A COMPLICATED FAMILY OF TREES WITH $\omega + 1$ LEVELS
SH331

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ABSTRACT. Our main aim is to prove that if T is a complete first-order theory, which is not superstable (no knowledge on this notion is required), included in a first-order theory T_1 then for any $\lambda > |T_1|$ there are 2^λ models of T_1 such that for any two of them, the $\tau(T)$-reducts of one is not elementarily embeddable into the $\tau(T)$-reduct of the other, thus completing the investigation of the 1978 author's book "*Classification Theory and the Number of Non-Isomorphic Models*". Note the difference with the case of unstable T: there $\lambda \geq |T_1| + \aleph_1$ suffices for the existence of 2^λ pairwise non-isomorphic such models.

As earlier, it suffices for every such λ to find a complicated enough family of trees with $\omega + 1$ levels of cardinality λ. If λ is regular this is done already in Chapter VIII of the author's book. The proof here (in sections 1 and 2) goes by dividing into cases, each with its own combinatorics. In particular, we have to use guessing clubs which was discovered for this aim.

In §3 we improve the combinatorics, an aim is to consider strongly $\aleph_\epsilon$-saturated models of stable T (so if you do not know stability better just ignore this). We also deal with separable reduced Abelian p-groups. We then deal with various improvements of the earlier combinatorial results.

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Was mostly ready in the early nineties, and public to some extent. This was written as Chapter VI of the book [Shear], which hopefully will materialize someday, but in meanwhile is it [Shei]. The intentions were: [Sheh] (revising [She86] for Chapter VI), [AGS] for Ch. II, [Shei] for Ch.III, [She22] for Ch.IV, [Shek] for Ch.V, [Shea] for Ch.VI, [Shec] for Ch.VII, [Sheg], revision of [She85] for Ch.VIII, [Shee] for Ch.IV [Shef] for Ch.X, [Shed] for the appendix, and probably [She04], and [Shej]. References like [Shed, q17 = Lc2] means that c2 is the label of 3.19 in [Shed], will only help the author if changes in the paper [Shed] will change the number. The reader should note that the version in author's website is usually more updated than the one in the mathematical archive.
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§ 0. Introduction

In [She78, Ch.VIII,§2] for unsuperstable (complete first order) theory T, it was proved that $\lambda > |T| + \aleph_1 \Rightarrow \dot{\mathbb{I}}(\lambda, T) = 2^\lambda$, in fact for every T_1 extending $T, \lambda > \mu := |T_1| + \aleph_0 \Rightarrow \dot{\mathbb{I}}(\lambda, T_1, T) = 2^\lambda$, and we have gone to considerable troubles to prove it for all cases, in ZFC (recall that $\dot{\mathbb{I}}(\lambda, T) = \dot{\mathbb{I}}(\lambda, T, T)$ where $\dot{\mathbb{I}}(\lambda, T_1, T)$ is the number of $\tau(T)$-reducts of models of T_1 of cardinality λ up to isomorphism, where $T \subseteq T_1$, and now both are first-order complete theories). $\dot{I}\dot{E}(\lambda, T_1, T)$ is the maximal number of such models no one elementarily embedded into another (pedantically, the supremum); see Definition [Shei, 1.4=L1.4new].

Now [She78, Ch.VIII,§2] gets results of the form $\dot{I}\dot{E}(\lambda, T_1, T) = 2^\lambda$ under some constraints on $\lambda > |T_1|$ but have not tried to exhaust. Later this was put in a more general framework (see [She83] or [Shei, §2]) with several applications and more cases for $\lambda > |T_1|$; the cases left open were:

(α) λ strong limit of cofinality $\aleph_0$

and

(β) λ not strong limit, $\neg(\exists \chi)[\mu \leq \chi = \chi^{\aleph_0} < \lambda \leq 2^\chi]$ (for example $\lambda < 2^{\aleph_0}$).

Looking through [She78, Ch.VIII] you may get the impression that the general case ($\lambda > |T_1|$) is obviously true, just needs a proof (as this holds in so many cases with diverse proofs). Now in addition to the accepted wisdom (at least among mathematicians) that such arguments are not proofs, there was until recently (i.e. before this was done in 1988) a reasonable argument for the other side: For most of the cases which were left open in [She78, Ch.VIII], their negations have been proved consistent (by [She80], [She89]). However, here we prove this in all the cases.

Here we replace the properties from [Shei, §2] with stronger ones (variants of "super unembeddable"), prove they imply the ones from [Shei, §2], look at their interrelations and mainly prove the existence of such families of trees for the various cardinalities. In 1.11–1.21 we have the parallel of old theorems in the present frame see 1.1 and 1.4; in §2 new ones. Lastly, in 2.34 we draw the conclusions.

For this we prove in ZFC theorems of the form "there is a club-guessing sequence" (continued, see [Shear, Ch.III], [She97] and more current summary in [Sheb]). Our main theorem is 2.34: for $\lambda > \mu, K^\omega_{\rm tr}$ has the full $(\lambda, \lambda, \mu, \aleph_0)$-super bigness property (defined in 1.1, 1.4 below). As a consequence, we here shall get that for $\lambda > \mu, K^\omega_{\rm tr}$ has the full strong $(\lambda, \lambda, \mu, \aleph_0)$-bigness property (see definition in [Shei, 2.2(3)=f5(3)], by 3.1(2)).

Lastly in 2.34 we sum up what we get for $K^\omega_{\rm tr}$ for every $\lambda > \mu$. The proof of 2.34 is split into cases (each being an earlier claim) using several combinatorial ideas (in some we get stronger combinatorial results than in others).

The results are phrased such that they apply to many non-first-order classes.

We conclude by deriving some further results dealing with some specific cases in §3. There, we begin by deducing the results on $\dot{I}\dot{E}(\lambda, T_1, T)$ being 2^λ when $\lambda > |T_1|, T \subseteq T_1$ are first order complete theories, T unsuperstable (in 3.1(1)). Then we get similar results for the number of strongly $\aleph_\varepsilon$-saturated models: the case which requires work is T stable not superstable $\lambda = \lambda(T) + \aleph_1, T = T_1$ this requires some knowledge of stability theory but is not used elsewhere; naturally this requires the so-called $\mathbf{F}^f_{\aleph_0}$-construction from [Shear, Ch. IV]. We then deal with the number of reduced separable abelian $\dot{p}$-groups on λ no one embeddable in another (not necessarily purely). We prove it by assuming $\lambda > 2^{\aleph_0}$ (in 3.28) for this we need to improve the main conclusion of §2 (in 3.26).

In §1 we, in a sense, redo results from [She90, §2,VIII] and [She83] restated in terms of super unembeddability in particular in 1.16 (so we prove somewhat more).

The results in §2 were presented in a mini course in Rutgers, fall 88; it contains "guessing clubs in ZFC", which because of the delay in publication was also represented and continued in [She94, Ch.III], see more [She97]; printed version exists since the early nineties.

The results on the number of strongly $\aleph_\varepsilon$-saturated model 3.3 improve Theorem [She87, 2.1] and [She88, 2.1] (see explanations below in 3.4), they assume knowledge of [She78] or [She90] but the reader can skip it as this theorem is not used later and move to 3.26; some definitions are recalled in 3.4 below. We refer to [Shed] for various combination facts, see history there (this will help if the book on non-structure will materialize).

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Convention 0.1. 1) $K_{\rm tr} = K^\omega_{\rm tr}$ (defined in [Shei, 1.9(4)=Lb11(4)] for $\kappa = \aleph_0$) is restricted (in this section) to the cases each $\mathrm{Suc}(\eta)$ is well ordered, so $K^\omega_{\rm tr}$ is the class of trees I with $\omega + 1$ levels expanded by a well ordering on each $\mathrm{Suc}_I(\eta)$ (see [Shei, 1.9 = Lb11(2)]), so getting a lexicographic order $<^I_{lx}$ on I, and P^I_α for $\alpha \le \omega$ is the set of elements of level α and if $\beta < \alpha \wedge \eta \in P^I$, then $\nu = \eta{\restriction}\beta$ is the unique $\nu \in P^I_\beta$, which is an initial segment of η.
2) Also, $\mathscr{M}_{\mu,\kappa}(J)$ from [Shei, 2.1 = Lf2] and κ regular for transparency.
3) Strong finitary, see Definition [Shei, 2.2 = Lf5], that is:

- a member a of $\mathscr{M}_{\mu,\kappa}(J)$ is called *strongly finitary* in $\mathscr{M}_{\mu,\kappa}(J)$ if there are n, a $\tau_{\mu,\kappa}$-term $\sigma(x_0, \dots, x_{n-1})$ with finitely-many sub-terms using only functions with finite arity are $\tau_0, \dots, \tau_{n-1} \in J$ such that $\mathscr{M}_{\mu,\kappa}(J) \models$"$a = \sigma(\tau_0, \dots, \tau_{n-1})$". Recall $\tau_{\mu,\kappa}$ is the vocabulary of $\mathscr{M}_{\mu,\kappa}(J)$.

Notation 0.2. 1) Let $\mathscr{H}(\chi)$ be the family of sets with transitive closure of cardinality $< \chi$ (so of cardinality $2^{<\chi}$) and we let $<^*_\chi$ be a well-ordering of $\mathscr{H}(\chi)$.

2) We use M, N for models, $\tau_M = \tau(M)$ is the vocabulary of M, and T for a theory in the vocabulary $\tau_T = \tau(T)$, complete if not said otherwise.

3) For a class K of models and a cardinal λ, let $\dot{\mathbb{I}}(\lambda, K)$ be the number of models $M \in K$ up to isomorphisms. We may write $\dot{\mathbb{I}}(\lambda, T)$ for a theory T (usually of first-order) instead of $K = \mathrm{Mod}_T$ and $\dot{\mathbb{I}}(\lambda, T_1, T)$ for theories $T_1 \supseteq T$ (usually first-order complete), instead of $K = \mathrm{PC}(T_1, T) = \{M{\restriction}\tau_T \colon M \text{ is a model of } T_1\}$.

Recall,

Definition 0.3. A complete first-order theory T is called *unsuperstable* <u>when</u> there is some φ witnessing it, which means:

(a) $\varphi = \langle \varphi_n(x, \bar{y}_n) \colon n < \omega \rangle$,
(b) $\varphi_n(x, \bar{y}_n) \in \mathbb{L}(\tau_T)$, i.e., it is a first-order formula in the vocabulary τ_T of T, and
(c) for every λ there are a model M of T, sequences $\bar{a}_\eta \in {}^{\lg(\bar{y}_m)}M$ for $\eta \in {}^n\lambda$ and elements b_η of M for $\eta \in {}^\omega\lambda$ such that if $\eta \in {}^\omega\lambda$, and $\nu \in {}^n\lambda$ then $M \models \varphi_n[b_\eta, \bar{a}_\nu] \Leftrightarrow \nu = \eta{\restriction}n$.

§ 1. PROPERTIES SAYING TREES ARE COMPLICATED

See also with [Shec, §1], 1.9.

Definition 1.1. We say $I \in K^\omega_{\rm tr}$ is (μ, κ)-super-unembeddable into $J \in K^\omega_{\rm tr}$ $\underline{\rm if}$: for every regular large enough χ^* (for which $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$, recalling 0.2(1)), and for simplicity $<^*_{\chi^*}$ is a well-ordering of $\mathscr{H}(\chi^*)$ and $x \in \mathscr{H}(\chi^*)$ we have:

- $(*)$ there are η, M_n, N_n (for $n < \omega$) such that:
  - (i) $M_n \prec N_n \prec M_{n+1} \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,
  - (ii) $M_n \cap \mu = M_0 \cap \mu$ and $\kappa \subseteq M_0$,
  - (iii) I, J, μ, κ and x belong to M_0,
  - (iv) $\eta \in P^I_\omega$, i.e. is of level ω in I,
  - (v) for each n for some $k, \eta{\restriction}k \in M_n, \eta{\restriction}(k+1) \in N_n \backslash M_n$,
  - (vi) for each $\nu \in P^J_\omega$, for every large enough $n < \omega$, and
$$\{\nu{\restriction}\ell : \ell < \omega\} \cap N_n \subseteq M_n.$$

Notation 1.2. We may write μ instead (μ, μ) and may omit it if $\mu = \aleph_0$.

Remark 1.3.

1. In Definition 1.1 the "and $x \in \mathscr{H}(\chi^*)$" and "and x" in clause (iii) can be omitted (and we get equivalent definition using a bigger χ^*). However, in using the definition, with x it is more natural: we construct something from a sequence of I's, we would like to show that "there are no objects such that..." and x will be such an undesirable object in a proof by contradiction.
2. We can also omit $<^*_{\chi^*}$ at the price of increasing χ^*.

Recall from [Shei].

Definition 1.4.

1. $K^\omega_{\rm tr}$ has the $(\chi, \lambda, \mu, \kappa)$-super-bigness property $\underline{\rm if}$: there are $I_\alpha \in K^\omega_{\rm tr}$ of cardinality λ for $\alpha < \chi$ such that for $\alpha \neq \beta, I_\alpha$ is (μ, κ)-super unembeddable into I_β.
2. $K^\omega_{\rm tr}$ has the full $(\chi, \lambda, \mu, \kappa)$-super-bigness property $\underline{\rm if}$:
there are $I_\alpha \in K^\omega_{\rm tr}$ of cardinality λ for $\alpha < \chi$ such that for $\alpha < \chi, I_\alpha$ is (μ, κ)-super unembeddable into $J_\alpha := \sum\limits_{\beta<\chi, \beta\neq\alpha} I_\beta$, which means, assuming for simplicity that the I_β-s are pairwise disjoint (except the common root ${\rm rt} = {\rm rt}_{I_\beta}$) that (recalling Definition [Shei, 1.9 = Lb11(4)] of $K^\omega_{\rm tr}$):
   - the set of elements of J_α is $\{s \colon s \in I_\beta$ for some $\beta < \chi$, $\beta \neq \alpha\}$,
   - if $s_\ell \in I_{\beta_\ell}$ for $\ell = 1, 2$ (and $\beta_1, \beta_2 \in \chi \setminus \{\alpha\}$) $\underline{\rm then}$:
$$s_1 <^{J_\alpha}_{ex} s_2 \Leftrightarrow \beta_1 < \beta_2 \vee (\beta_1 = \beta_2 \wedge s_1 <^{I_{\beta_1}}_{ex} s_2).$$
3. We may omit κ if $\kappa = \aleph_0$.

* * *

The next definition gives many variants of Definition 1.1 (clause $(\mathrm{D})^+$ repeat Definition 1.1); but the reader may understand the rest of the section without it; just

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ignore 1.5, 1.6, 1.7, 1.9(1); and from 1.9 on, ignore the superscript to "super" (we are getting stronger results).

Definition 1.5. We say $I \in K^\omega_{\rm tr}$ is (μ, κ)-super$^\ell$-unembeddable into $J \in K^\omega_{\rm tr}$ <u>if</u> one of the following holds (and in this paper, ℓ is always one of those):

- (A) $\ell = 1$ and for every regular large enough cardinal χ^* and $x \in \mathscr{H}(\chi^*)$ where $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$ and $f : I \to \mathscr{M}_{\mu,\kappa}(J)$, which is strongly finitary on P^I_ω [i.e. for $\nu \in P^I_\omega, f(\nu)$ is a strongly finitary member of $\mathscr{M}_{\mu,\kappa}(J)$; see 0.1(3)] and g a function from I (really P^I_ω) to finite sets of ordinals <u>there is</u> $\eta \in P^I_\omega$ such that,
    - $(*)$ letting $f(\eta) = \sigma(\nu_0, \dots, \nu_{n-1})$, for infinitely many $k < \omega$ there are M, N satisfying:
        - (i) $M \prec N \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,
        - (ii) $M \cap \mu = N \cap \mu$, and $\kappa \subseteq M$,
        - (iii) $\{I, J, \mu, \kappa, x\} \in M$,
        - (iv) $\eta{\restriction}k \in M$
        - (v) $\eta{\restriction}(k+1) \in N \setminus M$,
        - (vi) for each $m < n$:
            - (a) $\nu_m \in M$ or,
            - (b) for some $k_m < \lg(\nu_m), \nu_m{\restriction}k_m \in M, \nu_m(k_m) \notin N$ or,
            - (c) $\ell g(\nu_m) = \omega, \nu_m \notin N, (\forall \ell < \omega)[\nu_m{\restriction}\ell \in M]$,
        - (vii) $m < n$ and if $\alpha = \nu_m(k_m)$ (where $(vi)(b)$ holds for ν_m, k_m) or $\alpha \in g(\eta)$ then:
$$\mathrm{Min}[(\chi^* \setminus \alpha) \cap M] = \mathrm{Min}[(\chi^* \setminus \alpha) \cap N].$$
- (B) $\ell = 2$ and for every regular large enough χ^* satisfying $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$ and $x \in \mathscr{H}(\chi^*)$ there is $\eta \in P^I_\omega$ such that:
    - $(*)$ for any finite $w \subseteq \chi^*, n < \omega$ and $\nu_0, ..., \nu_{n-1} \in J$, for infinitely many $k < \omega$ there are models M, N satisfying:
        - (i) $M \prec N \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,
        - (ii) $M \cap \mu = N \cap \mu$, and $\kappa \subseteq M$,
        - (iii) $\{I, J, \mu, \kappa, x\} \in M$,
        - (iv) $\eta{\restriction}k \in M$,
        - (v) $\eta{\restriction}(k+1) \in N \backslash M$,
        - (vi) for each $m < n$ one of the following occurs:
            - (a) $\nu_m \in M$,
            - (b) for some $k_m < \lg(\nu_n), \nu_m{\restriction}k_m \in M, \nu_m{\restriction}(k_m+1) \notin N$,
            - (c) $\ell g(\nu_m) = \omega, \nu_m \notin N, (\forall \ell < \omega)[\nu_m{\restriction}\ell \in M]$.
        - (vii) for each α, if $\alpha = \nu_m(k_m)$ (where $m < n$, and ν_m, k_m satisfy sub-clause (b) of (vi)) or $\alpha \in w$ then[1] $\mathrm{Min}[(\chi^* \setminus \alpha) \cap M] = \mathrm{Min}[(\chi^* \setminus \alpha) \cap N]$.
- (C) $\ell = 3$ and for every regular large enough χ^*, and $x \in \mathscr{H}(\chi^*)$ such that $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$, there is $\eta \in P^I_\omega$ satisfying:
    - $(*)$ for any $n < \omega, \nu_0, \dots, \nu_{n-1} \in J$, there are sequences $\langle M_i, N_i : i < \omega\rangle, \langle \bar{k}^i : i < \omega\rangle$, where $\bar{k}^i = \langle k^i, k^i_0, \dots, k^i_{n-1}\rangle$. such that:

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[1] e.g. both can be undefined

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(i) $M_i \prec N_i \prec M_{i+1} \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,
(ii) $M_i \cap \mu = N_i \cap \mu$ and $\kappa \subseteq M_0$,
(iii) $\{I, J, \mu, \kappa, x\} \in M$,
(iv) $\eta \restriction k^i \in M_i$,
(v) $\eta \restriction (k^i + 1) \in N_i \setminus M_i$,
(vi) for each $i < \omega$ and $m < n$ one of the following occurs:
  (a) $\nu_m \in M_i$,
  (b) $k^i_m < \omega$, $\nu_m \restriction k^i_m \in M_i$, $\nu_m \restriction (k^i_m + 1) \notin N_i$,
  (c) $\lg(\nu_m) = \omega$; $\nu_m \notin N_i$, $(\forall \ell < \omega)[\nu_m \restriction \ell \in M]$,
(viii) for each α, if $\alpha = \nu_m(k_m)$ (where $m < n$, ν_m satisfies (b) of (vi)) then:
$$\mathrm{Min}[(\chi^* \setminus \alpha) \cap M] = \mathrm{Min}[(\chi^* \setminus \alpha) \cap N].$$

(D) $\ell = 4$ and for every regular large enough χ^* (for which $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$) and $x \in \mathscr{H}(\chi^*)$ we have:

($*$) there are $\eta, \dot{D}$ and M_n for $n < \omega$ such that:
(i) $M_n \prec M_{n+1} \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,
(ii) $M_n \cap \mu = M_0 \cap \mu$ and $\kappa \subseteq M_0$,
(iii) $\{I, J, \mu, \kappa\}$ and x belongs to M_0,
(iv) $\eta \in I$, in fact $\eta \in P^I_\omega$,
(v) $\dot{D}$ is a filter on ω containing the filter of all co-finite sets (usually it is equal to it),
(vi) $\{n < \omega$: for some $k, \eta \restriction k \in M_n, \eta \restriction (k+1) \in (M_{n+1} \setminus M_n)\}$ belongs to $\dot{D}$,
(vii) for every $\nu \in P^J_\omega$, we have $\{n$: for some $k < \omega, \nu \restriction k \in M_n, \nu \restriction (k+1) \in M_{n+1} \setminus M_n\}$ is $\emptyset \mod \dot{D}$.

(D^-) $\ell = 4^-$ and the condition of (D) holds just weakening (ii) to:
(ii)$'$ $\{n : M_n \cap \mu = M_{n+1} \cap \mu\} \in \dot{D}$ and $\kappa \subseteq M_0$,

(D^+) $\ell = 4^+$ and the condition of Definition 1.1 holds (so (μ, κ)-super$^{4^+}$-unembeddable will mean (μ, κ)-super-unembeddable).

(E) $\ell = 5$ and for every regular large enough χ^* (for which $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$) and $x \in \mathscr{H}(\chi^*)$ we have:

($*$) there are $\eta, \dot{D}, M_n$ (for $n < \omega$) such that:
(i) $M_n \prec M_{n+1} \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,
(ii) $\mu \subseteq M_n \in M_{n+1}$ (so $\kappa \subseteq M_0$),
(iii) $\{I, J, \mu, \kappa\}$ and x belong to M_0,
(iv) $\eta \in I$, in fact, $\eta \in P^I_\omega$,
(v) $\dot{D}$ is a filter on ω extending the filter of all co-finite sets (usually it is equal to it),
(vi) $\{n < \omega$: for some $k, \eta \restriction k \in M_n, \eta \restriction (k+1) \in (M_{n+1} \setminus M_n)\}$ belongs to $\dot{D}$,
(vii) for every $\nu \in P^J_\omega$ we have $\{n$: for some $k < \omega, \nu \restriction k \in M_n, \nu \restriction (k+1) \in M_{n+1} \setminus M_n\}$ is $\equiv \emptyset \mod \dot{D}$.

(F) $\ell = 6$ and for every regular large enough χ^* for which $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$, and $x \in \mathscr{H}(\chi^*)$ there are $\langle M_n : n < \omega\rangle, \eta$ such that:

$(*)$ (i) $M_n \prec M_{n+1} \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,

(ii) $M_n \cap \mu = M_0 \cap \mu$ and $\kappa \subseteq M_0$,

(iii) $\{I, J, \mu, \kappa, x\} \in M_0$,

(iv) $\eta \in P^I_\omega$,

(v) $\eta\restriction n \in M_n$,

(vi) $\eta\restriction(n+1) \notin M_n$

(vii) for every $\nu \in P^J_\omega$, for some $n, \{\nu\restriction\ell : \ell < \omega\} \cap (\bigcup\limits_{m<\omega} M_m) \subseteq M_n$.

(F^+) $\ell = 6^+$ and (i) - (vi) of (F) and:

$(vii)^+$ for every $\nu \in P^J_\omega$ we have:

$$\left[\bigwedge_\ell \nu\restriction\ell \in \bigcup_{n<\omega} M_n \Rightarrow \nu \in \bigcup_{n<\omega} M_n\right].$$

(F^-) $\ell = 6^-$ and the conditions in (F) hold but replace clause (v) by

$(v)^-$ $(\forall n)(\exists m)[\eta \restriction n \in M_m]$ but $(\forall m)(\exists n)[\eta \restriction n \notin M_m]$.

$(F^\pm)$ $\ell = 6^\pm$, and the condition in (F) when we make both changes (use $(vii)^+$ and $(v)^-$).

