HOMOGENEOUS FORCING
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Abstract. Assume $\kappa = \kappa^{<\kappa}$ (usually $\aleph_0$ or an inaccessible).

We shall deal with iterated forcings preserving ${}^{\kappa>}\mathrm{Ord}$ and not collapsing cardinals along a linear order L. A sufficient condition for this, which we will focus on, is for the forcings to have support $< \kappa$ and the κ^+-cc, and be strategically $< \kappa$-complete. The aim is to have homogeneous forcings, so that the iteration has many automorphisms.

In addition to the inherent interest, such iterations are helpful for considering some natural ideals on ${}^{\kappa}2$, in order to get a model of $\mathrm{ZF} + \mathrm{DC}_\kappa$ + "modulo this ideal, every set is equivalent to a κ-Borel one."

But here we only have many automorphisms of the index set L and therefore of the iteration of iterands $\mathbb{Q}$; we do not necessarily have homogeneity of $\mathbb{Q}$, and we do not have automorphisms mapping other names of $\mathbb{Q}$-reals onto each other. However, for some reasonable forcing notions, there are no other $\mathbb{Q}$-reals! This was the reason for introducing and investigating saccharinity in earlier works with Jakob Kellner and with Haim Horowitz.

Annotated Content

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References like (e.g.) [Sh:950, Th0.2$_{=\mathrm{Ly5}}$] mean that the internal label of Theorem 0.2 in Sh:950 is 'y5.' The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number 1257 on Saharon Shelah's list.

§ 0. Introduction

§ 0(A). **Aim.** We fix $\kappa = \kappa^{<\kappa}$ and consider homogeneous $(<\kappa)$-support iterations of $(<\kappa)$-complete forcing notions, with a version of κ^+-cc, preserving those properties. However, throughout this section we will concentrate on the classical case $\kappa := \aleph_0$.

To get homogeneity we intend to iterate along a linear order which is quite homogeneous (and therefore very much not well-ordered).

Ever since Solovay's celebrated work [Sol70], we know about the connection between the following two issues:

- $\bullet_1$ Forcing notions $\mathbb{P}$ with lots of automorphisms. E.g. for small $\mathbb{P}' \lessdot \mathbb{P}$ and two relevant $\mathbb{P}$-names $\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2$ of reals, generic for the same relevant forcing $\mathbb{Q}$ over $\mathbf{V}^{\mathbb{P}'}$, there is an automorphism of $\mathbb{P}$ over $\mathbb{P}'$ mapping $\underset{\sim}{\eta}_1$ to $\underset{\sim}{\eta}_2$.
- $\bullet_2$ Models of $\mathsf{ZF} + \mathsf{DC} +$ "every set of reals is equivalent to a Borel set modulo the null ideal (or other reasonable ideal)". (The most central forcings were Random Real forcings for the null ideal. The second prominent case was Cohen forcing, for the meagre ideal.)

Concerning the classical case of Lebesgue measurability, another formulation is "no non-measurable set is easily definable," formulated[1] as 'definable in $\mathbf{L}[\mathbb{R}]$.' See the history and more in [RS04], [RS06].

This applies to other ideals $\mathrm{id}(\mathbb{Q}, \underset{\sim}{\eta})$ for a definable forcing notion $\mathbb{Q}$ (mainly a ccc one) and a $\mathbb{Q}$-name $\underset{\sim}{\eta}$ of a real. Generally, it was not so easy to build such forcing notions: it required one to prove the existence of amalgamation in the relevant class of forcings. In Kellner-Shelah [KS11] it was suggested to look at so-called saccharine pairs $(\mathbb{Q}, \underset{\sim}{\eta})$, where $\mathbb{Q}$ is very non-homogeneous. (E.g. forcing with $\mathbb{Q}$ adds just one $(\mathbb{Q}, \underset{\sim}{\eta})$-generic, so we have few cases we need to build automorphisms for. The forcing notion here is proper but *not* ccc.)

Here we prove the existence of sufficiently homogeneous κ^+-cc $(<\kappa)$-complete iterated forcings (so, along a linear order with $(<\kappa)$-support).

Notation 0.1. 1)

- $\kappa, \lambda, \mu, \partial, \theta, \sigma$ are cardinals (infinite, if not explicitly said otherwise). $\lambda^+ = \lambda(+)$ will denote the successor of λ.
- $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ will denote ordinals; δ will be a limit ordinal if not stated otherwise.
- k, ℓ, m, n were originally natural numbers, but now just ordinals $< \kappa$.
- i and j have been used for indices $< \kappa$ throughout.

2) $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

3) Recall that $\mathbb{L}_{\kappa,\sigma}$ is defined like first-order logic, but allowing $\bigwedge_{i<\alpha} \varphi_i$ for $\alpha < \kappa$ and $(\exists \ldots x_i \ldots)_{i \in I}$ with I of cardinality $< \sigma$.

4) $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ will denote forcing notions.

[1] That is, $\bullet_2$ holds for an inner model $\mathbf{L}[\mathcal{P}(\kappa)]^{\mathbf{V}}$ with $\mathbf{V} \models \mathsf{ZFC}$, so in $\mathbf{V}$ all 'reasonable' sets are 'measurable' for this ideal.

Definition 0.2. 0) For κ a cardinal, the family of κ-Borel sets is the smallest family of subsets of 2^κ containing all basic sets of the form $\{\nu \in {}^\kappa 2 : \nu(\alpha) = i\}$ and closed under complements and unions of $\leq \kappa$-many sets.

1) For $\mathbb{Q}$ a forcing notion, $\underset{\sim}{\eta}$ a $\mathbb{Q}$-name of a member of 2^κ, and ∂ a cardinal, let $\mathrm{id}_{<\partial}(\mathbb{Q}, \underset{\sim}{\eta})$ be the ideal consisting of the unions of $< \partial$-many κ-Borel sets $\mathbf{B}$ such that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin \mathbf{B}$".

2) We say $\mathrm{id}_{<\partial}(\mathbb{Q}, \underset{\sim}{\eta})$ has *measurability* when for every $Y \subseteq {}^\kappa 2$ there exists a κ-Borel set $\mathbf{B}$ such that $Y \triangle \mathbf{B} \in \mathrm{id}_{<\partial}(\mathbb{Q}, \underset{\sim}{\eta})$.

3) Let $\mathrm{id}_{\leq\partial}(\mathbb{Q}, \underset{\sim}{\eta})$ be $\mathrm{id}_{<\partial^+}(\mathbb{Q}, \underset{\sim}{\eta})$.

§ 0(B). Background.

Discussion 0.3. 1) Comparing the forcing $\mathbb{Q}$ from [KS11] to the older results (such as Solovay [Sol70]), the forcings are Borel definable, proper, and of cardinality $2^{\aleph_0}$. In addition:

- $\bullet_{1.1}$ The forcing $\mathbb{Q}$ collapsed no cardinal (provided that CH held), but was not ccc; this[2] we consider a drawback.
- $\bullet_{1.2}$ The model, as in those older results, does satisfy $\mathsf{ZF} + \mathsf{DC}$.
- $\bullet_{1.3}$ The iteration was along a homogeneous linear order.
- $\bullet_{1.4}$ We get only a somewhat weaker version of measurability, the ideal being $\mathrm{id}_{\leq\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$ instead of $\mathrm{id}_{<\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$.

Alternatively (e.g. starting with the universe $\mathbf{L}$),

- $\bullet'_{1.4}$ Use $\mathrm{id}_{<\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta}) + X$, where X is the set $\{\underset{\sim}{\eta}[\mathbf{G}] : \mathbf{G} \subseteq \mathbb{Q}^{\mathbf{L}}$ is generic over $\mathbf{L}\}$.

2) The next step was Horowitz-Shelah [HS], where:

- $\bullet_{2.1}$ The forcing is ccc, which we consider a plus.
- $\bullet_{2.2}$ The model only satisfies ZF; we do not get DC or even $\mathsf{AC}_{\aleph_0}$ — not so good.
- $\bullet_{2.3}$ Again, the iteration is along a homogeneous linear order.
- $\bullet_{2.4}$ The ideal is again $\mathrm{id}_{\leq\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$ (or as in $\bullet'_{1.4}$ above).

Our intention is to regain both ccc (as in $\bullet_{2.1}$) as well as DC (as in $\bullet_{1.2}$) for the ideal $\mathrm{id}_{\leq\aleph_1}(\mathbb{Q}, \underset{\sim}{\eta})$. Moreover, we can demand $\mathsf{DC}_{\aleph_1}$ (or more; see §1) which is a significant plus.

* * *

We continue [She04], [She], but do not rely on them. Instead of defining iterations we introduce them axiomatically and allow $\kappa > \aleph_0$ (referenced in the completeness and the support). Unlike [She04], the present paper does not address forcing $\mathfrak{a} > \mathfrak{d}$. Earlier continuations of [She04] and [She] were the parallel papers, in preparation, with preliminary numbers F2009 and F2029 (and later, their descendants F2330 and F2329). In [She04] the set $\mathscr{P}^{\mathbf{m}}_s$ (see Def. 1.9) may be the whole power set, and we use more general definable forcing notions.

[2] Note that Solovay uses Levy collapse of an inaccessible, but the later versions use ccc ones (mainly for the meagre ideal).

In our iteration we are allowed to replace $\aleph_0$ by some $\kappa = \kappa^{<\kappa}$, so the forcing notions are $(<\kappa)$-complete κ^+-cc. For the $\kappa = \aleph_0$ case we intend to use the forcing notion $\mathbb{Q}^2_{\mathbf{n}}$ from [HS]. But we still need an analogue which works for any κ.

Explanation of the path chosen

(The reader may skip this subsection for now, as it will only make sense after reading §1-2.)

Consider a family $\mathbf{K}$ of κ^+-cc $(<\kappa)$-complete forcing notions, ordered by some $\leq_{\mathbf{K}}$. There are two natural ways to do this.

<u>THE FIRST WAY:</u>

We build a directed family $\langle \mathbb{P}_t : t \in I \rangle$ such that $s <_I t \Rightarrow \mathbb{P}_s \lessdot \mathbb{P}_t$ and $\mathbb{P} := \bigcup_{t \in I} \mathbb{P}_t$ is in $\mathbf{K}$ and has enough automorphisms.

Presently, this requires

(A) Start with $(\mathbf{P}, \leq_{\mathbf{P}})$ with the entire partial order $<^{\mathrm{pr}}_{\mathbf{P}}$, as in the present version of §2. (So no $\mathcal{B}_{\mathbf{q}}$!)

(B) We define $\mathbf{Q}$ as the set of $\mathbf{q}$ which consist of:

  (a) $W_{\mathbf{q}} \subseteq L_{\mathbf{q}}$ such that for every $s \in L_{\mathbf{q}}$ we have $\mathcal{A}^{\circ}_{\mathbf{q},s} \subseteq \mathcal{A}_{\mathbf{q},s}$ with $\lambda_{\mathbf{o}_s}$ members, which are pairwise disjoint sets all of cardinality $\leq \lambda_{\mathbf{o}_s}$, and are *dense* in some suitable sense.
  (Alternatively, allow 2^{λ_s}-many such that all possible isomorphism types appear.)

  (b) For each $s \in L_{\mathbf{q}} \setminus W_{\mathbf{q}}$ we have an ideal $\mathcal{I}_s$ on $\mathcal{A}^{\circ}_{\mathbf{q},s}$ such that

$$A \in \mathcal{A}_{\mathbf{q},s} \Rightarrow \{B \in \mathcal{A}^{\circ}_{\mathbf{q},s} : A \cap B \neq \varnothing\} \in \mathcal{I}_s.$$

  (c) If $s \in W_{\mathbf{q}}$ then $\bigcup \mathcal{A}_{\mathbf{q},s} \subseteq W_{\mathbf{q}}$.

  (d) If $\mathbf{q}_1 \leq_{\mathbf{Q}} \mathbf{q}_2$ and $s \in W_{\mathbf{q}_1}$, <u>then</u> $\mathcal{A}_{\mathbf{q}_1,s} = \mathcal{A}_{\mathbf{q}_2,s}$.

<u>THE WRONG WAY:</u>

Let $\chi = \chi^{<\chi} > \theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa)$ such that $\kappa = \kappa^{<\kappa}$ and $\alpha < \theta \Rightarrow |\alpha|^\kappa < \theta$. We force by $(\mathbf{K}_{<\theta} \cap \mathcal{H}(\chi), <_{\mathbf{K}})$, where $\mathbf{K} := \mathbf{P}$ and $\mathbf{K}_{<\theta} := \{\mathbb{P} \in \mathbf{K} : \|\mathbb{P}\| < \theta\}$.

To deal with $\mathbb{L}[\mathbb{R}^{<\theta}]$, it is enough to prove that there are enough amalgamation bases in $(\mathbf{K}_{<\theta}, \leq_{\mathbf{K}})$, which is not so hard. (It is enough to have "the union of a $\leq_{\mathbf{K}}$-increasing chain of length ∂ is a lub," for some $\partial \in [\kappa, \theta) \cap \mathrm{Reg}$.)

A suspicious point is how to define when a (not overly long) increasing sequence in $(\mathbf{P}, \leq_{\mathbf{P}})$ has an upper bound, even in the case $\kappa := \aleph_0$. However, for $(\mathbf{P}, \leq^M_{\mathbf{P}})$ this is not a problem.

Too good to be true? No; just that for homogeneity we need to return to the first way.

* * *

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§ 0(C). **Preliminaries.**

Hypothesis 0.4. 1) $\kappa = \kappa^{<\kappa}$ (mainly $\aleph_0$ or an inaccessible).

2) ∂ is a regular cardinal $> \kappa$.

3) D is a normal filter on κ^+ such that $S^{\kappa^+}_\kappa := \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\} \in D$.

For more on the condition defined below, see [She22].

Definition 0.5. 1) For D a normal filter on κ^+ containing $S^{\kappa^+}_\kappa$, we say the forcing notion $\mathbb{Q}$ satisfies $*^0_{\kappa,D}$ <u>when</u>:

$*^0_{\kappa,D}$ Given a sequence $\bar{p} = \langle p_i : i < \kappa^+\rangle$ of members of $\mathbb{Q}$, there is a set[3] $C \in D$, a sequence $\bar{p}^+ = \langle p^+_i : i < \kappa^+\rangle$ satisfying $(\forall i)[p_i \le p^+_i]$, and a regressive function $\mathbf{h}$ on C such that
$$\alpha, \beta \in C \wedge \mathbf{h}(\alpha) = \mathbf{h}(\beta) \Rightarrow \text{`}p^+_\alpha \text{ and } p^+_\beta \text{ have a lub.'}$$

2) For κ and D as above, we say $\mathbb{Q}$ satisfies $*^1_{\kappa,D}$ <u>when</u> $*^0_{\kappa,D}$ holds with $\bar{p}^+ := \bar{p}$.

3) We say a forcing notion $\mathbb{Q}$ is *Knaster*$^+$ <u>when</u>:

> If $p_i \in \mathbb{Q}$ for $i < \omega_1$ <u>then</u> there are $A \in [\omega_1]^{\aleph_1}$ and $\langle p^+_i : i \in A\rangle \subseteq \mathbb{Q}$, with each $p^+_i \ge_{\mathbb{Q}} p_i$, such that for every $i, j \in A$ the conditions p^+_i and p^+_j are compatible and have a least upper bound.

Notation 0.6. 1) $\mathbf{p}$ will be a member of $\mathbf{P}$ as in Definition 2.2.

2) $\mathbf{q}, \mathbf{r}$ will denote ATIs (*abstract template iterations*); i.e. members of $\mathbf{Q}_{\mathrm{pre}}$ (the weakest version — see Definitions 3.4, 4.1).

3) L is a linear order (usually $L \subseteq L_{\mathbf{m}}$) and $r, s, t \in L$.

L_+ is derived from L, with ∞ and $t, t(+) \in L_+$ for each $t \in L$. (See below in 1.9(2).)

4) $L_{\mathbf{m}}$ or $L_{\mathbf{q}}$ will be the relevant linear order for $\mathbf{m}$ or $\mathbf{q}$, etc.

5) $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ denote forcing notions as in Definition 0.5 (which means quasi-orders).

Definition 0.7. 1) We say TR is a κ-*trunk controller* <u>when</u> it consists of:

(A) A partial order $\le_{\mathrm{TR}}$ on a set $|\mathrm{TR}|$ of cardinality $\|\mathrm{TR}\|$.
(Abusing notation slightly, we may write '$s \in \mathrm{TR}$' instead of $s \in |\mathrm{TR}|$.)

(B) A partial function $\mathsf{plus}_{\mathrm{TR}} : {}^{\kappa>}|\mathrm{TR}| \to |\mathrm{TR}|$ giving a lub.

(C) Any increasing sequence of length $< \kappa$ belongs to $\mathrm{dom}(\mathsf{plus}_{\mathrm{TR}})$ (and hence has a lub).

