \rightarrow G_n$ is onto G_n). So (*₄) holds indeed.

(*₅) W.l.o.g. $|X| = k_*$, equivalently, X is independent.

(*₆) For $n < \omega$.

(a) Let $Y_n \subseteq T \cap \omega^n$ be finite and such that we have:

$$f_{(\omega, n)}(\langle X \rangle_G^*) \subseteq G_{(n, Y_n)} = \bigoplus \{K_\nu x_\nu : \nu \in Y_n\}.$$

(b) For $y \in X$, let $f_{(\omega, n)}(y) = \sum \{a_{(y, n, \nu)} x_\nu : \nu \in Y_n\}$, where $a_{(y, n, \nu)} \in K_\nu$.

(c) W.l.o.g. for every $n \geq n_*$ and $\nu \in Y_n$, there is $y \in X$ such that $a_{(y, n, \nu)} \neq 0$.

[Why? Obvious.]

(*₇) Let $Y_n^* = \{\nu \in Y_n : L(\nu) \text{ is finite}\}$.

(*₈) Recalling the definition of $\pi_{(n, Y_n^*)}$ from (*₁), we have that the sequence

$$(\text{rk}(\pi_{(n, Y_n^*)} \circ f_{(\omega, n)}(\langle X \rangle_G^*)) : n \geq n_*)$$

is eventually constant, say for $n \geq n_*$, and let k_* be this constant value.

(*₉) Assume $n_* \leq n_1 \leq n_2 < \omega$, then:

(·₁) if $\nu \in Y_{n_1}$, then there is $\rho \in Y_{n_2}$ such that $\nu \triangleleft \rho$;

(·₂) if $\nu \in Y_{n_1}^*$ and $\nu \triangleleft \rho \in Y_{n_2}$, then $\rho \in Y_{n_2}^*$ ($Y_{n_1}^*$ was defined in (*₇)).

[Why (*₉)(·₁)? By (*₆)(c).]

(*₁₀) W.l.o.g. $n_* = n_*$ (we shall use this freely).

(*₁₁) Let $(y_\ell : \ell < k_*)$ be a list of $|X|$ (recall (*₅)).

(*₁₂) For every $n < \omega$ and $\ell < k_*$, let:

$$y_{(\ell, n)} = \pi_{(n, Y_n^*)} \circ f_{(\omega, n)}(y_\ell),$$

where $\pi_{(n, Y_n^*)} : G_n \rightarrow \langle Y_n^* \rangle_{G_n}^*$ is the obvious projection onto $\langle Y_n^* \rangle_{G_n}^*$ (as in (*₁)).

(*₁₃) Hence, we have:

$$y_{(\ell, n)} = \sum \{a_{(y_\ell, n, \nu)} x_\nu : \nu \in Y_n^*\}.$$

(*₁₄) For $n \geq m \geq n_*$, let:

(a) $Y_{(n, m)}^* = \{\nu \in Y_n^* : \nu \upharpoonright m \in Y_m^*\}$;

(b) $y_{(\ell, n, m)} = \sum \{a_{(y_\ell, n, \nu)} x_\nu : \nu \in Y_{(n, m)}^*\}$.

(*₁₅) For $n \geq m \geq n_*$, $y_{(\ell, n, m)} = \pi_{(n, Y_{(n, m)}^*)}(y_\ell)$.

(*₁₆) $k_* = k_*(= |X|)$ (recall (*₅) and (*₈)).

We prove (*₁₆). Toward contradiction, assume that $k_* < k_*$. For $n \geq n_*$, the sequence $(y_{(\ell, n)} : \ell \leq k_*)$ cannot be independent, so for some $\ell(n) \leq k_*$ the element $y_{(\ell(n), n)}$ depends on $(y_{(\ell, n)} : \ell \neq \ell(n), \ell \leq k_*)_{G_n}$. But if $n_2 \geq n_1 \geq n_*$, then using $f_{(n_2, n_1)}$ and $y_{(\ell, n_2)}, y_{(\ell, n_2, n_1)}$ we get that $\ell(n_2)$ can serve as $\ell(n_1)$, and so w.l.o.g. $\ell(n) = \ell(n_*)$ for all $n \geq n_*$, so renaming w.l.o.g. we have that $\ell(n) = k_*$.