(G^-) $\ell = 7^-$ and for every regular large enough χ^* for which $\{I, J, \mu, \kappa\} \in \mathscr{H}(\chi^*)$ and $x \in \mathscr{H}(\chi^*)$ we have:

$(*)$ there are $M_n (n < \omega), \eta$ such that:

(i) $M_n \prec M_{n+1} \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,

(ii) $M_n \in M_{n+1}$ and $\mu \subseteq M_0$,

(iii) $\{I, J, \mu, \kappa, x\} \in M_0$,

(iv) $\eta \in P^I_\omega$,

(v) for every $k < \omega, \eta\restriction k \in \bigcup\limits_{n<\omega} M_n$,

(vi) for every n for some $k, \eta\restriction k \notin M_n$,

(vii) for every $\nu \in P^J_\omega$, for some n

$$\{\nu\restriction\ell : \ell < \omega\} \cap (\bigcup_{m<\omega} M_m) \subseteq M_n.$$

(G) $\ell = 7$ and (i) - (iv),(vii) of (G^-) and:

$(v)^+$ $\eta \restriction n \in M_n$,

$(vi)^+$ $\eta\restriction(n+1) \notin M_n$.

(G^+) $\ell = 7^+$ and (i) $-$ (iv) of (G^-) and, (v)$^+$, (vi)$^+$ of (G), and:

$(vii)^+$ for every $\nu \in P^J_\omega$,

$$\{\nu\restriction\ell : \ell < \omega\} \subseteq \bigcup_{m<\omega} M_m \Rightarrow \nu \in \bigcup_{m<\omega} M_m.$$

$(G^\pm)$ $\ell = 7^\pm$ and (i)-(iv) of (G)$^-$ of 1.1 and (vii)$^+$ of (G^+) and

$(vi)^{++}$ for every n, for some k we have $\eta\restriction k \in M_n, \eta\restriction(k+1) \in M_{n+1} \setminus M_n$.

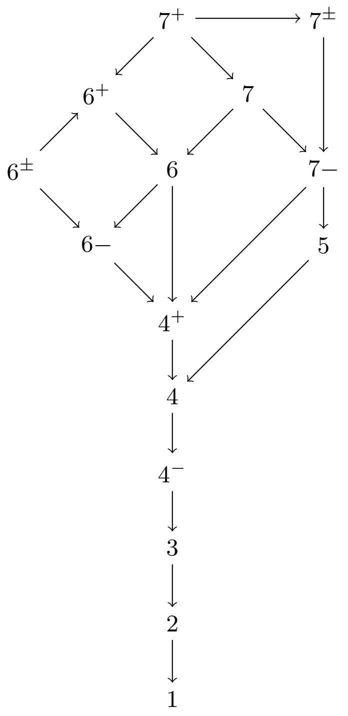

Figure 1. Implication diagram.

Definition 1.6. The $(\chi, \lambda, \mu, \kappa)$-super^l-bigness property and the full $(\chi, \lambda, \mu, \kappa)$-super^l-bigness property are defined in a similar way to Definition 1.4.

Fact 1.7.

1. (1) If $I \in K^\omega_{\rm tr}$ is (μ, κ)-super^m-unembeddable into $J \in K^\omega_{\rm tr}$ <u>then</u> I is (μ, κ)-super$^\ell$-unembeddable into J when $1 \leq \ell \leq m \leq 7, (\ell, m) \neq (5, 6), \ell, m \in \{1, 2, 3, 4, 5, 6, 7\}$ and when $(\ell, m) \in \{(3, 4^-), (4^-, 4), (4, 4^+), (4^+, 6), (6, 6^+), (4^+, 7^-), (7^-, 7), (7, 7^+), (7^-, 7^\pm), (7^\pm, 7^+), (6^+, 7^+), (6, 7), (6^-, 6^\pm), (6^\pm, 6^+)\}$.
2. (2) If $K^\omega_{\rm tr}$ has the $(\chi, \lambda, \mu, \kappa)$-super^m-bigness property <u>then</u> $K^\omega_{\rm tr}$ has the $(\chi, \lambda, \mu, \kappa)$-super$^\ell$-bigness property for (ℓ, m) as above.
3. (3) If $K^\omega_{\rm tr}$ has the full $(\chi, \lambda, \mu, \kappa)$-super^m-bigness property <u>then</u> $K^\omega_{\rm tr}$ has the full $(\chi, \lambda, \mu, \kappa)$-super$^\ell$-bigness property for (ℓ, m) as above.
4. (4) All those properties has obvious monotonicity properties: we can decrease μ, κ and χ and increase λ (if we add to I a well-ordered set in level 1, nothing happens).
5. (5) The notions
   "(μ, κ)-super$^{4^+}$-unembeddable" and
   "(μ, κ)-super-unembeddable" are the same; also
   "[full]$(\chi, \lambda, \mu, \kappa)$-super$^{4^+}$-bigness" and
   "[full]$(\chi, \lambda, \mu, \kappa)$-super bigness" are the same.

Proof. The proof follows by the definitions. $\square_{1.7}$

We shall now observe two things:

Observation 1.8. *1) First (1.9(1)) the "full" version (see Def. 1.4,1.6) is much stronger (increasing the χ) hence we shall later concentrate on it.*

2) Second (1.9(2)), the super version (from here) implies the "strong" version from [Shei, §2]*, hence, for example, implies the results on unsuperstable theories.*

Claim 1.9. *1) If $K^\omega_{\rm tr}$ has full $(\chi,\lambda,\mu,\kappa)$-super$^\ell$-bigness property, then $K^\omega_{\rm tr}$ has the $(2^{{\rm Min}\{\lambda,\chi\}},\lambda,\mu,\kappa)$-super$^\ell$-bigness property.*
2) If $K^\omega_{\rm tr}$ has the [full] $(\chi,\lambda,\mu^{<\kappa},2^{<\kappa})$-super$^\ell$-bigness property, $(\chi\ge\lambda)$ then $K^\omega_{\rm tr}$ has the [full] strong $(\chi,\lambda,\mu,\kappa)$-bigness property for $\varphi_{\rm tr}$ for functions f which are strongly finitary on P_ω.

Remark 1.10. (1) On "strongly finitary on P_ω" (see Definition [Shei, 2.2=Lf5] and 0.1(3) here).

(2) On "$(\chi,\lambda,\mu^{<\kappa},2^{<\kappa})$-super$^\ell$-bigness property, see Definition 1.4(1) with "full", see 1.4(2).

(3) On the "strong $(\chi,\lambda,\mu,\kappa)$-bigness property", see [Shei, 2.2 = Lf5(1), (2), pg. 21] and for the full version, see [Shei, 2.2=Lf5(3)].

Proof. 1) Easy (and similar in essence to [Shei, 2.20=Lj11(1)]). Suppose $\langle I_\alpha:\alpha<\chi\rangle$ witnesses the full $(\chi,\lambda,\mu,\kappa)$-super$^\ell$-bigness property, and (by 1.7(4)) without loss of generality $\chi\le\lambda$. Without loss of generality the I_α's have a common root $<>$, and except this are pairwise disjoint. We can find $\langle A_\alpha:\alpha<2^\chi\rangle$ such that:

$$A_\alpha\subseteq\chi,|A_\alpha|\le\lambda \text{ and } [\alpha\ne\beta\Rightarrow A_\alpha\nsubseteq A_\beta]$$

(use just $A\subseteq\lambda$ such that $[2\alpha\in A\Leftrightarrow 2\alpha+1\notin A]$).

Now, let

$$I^*_\alpha:=\sum_{i\in A_\alpha} I_i,$$

defined naturally: the universe is the union of the universes, where each I_i is a submodel of I^*_α when $i\in A_\alpha$, and the lexicographic order is such that:

$$i<j\ \&\ \eta\in I_i\setminus\{<>\}\ \&\ \nu\in I_j\setminus\{<>\}\Rightarrow\eta<_{lx}\nu.$$

[Note: $\chi>2^\lambda$ never occurs].
2) It suffices by 1.7(1), of course, to prove for the case $\ell=1$, and clearly, it suffices to show the following Subclaim. $\square_{1.9}$

Subclaim 1.11. *If $\mu_1=\mu^{<\kappa}$ and $\kappa_1=\sum\{(|\alpha|^{\aleph_0})^+:\alpha<\kappa\}$ and, if $I\in K^\omega_{\rm tr}$ is (μ_1,κ_1)-super^1-unembeddable into $J\in K^\omega_{\rm tr}$ then I is strongly $\varphi_{\rm tr}$-unembeddable for (μ,κ) into J, for functions f which are strongly finitary on P^I_ω (see* [Shei, 2.2=Lf5(1), (4), (5)]*).*

Proof. Recalling Definition 1.5(A), without loss of generality I,J are subsets of ${}^{\omega\ge}\theta$ for some cardinal θ (see 0.1 and [Shei, 1.9(2)=Lb11(2)]), and let $<^*$ be a well-ordering of $\mathscr{M}_{\mu,\kappa}(J)$ (respecting being a subterm, i.e. if a is a subterm of b then $a\le^* b$). Suppose f is a function from I into $\mathscr{M}_{\mu,\kappa}(J)$, so

$(*)_1$ for $\varrho\in I$, we have $f(\varrho)=\sigma_\varrho(\nu_{\varrho,0},\dots,\nu_{\varrho,i},\dots)_{i<\alpha_\varrho}$ for some $\alpha_\varrho<\kappa,\nu_{\varrho,i}\in J$.

Recalling that we are assuming f is strongly finitary on P^I_ω we have,

$(*)_2$ $\varrho\in P^I_\omega\Rightarrow\alpha_\varrho<\omega\,\&$ $[\sigma_\varrho$ has finitely many subterms].

We should now find $\bar{s}$, $\bar{t}$ as in [Shei, 2.2=Lf5(1)]. Let χ be regular large enough, $x = \langle \mu, \kappa, I, J, f \rangle$ and define for $\varrho \in P^I_\omega$,

$(*)_3$ $g(\varrho) = \{\alpha$: the α-th element by $<^*$ is a subterm of $f(\varrho)\}$,

which is finite (so we use "the strongly finitary" so that $g(\eta)$ is finite, this is the only use). Now we shall use Definition 1.5(A).

So let η, k, M, N be as in (A) of Definition 1.5 (for the given χ, f, g we use just one k), hence noting $(*)_1$, $(*)_2$ apply in particular for $\varrho = \eta$, and clearly $\sigma_\eta(\nu_{\eta,0}, \ldots, \nu_{\eta,i}, \ldots)_{i<\alpha_\eta}$ is well defined and equal to $f(n)$. So by reordering $\nu_{\eta,\ell}(\ell < \alpha_\eta)$ we can have:

Subfact 1.12. There are $n_0 \le n_1 \le n_2 = \alpha_\eta$ such that:

$(*)_4$ (a) for $\ell < n_0, \nu_{\eta,\ell} \in M$, and let $j_\ell = \ell g(\nu_{\eta,\ell})$,

(b) for $\ell \in [n_0, n_1)$ for a (unique) $j_\ell, \nu_{\eta,\ell} \restriction j_\ell \in M, \nu_{\eta,\ell}(j_\ell) \notin N$ and

$$\begin{aligned}\gamma_\ell \; &:= \min\{\gamma : \gamma \text{ an ordinal from } M, \nu_{\eta,\ell}(j_\ell) \le \gamma\} \\ &= \min\{\gamma : \gamma \text{ an ordinal from } N, \nu_{\eta,\ell}(j_\ell) \le \gamma\},\end{aligned}$$

(c) for $\ell \in [n_1, n_2), \nu_{\eta,\ell} \notin N$, $\lg(\nu_{\eta,\ell}) = \omega$ but $\{\nu_{\eta,\ell} \restriction m : m < \omega\} \subseteq M$.

Proof. The sets $u_3 := \{\ell < \alpha_\eta \colon \nu_{\eta,\ell} \notin N, \lg(\nu_{\eta,\ell}) = \omega$ and $\{\nu_{\eta,\ell} \restriction j \colon j < \omega\} \subseteq M\}$ and $u_1 := \{\ell < \alpha_\eta \colon \nu_{\eta,\ell} \in M\}$ are disjoint. So letting $u_2 = \alpha_\eta \setminus (u_1 \cup u_3)$, clearly (u_1, u_2, u_3) is a partition of α_η, so remaining $u_1 = [0, n_0]$, $u_2 = [n_0, n_1]$, $u_3 = [n_1, n_2]$, where $n_2 = \alpha_\eta$. So checking $(*)_4$, the least trivial point is why the two definitions of γ_ℓ (for $\ell \in u_2$) are equivalent, which holds by clause (vii) of Definition 1.5(A). $\square_{1.12}$

Clearly k was chosen together with η, M, N and the sequence $\langle \nu_{\eta \restriction (k+1),i} : i < \alpha_{\eta \restriction (k+1)} \rangle$ evidently belong to N (as $f \in N$ and $\eta \restriction (k+1)$ belongs to N).

Now,

Fact 1.13. We have:

$(*)_5$ for each $\ell \in [n_1, n_2)$ for some $k_\ell < \omega$ we have: $\nu_{\eta,\ell} \restriction k_\ell \notin A_1 := B_0 \cup B_1 \cup B_2$ where:

- $B_0 = \{\nu_{\eta \restriction (k+1),i} \restriction m \colon i < \alpha_{\eta \restriction (k+1)}, m \le \ell g(\nu_{\eta \restriction (k+1),i}), i < \alpha_{\eta \restriction (k+1)}\}$
- $B_1 = \{\nu_{\eta,j} \restriction m : j < n_0, m \le \ell g(\nu_{\eta,j})\}$
- $B_2 = \{(\nu_{\eta,\ell} \restriction j_\ell)\hat{\ }\langle \gamma_j \rangle : \ell \in [n_0, n_1)\}$.

[Recall γ_ℓ is from clause (b) of 1.12.]

Proof. Case 1: $\kappa > \aleph_0$

So for each $\ell \in [n_1, n_2)$ if no such k_ℓ exists, so by clause (e) of 1.12, we have $\{\nu_{\eta,\ell} \restriction m \colon m < \omega\}$ is a subset of the set A_1 appearing in the right side above, which belongs to N.

[Why? The first set B_0 in the union belongs to N as $\eta \restriction (k+1) \in N$ by the choice of η, k, M, N. The second set B_1 as $j < n_0 \nRightarrow \nu_{\eta,j} \in M$ by clause (a) of 1.12, i.e. the choice of n_0 and the third set B_2 by the choice of γ_j in clause (b) of 1.12.]

Now A_1 has cardinality $< \kappa$ (as $\alpha_{\eta \restriction (k+1)} < \kappa \wedge \aleph_0 < \kappa$); hence $A_1 \subseteq N$ and (as $\kappa_1 + 1 \subseteq M$ because κ_1 in 1.11 plays the role of κ in Definition 1.6 recalling we are assuming $\kappa > \aleph_0$) not only is included in it, but every ω-sequence from it belongs to N, hence $\nu_{\eta,\ell} \in N$, contradicting $\ell \in [n_1, n_2)$.

Case 2: $\kappa = \aleph_0$

So $\alpha_{\eta\restriction(k+1)} < \omega$ and let $\ell \in [n_1, n_2)$; toward contradiction assume the conclusion in 1.13 fails for ℓ. So one of the following possibly occurs. First, if $(\exists^\infty m)\nu_{\eta,\ell}\restriction m \in B_0$, then for some $i < \alpha_{\eta\restriction(k+1)}$ for infinitely many $m < \omega, \nu_{\eta,\ell}\restriction m = \nu_{\eta\restriction(k+1),i}\restriction m$. As $\ell g(\nu_{\eta,\ell}) = \omega$ (remembering $\ell \in [n_1, n_2)$) this implies $\nu_{\eta,\ell} = \nu_{\eta\restriction(k+1),i}$, but $\nu_{\eta\restriction(k+1),i}$ belongs to N whereas $\nu_{\eta,\ell}$ does not belong to N (remembering $\ell \in [n_1, n_2)$), contradiction. Second, if $(\exists^\infty m)(\nu_{\eta,\ell}\restriction m \in B_1)$, similarly some $j < n_0, \nu_{\eta,\ell} = \nu_{\eta,j}$ but $j < n_0 < \ell$, contradiction to clause (a) of 1.12. Third, if $(\exists^\infty n)(\nu_{\eta,\ell}\restriction m \in B_2)$ but B_2 is finite, contradiction. So 1.13 holds indeed. $\square_{1.13}$

Note:

Fact 1.14. We have:

$(*)_6$ $\sigma_{\eta\restriction(k+1)}$ belongs to M.

Proof. Why? It belongs to N (as $f, \eta\restriction(k+1) \in N$) and it belongs to a set of cardinality $\mu_1 := \mu^{<\kappa}$ from M (the set of $\tau_{\mu,\kappa}$-terms) and $M \cap \mu^{<\kappa} = N \cap \mu^{<\kappa}$ by clause (ii) of Definition 1.5(A) as in subclaim 1.11 the cardinal $\mu^{<\kappa}$ play the role of μ in 1.5(A)]. $\square_{1.14}$

Continuation of the proof of 1.9(2). Now recalling the definition of $\varphi_{\rm tr}$ (in [Shei, 2.6=Lg5, pg. 23]) and of unembeddable (in [Shei, 2.2(1)=Lf5(1)]) clearly, it is enough to show:

$(*)_7$ there is ρ such that:

(A) $\rho \in P^I_{k+1}, \rho(k) \neq \eta(k), \rho\restriction k = \eta\restriction k, \rho \in M$ and $\sigma_\rho = \sigma_{\eta\restriction(k+1)}$ (so $\alpha_\rho = \alpha_{\eta\restriction(k+1)}$),

(B) the sequence $\langle \nu_{\rho,i} : i < \alpha_\rho \rangle$ is similar (i.e. realizes the same quantifier free type in J) to $\langle \nu_{\eta\restriction(k+1),i} : i < \alpha_{\eta\restriction(k+1)} \rangle$ over the set

$$A_2 = \{\nu_{\eta,\ell} : \ell < n_0\} \quad \cup\{(\nu_{\eta,\ell}\restriction j_\ell)\char`^\langle \gamma_\ell \rangle : \ell \in [n_0, n_1)\} \\ \cup\{\nu_{\eta,\ell}\restriction k_\ell : \ell \in [n_1, n_2)\}$$

(C) Let $A_{3,\rho} = A_{3,\eta\restriction(k+1)}$ and $A_{4,\rho,\eta} = A_{4,\eta\restriction(k+1),\eta}$ where for $\rho \in I$

- $A_{3,\rho} = \{(\sigma^1, \sigma^2) : \sigma^1, \sigma^2$ subterms of σ_ρ and $\sigma^1(\ldots, \nu_{\rho,i}, \ldots) <^* \sigma^2(\ldots, \nu_{\eta,\rho,i}, \ldots)\}$

and

- $A_{4,\rho,\eta} = \{(\iota, \sigma^1, \sigma^3) : \iota \in \{0,1\}, \sigma^1$ subterm of σ_ρ, σ^3 a subterm of σ_η, we have that

$$\iota = 0 \Rightarrow \sigma^1(\ldots, \nu_{\eta\restriction(k+1),i}, \ldots) <^* \sigma^3(\ldots, \nu_{\eta,i}, \ldots), \text{ and}$$
$$\iota = 1 \Rightarrow \text{they are equal}\}.$$

$(*)_8$ the sets $A_{3,\rho}, A_{4,\rho,\eta}$ belongs to M.

[Why? Like the proof of 1.14, remembering 1.14.

Now, the set A_2 is a finite subset of M by clause (a) of 1.12, the choice of k_ℓ in clause (b) of 1.12 and the choice of k_ℓ in 1.13. Also the "similarly type in J" of $\langle \nu_{\eta\restriction(k+1),i} : i < \alpha_{\eta\restriction(k+1)} \rangle$ over A_2 belongs to M (in whatever reasonable way we represent it), as the set of such similarly types over A_2 is of cardinality $\leq 2^\kappa$ and it belongs to M, hence there is a first-order formula $\psi(x)$ in the vocabulary of

$(\mathscr{H}(\chi), \in, <^*_\chi)$, with parameters from M saying $x \in I$ is an immediate successor of $\eta \restriction k, \sigma_x = \sigma_{\eta \restriction (k+1)}$, and $\langle \nu_{x,i} : i < \alpha_x \rangle$ is similar to $\langle \nu_{\eta \restriction (k+1),i} : i < \alpha_{\eta \restriction (k+1)} \rangle$ over A_2 in J and ψ express what clauses of (C) from $(*)_7$ says (using the choice of g and (vii) of 1.5 clause (A)). So there is a solution to ψ in M (as $M \prec N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$), now $\eta \restriction (k+1)$ cannot be the $<^*_\chi$-first in $\{x : \psi(x)\}$, but the first is in M, hence there is an $x \in M$ such that $\psi(x) \,\&\, x <_{lx} \eta \restriction (k+1)$.]

So we have finished. $\square_{1.11}$

Remark 1.15. In 1.11, if we weaken the conclusion "I is strongly $\varphi_{\rm tr}$-unembeddable..." to "I is $\varphi_{\rm tr}$-unembeddable" (see middle of Definition [Shei, 2.1(1)=Lf5(1)]) then we can weaken the demand on f replacing "f is strongly finitary for $\eta \in P^I_\omega$" to "$f(\eta)$ is finitary for $\eta \in P^I_\omega$".

Claim 1.16. *1) If λ is regular $> \mu$ <u>then</u> $K^\omega_{\rm tr}$ has the full $(\lambda, \lambda, \mu, \mu)$-super$^{7^\pm}$-bigness property.*
2) If λ is singular $> \chi = \chi^\kappa$ and $2^\chi \geq \lambda$ <u>then</u> $K^\omega_{\rm tr}$ has the full $(\lambda, \lambda, \chi, \aleph_0)$-super^6-bigness property (even the full $(2^\chi, \lambda, \chi, \kappa)$-super^6 bigness property) getting M_n's such that $(\forall \theta)[\kappa^\theta = \kappa \Rightarrow {}^\theta(M_n) \subseteq M_n]$; so if $\kappa^{\aleph_0} = \kappa$ we actually have the full $(\lambda, \lambda, \chi, \kappa)$-super$^{6^+}$-bigness property (and even the full $(2^\chi, \lambda, \chi, \kappa)$-super$^{6^+}$ bigness property).
2A) If λ is singular $> \chi = \chi^\kappa$ and $2^\chi \geq \lambda$, <u>then</u> $K^\omega_{\rm tr}$ has the full $(\lambda, \lambda, \kappa, \kappa)$-supper^7-bigness-property, and if $\kappa > \kappa^{\aleph_0}$ then even the full $(\lambda, \lambda, \kappa, \kappa)$-supper$^{7^+}$-bigness-property.
3) If λ is strong limit singular of cofinality $> \kappa \geq \aleph_0, \kappa \leq \mu < \lambda$ <u>then</u> $K^\omega_{\rm tr}$ has the full $(\lambda, \lambda, \mu, \kappa)$-super^6-bigness property and even the full $(\lambda, \lambda, \mu, \kappa)$-super^6-bigness property.
4) We can, in part (3), weaken "λ strong limit" to $(\forall \theta < \lambda)[\theta^\kappa < \lambda]$.

Remark 1.17. On part (1) see also 2.16.

Proof. 1) Earlier relative is the proof of [She78, Ch.VIII 2.2]), latter relatives is 3.26(1) case 1, (and see in [Shec]) but we give it fully.

Let $S = \{\delta < \lambda : {\rm cf}(\delta) = \aleph_0\}$, let $\langle S_\zeta : \zeta < \lambda \rangle$ be a sequence of pairwise disjoint stationary subsets of S. For each ζ we can find $\bar{C} = \langle C_\delta : \delta \in S_\zeta \rangle$ such that:

- (a) C_δ is an unbounded subset of δ
- (b) ${\rm otp}(C_\delta) = \omega$.

For $\delta \in S_\zeta$ let $\eta_\delta \in {}^\omega\lambda$ be defined by:

$$\eta_\delta(n) \text{ is the (2n)-th member of } C_\delta.$$

For $\zeta < \lambda$, let $I_\zeta = \{\langle \zeta \rangle {}^\frown \nu : \nu \in {}^{\omega >}\lambda\} \cup \{\langle \zeta \rangle {}^\frown \eta_\delta : \delta \in S_\zeta\} \cup \{\langle\, \rangle\}$, and we shall show that $\langle I_\zeta : \zeta < \lambda \rangle$ exemplify the conclusion (for super$^{7^\pm}$). For this, we use $\langle \zeta \rangle {}^\frown \nu$ to make the I_ζ-s essentially pairwise disjoint.

So let $\zeta(*) < \lambda, I := I_{\zeta(*)}$ and $J =: \sum\limits_{\zeta \neq \zeta(*)} I_\zeta$ and we should prove that $I \in K^\omega_{\rm tr}$ is (μ, κ)-super-7^+-embeddable into $J \in J^\omega_{\rm tr}$.