(D) If $\zeta < \kappa$ and $\mathbf{x}_{\alpha,\varepsilon} \in \mathrm{TR}$ for $\alpha < \kappa^+$ and $\varepsilon < \zeta$, <u>then</u> for some $\alpha < \beta$ we have
$$\bigwedge_{\varepsilon<\zeta} \big[\langle \mathbf{x}_{\alpha,\varepsilon}, \mathbf{x}_{\beta,\varepsilon}\rangle \in \mathrm{dom}(\mathsf{plus}_{\mathrm{TR}})\big].$$

(E) $\varnothing \in \mathrm{TR}$, and $(\forall \mathbf{x} \in \mathrm{TR})\big[\varnothing \le_{\mathrm{TR}} \mathbf{x}\big]$.

[3] Yes! Not just '$C \in D^+$;' see [She22].

(F) $S_{\mathrm{TR}} \subseteq \mathcal{H}_{<\kappa}(\mathrm{Ord})$ and

$$\mathrm{val}_{\mathrm{TR}} : |TR| \times S_{\mathrm{TR}} \to 2$$

is a partial function.

Specifically, for each $\mathbf{x} \in |\mathrm{TR}|$, $\mathrm{val}(\mathbf{x}) = \mathrm{val}(\mathbf{x}, -)$ is a partial function from S_{TR} to $\{0,1\}$, increasing with $\mathbf{x}$.

1A) We call TR a *simple* κ-trunk controller <u>when</u>:

(A) TR is a κ-trunk controller.

(B) $(\forall \mathbf{x} \in \mathrm{TR})\big[|\mathbf{x}| \leq \kappa\big]$ (We usually have $|\mathrm{TR}| \subseteq \mathcal{H}(\kappa)$, or at least $\mathcal{H}_{<\kappa}(\lambda)$ for some λ.)

(C) $\leq_{\mathrm{TR}} := \subseteq$

(D) If $\delta < \kappa$ is limit and $\langle \mathbf{x}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathrm{TR}}$-increasing, <u>then</u>

$$\mathsf{plus}_{\mathrm{TR}}(\langle \mathbf{x}_\alpha : \alpha < \delta \rangle) := \bigcup_{\alpha<\delta} \mathbf{x}_\alpha.$$

(E) $S_{\mathrm{TR}} := |\mathrm{TR}|$, and for each $\mathbf{x} \in \mathrm{TR}$, we define the function $\mathrm{val}_{\mathrm{TR}}(\mathbf{x})$ by demanding that $\mathrm{val}_{\mathrm{TR}}(\mathbf{x})(\mathbf{y})$ is 1 if $\mathbf{y} \leq_{\mathrm{TR}} \mathbf{x}$ and is 0 if $\mathbf{x}$ and $\mathbf{y}$ are incompatible (and is undefined otherwise).

2) We say TR is a $(\kappa, \Theta, \Upsilon)$-trunk controller <u>when</u>:

- $\bullet_1$ Θ is a set of regular cardinals $< \kappa$.
- $\bullet_2$ $\kappa > \aleph_0 \Leftrightarrow \Theta \neq \varnothing$
- $\bullet_3$ $\Upsilon \in [2, \kappa]$.
- $\bullet_4$ If $\Theta = \mathrm{Reg} \cap \kappa$ then we may omit it; similarly if $\Upsilon = 2$.

(A), (B), (C), (E) As above.

(D)′ Any $\leq_{\mathrm{TR}}$-increasing sequence $\langle \mathbf{x}_i : i < j \rangle$ with $j < \kappa$ has an upper bound, <u>but</u> if $j \in \Theta$ then it has a lub $\mathsf{plus}_{\mathrm{TR}}(\langle \mathbf{x}_i : i < j \rangle)$.

(F) If $j < 1 + \Upsilon$, $\zeta < \kappa$, and $\mathbf{x}_{\alpha,\varepsilon} \in \mathrm{TR}$ for $\alpha < \kappa^+$ and $\varepsilon < \zeta$, <u>then</u> for some increasing sequence of ordinals $\langle \alpha_i : i < j \rangle$ bounded above by κ^+, for each $\varepsilon < \zeta$, we have

$$\langle \mathbf{x}_{\alpha_i,\varepsilon} : i < j \rangle \in \mathrm{dom}(\mathsf{plus}_{\mathrm{TR}}).$$

3) For $\langle \mathrm{TR}_s : s \in I \rangle$, with each TR_s as in (1A) or (2), we define $\mathrm{TR} = \prod\limits_{s \in I}^{<\kappa} \mathrm{TR}_s$, the $(<\kappa)$-*support product*, as follows.

(A) $f \in \mathrm{TR}$ <u>iff</u> $f \in \prod\limits_{s\in J} \mathrm{TR}_s$ for some $J \in [I]^{<\kappa}$.

(B) $f \leq_{\mathrm{TR}} g$ <u>iff</u> ($f, g \in \mathrm{TR}$ and)
  (a) $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$
  (b) $f(s) \leq_{\mathrm{TR}_s} g(s)$ for all $s \in \mathrm{dom}(f)$.

(C) Let $\bar{f} = \langle f_\varepsilon : \varepsilon < \zeta \rangle$, with $\zeta < \kappa$.
  We have $\mathsf{plus}_{\mathrm{TR}}(\bar{f}) = g$ <u>when</u>
  (a) $f_\varepsilon \in \mathrm{TR}$ for all $\varepsilon < \zeta$.
  (b) $\mathrm{dom}(g) = \bigcup\limits_{\varepsilon<\zeta} \mathrm{dom}(f_\varepsilon)$
  (c) For all $s \in \mathrm{dom}(g)$, if $u = u_{\bar{f},s} := \big\{\varepsilon < \zeta : s \in \mathrm{dom}(f_\varepsilon)\big\}$, <u>then</u>

$$g(s) = \mathsf{plus}_{\mathrm{TR}_s}(\langle f_\varepsilon(s) : \varepsilon \in u \rangle).$$

(D) (a) $S_{\rm TR} := \big\{(s,\mathbf{x}) : s \in I,\ \mathbf{x} \in S_{{\rm TR}_s}\big\}$

(b) For $f \in {\rm TR}$, define ${\rm val}(f)$ as the function with domain

$$\big\{(s,\mathbf{x}) : s \in {\rm dom}(f),\ \mathbf{x} \in {\rm dom}({\rm val}(f(s)))\big\}$$

which sends $(s,\mathbf{x}) \mapsto {\rm val}_{{\rm TR}_s}(f(\mathbf{x}))$.

Remark 0.8. 1) For TR a (κ,Θ,Υ)-trunk controller, the function $\mathsf{plus}_{\rm TR}$ will not really be used for sequences of length $j \in \kappa \cap {\rm Reg} \setminus \Theta$; only the existence of such a value will be necessary. (See more on this in §5.)

2) If ${\rm TR}_s$ is a κ-trunk controller for $s \in I$, then $\prod\limits_{s\in I}^{<\kappa} {\rm TR}_s$ is not *guaranteed* to be a κ-trunk controller, as 0.7(1)(D) may fail.

Example 0.9. We define an explicit κ-trunk controller ${\rm TR} = {\rm TR}_\kappa$ as follows.

(A) The set of elements is $\{f \in \mathcal{H}(\kappa) : f \text{ is a function}\}$.
(B) $\le_{\rm TR} := \subseteq$
(C) $\mathsf{plus}_{\rm TR}(\langle f_i : i < j\rangle) := g$ <u>iff</u> $\bigwedge\limits_{i<j} [f_i = g \in {\rm TR}]$.
(D) $S_{\rm TR} := |{\rm TR}|$ and

$${\rm dom}({\rm val}_{\rm TR}(f)) := \big\{g \in |{\rm TR}| : {\rm dom}(g) \subseteq {\rm dom}(f)\big\}$$

with

$${\rm val}_{\rm TR}(f)(g) := \begin{cases} 1 & \text{if } g \subseteq f \\ 0 & \text{otherwise.}\end{cases}$$

Claim 0.10. 1) *Assume $\kappa = \kappa^{<\kappa}$. If I is a set and ${\rm TR}_s$ is a κ-trunk controller for each $s \in I$, <u>then</u> ${\rm TR} := \prod\limits_{s\in I}^{<\kappa} {\rm TR}_s$ is a κ-trunk controller <u>iff</u> $\prod\limits_{s\in J}^{<\kappa} {\rm TR}_s$ is a κ-trunk controller for all $J \in [I]^{<\kappa}$.*

2) *Above, if ${\rm TR}_s$ is constant in s <u>then</u> the product is also a κ-trunk controller.*

3) ${\rm TR}_\kappa$ *from Example* 0.9 *is indeed a κ-trunk controller.*

4) *Every κ-trunk controller is a $(\kappa, {\rm Reg}\cap\kappa, 2)$-trunk controller.*

Proof. Straightforward. $\square_{0.10}$

§ 1. The frame

To explain our framework, we shall first concentrate on the most simple version: every member of **TR** is simple (see 0.7(1A)) and each $\mathcal{A}_t$ is a family of pairwise disjoint sets.

Hypothesis 1.1. For the rest of this work, we shall assume the following:

1) $\kappa = \kappa^{<\kappa}$

2) **TR** is a non-empty set (or class) of definitions of κ-trunk controllers, such that the product of any $< \kappa$ of them is a κ-trunk controller.

3) **TR** is *simple*; that is, every member of **TR** is a simple κ-trunk controller (see 0.7(1A)).

But we may use S_{TR} (or $S_{\mathbf{o}}$) and $\mathrm{val}_{\mathbf{o}}$ below (see 2.5).

4) $\mathbf{TR}^+ = \mathbf{TR}(+)$ is the closure of **TR** under products of length $< \kappa$.

5) If $(\forall s \in I)[\mathrm{TR}_s \in \mathbf{TR}]$ then $\prod\limits_{s \in I}^{<\kappa} \mathrm{TR}_s$ is a κ-trunk controller. (Note that this is an easy consequence of part (2) above.)

6) $\mathbf{O}$ is a subset (or subclass) of $\mathbf{O}_{\mathbf{TR}}$ — see Definition 1.2.

Definition 1.2. $\mathbf{O}_{\mathbf{TR}}$ is the class of objects $\mathbf{o}$ which consist of:[4]

(A) $\mathrm{TR}_{\mathbf{o}} = \mathrm{TR}[\mathbf{o}] = (|\mathrm{TR}_{\mathbf{o}}|, \leq_{\mathbf{o}}, \mathsf{plus}_{\mathbf{o}})$ is a κ-trunk controller from **TR** (and hence is simple, *per* our hypothesis).

(B) The *atomic forcing* $\mathbb{Q}^{\mathrm{at}}_{\mathbf{o}} = (Q^{\mathrm{at}}_{\mathbf{o}}, \leq^{\mathrm{at}}_{\mathbf{o}})$, which is a definition of a forcing notion (i.e. a quasiorder).

(C) (a) A function $\mathrm{tr}_{\mathbf{o}} : Q^{\mathrm{at}}_{\mathbf{o}} \to |\mathrm{TR}_{\mathbf{o}}|$.
  (b) A non-decreasing function $\sigma_{\mathbf{o}} : |\mathrm{TR}_{\mathbf{o}}| \to [2, \kappa]_{\mathrm{Card}}$.
  (c) A function $A = A_{\mathbf{o}}$ with domain $\mathbb{Q}^{\mathrm{at}}_{\mathbf{o}}$ and range $\subseteq \mathcal{P}(\mathrm{TR}_{\mathbf{o}})$. We may write $A_{\mathbf{o},p}$ in place of $A_{\mathbf{o}}(p)$.
  (So again, these are *definitions of functions.*)

(D) **[Notation:]**
  (a) Let $\sigma_{\mathbf{o}}(p) := \sigma_{\mathbf{o}}(\mathrm{tr}_{\mathbf{o}}(p))$ for $p \in Q^{\mathrm{at}}_{\mathbf{o}}$.
  (b) $\lambda_{\mathbf{o}} := \big(|Q^{\mathrm{at}}_{\mathbf{o}}| + \|\mathrm{TR}_{\mathbf{o}}\| + \kappa\big)^{\kappa}$
  (c) For $\mathbf{O} \subseteq \mathbf{O}_{\mathbf{TR}}$, let $\lambda_{\mathbf{O}} := \big(\sum\limits_{\mathbf{o} \in \mathbf{O}} \lambda_{\mathbf{o}}\big)^{\kappa}$.

In addition, we make the following demands that the constituents of $\mathbf{o}$ must satisfy.

(E) (a) Each $p \in Q^{\mathrm{at}}_{\mathbf{o}}$ is a subset of $\mathcal{H}(\kappa)$ (or just $\subseteq \mathcal{H}_{<\kappa}(\lambda_{\mathbf{o}})$).
  (b) If $p \leq_{\mathbf{o}} q$ then $\mathrm{tr}(p) \leq_{\mathrm{TR}[\mathbf{o}]} \mathrm{tr}(q)$.

(F) (a) Let us say that a sequence $\bar{p}$ is $\sigma_{\mathbf{o}}$-*truly increasing* when

$$i < j < \ell g(\bar{p}) \Rightarrow \big[\sigma_{\mathbf{o}}(p_i) < \sigma_{\mathbf{o}}(p_j) \vee \sigma_{\mathbf{o}}(p_i) = \sigma_{\mathbf{o}}(p_j) = \kappa\big].$$

[4] In the future, we may demand that $\mathbf{o}$ has a sufficiently absolute definition from the parameter $v \in {}^{W(\mathbf{o})}2$ — but for now, $W(\mathbf{o}) := \varnothing$ is enough.

(b) If $\delta < \kappa$ is a limit ordinal and $\bar{p} = \langle p_\varepsilon : \varepsilon < \delta \rangle$ is $\sigma_{\mathbf{o}}$-truly increasing, <u>then</u> $\bar{p}$ has a $\leq_{\mathbf{o}}$-lub
$$\bigcup \bar{p} := \big(\mathsf{plus}_{\mathbf{o}}(\langle \mathrm{tr}(p_\varepsilon) : \varepsilon < \delta \rangle), \bigcup_{\varepsilon<\delta} p_\varepsilon\big).$$
(c) $A_{\mathbf{o},p} \subseteq \{\mathrm{tr}(q) : q \geq_{\mathbf{o}} p\}$

(G) Case 1: $[\mathrm{TR}_{\mathbf{o}} \subseteq \mathcal{H}(\kappa)$ for all $\mathbf{o} \in \mathbf{O}.]$
If $\mathbf{x} \in \mathrm{TR}_{\mathbf{o}}$ and $\mathrm{tr}_{\mathbf{o}}(p_i) = \mathbf{x}$ for all $i < i_* < \sigma_{\mathbf{o}}(\mathbf{x})$, <u>then</u> $\{p_i : i < i_*\}$ has a common upper bound q with $\mathrm{tr}(q) \in \bigcap_{i<i_*} E_{\mathbf{o},p_i}$.

Case 2: [Holds in general.]
If $\mathbf{x}_i \in \mathrm{TR}_{\mathbf{o}}$, $\langle \sigma_{\mathbf{o}}(\mathbf{x}_i) : i < i_* \rangle$ is constant, $\mathrm{tr}_{\mathbf{o}}(p_i) = \mathbf{x}_i$ for all $i < i_* < \sigma_{\mathbf{o}}(\mathbf{x}_0)$, and $\langle \mathbf{x}_i : i < i_* \rangle \in \mathrm{dom}(\mathsf{plus}_{\mathbf{o}})$, <u>then</u> $\{p_i : i < i_*\}$ has a common upper bound q (again with $\mathrm{tr}(q) \in \bigcap_{i<i_*} E_{\mathbf{o},p_i}$).

(H) **[Absoluteness:]** If $\mathbb{R}_1 \lessdot \mathbb{R}_2$ such that $\mathbb{R}_1$ and $\mathbb{R}_2/\mathbb{R}_1$ are strategically $(<\kappa)$-complete κ^+-cc forcings, <u>then</u>
(a) In $\mathbf{V}^{\mathbb{R}_2}$, the definitions in $\mathbf{o}$ satisfy clauses (A)-(G).
(b) In $\mathbf{V}^{\mathbb{R}_2}$ we have $\mathbb{Q}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_1]} \subseteq \mathbb{Q}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_2]}$, $\mathrm{tr}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_1]} = \mathrm{tr}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_2]}$, $\sigma_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_1]} \subseteq \sigma_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_2]}$, and $A_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_1]} \subseteq A_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_2]}$.
(c) If $p_1 \in \mathbb{Q}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_1]}$, $p_2 \in \mathbb{Q}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_2]}$, $\mathbb{Q}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_2]} \models$ '$p_1 \leq p_2$', and $\mathbf{y} := \mathrm{tr}(p_2)$, <u>then</u> there is $p_3 \in \mathbb{Q}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_1]}$ such that $\mathrm{tr}(p_3) = \mathbf{y}$ and $\mathbb{Q}_{\mathbf{o}}^{\mathbf{V}[\mathbb{R}_1]} \models$ '$p_1 \leq p_3$'.