- (*16.1) Let $\sum\{b_\ell^n y_{(\ell,n)} : \ell \leq k_\star\} = 0$, where, for $\ell \leq k_\star$, $b_\ell^n \in \mathbb{Z}$ and $b_{k_\star}^n \neq 0$.
- (*16.2) W.l.o.g. for every $n_2 \geq n_1 \geq n_\star$ we have that, for every $\ell \leq k_\star$, $b_\ell^{n_2} = b_\ell^{n_1}$, call this number b_ℓ , so that in particular $b_{k_\star} \neq 0$.
- (*16.3) However, in G we have that $z := \sum\{b_\ell y_\ell : \ell \leq k_\star\} \neq 0$ and $z \notin \langle y_\ell : \ell < k_\star \rangle_G^*$, so there is $n_+ > n_\star$ such that $f_{(\omega, n_+)}(z) \neq 0$, so we can find $\nu \in Y_{n_+}$ such that $a_{(z, n_+, \nu)} \neq 0$.

(*16.4) $\nu \notin Y_{n_+}^*$.

[Why? Because $\pi_{(n_+, Y_{n_+}^*)} \circ f_{(\omega, n_+)}(z) = \sum\{b_\ell^n y_{(\ell,n)} : \ell \leq k_\star\} = 0$ by (*16.1).]

(*16.5) There are n_\dagger and ρ such that:

- (a) $n_\dagger \geq n_+ \geq n_\star$;
- (b) $\rho \in Y_{n_\dagger}$ and $\nu \triangleleft \rho$ (where ν is as in (*16.3), (*16.4));
- (c) $\rho \in Y_{n_\dagger}^*$;
- (d) $a_{(z, n_\dagger, \rho)} \neq 0$.

Why does (*16.5) hold? We choose ν_n by induction on $n \geq n_+$ such that:

- (a) $\nu_{n_+} = \nu \in Y_{n_+}$;
- (b) $\nu_n \in Y_n$;
- (c) if $n = m + 1 > n_+$, then $\nu_m \triangleleft \nu_n$;
- (d) $a_{(z, n, \nu_n)} \neq 0$.

As in the beginning of the proof we can carry the induction. Now, as we are assuming that S has no ω -branch, clearly for some $n_\dagger > n_+$ we have that $\nu_{n_\dagger} \notin S$, but then necessarily $L(\nu_{n_\dagger})$ has to be finite, i.e., $\nu_n \in Y_{n_\dagger}^*$, since by assumption S is a subtree of T and we defined, for $\eta \in T$, $L(\eta) = \{p_\nu : (\nu \triangleleft \eta \text{ or } \eta \triangleleft \nu) \text{ and } \nu \in S^+\}$.

(*16.6) $\pi_{(n_\dagger, Y_{n_\dagger}^*)} \circ f_{(\omega, n_\dagger)}(z) \neq 0$.

[Why? By (*16.5)(d).]

(*16.7) $\pi_{(n_\dagger, Y_{n_\dagger}^*)} \circ f_{(\omega, n_\dagger)}(z) = 0$.

[Why? Notice that:

$$\begin{aligned} \pi_{(n_\dagger, Y_{n_\dagger}^*)} \circ f_{(\omega, n_\dagger)}(z) &= \pi_{(n_\dagger, Y_{n_\dagger}^*)} \circ f_{(\omega, n_\dagger)}(\sum\{b_\ell y_\ell : \ell \leq k_\star\}) \\ &= \sum\{b_\ell^{n_\dagger} y_{(n_\dagger, \ell)} : \ell \leq k_\star\} \\ &= \sum\{b_\ell y_{(n_\dagger, \ell)} : \ell \leq k_\star\} \\ &= 0 \end{aligned}$$

where the first equality is by the definition of z in (*16.3); the second equality is by (*12), the third equality is by (*16.2) and the fourth equality is by (*16.1).]

But (*16.7) contradicts (*16.6). Hence, (*16) holds indeed.

(*17) (a) Let $f_{(\omega, n)}^* = \pi_{(n, Y_n^*)} \circ f_{(\omega, n)}$;

(b) $f_{(\omega, n)}^*$ is a homomorphism from G into $\langle Y_n^* \rangle_{G_n}^* \leq G_n$.