Let $\chi^* = \chi(*)$ be regular large enough, $<^*_{\chi(*)}$ a well-ordering of $\mathscr{H}(\chi^*)$ and $x \in \mathscr{H}(\chi^*)$. We choose by induction on $\alpha < \lambda, M^*_\alpha$ such that:

$\boxplus$ (α) $M^*_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ increasing and continuous with α,

- (β) $\|M^*_\alpha\| < \lambda$,
- (γ) $M^*_\alpha \cap \lambda$ an ordinal,
- (δ) $\langle M^*_\beta : \beta \le \alpha\rangle$ belongs to $M^*_{\alpha+1}$,
- (ε) $\mu \subseteq M^*_0$,
- (ζ) $\{\lambda, \mu, I, J, x, \langle\langle \eta_\delta : \delta \in S_\zeta\rangle : \zeta < \lambda\rangle, \langle I_\zeta : \zeta < \lambda\rangle, \zeta(*)\}$ belongs to M^*_0.

Let $E = \{\delta < \lambda : M^*_\delta \cap \lambda = \delta\}$, it is a club of λ (by clauses (α), (γ), and (δ)), so for some $\delta(*) \in S_{\zeta(*)}$ we have $\delta(*) \in \mathrm{acc}(E)$. Let $\langle m_\ell : \ell < \omega\rangle$ be a strictly increasing sequence of natural numbers such that letting k_ℓ be minimal satisfying $\eta_{\delta(*)}(k_\ell) \ge \lambda \cap M^*_{\eta_{\delta(*)}(m_\ell)}$, we have $k_\ell < m_{\ell+1}$ (or just $\eta_{\delta(*)}(k_\ell) \in M^*_{\eta_{\delta(*)}(m_{\ell+1})}$ which follows as $\alpha \subseteq M^*_\alpha$).

Let $M_n = M^*_{\eta_{\delta(*)}(m_n)}$ and $\eta = \eta_{\delta(*)}$.

Let us check the conditions in $(*)$ of Def.1.5$(G^\pm)$ (see (i)-(iv) of $(G)^-$, and $(v)^+$, $(vi)^{++}$ of (G)$^\pm$, and $(v)^+$, $(vii)^+$ from 1.5(G)$^+$) hold for those M_n, η.

<u>Clause (i)</u>: is obvious, as $\eta_{\delta(*)}(n)$ is strictly increasing, and M^*_α is $\prec$-increasing with α.

<u>Clause (ii)</u>: Now $\mu + 1 \subseteq M_n$ as $M_n \cap \lambda$ is an ordinal (by clause (γ) of $\boxplus$) and $\mu \in M_n$ (by clause (ζ) of $\boxplus$) and $\mu < \lambda$ by assumption. Also $M_n \in M_{n+1}$ by clauses $(\gamma) + (\delta)$ of $\boxplus$.

<u>Clause (iii)</u>: by clause (ζ) of $\boxplus$ above.

<u>Clause (iv)</u>: $\eta \in P^I_\omega$ as $I = I_{\zeta(*)}$, $\delta(*) \in S_{\zeta(*)}, \eta = \eta_{\delta(*)}$ and our definitions.

<u>Clause (vi)$^{++}$</u>: By our choice of k_ℓ and of M_0.

<u>Clauses (vii)$^+$</u>: Note: if $\nu \in P^J_\omega, \alpha < \lambda$ and $\{\nu{\restriction}\ell : \ell < \omega\} \subseteq M^*_\alpha$ then for some $\xi < \lambda$ and $\delta \in S_\xi$ we have $\xi \ne \zeta(*), \nu = \langle\xi\rangle{}^\frown\eta_\delta$, so $\mathrm{cf}(\delta) = \aleph_0$ and $\delta = \sup(\delta \cap M^*_\alpha)$ but $\langle S_\xi : \xi < \lambda\rangle$ are pairwise disjoint so for every $\alpha, \delta < \lambda$ we have at most one such ν, so $\{\nu \in P^J_\omega : \bigwedge\limits_{\ell<\omega} \nu{\restriction}\ell \in M^*_\alpha\}$ has cardinality $\le \|M^*_\alpha\|$ hence is a subset of $M^*_{\alpha+1}$ (as $M^*_\alpha, J \in M^*_{\alpha+1}$). Moreover $\delta \in S_\xi$.

To prove Clause (vii)$^+$ we assume $\nu \in P^J_\omega$; we should prove that $\{\nu{\restriction}\ell : \ell < \omega\} \subseteq \bigcup\limits_{m<\omega} M_m \Rightarrow \nu \in \bigcup\limits_{n<\omega} M_n$, but this union is equal to $M^*_{\delta(*)}$. So using $\alpha = \delta(*)$ above we have (ξ, δ) as there and one of the following cases occurs, and it suffice to check the implication in each of them.

<u>Case 1</u>: $\xi < \delta(*)\,\&\,\delta < \delta(*)$

Clearly $\mathrm{Rang}(\nu) \subseteq ((\xi+1) \cup \delta)$ and so $\nu \in M^*_{((\xi+1)\cup\delta)+1}$ hence $\nu \in M^*_{\delta(*)} = \bigcup\limits_{n<\omega} M_n$.

<u>Case 2</u>: $\xi \ge \delta(*)$

So even $\nu \restriction 1 \notin M_{\delta(*)}$ hence

$$(\exists k < \omega)(\nu(k) \notin \bigcup_{n<\omega} M_n \,\&\, \nu \restriction k \in \bigcup_{n<\omega} M_n)$$

<u>Case 3</u>: $\xi < \delta(*) \le \delta$

So as $\delta(*) \in S_{\zeta(*)}$ necessarily $\delta \notin S_{\zeta(*)}$, hence $\delta(*) < \delta$. Recall that $\nu = \eta_\delta$, so for some k we have $(\forall \ell < k)\eta_\delta(\ell) < \delta(*)$ and $\eta_\delta(k) \ge \delta(*)$. So $\eta_\delta \restriction k \in M^*_{\delta(*)}$, and

$\eta_\delta(k) \geq \delta(*)$, hence as $M^*_{\delta(*)} \cap \lambda = \delta(*)$ because $\delta(*) \in E$ we have $\eta_\delta \restriction (k+1) \notin M^*_{\delta(*)}$, but $M^*_{\delta(*)} = \cup\{M_n : n < \omega\}$, so we are done.
2) Compare with the earlier version [She83, 2.7,pg.116,§3], it is easier than the proof of part (3).

We are assuming $\chi = \chi^\kappa$, now there are subsets A_i of χ for $i < 2^\chi$ ($i < \lambda$ is enough) such that (see [Shed, 3.13=L4.EK], i.e. by Engelking-Karlowicz [EK65]):

- $(*)$ if $w \subseteq \chi$ has cardinality $\leq \kappa$ and $i \in \chi \setminus w$ then $A_i \nsubseteq \bigcup\limits_{j \in w} A_j$.

Let $S^\zeta \subseteq \{\delta < \chi^+ : \mathrm{cf}(\delta) = \aleph_0\}$ be stationary pairwise disjoint for $\zeta < \chi$. For $i < 2^\chi$ let $S_i = \bigcup\limits_{\zeta \in A_i} S^\zeta$ and

$$I_i = \{\langle\rangle\} \cup \{\langle i \rangle {}^\frown \nu \colon \nu \in {}^{\omega>}\lambda\} \cup \{\langle i \rangle {}^\frown \eta \colon \eta \in {}^\omega(\chi^+) \text{ and } \eta \text{ strictly increasing with limit } \sup_{n<\omega} \eta(n) \in S_i\}.$$

We shall show that $\langle I_i : i < 2^\chi \rangle$ is as required, so for $j < 2^\chi$ let $J_j = \sum\limits_{i < 2^\chi, i \neq j} I_i$ and χ^* large enough, $x \in \mathscr{H}(\chi^*)$, so for the rest of the proof of 1.16(2),

- $(*)$ we fix $i_* = i(*) < 2^\chi$ and will prove $I_{i(*)}$ is $(\kappa, \aleph_0)$-super$^{6^+}$-unembeddable into $J_{i(*)}$.

By [Shed, 1.17(2)=La48(2), pg. 10] with χ^+ here standing for λ there, we can find a sequence $\langle N_\eta : \eta \in \mathscr{T} \rangle$, such that:

$\boxplus_1$(a) $\langle\rangle \in \mathscr{T} \subseteq {}^{\omega>}(\chi^+)$,
(b) $\nu \triangleleft \eta \in \mathscr{T} \Rightarrow \nu \in \mathscr{T}$,
(c) $(\forall \eta \in \mathscr{T})(\exists^{\chi^+} \alpha < \chi^+)[\eta {}^\frown \langle \alpha \rangle \in \mathscr{T}]$,
(d) $N_\eta \prec (\mathscr{H}(\chi^*), \in, <^*_\chi)$, $\|N_\eta\| = \kappa, \kappa \subseteq N_\eta$,
(e) $N_\eta \cap N_\nu = N_{\eta \cap \nu}$, where $\eta \cap \nu$ is the maximal ρ such that $\rho \trianglelefteq \eta \wedge \rho \triangleleft \nu$,
(f) $\eta \in N_\eta$,
(g) $(\forall \theta)[\kappa = \kappa^\theta \Rightarrow {}^\theta(N_\eta) \subseteq N_\eta]$,
(h) $\{i_*, \kappa, \chi, x, \langle S_i \colon i < 2^\chi \rangle, \langle S^\varepsilon \colon \varepsilon < \chi^+ \rangle, \langle (A_i, J_i) \colon i < 2^\chi \rangle\}$ belongs to $N_{<>}$,
(i) $N_\eta \cap \chi = N_{\langle\rangle} \cap \chi$.

Clearly $(2^\chi \setminus \{i_*\}) \cap N_{\langle\rangle}$ has cardinality $\leq \kappa$.

For each $\eta \in \lim(\mathscr{T}) := \{\eta \in {}^\omega(\chi^+) \colon$ if $n < \omega$ then $\eta \restriction n \in \mathscr{T}\}$, clearly $N_\eta := \bigcup\limits_{\ell < \omega} N_{\eta \restriction \ell}$ has cardinality κ so there is $\epsilon_\eta \in A_{i(*)} \setminus \bigcup\{A_j \colon j \in 2^\chi \cap N_{\langle\rangle}$ but $j \neq i_*\}$ $(\subseteq \chi)$. For each $\epsilon < \chi$, let Y_ϵ be the set of $\eta \in \lim(\mathscr{T})$ such that:

- η is strictly increasing with limit ε, and
- $\epsilon \in A_{i(*)} \setminus \bigcup\{A_j : j \in N_\eta$ and $j \neq i(*)\}$.

Clearly Y_ε is a closed subset of $\lim(\mathscr{T})$, (for the topology where $\mathscr{U} \subseteq \lim(\mathscr{T})$ is open iff for every $\eta \in \mathscr{T}$, there is $n < \omega$ such that $(\forall \nu)(\eta \restriction \nu \triangleleft \nu \in \lim(\mathscr{T}) \Rightarrow \nu \in \mathscr{U})$). Now those χ closed sets $\langle Y_\epsilon \colon \epsilon < \chi \rangle$ cover $\lim(\mathscr{T})$ as $\eta \in \lim(\mathscr{T}) \Rightarrow \varepsilon_\eta$ well defined by a previous paragraph, so possibly shrinking $\mathscr{T}$, by [Shed, 1.17(1)=La48(1), pg. 10], without loss of generality $\epsilon_\eta = \epsilon(*)$ for every $\eta \in \lim(\mathscr{T})$.

Now the set

$$C = \{\delta < \chi^+ : \nu \in {}^{\omega>}\delta \Rightarrow \sup(N_\nu \cap \chi^+) < \delta\}$$

is a club of χ^+ and we can find $\rho \in \lim(\mathscr{T})$, strictly increasing with limit $\delta \in C \cap S^{\epsilon(*)}$. Now let $M_n = N_{\rho \restriction n}$ and choose by induction on n an ordinal $\alpha_n \in (M_{n+1} \setminus M_n) \cap \chi^+$ and let $\eta = \langle \alpha_n : n < \omega \rangle$, and we can prove as in the proof of part (1) of 1.16 that it is as required.

2A) Until $\boxplus_1$, we repeat the proof of part (2), but replace $\boxplus_1$ by:

$\boxplus_2$ there is $\bar{N}^* = \langle N^*_\eta \colon \eta \in \mathscr{T}_* \rangle$ satisfying clauses (a)-(d), (f)-(h) of $\boxplus_1$ and:
(e$'$) if $\nu \triangleleft \eta \in \mathscr{T}_*$, <u>then</u> $N_\nu \in N_\eta$ (so $N_\nu \prec N_\eta$).

As we waive $\boxplus_1$(e), there is no problem doing it. As above, we can find

$\boxplus_3$ there is $\bar{N} = \langle N_\eta \colon \eta \in \mathscr{T} \rangle$ satisfying $\boxplus_1$ such that $\mathscr{T} \subseteq \mathscr{T}_*$ and $\bar{N}^* \in N_\eta$ for $\eta \in \mathscr{T}$.

Clearly,

$\boxplus_a$ if $\eta \in \mathscr{T}$, then $\eta \in N^*_\eta \prec N_\eta$.

Continuing the proof of part (2), getting ρ, δ, we let $M_\eta = N^*_{\rho \restriction n}$, choose $\alpha_n \in M_{n+1} \cap \chi^+$, $\sup(M_\eta \cap \chi^+)$, $\eta = \langle \alpha_n \colon n < \omega \rangle$, we get the desired conclusion for the $(\lambda, \lambda, \kappa, \kappa)$-supper^7-stability-property. The "super$^{7^+}$" variant is similar.

3)-4) Choose an increasing continuous sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ of cardinals, such that:

(a) $\lambda = \sum\limits_{i < \mathrm{cf}(\lambda)} \lambda_i$

(b) i non-limit $\Rightarrow \lambda_i = \mu_i^+ \;\&\; \mu_i^\kappa = \mu_i$ for some $\mu_i > \mu$,

(c) $\lambda_0 > \mu^\kappa + \mathrm{cf}(\lambda)$.

Choose further for any

$$\delta \in S =: \{ i : i < \mathrm{cf}(\lambda) \text{ and } \mathrm{cf}(i) = \aleph_0 \}$$

a sequence $\langle \lambda_{\delta,n} : n < \omega \rangle$ such that:

$$\lambda_{\delta,n} \in \{ \lambda_{j+1} : j < \delta \} \text{ and } \lambda_{\delta,n} < \lambda_{\delta,n+1} \text{ and } \lambda_\delta = \sum_{n<\omega} \lambda_{\delta,n}.$$

For $\delta \in S$, let $\mathfrak{s}^0_\delta$ be the family of sequences $\bar{N} = \langle N_\eta : \eta \in \mathscr{T} \rangle$ satisfying:

(A) $\mathscr{T}$ is a subset of $\bigcup\limits_{n<\omega} \prod\limits_{\ell<n} \lambda_{\delta,\ell}$, closed under initial segments, $\langle \rangle \in \mathscr{T}, [\eta \in \mathscr{T} \;\&\; \ell g(\eta) = n \Rightarrow (\exists^{\lambda_{\delta,n}} \alpha)(\eta \hat{\ } \langle \alpha \rangle \in \mathscr{T})]$.

(B) for some countable vocabulary $\tau = \tau_{\bar{N}}$ where $<_*$ belongs to τ, each N_η is a τ-model with universe a bounded subset of λ_δ of cardinality $\kappa, \kappa + 1 \subseteq N_\eta, \{\lambda_{\delta,n} : n < \omega\} \subseteq N_\eta, <_*^{N_\eta} = < \restriction N_\eta$, $N_{\eta \restriction k} \prec N_\eta$ and $N_\eta \cap N_\nu \prec N_\eta$ and[2] $(\forall \ell < \ell g(\ell))[\eta(\ell) \in N_\eta$ and $\eta \in \lim(\mathscr{T}) \Rightarrow \sup\{N_{\eta \restriction n} \colon n < \omega\} = \delta]$.

For $\mathscr{T}$ as in clause (A), recall

$$\lim(\mathscr{T}) = \{\eta : \; \eta \text{ an } \omega\text{-sequence such that every proper initial segment of } \eta \text{ is in } \mathscr{T}\}.$$

For a given $\bar{N} \in \mathfrak{s}^0_\delta$, and $\eta \in \lim(\mathscr{T})$ we use freely N_η as $\bigcup\limits_{\ell<\omega} N_{\eta \restriction \ell}$, (clearly still N_η is a τ-model of cardinality κ with universe $\subseteq \lambda_\delta, \kappa + 1 \subseteq N_\eta$ and $N_{\eta \restriction \ell} \prec N_\eta$).

Recall that $\eta \cap \nu$ is the largest common initial segment of η and ν.

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[2] if $\eta \in \mathscr{T} \Rightarrow N_\eta \prec \mathfrak{C}$ and $\mathfrak{C}$ has Skolem functions then $N_\eta \cap N_\nu \prec N_\eta$ follows, we can add $\kappa^\theta = \kappa \Rightarrow [N_\eta]^{\leq \theta} \subseteq N_\eta$

Let $\mathfrak{s}^1_{\delta,\mu}$ be the family of $\bar{N} = \langle N_\eta : \eta \in \mathscr{T}\rangle$ satisfying (in addition to being in $\mathfrak{s}^0_\delta$):

(C) (i) if $\eta, \nu \in \lim(\mathscr{T})$ then[3] $N_\eta \cap N_\nu = N_{\eta\cap\nu}$

(ii) if $\eta, \nu \in \mathscr{T}$ and $\mathrm{Rang}(\eta) \subseteq N_\nu$ then $\eta \trianglelefteq \nu$

(iii) $N_\eta \cap \mu = N_{<>} \cap \mu$.

Before finishing the proof of 1.16, we prove the following subfacts:

Subfact 1.18. (Recall λ be strong limit singular, $\mathrm{cf}(\lambda) > \kappa$.) Suppose M^* is a model with countable vocabulary and universe λ and $<^{M^*}_* = < \restriction\lambda$. Then for some club C of $\mathrm{cf}(\lambda)$, for every $\delta \in S \cap C$ we have:

$(*)^\delta_1$ for some $\langle N_\eta : \eta \in \mathscr{T}\rangle \in \mathfrak{s}^0_\delta$ we have:

$$\text{for every } \eta \in \mathscr{T}, \quad N_\eta \prec M^*.$$

Proof. Define a function f from ${}^{\omega>}\lambda$ to $\{A : A \subseteq \lambda, |A| = \kappa < \mathrm{cf}(\lambda)\}$ by: $f(\eta)$ is the (universe of the) Skolem Hull of $(\mathrm{Rang}(\eta)) \cup \{i : i \leq \kappa\} \cup \{\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle, \langle \lambda_{\delta,n} : \delta \in S, n < \omega\rangle\}$ in $(\mathscr{H}(\chi^*), \in, <_{\chi^*})$. Now apply [Shed, 1.23=L1.17].

Subfact 1.19. In 1.18 we can strengthen $(*)^\delta_1$ to:

$(*)^\delta_2$ for some $\langle N_\eta : \eta \in \mathscr{T}\rangle \in \mathfrak{s}^1_{\delta,\mu}$ we have:

$$\text{for every } \eta \in \mathscr{T}, N_\eta \prec M^*.$$

Proof. Let $\langle N_\eta : \eta \in \mathscr{T}\rangle$ be a member of $\mathfrak{s}^0_\delta$ satisfying $\eta \in \mathscr{T}$ implies $N_\eta \prec M^*$.

We now will apply [Shed, 1.19=La54] with $|N_\eta|$ here standing for A_n there.

So there is $\mathscr{T}' \subseteq \mathscr{T}$ such that $\langle N_\eta : \eta \in \mathscr{T}'\rangle$ is a Δ-system; i.e.

(i) $\mathscr{T}' \subseteq \mathscr{T}$ satisfies (A)

(ii) there is a function $\mathbf{h}$ with domain $\mathscr{T}' \times \omega \times \omega$ such that for all incomparable $\eta, \nu \in \mathscr{T}'$ we have:

$$N_\eta \cap N_\nu = \mathbf{h}(\eta \cap \nu, \ell g(\eta), \ell g(\nu)).$$

Let

$$\mathbf{h}^+(\eta) := \bigcup_{n,m > \ell g(\eta)} \mathbf{h}(\eta, n, m)$$

so $\mathbf{h}^+(\eta)$ is a subset of λ_δ of cardinality κ; as $[\eta \triangleleft \nu \Rightarrow N_\eta \prec N_\nu]$ clearly:

$(*)$ if $\eta \neq \nu \in \lim(\mathscr{T}')$ then $\mathbf{h}^+(\eta \cap \nu) = N_\eta \cap N_\nu$.

As M^* has definable Skolem functions, if $\eta, \nu \in \lim(\mathscr{T}')$ then

$$M_{\eta\cap\nu} := N_\nu \restriction \mathbf{h}^+(\eta \cap \nu) = N_\eta \restriction \mathbf{h}^+(\eta \cap \nu)$$

is an elementary submodel of N_η, N_ν (remember: $<^{N_\eta} = < \restriction N_\eta$ is a well ordering). So it is easy to check $\langle M_\eta : \eta \in \mathscr{T}'\rangle$ is almost as required. The missing point is $M_\eta \cap \mu = M_{<>} \cap \mu$ for every $\eta \in \mathscr{T}'$. As $\langle N_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in \mathscr{T}'\rangle$ are pairwise disjoint and $\lambda_{\delta,\ell g(\eta)} > \mu$ for some $\alpha_\eta < \lambda_{\delta,\ell g(\eta)}$ we have $\eta\hat{}\langle\alpha_\eta\rangle \in \mathscr{T}' \Rightarrow N_{\eta\hat{}\langle\alpha_\eta\rangle} \cap \mu = N_\eta \cap M$. So by throwing away enough members of $\mathscr{T}'$ (i.e. we choose $\{\nu \in \mathscr{T}' : \ell g(\nu) = n\}$ by induction on n) we can manage. $\square_{1.19}$

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[3] this simplifies the clause before last in (B) above

Subfact 1.20. We can find $\langle \eta^{\delta,\alpha}, \langle M_n^{\delta,\alpha} : n < \omega \rangle : \delta \in S, \alpha < 2^{\lambda_\delta} \rangle$ such that:

- (i) $M_n^{\delta,\alpha}$ is a model of power κ, countable vocabulary $\subseteq \mathscr{H}(\aleph_0)$ including the predicate $<_*$, universe including $\kappa + 1$ and being included in λ_δ
- (ii) $M_n^{\delta,\alpha} \prec M_{n+1}^{\delta,\alpha}$ and $M_n^{\delta,\alpha}$ is a proper initial segment of $M_{n+1}^{\delta,\alpha}$
- (iii) $M_n^{\delta,\alpha} \cap \mu = M_0^{\delta,\alpha} \cap \mu$
- (iv) $\eta^{\delta,\alpha} \in \prod\limits_n \lambda_{\delta,n}$ and $\eta^{\delta,\alpha} \restriction (n+1)$ belongs to $M_{n+1}^{\delta,\alpha}$ but not to $M_n^{\delta,\alpha}$
- (v) for some increasing sequence $\bar{k}_{\delta,\alpha} = \langle k_{\delta,\alpha}(\ell) \colon \ell < \omega \rangle$ from ${}^\omega\omega$ we have $\bigcup\limits_{\ell < n} \lambda_{\delta,k_{\delta,\alpha}(\ell)} < \eta^{\delta,\alpha}(n) < \lambda_{\delta,k_{\delta,\alpha}(n)}$ [hence $\lambda_\delta = \bigcup\limits_n \eta^{\delta,\alpha}(n)$],
- (vi) if $\alpha < \beta < 2^{\lambda_\delta}$ and $\delta \in S$ then for some $m < \omega$ we have

$$(\bigcup_{n<\omega} M_n^{\delta,\beta}) \cap (\bigcup_{n<\omega} M_n^{\delta,\alpha}) \subseteq M_m^{\delta,\beta}$$

hence

- (vii) for $\alpha < 2^{\lambda_\delta}, \delta \in S$ we have $\sup(M_n^{\delta,\alpha} \cap \lambda_{\delta,n}) = \sup(M_{n+1}^{\delta,\alpha} \cap \lambda_{\delta,n})$
- $(viii)$ if M^* is a model with countable vocabulary $\subseteq \mathscr{H}(\aleph_0)$ and universe λ with $<_*^{M^*} = < \restriction \lambda$ then for some [4] $\delta \in S$ and $\alpha < 2^{\lambda_\delta}$ we have $\bigwedge_n M_n^{\delta,\alpha} \prec M^*$
- (ix) if $\delta \in S, \alpha \neq \beta$ are $< 2^{\lambda_\delta}$ then $\{\eta^{\delta,\alpha} \restriction n : n < \omega\} \nsubseteq \cup \{M_n^{\delta,\beta} : n < \omega\}$.