(A) $\underset{\sim}{\eta}_{\mathbf{o}}$ is the $\mathbb{Q}_{\mathbf{o}}^{\mathrm{at}}$-name of the function
$$\big\{(s,\iota) \in S_{\mathbf{o}} \times 2 : (\exists p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\mathbf{o}}})[\mathrm{val}(\mathrm{tr}(p), s) = \iota]\big\}.$$

The following definition will make more sense when used in our iterations.

Definition 1.3. For $\mathbf{o} \in \mathbf{O}_{\mathbf{TR}}$, we define $\mathbb{Q}_{\mathbf{o}} = (Q_{\mathbf{o}}, \leq_{\mathbf{o}})$ as follows.

(A) We let $\mathrm{tr}_{\mathbf{o}}(p) := \mathrm{tr}(p)$ and $\sigma_{\mathbf{o}}(p) := \sigma_{\mathbf{o}}(\mathrm{tr}_p)$.
(B) Each $p \in Q_{\mathbf{o}}$ is of the form $(\mathrm{tr}_p, \mathcal{F}_p)$, where $\mathrm{tr}_p = \mathrm{tr}(p) \in \mathrm{TR}_{\mathbf{o}}$ and $\mathcal{F}_p$ is a set of members of $Q_{\mathbf{o}}^{\mathrm{at}}$ of cardinality $< \sigma_{\mathbf{o}}(\mathrm{tr}_p)$ such that (recalling 1.2(F)(c))
$$q \in \mathcal{F}_p \Rightarrow \mathrm{tr}_{\mathbf{o}}(q) \leq_{\mathrm{TR}[\mathbf{o}]} \mathrm{tr}_{\mathbf{o}}(p) \in A_{\mathbf{o},q}.$$
(C) For $p, q \in Q_{\mathbf{o}}$, we say $p \leq_{\mathbf{o}} q$ <u>iff</u> $\mathrm{tr}(p) \leq_{\mathrm{TR}} \mathrm{tr}(q)$ and[5] $\mathcal{F}_p \subseteq \mathcal{F}_q$.
(D) When we say '$\langle p_i : i < \delta \rangle$ is $\leq_{\mathbf{o}}$-*truly increasing* in $\mathbb{Q}_{\mathbf{o}}$,' we mean it is increasing in $\mathbb{Q}_{\mathbf{o}}$ and $\sigma_{\mathbf{o}}$-truly increasing as defined in 1.2(F)(a).
(E) We define $\underset{\sim}{\eta}_{\mathbf{o}}$ as in Definition 1.2. (Abusing our notation slightly, we will not distinguish between the two uses.)

Claim 1.4. *Assume* $\mathbf{o} \in \mathbf{O}_{\mathbf{TR}}$.

1) $\mathbb{Q}_{\mathbf{o}}$ *is a strategically* $(<\kappa)$*-complete* κ^+*-cc forcing.*

2) *Assume* $(\forall \mathbf{o} \in \mathbf{TR})\big[\mathrm{TR}_{\mathbf{o}} \subseteq \mathcal{H}(\kappa)\big]$ *(*or *just that for every* $\mu < \kappa$*, the set* $\{s \in \mathrm{TR}_{\mathbf{o}} : \sigma_{\mathbf{o}}(s) \geq \mu\}$ *is equal to* $\bigcup_{\varepsilon<\kappa} Y_\varepsilon$ *such that*[6] $\bar{t} \in {}^{\mu>}Y_\varepsilon \Rightarrow \bar{t} \in \mathrm{dom}(\mathsf{plus}_{\mathbf{o}})$*).*

[5] Why not just "for every $r_1 \in \mathcal{F}_p$ there exists $r_2 \in \mathcal{F}_q$ such that $r_1 \leq_{\mathbf{o}} r_2$?" Then we would only get $(<\kappa)$-strategic completeness.

[6] A typical case is when $\mathrm{tr}(p)$ is a function from some $u \in [2^\kappa]^{<\kappa}$ to $\mathcal{H}(\kappa)$ and $s \in \mathrm{TR}_{\mathbf{o}} \Rightarrow \sigma_{\mathbf{o}}(s) = \min\{\sigma_{\mathbf{o}}(s \restriction \{\alpha\}) : \alpha \in \mathrm{dom}(s)\}$.

For every $\mu < \kappa$, there exists a dense open subset of $\mathbb{Q}_{\mathbf{o}}$ which is the union of κ-many sets such that any $< \mu$ members of any one set has a common upper bound.

3) *If $\delta < \kappa$ is a limit ordinal and $\bar{p} = \langle p_i : i < \delta \rangle$ is truly $\leq_{\mathbf{o}}$-increasing in $\mathbb{Q}_{\mathbf{o}}$, then $\bar{p}$ has a $\leq_{\mathbb{Q}_{\mathbf{o}}}$-lub (call it p_δ) such that*

- $\mathrm{tr}(p_\delta) = \mathsf{plus}_{\mathrm{TR}_{\mathbf{o}}}(\langle \mathrm{tr}_{\mathbf{o}}(p_i) : i < \delta \rangle)$
- $\mathcal{F}_{p_\delta} = \bigcup\limits_{i<\delta} \mathcal{F}_{p_i}$.

4) *If $p_i \in Q_{\mathbf{o}}$, $\mathrm{tr}_{\mathbf{o}}(p_i) = \mathbf{x}_i$ for $i < i_*$, $\langle \mathbf{x}_i : i < i_* \rangle \in \mathrm{dom}(\mathsf{plus}_{\mathbf{o}})$, and $i_* < \sigma_{\mathbf{o}}(\mathbf{x}_0)$, then there is a $\mathbf{y} \in \mathrm{TR}_{\mathbf{o}}$ such that $\big(\mathbf{y}, \bigcup\limits_{i<i_*} \mathcal{F}_{p_i}\big) \in Q_{\mathbf{o}}$ is a common upper bound of $\{p_i : i < i_*\}$ in $\mathbb{Q}_{\mathbf{o}}$.*

5) *The set $\{p \in Q_{\mathbf{o}} : |\mathcal{F}_p| \leq 1\}$ is dense in $\mathbb{Q}_{\mathbf{o}}$.*

Proof. Straightforward. E.g. for part (1),

1) [$(<\kappa)$**-complete:**] Obvious.

[κ^+**-cc:**]

For each $p \in \mathbb{Q}_{\mathbf{o}}$, choose $\nu = \nu_p \in \mathrm{TR}$ such that $\mathrm{tr}(p) \leq_{\mathrm{TR}_{\mathbf{o}}} \mathbf{x}_p$ and $\sigma_{\mathbf{o}}(\mathbf{x}_p) > \mu \cdot |\mathcal{F}_p|$. Suppose $\langle p_\alpha : \alpha < \kappa^+ \rangle$ is a sequence of members of $\mathbb{Q}_{\mathbf{o}}$ and $\mathbf{x}_\alpha := \mathbf{x}_{p_\alpha}$ for $\alpha < \kappa^+$.

By the demand on $\mathrm{TR}_{\mathbf{o}}$, there are $\alpha_0 < \ldots < \alpha_{n-1}$ (for $n \geq 2$) such that $\mathsf{plus}_{\mathbf{o}}(\langle \mathbf{x}_{\alpha_\ell} : \ell < n \rangle)$ is well defined so easily $\{p_{\alpha_\ell} : \ell < n\}$ has a common upper bound q such that $\mathrm{tr}(q)$ is a lub of $\langle \mathbf{x}_{\alpha_\ell} : \ell < n \rangle$. $\square_{1.4}$

Claim 1.5. 1) *If $\mathbf{o} \in \mathbf{O}$ then any $\sigma_{\mathbf{o}}$-truly increasing sequence of length $< \kappa$ has a lub. (See* 1.2(G)(c),(d).*)*

2) *If $\mathbf{o}_s \in \mathbf{O}$ (or just $\mathbf{o}_s \in \mathbf{O}^+$) for $s \in I$, then $\mathbf{o} := \prod\limits_{s \in I}^{<\kappa} \mathbf{o}_s$, naturally defined, belongs to $\mathbf{O}_{\mathbf{TR}(+)}$, and $\mathrm{TR}_{\mathbf{o}} = \prod\limits_{s \in I}^{<\kappa} \mathrm{TR}_s$.*

3) *Assume $\mathbf{o} \in \mathbf{O}$.*

- (*A*) $(\mathbf{x}, \mathcal{F}) \in \mathbb{Q}_{\mathbf{o}}$ *iff*
  - (*a*) $\mathbf{x} \in \mathrm{TR}_{\mathbf{o}}$
  - (*b*) $|\mathcal{F}| < \sigma_{\mathbf{o}}(\mathbf{x})$
  - (*c*) $f \in \mathcal{F} \Rightarrow \big(\mathbf{x}, \{f\}\big) \in \mathbb{Q}_{\mathbf{o}}^{\mathrm{at}}$.
- (*B*) $(\mathbf{x}_1, \mathcal{F}_1) \leq_{\mathbf{o}} (\mathbf{x}_2, \mathcal{F}_2)$ *iff*
  - (*a*) $(\mathbf{x}_1, \mathcal{F}_1), (\mathbf{x}_2, \mathcal{F}_2) \in \mathbb{Q}_{\mathbf{o}}$
  - (*b*) $\mathcal{F}_1 \subseteq \mathcal{F}_2$
  - (*c*) $f \in \mathcal{F}_1 \Rightarrow \big(\mathbf{x}_1, \{f\}\big) \leq_{\mathbf{o}} \big(\mathbf{x}_2, \{f\}\big)$.

Proof. Easy. $\square_{1.5}$

Example 1.6. 1) Let $\mathbf{n}$ and $\mathbb{Q}^2_{\mathbf{n}}$ be as in [HS, Def. 2.5].

(A) We define an $\aleph_0$-trunk controller $\mathrm{TR} = \mathrm{TR}_{\mathbf{n}} = \mathrm{TR}[\mathbf{n}]$ as follows.
  (a) $\kappa := \aleph_0$
  (b) $\mathrm{TR}_{\mathbf{n}}$ has set of elements $T_{\mathbf{n}}$ (as in [HS]).
  (c) $\sigma_{\mathrm{TR}_{\mathbf{n}}}$ will be the function $\eta \mapsto \mu_\eta$ (again, see there).

(B) We define $\mathbf{O}_{\mathbf{n}} := \{\mathbf{o} : \mathbb{Q}_{\mathbf{o}} = \mathbb{Q}^2_{\mathbf{n}}\}$ (so it is a singleton), and $\mathbf{O}_{1067}$ will be the union of all $\mathbf{O}_{\mathbf{n}}$-s for $\mathbf{n}$ as in [HS, Def. 2.5].

2) Let κ be inaccessible and $\mathbb{Q}_{\bar\lambda}$ be defined similarly to that in [She23, 0.2=Lz23] (but there $\lambda_\eta := \theta_{\ell g(\eta)}$). It will be of the form $\mathbb{Q}_{\mathbf{o}}$, where $\mathbf{o} = \mathbf{o}_{\bar\lambda} \in \mathbf{O}_{\mathbf{TR}}$.

More fully,

- $\bullet_1$ $\bar\lambda = \langle \lambda_\eta, \lambda^0_\eta, D_\eta : \eta \in \mathcal{T}\rangle$
- $\bullet_2$ $\mathcal{T} := \{\eta \in {}^{\kappa >}\kappa : \varepsilon < \ell g(\eta) \Rightarrow \eta(\varepsilon) < \lambda_{\eta\restriction\varepsilon}\}$
- $\bullet_3$ If $\eta \neq \nu \in \mathcal{T}$ then $\lambda_\eta \neq \lambda_\nu$.
- $\bullet_4$ If $\ell g(\eta) < \ell g(\nu)$ then $\lambda_\eta < \lambda^0_\eta \leq \lambda_\nu < \kappa$.
- $\bullet_5$ If $\eta \in \mathcal{T}$ then $\mathrm{cf}(\lambda_\eta) \geq \lambda^0_\eta = \mathrm{cf}(\lambda^0_\eta) > \prod\limits_{\lambda_\nu < \lambda_\eta} \lambda_\nu$.
- $\bullet_6$ D_η is a λ^0_η-complete uniform filter on λ_η (and naturally $A \in [\lambda_\eta]^{<\lambda_\eta} \Rightarrow \lambda_\eta \setminus A \in D_\eta$).

Now let us work towards defining $\mathbf{o} = \mathbf{o}_{\bar\lambda}$.

(A) $\mathrm{TR} = \mathrm{TR}_{\bar\lambda} = \mathrm{TR}[\bar\lambda]$ is defined as follows.
  (a) $\kappa_{\mathrm{TR}} := \kappa$
  (b) $|\mathrm{TR}| := \big\{\eta \in {}^{\kappa >}\kappa : \varepsilon < \ell g(\eta) \Rightarrow \eta(\varepsilon) < \lambda_{\eta\restriction\varepsilon}\big\}$
  (c) $\sigma_{\mathrm{TR}}(\eta) := \lambda^0_\eta$.

(B) $p \in \mathbb{Q}^{\mathrm{at}}_{\bar\lambda}$ iff
  (a) p is a $(<\kappa)$-complete subtree of TR.
  (b) $\mathrm{tr}(p)$ is the trunk of p.
  (c) If $\mathrm{tr}(p) \trianglelefteq \eta \in p$, then $\{\alpha < \lambda_\eta : \eta\hat{\ }\langle\alpha\rangle \in p\} \in D_\eta$.

(C) $p \leq_{\mathbb{Q}_{\bar\lambda}} q$ iff $p \supseteq q$.

3) Assume κ is inaccessible, and suppose $\mathcal{T} = \mathcal{T}_{\bar\lambda}$ is a subtree of $({}^{\kappa>}\kappa, \triangleleft)$ (so it is closed under initial segments and includes $\langle\,\rangle$). Further suppose $\mathcal{T}$ has no maximal node, and is closed under unions of $\triangleleft$-increasing chains of length $< \kappa$.

Let $\bar\lambda = \langle \lambda_\eta, \lambda^0_\eta, D_\eta : \eta \in \mathcal{T}\rangle$ be such that '$\eta \in \mathcal{T} \Rightarrow (A)_\eta \vee (B)_\eta$' holds, where

$(A)_\eta$ $\lambda_\eta = \lambda^0_\eta = 1$ and $D_\eta = \{0\}$.

$(B)_\eta$ λ^0_η is an infinite regular cardinal $\leq \lambda_\eta < \kappa$, and D_η is a λ^0_η-complete filter on λ_η.

Furthermore, we demand that clause (2) $\bullet_2$ holds.

Now define $\mathbf{o}_{\bar\lambda}$ as in clause (2).

We say that $\bar\lambda$ (or $\mathcal{T}_{\bar\lambda}$, or $\mathbf{o}_{\bar\lambda}$) is a κ-*candidate* when it is as above.

4) We say that $\bar\lambda$ (and $\mathbf{o}_{\bar\lambda}$) is κ-*active* when, in addition to the above, we have the following:

- For every $\eta \in \lim(\mathcal{T})$, the set
$\{\delta < \kappa : \lambda_{\eta\restriction\delta} > 1 \text{ and } (\forall\nu \in \mathcal{T} \cap {}^{\delta}\kappa)[\nu \neq \eta \restriction \delta \Rightarrow \lambda_{\eta\restriction\delta} < \lambda_\nu \vee \lambda_\nu = 1]\}$
is stationary.
- If $\eta, \nu \in \mathcal{T}$ with $\ell g(\eta) < \ell g(\nu)$, then $\lambda_\nu \neq 1 \Rightarrow \lambda_\eta < \lambda^0_\nu$.

Observation 1.7. *If κ is inaccessible (or weakly inaccessible) and $\diamondsuit_\kappa$ holds, then there is a κ-active $\bar{\lambda}$.*

Remark 1.8. We will return to this example in §3.

Definition 1.9. 0) Let $\mathbf{M}$ be the class of combinatorial templates (defined below).

1) A *combinatorial template* (or CT) $\mathbf{m}$ consists of:

(a) A linear order $L_{\mathbf{m}}$ (we could have used 'partial order'; it does not really matter for our purposes).
We may write $s \in \mathbf{m}$ instead of $s \in L_{\mathbf{m}}$, or $s <_{\mathbf{m}} t$ instead of $s <_{L_{\mathbf{m}}} t$.