(*18) If $n \geq n_\star$, then $g_0 = f_{(\omega, n)}^* \upharpoonright \langle X \rangle_G^*$ is 1-to-1 from $\langle X \rangle_G^*$ onto $\langle Y_n^* \rangle_{G_n}^*$.

[Why? By (*16) and (*17), as in the proof of (*4).]

(*19) The conditions from (*1) holds and so the proof is complete.

[Why? Let $n \geq n_\star$ and let Y from (*1) be Y_n^* , then by (*18) we are done.] ■

4. PROOF OF MAIN THEOREMS

In this section we show how to use our machinery to prove Theorems 1.2, 1.3.

4.1. Proof of Theorem 1.2

Definition 4.1. Let S be the following well-founded tree. For $n < \omega$, let T_n be a copy of $\omega^{<n}$ such that for $n < m < \omega$ we have that $T_n \cap T_m = \emptyset$. Let S be the tree obtained adding a minimum x_* to $\bigcup\{T_n : n < \omega\}$, so x_* is the root of the tree S . Easily, S can be realized as a subtree of $\omega^{<\omega}$ satisfying condition 3.9(2), and S is well-founded. Fix an enumeration of S of order type ω , say $(\eta_i : i < \omega)$. Let \mathbb{P}_* be an infinite set of primes, enumerate \mathbb{P}_* in increasing order as $(p_i^{\mathbb{P}_*} : i < \omega)$ and let $p_{\eta_i} = p_i^{\mathbb{P}_*}$. Then S and $\bar{p} = (p_{\eta_i} : i < \omega)$ are as in Definition 3.9. W.l.o.g. we assume that if η is not the root of the tree, then $\eta(0) = n$ if and only if $\eta \in T_n$.

Observation 4.2. By 3.14, for $(\omega^{<\omega}, L_{\mathbb{P}_*})$ as in 4.1, $G(\omega^{<\omega}, L_{\mathbb{P}_*})$ is \aleph_1 -free.

Lemma 4.3. Let \mathbb{P}_1 and \mathbb{P}_2 be infinite sets of primes such that $\mathbb{P}_1 \cap \mathbb{P}_2$ is finite, and let $(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ be as in 4.1 and $G_\ell = G(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ (recall 3.2). Then there is no topological isomorphism from G_1 onto G_2 .

Proof. Let $A = G_1 = \varprojlim(G_{(1,n)})$ and $B = G_2 = \varprojlim(G_{(2,n)})$, where the $G_{(\ell,n)}$'s are from the inverse system $\text{inv}(\omega^{<\omega}, L_{\mathbb{P}_\ell})$, of course. Let $G_{(1,n)} = A_n$ and $G_{(2,n)} = B_n$. Also, for $n \leq m < \omega$, denote the bonding map from A_n onto A_m by $t_{(n,m)}$, and the bonding map from B_n onto B_m by $s_{(n,m)}$. Suppose that there is a topological isomorphism from A onto B , then, there are ϕ, ψ, f_n and g_n , for $n < \omega$, such that:

- (a) $\phi : \omega \rightarrow \omega$ is increasing;
- (b) $\psi : \omega \rightarrow \omega$ is increasing;
- (c) $f_n : A_{\phi(n)} \rightarrow B_n$;
- (d) $g_n : B_{\psi(n)} \rightarrow A_n$;
- (e) for every $n \leq k < \omega$, the following diagram is commutative:

$$\begin{array}{ccc} A_{\phi(k)} & \xrightarrow{f_k} & B_k \\ \downarrow t_{(\phi(k), \phi(n))} & & \downarrow s_{(k,n)} \\ A_{\phi(n)} & \xrightarrow{f_n} & B_n \end{array}$$

- (f) for every $n \leq k < \omega$, the following diagram is commutative:

$$\begin{array}{ccc} B_{\psi(k)} & \xrightarrow{g_k} & A_k \\ \downarrow s_{(\psi(k), \psi(n))} & & \downarrow t_{(k,n)} \\ B_{\psi(n)} & \xrightarrow{g_n} & A_n \end{array}$$

- (g) for every $n < \omega$, $f_n \circ g_{\phi(n)} : B_{\psi(\phi(n))} \rightarrow B_n$ is equal to $s_{(\psi(\phi(n)), n)}$;
- (h) for every $n < \omega$, $g_n \circ f_{\psi(n)} : A_{\phi(\psi(n))} \rightarrow A_n$ is equal to $t_{(\phi(\psi(n)), n)}$.
- (*₁) If $\eta \in \omega^{\phi(k)+1}$ is not maximal in S , then $f_k(x_\eta) = 0_{B_k}$.