Proof. Straightforward from 1.19 (and diagonalizing). $\square_{1.20}$

Proof of 1.16(3): Should be clear now using [Shed, 1.16=La45] and similar to [CS19, Th. 2.6, pg. 113] (and see the proof of 2.1(1) below).

Proof of 1.16(4). Similar (and not used for 2.34). $\square_{1.16}$

$\square_{1.18}$

$\square_{1.16}$

For our main conclusion 2.34 we shall not actually use 1.21 (as other cases cover it).

Claim 1.21. *Suppose λ is singular, $\mu < \lambda$ and for arbitrarily large $\theta < \lambda$ at least one of the conditions $(*)^1_\theta$, $(*)^2_\theta$ below holds. Then K^ω_{tr} has the full $(\lambda, \lambda, \mu, \mu)$-super[7]-bigness property*

- $(*)^1_\theta$ *θ singular, $\text{pp}(\theta) > \theta^+$ (see Definition* [Shed, 3.16=Lprf.2]*)*
- $(*)^2_\theta$ *there is a set $\mathfrak{a}$ of regular cardinals $< \theta$ unbounded below $\theta, |\mathfrak{a}| < \theta$, such that $\partial < \theta \Rightarrow \max \text{pcf}(\mathfrak{a} \setminus \partial) > \theta^+$.*

Proof. First, by [Shed, 3.22=Lpcf.8] we have $(*)^1_\theta \Rightarrow (*)^2_\theta$, second, by [Shed, 3.20=Lpcf.6a] without loss of generality $\text{cf}(\theta) = \aleph_0$; third, by [Shed, 3.22=Lpcf.8] without loss of generality $\mathfrak{a}$ has order type ω and J is the ideal of bounded subsets of $\mathfrak{a}$, lastly by [Shed, 3.10=Lpcf.1] (and easy manipulation) $(*)^2_\theta \Rightarrow (*)^3_{\theta^+}$, where

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[4] really for a club of $\delta \in S$ for "many" $\alpha < 2^{\lambda_\delta}$ this holds

$(*)^3_\partial$ ∂ regular, there is a stationary $S \subseteq \{\delta < \partial : \mathrm{cf}(\delta) = \aleph_0\}$ and η_δ, an increasing ω-sequence converging to δ, for $\delta \in S$, such that for every $\alpha < \partial$, for some $h : S \cap \alpha \to \omega$ we have: $\{\eta_\delta{\restriction}\ell : h(\delta) < \ell < \omega\} : \delta \in S \cap \alpha\}$ are pairwise disjoint.

So assume $\langle \theta_i : i < \mathrm{cf}(\lambda)\rangle$ is strictly increasing,

$$\mu < \theta_i < \sum_{j<\mathrm{cf}\lambda} \theta_j = \lambda,$$

each θ_i regular and for each i, $\langle \eta^i_\delta : \delta \in S_i\rangle$ is as required in $(*)^3_{\theta_i^+}$. Let $\langle S_{i,\alpha} : \alpha < \theta_i\rangle$ be a partition of S_i to (pairwise disjoint) stationary sets. For $\bigcup\limits_{j<i} \theta_j \leq \alpha < \theta_i$, let $I_\alpha = {}^{\omega>}\lambda \cup \{\eta^i_\delta : \delta \in S_{i,\alpha}\}$. The rest is as in 1.16(1) above (or the proof of 3.26, Case 1 below). $\square_{1.21}$

Remark 1.22. In 1.21 we can use $(*)^3_\partial, \partial$ regular arbitrarily large $< \lambda$.

§ 2. FURTHER CASES OF SUPER UNEMBEDDABILITY

Lemma 2.1. *1) Suppose $\lambda \geq \mu^+ + \chi^{+2}$ <u>then</u> $K^\omega_{\rm tr}$ has the full $(\chi^{\aleph_0}, \lambda, \mu, \mu)$-super-bigness property.*
2) In addition $K^\omega_{\rm tr}$ has the full $(\chi^{\aleph_0}, \lambda, \mu, \mu)$-super^5-bigness property (with $\dot D = D^{\rm cbe}_\omega = \{A \subseteq \omega$: every large enough even number belongs to $A\}$).
3) In part (2), we can add in Definition 1.5, Case E the requirement:

$$\circledast \ \text{ if } \nu \in P^I_\omega \cup P^J_\omega \text{ and } \{\nu{\restriction}k : k < \omega\} \subseteq M_n \text{ then } \nu \in M_n.$$

This claim will be proved later (after 2.17). Towards this we develop "guessing of clubs" in ZFC, in fact for this it was introduced.

Claim 2.2. *Suppose κ, λ are regular cardinals, $\kappa^+ < \lambda$, and*

$$S \subseteq \{\delta < \lambda : {\rm cf}(\delta) = \kappa\}$$

is a stationary subset of λ.
<u>Then</u> we can find $\langle C_\delta : \delta \in S\rangle$ such that:

- *(a) C_δ is a club of δ of order type κ (if $\kappa = \aleph_0, C_\delta$ is just an unbounded subsetd of δ and ${\rm otp}(C_\delta) = \omega$)*
- *(b) for every club C of λ, the set $\{\delta \in S : C_\delta \subseteq C\}$ is stationary.*

Proof. Toward contradiction, suppose that such $\langle C_\delta : \delta \in S\rangle$ does not exist. Let $\langle C^*_\delta : \delta \in S\rangle$ satisfy (a). We choose E_ζ by induction on $\zeta < \kappa^+$ such that:

- (i) E_ζ is a club of λ, and $0 \notin E_\zeta$,
- (ii) $\xi < \zeta \Rightarrow E_\zeta \subseteq E_\xi$,
- (iii) for no $\delta \in S$ does $C^\zeta_\delta \subseteq E_{\zeta+1}$ hence $\delta = \sup(E_{\zeta+1} \cap \delta)$ where
$$C^\zeta_\delta := \{\sup(\alpha \cap E_\zeta) : \alpha \in C^*_\delta, \alpha > \min(E_\zeta)\}.$$

For $\zeta = 0, \zeta$ limit: we have no problem. For $\zeta = \xi + 1$, first define C^ξ_δ for $\delta \in S$ as in clause (iii). Then, let E'_ζ be the set of accumulation points of E_ξ, so clearly $\delta \in E'_\zeta \Rightarrow \delta = \sup(C^\xi_\delta) \wedge \kappa = {\rm otp}(C^\xi_\delta)$. By our assumption toward contradictions, $\langle C^\xi_\delta \colon \delta \in E'_\zeta \cap S\rangle$ does not satisfy both properties (a) and (b), but obviously it satisfies clause (a), so it fails to satisfy clause (b) (it does not matter what we do for $\delta \in S \setminus E'_\zeta$). So for some club E''_ζ of λ, the set $A_\zeta = \{\delta \in E'_\zeta \cap S : C^\xi_\delta \subseteq E''_\zeta\}$ is not stationary, so it is disjoint to some club E_ζ and without loss of generality E_ζ is a subset of $E''_\zeta \cap E'_\zeta$. So we have carried the induction.

In the end as $\kappa^+ < \lambda = {\rm cf}(\lambda)$, clearly $E^+ = \bigcap\limits_{\zeta < \kappa^+} E_\zeta$ is a club of λ, choose $\delta(*) \in S$ which is an accumulation point of E^+; so $\delta(*) \in E_\zeta \cap S$ for every $\zeta < \kappa^+$. Now for each $\alpha \in C^*_{\delta(*)}$, which is $> \min(E^+)$, the sequence

$$\langle \sup(\alpha \cap E_\zeta) : \zeta < \kappa^+\rangle$$

is a non-increasing sequence of ordinals $\leq \alpha$, hence is eventually constant. As κ^+ is regular $> \kappa = |C^*_{\delta(*)}|$, for some $\zeta(*) < \kappa^+$, for every $\zeta \in [\zeta(*), \kappa^+)$ and $\alpha \in C^*_{\delta(*)}$, which is above $\min(E^+)$, we have

$$\sup(\alpha \cap E_\zeta) = \sup(\alpha \cap E_{\zeta(*)}).$$

Hence $C^{\zeta(*)}_{\delta(*)} = C^{\zeta(*)+1}_{\delta(*)}$, and we get a contradiction to the choice of $E_{\zeta(*)+1}$. $\square_{2.2}$

Remark 2.3. If $\kappa > \aleph_0$, the proof is simpler, just $C^\zeta_\delta = C^*_\delta \cap E_\zeta$ is O.K.

Claim 2.4. *Suppose that in 2.2 we have also $\kappa < \theta = \mathrm{cf}(\theta) < \lambda$; then we can add*

(*c*) *for some club C of λ, if $\delta \in S \cap C, \alpha \in C_\delta$ and $\alpha > \sup(\alpha \cap C_\delta)$ then $\mathrm{cf}(\alpha) \geq \theta$.*

Proof. Let $\bar{C} = \langle C_\delta \colon \delta \in S\rangle$ be as in 2.2. Let $S^+ =: \{\delta < \lambda : \mathrm{cf}(\delta) < \theta\}$ (so $S \subseteq S^+$). For each $\delta \in S^+$ choose a club C^*_δ of δ of order type $\mathrm{cf}(\delta)$. Assume that the conclusion fails, that is, we cannot find $\bar{C}'$ satisfying clauses (a), (b), (c). Above me may weaken the demand "$\mathrm{otp}(C'_\delta) = \kappa$" as we can shrink the C'_δ-s appropriately.

We define by induction on $\zeta < \theta, E_\zeta$ such that:

(*i*) E_ζ is a club of $\lambda, 0 \notin E_\zeta$

(*ii*) $\xi < \zeta$ implies $E_\zeta \subseteq E_\xi$

(*iii*) for no $\delta \in S$ do we have $C^\zeta_\delta \cap E_\zeta \subseteq E_{\zeta+1}, \delta = \sup(C^\zeta_\delta \cap E_\zeta)$ where $C^\zeta_\delta = C^{\zeta,\omega}_\delta$ and by recursion on $n \leq \omega$, we define $C^{\zeta,n}_\delta$ as follows:
  (a) $C^{\zeta,0}_\delta := \{\sup(\alpha \cap E_\zeta) : \alpha \in C_\delta, \alpha > \min(E_\zeta)\}$,
  (b) $C^{\zeta,n+1}_\delta := C^{\zeta,n}_\delta \cup \{\sup(\alpha \cap E_\zeta)\colon$ for some $\beta \in C^{\zeta,n}_\delta$ we have $\mathrm{cf}(\beta) < \theta$ and $\alpha \in C^*_\beta, \alpha > \min(E_\zeta)$ and $\alpha > \sup[C^{\zeta,n}_\delta \cap \beta]\}$, and
  (c) $C^\zeta_\delta := \bigcup\limits_{n<\omega} C^{\zeta,n}_\delta$.

For $\zeta = 0, \zeta$ limit: we have no problems. For $\zeta = \xi + 1$, we first define $C^{\xi,0}_\delta$ and then $C^{\xi,n}_\delta$ (by induction on n) and lastly C^ξ_δ. We can show by induction on n that $C^{\xi,n}_\delta \subseteq E_\zeta$ and

$(*)_0$ $C^{\zeta,n}_\delta$ is a closed subset of δ of cardinality $< \theta$.

Now,

$(*)_1$ (a) C^ξ_δ is of cardinality $< \theta$ (as $\aleph_0 < \theta = \mathrm{cf}(\theta)$),
  (b) if $\gamma = \sup(C^\xi_\delta \cap \gamma) < \delta$, then
    - γ limit of cofinality $\leq |C^\xi_\delta| < \theta$,
    - $\langle \min(C^{\xi,n}_\delta \setminus \gamma)\colon n < \omega\rangle$ is $\leq$-decreasing, hence,
    - for some $n(*) < \omega$, we have:
      $$n \geq n(*) \Rightarrow \min(C^{\xi,n}_\delta \setminus \gamma) = \min(C^{\xi,n(*)}_\delta),$$
    - so by the choice of $C^{\xi,n(*)+1}_\delta$ necessarily $\gamma \in C^{\xi,n(*)}_\delta$.

Together,

$(*)_2$ C^ξ_δ is a closed subset of δ.

Hence,

$(*)_3$ if δ is an accumulation point of E^ξ_δ, then C^ξ_δ is a club of δ.

If "for every club E of λ for some $\delta \in S \cap \mathrm{acc}(E_\xi), C^\xi_\delta \subseteq E$" then we can shrink the club E; i.e. deduce C^ξ_δ is included in the set of accumulation points of $E \cap E_\xi$ hence $\langle C^\xi_\delta : \delta \in S \cap \mathrm{acc}(E_\xi)\rangle$ satisfies "for every club E of λ for some $\delta \in S \cap E_\xi$, we have $C^\xi_\delta \subseteq E$ and $(\forall\alpha)[\alpha \in C^\delta_\xi \mathbin{\&} \alpha > \sup(C^\xi_\delta \cap \alpha) \Rightarrow \mathrm{cf}(\alpha) \geq \theta]$" so the desired conclusion holds.

Hence we can assume that for some club E^1_ζ of λ, for no $\delta \in S \cap E^1_\zeta \cap \mathrm{acc}(E_\xi)$ does $C^\xi_\delta \subseteq E^1_\zeta$; let E_ζ be the set of accumulation points of $E^1_\zeta \cap E_\xi$. So we have finished defining $C^{\xi,n}_\delta$ for $n < \omega$ and C^ξ_δ.

So we have finished defining $\langle E_\zeta : \zeta < \theta\rangle$ and we choose $\delta(*) \in S$ a accumulation point of $\bigcap\limits_{\zeta<\theta} E_\zeta$. Again as in the proof of 2.2, for some $\zeta(0) < \theta$, we have

$(\oplus)$ $\zeta(0) \le \zeta < \theta \Rightarrow C^{\zeta,0}_{\delta(*)} = C^{\zeta(*),0}_{\delta(*)}$.

Similarly we can prove by induction on n that for some $\zeta(n) < \theta$:

$\oplus_2$ $\zeta(n) \le \zeta < \theta \Rightarrow C^{\zeta,n}_{\delta(*)} = C^{\zeta(*),n}_{\delta(*)}$.

Let $\zeta(*) = \bigcup\limits_{n<\omega} \zeta(n)$, and we get contradiction as in the proof of 2.2. $\square_{2.4}$

Recall (see [She22, 3.8] or [Sheb, 0.4=L0.2A]).

Definition 2.5. 1) We say $\bar C$ is a pre-partial square sequence of λ: (omitting λ means $(\exists\lambda)[\bar C$ is a partial square of $\lambda])$ when: $\bar C$ has the form $\langle C_\alpha : \alpha \in S\rangle$ and satisfies:

- (a) $S \subseteq \lambda$
- (b) C_α is a closed subset of α
- (c) $C_\alpha \subseteq S$
- (d) if $\beta \in C_\alpha, \alpha \in S$ then $C_\beta = C_\alpha \cap \beta$.

2) We say $\bar C$ is partial square when in addition:

- (e) if $\alpha \in S$ is a limit ordinal then $\alpha = \sup(C_\alpha)$.
- (f) $\mathrm{otp}(C_\alpha) < \alpha$ if $0 < \alpha \in S$.

We next state a consequence of the failure of partial square (the reader may suspect the assumption (1) is vacuous, e.g., by 2.9, but e.g. there we speak on successor cardinals).

Conclusion 2.6. *If $\lambda > \kappa$ are regular, then (A) $\Rightarrow$ (B), where:*

(A) *There are no pairs $(S^+, \bar C)$ such that:*
  - $S^+ \subseteq \lambda$,
  - $\bar C = \langle C_\delta : \delta \in S^+\rangle$ *is a partial square, and*
  - $\{\delta \in S^+ : \mathrm{cf}(\delta) = \kappa\}$ *is stationary.*

(B) *For every regular $\lambda_1 \ge \lambda$ there is a stationary $S \subseteq \{\delta < \lambda_1^+ : \mathrm{cf}(\delta) = \kappa\}$ which does not reflect in any $\delta < \lambda_1^+$ of cofinality λ, (really one $S \subseteq \lambda_1^+$ works for all such $\lambda, \kappa < \lambda < \lambda_1$).*

Proof. The case $\kappa = \aleph_0$ is trivial because clause (A) never holds. In detail, we can choose $\langle C_\alpha : \alpha < \lambda$ is a successor ordinal$\rangle$ such that:

$(*)$ (a) C_α is a finite subset of α,
  (b) $\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta$,
  (c) if C is a finite subset of α, then for some $\beta < \alpha + |\alpha| + |\alpha|$ we have $C_{\beta+1} = C$.

[Why? By induction on α belonging to the club $E := \{\beta < \lambda : \beta$ is infinite divisible by $|\beta|\}$, we can choose $\langle C_{\beta+1} : \beta < \alpha\rangle$ satisfying the above (actually is "$\beta < \alpha + |\alpha|$" suffice).]

Let $S \coloneqq \{\alpha \in \mathrm{Ord} \colon \mathrm{cf}(\alpha) = \aleph_0 \wedge \alpha = \sup(E \cap \alpha)\}$ and

$$S^+ \coloneqq \{\alpha \in \mathrm{Ord} \colon \alpha \text{ is a successor or } \alpha \in S\}.$$

For $\alpha \in S$, let $\langle \beta_{\alpha,n} \colon n < \omega \rangle$ be increasing with limit α, and choose $\gamma_{\alpha,n}$ by induction on $n < \omega$ increasing with n such that $\gamma_{\alpha,n} \in (\beta_{\alpha,n}, \alpha)$ is a successor ordinal and $C_{\gamma_{\alpha,n}} = \{\gamma_{\alpha,m} \colon m < n\}$ and lastly, let $C_\alpha = \{\gamma_{\alpha,n} \colon n < \omega\}$.

So assume $\kappa > \aleph_0$. By clauses (c) and (d) of [She22, 3.8(2)=L6.4,3.9=L6.4B] in the arXiv version, [She22, 4.8(2)] in the published one, we can find $S^+ \subseteq \lambda_1^+$ and a partial square $\bar{C} = \langle C_\delta : \delta \in S^+ \rangle$ such that $\delta \in S^+ \Rightarrow \mathrm{otp}(C_\delta) \leq \kappa$, and

$$S = \{\delta \in S^+ : \mathrm{otp}(C_\delta) = \kappa\} = \{\delta \in S^+ : \mathrm{cf}(\delta) = \kappa\} \text{ is stationary.}$$

Towards a contradiction, assume that S reflect in some $\delta, \mathrm{cf}(\delta) = \lambda$, let $\langle \alpha_\zeta \colon \zeta < \lambda \rangle$ be a strictly increasing continuous sequence with limit δ.

Let $S^+_\lambda = \{\zeta < \lambda : \alpha_\zeta \in S^+\}$ and for $\zeta \in S^+_\lambda$, $C^\lambda_\zeta = \{\epsilon < \zeta : \alpha_\epsilon \in C_{\alpha_\zeta}\}$; now $\langle C^\lambda_\zeta : \zeta \in S^+_\lambda \rangle$ essentially show that for λ satisfying the assumption, the conclusion holds; note possibly for some $\zeta \in S^+_\lambda$ we have $\sup(C^\lambda_\zeta) < \zeta$ but then $\mathrm{cf}(\zeta) = \aleph_0$ and for no $\varepsilon \in S^+_\lambda$ do we have $\zeta \in C^\lambda_\varepsilon$. To correct this, let

$$S'_\lambda = \{\zeta \in S_\lambda \colon \zeta = \sup(C^\lambda_\zeta)\} \cup \{\zeta + 1 : \zeta \in S_\lambda \text{ and } \zeta > \sup(C^\lambda_\zeta)\}$$

and redefine C^λ_ζ accordingly. $\square_{2.6}$

Claim 2.7. *Suppose $\lambda = \theta^+, \theta$ regular uncountable, $S^0 = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$. Then we can find $\langle C_\delta : \delta \in S^0 \rangle$ such that:*

- *(a) C_δ is a club of δ of order type θ,*
- *(b) for any club E of λ, the set*

$$\{\delta \in S^0 : \delta = \sup\{\alpha : \alpha \in C_\delta, \alpha > \sup(\alpha \cap C_\delta) \text{ and } \alpha \in E\}\}$$

is stationary.

Equivalently,

- *(c) for any club E of λ*

$$\{\delta \in S^0 : \{\zeta < \theta : \text{ the } (\zeta{+}1)\text{-th member of } C_\delta \text{ is in } E\} \text{ is an unbounded subset of } \theta\}$$

is a stationary subset of λ.

Remark 2.8. In clause (c) above, obviously for a club of $\zeta < \theta$, the ζ-th member of C_δ is in E.

Proof. Like the proof of 2.2, again we continue ω times, and assume the failure of the statement here, noting: if C_δ is a club of δ of order type κ, E is a club of δ such that $\mathrm{otp}(E)$ is divisible by κ^2, then for unboundedly many $\alpha \in C_\delta$ we have $\alpha > \sup(C_\delta \cap \alpha)$. $\square_{2.7}$

Now we give a small improvement of 2.2.

Claim 2.9. *Suppose $\lambda > \kappa + \aleph_1$ where λ and κ are regular cardinals, and $\epsilon(*)$ is a limit ordinal $< \lambda$ of cofinality κ. Then if S_* satisfies (A) then we can find $\bar{C} = \langle C_\delta \colon \delta \in S_* \rangle$ satisfying[5] (B), where:*

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[5] As in 2.10(3) we can add to clause (B)(d), there is a partial square $\langle C'_\delta \colon \delta \in S' \rangle$ where $S \subseteq S' \subseteq \lambda$ and $\bar{C}' \restriction S = \bar{C}$.

(A) (a) $S_* \subseteq \{\delta < \lambda^+ \colon \mathrm{cf}(\delta) = \kappa$ *and* δ *is divisible*[6] *by* $\lambda \times \kappa\}$,
(b) S_* *is a stationary subset of* λ^+.
(B) (a) C_δ *is a closed unbounded subset of* δ,
(b) $\mathrm{otp}(C_\delta) = \epsilon(*)$,
(c) *for every club* E *of* λ^+, $\{\delta \in S_* \colon C_\delta \subseteq E\}$ *is stationary.*

Before we prove 2.9, we phrase a further claim:

Claim 2.10. *In 2.9:*
1) E.g.

$$S_* := \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa \text{ and } \delta \text{ is divisible by } \lambda \times \kappa\}$$

is as required.
2) We can add:

(*d*) $\alpha \in C_\delta \,\&\, \alpha > \sup(C_\delta \cap \alpha) \Rightarrow \mathrm{cf}(\alpha) = \lambda$.

3) For any sequence $\bar\epsilon = \langle \epsilon_\zeta(*) : \zeta < \lambda\rangle$, *where* $\epsilon_\zeta(*) < \lambda$ *is a limit ordinal let* $\kappa_\zeta = \mathrm{cf}(\varepsilon_\zeta(*))$, *we can find* $\langle (S^*_\zeta, S^+_\zeta, \bar C^\zeta) : \zeta < \lambda\rangle$ *such that:*

(*i*) $\{\alpha : \alpha < \lambda^+, \mathrm{cf}(\alpha) < \lambda\} = \bigcup\limits_{\zeta<\lambda} S^+_\zeta$,
(*ii*) $S^*_\zeta \subseteq S^+_\zeta \subseteq \{\alpha : \alpha < \lambda^+, \mathrm{cf}(\alpha) < \lambda\}$,
(*iii*) $S^*_\zeta \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa_\zeta\}$,
(*iv*) $\delta \in S^*_\zeta \Rightarrow \mathrm{otp}(C^\zeta_\delta) = \epsilon_\zeta(*)$,
(*v*) $\bar C^\zeta = \langle C^\zeta_\alpha : \alpha \in S^+_\zeta\rangle$ *satisfies (a), (c) of 2.9(B) (and (b) - for* $\epsilon_\zeta(*)$*),*
(*vi*) $\bar C^\zeta$ *is a partial square,*
(*vii*) $\langle S^*_\zeta \colon \zeta < \lambda\rangle$ *is a sequence of pairwise disjoint sets.*

Proof. Proof of 2.9:
We know here essentially by [She22, 3.8(2)=L6.4(2)] (in the arXiv version, [She22, 4.8(2)]) in the published one and by [She91, §4] that there are $\langle S_\zeta : \zeta < \lambda\rangle$ and $\bar C^\zeta = \langle C^\zeta_\alpha \colon \alpha \in S_\zeta\rangle$ for $\zeta < \lambda$ such that:

$(*)_1$ (α) $\bar C^\zeta$ is a pre-partial square sequence of λ^+,
(β) if $\alpha > \sup(C^\zeta_\alpha)$ then $\mathrm{cf}(\alpha) \in \{1, \lambda\}$,
(γ) $\lambda^+ = \bigcup\limits_{\zeta<\lambda} S_\zeta$,
(δ) $|C^\zeta_\alpha| < \lambda$,
(ε) if $\alpha < \lambda^+$, then for some $\xi < \lambda$ we have:
- if $\zeta < \lambda$ then $\alpha \in S_\zeta \Leftrightarrow \zeta \geq \xi$,
- $\langle C^\zeta_\alpha : \zeta \in [\xi, \lambda]\rangle$ is $\subseteq$-increasing,
- if $\mathrm{cf}(\zeta) = \aleph_1 \wedge \zeta > \xi$, then C^ζ_α is the closure of $\cup\{C^\varepsilon_\alpha : \varepsilon \in [\xi, \zeta)\}$ in α,
- $\cup\{C^\zeta_\alpha : \zeta \in [\xi, \lambda)\}$ is equal to α.