(b) $\bullet_1$ $\overline{\mathcal{A}} = \langle \mathcal{A}_t : t \in L_{\mathbf{m}} \rangle$ (called the *memory*), where $\mathcal{A}_t \subseteq \mathcal{P}(L_{\mathbf{m},t})$ for each $t \in L_{\mathbf{m}}$ and $L_{\mathbf{m},t} := \{s \in L_{\mathbf{m}} : s <_{L_{\mathbf{m}}} t\}$.
$\bullet_2$ We denote $\mathcal{A} := \bigcup\limits_{t \in L_{\mathbf{m}}} \mathcal{A}_t$.

(c) Each $\mathcal{A}_t$ is a family of subsets of $L_{\mathbf{m},t}$ with $\varnothing \in \mathcal{A}_t$. We let $\mathcal{A}^+_t := \mathcal{A}_t \setminus \{\varnothing\}$.

2) For $\mathbf{m} \in \mathbf{M}$, we add new objects $t(+)$ for all $t \in L_{\mathbf{m}}$, as well as ∞, and define $L^+_{\mathbf{m}}$, $L_{\mathbf{m},x}$, $L^+_{\mathbf{m},x}$, etc. as follows.

(a) $L^+_{\mathbf{m}} := \{t, t(+) : t \in L_{\mathbf{m}}\} \cup \{\infty\}$
(b) Naturally, $\langle t : t \in L_{\mathbf{m}} \rangle\hat{\ }\langle t(+) : t \in L_{\mathbf{m}} \rangle\hat{\ }\langle \infty \rangle$ is without repetition.
(c) $<_{L^+_{\mathbf{m}}}$ is the closure, to a linear order, of the set
$$\{t < t(+) : t \in L_{\mathbf{m}}\} \cup \{s(+) < t : s <_{L_{\mathbf{m}}} t\} \cup \{t(+) < \infty : t \in L_{\mathbf{m}}\}.$$
(d) For $t \in L^+_{\mathbf{m}}$, let $L_{\mathbf{m},t} := \{s \in L_{\mathbf{m}} : s <_{L^+_{\mathbf{m}}} t\}$ and
$$L^+_{\mathbf{m},t} := \{s \in L^+_{\mathbf{m}} : s <_{L^+_{\mathbf{m}}} t\}.$$

3) For $L \subseteq L_{\mathbf{m}}$, we define $\mathbf{m} \restriction L \in \mathbf{M}$ as follows.

$\bullet_1$ $L_{\mathbf{m}\restriction L} := L$
$\bullet_2$ $\mathcal{A}^{\mathbf{m}\restriction L}_t := \mathcal{A}^{\mathbf{m}}_t \cap \mathcal{P}(L)$.

4) For $t \in L^+_{\mathbf{m}}$, let $\mathbf{m} \restriction t := \mathbf{m} \restriction L^+_{\mathbf{m},t}$.

5) We say π is an *isomorphism from* $\mathbf{m}_1$ *onto* $\mathbf{m}_2$ (for $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$) <u>when</u>
$$\pi : L_{\mathbf{m}_1} \to L_{\mathbf{m}_2}$$
is an order-preserving surjection mapping $\mathcal{A}^{\mathbf{m}_1}_t$ onto $\mathcal{A}^{\mathbf{m}_2}_{\pi(t)}$ for each $t \in L_{\mathbf{m}_1}$.

Remark 1.10. If $t \in L_{\mathbf{m}}$ and $L \subseteq L_{\mathbf{m}}$, we may abuse notation and write L_t in place of $L \cap L_{\mathbf{m},t}$.

Definition 1.11. We define a two-place relation $\leq_{\mathbf{M}}$ (obviously a partial order) on the class of combinatorial templates by:

$$\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2 \text{ \underline{iff}}$$

(a) $L_{\mathbf{m}_1} \subseteq L_{\mathbf{m}_2}$ as linear orders.
(b) $\mathbf{m}_1$ and $\mathbf{m}_2$ use the same ∞ and the same $t(+)$ for all $t \in L_{\mathbf{m}_1}$.
(c) $t \in L_{\mathbf{m}_1} \Rightarrow \mathcal{A}_{\mathbf{m}_1,t} \subseteq \mathcal{A}_{\mathbf{m}_2,t}$

Claim 1.12. 1) *$\leq_{\mathbf{M}}$ is indeed a partial order on $\mathbf{M}$.*

2) *If $\langle \mathbf{m}_\varepsilon : \varepsilon < \delta \rangle$ is $\leq_{\mathbf{M}}$-increasing then $\mathbf{m} := \bigcup_{\varepsilon<\delta} \mathbf{m}_\varepsilon$ (naturally defined) exists, is a $\leq_{\mathbf{M}}$-lub, and is unique. In particular,*

$\boxplus_2$ (a) $t \in L_{\mathbf{m}}$ *iff* $(\exists \varepsilon < \delta)[t \in L_{\mathbf{m}_\varepsilon}]$.
(b) $s <_{\mathbf{m}} t$ *iff* $(\exists \varepsilon < \delta)[s, t \in L_{\mathbf{m}_\varepsilon} \wedge s <_{\mathbf{m}_\varepsilon} t]$.
(c) $\zeta < \delta \wedge t \in L_{\mathbf{m}_\zeta} \Rightarrow \mathcal{A}_{\mathbf{m},t} = \bigcup_{\varepsilon \in [\zeta,\delta)} \mathcal{A}_{\mathbf{m}_\varepsilon,t}$.

3) *There exists a $\leq_{\mathbf{M}}$-minimal element of $\mathbf{M}$. (Specifically, it is an $\mathbf{m}$ with $L_{\mathbf{m}} = \varnothing$.)*

4) *$(\mathbf{M}, \leq_{\mathbf{M}})$ has amalgamation, and hence the JEP. Moreover, if $\mathbf{m}_0 \leq \mathbf{m}_\ell$ for $\ell = 1, 2$ and (for transparency) $L_{\mathbf{m}_0} = L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2}$, then $\mathbf{m}_1$ and $\mathbf{m}_2$ have a* canonical amalgamation *over $\mathbf{m}_0$ — call it $\mathbf{m}_*$.*

By this we mean

$\boxplus_4$ (a) $\mathbf{m}_* \in \mathbf{M} \wedge \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_* \wedge \mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}_*$
(b) $s \in L^+_{\mathbf{m}_*}$ *iff* $s \in L^+_{\mathbf{m}_1} \cup L^+_{\mathbf{m}_2}$
(c) $L^+_{\mathbf{m}_*} \models$ '$r <_{\mathbf{m}} s$' *iff* '$\bullet_0 \vee \bullet_1 \vee \bullet_2$,' *where*
$\bullet_0$ $L_{\mathbf{m}_\ell} \models$ '$s < t$' *for some* $\ell \in \{0, 1\}$.
$\bullet_1$ $\big(\exists t \in L_{\mathbf{m}_0}\big)\big(\exists k, \ell \in \{1, 2\}\big)\big[r \leq_{\mathbf{m}_\ell} t \wedge t \leq_{\mathbf{m}_k} s \wedge r \neq s\big]$
$\bullet_2$ $r \in L^+_{\mathbf{m}_1} \setminus L^+_{\mathbf{m}_0}$, $s \in L^+_{\mathbf{m}_2} \setminus L^+_{\mathbf{m}_0}$, *and*
$$(\forall t \in L^+_{\mathbf{m}_0})[t <_{\mathbf{m}_1} r \wedge t <_{\mathbf{m}_2} s].$$
(d) $$\mathcal{A}_{\mathbf{m},s} = \begin{cases} \mathcal{A}_{\mathbf{m}_\ell,s} & \text{if } \ell \in \{1,2\} \wedge s \in L_{\mathbf{m}_\ell} \setminus L_{\mathbf{m}_0} \\ \mathcal{A}_{\mathbf{m}_1,s} \cup \mathcal{A}_{\mathbf{m}_2,s} & \text{if } s \in \mathcal{A}_{\mathbf{m}_0} \end{cases}$$

Proof. 1) Obvious.

2) Straightforward.

3) Trivial.

4) We have to check all the clauses in Definition 1.9(1) to verify '$\mathbf{m}_* \in \mathbf{M}$,' and all clauses in 1.12 for $\mathbf{m}_\ell \leq_{\mathbf{M}} \mathbf{m}_*$.

<u>$\mathbf{m}_* \in \mathbf{M}$:</u>

Clauses 1.9(1)(a),(b): Obvious: these were just definitions and notation.

Clause (c): Also a definition; obvious.

$\underline{\mathbf{m}_\ell \leq_{\mathbf{M}} \mathbf{m}_*}$ (for $\ell \leq 2$).

Clauses 1.11(a)-(c) are all obvious. $\square_{1.12}$

§ 2. Investigating $\mathbf{p} \in \mathbf{P}$

Definition 2.1. $\mathbf{P}_{\mathbf{m}}$ is the class of $\mathbf{m}$-ATIs (see below), and

$$\mathbf{P} := \bigcup_{\mathbf{m} \in \mathbf{M}} \mathbf{P}_{\mathbf{m}}.$$

(ATI stands for *abstract template iterations.*)

Definition 2.2. For $\mathbf{m}$ a combinatorial template, we say $\mathbf{p}$ is an $\mathbf{m}$-*ATI* when it consists of the objects

$\boxplus$ (a) $\mathbf{m} \in \mathbf{M}$ (We may write $L_{\mathbf{p}}$ and $L_{\mathbf{p},t}$ instead of $L_{\mathbf{m}}$ and $L_{\mathbf{m},t}$, etc.)
(b) Forcing notions $\mathbb{P}$ and $\mathbb{P}_s$ for $s \in L^+_{\mathbf{p}}$.
(c) $\mathbf{o}_t \in \mathbf{O}_{\mathrm{TR}}$ (an object, not a name), $\underset{\sim}{\mathbb{Q}}_t$, and $\underset{\sim}{\eta}_t$ for $t \in L_{\mathbf{p}}$. We let $S_t = S[t] := S_{\mathbf{o}_t}$.
(d) $\mathrm{TR}_t = (\mathrm{TR}_{\mathbf{o}_t}, \leq_{\mathbf{o}_t}, \mathsf{plus}_{\mathbf{o}_t}) = (\mathrm{TR}_t, \leq_t, \mathsf{plus}_t)$ is from $\mathbf{V}$.

We demand that the following conditions are satisfied:

(A) $\prod\limits_t^{<\kappa} \mathrm{TR}_{\mathbf{o}_t}$ is a κ-trunk controller.
(B) (a) $\langle \mathbb{P}_s : s \in L^+_{\mathbf{p}} \rangle$ is $\lessdot$-increasing with s.
(b) Furthermore, we demand $\mathbb{P} = \mathbb{P}_\infty$.
(c) $\underset{\sim}{\eta}_t$ is a $\mathbb{P}_{t(+)}$-name of a member of ${}^{S[t]}2$,[7] and $\underset{\sim}{\bar{\eta}} := \langle \underset{\sim}{\eta}_t : t \in L_{\mathbf{p}} \rangle$.
(C) We require that $p \in \mathbb{P}$ <u>iff</u>
(a) p is a function with $\mathrm{dom}(p) \in [L_{\mathbf{p}}]^{<\kappa}$.
(b) For $s \in \mathrm{dom}(p)$, $p(s)$ is a $\mathbb{P}_s$-name of a member of $\underset{\sim}{\mathbb{Q}}_s$ of the form $(\mathrm{tr}_{p(s)}, \mathcal{F}_{p(s)})$.
(c) $\mathrm{tr}_{p(s)} \in \mathrm{TR}_t$ is an object, not just a $\mathbb{P}_s$-name.
(d) $\mathcal{F}_{p(s)} := \{p_{s,A} : A \in \mathcal{A}_{p(s)}\}$, where $|\mathcal{A}_{p(s)}| < \kappa$ But abusing our notation slightly, we allow $A_1 = A_2$ but $p_{s,A_1} \neq p_{s,A_2}$ (but still $|\mathcal{F}_{p(s)}| < \kappa$ as well).
(e) $\mathfrak{C}_{p(s)} \subseteq [\mathcal{A}_s]^{<\sigma(\mathrm{tr}_{p(s)})}$ (see clause (G)(b) below) and

$$p_{s,A} = \mathbf{B}(\ldots, \underset{\sim}{\eta}_{t[j,p_{s,A}]}(\varepsilon_j^{p_{s,A}}), \ldots)_{j<j_{p_{s,A}}}$$

<u>where</u> $t_j^{p_{s,A}} = t[j, p_{s,A}] \in {}^\kappa A$ and $\mathbf{B} = \mathbf{B}_{s,A} = \mathbf{B}_{p,s,A}$, and we make the following demands:

- $\bullet_1$ $t_j^{p_{s,A}} \in A$, $\varepsilon_j^{p_{s,A}} \in S_{t_j}$ and $j_{p_{s,A}} \leq \kappa$.
- $\bullet_2$ $\mathbf{B}$ is a κ-Borel function[8] from

$$\prod_{j<j_{p_{s,A}}} S_{t[j,p_{s,A}]}$$

to S_s such that the image has cardinality $\leq \kappa$.
More concretely, there is (in $\mathbf{V}$) an $S'_{p_{s,A}} \in [S_s]^{\leq\kappa}$ which contains the image of $\mathbf{B}$.
- $\bullet_3$ $\mathrm{tr}_{p_{s,A}} \leq_{\mathrm{TR}_t} \mathrm{tr}_{p(s)}$.
- $\bullet_4$ If $A \in \mathcal{A}_{p(s)}$ then $p \restriction L_s \Vdash_{\mathbb{P}_s}$ "$\mathrm{tr}_{p(s)} \in \mathcal{E}_{p_{s,A}}$".

[7] So in the simple case (which we have currently adopted), this will be equal to ${}^{\mathrm{TR}_t}2$.
[8] That is, a function where the pre-image of every element of S_s is a $\leq\kappa$-Borel set. (The point here is absoluteness.)

(f) $S = S_{\mathbf{p}} := \bigcup_{t \in L_{\mathbf{p}}} S_t$.

(D) (a) Given $p \in \mathbb{P}$ and $s \in \mathrm{dom}(p)$, let $\mathrm{supp}(p(s))$ be the set of all coordinates used in the Borel function $p(s)$ (i.e. the $t_j^{p_{s,A}}$-s). So $\big|\mathrm{supp}(p(s))\big| \leq \kappa$.

(b) For $p \in \mathbb{P}$, we define $\mathrm{supp}(p) := \mathrm{dom}(p) \cup \bigcup_{s \in \mathrm{dom}(p)} \mathrm{supp}(p(s)) \in [L_{\mathbf{p}}]^{\leq \kappa}$.

(c) Note that $\mathrm{supp}(p) \subseteq L_{\mathbf{p},t}$ if $\mathrm{dom}(p) \subseteq L_{\mathbf{p},t}$ (i.e. if $p \in \mathbb{P}_t$).

(d) For $L \subseteq L_{\mathbf{p}}$, we set

$$\mathbb{P}_L := \mathbb{P} \restriction \{p \in \mathbb{P} : \mathrm{supp}(p) \subseteq L\}$$

and $\mathbb{P}_s := \mathbb{P}_{L_{\mathbf{m},s}}$.

(e) If $\ell = 1, 2$, $s \in I$, $\mathrm{dom}(p^\ell) = \{s\}$, and

$$p^\ell\Big(\mathbf{x}_\ell, \big\{\mathbf{B}(\ldots, \underset{\sim}{\eta}_{t[y,(p^\ell)_{s,A_0}]}(\varepsilon[(p^\ell)_{s,A_\ell}]), \ldots)_{i < j_{(p^\ell)_{s,A_\ell}}}\big\}\Big)$$

is as in clause (C)(e) above, <u>then</u> there exists some κ-Borel function

$$\mathbf{B} = \mathbf{B}_{\mathbf{x}_1,\mathbf{x}_2,t}(\ldots, \underset{\sim}{\eta}_{t_{1,i}}(\varepsilon_{1,i}), \ldots; \ldots, \underset{\sim}{\eta}_{2,j}(\varepsilon_{2,j}), \ldots)_{\substack{i < \iota_1 \\ j < \iota_2}}$$

from $\{t_{\ell,i} : \ell \in \{1,2\} \text{ and } i < \iota_\ell\}$ to $\{0,1\}$ which gives the truth value of '$p^1 \leq_{\underset{\sim}{\mathbb{Q}}_t} p^2$'.

(E) For $L \subseteq L_{\mathbf{m}}$ and $p \in \mathbb{P}$, we define $p \restriction L$ to mean

(a) $\mathrm{dom}(p \restriction L) := \mathrm{dom}(p) \cap L$

(b) For all s in the domain,

$$(p \restriction L)(s) := \big(\mathrm{tr}_{p(s)}, \big\{p_{s,A} : A \in \mathcal{A}_{p(s)} \cap \mathcal{P}(L)\big\}\big).$$

(F) For $t \in L_{\mathbf{p}}$,

(a) $\underset{\sim}{\mathbb{Q}}_t$ is the $\mathbb{P}_t$-name of $\underset{\sim}{\mathbb{Q}}_{\mathbf{o}_t}$.