[Why (*₁)? By our choice η is not maximal in S , hence we have that:

$$A_{\phi(k)+1} \models \text{“}x_\eta \text{ is divisible by infinitely many primes in } \mathbb{P}_1\text{”},$$

but since $\mathbb{P}_1 \cap \mathbb{P}_2$ is finite, necessarily $f_k(x_\eta) = 0_{B_k}$, by 3.7.]

Let $n = \phi(\psi(0))$ and let $\eta \in \omega^{n+1}$ be such that η is not maximal in the tree S . Then by (*₁) we have that $f_{\psi(n)}(x_\eta) = 0_{B_{\psi(n)}}$ and so necessarily $g_0 \circ f_{\psi(0)}(x_\eta) = 0_{A_0}$, but this contradicts item (h) above, in fact, since $g_n \circ f_{\psi(n)} : A_{\phi(\psi(n))} \rightarrow A_n$ is equal to $t_{(\phi(\psi(n)), n)}$ we also have that $g_0 \circ f_{\psi(0)}(x_\eta) = x_\theta$ and $x_\theta \neq 0_{A_0}$, a contradiction. ■

Proof of 1.2. Relying on 4.3, it suffices to take a collection $\{\mathbb{P}_\alpha : \alpha < 2^{\aleph_0}\}$ of almost disjoint subsets of the set of prime numbers, where A, B are almost disjoint if $A \cap B$ is finite. \blacksquare

4.2. Proof of Theorem 1.3

Definition 4.4. Let $A \subseteq \omega^{<\omega}$ be an infinite tree (in this case, crucially, possibly with leaves). For $\eta \in A$, let $\eta^{+1} = (\eta(n) + 1 : n < \omega)$ and let $A^{+1} = S = \{\eta^{+1} : \eta \in A\}$. Let $\bar{p} = \{p_\eta : \eta \in S\}$ and $T = \omega^{<\omega}$. And let $G(\omega^{<\omega}, L_A)$ be defined as in 3.9 for this choice of S and \bar{p} .

Lemma 4.5. For $(\omega^{<\omega}, L_A)$ as in 4.4, letting $G_A = G(\omega^{<\omega}, L_A)$ we have:

- (1) if A is well-founded, then G_A is separable, and so torsionless, and so \aleph_1 -free, and so \mathbb{Z} -homogeneous;
- (2) if A is not well-founded, then G_A is not \mathbb{Z} -homogeneous, and so not \aleph_1 -free, and so not torsionless, and so not separable.

Proof. This is by 3.12 and 3.14. \blacksquare

Proof of 1.3. First of all, recall that, as well-known, well-founded trees form a complete co-analytic set. This is by 2.13 and 4.5, together with the observation that the map $A \mapsto \text{inv}(\omega^{<\omega}, L_A)$ is Borel, when considering models whose domain is $\subseteq \omega$. \blacksquare

5. OTHER RESULTS

5.1. Separating \mathbb{Z} -homogeneous and \aleph_1 -free

In this section we prove Theorem 1.4. This result is of independent interest.

Fact 5.1. For every $2 \leq \mathbf{n} < \omega$ and \mathbb{P}_* an infinite set of primes, there are $K_{\mathbf{n}}$, $\bar{x}_{\mathbf{n}} = (x_i : i < \mathbf{n})$ and $\bar{y}_{\mathbf{n}} = (y_p : p \in \mathbb{P}_*)$ such that:

- (1) $Z_{\mathbf{n}} := \mathbb{Z}^{\mathbf{n}} \leq K_{\mathbf{n}} \leq \mathbb{Q}^{\mathbf{n}} := H_{\mathbf{n}}$;
- (2) for every $p \in \mathbb{P}_*$, there is $y_p \in K_{\mathbf{n}} \setminus Z_{\mathbf{n}}$ such that $py_p \in H_{\mathbf{n}}$;
- (3) for every $X \subseteq K_{\mathbf{n}}$ of size $< \mathbf{n}$ we have that $\langle X \rangle_{K_{\mathbf{n}}}^*$ is free;
- (4) $\langle \bar{x} \rangle_{K_{\mathbf{n}}}^*$ is not free, in fact, more strongly, if $\mathbb{P}_0 \subseteq \mathbb{P}_*$, then the group

$$K_{(\mathbf{n}, \mathbb{P}_0)} = \langle x_\ell, y_p : \ell < \mathbf{n} \text{ and } p \in \mathbb{P}_0 \rangle_{K_{\mathbf{n}}} \leq \langle \bar{x} \rangle_{K_{\mathbf{n}}}^*$$

is not free and disjoint from $\{y_p : p \in \mathbb{P}_* \setminus \mathbb{P}_0\}$.

Claim 5.2. Let $G = \prod_{n < \omega} K_n$. Then G is \mathbb{Z} -homogeneous but not \aleph_1 -free.

Proof. Easy. \blacksquare

Crucially the group G from 5.2 is not an inverse limit of completely decomposable groups, we want to strengthen 5.2 so as to have in addition this.

- Convention 5.3.** (1) $\mathbf{n} \in [2, \omega)$;
- (2) $K_{\mathbf{n}}, H_{\mathbf{n}}, Z_{\mathbf{n}}$ are as in 5.1;
 - (3) $(p_n : n < \omega)$ lists \mathbb{P}_* in increasing order.

Definition 5.4. For $n < \omega$, let $G_n^* = \langle x_\ell, y_{p_i} : \ell < \mathbf{n} \text{ and } i < n \rangle_{K_{\mathbf{n}}} \leq K_{\mathbf{n}} \leq \langle \bar{x} \rangle_{H_{\mathbf{n}}}^*$.

- Claim 5.5.** (1) For every $n < \omega$, G_n^* is free and $G_n^* \leq H_{\mathbf{n}}$;
- (2) For every $n < \omega$, G_n^* is completely decomposable;
 - (3) $(G_n^* : n < \omega)$ is \leq -decreasing with intersection $K_{\mathbf{n}}$.

Proof. It suffices to consider the following group:

$$G_n^* = \left\langle \frac{1}{\prod_{i < n} p_i} x_0, \dots, \frac{1}{\prod_{i < n} p_i} x_{n-1} \right\rangle_{H_n}$$

and to observe that $G_n^* \leq G_n^*$ and that G_n^* is free. \blacksquare

Claim 5.6. There is an inverse system $(G_n, f_{(n,m)} : m \leq n < \omega)$ such that:

- (a) for every $n < \omega$, $G_n = \bigoplus \{G_{(n,j)} : j < \omega\}$;
- (b) $G_{(n,0)} = G_n^*$ and, for $j < \omega$, $G_{(n,j+1)} = \mathbb{Z}x_{(n,j+1)}$;
- (c) for each $n < \omega$, there is $g_n : \omega \rightarrow \omega$ onto such that for $j < \omega$:

$$h_{(n,j)} := f_{(n+1,n)} \upharpoonright G_{(n+1,j)} \in \text{Hom}(G_{(n+1,j)}, G_{(n,g_n(j))});$$

- (d) for each $n < \omega$, $h_{(n,0)}$ is the inclusion map from G_{n+1}^* into G_n^* , recalling that $G_{n+1}^* \leq G_n^*$;
- (e) $g_n(2j+1) = 0$ and $(h_{(n,2j+1)}(x_{(n+1,2j+1)})) : j < \omega$ lists the elements of $G_{(n,0)}$;
- (f) $g_n \upharpoonright \{2j+2 : j < \omega\}$ is 1-to-1 onto $\omega \setminus \{0\}$ and $h_{(n,2j+2)}(x_{(n+1,2j+2)}) = x_{g_n(2j+2)}$.

Claim 5.7. For $\mathfrak{s} = (G_n, f_{(n,m)} : m \leq n < \omega)$ and $G = \varprojlim(\mathfrak{s})$ as in 5.6 we have:

- (1) each G_n is completely decomposable (even free);
- (2) G is \mathbb{Z} -homogeneous;
- (3) G is not \aleph_1 -free.