Obviously,

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[6] "δ *divisible* by $\lambda \times \kappa$" means $(\forall \alpha < \delta)(\alpha + \lambda \times \kappa \leq \delta)$

$(*)_2$ for every $\epsilon, \xi < \lambda$ and club E^0 of λ^+ for some club E^1 of λ^+, for each $\delta \in E^1$ of cofinality $< \lambda$, for some $\zeta < \lambda$ above ξ we have: $\delta = \sup(E^0 \cap C^\zeta_\delta)$ and $\mathrm{otp}(E^0 \cap C^\zeta_\delta)$ is divisible by ϵ.

Next, (relying on $(*)_1 + (*)_2$ above)

Fact 2.11. We have:

$(*)_3$ (a) for every stationary $S \subseteq S^{\lambda^+}_{<\lambda}$ and $\xi, \varepsilon < \lambda$ and for some $\zeta(*) < \lambda, \zeta(*) > \xi$ and $S \cap S_{\zeta(*)}$ is stationary, moreover:

(b) above, if E is a club of λ^+, then $\mathrm{set}^1_{\zeta(*),\varepsilon}(E, S)$ is stationary where $\mathrm{set}^1_{\zeta(*),\varepsilon}(E, S)$ is the set of δ such that:

- $\delta \in S \cap S_{\zeta(*)} \cap E$,
- $\delta = \sup(C^{\zeta(*)}_\delta \cap E)$,
- $\mathrm{otp}(C^{\zeta(*)}_\delta \cap E)$ is divisible by $\epsilon \cdot \omega$ (ordinal multiplication),
- if $\alpha \in C^{\zeta(*)}_\delta \wedge \alpha > \sup(C^{\zeta(*)}_\alpha)$ then[7] $\mathrm{cf}(\alpha) \in \{1, \lambda\}$,
- if $\alpha \in C^{\zeta(*)}_\delta$ and $\alpha \in E \wedge (\alpha > \sup(\alpha \cap E) > \sup(C^{\zeta(*)}_\delta \cap E)$ then[8] $\mathrm{cf}(\alpha) = \lambda$.

Proof. If not, then for every $\zeta \in [\xi, \lambda)$ there is a club E^1_ζ of λ^+ such that $\mathrm{set}^1_{\zeta,\varepsilon}(E^1_\zeta, S)$ is not stationary, hence there is a club E^2_ζ of λ^+ disjoint to it. Let $E := \cap\{E^1_\zeta \cap E^2_\zeta : \zeta \in [\xi, \lambda)\}$, it is a club of λ^+. As E is a club of the regular uncountable cardinal λ, necessarily there is $\delta_* \in S$ such that $\delta_* = \mathrm{otp}(E \cap \delta_*)$. Now for some club E' of λ^+, for every $\zeta \in E'$ of cofinality $\aleph_1$ we get a contradiction to $(*)_2$, therefore $(*)_3$ holds. $\square_{2.11}$

Fact 2.12. For regular $\kappa < \lambda$ and stationary $S \subseteq \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) = \kappa\}$ by induction on $\zeta < \lambda$ we can choose $\xi(\zeta, S) \in S$ such that:

$(*)_{4,S,\zeta}$ $\xi(\zeta, S)$ is an ordinal $> \xi(\zeta_1, S)$ for every $\zeta_1 < \zeta$ but $< \lambda$ hence $\xi(\zeta, S) \in [\zeta, \lambda)$ such that:

- if E is a club of λ^+, then $\mathrm{set}^1_{\xi(\zeta)}(E)$ from $(*)_3$ is a stationary subset of λ^+.

Claim 2.13. *We have:*

$(*)_5$ *for every regular $\kappa < \lambda$, stationary $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$ and $\zeta < \lambda$ there are a club E_* of λ^+ and ordinals $\delta_*, \delta_1(*), \delta_2(*)$ of cofinality κ divisible by ζ such that: if E is a club of λ^+ then the following is a stationary subset of λ^+:*

$$\begin{aligned}\mathrm{set}^2_{\zeta,\delta_*}(E, E_*, S) := \{\delta : \quad & \delta \in S \cap S_{\xi(\zeta,\delta)}, \delta = \sup(C^{\xi(\zeta,\delta)}_\delta \cap E_* \cap E), \\ & \mathrm{otp}(C^{\xi(\zeta,\delta)}_\delta) = \delta_1(*), \mathrm{otp}(C^{\xi(\zeta,\delta)}_\delta \cap E_*) = \delta_2(*) \textit{ and} \\ & C^{\xi(\zeta,\delta)}_\delta \cap (E_* \cap E) = C^{\xi(\zeta,\delta)}_\delta \cap E_*\}.\end{aligned}$$

Proof. Let $\langle \delta_i : i < \lambda \rangle$ list the ordinals $< \lambda$ of cofinality κ divisible by ζ, each appearing stationarily often. We choose by induction on $i < \lambda$, a club E_i of λ, decreasing with i such that E_{i+1} exemplifying (E_i, δ_i) are not as required on (E_*, δ_*), moreover E_{i+1} is disjoint to $\mathrm{set}^2_{\zeta,\delta_*}(E_{i+1}, E_i, S)$, therefore $(*)_5$ holds. $\square_{2.13}$

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[7] This is not the same as $(*)_2(\beta)$

[8] used only for (d) from 2.10(2)

Now we can prove Fact 2.9: apply $(*)_{4,S,\zeta}$ for κ, S being the κ, S_* from 2.9 and ζ being $\varepsilon(*) \times \omega$ and get $\xi := \xi(\zeta, S)$ as there, and let (E_*, δ_*) be as in $(*)_5$ of 2.13.

Clearly δ_* has a closed unbounded subset C_* of order type $\varepsilon(*)$, as $\mathrm{cf}(\delta_*) = \kappa = \mathrm{cf}(\varepsilon(*))$ and $\varepsilon(*) \cdot \omega$ divides δ_*).

Continuing the proof of 2.10:

Now for each $\delta \in S_*$ we choose C_δ as follows:

- if $\delta = \sup(C^\xi_\delta \cap E_*)$ and $\delta_* = \mathrm{otp}(C^\xi_\delta)$ then $C_\delta = \{\beta \in C^\xi_\delta \cap E_*, \mathrm{otp}(\beta \cap C^\xi_\delta \cap E_*) \in C_*\}$ and if otherwise let C_δ be any closed unbounded subset of δ, possible by the assumption on S_* in 2.9.

Considering Claim 2.10(1) is obvious.

Considering Claim 2.9(2) stated below use the last clause in $(*)_3(b)$.

We are left with 2.10(3).

3) We can find a sequence $\langle S[\zeta] \colon \zeta < \lambda \rangle$ such that:

$(*)$ (a) $S[\zeta] \subseteq \{\alpha < \lambda^+ \colon \mathrm{cf}(\alpha) = \kappa_\varepsilon$ and $\kappa \times \varepsilon$ divide $\delta\}$,
(b) $S[\zeta]$ is a stationary subset of λ,
(c) the $S[\zeta]$-s are pairwise disjoint.

Toward this we choose $\xi(\zeta), E^*_\zeta, \delta_1(\zeta), \delta_2(\zeta)$ by induction on ζ such that:

$\boxplus$ (a) E^*_ζ is a club of λ^+
(b) if $\alpha \in E^*_\zeta$ and $\alpha > \sup(\alpha \cap E^*_\zeta)$ then $\mathrm{cf}(\alpha) \in \{1, \lambda\}$
(c) if $\zeta(1) < \zeta$ then $E^*_\zeta \subseteq E^*_{\zeta(1)}$
(d) $(\zeta, E^*_\zeta, \xi(\zeta), \varepsilon_\zeta(*), \delta_1(\zeta), \delta_2(\zeta))$ are as $(\zeta, E_*, \xi, \varepsilon, \delta_1(*), \delta_2(*))$ in $(*)_5$ for $S[\zeta]$.

Then we can find a club E_* of λ^+ which is $\subseteq \cap\{E^*_\zeta : \zeta < \lambda\}$ and satisfies $\boxplus(b)$. We shall define $\langle (S^*_\zeta, S^+_\zeta, \bar{C}_{\bar{\zeta}}) : \zeta < \lambda \rangle$ as required in 2.10(3) except that:

- $\bar{C}_\zeta$ is now $\langle C_{\zeta,\alpha} : \alpha \in S^+_\zeta \rangle$
- we replace $\lambda = \{\alpha : \alpha < \lambda^+\}$ by E_* so renaming we get the promised result filter.

For each ζ let C^*_ζ be a club of $\delta_2(*)$ of order type $\varepsilon_\zeta(*)$ such that $\alpha \in C^*_\zeta \wedge \alpha > \sup(\alpha \cap C^*_\zeta), 1 = \mathrm{cf}(\alpha)$.

We let

- $S^+_\zeta = S_{\xi(\zeta)} \cap E_*$
- if $\alpha \in S^+_\zeta$ and $\mathrm{otp}(C^{\xi(\zeta)}_\alpha \cap E_* > \delta_2(\zeta)$ then $C^*_{\zeta,\alpha} = \{\beta \in C^{\xi(\zeta)}_\alpha \cap E_* : \mathrm{otp}(\beta \cap C^{\xi(\zeta)}_\alpha \cap E_*) > \delta$
- if $\alpha \in S^+_\zeta$ and $\mathrm{otp}(C^{\xi(\zeta)}_\alpha \cap E_*)$ is $< \delta_2(\zeta)$ and $\notin C^*_\zeta$ then $C_{\zeta,\alpha} = \{\beta \in C^{\xi(\zeta)}_\alpha \cap E_* : \mathrm{otp}(\beta \cap C^{\xi(\zeta)}_\alpha \cap E_*)$ is $> \sup(C^*_\zeta \cap \mathrm{otp}(C^{\xi(\zeta)}_\alpha \cap E_*)\}$
- if $\alpha \in S^+_\zeta$ and $\mathrm{otp}(C^{\xi(\zeta)}_\alpha \cap E_*) \in C^*_\zeta \cup \{\delta_2(\zeta)\}$ then $C_{\zeta,\alpha} = \{\beta \in C^{\xi(\zeta)}_\alpha \cap E_* : \mathrm{otp}(C^{\xi(\zeta)}_\beta \cap E_*) \in C^*_\zeta\}$
- $S^*_\zeta = \{\alpha \in S^+_\zeta : \mathrm{otp}(C_{\zeta,\alpha}) = \varepsilon_\zeta(*)\}$.

Now check. $\square_{2.10}$

Remark 2.14. We may start with a partial square $\langle C^1_\delta : \delta \in S^1 \rangle, S^1 \subseteq \mu$ such that: C_δ is a club of δ, and

$$S^2 =: \{\delta \in S^1 : \quad \begin{array}{l}\{\alpha : \alpha \in C^1_\delta \cap S^1 \text{ and } \mathrm{cf}(\alpha) = \kappa\} \\ \text{is a stationary subset of } \delta\}\end{array}$$

is a stationary subset of $\mu, \mu = \mathrm{cf}(\mu) > \epsilon(*), \mathrm{cf}(\epsilon(*)) = \kappa$ and find $S \subseteq \{\delta \in S^1 : \mathrm{cf}(\delta) = \kappa\}$ stationary in $\mu, \bar{C} = \langle C_\delta : \delta \in S \rangle, C_\delta$ a club of δ of order type $\epsilon(*)$, such that for every club E of μ for some $\delta \in S, C_\delta \subseteq E$. See also [Shei, §2], [She03, §3].

Claim 2.15. *Suppose λ is regular, $\langle C_\alpha : \alpha \in S \rangle$ is a partial square ($S \subseteq \lambda$ stationary), $\kappa = \mathrm{cf}(\kappa) < \lambda, \varepsilon(*) < \lambda$ and*

$$S_1 \subseteq \{\delta \in S : \mathrm{cf}(\delta) = \kappa, \text{ and } \mathrm{otp}(C_\delta) < \delta\} \text{ is stationary.}$$

Then we can find S_2 and E such that:

- *(i) $S_2 \subseteq S_1, S_2$ stationary in λ, E a club of λ,*
- *(ii) for some $\epsilon(*)$ for all $\delta \in S_2, \mathrm{otp}(C_\delta) = \epsilon(*)$*
- *(iii) $\langle C_\delta \cap E : \delta \in S \cap E, \delta = \sup C_\delta \cap E \rangle$ satisfies (a) + (c) of 2.9*
- *(iv) letting*

$$C'_\delta = \begin{cases} C_\delta \cap E & \text{if } C_\delta \cap S_2 = \emptyset \\ C_\delta \cap E \setminus [\min(S_2 \cap C_\delta) + 1] & \text{if } C_\delta \cap S_2 \neq \emptyset \end{cases}$$

we have $\langle C'_\delta : \delta \in S \cap E \rangle$ is a partial square.

Proof. Straightforward if you read 2.2–2.14. $\square_{2.15}$

We now go back to bigness properties, first an easy improvement of 1.16, and then to the promises from the beginning of this section.

Claim 2.16. *If $\lambda = \mathrm{cf}(\lambda) > \mu + \aleph_1$ then K^ω_{tr} has the full $(\lambda, \lambda, \mu, \mu)$-super$^{7^+}$-bigness property.*

Proof. For each stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ let $\langle C^S_\delta : \delta \in S \rangle$ be as in 2.2 with $\kappa = \aleph_0$. Now repeat the proof of 1.16(1) only now, for $\delta \in S_\zeta$ the sequence η_δ list the set $C^{S_\zeta}_\delta$ in increasing order. See also the proof of case 1 in 3.26. $\square_{2.16}$

Remark 2.17. Note: to define partial square on a club of λ or on the set of all limit ordinals, usually makes minor difference (only for non-Mahlo λ, limit of inaccessible, we can get $\mathrm{otp}(C_\delta) < \delta$ more easily in the first case).

Proof. Proof of 2.1: We can find λ_1 a successor of regular cardinal satisfying $\mu + \chi^+ < \lambda_1 \leq \lambda$ and $\chi^+ < \lambda_1$ (just let $\lambda_1 = \mu^+ + \chi^{++}$ if μ is regular and let $\lambda_1 = \mu^{++} + \chi^{++}$ if μ is singular, hence $\lambda_1 \geq \mu^+ \Rightarrow \lambda \geq \mu^{++}$).

Also without loss of generality $\mathrm{cf}(\chi) = \aleph_0$.
[Why? As letting $\chi_1 = \min\{\chi_0 : \chi_0 \geq \aleph_0 \text{ and } \chi_0^{\aleph_0} = \chi^{\aleph_0}\}$, we have $\chi_1 \leq \chi, \chi_1^{\aleph_0} = \chi^{\aleph_0}, \mathrm{cf}(\chi_1) = \aleph_0$ and: $(\forall \alpha < \chi_1)[|\alpha|^{\aleph_0} < \chi_1]$ or $\chi_1 = \aleph_0$ (Notice that, instead changing χ we can use below in clause (a) the ordinal $\chi \times \omega$).]

By Fact 2.9 there are a stationary set $S \subseteq \{\delta < \lambda_1 : \mathrm{cf}(\delta) = \aleph_0\}$ and a sequence $\langle C_\delta : \delta \in S \rangle$ such that:

$(*)_1$ (a) C_δ is a club of δ of order type χ,
(b) for every club E of λ_1 for stationary many $\delta \in S, C_\delta \subseteq E$.

For any $\rho \in {}^{\omega}\chi$ we define

$(*)_2$ $I_\rho = {}^{\omega>}\lambda \cup \{\rho^{[\delta]} : \delta \in S\}$

where $\rho^{[\delta]} \in {}^{\omega}(\lambda_1)$ is defined by $\rho^{[\delta]}(n) =$ the $\rho(n)$-th member of C_δ.

Easily there are $\Upsilon, \bar\chi$ such that

$(*)_3$ (a) $\Upsilon \subseteq {}^{\omega}\chi$ have cardinality $\chi^{\aleph_0}$,

(b) each $\rho \in \Upsilon$ is increasing with limit χ

(c) for $\rho_1 \neq \rho_2$ from $\Upsilon, \mathrm{Rang}(\rho_1) \cap \mathrm{Rang}(\rho_2)$ is finite

(d) $\bar\chi = \langle \chi_n : n < \omega\rangle$ is a strictly increasing sequence of χ-s,

(e) $\chi = \bigcup\limits_{n<\omega} \chi_n$

(f) $\rho \in \Upsilon \Rightarrow \rho(n) \in (\chi_n, \chi_{n+1})$.

We shall show that $\{I_\rho : \rho \in \Upsilon\}$ exemplifies the desired conclusion: the full $(\chi^{\aleph_0}, \lambda, \mu, \mu)$-super-bigness property.

Suppose $\rho \in \Upsilon, J = J_\rho := \sum\{I_\nu : \nu \in \Upsilon \backslash \{\rho\}\}$, for example let $\Upsilon = \{\rho_i : i < |\Upsilon|\}$ and $J = \{\langle\rangle\} \cup \{\langle\zeta\rangle \underset{\lambda}{\otimes} \nu : \zeta < |\Upsilon|, \rho_\zeta \neq \rho \text{ and } \nu \in I_{\rho_i}\}$, where:

⊞ for ρ a sequence of ordinals and $\zeta < \lambda$ let $\langle\zeta\rangle \underset{\lambda}{\otimes} \rho$ or $\zeta \underset{\lambda}{\otimes} \rho$ be the sequence ρ' of length $\ell g(\rho), \rho'(0) = \lambda \times \zeta + \rho(0), \rho'(1+\gamma) = \rho(1+\gamma)$.

Clearly

⊕ It suffice to show that $I_\rho \in K^\omega_{\mathrm{tr}}$ is (μ, κ)-super-unembedabble into $J_\rho \in K^\omega_{\mathrm{tr}}$. Let χ^* be regular large enough and $<^*$ a well ordering of $\mathscr{H}(\chi^*)$ and $x \in \mathscr{H}(\chi^*)$.

We choose by induction on $\alpha < \lambda_1, M^*_\alpha$ such that:

$(*)_4$ (a) $M^*_\alpha \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$,

(b) M^*_α is increasing continuous,

(c) $\|M^*_\alpha\| < \lambda_1$,

(d) $M^*_\alpha \cap \lambda_1$ is an ordinal,

(e) $\langle M^*_\beta : \beta \leq \alpha\rangle \in M^*_{\alpha+1}$,

(f) $\mu + 1$ is a subset of M^*_0,

(g) $\mu, I_\rho, x, J = \sum\limits_{\nu \in \Upsilon\backslash\{\rho\}} I_\nu$ belong to M^*_0.

Let

$(*)_5$ $E := \{\delta < \lambda_1 : M^*_\delta \cap \lambda_1 = \delta\}$.

Clearly E is a club of λ_1. So, by the choice of $\langle C_\delta : \delta \in S\rangle$, for some $\delta(*) \in S$ we have $C_{\delta(*)} \subseteq E$.

We shall show that $\eta := \rho^{[\delta(*)]}, M_n := M^*_{\eta(n)}, N_n := M^*_{\eta(n)+1}$ are as required in Definition 1.1.

Note:

$(*)_6$ $\eta(n) + 1 \leq \chi_{n+1} < \eta(n+1)$.

Hence,

$(*)_7$ (a) $M_n \in N_n$,

(b) $N_n \in M_{n+1}$,

(c) $\eta{\restriction}n \in M_n$,

(d) $\eta\restriction(n+1) \notin N_n$.

[E.g. Why clause (c)? It suffices to prove $\ell < n \Rightarrow \eta(\ell) \in M_n$ because $\eta(\ell) < \eta(n) = \lambda_1 \cap M_n \subseteq M^*_{\eta(n)} = M_n$ recalling $C_{\delta(*)} \subseteq E$ and the definition of E. Why clause (d) holds? By $(*)_6$.]

Of course,

$(*)_8$ $\mu \subseteq M_n \prec N_n \prec M_{n+1} \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$.

So clearly clauses (i)-(v) of $(*)$ of 1.1 holds and we are left with proving clause (vi).

Let $\nu \in P^J_\omega$, and choose an ordinal $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\} < \delta(*)$ where:

$(*)_9$ (a) if $\nu \in M^*_{\delta(*)}$ then $\alpha_1 < \delta(*)$ is such that $\nu \in M^*_{\alpha_1}$,

(b) if for some $m < \ell g(\nu), \nu\restriction m \in M_{\delta(*)}, \nu\restriction(m+1) \notin M_{\delta(*)}$, then $\alpha_2 < \delta(*)$ is large enough such that $\nu\restriction m \in M^*_{\alpha_2}$,

(c) if $\nu = \langle i\rangle \underset{\lambda}{\otimes} \rho_i^{[\delta(*)]}$ (so $\rho_i \neq \rho$), let $\alpha_3 \in C_{\delta(*)}$ be such that $(\mathrm{Rang}(\rho)) \cap \mathrm{Rang}(\rho_i) \subseteq \alpha_3 \cap C_{\delta(*)}$ (exists by the choice of Υ).

It is easy to check that every $n < \omega$ such that $\rho^{[\delta(*)]}(n) > \alpha$ is as required in Definition 1.1(vi), but every large enough $n < \omega$ is like that by the choice of α.

So indeed $I_\rho \in K^\omega_{\mathrm{tr}}$ is (μ, κ)-super-unembeddable into $J_\rho \in K^\omega_{\mathrm{tr}}$.

So we have proved 2.1(1).

For proving 2.1(2) let $M'_{2n} = M_n, M'_{2n+1} = N_n$.

As for proving 2.1(3), (using again M'_n) make the following changes. <u>First</u> in proving 2.9 we can guarantee

$$[\sup(C_\delta \cap \alpha) < \alpha \in C_\delta \Rightarrow \mathrm{cf}(\alpha) > \aleph_0],$$

(apply e.g. 2.9 to $\omega_1 \times \epsilon(*)$ getting C'_δ, and let

$$C_\delta = \{\zeta \in C_\delta : \mathrm{otp}(C_\delta \cap \zeta) \text{ divisible by } \omega_1\}).$$

Second choosing Υ in $(*)_3$ above (in the proof of 2.1(11)), we can demand:

$$\eta \in \Upsilon \Rightarrow \mathrm{Rang}(\eta) \text{ consists of successor ordinals only}).$$

Then the requirement holds — check. $\square_{2.16}$

Lemma 2.18. *Suppose λ is singular, $\lambda > \mu, \lambda > \theta > \mathrm{cf}(\theta) = \aleph_0, \theta \geq \mu + \mathrm{cf}(\lambda), \mathfrak{a}_\epsilon$ for $\epsilon < \mathrm{cf}(\lambda)$ is a set of regular uncountable cardinals, $\omega = \mathrm{otp}(\mathfrak{a}_\epsilon), \theta = \sup(\mathfrak{a}_\epsilon)$, they are pairwise almost disjoint (i.e. for $\epsilon < \zeta < \mathrm{cf}(\lambda)$, $\mathfrak{a}_\epsilon \cap \mathfrak{a}_\zeta$ is finite) and $\max \mathrm{pcf}_{J^{bd}_{\mathfrak{a}_\epsilon}}(\mathfrak{a}_\epsilon) = \theta^+ < \lambda$, see Definition* [Shed, 3.16=Lprf.2].