(b) If $p \in \mathbb{P}_t$ then $\mathrm{dom}(p) \subseteq \mathrm{supp}(p) \in [L_{\mathbf{p},t}]^{\leq \kappa}$. [This follows.]

(c) $\mathbb{P}_t$ (and $\mathbb{P}_A$ and $\mathbb{P}_L$, for $A \in \mathcal{A}_{\mathbf{p}}$ and $L \subseteq L_{\mathbf{m}}$) is a $({<}\,\kappa)$-complete κ^+-cc forcing notion.[9]

(d) $\underset{\sim}{\eta}_t$ is the generic of $\underset{\sim}{\mathbb{Q}}_t$ (which we may identify with the subset $\underset{\sim}{\eta}_t^{-1}(\{1\}) \subseteq S_t$).

(G) (a) Note that a $\mathbb{P}_A$-generic filter (for $A \in \mathcal{A}_{\mathbf{m},s}$) lets us evaluate the $\mathbb{P}_{r(+)}$-names $\underset{\sim}{\eta}_r$ for $r \in A$, and therefore the value of the κ-Borel function $p_{s,A}$.
This way we get a $\mathbb{P}_s$-name for the value, which we may write as $p_{s,A}[\mathbf{G}_{\mathbb{P}_A}]$ or as $p(s)(\bar{\underset{\sim}{\eta}} \restriction A)$.

(b) We require that $\underset{\sim}{\eta}_t^{-1}(\{1\}) = \bigcup\{\mathrm{val}_t(\mathrm{tr}_{p(t)}) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{t(+)}}\}$ (recalling Definition 0.7(1)(F)).

(c) If $\kappa := \aleph_0$ then we demand that for every $p \in \mathbb{P}_t$ and $m < \kappa$ there exists a $q > p$ such that $|\mathcal{F}_{q(s)}| \cdot m < \sigma_{\mathbf{o}_s}(\mathrm{tr}(q(s)))$ for all $s \in \mathrm{dom}(q)$.

(H) We require that $p < q$ in[10] $\mathbb{P}$ <u>iff</u>

(a) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$

(b) If $s \in \mathrm{dom}(p)$ <u>then</u> $\mathrm{tr}_{p(s)} \leq_{\mathbf{o}_s} \mathrm{tr}_{q(s)}$, $\sigma_{\mathbf{o}_s}(\mathrm{tr}_{p(s)}) < \sigma_{\mathbf{o}_s}(\mathrm{tr}_{q(s)})$, and $\mathcal{F}_{p(s)} \subseteq \mathcal{F}_{q(s)}$.

---

[9] The κ^+-cc follows from clause (B)(a).

[10] Yes! <u>Not</u> '$p \leq q$!'

(Note that for $p \in \mathbb{P}$ and $s \in L^+_{\mathbf{p}}$, we have '$p \restriction L_{\mathbf{p},s} \in \mathbb{P}_s$' by clause (E).)

Also observe that this is a requirement and *not* a definition, unlike the classical case.

(I) When dealing with different ATIs, instead of $\mathbb{P}$, $\leq$, $\mathbb{P}_t$, S_t, $\underset{\sim}{\mathbb{Q}}_t$, etc., we may write $\mathbb{P}_{\mathbf{p}}$, $\leq_{\mathbf{p}}$, $\mathbb{P}_{\mathbf{p},t}$, $S_{\mathbf{p},t}$, $\underset{\sim}{\mathbb{Q}}_{\mathbf{p},t}$, etc., to indicate that we mean the component of the respective $\mathbf{p}$.

Remark 2.3. 1) Recall that $L_{\mathbf{p}}$ is just a linear order and not necessarily a well-ordering. More concretely, we do not even exclude the possibility that

- There is an infinite sequence $(s_n)_{n\in\omega}$ with $s_{n+1} \in A_n \in \mathcal{A}_{s_n}$.

(unlike [She04], [She]).

2) As a consequence: Given $L_{\mathbf{m}}$ and a sequence of (definitions for) $\mathbb{Q}_s$ for $s \in L_{\mathbf{m}}$, it is neither clear that there is an iteration $\mathbb{P}$ as above *nor* that it is unique.

(In contrast, the usual forcing iteration assumes that the index set is well-ordered, and we always get a well-defined iteration from a sequence of iterands.)

3) But if $\mathbf{m}$ fails the demand from the bullet in part (1) (so $L_{\mathbf{m}}$ is as in [She04, §2]), then it is unique.

4) The following is true:

⊛ If $A \in \mathcal{A}_t$ then $\Vdash_{\mathbb{P}_{\mathbf{p}}}$ "$\underset{\sim}{\eta}_t$ is generic for $(\mathbb{Q}_{\mathbf{o}}, \underset{\sim}{\eta}_{\mathbf{o}})$ over $\mathbf{V}[\underset{\sim}{\bar{\eta}} \restriction A]$".

Claim 2.4. 1) *If* $\mathbf{p} \in \mathbf{P}$, *then* $\mathbb{P}_{\mathbf{p}}$ *is* κ^+*-cc and* $(<\kappa)$*-complete.*

2) *In Definition* 2.2, $\underset{\sim}{\mathbb{Q}}_t$, S_t, *and* TR_t *are derivable from the other objects.*

3) *Any* $\leq_{\mathbb{P}_{\mathbf{p}}}$*-increasing*[11] *sequence* $\bar{p} = \langle p_\varepsilon : \varepsilon < \delta\rangle$ *(where* $\delta < \kappa$*) has a* $\leq_{\mathbb{P}_{\mathbf{p}}}$*-lub* $\lim(\bar{p})$.

Proof. 1) $(<\kappa)$-complete:

Let $\delta < \kappa$ be a limit ordinal and $\bar{p} = \langle p_i : i < \delta\rangle$ be increasing in $\mathbb{P}_{\mathbf{p}}$.

Recalling Definition 2.2(H)(b), we know that either $\bar{p} \restriction [j, \delta)$ is constant for some $j < \delta$ or $\bar{p} \restriction u$ is $<_{\mathbb{P}_{\mathbf{p}}}$-increasing for some unbounded $u \subseteq \delta$. In the first case, p_j is a lub. In the latter case,

⊞₁ For each $\varepsilon \in u$ and $s \in \mathrm{dom}(p_\varepsilon)$,

$$\Vdash_{\mathbb{P}_s} \text{“}\langle p_\zeta(s) : \zeta \in u\rangle \text{ is truly } \leq_{\underset{\sim}{\mathbb{Q}}_{\mathbf{o}_s}}\text{-increasing”}$$

(as defined in the footnote inside clause (3)).

Define the function $p_\delta = \lim(\bar{p})$ as follows:

⊞₂ (a) $\mathrm{dom}(p_\delta) := \bigcup_{i<\delta} \mathrm{dom}(p_i)$

[11] What would occur if we weakened the order by removing the demand '$\sigma_{\mathbf{o}_s}(\mathrm{tr}_{p(s)}) < \sigma_{\mathbf{o}_s}(\mathrm{tr}_{q(s)})$' from 2.2(H)(b)?

Then we would have add it as a caveat to this statement — perhaps by stipulating "$\bar{p}$ is truly $\mathbb{P}_{\mathbf{p}}$-increasing," where *truly* means $s \in \mathrm{dom}(p_\varepsilon) \wedge \varepsilon < \zeta < \delta \Rightarrow \sigma(\mathrm{tr}(p_\varepsilon(s))) < \sigma(\mathrm{tr}(p_\zeta(s)))$.

(b) For $s \in \mathrm{dom}(p_\delta)$, let

$$p_\delta(s) := \Big(\mathsf{plus}_{\mathrm{TR}_s}(\langle \mathrm{tr}_{p_i(s)} : i \in [i_s,\delta)\rangle), \bigcup_{i\in[i_s,\delta)} \mathcal{F}_{p_i(s)}\Big),$$

where $i_s := \min\{i < \delta : s \in \mathrm{dom}(p_i)\}$.

Now check that p_δ is as required, recalling 1.2(F)(a) and 2.2(H).

<u>κ^+-cc:</u>

Let $\langle p_\alpha : \alpha < \kappa^+\rangle \subseteq \mathbb{P}_{\mathbf{p}}$. Without loss of generality, each p_α is like q from 2.2(G)(c) (for $m = 2$). There exists a stationary $S_1 \subseteq S^{\kappa^+}_\kappa$ such that $\langle \mathrm{dom}(p_\alpha) : \alpha \in S_1\rangle$ is a Δ-system with heart W_*. Recalling that $\prod\limits_{s\in W_*} \mathrm{TR}_{\mathbf{o}_s}$ is a κ-trunk controller, there exist $\alpha < \beta$ in S_1 such that

$$s \in W_* \Rightarrow \langle \mathrm{tr}_{p_\alpha(s)}, \mathrm{tr}_{p_\beta(s)}\rangle \in \mathrm{dom}(\mathsf{plus}_{\mathrm{TR}_s}).$$

If $\kappa > \aleph_0$ then the rest is clear by 1.2(G).

[Why? Because $\sigma_{\mathbf{o}_s}(\mathrm{tr}_{p_\varepsilon(s)})$ is a regular cardinal for some $s \in W_*$.]

So assume $\kappa := \aleph_0$. Let $\langle t_\ell : \ell < n\rangle$ list W_* in $<_{L_{\mathbf{p}}}$-increasing order, and $t_n := \infty \in L^+_{\mathbf{p}}$. Now fix $\alpha < \beta$ from S_1, and choose $q_\ell \in \mathbb{P}_{t_\ell}$ above

$$\{p_\alpha \restriction L_{t_\ell}, p_\beta \restriction L_{t_\ell}\}$$

by induction on $\ell \le n$. In the induction step, we use our assumption that

$$\sigma_{\mathbf{o}_s}\big(\mathrm{tr}(q_\ell(s))\big) > 2\big|\mathcal{F}_{q_\ell(s)}\big|.$$

2) Easy.

3) Covered in the proof of part (1). $\square_{2.4}$

Definition 2.5. 1) We say that L is $\mathbf{p}$-closed <u>when</u>

(A) $L \subseteq L_{\mathbf{p}}$
(B) If $A \in \mathcal{A}_t$ for some $t \in L$, then $A \subseteq L$.

2) For $L \subseteq L_{\mathbf{p}}$, we define

(A) $\mathbb{P}_L = \mathbb{P}_{\mathbf{p},L} := \mathbb{P}_{\mathbf{p}} \restriction \{p \in \mathbb{P}_{\mathbf{p}} : \mathrm{supp}(p) \subseteq L\}$ (as in 2.2(G)(d)).
(B) $\mathbf{p}' = \mathbf{p} \restriction L$ is defined naturally, with the intention that $\mathbf{p}' \in \mathbf{P}$. That is,
  (a) $\mathbb{P}_{\mathbf{p}'} := \mathbb{P}_{\mathbf{p},L}$
  (b) $\mathbf{o}_{\mathbf{p}',t} := \mathbf{o}_{\mathbf{p},t}$ for all $t \in L$.
  (c) $\mathcal{A}_{\mathbf{p}',t} := \mathcal{A}_{\mathbf{p},t} \cap \mathcal{P}(L)$.

3) We define "π is an isomorphism from $\mathbf{p}'$ onto $\mathbf{p}$" naturally. That is,

(A) $\pi : L_{\mathbf{p}'} \to L_{\mathbf{p}}$ is an isomorphism.
(B) If $t \in L_{\mathbf{p}'}$ then $\mathbf{o}_{\mathbf{p},\pi(t)} := \mathbf{o}_{\mathbf{p}',t}$.
(C) If $t \in L_{\mathbf{p}'}$ then $\mathcal{A}_{\mathbf{p},\pi(t)} := \{\pi[A] : A \in \mathcal{A}_{\mathbf{p}',t}\}$.

4) We define the partial order $\le_{\mathbf{P}}$ on $\mathbf{P}$ as $\mathbf{p}' \le_{\mathbf{P}} \mathbf{p}$ <u>iff</u> $\mathbf{p}' = \mathbf{p} \restriction L_{\mathbf{p}'}$.

5) If $p \in \mathbb{P}_{\mathbf{p}}$ and $L \subseteq L_{\mathbf{p}}$, then $p \restriction L$ is defined as follows.

- $\mathrm{dom}(p \restriction L) := \mathrm{dom}(p) \cap L$

- For all s in the domain, $p \upharpoonleft L$ sends
$$s \mapsto \big(\mathrm{tr}_{p(s)}, \{r \in \mathcal{F}_{p(s)} : \big(\exists A \in \mathcal{A}_s \cap \mathcal{P}(L)\big)\big[\mathrm{supp}(r) \subseteq A\big]\}\big).$$

Observation 2.6. 1) *If $p \in \mathbb{P}$ and $L \subseteq \mathrm{dom}(p)$, then:*

(A) *$p \upharpoonleft L$ and $p \restriction L$ belong to $\mathbb{P}$, with $p \upharpoonleft L \leq_{\mathbb{P}} p \restriction L \leq_{\mathbb{P}} p$.*
(B) *Moreover, $p \upharpoonleft L \in \mathbb{P}_L$.*

2) *If $L \subseteq L_{\mathbf{p}}$ <u>then</u>*

(A) $\mathbf{p} \restriction L \in \mathbf{P}$
(B) *For p, q in $\mathbb{P}_L$, we have $p \leq q$ <u>iff</u> 2.2(H) holds for $\mathbb{P}_L$; i.e. iff*
  - *$\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $\mathcal{F}_p \subseteq \mathcal{F}_q$.*
  - *If $s \in \mathrm{dom}(p)$ and $r \in \mathcal{F}_{p(s)}$ with $\mathrm{supp}(r) \subseteq A \in \mathcal{A}_{\mathbf{p},s}$ (so $A \subseteq L$), <u>then</u>*
$$q \upharpoonleft A \Vdash_{\mathbb{P}_A} \text{“}\mathbb{Q}_{\mathbf{o}_t} \models (\mathrm{tr}_p, \{r\}) \leq (\mathrm{tr}_q, \{r\})\text{”}.$$

Proof. Easy. $\square_{2.6}$

Claim 2.7. 1) *If L is a $\mathbf{p}$-closed subset of $L_{\mathbf{p}}$, <u>then</u> $\mathbb{P}_{\mathbf{p},L} \lessdot \mathbb{P}_{\mathbf{p}}$.*

2) *If L is a $\mathbf{p}$-closed subset of $L_{\mathbf{p}}$, <u>then</u> for all $p \in \mathbb{P}_{\mathbf{p}}$, letting $r := p \restriction L \in \mathbb{P}_{\mathbf{p},L}$, we have*

- $\bullet_1$ $\mathrm{dom}(r) = \mathrm{dom}(p) \cap L$
- $\bullet_2$ *If $r \leq q \in \mathbb{P}_{\mathbf{p},L}$ <u>then</u>*
$$\big(p \restriction (\mathrm{dom}(p) \setminus L)\big) \cup q$$
is a lub of p and q (in $\mathbb{P}_{\mathbf{p}}$).

3) *If $L_1 \subseteq L_2$ are $\mathbf{p}$-closed subsets of $L_{\mathbf{p}}$, <u>then</u> $\mathbb{P}_{\mathbf{p},L_2} / \mathbb{P}_{\mathbf{p},L_1}$ is $(<\kappa)$-complete and κ^+-cc.*

Proof. 1) Follows by part (2).

2) Easy.

3) Straightforward. $\square_{2.7}$

We shall freely use the following claim:

Claim 2.8. *Assume $\mathbf{p} \in \mathbf{P}$ and $\mathbf{p} \subseteq \mathcal{H}(\chi)$, where $\chi > \lambda = \lambda^\kappa$. Suppose $N \prec (\mathcal{H}(\chi), \in)$ is of cardinality λ and $[N]^{\leq\kappa} \subseteq N$. <u>Then</u> $\mathbf{p}' := \mathbf{p}^N$ (naturally defined) belongs to $\mathbf{P}$ and has cardinality λ.*

Proof. Straightforward. $\square_{2.8}$

Definition 2.9. 1) We now define the two-place relation $\leq_{\mathbf{P}}$ as follows.

Let $\mathbf{p}_1 \leq_{\mathbf{P}} \mathbf{p}_2$ mean:

(a) $\mathbf{p}_\ell$ is an $\mathbf{m}_\ell$-ATI for $\ell = 1, 2$ (where $\mathbf{m}_\ell := \mathbf{m}_{\mathbf{p}_\ell}$; recall that $\mathbf{p}_\ell$ determines $\mathbf{m}_\ell$).