Proof. Item (1) is clear by 5.5 and 5.6. Concerning (3), let $G_{(\omega,0)}$ be the subgroup of G consisting of the elements $(g_n : n < \omega) \in G$ such that for every $n < \omega$, $g_n \in G_{(n,0)}$. Then $G_{(\omega,0)} \cong K_{\aleph_1}$ (recall 5.5(c)) and so obviously G is not \aleph_1 -free. Finally, we show (2), i.e., that G is \mathbb{Z} -homogeneous. Let $g = (g_n : n < \omega) \in G \setminus \{0\}$. We distinguish two cases.

Case 1. For some $n < \omega$, $g_n \notin G_{(n,0)}$.

If this is the case, then for some $n_* < \omega$ we have that, for every $m \geq n_*$, $f_{(\omega,m)}(g) \neq 0$. But then, as in earlier proofs, we can find $h_* \in \text{Hom}(G, G_{(n_*,j_*)})$, for some $j_* > 0$, such that $h_*(g) = x_{(n_*,j_*)}$ and so easily $\langle g \rangle_G^*$ is free.

Case 2. For every $n < \omega$, $g_n \in G_{(n,0)}$.

If this is the case, then $g \in G_{(\omega,0)}$ and so as $G_{(\omega,0)}$ is pure in G it suffices to observe that $G_{(\omega,0)} \cong K_{\aleph_1}$ is \mathbb{Z} -homogeneous. \blacksquare

5.2. For every $\eta \in [T]$, $\bigcap \{L(\eta \upharpoonright n)\}$ is finite is not sufficient

In a previous version of this paper we were claiming that if (T, L) is as in 3.2 and in addition, for every $\eta \in [T]$, $\bigcap \{L(\eta \upharpoonright n)\}$ is finite, then $G(T, L)$ is \aleph_1 -free. The next example shows that this is *not* the case. This motivated the introduction of nice pairs in Section 3.2. We include the example for completeness of exposition.

Claim 5.8. Let $(p_n : n \in \omega)$ be distinct primes. Then, for every $n < \omega$, there are $(\bar{p}_n, \bar{a}_n, \bar{Q}_n)$ such that the following conditions hold:

- (a) $\bar{p}_n = \{p_\eta : \eta \in 2^n\}$ where $p_\eta \in \{p_\ell : \ell \leq n\}$;
- (b) $\bar{a}_n = \{a_\eta : \eta \in 2^n\}$, where $a_\eta \in \mathbb{Z} \setminus \{0\}$;
- (c) $\bar{Q}_n = \{Q_\eta : \eta \in 2^n\}$, $Q_\eta \subseteq \{p_\nu : \nu \in 2^n\} \setminus \{p_\eta\}$;
- (d) $(\prod Q_\eta)$ divides a_η in \mathbb{Z} (when η is the root of the tree, we let $\prod Q_\eta = \{1\}$);
- (e) p_η does not divide a_η ;
- (f) If $n = m + 1$ and $\eta \in 2^m$ then:
 - (i) $p_{\eta \frown (0)} = p_\eta$;
 - (ii) $p_{\eta \frown (1)} = p_{n+1}$

- (iii) $\bar{Q}_{\eta \smallfrown (0)} = \bar{Q}_\eta \cup \{p_{n+1}\}$;
- (iv) $\bar{Q}_{\eta \smallfrown (1)} = \bar{Q}_\eta \cup \{p_\eta\}$;
- (v) $a_\eta = a_{\eta \smallfrown (0)} + a_{\eta \smallfrown (1)}$.

Proof. Case 1. $n = 0$

Let $a_{()} = 1$, $Q_{()} = \emptyset$, $P_{()} = p_0$.

Case 2. $n = m + 1$ Now $(p_\eta, Q_\eta : \eta \in 2^m)$ are determined by the other conditions, but we have to find for $\eta \in 2^m$, the numbers $a_{\eta \smallfrown (0)}, a_{\eta \smallfrown (1)} \in \mathbb{Z} \setminus \{0\}$ satisfying (d)-(f). To this end, choose $a'_0, a'_1 \in \mathbb{Z}$ such that:

- $p_\eta \nmid a'_0$ and $p_{n+1} \nmid a'_1$;
- $p_\eta a'_0 + p_{n+1} a'_1 = 1$.