<u>Then</u> K^ω_{tr} has the full $(\lambda, \lambda, \mu, \mu)$-super-bigness property.

Remark 2.19. We shall repeat this proof with some changes in 3.26 case 3.

Proof. Let $\langle \mu_\epsilon : \epsilon < \mathrm{cf}(\lambda)\rangle$ be a strictly increasing sequence of regular cardinals such that $\sum\limits_{\epsilon < \mathrm{cf}(\lambda)} \mu_\epsilon = \lambda$, and $\mu + \theta^+ < \mu_\epsilon$. For $\epsilon < \mathrm{cf}(\lambda)$, let $\lambda_\epsilon = \mu_\epsilon^{+3}$.

We now apply 2.10(3) with λ_ε, $\langle \theta \colon \zeta < \lambda\rangle$, $\langle \aleph_0 \colon \zeta < \lambda_\varepsilon\rangle$ here standing for λ, $\langle \varepsilon_\zeta(*) \colon \zeta < \lambda\rangle$, $\langle \kappa_\zeta \colon \zeta < \lambda\rangle$ there. Let us get $\bar{S}_\varepsilon = \langle S_{\epsilon,\zeta} : \zeta < \lambda_\epsilon\rangle$, $\bar{C}_\varepsilon = \langle C^\epsilon_\delta \colon \delta \in S_\epsilon\rangle$ here standing for $\langle S_\zeta \colon \zeta < \lambda\rangle$, $\langle \rho^\zeta_\delta \colon \delta \in S_1, \zeta < \lambda\rangle$ there, hence:

$(*)_1$ (i) $\langle S_{\epsilon,\zeta} : \zeta < \lambda_\epsilon\rangle$ is a partition of S_ε,

(ii) $S_{\epsilon,\zeta} \subseteq \{\delta < \lambda_\epsilon : \mathrm{cf}(\delta) = \aleph_0\}$ are stationary subsets of λ_ϵ, (*iii*) if $\delta \in S_\epsilon$ then C^ϵ_δ is a club of δ and C^ϵ_δ has order type θ (recall that $\mathrm{cf}(\theta) = \aleph_0$ by the claim's assumptions and $\delta \in S_{\varepsilon,\zeta}$ implies $\mathrm{cf}(\delta) = \aleph_0$, so δ is divisible by θ),

(*iv*) for every club E of λ_ϵ and $\zeta < \lambda_\epsilon$, the set $\{\delta \in S_{\epsilon,\zeta} : C^\epsilon_\delta \subseteq E\}$ is stationary

(*v*) $\langle C^\epsilon_\delta : \delta \in S_\epsilon \rangle$ is a partial square.

For transparency, S_ϵ is disjoint to $\bigcup\limits_{\xi<\epsilon} \lambda_\xi$ and let $\langle \kappa_{\varepsilon,n} : n < \omega \rangle$ list $\mathfrak{a}_\varepsilon$ in increasing order.

For each $\epsilon < \mathrm{cf}(\lambda)$ we can find $\langle \rho_{\epsilon,i} : i < \theta^+ \rangle$ such that:

$(*)_2$ (*i*) $\rho_{\epsilon,i} \in \Pi\mathfrak{a}_\epsilon$ is (strictly) increasing

(*ii*) $i < j < \theta^+ \Rightarrow \rho_{\epsilon,i} <_{J^{\mathrm{bd}}_{\mathfrak{a}_\epsilon}} \rho_{\epsilon,j}$; (i.e. for every large enough $\kappa \in \mathfrak{a}_\epsilon, \rho_{\epsilon,i}(\kappa) < \rho_{\epsilon,j}(\kappa) < \kappa$),

(*iii*) for every $\rho \in \Pi\mathfrak{a}_\epsilon$ for some $i < \theta^+$ we have $\rho <_{J^{\mathrm{bd}}_{\mathfrak{a}_\epsilon}} \rho_{\epsilon,i}$

(*iv*) $\rho_{\epsilon,i}(k_{\varepsilon,n})$ is a limit ordinal of uncountable cofinality

(*v*) $\rho_{\epsilon,i}(\kappa) > \sup(\mathfrak{a}_\varepsilon \cap \kappa)$ hence $\theta = \cup\{\rho_{\epsilon,i}(\kappa) : \kappa \in \mathfrak{a}_\varepsilon\}$.

Let $\Upsilon_\epsilon := \{\rho_{\epsilon,i} : i < \theta^+\}$.

For $\epsilon < \mathrm{cf}(\lambda)$, and ζ such that $\bigcup\limits_{\xi<\epsilon} \lambda_\xi \le \zeta < \lambda_\epsilon$ let $I_\zeta = {}^{\omega>}\lambda \cup \{\rho^{[\delta]} : \rho \in \boldsymbol{\Upsilon}_\epsilon, \delta \in S_{\epsilon,\zeta}\}$ recalling $\rho^{[\delta]}$ is an ω-sequence of ordinals: $\rho^{[\delta]}(n) =$ the $\rho(\kappa_{\varepsilon,n})$-th element of C^ϵ_δ (now $\rho^{[\delta]}$ depend on C^ϵ_δ and ρ so on δ, ϵ, ρ, but ϵ can be reconstructed from δ, as $S_\epsilon \subseteq [\bigcup\limits_{\xi<\epsilon} \lambda_\xi, \lambda_\epsilon))$.

We shall show that $\langle I_\zeta : \zeta < \lambda \rangle$ exemplify the desired conclusion, this suffices. So let $\epsilon(*) < \mathrm{cf}(\lambda)$, $\bigcup\limits_{\xi<\epsilon(*)} \lambda_\xi \le \zeta(*) < \lambda_{\epsilon(*)}, \chi^*$ regular large enough and $x \in \mathscr{H}(\chi^*)$, and let $J = \sum\limits_{\xi<\lambda, \xi\ne\zeta(*)} I_\xi$. We can choose by induction on $\alpha < \lambda_{\epsilon(*)}$ a model M^*_α such that:

$(*)_3$ (*a*) $M^*_\alpha \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$

(*b*) M^*_α increasing continuous in α

(*c*) $\langle M^*_\beta : \beta \le \alpha \rangle \in M^*_{\alpha+1}$

(*d*) $\|M^*_\alpha\| < \lambda_{\epsilon(*)}$

(*e*) $M^*_\alpha \cap \lambda_{\epsilon(*)}$ is an ordinal $> \mu^{+2}_\epsilon > \mu + \mathrm{cf}(\lambda) + \sum\limits_{\zeta<\epsilon(*)} \lambda_\zeta$

(*f*) the objects $I_{\varepsilon(*)}, J$ and $\langle < \rho_{\epsilon,j} : j < \theta^+ >: \epsilon < \mathrm{cf}(\lambda) \rangle, \epsilon(*), \langle \lambda_\epsilon : \epsilon < \mathrm{cf}(\lambda) \rangle, \langle < S_{\epsilon,\zeta} : \zeta < \lambda_\epsilon >: \epsilon < \mathrm{cf}(\lambda) \rangle$ and $\langle < C^\epsilon_\delta : \delta \in S_\epsilon >: \epsilon < \mathrm{cf}(\lambda) \rangle$ belong to M^*_α.

Let $E = \{\delta < \lambda_{\epsilon(*)} : M^*_\delta \cap \lambda_{\epsilon(*)} = \delta\}$; clearly E is a club of $\lambda_{\epsilon(*)}$. So for some $\delta(*) \in E \cap S_{\epsilon(*),\zeta(*)}$, we have $C^{\epsilon(*)}_{\delta(*)} \subseteq E$.

We can find $N \prec M^*_{\delta(*)}$ such that:

$(*)_4$ (α) $\|N\| = \theta$, $\theta + 1 \subseteq N$ hence $\mu + 1 \subseteq N, \{\mu, \kappa\} \subseteq N$, and $C^{\epsilon(*)}_{\delta(*)} \subseteq N$;

(β) if $\delta \in M^*_{\delta(*)}, \mathrm{cf}(\delta) = \aleph_0, \delta = \sup(N \cap \delta)$ then $\delta \in N$;

(γ) the following objects belong to N

- $\langle < \rho_{\epsilon,j} : j < \theta^+ >: \epsilon < \mathrm{cf}(\lambda)\rangle$,
- $I_{\zeta(*)}, J, x$,
- $\epsilon(*), \langle \lambda_\epsilon : \epsilon < \mathrm{cf}(\lambda)\rangle$,
- $\langle < S_{\epsilon,\zeta} : \zeta < \lambda_\epsilon >: \epsilon < \mathrm{cf}(\lambda)\rangle$,
- $\langle < C^\epsilon_\delta : \delta \in S_\epsilon >: \epsilon < \mathrm{cf}(\lambda)\rangle$

(δ) $\langle M^*_\alpha : \alpha < \gamma\rangle \in N$ for $\gamma \in C^{\epsilon(*)}_{\delta(*)}$.

Let

$\boxplus_1$ (a) $W := \{(\epsilon,\zeta,\delta) : \epsilon < \mathrm{cf}(\lambda), \zeta \neq \zeta(*), \bigcup\limits_{j<\epsilon} \lambda_j \le \zeta < \lambda_\epsilon, \delta \in S_{\epsilon,\zeta}$ and $\zeta \in N, \delta = \sup(N \cap \delta) \notin N$ but $C^\epsilon_\delta \subseteq N\}$,

(b) $W_1 := \{(\epsilon,\zeta,\delta) \in W : \epsilon > \epsilon(*)\}$.

It is enough to show that for some $\rho \in \boldsymbol{\Upsilon}_{\epsilon(*)}$ we have:

$\boxplus_{2,\rho}$ $\eta_\rho := \rho^{[\delta(*)]}, M^\rho_n := N \cap M^*_{\rho^{[\delta(*)]}(n)}, N^\rho_n := N \cap M^*_{\rho^{[\delta(*)]}(n)+1}$

(for $n < \omega$) satisfy the requirement $(*)$ of Definition 1.1.

Now, for every $\rho \in \boldsymbol{\Upsilon}_{\epsilon(*)}$, consider the conditions (from $(*)$ of 1.1):

- $(i), (iii), (iv)$ are trivial
- (v) holds by the definition, in fact for every $n, \eta_\delta{\restriction}\Upsilon \in M^\ell_n, \eta_\delta{\restriction}(n+1) \in N^\rho_n \backslash M^\rho_n$
- (ii) holds as $\mu + 1 \subseteq M_0$ because $\mu \le \theta \subseteq N$.

The main point is condition (vi) and we shall show that for some $\rho \in \boldsymbol{\Upsilon}_{\epsilon(*)}$ it holds

$\boxplus_3$ let $\Lambda = \{\rho \in \Upsilon_{\epsilon(*)}:$ clause (vi) of Definition 1.1 fails for $\eta_\rho = \rho^{[\delta(*)]}, M^\rho_n, N^\rho_n (n < \omega)\}$

$\boxplus_4$ if $\rho \in \Lambda$, then let Λ_ρ be the set of $\nu \in P^J_\omega$ such that: $\{\nu{\restriction}\ell : \ell < \omega\} \subseteq N$ but for no $\alpha < \delta(*)$ do we have $\{\nu{\restriction}\ell : \ell < \omega\} \subseteq N \cap M^*_\alpha$ and for infinitely many n for some k we have $\nu{\restriction}k \in M^\rho_n, \nu(k) \in N^\rho_n \setminus M^\rho_n$.

(Recall that N is from $(*)_4$).

$\boxplus_5$ it suffice to find $\rho \in \Lambda$ such that $\Lambda_\rho = \emptyset$, so towards a contradiction, assume that:

- if $\rho \in \Lambda$, then we choose $(\nu, \varrho, \epsilon, \zeta, \delta) = (\nu_\rho, \varrho_\rho, \epsilon_\rho, \zeta_\rho, \delta_\rho)$ such that (but if ρ is clear from the context, we may omit the subscript ρ):
  - (a) $\nu_\rho \in \Lambda_\rho$,
  - (b) $\zeta_\rho \in \lambda \setminus \{\zeta(*)\}$,
  - (c) ε_ρ is the unique $\varepsilon < \mathrm{cf}(\lambda)$ such that $\bigcup_{\xi<\epsilon} \lambda_\xi \le \zeta_\rho < \lambda_\epsilon$,
  - (d) $\delta_\rho = \delta$,
  - (e) $\varrho_\rho \in \Lambda_{\varepsilon_\rho}$ is such that $\nu_\rho = \varrho^{[\delta(*)]}$, see after $(*)_2$.

[Why such a quintuple exists? If no such $\nu_\rho \in \Lambda_\rho$ exists, then we get the desired contradiction. By the definition of Λ_ρ there is ϱ such that $\nu = \langle\zeta\rangle \underset{\lambda}{\otimes} \varrho^{[\delta]}$ (if we use the first version in the proof of 2.16, or $\langle\zeta\rangle{}^\frown\varrho$ if we use another one there) and $\varrho \in \boldsymbol{\Upsilon}_\epsilon, \delta \in S_{\epsilon,\zeta}, \bigcup\limits_{\xi<\epsilon} \lambda_\xi < \zeta < \lambda_\epsilon, \zeta \neq \zeta(*)$; hence without loss of generality $(\nu, \varrho, \epsilon, \zeta, \delta) = (\nu_\rho, \varrho_\rho, \epsilon_\rho, \zeta_\rho, \delta_\rho)$.]

$\boxplus_6$ if $\rho \in \Lambda$ then $\epsilon < \epsilon(*)$ is impossible.

Why? In this case $\lambda_\epsilon \subseteq M^*_0$ (see condition $(*)_3(e)$ on the M^*_α's, hence $N \cap \{\nu{\restriction}\ell : \ell < \omega\} \subseteq M^*_0$, contradiction.

Next,

Fact 2.20. We have:

$\boxplus_7$ if $\rho \in \Lambda$ then $\epsilon = \epsilon(*)$ is impossible.

Proof. As $\nu \in J$ necessarily $\zeta \neq \zeta(*)$. As $\delta \in S_{\epsilon,\zeta}$, clearly $S_{\epsilon,\zeta} \cap S_{\epsilon(*),\zeta(*)} = \emptyset$ so necessarily $\delta \neq \delta(*)$. If $\delta > \delta(*)$, as $\langle \nu(n) : 1 \leq n < \omega \rangle$ is strictly increasing with limit δ, for some $n, \lambda_{\epsilon(*)} > \nu(n) > \delta(*)$ hence $\nu{\restriction}(n{+}1) \notin M^*_{\delta(*)}$ hence $\nu{\restriction}(n{+}1) \notin N$, contradiction. If $\delta < \delta(*)$ then for some $\alpha < \delta(*), \{\nu{\restriction}\ell : \ell < \omega\} \subseteq N \cap M^*_\alpha$, (remember that $\theta \subseteq N$ by $(*)_4(\alpha)$ and $\{\nu{\restriction}\ell : \ell < \omega\} \subseteq N$ by the assumption on ν); again impossible by $\boxplus_4$, so $\boxplus_7$ holds. $\square_{2.20}$

Subfact 2.21. We have:

$\boxplus_8$ $(\varepsilon, \zeta, \delta) \in W_1$.

Proof. By $\boxplus_6, \boxplus_7$ we have $\epsilon > \epsilon(*)$. Now (remembering $\bar{C}^\epsilon$ is a partial square), for $1 \leq n < m < \omega, C^\epsilon_{\nu(n)} = C^\epsilon_{\nu(m)} \cap \nu(n)$, and as $\nu(n) \in N$ by $(*)_4(\gamma)$ necessarily $C^\epsilon_{\nu(n)} \in N$, so as $\theta \subseteq N \wedge |C^\epsilon_{\nu(n)}| \leq |C^{\epsilon(*)}_\delta| = \theta$ clearly $C^\epsilon_{\nu(n)} \subseteq N$.

It follows that $C^\epsilon_\delta = \bigcup_{1<n<\omega} C^\epsilon_{\nu(n)}$, hence $C^\epsilon_\delta \subseteq N$, so $\delta = \sup(\delta \cap N)$. Next,

Subfact 2.22. $\delta \notin N$.

Proof. As otherwise for some $\alpha < \delta(*), \delta \in M^*_\alpha$, hence $C^\epsilon_\delta \subseteq M^*_\alpha$; now from $\nu{\restriction}1 \in N$ it follows that $\zeta \in N$ but $\epsilon < \mathrm{cf}(\lambda) \subseteq \theta \subseteq N$ so also $\epsilon \in N$ and $\Upsilon_\epsilon \in N$. Hence $\Lambda = \{\langle \gamma, \eta, \eta^{[\gamma]} \rangle : \gamma \in S_{\epsilon,\zeta}$ and $\eta \in \boldsymbol{\Upsilon}_\epsilon\} \in N$ but we are assuming $\delta \in N$, and hence $\{\nu(n) \colon n < \omega\} \subseteq C^\epsilon_\delta \subseteq M^*_\alpha$ but also $\{\nu(n) \colon n < \omega\} \subseteq N$ by $\oplus_4$, hence $\{\nu(n) \colon n < \omega\} \subseteq N \cap M_\alpha$, therefore by $\oplus_{2,\rho}$ is $\subseteq N_n$ for n large enough, which contradicts $\oplus_4$. So really $\delta \notin N$]. $\square_{2.22}$

Continuing the proof of 2.21: By clause $(*)_4(\beta)$ in the choice of N necessarily $\delta \notin M_{\delta(*)}$ and recalling W is defined in $\boxplus_1$ above clearly $(\epsilon, \zeta, \delta) \in W$.

Clearly $(\epsilon, \zeta, \delta) \in W_1$ as we have shown $\epsilon > \epsilon(*)$ by $\boxplus_6 + \boxplus_7$, so $\boxplus_8$ holds indeed. $\square_{2.21}$

Note that

$\boxplus_9$ $|W_1| \leq |W| \leq \theta$.

This is because

- $\bullet_1$ the set $W_{1,1}$ has cardinality $\leq \theta$, where $W_{1,1} := \{\epsilon \colon$ for some ζ, δ we have $(\epsilon, \zeta, \delta) \in W_1\}$ because $\epsilon \in W_{1,1} \Rightarrow \epsilon \leq \mathrm{cf}(\lambda) \leq \theta$.
- $\bullet_2$ the set $W_{1,2}$ has cardinality $\leq \theta$, where $W_{1,2} = \{\zeta \colon$ for some ϵ, δ we have $(\epsilon, \zeta, \delta) \in W_1\}$ because $\zeta \in W_{1,2} \Rightarrow \zeta \in N$ so $|W_{1,2}| \leq \|N\| = \theta$.
- $\bullet_3$ the set $W_{1,3}$ has cardinality $\leq \theta$, where $W_{1,3} := \{\delta \colon$ for some ϵ, ζ we have $(\epsilon, \zeta, \delta) \in W_1\}$ because $\|N\| = \theta$, and a well ordering of cardinality $\leq \theta$ has $\leq \theta$ Dedekind cuts and $\delta = \sup(\delta \cap N) > \sup(\delta \cap M_\alpha)$ for $\alpha < \delta$ (see $(*)_4(\beta)$ in choice of N).

By $\bullet_1$, $\bullet_2$ and $\bullet_3$, clearly $\boxplus_5$ holds indeed.

Remember, we are trying to show only that for some $\rho \in \boldsymbol{\Upsilon}_{\epsilon(*)}$ we have $\eta_\rho =: \rho^{[\delta(*)]}, M^\rho_n, N^\rho_n$ $(n < \omega)$ are as required, we shall prove more,

> $\oplus_1$ if $(\epsilon, \zeta, \delta) \in W_1$ <u>then</u> $\Omega_{(\varepsilon,\zeta,\delta)}$ has cardinality $\leq \theta$ where $\Omega_{(\varepsilon,\zeta,\delta)} := \{\nu_\rho : \rho \in \Lambda$ and $(\epsilon_\rho, \zeta_\rho, \delta_\rho) = (\varepsilon, \zeta, \delta)\}$.

as $|W_1| \leq \theta < \theta^+ = |\boldsymbol{\Upsilon}_{\epsilon(*)}|$, this will be enough.

So let $y = (\epsilon, \zeta, \delta) \in W_1$ we know that $\epsilon > \epsilon(*)$ hence $\mathfrak{a}_\epsilon \cap \mathfrak{a}_{\epsilon(*)}$ is finite. Let for $\alpha \in C^\epsilon_\delta$:

$$\gamma[\alpha] = \min\{\gamma \in C^{\epsilon(*)}_{\delta(*)} : \alpha \text{ belongs to } M^*_\gamma, (\text{equivalently: } C^\epsilon_\alpha \in N \cap M^*_\gamma)\}.$$

Now, $\gamma[\alpha]$ is well defined because $C^\varepsilon_\delta \subseteq N$ (by $(*)_4$) and $N \subseteq M_{\delta(*)} = \bigcup\{M^*_\gamma \colon \gamma \in C^{\varepsilon(*)}_{\gamma(*)}\}$ because $C^{\varepsilon(*)}_{\delta(*)}$ is unbounded in $\delta(*)$. Next,

Fact 2.23. $\langle \gamma[\alpha] : \alpha \in C^\epsilon_\delta \rangle$ is a non-decreasing sequence of ordinals which are non-accumulation members of $C^{\epsilon(*)}_{\delta(*)}$, (with limit $\delta(*)$).

Proof. If $\alpha \in C^\epsilon_\delta$ then $C^\epsilon_\alpha = C^\varepsilon_\delta \cap \alpha \subseteq N \cap M^*_{\gamma[\alpha]}$ hence $\beta \in C^\epsilon_\alpha \Rightarrow C^\epsilon_\beta = C^\epsilon_\alpha \cap \beta \subseteq C^\epsilon_\alpha \subseteq N \cap M^*_{\gamma[\alpha]} \Rightarrow \gamma[\beta] \leq \gamma[\alpha]$ so $\beta < \alpha$ & $\alpha \in C^\epsilon_\delta$ & $\beta \in C^\epsilon_\delta \Rightarrow \gamma[\beta] \leq \gamma[\alpha]$. Being non-accumulation points is trivial by the definition. $\square_{2.23}$

<u>Continuation of the proof of 2.18</u>:

For $\kappa \in \mathfrak{a}_{\epsilon(*)}$, let:

- $\beta^{\epsilon(*)}(\kappa)$ be the supremum of the set

  $\{\gamma[\alpha] : \alpha \in C^\epsilon_\delta \text{ and } \mathrm{otp}(\alpha \cap C^\epsilon_\delta) \leq \sup(\mathfrak{a}_\epsilon \cap \kappa) \text{ andotp}(\gamma[\alpha] \cap C^{\epsilon(*)}_{\delta(*)}) < \kappa\}$,
- $\gamma^{\epsilon(*)}(\kappa) = \mathrm{otp}(C^{\epsilon(*)}_{\delta(*)} \cap \beta^{\epsilon(*)}(\kappa))$.

Note: the supremum is taken over a set of $\leq \sup(\mathfrak{a}_\epsilon \cap \kappa)$ ordinals $< \kappa$ but $\mathfrak{a}_\epsilon \cap \kappa$ is a countable (even finite) set of cardinals $< \kappa, \kappa$ regular uncountable so $\sup(\mathfrak{a}_\epsilon \cap \kappa) < \kappa$ hence clearly $\gamma^{\epsilon(*)}(\kappa) < \kappa$.

So $\langle \gamma^{\epsilon(*)}(\kappa) : \kappa \in \mathfrak{a}_{\epsilon(*)} \rangle$ belongs to $\Pi \mathfrak{a}_{\epsilon(*)}$ hence for some $j(y) < \partial_{\varepsilon(*)}$, we have:

> $\oplus_{1.1}$ $\rho_{\epsilon(*),j(y)}(\kappa) > \gamma^{\epsilon(*)}(\kappa)$ for every large enough $\kappa \in \mathfrak{a}_{\epsilon(*)}$.

For $\kappa \in \mathfrak{a}_\epsilon$, let:

- $\beta^\epsilon(\kappa)$ be the supremum of the set of the ordinals $\alpha \in C^\epsilon_\delta$ such that:
  - for some $\kappa_1 \in \mathfrak{a}_{\epsilon(*)}, \mathrm{otp}(\gamma[\alpha] \cap C^{\epsilon(*)}_{\delta(*)}) \leq \kappa_1$,
  - $(\forall \beta < \alpha)[\mathrm{otp}(\gamma[\beta] \cap C^{\epsilon(*)}_{\delta(*)}) < \kappa_1]$, and
  - $\mathrm{otp}(\alpha \cap C^\epsilon_\delta) < \kappa$.
- $\gamma^\epsilon(\kappa) = \mathrm{otp}(C^\epsilon_\gamma \cap \beta^\epsilon(\kappa))$.

again, the supremum is taken over a set of $< \kappa$ ordinals $< \kappa$, hence clearly $\gamma^\epsilon(\kappa) < \kappa$.