(b) $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$

(c) For all $t \in L_{\mathbf{m}_1}$ we have $\mathbf{o}_{\mathbf{p}_1,t} = \mathbf{o}_{\mathbf{p}_2,t}$ (hence $S_{\mathbf{p}_2,t} = S_{\mathbf{p}_1,t}$ and $\underset{\sim}{\mathbb{Q}}_t^{\mathbf{p}_1} = \underset{\sim}{\mathbb{Q}}_t^{\mathbf{p}_2}$, etc.).

(d) $\mathbb{P}_{\mathbf{p}_1} \lessdot \mathbb{P}_{\mathbf{p}_2}$, which implies $\mathbb{P}_{\mathbf{p}_1,t} \lessdot \mathbb{P}_{\mathbf{p}_2,t}$ for $t \in L^+_{\mathbf{m}_1}$. (This follows by 2.7.)

(e) **[Also follows:]** $\Vdash_{\mathbb{P}_{\mathbf{p}_2}}$ "$\underset{\sim}{\eta}_t^{\mathbf{p}_1} = \underset{\sim}{\eta}_t^{\mathbf{p}_2}$" for $t \in L_{\mathbf{m}_1}$.

2) If $\mathbf{r} \leq_{\mathbf{P}} \mathbf{p}$ and $p \in \mathbb{P}_{\mathbf{p}}$, then we define $p \restriction \mathbf{r}$ as $p \restriction L_{\mathbf{r}}$.

3) If $\langle \mathbf{p}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathbf{P}}$-increasing then "$\mathbf{p} := \bigcup\limits_{\alpha<\delta} \mathbf{p}_\alpha$" will mean the following:

(a) $\mathbf{p} \in \mathbf{P}$

(b) $\mathbf{m}_{\mathbf{p}} := \bigcup\limits_{\alpha<\delta} \mathbf{m}_{\mathbf{p}_\alpha}$

(c) $\mathbf{p}_\alpha \leq_{\mathbf{P}} \mathbf{p}$ for all $\alpha < \delta$.

(d) **[Follows:]** If $s \in L_{\mathbf{p}_\alpha}$ <u>then</u> $\mathbf{o}_{\mathbf{p},s} = \mathbf{o}_{\mathbf{p}_\alpha,s}$.

4) We say $\overline{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \alpha_* \rangle$ is $\leq_{\mathbf{P}}$-*increasing continuous* <u>if</u> it is $\leq_{\mathbf{P}}$-increasing and $\mathbf{p}_\delta = \bigcup\limits_{\alpha<\delta} \mathbf{p}_\alpha$ for every limit $\delta < \alpha_*$.

Remark 2.10. 1) Note that in parts (3),(4) of Definition 2.9, for a given $\langle \mathbf{p}_\alpha : \alpha < \delta \rangle$, it is not *a priori* clear that such $\mathbf{p}$ exists — and even if it does, whether it is unique.

2) Regarding 2.9(1)(d), does "$\mathbb{P}_{\mathbf{p}_1} \lessdot \mathbb{P}_{\mathbf{p}_2}$" follow from the rest by 2.2(G)(d) by Definition 1.11? This is not clear. (See 2.3(2).)

3) We can only show that given $\mathbf{p}$ and a $\mathbf{p}$-closed $L \subseteq L_{\mathbf{p}}$, we have $(\mathbf{p} \restriction L) \leq_{\mathbf{P}} \mathbf{p}$.

Claim 2.11. 1) *Clause* 2.9(1)(d) *does indeed follow from Claim* 2.7.

1A) *Clause* 2.9(1)(e) *does indeed follow from the rest of* 2.9(1).

2) $\leq_{\mathbf{P}}$ *is indeed a partial order on* $\mathbf{P}$.

3) *If* $\langle \mathbf{p}_\alpha : \alpha < \delta \rangle$ *is a* $\leq_{\mathbf{P}}$*-increasing continuous sequence of* (∂, κ)*-combinatorial templates (Note: when* $\kappa > \aleph_0$ *this does* <u>*NOT*</u> *mean that* $\langle \mathbb{P}_{\mathbf{p}_\alpha} : \alpha < \delta \rangle$ *is* $\subseteq$*-increasing continuous!) and* $\mathrm{cf}(\delta) \geq \kappa$*,* <u>*then*</u> $\bigcup\limits_{\alpha<\delta} \mathbf{p}_\alpha$ *exists and is unique.*

4) *If* $\mathbf{p} \in \mathbf{P}$ *and* L *is* $\mathbf{p}$*-closed,* <u>*then*</u> $\mathbf{p} \restriction L \in \mathbf{P}$ *and* $\mathbf{p} \restriction L \leq_{\mathbf{P}} \mathbf{p}$.

Proof. Straightforward. $\square_{2.11}$

§ 3. From $\mathbf{M}$ and $\mathbf{P}$ to $\mathbf{N}$ and $\mathbf{Q}$

For the classes $\mathbf{P}$ and $\mathbf{M}$, we do not know how to prove the existence of a sufficiently homogeneous member, because when taking unions it is not clear how to define or prove (e.g.) the κ^+-cc. To solve this, we introduce the classes $\mathbf{Q}$ and $\mathbf{N}$.

Definition 3.1. We let $\mathbf{N}$ be the class of objects $\mathbf{n}$ consisting of[12]

(A) $\mathbf{m} \in \mathbf{M}$

(B) (a) $W_\mathbf{n} \subseteq L_\mathbf{n}$ $(:= L_\mathbf{m})$

(b) If $t \in W_\mathbf{n}$ <u>then</u>

- $\bullet_1$ $\lambda_t = \lambda_t^\kappa$
- $\bullet_2$ $\mathcal{B}_t := \mathcal{A}_t$ has cardinality $\leq \lambda_{\mathbf{o}_t}$. (See 1.2(D)(b).)
- $\bullet_3$ If $A \in \mathcal{A}_{\mathbf{m},t}$ then A has cardinality $\leq \lambda_{\mathbf{o}_t}$ and is a subset of $W_\mathbf{n}$.

(C) For each $t \in L_\mathbf{n} \setminus W_\mathbf{n}$, it also contains $\mathcal{B}_t = \mathcal{B}_{\mathbf{n},t}$ and $\mathcal{I}_t = \mathcal{I}_{\mathbf{n},t}$ satisfying the following:

(a) $\mathcal{B}_t \subseteq \mathcal{A}_t$, a family of subsets of $W_\mathbf{n}$.

(b) If $A \in \mathcal{B}_t$ then $|A| \leq \lambda_t$.

(c) $|\mathcal{B}_{\mathbf{m},t}| \leq 2^{\lambda_{\mathbf{o}_t}}$
(Really, $\leq \lambda_{\mathbf{o}_t}$ would suffice.)

(d) $\mathcal{I}_{\mathbf{n},t}$ is a κ-complete ideal on $\mathcal{B}_{\mathbf{n},t}$.

(e) $A \in \mathcal{A}_{\mathbf{m},t} \Rightarrow \{B \in \mathcal{B}_{\mathbf{m},t} : A \cap B \neq \varnothing\} \in \mathcal{I}_{\mathbf{n},t}$.

Definition 3.2. 1) We define a partial order $\leq_\mathbf{N}$ on $\mathbf{N}$ as follows.

$\mathbf{n}_1 \leq_\mathbf{N} \mathbf{n}_2$ <u>iff</u>

(A) $\mathbf{m}_{\mathbf{n}_1} \leq_\mathbf{M} \mathbf{m}_{\mathbf{n}_2}$

(B) $W_{\mathbf{m}_1,s} = W_{\mathbf{m}_2,s} \cap L_{\mathbf{m}_1}$

(C) If $s \in L_{\mathbf{m}_1}$ then

(a) $\lambda_{\mathbf{m}_1,s} = \lambda_{\mathbf{m}_2,s}$

(b) $\mathcal{B}_{\mathbf{m}_1,s} = \mathcal{B}_{\mathbf{m}_2,s}$

(c) $\mathcal{A}_{\mathbf{m}_1,s} \subseteq \mathcal{A}_{\mathbf{m}_2,s}$

(d) $\mathcal{I}_{\mathbf{m}_1,s} = \mathcal{I}_{\mathbf{m}_2,s} \cap \mathcal{P}(L_{\mathbf{m}_1})$.

2) For $\mathbf{n} \in \mathbf{N}$, we say $L \subseteq L_\mathbf{n}$ is $\mathbf{n}$-*closed* <u>when</u> $A \in \bigcup\limits_{s \in L} \mathcal{B}_s \Rightarrow A \subseteq L$.

3) If $\mathbf{n} \in \mathbf{N}$ and L is $\mathbf{q}$-closed, then $\mathbf{n}_\bullet := \mathbf{n} \restriction L$ is defined naturally: $L_{\mathbf{n}_\bullet} := L$, and for all $t \in L$ we have

- $\mathcal{B}_{\mathbf{n}_\bullet,t} := \mathcal{B}_{\mathbf{n},t}$
- $\lambda_{\mathbf{n}_\bullet,t} := \lambda_{\mathbf{n},t}$
- $\mathcal{A}_{\mathbf{n}_\bullet,t} := \mathcal{A}_{\mathbf{n},t} \cap \mathcal{P}(L)$
- $\mathcal{I}_{\mathbf{n}_\bullet,t} := \mathcal{I}_{\mathbf{n},t}$.

[12] So $\mathbf{m} = \mathbf{m}_\mathbf{n}$, $L_\mathbf{m} = L_\mathbf{n}$, etc.

Claim 3.3. 1) *If* $\mathbf{n}_1 \le_{\mathbf{N}} \mathbf{n}_2$ *then* $L_{\mathbf{n}_1}$ *is* $\mathbf{n}_2$*-closed.*

2) *If* L *is* $\mathbf{n}$*-closed then* $\mathbf{n} \restriction L \in \mathbf{N}$ *and* $\mathbf{n} \restriction L \le_{\mathbf{N}} \mathbf{n}$.

3) $\le_{\mathbf{N}}$ *is indeed a partial order on* $\mathbf{N}$.

4) *If* δ *is a limit ordinal and* $\langle \mathbf{n}_\alpha : \alpha < \delta \rangle$ *is* $\le_{\mathbf{N}}$*-increasing, then it has a* $\le_{\mathbf{N}}$*-least upper bound* $\mathbf{n}$.

That is:

(a) $\mathbf{m}_{\mathbf{n}} = \bigcup\limits_{\alpha<\delta} \mathbf{m}_{\mathbf{n}_\alpha}$

(b) *For* $\alpha < \delta$, $s \in L_{\mathbf{n}_\alpha} \Rightarrow \mathcal{B}_{\mathbf{n},s} = \mathcal{B}_{\mathbf{n}_\alpha,s}$.

(c) *For* $\alpha < \delta$ *and* $s \in L_{\mathbf{n}_\alpha}$, $\mathcal{I}_{\mathbf{n},s}$ *is the closure of* $\bigcup\limits_{\beta\in[\alpha,\delta)} \mathcal{I}_{\mathbf{n}_\beta,s}$ *under unions of* $< \kappa$ *members.*

Proof. Straightforward. $\square_{3.3}$

Definition 3.4. We let $\mathbf{Q}$ be the class of objects $\mathbf{q}$ which consist of

(A) $\mathbf{n} = \mathbf{n}_{\mathbf{q}} \in \mathbf{N}$

(B) $\mathcal{A}_{\mathbf{q}} := \bigcup\limits_{t\in L_{\mathbf{n}}} \mathcal{A}_{\mathbf{n},t}$

(C) $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{m}_{\mathbf{p}} = \mathbf{m}_{\mathbf{n}}$ and $\lambda_{\mathbf{o}_{\mathbf{p}},t} = \lambda_{\mathbf{n},t}$.

We demand that $\mathbf{q}$ satisfies

(D) If $t \in L_{\mathbf{q}} \setminus W_{\mathbf{q}}$ and $\mathbf{p}_\bullet \in \mathbf{P}$ has cardinality $\le (|\mathbb{P}_{\mathbf{p}}| + \kappa)^\kappa$, then the set
$$\mathbf{Y}_{\mathbf{p}_\bullet,t} := \{A \subseteq W_{\mathbf{q}} \cap L_{\mathbf{q},t} : \mathbf{p}_{\mathbf{q}} \restriction A \cong \mathbf{p}_\bullet\}$$
belongs to $\mathcal{I}^+_{\mathbf{q},t}$.
(If in (C) we demand only "$\mathcal{B}_{\mathbf{q},t}$ has cardinality $\le \lambda_{\mathbf{o}_t}$," then we should weaken the demand on $A \in \mathbf{Y}_{\mathbf{p}_\bullet,t}$.)

Definition 3.5. 1) We define a partial order $\le_{\mathbf{Q}}$ on $\mathbf{Q}$ as follows.

$\mathbf{q}_1 \le_{\mathbf{Q}} \mathbf{q}_2$ iff

(A) $\mathbf{n}_{\mathbf{q}_1} \le_{\mathbf{N}} \mathbf{n}_{\mathbf{q}_2}$

(B) If $s \in L_{\mathbf{q}_1}$ then $\mathbf{o}_{\mathbf{q}_1,s} = \mathbf{o}_{\mathbf{q}_2,s}$.

(C) $\mathbb{P}_{\mathbf{q}_1} \lessdot \mathbb{P}_{\mathbf{q}_2}$. (This follows by 3.4(D).)

2) For L a $\mathbf{q}$-closed (i.e. $\mathbf{n}_{\mathbf{q}}$-closed) subset of $L_{\mathbf{q}}$, let $\mathbf{q} \restriction L$ be such that $\mathbb{P}_{\mathbf{q}\restriction L} := \mathbb{P}_{\mathbf{q},L}$ and $W_{\mathbf{q}\restriction L} := W_{\mathbf{q}} \cap L$.

Claim 3.6. *Let* $\mathbf{q} \in \mathbf{Q}$ *and* L *be* $\mathbf{q}$*-closed (that is,* $\mathbf{n}_{\mathbf{q}}$*-closed).*

1) $\mathbb{P}_{\mathbf{q},L} = \mathbb{P}_{\mathbf{p}_{\mathbf{q}},L} \lessdot \mathbb{P}$ *and* $\mathbf{q} \restriction L \in \mathbf{Q}$.

2) *If* $A \in \mathcal{A}_{\mathbf{q}}$ *and* $q \in \mathbb{P}_{\mathbf{q}}$, *then there exists* $p \in \mathbb{P}_{\mathbf{q},A}$ *such that if* $\mathbb{P}_{\mathbf{q},A} \models$ '$p \le r$' *then* $q_* = r \oplus q \in \mathbb{P}_{\mathbf{q}(+)}$, *as defined below, is a common upper bound of* p *and* q.

- $\mathrm{dom}(q_*) := \mathrm{dom}(r) \cup \mathrm{dom}(q)$

- $q_*(s) := \begin{cases} r(s) & \textit{if } s \in \mathrm{dom}(r) \setminus \mathrm{dom}(q) \\ q(s) & \textit{if } s \in \mathrm{dom}(q) \setminus \mathrm{dom}(r) \\ \big(tr(s), \mathcal{F}_{p(s)} \cup \mathcal{F}_{q(s)}\big) & \textit{if } s \in \mathrm{dom}(q) \cap \mathrm{dom}(r). \end{cases}$

Proof. 1) Follows from part (2).

2) By 2.8 and Definition 3.4(2). $\square_{3.6}$

§ 4. Homogeneity

Definition 4.1. 1) We say $\mathbf{q} \in \mathbf{Q}_{\text{pre}}$ when it consists of

(a) $\mathbf{n_q} \in \mathbf{N}$
(Hence $\mathbf{m_q} = \mathbf{m}_{\mathbf{n_q}} \in \mathbf{M}$, $L_\mathbf{q} = L_\mathbf{m}$, $\mathcal{A}_\mathbf{q} = \mathcal{A}_{\mathbf{n_q}}$, etc. — see 3.4(A).)

(b) $\bar{\mathbf{o}}_\mathbf{q} = \langle \mathbf{o}_{\mathbf{q},t} : t \in L \rangle$ as in Definition 2.2.

We demand the following:

(c) For all $A \in \mathcal{A}_\mathbf{q}$ there exists a $\mathbf{q}_A \in \mathbf{Q}$ with $\mathbf{n}_{\mathbf{q}_A} = \mathbf{n_q} \restriction A$ and $\bar{\mathbf{o}}_{\mathbf{q}_A} = \bar{\mathbf{o}}_\mathbf{q} \restriction A$.

(d) If $A \subseteq B$ are members of $\mathcal{A}_\mathbf{q}$ then $\mathbf{q}_A = \mathbf{q}_B \restriction A$.

($\mathbf{q} \restriction A$ is defined in 2.5(2)(B).)

2) Let $\leq^{\text{pre}}_\mathbf{Q}$ be the natural order on $\mathbf{Q}_{\text{pre}}$ (see 2.5(2)(B)).

3) For $\mathbf{q} \in \mathbf{Q}$, let $\text{pre}(\mathbf{q})$ denote $\mathbf{q}$ viewed as a member of $\mathbf{Q}_{\text{pre}}$.