Let then:

- $a_{\eta \smallfrown (0)} = a_\eta p_\eta a'_0$;
- $a_{\eta \smallfrown (1)} = a_\eta p_{n+1} a'_1$.

■

Claim 5.9. Let $T = 2^{<\omega}$ and, for $\eta \in T$, let $L(\eta) = \{p_\nu : \eta \triangleleft \nu \in T\}$. Then (T, L) is as in 3.2 and we have that there is $y \in G$ such that:

- (i) $f_{(\omega, n)}(y) = \sum \{a_\eta x_\eta : \eta \in 2^n\}$;
- (ii) in G we have that $\prod_{\ell < n} p_\ell$ divides y , for every $n < \omega$;
- (iii) G is not \mathbb{Z} -homogenous.

Proof. Let $y_n = \sum \{a_\eta x_\eta : \eta \in 2^n\}$. Now, clearly $y_n \in G_n$. Furthermore, if $n < m < \omega$, then $f_{(m, n)}(y_m) = y_n$. To see this it suffices to prove it for $n = m + 1$. For this purpose, notice:

$$\begin{aligned}
 f_{(n+1, n)}(y_n) &= f_{(n+1, n)}(\sum \{a_\eta x_\eta : \eta \in 2^{n+1}\}) \\
 &= \sum \{a_\eta f_{(n+1, n)}(x_\eta) : \eta \in 2^{n+1}\} \\
 &= \sum \{a_\eta x_{\eta \upharpoonright n} : \eta \in 2^{n+1}\} \\
 &= \sum \{\sum \{a_\eta : \nu \triangleleft \eta \in 2^{n+1}\} : \nu \in 2^n\} \\
 &= \sum \{(a_{\nu \smallfrown (1)} + a_{\nu \smallfrown (0)})x_\nu : \nu \in 2^n\},
 \end{aligned}$$

but, by 5.8(f)(v), $a_{\nu \smallfrown (1)} + a_{\nu \smallfrown (0)} = a_\nu$ and so we are done. ■

REFERENCES

- [1] H. Becker and A. Kechris. *The descriptive set theory of Polish group actions*. London Math. Soc. Lecture Notes Ser. 232, Cambridge University Press, 1996.
- [2] J. Bergfalk, M. Lupini and A. Panagiotopoulos. *The definable content of homological invariants I: Ext & lim1*. Proc. Lond. Math. Soc. (3) **129** (2024), no. 03.
- [3] J. Bergfalk, M. Lupini and A. Panagiotopoulos. *The definable content of homological invariants II: cohomology and homotopy classification*. Forum Math. Pi (2024), paper no. e12.
- [4] L. Ding and S. Gao. *Non-Archimedean abelian Polish groups and their actions*. Adv. Math. **307** (2017), 312-343.
- [5] R. M. Dudley. *Continuity of homomorphisms*. Duke Math. J. **28** (1961), no. 4, 34-60.
- [6] P.C. Eklof and A.H. Mekler. *Almost free groups*. North Holland 2002.
- [7] L. Fuchs. *Abelian groups*. Springer International Publishing Switzerland 2015.
- [8] S. Gao. *Invariant descriptive set theory*. Taylor & Francis Inc, 2008.
- [9] A. Kechris, A. Nies and K. Tent. *The complexity of topological group isomorphism*. J. Symb. Log. **83** (2018), no. 03, 1190-1203.
- [10] A. Khelif. *Uncountable homomorphic images of Polish groups are not \aleph_1 -free groups*. Bull. London Math. Soc. **37** (2005), 54-60.
- [11] S. Mardesic and J. Segal. *Shape and shape theory*. Springer Berlin, Heidelberg, 1978.

- [12] G. Paolini and S. Shelah. *Polish topologies for graph products of groups*. J. Lond. Math. Soc. **100** (2019), no. 02, 383-403.
- [13] R. J. Nunke. *On direct products of infinite cyclic groups*. Proceedings of the American Mathematical Society Vol. 13, No. 1 (Feb., 1962), pp. 66-71.
- [14] S. Shelah. *A countable structure does not have a free uncountable automorphism group*. Bull. London Math. Soc. **35** (2003), 1-7.
- [15] S. Shelah. *Polish algebras, shy from freedom*. Israel J. Math. **181** (2011), 477-507.

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