So $\langle \gamma^\epsilon(\kappa) : \kappa \in \mathfrak{a}_\epsilon \rangle$ belongs to $\Pi \mathfrak{a}_\epsilon$ hence for some $i(y) < \theta^+$, we have: $\rho_{\epsilon,i(y)}(\kappa) > \gamma^\epsilon(\kappa)$ for every large enough $\kappa \in \mathfrak{a}_\epsilon$.

Now recall $\boldsymbol{\Upsilon}_\epsilon = \{\rho_{\epsilon,i} : i < \theta^+\}$ and similarly for $\epsilon(*)$, so clearly if $i(y) < i < \theta^+$ & $i(y) < j < \theta^+$ then $\rho^{[\delta]}_{\epsilon,j}$ cannot "hurt" $\rho^{[\delta(*)]}_{\epsilon(*),i}$, that is, $\nu_{\rho_{\varepsilon(*),i}} \in \{\rho^{[\delta]}_{\epsilon,j} : i(y) < j < \theta^+\}$ so $|\Omega_y| \leq |i(y)|$ so $\oplus_1$ holds.

Now we shall show that each $\nu = \rho^{[\delta]}_{\epsilon,j}$ (for $j \leq i(\epsilon)$) can hurt at most θ (also $\leq 2^{\aleph_0}$) many $\rho \in \boldsymbol{\Upsilon}_{\epsilon(*)}$; that is:

$\oplus_2$ if $\nu \in \Omega_y$ then $\Lambda_{y,\nu} = \{\rho \in \Lambda : (\varepsilon_\rho, \zeta_\rho, \delta_\rho) = y$ and $\nu_\rho = \nu\}$ has cardinality $\leq \theta$, recall ν_ρ is from $\boxplus_5$.

Now $\mathrm{Rang}(\rho^{[\delta(*)]})$ has infinite intersection with

$$B := \{\alpha < \delta(*) : \text{ for some } \ell, \nu{\restriction}\ell \in M^*_{\alpha+1} \setminus M_\alpha\}$$

so let for $\kappa \in \mathfrak{a}_{\epsilon(*)}$:

$$\beta^*_\kappa = \sup\{\mathrm{otp}(C^{\epsilon(*)}_{\delta(*)} \cap \alpha) : \alpha \in B, \mathrm{otp}(C^{\epsilon(*)}_{\delta(*)} \cap \alpha) < \kappa\}.$$

So for some $i(*) < \theta^+, \beta^*_\kappa < \rho_{\epsilon(*),i(*)}(\kappa)$ for every $\kappa \in \mathfrak{a}_{\epsilon(*)}$ large enough; so for every i, if $i(*) < i < \theta^+$, then $\rho_{\epsilon(*),i}$ is not hurt, that is, $\rho_{\varepsilon(*),i(*)} \notin \Lambda_{y,\nu}$ so $\oplus_2$ holds.

We can conclude

$\oplus_3$ if $y = (\varepsilon, \zeta, \delta) \in W$ then $\Lambda_y = \{\rho \in \Lambda : (\varepsilon_\rho, \zeta_\rho, \delta_\rho) = y\}$ has cardinality $\leq \theta$.

[Why? By $\oplus_1 + \oplus_2$.]

Fact 2.24. <u>But</u> we have:

$\oplus_4$ Λ has cardinality $\leq \theta$.

Proof. As $|W_1| \leq \theta$ and $\Lambda = \cup\{\Lambda_y : y \in W_1\}$ and each Λ_y has cardinality $\leq \theta\}$. $\square_{2.24}$

So necessarily $\boldsymbol{\Upsilon}_{\varepsilon(*)} \nsubseteq \Lambda$ and so for some $\rho \in \boldsymbol{\Upsilon}_{\varepsilon(*)} \setminus \Lambda$. Definition 1.1 is exemplified by $\eta_\rho = \rho^{[\delta(*)]}, \mu^\rho_n, N^\rho_n$ (for $n < \omega$), so we finish. $\square_{2.18}$

* * *

Lemma 2.25. *Suppose λ is strong limit, $\lambda = \kappa^{+\omega} > \mu$.* <u>*Then*</u> *$K^\omega_{\mathrm{tr}}$ has the full $(\lambda, \lambda, \mu, \aleph_0)$-super^6-bigness property.*

Remark 2.26. We use variants of this proof in case 6 of the proof of 3.26.

Proof. Let $\chi > 2^\lambda$ be large enough.

Without loss of generality $\kappa > \mu$ and $\kappa^\mu = \kappa$. Let $\langle C_\delta : \delta < \lambda\rangle$ be such that C_δ is a club of δ of order type $\mathrm{cf}(\delta)$. If $(\kappa^{+n})^{\kappa^{++}} = \kappa^{+n}$ then we choose a function cd_n from $\{M \in \mathscr{H}(\chi)\colon M$ a model, $\|M\| \leq \kappa^{++}, |\tau(M)| \leq \kappa^{++}$ and $\tau(M) \in \mathscr{H}_{<\kappa^{+3}}(\kappa^{+n})\}$ to κ^{+n} such that:

$\oplus_1$ $\mathrm{cd}_n(M_1) = \mathrm{cd}_n(M_2)$ iff $M_1 \cong M_2 \,\&\, M_1 \cap \kappa^{+n} = M_2 \cap \kappa^{+n}$.

As λ is strong limit, $2^{\kappa^{++}} < \lambda = \kappa^{+\omega}$ hence cd_n is well defined for every n large enough, so choose $n_0 > 3$ such that cd_n is well-defined for every $n \geq n_0$. Without loss of generality cd_n is definable in $(\mathscr{H}(\chi), \in, <^*_\chi)$. We call $\mathrm{cd}_n(M)$ the n-code of M or a code of M.

Also for every $n > 0$ there are f_n, g_n (definable in $(\mathscr{H}(\chi), \in, <^*_\chi)$), two place functions from κ^{+n} to κ^{+n} such that for $\alpha < \kappa^{+n}$ if $\alpha \geq \kappa^{+(n-1)}$ then:

$\oplus_2$ $\{f_n(\alpha, i) : i < \kappa^{+(n-1)}\} = \alpha$ and $i < \kappa^{+(n-1)} \Rightarrow g_n(\alpha, f_n(\alpha, i)) = i$.

For any $n \geq 3$, let

$\oplus_3$ $\mathscr{S}_n$ is the set of the sets A such that:

$\boxplus_1$ $A \subseteq \kappa^{+n}$ and $|A| = \kappa^{++}$,

$\bullet_2$ $\{\kappa^{+\ell} + i \colon i < \omega, \ell \leq n\} \cup \kappa^{++}$, $\kappa^{++} \subseteq A$,

- $\bullet_3$ letting $\delta_\ell(A) = \sup(A \cap \kappa^{+\ell})$, (for $\ell = 3, ..., n$), we have: $\delta_\ell(A) > \kappa^{+(\ell-1)}$ as $\kappa^{+\ell} + 1 \in A$,
- $\bullet_4$ $\mathrm{cf}(\delta_\ell(A)) = \aleph_0$
- $\bullet_5$ $\ell \le n \Rightarrow C_{\delta_\ell(A)} \subseteq A$,
- $\bullet_6$ A is the closure of $\{i : i < \kappa^{++}\} \cup \bigcup\limits_{\ell=3}^{n} C_{\delta_\ell(A)}$ under the functions $f_\ell, g_\ell, (\ell = 3, ..., n)$.

Note that if $A \in \mathscr{S}_n$, then n can be reconstructed from A.

We can prove by induction on $n \ge 3$ that:

$\oplus_4$ for every $x \in \mathscr{H}(\chi)$ there is a sequence $\langle M_m : m < \omega\rangle$, such that $M_m \prec (\mathscr{H}(\chi), \in, <_\chi), \|M_m\| = \kappa^{++}, \kappa^{++} + 1 \subseteq M_0, x \in M_0, M_m \prec M_{m+1}, M_m \in M_{m+1}$ (hence $\mathrm{cd}(M_m) \in M_{m+1}$) and

$$\bigcup_{m<\omega} M_m \cap \kappa^{+n} \in \mathscr{S}_n.$$

Hence for $n \ge n_0$ we know that $\diamondsuit_{\mathscr{S}_n}$ holds (see [Shed, 4.5(2)=Ld14]), in fact

$\boxplus_1$ for $n \ge n_0$ there is $\bar{N}_n$ such that:

- (a) $\bar{N}_n = \langle N_A : A \in \mathscr{S}_n\rangle$,
- (b) N_A a model with universe A,
- (c) if (α) then $(\beta) + (\gamma)$, where:
  - (α) N is a model with universe κ^{+n} and vocabulary $\tau(N)$ of cardinality $\le \mu$ included in $\mathscr{H}(\mu^*)$ and satisfying $<_*$ is a member of $\tau(N)$, the vocabulary of N, $<_*^N = < \restriction N$,
  - (β) the set $\mathscr{S}_n[N] = \{A \in \mathscr{S}_n : N_A = N \restriction A\}$ is a stationary subset of $[\kappa^{+n}]^{\le \kappa^{++}}$,
  - (γ) $N_A = \bigcup\limits_{\ell<\omega} N_A^\ell$, where for each ℓ, some code α_A^ℓ of N_A^ℓ, belongs to N_A and $N_A^\ell \prec N_A$.

By [Shed, 1.19=La54] there are $\langle N_A^\eta : \eta \in \mathscr{T}_A\rangle$ for $A \in \mathscr{S}_n$ such that:

$\boxplus_2$
- (a) $\mathscr{T}_A \subseteq {}^{\omega>}(\kappa^{++}), \mathscr{T}_A$ closed under initial segments, $\langle\rangle \in \mathscr{T}_A, \eta \in \mathscr{T}_A \Rightarrow (\exists^{\kappa^{++}} \alpha)[\eta\char`^\langle\alpha\rangle \in \mathscr{T}_A]$,
- (b) if $\eta \in \mathscr{T}_A$ then $N_A^\eta \prec N_A, \eta \in N_A^\eta$,
- (c) N_A^η countable,
- (d) $N_A^\eta \cap \kappa = N_A^{<>} \cap \kappa$,
- (e) $N_A^\eta \cap N_A^\nu = N_A^{\eta \cap \nu}$,
- (f) $[\eta \ne \nu \in \mathscr{T}_A \Rightarrow N_A^\eta \ne N_A^\nu]$ and $[\neg(\eta \trianglelefteq \nu) \Rightarrow \eta \notin N_A^\nu]$,
- (g) $\{\alpha_A^\ell : \ell < \omega\} \cup \bigcup\limits_{\ell=3}^{n} C_{\delta_\ell(A)} \subseteq N_A^{<>}$, recalling that α_A^ℓ is from $\boxplus_1$(c)(γ),
- (h) $\eta \triangleleft \nu \Rightarrow N_A^\eta \cap \kappa^{++}$ is an initial segment of $N_A^\nu \cap \kappa^{++}$.

$\boxplus_3$
- (a) We let $N_A^\eta = \bigcup\limits_{\ell<\omega} N_A^{\eta\restriction\ell}$ when $\eta \in \lim(\mathscr{T}_A)$.
- (b) Without loss of generality, if N_A, N_B are isomorphic then $\mathscr{T}_A = \mathscr{T}_B$ and the (unique) isomorphisms from N_A onto N_B carry N_A^η to N_B^η for each $\eta \in \mathscr{T}_A$.

(c) For $\nu \in \lim(\mathscr{T}_A)$, let $\eta^\nu_A \in {}^\omega(N^\nu_A)$ just list exactly the members of N^ν_A and satisfies $\alpha^\ell_A = \eta^\nu_A(3\ell)$ (for $\ell < \omega$)[9].

Let

$$\langle S^n_{<\gamma_3,\gamma_4,\dots,\gamma_n>} : n < \omega \text{ and } \ell \in \{3,\dots,n\} \Rightarrow \gamma_\ell < \kappa^{+3}\rangle$$

be a sequence of stationary subsets of $\{\delta < \kappa^{++} : \mathrm{cf}(\delta) = \aleph_0\}$, any two have a bounded intersection (exist, see [Shed, 4.1=Ld4] (which prove more))[10].

Fact 2.27. We have:

$\boxplus_4$ We can choose $\bar{\mathscr{S}}$, $\bar{N}$ such that:
- (a) $\bar{\mathscr{S}} = \langle \mathscr{S}_n : n \in [n_0, \omega)\rangle$, where n_0 was chosen applying $(*)_1$,
- (b) $\bar{\mathscr{S}}_n = \langle \mathscr{S}_{n,\zeta} : \zeta < \kappa^{+n}\rangle$ (for $n \geq n_0$) is a partition of $\mathscr{S}_n$.
- (c) $\bar{N} = \langle \bar{N}^n : n \in [n_0, n\rangle$, where $\bar{N}^n = \langle N_A : A \in \mathscr{S}_n\rangle$,
- (d) for each n, ζ the sequence $\langle N_A : A \in \mathscr{S}_{n,\zeta}\rangle$ is a diamond sequence.

Proof. E.g. let $P_* \in \mathscr{H}(\mu^+)$ serve as a unary predicate and for every $\zeta < \kappa^{+n}$ let $\mathscr{S}'_{n,\zeta} = \{A \in \mathscr{S}_n : P \in \tau(N_A) \text{ and } P^{N_A}_* = \{\zeta\}\}$ and for $A \in \mathscr{S}'_{n,\zeta}$ let $N'_A = N_A{\restriction}(\tau(N_A))\backslash\{P_*\}$; renaming the vocabularies and adding $\mathscr{S}_n \backslash \cup\{\mathscr{S}'_{n,\zeta} : \zeta < \lambda\}$ to $\mathscr{S}'_{n,\zeta}$, we can finish proving 2.27. $\square_{2.27}$

Fact 2.28. If $A \in \mathscr{S}_{n,\zeta}$ then $\mathrm{otp}(N_A \cap \kappa^{+\ell}) < \kappa^{+3}$.

Proof. This holds because $\|N_A\| = \kappa^{++}$ as $A \in \mathscr{S}_n$, see its definition. $\square_{2.28}$

Now, for $\zeta \in [\kappa^{+(n-1)}, \kappa^{+n})$, where $n > n_0$, let:

$\oplus$ $I_\zeta = {}^{\omega>}\lambda \cup \{\eta^\nu_A : A \in \mathscr{S}_{n,\zeta} \text{ and } \nu \in Y^n_{A,\langle \mathrm{otp}(N_A\cap\kappa^{+\ell}):3\leq\ell\leq n\rangle}\}$.

where (recall that $\lim(\mathscr{T}_A) := \{\eta \in \omega^{\mathrm{Ord}} :\ \text{if } n < \omega \text{ then } \eta{\restriction}n \in \mathscr{T}_A\}$):

$$Y^n_{A,\gamma_3,\dots,\gamma_n} = \{\nu : \nu \in \lim(\mathscr{T}_A),\ \text{increasing with limit in } S^n_{<\gamma_3,\dots,\gamma_n>}\} :$$

$\boxplus_5$ We shall prove that the sequence $\langle I_\zeta : \kappa^{+n_0} \leq \zeta < \lambda\rangle$ is as required; this suffices.

$\boxplus_6$ For this suppose that:
- (a) $x \in \mathscr{H}(\chi)$, where χ regular large enough,
- (b) $\zeta \in [\kappa^{+n_0}, \lambda)$,
- (c) $J_\zeta = \sum\{I_\xi : \xi \neq \zeta \text{ and } \xi \in [\kappa^{+n_0}, \lambda)\}$, and
- (d) let n be such that $\kappa^{+(n-1)} \leq \zeta < \kappa^{+n}$. Let $M \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ have cardinality κ^{+n} and be such that $\kappa^{+n} + 1 \subseteq M, \{x, I_\zeta, J_\zeta, \mu\} \in M$ and $\langle C_\delta : \delta < \lambda\rangle, \langle \mathrm{cd}_n, f_n, g_n : n < \omega\rangle$ belong to M.

$\boxplus_7$
- (a) Let h be a one to one function from $|M|$ onto κ^{+n},
- (b) let N^+ be the model with universe κ^{+n} and all relations and functions on κ^{+n} definable (with no parameters) in (M, h),
- (c) In particular we may use F, F_1, F_2 such that $x = \langle y, z\rangle \in M \Rightarrow F^{N^+}(h(y), h(z)) = h(x), F^{N^+}_1(h(x)) = h(y), F^{N^+}_2(h(x)) = h(z)$.

So,

$\boxplus_8$
- (a) For some $A \in \mathscr{S}_{n,\zeta}$ we have $N_A \prec N^+$.

---

[9] Actually it suffices if it lists $\cup\{C_{\delta_\ell(A)} : 3 \leq \ell < n\} \cup \{\alpha^\ell_A : \ell < \omega\} \cup \{\nu(\ell) : \ell < \omega\}$; this change is needed for 2.33.

[10] We can assume $\cup\{S^n_{<\gamma_\ell:\ell=3,\dots,n>} : \gamma_\ell < \kappa^{+2}\}$ for $n < \omega$ are pairwise disjoint.

(b) We shall show that for some $\nu \in Y^n_{A,\langle \mathrm{otp}(N_A \cap \kappa^{+\ell}):3 \le \ell \le n\rangle}$ we have: η^ν_A, N^ν_A, are as required.

Let M_A, M^ν_A for ($\nu \in \lim(\mathscr{T}_A)$) be the Skolem Hull of N_A, N^ν_A respectively in (M, h). Note: $|M_A| \cap \kappa^{+n} = |N_A|, |M^\eta_A| \cap \kappa^{+n} = |N^\eta_A|$. For $\nu \in \lim(\mathscr{T}_A)$, let Z_ν be the set of triples (ξ, B, ρ) such that for some $m = m(\xi) > n_0 : \xi \neq \zeta, B \in \mathscr{S}_{m,\xi}, \xi \in [\kappa^{+(m-1)}, \kappa^{+m}), \rho \in \lim(\mathscr{T}_B)$ and $< \xi > \hat{\ } \eta^\rho_B \notin M^\nu_A$ but $\{(< \xi > \hat{\ } \eta^\rho_B) \restriction \ell : \ell < \omega\} \subseteq M^\nu_A$.

We now make some important observations:

Fact 2.29. We have:

$(*)_1$ if $(\xi, B, \rho) \in Z_\nu, \xi \in [\kappa^{+(n-1)}, \kappa^{+n})$ (i.e. $m(\xi) = n$) then $\mathrm{otp}(N_B \cap \kappa^{+\ell}) \le \mathrm{otp}(N_A \cap \kappa^{+\ell})$ for $\ell \in [3, n]$; and for at least one ℓ the inequality is strict and $B \subseteq A$.

Proof. As $C_{\delta_\ell(B)} \subseteq \mathrm{Rang}(\eta^\rho_B)$ we have $\bigcup\limits_{\ell=3}^{n} C_{\delta_\ell(B)} \subseteq N^\nu_A \subseteq A$, hence (see the definition of $\mathscr{S}_n$, using the $\langle f_\ell, g_\ell : \ell = 3, \dots, n-1\rangle$ we get $B \subseteq A$ so the inequality "$\mathrm{otp}(N_B \cap \kappa^{+\ell}) \le \mathrm{otp}(N_A \cap \kappa^{+\ell})$" follows; but necessarily $B \neq A$ (as $\langle \xi \rangle \hat{\ } \eta^\rho_B \in J_\zeta$ and $\mathscr{S}_{n,\xi} \cap \mathscr{S}_{n,\zeta} = \emptyset$) and if $\neg(\exists \ell)(\delta_\ell(B) < \delta_\ell(A))$ then we have: $\kappa^{++} \subseteq B$, and for $\ell \le$ and $n \ge 3$

$$\sup(B \cap \kappa^{+\ell}) = \sup(A \cap \kappa^{+\ell}) = \sup(A \cap B \cap \kappa^{+\ell});$$

now use the choice of f_n, g_n. You can show, using $B \subseteq A$, by induction on $\ell \le n$ that $B \cap \kappa^{+\ell} = A \cap \kappa^{+\ell}$; for $\ell = n$ we get a contradiction. $\square_{2.29}$

Next, we have:

Fact 2.30. $(*)_2$ if $(\xi, B, \rho) \in Z_\nu$ then $\{\delta_\ell(B) : 3 \le \ell \le m(\xi)\}$ is included in the closure of $|M^\nu_A|$ in the order topology, which is a countable set of ordinals; also $B \subseteq M_A$.

Proof. Similar argument; for $B \subseteq M_A$ use $\eta^\rho_B(3\ell) = \alpha^\ell_B$. $\square_{2.30}$

Fact 2.31. $(*)_3$ So if $Y \subseteq \lim(\mathscr{T}_A)$ is closed with countable density, and no isolated points, then for some $\nu \in Y$ (really for a co-countable set of ν's):

$\otimes$ $(\xi, B, \rho) \in Z_\nu \Rightarrow (\exists k)[\{\alpha^\ell_B : \ell < \omega\} \subseteq M^{\nu \restriction k}_A]$.

Proof. Why? The point is that $\{(\xi, B) : (\exists \nu \in Y)(\exists \rho)[(\xi, B, \rho) \in Z_\nu]\}$ is countable (as for each $(\xi, B, \rho) \in Z_\nu$ the ordinals $\delta_\ell(B), 3 \le \ell \le m(\xi)$, are all accumulation points of $\bigcup\limits_{\nu \in Y} M^\nu_A$, which is countable and $\langle \delta_\ell(B) : 3 \le \ell \le m(\xi)\rangle$ determine B hence ξ, and for each such (ξ, B) the set of $\nu \in Y$ for which $\otimes$ fails is at most a singleton, using clause (e) above and the last clause in the definition of Z_ν. $\square_{2.31}$

Lastly,

Fact 2.32. We have:

$(*)_4$ C^* is a club of κ^{++}, where C^* is the set of $\delta < \kappa^{++}$ such that $\delta \in C'$ where:

- $\{\alpha^\ell_B : \ell < \omega\} \subseteq M^\nu_A$ for some $\nu \in {}^{\omega>}\delta$, and $m < \omega$ and $B \in \mathscr{S}_{m,\xi}$,
- C' is the $<^*_\chi$-first club disjoint to $S^m_{\langle \mathrm{otp}(A \cap \kappa^{+\ell}):\, 4 \le \ell \le n\rangle} \cap S^m_{\langle \mathrm{otp}(B \cap \kappa^{+\ell}):\, 3 \le \ell \le m\rangle}$.

Proof. Why? Note that $\kappa^\mu = \kappa$ hence $(\kappa^+)^{\aleph_0} = \kappa^+$, so the number of possible B's for each $\nu \in {}^{\omega>}(\kappa^{++})$ is $\le \|M^\nu_A\|^{\aleph_0} \le \kappa^+$, and use diagonal intersection. $\square_{2.32}$

$(*)_5$ if $\nu \in \lim(\mathscr{T}_A), \nu$ increasing with limit $\delta \in C^* \cap S^n_{\langle \mathrm{otp}(A \cap \kappa^{+\ell}) : 3 \le \ell \le n \rangle}$ then

$$(\xi, B, \rho) \in Z_\nu \Rightarrow \neg \exists k[\{\alpha^\ell_B : \ell < \omega\} \subseteq M^{\nu \restriction k}_A].$$

[Why? Easy by the choice of C^*.]

Together we finish: by $(*)_4$, we can find δ as in $(*)_5$ and hence we can find a perfect set $Y \subseteq \mathscr{T}_A$ of sequences with limit δ; now $(*)_3, (*)_5$ give contradictory conclusions (alternatively, see the proof of 2.33). $\square_{2.25}$

Claim 2.33. *In fact in 2.25 we can get (under the assumptions of 2.25) that* K^ω_{tr} *has the full* $(\lambda, \lambda, \mu, \mu)$*-super*$^{6^+}$*-bigness property (and moreover in 1.5(F) in clause (ii) there we get "*$\mu + 1 \subseteq M_n$*" and* $[M_n]^{\aleph_0} \subseteq M_n$ *which implies (vii)* $\Leftrightarrow$ *(vii)*$^+$ *there).*

Proof. For this, we have to make several changes in the proof of 2.25. What more do we prove? we get $\mu + 1 \subseteq M_0$ and $[M_n]^{\aleph_0} \subseteq M_n$. Without loss of generality $\kappa^\mu = \kappa, \mu^{\aleph_0} = \mu$.