Claim 4.2. 1) *If $\bar{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha < \delta \rangle$ is $\leq_\mathbf{Q}$-increasing continuous (see 2.9(4)) then $\mathbf{q}_\delta := \bigcup_{\alpha<\delta} \mathbf{q}_\alpha$ exists and is unique, belongs to $\mathbf{Q}$, and $\bar{\mathbf{q}}\char`^\langle \mathbf{q}_\delta \rangle$ is $\leq_\mathbf{Q}$-increasing continuous.*

2) *Every $\mathbf{q} \in \mathbf{Q}_{\text{pre}}$ can be expanded to some $\mathbf{q}^+ = \mathbf{q}(+) \in \mathbf{Q}$. (By this we mean that $\mathbf{n}_{\mathbf{q}^+} = \mathbf{n_q}$, $\bar{\mathbf{o}}_{\mathbf{q}^+} = \bar{\mathbf{o}}_\mathbf{q}$, $\mathcal{A}_{\mathbf{q}^+} = \mathcal{A}_\mathbf{q}$, and $(\forall A \in \mathcal{A}_\mathbf{q})\big[\mathbf{q}^+ \restriction A = \mathbf{q} \restriction A\big]$.)*

3) *If $\mathbf{r}, \mathbf{q} \in \mathbf{Q}_{\text{pre}}$ with $\mathbf{r} \leq^{\text{pre}}_\mathbf{Q} \mathbf{q}$, then $\mathbf{r}^+ \leq_\mathbf{Q} \mathbf{q}^+$.*

Proof. 1) Let $\mathbf{q}^*_\alpha := \text{pre}(\mathbf{q}_\alpha)$; that is, a member of $\mathbf{Q}_{\text{pre}}$.

Clearly,

$(*)_1$ $\langle \mathbf{q}^*_\alpha : \alpha < \delta \rangle$ is $\leq^{\text{pre}}_\mathbf{Q}$-increasing.

$(*)_2$ If I is a directed partial order and $\bar{\mathbf{q}}^* = \langle \mathbf{q}^*_s : s \in I \rangle$ is $\leq^{\text{pre}}_\mathbf{Q}$-increasing continuous, then the union $\mathbf{r} := \bigcup \bar{\mathbf{q}}^*$ (naturally defined[13]) is well-defined and is a $\leq^{\text{pre}}_\mathbf{Q}$-lub of the sequence.

Together with part (2), we are done.

2) First we will define $\mathbb{P}_{\mathbf{q}^+}$. Let $\mathbf{n} := \mathbf{n}_{\mathbf{q}^+} = \mathbf{n_q}$.

$(*)_3$ $p \in \mathbb{P}_{\mathbf{q}^+}$ iff

(a) p is a function with domain $\in [L_\mathbf{m}]^{<\kappa}$.

(b) If $s \in \text{dom}(p)$ then $p(s)$ is of the form $(\text{tr}(p(s)), \mathcal{F}_{p(s)})$, where $\text{tr}(p(s)) \in \text{TR}_{\mathbf{o}_s}$ and $|\mathcal{F}_{p(s)}| < \kappa$.

(c) If $\underset{\sim}{f} \in \mathcal{F}_{p(s)}$, then for some $A \in \mathcal{A}_{\mathbf{n},s}$ we have $\big(\text{tr}(p(s)), \underset{\sim}{f}\big) \in \mathbb{P}_{\mathbf{q}\restriction A}$.

$(*)_4$ The order $\leq_{\mathbb{P}_{\mathbf{q}^+}}$ is defined as follows.

$p \leq_{\mathbb{P}_{\mathbf{q}(+)}} q$ iff

[13] In particular:
- $\mathbf{m_r} = \bigcup_{s\in I} \mathbf{m}_{\mathbf{q}^*_s}$ (so $\mathcal{A}_\mathbf{r} = \bigcup_{s\in I} \mathcal{A}_{\mathbf{q}^*_s}$).
- $\mathbf{n_r} = \bigcup_{s\in I} \mathbf{n}_{\mathbf{q}^*_s}$ (so $W_\mathbf{r} = \bigcup_{s\in I} W_{\mathbf{q}^*_s}$).

(a) $p, q \in \mathbb{P}_{\mathbf{q}^+}$
(b) If $A \in \mathcal{A}_{\mathbf{q}^+}$ then $(p \restriction A) \leq_{\mathbb{P}_{\mathbf{q} \restriction A}} (q \restriction A)$.

Now check.

3) Easy. $\square_{4.2}$

Definition 4.3. We define paut($\mathbf{q}$) (the *partial automorphisms* of $\mathbf{q}$) as the set of π such that for some $\mathbf{q}$-closed sets L_1 and L_2, π is an isomorphism from $\mathbf{q} \restriction L_1$ onto $\mathbf{q} \restriction L_2$.

Claim 4.4. 1) *If $s \in L_{\mathbf{q}}$ then $\bigcup \mathcal{B}_{\mathbf{q},s} \cup \{s\}$ is a $\mathbf{q}$-closed set (recalling 3.2(2)).*

2) *If $s, t \in L_{\mathbf{q}}$ and $\mathbf{o}_s = \mathbf{o}_t$, then $\{(s,t)\} \in \mathrm{paut}(\mathbf{q})$.*

3) *$(\mathbf{Q}, \leq_{\mathbf{Q}})$ has disjoint amalgamation (as in 1.12(4)):*

If $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{Q}$ and $L_* := L_{\mathbf{q}_1} \cap L_{\mathbf{q}_2}$ is $\mathbf{q}_1$-closed and $\mathbf{q}_2$-closed such that $\mathbf{q}_1 \restriction L_* = \mathbf{q}_2 \restriction L_*$, then there is $\mathbf{p} \in \mathbf{Q}$ such that $L_{\mathbf{p}} = L_{\mathbf{q}_1} \cup L_{\mathbf{q}_2}$ and $\bigwedge\limits_{\ell=1,2} \big[\mathbf{q}_\ell \leq_{\mathbf{Q}} \mathbf{p}\big]$.

4) *If $\mathbf{o} \in \mathbf{O}$ then there exists $\mathbf{q} \in \mathbf{Q}$ such that for some t, $W_{\mathbf{q}} = \bigcup \mathcal{B}_{\mathbf{q},t}$, $L_{\mathbf{q}} = W_{\mathbf{q}} \cup \{t\}$, and $\mathbf{o}_t = \mathbf{o}$.*

5) *If $\mathbf{q} \in \mathbf{Q}$, $\mathbf{o} \in \mathbf{O}$, L_* is an initial segment of $L_{\mathbf{q}}$, and $\mathcal{A} \subseteq \mathcal{P}(L_*)$, then there exist $\mathbf{r}$ and s such that:*

- $\bullet_1$ $\mathbf{q} \leq_{\mathbf{Q}} \mathbf{r}$
- $\bullet_2$ $L_{\mathbf{r}} \setminus L_{\mathbf{q}} = \bigcup \mathcal{B}_{\mathbf{q},s} \cup \{s\}$
- $\bullet_3$ $\mathbf{o}_{\mathbf{r},s} = \mathbf{o}$
- $\bullet_4$ *If $r \in \bigcup \mathcal{B}_{\mathbf{r}} \cup \{s\}$ and $t \in L_{\mathbf{q}}$, then $L_{\mathbf{r}} \models$ "$r < t \Leftrightarrow r \in L_*$".*
- $\bullet_5$ $\mathcal{A}_{\mathbf{r},s} = \mathcal{A} \cup \mathcal{B}_{\mathbf{q},s}$.

6) *In part (5), we may allow $\mathcal{A} \subseteq \mathcal{P}(L_* \cup \mathcal{B}_{\mathbf{q}})$.*

Proof. Straightforward. $\square_{4.4}$

Claim 4.5. *Let $\partial = \mathrm{cf}(\partial) \leq \lambda := |\mathbf{O}| + \sum\limits_{\mathrm{TR} \in \mathbf{TR}} \|\mathrm{TR}\| + \sum\limits_{\mathbf{o} \in \mathbf{O}} \lambda_{\mathbf{o}}$, where $\lambda_{\mathbf{o}} = \lambda_{\mathbf{o}}^\kappa \geq \|\mathrm{TR}_{\mathbf{o}}\|$ and $\mathbf{O} \neq \varnothing$. Suppose $\mathbf{q} \in \mathbf{Q}$ with $\|L_{\mathbf{q}}\| \leq \lambda$ such that $L_{\mathbf{q}} \Rightarrow \lambda_{\mathbf{q},s} \leq \lambda_{\mathbf{o}}$.*

Then for some $\mathbf{r} \in \mathbf{Q}$:

(a) *$|L_{\mathbf{r}}| = \lambda$ and $\mathbf{q} \leq_{\mathbf{Q}} \mathbf{r}$.*
(b) *For every $\mathbf{o} \in \mathbf{O}$, the set $\{s \in L_{\mathbf{r}} : \mathbf{o}_{\mathbf{r},s} = \mathbf{o}\}$ is dense in $L_{\mathbf{r}}$.*
(c) *There exists an increasing sequence $\bar{L} = \langle L_\varepsilon : \varepsilon < \partial \rangle$ of subsets of $L_{\mathbf{r}}$, with $\bigcup \bar{L} = L_{\mathbf{r}}$, $L_0 := \varnothing$, $L_1 := L_{\mathbf{q}}$, and $\mathbf{r} \restriction L_1 = \mathbf{q}$.*
(d) *If $s, t \in L_{\mathbf{r}}$ with $\mathbf{o}_{\mathbf{r},s} = \mathbf{o}_{\mathbf{r},t} = \mathbf{o}$, then there is an automorphism π of $\mathbf{r}$ such that $\pi(s) = t$ and $\pi[L_\varepsilon] = L_\varepsilon$ for every $\varepsilon < \partial$ large enough.*
(e) *If $\kappa \leq \theta < \partial$ is such that $\lambda = \lambda^\theta$, $\varepsilon < \partial$, $L_*, L_{**} \in [L_{\mathbf{r}}]^\theta$, and π is an isomorphism from $\mathbf{r} \restriction (L_\varepsilon \cup L_*)$ onto $\mathbf{r} \restriction (L_\varepsilon \cup L_{**})$ extending $\mathrm{id}_{L_\varepsilon}$, then there exists an automorphism $\hat{\pi} : \mathbf{r} \to \mathbf{r}$ extending π.*

Proof. We choose $(\mathbf{q}_\alpha, \bar{\pi}^\alpha)$ by induction on $\alpha < \lambda^+$ such that:

$(*)_\alpha$ (a) $\mathbf{q}_\alpha \in \mathbf{Q}$
(b) $\mathbf{q}_0 := \mathbf{q}$ (and $L_{\mathbf{q}_0} := \varnothing$ for notational simplicity).
(c) $\bar{\pi}^\alpha = \langle \pi_{\beta,i} : \beta < \alpha,\ i < \lambda\beta + \lambda \rangle$
(d) $\langle \mathbf{q}_\beta : \beta \leq \alpha \rangle$ is $\leq_{\mathbf{Q}}$-increasing continuous.
(e) If $\beta < \alpha$ and $i < \lambda\beta + \lambda$, then $\pi_{\beta,i} \in \mathrm{paut}(\mathbf{q}_\alpha)$.
(f) If $\beta < \alpha$ and $i \in [\lambda\beta, \lambda\beta + \lambda)$, then $\big\langle \pi_{\gamma,i} : \gamma \in [\beta, \alpha) \big\rangle$ is $\subseteq$-increasing.
(g) If $\gamma + 1 \in [\beta, \alpha)$ and $i \in [\lambda\beta, \lambda\beta + \lambda)$, then

$$L_{\mathbf{q}_\gamma} \subseteq \mathrm{dom}(\pi_{\gamma+1,i}) \cap \mathrm{rang}(\pi_{\gamma+1,i}).$$

(h) Let $\beta < \alpha$. Each π which satisfies '$\bullet_1 \vee \bullet_2$' below will appear somewhere in the sequence $\big\langle \pi_{\beta,i} : i \in [\lambda\beta, \lambda\beta + \lambda) \big\rangle$.
  - $\bullet_1$ $\pi = \{(s,t)\}$, where $s,t \in L_{\mathbf{q}_\beta}$ and $\mathbf{o}_{\mathbf{q}_\beta,s} = \mathbf{o}_{\mathbf{q}_\beta,t}$.
  - $\bullet_2$ There exist $\gamma < \beta$ and $\mathbf{q}_\beta$-closed L_*, L_{**} such that $L_* \cup L_{**} \subseteq L_{\mathbf{q}_\beta}$, $\lambda = \lambda^{|L_*|}$, and π is an isomorphism from $\mathbf{q}_\beta \restriction (L_{\mathbf{q}_\gamma} \cup L_*)$ onto $\mathbf{q}_\beta \restriction (L_{\mathbf{q}_\gamma} \cup L_{**})$ with $\pi \restriction L_{\mathbf{q}_\gamma} = \mathrm{id}_{L_{\mathbf{q}_\gamma}}$.

There is no problem in carrying the induction. Now for a club of $\delta < \lambda^+$, if $\mathrm{cf}(\delta) = \partial$ then $\mathbf{q}_\delta$ is as required. $\square_{4.5}$

* * *

We now turn to saccharinity (using a single $\mathbf{o}$ for transparency).

Definition 4.6. Let κ be an inaccessible cardinal (or just weakly inaccessible). We say $\mathbf{d}$ is a *κ-parameter* when it consists of

(a) $\bar{\lambda} = \langle \lambda_\eta, \lambda^0_\eta, D_\eta : \eta \in \mathcal{T} \rangle$ as in 1.6(4) (so $\mathcal{T}$ is a subtree of $({}^{\kappa >}\kappa, \triangleleft)$).
(b) $\mathbf{o}_{\bar{\lambda}}$ as in 1.6(4).
(c) $\lambda = \mathrm{cf}(\lambda) > \kappa$
(d) $\bar{\rho} = \langle \rho_\alpha : \alpha < \lambda \rangle \subseteq \prod\limits_{\eta \in \mathcal{T}} \lambda_\eta$.

Additionally, we demand the following:

(e) **[Notation:]** For any L, let
  - $X_{L,\bar{\lambda}} :=$
$$\big\{ \mathbf{i} = (\zeta, u, g, \nu) : \zeta < \kappa,\ u \subseteq [\kappa]^{<\kappa},\ g : u \to \mathcal{T} \cap {}^\zeta\kappa,\ \lambda_\nu \neq 1, \text{ and } \eta \in \mathcal{T} \cap {}^\zeta\kappa \setminus \{\nu\} \Rightarrow \lambda_\eta = 1] \big\}$$
  - We define the following partial order on $X_{L,\bar{\lambda}}$:
    $(\zeta_1, u_1, g_1, \nu_1) \leq_{L,\bar{\lambda}} (\zeta_2, u_2, g_2, \nu_2)$ <u>iff</u>
    $\zeta_1 < \zeta_2$, $\nu_1 \triangleleft \nu_2$, $u_1 \subseteq u_2$, and $\beta \in u_1 \Rightarrow g_1(\beta) \trianglelefteq g_2(\beta)$.
  - Let $Y_{L,\bar{\lambda}} := \mathrm{inc}_\kappa(X_{L,\bar{\lambda}})$.
    (That is, $\leq_{L,\bar{\lambda}}$-increasing sequences from $X_{L,\bar{\lambda}}$ of length κ.)
  - For $\bar{\mathbf{i}} \in Y_{L,\bar{\lambda}}$, let $E_{\mathbf{i}} := \{\varepsilon < \kappa : \zeta_{\mathbf{i}_\varepsilon} = \varepsilon = u_{\mathbf{i}_\varepsilon}\}$; this is a club of κ.

(f) For any $h : \lambda \to Y_{L,\bar{\lambda}}$ we can find $(\zeta_*, u_*, g_*, \nu_*) \in X_{L,\bar{\lambda}}$ and $\mathsf{w} \subseteq \lambda$ such that:
  - $\mathrm{otp}(\mathsf{w}) = \lambda_{\nu_*}$
  - $(\zeta_{\mathbf{i}_\alpha}, \nu_{\mathbf{i}_\alpha}) = (\zeta_*, \nu_*)$ and $\zeta_* \in E_{h(\alpha)}$ for $\alpha \in \mathsf{w}$.

- $\langle u_{\mathbf{i}_\alpha} : \alpha \in \mathsf{w}\rangle$ is a Δ-system with heart u_*.
- $g_{\mathbf{i}_\varepsilon} \restriction u_* = g_*$
- $\langle \rho_\alpha(\nu_*) : \alpha \in \mathsf{w}\rangle \in {}^{\mathsf{w}}\lambda_{\nu_*}$ is $<_{L,\bar\lambda}$-increasing.

Claim 4.7. *Let κ be inaccessible (or weakly inaccessible) such that $\diamondsuit_\kappa$ holds.*

1) *If $\kappa^+ = 2^\kappa$ <u>then</u> there exists a κ-parameter $\mathbf{d}$.*

2) *If $\lambda > \kappa$, $\mathbb{P} := \mathsf{Cohen}_\kappa(\lambda)$ (the forcing notion adding λ-many κ-reals), and $\bar\lambda$, $\mathbf{o}_{\bar\lambda}$ are as in 1.6(2), then*

$$\Vdash_{\mathbb{P}} \text{“there is a } \kappa\text{-parameter } \mathbf{d} \text{ such that } \bar\lambda_{\mathbf{d}} = \bar\lambda \text{ and } \bar{\mathbf{o}}_{\mathbf{d}} = \mathbf{o}\text{”}.$$

Proof. Easy. $\square_{4.7}$

Claim 4.8. *We have ‘(A) $\Rightarrow$ (B),’ where*

(A) (a) $\mathbf{q} \in \mathbf{Q}$
(b) *κ is inaccessible.*
(c) *$\mathbf{d}$ is a κ-parameter.*
(d) *$\bar\lambda$ is κ-active (see 1.6(4)).*
(e) $s \in L_{\mathbf{q}} \Rightarrow \mathbf{o}_{\mathbf{q},s} = \mathbf{o}_{\bar\lambda}$

(B) *If $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$ is generic over $\mathbf{V}$, <u>then</u> in $\mathbf{V}[\mathbf{G}]$, the set of members $\eta \in \lim(\mathcal{T}_{\bar\lambda})$ which are generic for $(\mathbb{Q}_{\bar\lambda}, \underset{\sim}{\eta}_{\bar\lambda})$ over $\mathbf{V}$ is exactly $\{\underset{\sim}{\eta}_{\mathbf{q},s} : s \in L_{\mathbf{q}}\}$.*

Proof. $(*)_1$ In $\mathbf{V}$ and $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$, we have: ‘if $\rho \in \prod\limits_{\eta\in\mathcal{T}_{\bar\lambda}} \lambda_\eta$ <u>then</u> $\underset{\sim}{\mathbf{B}}_\rho \in \mathrm{id}_{\leq\kappa}(\mathbb{Q}_{\bar\lambda}, \underset{\sim}{\eta}_{\bar\lambda})$,’ where

$$\mathbf{B}_\rho := \big\{\nu \in \lim \mathcal{T}_{\bar\lambda} : (\exists^\kappa \varepsilon < \kappa)\big[\nu(\varepsilon) < \rho(\varepsilon)\big]\big\}.$$

This is easily seen to be true. Now assume:

$(*)_2$ $\underset{\sim}{\eta}$ is a $\mathbb{P}_{\mathbf{q}}$-name of a member of $\lim \mathcal{T}_{\bar\lambda}$, and $p \in \mathbb{P}_{\mathbf{q}}$ is such that

$$p \Vdash_{\mathbb{P}_{\mathbf{q}}} (\forall^\infty \varepsilon < \kappa)\big[\underset{\sim}{\eta}(\varepsilon) > \rho_\alpha(\underset{\sim}{\eta} \restriction \varepsilon)\big]$$

for all $\alpha < \lambda$.

For $\alpha < \lambda$, we can choose $\big\langle(p_{\alpha,\varepsilon}, \zeta_{\alpha,\varepsilon}, \Lambda_{\alpha,\varepsilon}, \nu_{\alpha,\varepsilon}) : \varepsilon < \kappa\big\rangle$ by induction on ε such that:

$(*)_3$ (a) $p_{\alpha,\varepsilon} \in \mathbb{P}_{\mathbf{q}}$ is increasing continuously with ε.
(b) $p \leq p_{\alpha,\varepsilon}$ and $\zeta_* \leq \zeta_{\alpha,\varepsilon}$.
(c) $(\zeta_{\alpha,\varepsilon}, u_{\alpha,\varepsilon}, g_{\alpha,\varepsilon}, \nu_{\alpha,\varepsilon}) \in X_{L,\bar\lambda}$, where $\Lambda_{\alpha,\varepsilon} := \mathrm{rang}(g_{\alpha,\zeta})$.
(d) $\Lambda_{\alpha,\varepsilon} := \big\{\mathrm{tr}(p_{\alpha,\varepsilon}(\gamma)) : \gamma \in \mathrm{dom}(p_{\alpha,\varepsilon})\big\}$
(e) $\langle \zeta_{\alpha,\varepsilon} : \varepsilon < \kappa\rangle$ is increasing continuous.
(f) $p_{\alpha,\varepsilon} \Vdash$ “$\underset{\sim}{\eta} \restriction \zeta_{\alpha,\varepsilon} = \nu_{\alpha,\varepsilon}$”
(g) $p_{\alpha,0} \Vdash_{\mathbb{P}_{\mathbf{q}}}$ “if $\varepsilon \in [\zeta_{\alpha,0}, \kappa)$ then $\underset{\sim}{\eta}(\varepsilon) > \rho_\alpha(\underset{\sim}{\eta} \restriction \varepsilon)$”.

[Why? For $\varepsilon = 0$ this is easy, by $(*)_2$. For ε limit use limits, and for ε a successor ordinal recall that every increasing sequence in $\mathbb{P}_{\mathbf{q}}$ of length $< \kappa$ has a lub.]

$(*)_4$ (a) Let $h : \lambda \to Y_{L_{\mathbf{q}},\bar\lambda}$ be defined by $\beta \mapsto \langle \bar{\mathbf{i}}_\alpha : \alpha < \beta\rangle$.

(b) Applying 4.6(f) to our choice of h, we can choose w and $(\zeta_*, u_*, g_*, \nu_*)$ as there.

Lastly, we define q.

(A) $\mathrm{dom}(q) := \bigcup_{\alpha \in \mathsf{w}} \mathrm{dom}(p_{\alpha,\zeta_*})$

(B) $q(s) := \begin{cases} p_\alpha(s) & \text{if } s \in \mathrm{dom}(p_{\alpha,\zeta_*}) \setminus u_* \\ \big(\mathrm{tr}(p_{\alpha_*}(s)), \bigcup_{\alpha \in \mathsf{w}} \mathcal{F}_{p_\alpha(s)}\big) & \text{if } s \in u_*, \end{cases}$

where we fix α_* as (e.g.) $\min u_*$.

[Why is $q \in \mathbb{P}_{\mathbf{q}}$? Because if $\alpha \in \mathsf{w}$ and $\varrho \in \mathcal{T} \cap \bigcup_{\varepsilon \leq \zeta_*} {}^{\varepsilon}\kappa$, then $\varrho \neq \nu_* \Rightarrow \lambda_{\nu_*} < \lambda^0_\varrho$.]

Also, q is a $\leq_{\mathbb{P}_{\mathbf{q}}}$-upper bound of $\{p_{\alpha,\zeta_*} : \alpha \in \mathsf{w}\}$, but this implies

$$q \Vdash \text{`}\alpha \in \mathsf{w} \Rightarrow \underaccent{\tilde}{\eta}(\zeta_*) > \rho_\alpha(\zeta_*)\text{'}.$$

As $\langle \rho_\alpha(\zeta_*) : \alpha < \lambda_{\nu_*} \rangle \in {}^{\lambda_{\nu_*}}\lambda_{\nu_*}$

is increasing, we get a contradiction to $p_\alpha \Vdash \text{`}\underaccent{\tilde}{\eta}(\zeta_*) < \lambda_{\underaccent{\tilde}{\eta}\restriction\zeta_*}\text{'}$ and $p_\alpha \Vdash \text{`}\lambda_{\underaccent{\tilde}{\eta}\restriction\zeta_*} = \lambda_{\nu_*}\text{'}$. $\square_{4.8}$

Fact 4.9. The conclusion of 4.8 holds for any inaccessible κ (allowing a preliminary forcing) without any extra set-theoretic assumptions.

Proof. If we force by $\mathbb{P} := \mathsf{Levy}(\kappa^+, 2^\kappa)$ <u>then</u>

$$\Vdash_{\mathbb{P}} \text{“}2^\kappa = \kappa^+ \text{ and the desired conclusion holds”},$$

so we have $\langle \underaccent{\tilde}{p}_\alpha : \alpha \in \underaccent{\tilde}{S} \rangle$: names of the desired objects. But $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\mathbf{q}}^{\mathbf{V}[\mathbb{P}]}$, so we can easily finish. $\square_{4.9}$

As in 4.8 and 4.9,

Claim 4.10. *We have ‘(A) $\wedge$ (B) $\Rightarrow$ (C)’, where*

(A) $\kappa := \aleph_0 < \lambda = \lambda^{\aleph_0} \leq 2^\kappa$

(B) (a) $\mathbf{q} \in \mathbf{Q}$

(b) $\mathbf{o} = \mathbf{o}^2_{\mathbf{n}}$ *is as in* 1.6(1).

(c) $s \in L_{\mathbf{q}} \Rightarrow \mathbf{o}_{\mathbf{q},s} = \mathbf{o}$.

(C) *If* $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$ *is generic over* $\mathbf{V}$, <u>*then*</u> *in* $\mathbf{V}[\mathbf{G}]$, *the set of members* $\eta \in \lim(\mathcal{T}_{\bar{\lambda}})$ *which are generic for* $(\mathbb{Q}^2_{\mathbf{n}}, \underaccent{\tilde}{\eta}^2_{\mathbf{n}})$ *over* $\mathbf{V}$ *is exactly* $\{\eta_{\mathbf{q},s} : s \in L_{\mathbf{q}}\}$.

Proof. Essentially by [HS, §6, Claim 21]. There the iteration is FS, but this does not cause any serious problem. Also, we can prove this similarly to the claims above. $\square_{4.10}$

§ 5. Variants, and the theorem

Discussion 5.1. 1) In §1-4 we chose the simplest version of the iteration $\mathbf{q}$: specifically, $\mathrm{TR}_{\mathbf{q}}$ was simple. Here we will present a more general version.

2) Considering a $\leq_{\mathbb{P}_{\mathbf{q}}}$-increasing sequence $\langle p_i : i < \delta \rangle$, where $\delta < \kappa$ is a limit ordinal, we do not always have a least upper bound. This occurs when $\mathrm{cf}(\delta) \in \Theta$: otherwise we would have to add the existence of lubs in the demands in 1.2.

3) Also, 2.4(3) has to be changed.

4) We have the option of generalizing 'κ-trunk controller' to '$(\kappa, \Theta, \Upsilon)$-trunk controller,' but to save on parameters we will not pursue this. (But we will remark when the proof requires changes.)

5) In the iteration, we did not deal with the case where $\underset{\sim}{\mathbb{Q}}_{\mathbf{p},t}$ is defined from a parameter which is a $\mathbb{P}_{\mathbf{p},t}$-name (even in this case it is a $\mathbb{P}_{\mathbf{p},t}$-name of a real).

Fact 5.2. We can repeat §1-4 with the following changes:

1) In 1.1(3),(4),

- $\bullet_1$ Every $\mathrm{TR} \in \mathbf{TR}$ is a $(\kappa, \Theta, \Upsilon)$-trunk controller (see 0.7(2)) so $\mathbf{TR}$ is not necessarily simple.
- $\bullet_2$ Again, we demand that the $< \kappa$-support product of a sequence of members of $\mathbf{TR}$ is always a κ-trunk controller. (This is not crucial.)
- $\bullet_3$ $\theta = \mathrm{cf}(\theta) > \kappa$ and $\alpha < \kappa \Rightarrow |\alpha|^{\kappa} < \theta$

2) In 1.2(A),

$$\mathrm{TR}_{\mathbf{o}} = (|\mathrm{TR}_{\mathbf{o}}|, \leq_{\mathbf{o}}, \mathsf{plus}_{\mathbf{o}}, S_{\mathbf{o}}, \mathrm{val}_{\mathbf{o}}).$$

3) In 1.2(F)(b):

- In the conclusion, generally only an upper bound exists.
- But if $\mathrm{cf}(\delta) \in \Theta$ then the full conclusion holds.

4) In 1.5(2), we demand only

- Any truly increasing sequence in $\mathbb{Q}_{\mathbf{o}}$ of length $< \kappa$ has an upper bound.
- Any truly increasing sequence of length $\delta < \kappa$ with $\mathrm{cf}(\delta) \in \Theta$ has a least upper bound.

5) We replace 1.9(1)(c) by

(c)$'$ $\mathcal{A}_t \subseteq [L_{\mathbf{q},t}]^{<\theta}$

6) In Definition 1.11, we omit clause (f).

7) In 2.4(3), if $\mathrm{cf}(\delta) \in \Theta$ then yes, there are lubs. (Otherwise, we just have upper bounds.)

8) In $(*)_{s,i}$ in the proof of 2.7(2), we replace clause (c) with

(c)$'$ $A^*_{s,i} \cap A^*_{s,j} = \varnothing$ for all $j < i$, and $A^*_{s,i} \cap A = \varnothing$ for all $A \in \mathcal{A}_{p(s),i}$.

9) Nevertheless, if we weaken clause (1) $\bullet_2$ then in 4.5(b) we have to add the assumption that $\mathrm{TR}_{\mathbf{o}} \times \prod\limits_{t\in L_{\mathbf{q}}}^{<\kappa} \mathrm{TR}_{\mathbf{o}_t} \notin \mathbf{O}^+$ is a κ-trunk controller.

Theorem 5.3. *Adopting the context of* 5.1, *we have* $(A) \Rightarrow (B)$ *and* $(A) \wedge (B) \Rightarrow (C)$, *where*

(*A*) (*a*) *We have '* $\bullet_1 \vee \bullet_2$,*' where*
- $\bullet_1$ κ *is inaccessible and* $\mathbf{o} := \mathbf{o}_{\bar\lambda}$ *(so* $\mathbb{Q}_{\mathbf{o}} = \mathbb{Q}_{\bar\lambda}$*).*
- $\bullet_2$ $\kappa := \aleph_0$ *and* $\mathbf{o} := \mathbf{o}^2_\kappa$ *as in* 1.6(1).

(*b*) $\lambda > \theta = \mathrm{cf}(\theta) > \kappa$

(*B*) *There exist* $\bar{\mathbf{q}}$ *and* $\bar t$ *such that*
- (*a*) $\bar{\mathbf{q}} = \langle \mathbf{q}_\alpha : \alpha \leq \theta\rangle \subseteq \mathbf{Q}$*. Let* $\mathbf{q} := \mathbf{q}_\theta$*.*
- (*b*) $|L_{\mathbf{q}_\alpha}| = \lambda$
- (*c*) $\bar t = \langle t_\alpha : \alpha < \theta\rangle$
- (*d*) $t_\alpha \in L_{\mathbf{q}_{\alpha+1}}$ *and* $L_{\mathbf{q}_\alpha} \in \mathcal{A}_{\mathbf{q},t_\alpha}$
- (*e*) $\mathbf{o}_{\mathbf{q},t} := \mathbf{o}$
- (*f*) *For all* $r, s \in L_{\mathbf{q}}$ *there exists a* $\pi \in \mathrm{aut}(L_{\mathbf{q}})$ *such that* $\pi(r) = s$*.*
- (*g*) *Moreover, if* $s <_{L_{\mathbf{q}}} t_1$ *and* $s <_{L_{\mathbf{q}}} t_2$*, then there exists a* $\pi \in \mathrm{aut}(L_{\mathbf{q}})$ *such that* $\pi(t_1) = t_2$ and $\pi \restriction L_{\leq s}$ *is the identity.*

(*C*) *Letting* $\underset{\sim}{X} := \{\underset{\sim}{\eta}_{\mathbf{q},s} : s \in L_{\mathbf{q}}\}$ *be a* $\mathbb{P}_{\mathbf{q}}$*-name, we have that*

$$\mathbf{V}_1 := \mathrm{HOD}(\mathbf{V} \cup \underset{\sim}{X})^{\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}}$$

satisfies the following:
- (*a*) *It is a model of* ZF*.*
- (*b*) $\mathrm{id}_{<\theta}(\mathbb{Q}, \underset{\sim}{\eta})$ *has measurability.*
- (*c*) $\mathbf{V}_1$ *has* $\mathsf{DC}_{<\theta}$*.*

Proof. Straightforward. $\square_{5.3}$

Remark 5.4. We may consider an alternative to 5.2, retaining "$\mathcal{A}_{\mathbf{m},t}$ is a collection of pairwise disjoint sets." But when defining $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$, we replace "$\mathcal{A}_{\mathbf{m},s} \subseteq \mathcal{A}_{\mathbf{n},s}$" by

- For every $A \in \mathcal{A}^+_{\mathbf{m},s}$ there exists a unique $B \in \mathcal{A}^+_{\mathbf{n},s}$ such that $B \supseteq A$.

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