Considering models N with universe κ^{+n} we demand that $P_{\mathrm{or}}, <_{\mathrm{or}}$ belong to $\tau(N)$ where we let $P_{\mathrm{or}}, <_{\mathrm{or}}$ be fixed one and two place predicates and we demand that $<^N_{\mathrm{or}}$ is a well-ordering of the subset P^N_{or} of κ^{+n}. Parallel restriction applies to N_A for $A \in \mathscr{S}_n$. Latter having M and h, we demand $P^{N^+}_{\mathrm{or}} = \{h(\alpha) : \alpha \in M$ an ordinal$\}, <^{N+}_{\mathrm{or}} = \{(h(\alpha), h(\beta)) : \alpha < \beta$ are ordinals from $M\}$. For any $A \in \mathscr{S}_n$, we choose a two place function g_A such that:

$\oplus$ for every $\alpha \in P^{N_A}_{\mathrm{or}}$, for some regular [11] $\theta \le \kappa^{++}$
- (i) $(\forall \alpha < \theta)(\forall \beta < \gamma)[g_A(\alpha, \beta) <_{\mathrm{or}} g_A(\alpha, \gamma)]$
- (ii) $(\forall \beta)(\exists \gamma)[\beta <_{\mathrm{or}} \alpha \to \gamma < \theta \,\&\, \beta \le_{\mathrm{or}} g_A(\alpha, \gamma)]$
- (iii) $(\forall \beta)[\theta \le \beta \Rightarrow g_A(\alpha, \beta) = \alpha]$.

Of course we demand that if $N_A \cong N_B, A, B \in \mathscr{S}_n$ then the (unique) isomorphic maps g_A to g_B.

When we choose M, we demand (note that: if $\|M\|^{\aleph_0} > \|M\|$):

$$[a \subseteq M \,\&\, \|M\|^{|a|} = \|M\| \Rightarrow a \in M].$$

When we choose $\langle N^\eta_A : \eta \in \mathscr{T}_A \rangle$ we replace condition (c) in $\boxplus_2$ by

$(c)''$ N^η_A has cardinality μ and include $\mu + 1$ and

$$[a \subseteq N^\eta_A \,\&\, \|N^\eta_A\|^{|a|} = \|N^\eta_A\| \Rightarrow a \subseteq N^\eta_A]$$

(the partition theorem on trees still holds) and add, i.e. we now use [Shed, 1.16=La48]

(i) if $\eta \triangleleft \nu$ are from $\mathscr{T}_A, <^{N_A}_{\mathrm{or}}$ is a well ordering of $P^{N_A}_{\mathrm{or}} (\subseteq A)$ then for any $x \in P^{N_A}_{\mathrm{or}} \cap N^\eta_A$:

(α) if $\kappa^{++} > \mathrm{cf}(\{y : y \in P^{N_A}_{\mathrm{or}}, y <^{N_A}_{\mathrm{or}} x\}, <^{N_A}_{\mathrm{or}})$ <u>then</u>

$$N^\eta_A \cap \{y : y \in P^{N_A}_{\mathrm{or}}, y <^{N_A}_{\mathrm{or}} x\}$$

is an unbounded subset of

$$(\{y : y \in P^{N_A}_{\mathrm{or}}, y \in N^\nu_A, y <^{N_A}_{\mathrm{or}} x\}, <^{N_A}_{\mathrm{or}})$$

---

[11] or $\theta = 1$ or $\theta = 0$, cases which still fit.

(β) if $\kappa^{++} = \mathrm{cf}(\{y : y \in P^{N_A}_{\mathrm{or}}, y <^{N_A}_{\mathrm{or}} x\}, <^{N_A}_{\mathrm{or}})$ then for any $y \in P^{N_A}_{\mathrm{or}}, y <^{N_A}_{\mathrm{or}} x$, for some $\alpha < \kappa^{++}$ we have: $\eta \triangleleft \rho \in \mathscr{T}_A \,\&\, \rho(\ell g(\eta)) > \alpha \,\&\, y^* \in N_A \cap P^{N_A}_{\mathrm{or}} \,\&\, y^* <^{N_A}_{\mathrm{or}} x \,\&\, (\forall z)[z \in N^\eta_A \,\&\, z <^{N_A}_{\mathrm{or}} x \to z <^{N_A}_{\mathrm{or}} y^*] \Rightarrow y <^{N_A}_{\mathrm{or}} y^*$.

Note that as $<^{N_A}_{\mathrm{or}}$ well order $P^{N_A}_{\mathrm{or}}$, this is possible — see [Shed, 1.16=La48] and apply it to (M_A, g).

But now we cannot demand "η^ν_A list the members of N^ν_A"; so we just require

$\boxplus$ (a) $\alpha^\ell_A = \eta^\nu_A(3\ell)$,

(b) $\langle \eta^\nu_A(3\ell+1) : \ell < \omega\rangle$ list $\bigcup\limits_{\ell=3}^{n} C_{\delta_\ell(A)}$ and

(c) $\langle \eta^\nu_A(3\ell+2) : \ell < \omega\rangle$ is $\langle \nu(\ell) : \ell < \omega\rangle$.

Note that this holds for all $\nu \in \mathscr{T}_A$.

This, of course, "kills" $(*)_3$ in the proof of 2.25. Now if $(\xi, B, \rho) \in Z_\nu$, for $\ell = 3, \dots, m(\xi)$ define $\beta_\ell = \sup[\kappa^{+\ell} \cap \mathrm{rang}(\rho)]$, and define $\gamma[\beta_\ell] = \min(M^\nu_A \cap \lambda \setminus \beta)$, so for some $k(*) < \omega$ we have $\bigwedge\limits_{\ell \in [3, m(\xi)]} \gamma[\beta_\ell] \in M^{\nu \restriction k(*)}_A$. So by condition (i) above for each $\ell \in [3, m(\xi)]$, either $\circledast^1_\ell$ holds or $\circledast^2_\ell$ holds where:

$\circledast^1_\ell$ $\mathrm{cf}(\gamma[\beta_\ell]) < \kappa^{++}, \sup[\gamma[\beta_\ell] \cap M^{\nu\restriction k(*)}_A] = \sup[\gamma[\beta_\ell] \cap M^{\eta'}_A]$ whenever $\nu \restriction k(*) \triangleleft \eta' \in \mathscr{T}_A \cup \lim(\mathscr{T}_A)$

$\circledast^2_\ell$ $\kappa^{++} = \mathrm{cf}(\gamma[\beta_\ell])$ and there is $h_{\gamma[\beta_\ell]} : \kappa^{++} \to \gamma(\beta)$ increasing continuous with limit $\gamma[\beta_\ell]$ such that
- $\nu \restriction k(\beta) \triangleleft \eta' \in \lim(\mathscr{T}_A) \Rightarrow \sup(N^{\eta'}_A \cap \gamma[\beta_\ell])$
- $\sup(M^{\eta'}_A \cap \mathrm{Rang}(h_{\gamma[\beta_\ell]})) = h(\sup[M^{\eta'}_A \cap \kappa^{++}])$.

As $\mu \le \kappa$, we can finish easily: we can find a club

$$C' = \{\delta \in C^* : \text{ if } \nu \in {}^{\omega >}\delta, \ell \in [3, \omega) \text{ and } \gamma \in N^\nu_A \text{ then } \delta \text{ is closed under } h_\gamma\}.$$

of κ^{++} and choose $\delta \in C'$. $\square_{2.33}$

Theorem 2.34. *1) If $\lambda > \mu$ then K^ω_{tr} has the full $(\lambda, \lambda, \mu, \aleph_0)$-super-bigness property and also the $(2^\lambda, \lambda, \mu, \aleph_0)$-super bigness property.*
2) Similarly replacing $\aleph_0$ by μ.

Proof. We prove both parts of 2.34 together. The first phase implies the second by 1.9(1) hence we concentrate on the first phrase. This will follow by combining the previous Lemmas. We shall use all the time 1.7(1) to get "our super", the one from Definition 1.1, i.e. super^{4^+}. If λ is regular, use 1.16(1) so assume λ singular; if $(\exists \mu_1)[\mu \le \mu_1 = \mu_1^{\aleph_0} < \lambda \le 2^{\mu_1}]$ use 1.16(2), for part (2) note "(even the full ...") and if $(\exists \theta)[\theta < \lambda \le \theta^{\aleph_0}]$ let χ be minimal such that $\lambda \le \chi^{\aleph_0}$; so $< \lambda$ hence $\mu + \chi < \lambda$, but λ is a limit cardinal so $\mu^+ + \chi^{++} \le \lambda$ and use 2.1. So assume the last two cases fail, hence λ is singular strong limit. If $\mathrm{cf}(\lambda) > \aleph_0$ use 1.16(3), if $\mathrm{cf}(\lambda) = \aleph_0, \lambda = \aleph_\delta, \delta$ divisible by ω^2, choose $\theta, \mu < \theta < \lambda, \mathrm{cf}(\theta) = \aleph_0$ and apply 2.18, ($\langle \mathfrak{a}_\epsilon : \epsilon < \mathrm{cf}(\lambda)\rangle$ exists by [Shed, 3.22=Lpcf.8]). The remaining case is $\lambda = \aleph_\delta = \aleph_{\alpha+\omega}$ strong limit and use 2.25 for part (1), use 2.33 for part (2). $\square_{2.34}$

§ 3. Applications and generalizations

Conclusion 3.1(1) (though not 3.1(2),(3)) tell us that unstable and unsuperstable has many models, and the proof use only a version of the definition from [Shei]. Theorem 3.3 tell us more in this direction but the proof of 3.3 in case $\lambda = \lambda(T), T_1 = T$ stable require knowledge of stability theory (and is not used later), this case appear as end-segment of the proof of 3.3, i.e. starting with the third paragraph of the proof of 3.3 and with 3.25). We restart in 3.26 resuming our investigations of bigness properties and then deal with abelian separable $\dot{p}$-group.

§ 3(A). **The Many pairwise Unembeddable Models.**

Conclusion 3.1. *1) If $T \subseteq T_1$ are complete first order theories and $\lambda > |T_1|$ then $\dot{I}\dot{E}(\lambda, T_1, T) = 2^\lambda$ whenever T is unsuperstable.*
2) If $\lambda > \mu$ then $K^\omega_{\rm tr}$ has the full strong $(\lambda, \lambda, \mu, \aleph_0)$-bigness property and $(2^\lambda, \lambda, \mu, \aleph_0)$-bigness property (see Definition [Shei, 2.5(3)=L2.3(3)]*).*
3) If $\Phi, \langle \varphi_n(x, \bar{y}) : n < \omega \rangle$ are as in [Shei, 1.11=Lb17(2)]*, and $\lambda > |\tau(\Phi)|$, see 3.2 then we can find $I_\alpha \in K^\omega_{\rm tr}$ of cardinality λ for $\alpha < 2^\lambda$ such that letting $M_\alpha = {\rm EM}_{\tau(T)}(I_\alpha, \Phi)$, for any $\alpha \neq \beta$, there is no function from M_α into M_β preserving the $\pm\varphi_n$.*

Remark 3.2. Recall that $\tau(\Phi)$ is the vocabulary of Φ, that is, for a linear order I, the Ehrenfeucht-Mostoswki model, ${\rm EM}(I, \Phi)$ has the vocabulary $\tau(\Phi)$. Also for $\tau \subseteq \tau(\Phi)$, ${\rm EM}_\tau(I, \Phi)$ is the τ-reduct of ${\rm EM}(I, \Phi)$, recall $\tau = \tau(T)$ is the vocabulary of the theory T.

Proof. 1) Let Φ be a template proper for $K^\omega_{\rm tr}$ as in [Shei, 1.11=Lb17(2)]; i.e. $|\tau(\Phi)| = |T_1| + \aleph_0, \tau_{T_1} \subseteq \tau(\Phi)$, every ${\rm EM}(I, \Phi)$ is a model of T_1 and for some first order formulas $\varphi_n(x, \bar{y}_n)$ of $\mathbb{L}_{\omega,\omega}(\tau_T)$ for $s \in P^I_n, t \in P^I_\omega, I \in K^\omega_{\rm tr}$ we have ${\rm EM}(I, \Phi) \vDash \varphi(a_t, \bar{a}_s)$ iff $I \models s \lhd t$. By 3.1(2) (which is proved below) and the definition, the conclusion follows reading the definition of $\dot{I}\dot{E}$ (see [Shei, 1.4=L1.4new]) and the definition of the bigness property.
2) By 2.34 and 1.9(2).
3) Included in the proof above. $\square_{3.1}$

Theorem 3.3. *Suppose T is (a first-order, complete) unsuperstable theory and $\lambda \geq \lambda(T) + \aleph_1$ (see below 3.4(1)).*
1) T has 2^λ pairwise non-isomorphic strongly $\aleph_\epsilon$-saturated models of cardinality λ, see 3.4(2),(3).
2) If in addition T is stable or $\lambda > \lambda(T) + \aleph_0$, then T has $2^\lambda, \aleph_\epsilon$-saturated (see 3.4(21)) models of power λ no one elementarily embeddable into another.
3) We can in part (2) weaken the assumption to $\lambda > |T| + \aleph_0$ but then have to weaken the conclusion to "strongly $\aleph_0$-homogeneous (see 3.4(3) below) models of cardinality λ (omitting the "$\aleph_\epsilon$-saturated"; interesting when $\lambda = |T| + \aleph_1, T$ stable).
4) If $T \subseteq T_1, T_1$ first order, we can demand above that the models are in ${\rm PC}_{\tau(T)}(T_1, T)$, that is are reducts of models of T_1, provided that: in 3.3(1)+(2) we demand $\lambda > \lambda(T) + |T_1| + \aleph_0$, in 3.3(3) we demand $\lambda > |T_1| + \aleph_0$.

Remark 3.4. 0) In the notion of "M is $\aleph_\varepsilon$-saturated" the ε is not a variable, it try to say "a little more than being $\aleph_0$-saturated"; see 3.4(2) (see [She90, IV]).

1) $\lambda(T)$ can be defined as the minimal cardinality of an $\aleph_\varepsilon$-saturated model of T, see (2) below.

2)

(a) M is $\aleph_\epsilon$-saturated when it is $\mathbf{F}^a_{\aleph_0}$-saturated, where:

(b) A model M is $\mathbf{F}^a_\kappa$-saturated <u>if</u> for every $N, M \prec N$ and $A \subseteq M$ of cardinality $< \kappa$ and (finite) $\bar b \in N$ there is $\bar b' \in M$ realizing $\{\vartheta(\bar x, \bar b, \bar a) \colon \vartheta(\bar x, \bar y, \bar z)$ is a first-order formula and in N, $\bar a \in {}^{\lg(\bar z)}A$, and the formula $\vartheta(\bar x, \bar y, \bar a)$ is an equivalence relation with finitely many equivalent classes$\}$.

3) M is strongly $\aleph_\varepsilon$-saturated <u>if</u> in addition it is strongly $\aleph_0$-homogeneous which means that for any $\bar a, \bar b \in {}^{\omega>}M$ realizing the same type, there is an automorphism of M mapping $\bar a$ to $\bar b$.
4) The restrictions in 3.3 are reasonable as, e.g. by [She80]: it is consistent with ZFC that for T the theory of dense linear orders (which is an unstable one) there is $T_1 \supseteq T$ (first order complete theory) of cardinality $\aleph_1$ such that for any models M_1, M_2 in $\mathrm{PC}(T_1, T)$ of cardinality $\aleph_1, M_1$ can be embeddable into M_2.
5) Recall $\mathfrak{C}^{\mathrm{eq}}$ is extending $\mathfrak{C}$ by giving names to equivalence classes, see [She90]. Let us say that $M^{\mathrm{eq}} \prec \mathfrak{C}^{\mathrm{eq}}$ is strongly$^+$-$\aleph_\varepsilon$-saturated if it is $\aleph_\varepsilon$-saturated and: for any finite $A, B \subseteq M^{\mathrm{eq}}$ and $(M^{\mathrm{eq}}, M^{\mathrm{eq}})$-elementary mapping $\mathbf{f}$ from $\mathrm{acl}(A, M^{\mathrm{eq}})$ onto $\mathrm{acl}(B, M^{\mathrm{eq}})$ there is an automorphism $\mathbf{f}^+$ of M^{eq} extending $\mathbf{f}$.
6) The reader may wonder if we can get in 3.3, models which are strongly$^+$ $\aleph_\varepsilon$-saturated.

Let $\lambda'(T)$ be the first $\lambda \geq \lambda(T)$ such that for any M^{eq}, A, B as above, the number of $\mathbf{f}$ as above is $\leq \lambda$.

Now we can in 3.3 demand the models to be strongly$^+$ $\aleph_\varepsilon$-saturated if $\lambda \geq \lambda'(T) + \aleph_1$ (or $\lambda > \lambda'(T) + \aleph_0$, as natural). The proof is essentially the same.
7) In fact the proof indicated in 3.4(6) is simpler and gives in some respect more information. We can easily prove:

$(*)_1$ if $A \subseteq \mathfrak{C}^{\mathrm{eq}}, |A| \leq \lambda$ then there is an $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}$ (see context 3.6, Definition 3.8),such that:
  - (i) $A_0[\mathscr{A}] = A$,
  - (ii) $\mathrm{lng}(\mathscr{A})$ is divisible by λ and $\mathrm{cf}(\mathrm{lng}(\mathscr{A})) \geq \kappa$
  - (iii) if $D \in \mathscr{P}$ and $i < \mathrm{lng}(\mathscr{A}), B_1 \subseteq A_i[\mathscr{A}], B_2 \subseteq D$ and $\mathbf{f}$ is an element mapping from $\mathrm{acl}(B_2, \mathfrak{C}^{\mathrm{eq}})$ onto $\mathrm{acl}(B_1, \mathfrak{C}^{\mathrm{eq}}), |B_1| < \kappa, |B_2| < \kappa$ then $\mathrm{lng}(\mathscr{A}) = \mathrm{otp}\{\beta < \mathrm{lng}\mathscr{A} \colon$ there is an elementary mapping $\mathbf{f}'$ from D onto $D_\beta[\mathscr{A}]$ extending $\mathbf{f}, B_\beta[\mathscr{A}] = B_1\}$

$(*)_2$ if $\mathscr{A}^1, \mathscr{A}^2$ are as in $(*)_1$, and $A_0[\mathscr{A}^1] = \emptyset = A_0[\mathscr{A}^2]$ then $A[\mathscr{A}^1], A[\mathscr{A}^2]$ are isomorphic $\mathbf{F}^a_\kappa$-saturated models (see 3.4(21)).

This replaces 3.9-3.16, (but use some of those proofs). After that, we can continue as in 3.18.

8) For the case $T_1 = T, \kappa = \mathrm{cf}(\kappa) \leq \kappa_r(T)$ we can replace in the proof $\aleph_\epsilon$-saturated by $\mathbf{F}^a_\kappa$-saturated, etc.

9) Recall that for a formula φ and statement **stat**, $\varphi^{\mathbf{if(stat)}}$ mean φ when the statement is true and mean $\neg\varphi$, the negation of φ when the statement is false.

Proof. Let $\tau = \tau_T$.

First assume T is unstable; note:

Fact 3.5. There is a template Φ, proper for linear orders, $|\tau_\Phi| = \lambda(T)$ such that every model M of the form $\mathrm{EM}_\tau(I, \Phi)$ is an $\aleph_\epsilon$-saturated model of T and $M \models \varphi[\bar a_s, \bar a_t]$ iff $s <_I t$ for $s, t \in I$, where I is a linear order.

Proof. Apply [Shei, 1.26=L1.24new] as follows. As T is unstable there are $\varphi(\bar{x},\bar{y}), \bar{a}_\ell$ ($\ell<\omega$) and M such that M is a model of $T, \bar{a}_\ell \in M, n = \lg(\bar{x}) = \lg(\bar{y}) = \lg(\bar{a}_\ell)$ and $M \models \varphi(\bar{a}_\ell,\bar{a}_k)^{\mathbf{if}(\ell<k)}$ (recall 3.3(9)). We can also find a vocabulary $\tau_1, \tau \subseteq \tau_1, |\tau_1| = \lambda(T)$ and $\psi \in \mathbb{L}_{|T|^+,\aleph_0}(\tau_1)$ such that a model of T is $\aleph_\epsilon$-saturated iff it can be expanded to a model of ψ.

For every λ we can find a strongly $|T|^+$-saturated model M_λ of T and $\bar{a}^\lambda_\alpha \in M_\lambda$ such that $M_\lambda \models \varphi(a^\lambda_\alpha, a^\lambda_\beta) \Leftrightarrow (\alpha<\beta)$, hence there is an expansion M^+_λ of M_λ to a model of ψ. Lastly, check that [Shei, 1.26=L1.24new] gives the desired conclusion. $\square_{3.5}$

Continuing the proof of 3.3:

Now part (1) (of 3.3) holds by [Shei, 3.19=L3.9] (with M_I being $\mathrm{EM}(I,\Phi)$, it is as required in [Shei, 3.19=L3.9A] by [Shei, 3.8=L3.4]).

Also part (2) (of 3.3) holds by 3.18 (interpreting $I \in K^\omega_{\mathrm{tr}}$ as a linear order as in [Shei, 2.4=L2.2]) noting that we have $\lambda > \lambda(T) + \aleph_0 = |\tau_\Phi|$ as we are assuming T is unstable. The proof of part (3) is similar, replacing τ_Φ by τ' of cardinality $\lambda + \aleph_1, |T_1| + \aleph_1$. Lastly for part (4) without loss of generality every model $\mathrm{EM}_\tau(I,\Phi)$ is a reduct of a model of T_1, so we are done by 3.1(3).

So without loss of generality, T is stable. As T is unsuperstable, by [She87, Proof of 2.1] (or a proof similar to the first paragraph),

- $(*)$ There is a template Φ proper for $K^\omega_{\mathrm{tr}}, |\tau_\Phi| = \lambda(T)$ as in [Shei, 1.11(2)=L1.8(2)] such that every $\mathrm{EM}_\tau(I,\Phi)$ is strongly $\aleph_\epsilon$-saturated.

If $\lambda > \lambda(T)$, note that 3.3(1) follows by 3.1(3) and 3.3(3) by decreasing τ_Φ.

In all those proofs we can restrict ourselves to models of T which are reducts of models of T_1, i.e. demand that for any suitable $I \in K^\omega_{\mathrm{tr}}$, the model $\mathrm{EM}(I,\Phi)$ is a model of T_1 so part (4) follows. We are left with part (2) the case T is stable, and the proof is restricted to elementary classes: the proof needs some knowledge of forking, but it is not used later, so a reader can skip it. We also use the notation of [She90].

- $\boxplus_1$ Let $\varphi_n(\bar{x},\bar{y}_n)$ (for $n<\omega$), $\bar{a}_\eta (\eta \in {}^{\omega\ge}\lambda)$ witness unsuperstability, i.e. be as in [She78, Ch.III,§3], so there is $\langle \bar{a}_\eta : \eta \in {}^{\omega>}\lambda\rangle$ which is a non-forking tree (that is $\eta \in {}^{\omega>}\lambda \Rightarrow \mathrm{tp}(\bar{a}_\eta, \cup\{\bar{a}_\nu : \neg(\eta \trianglelefteq \nu), \nu \in {}^{\omega>}\lambda\})$ does not fork over $\cup\{\bar{a}_{\eta\restriction\ell} : \ell < \lg(\eta)\}$), and for $\eta \in {}^\omega\lambda, \mathrm{tp}(\bar{a}_\eta, \cup\{\bar{a}_\nu : \nu \in {}^{\omega>}\lambda\})$ does not fork over $\bigcup\limits_{\ell<\omega} \bar{a}_{\eta\restriction\ell}$ and $\mathrm{tp}(\bar{a}_\eta, \bigcup\limits_{\ell<k} \bar{a}_{\eta\restriction\ell})$ forks over $\bigcup\limits_{\ell<k} \bar{a}_{\eta\restriction\ell}$ for $k<\omega$.

Let $I \subseteq {}^{\omega\ge}\lambda$ be closed under initial segments, $|I| = \lambda$ and we shall construct a model M_I. We work in $\mathfrak{C}^{\mathrm{eq}}$, so without loss of generality $\bar{a}_\eta = \langle a_\eta\rangle$ so the a_η's are pairwise distinct.

By induction on $\alpha < \lambda^+$ we choose $(\bar{A}^\alpha, \bar{f}^\alpha) \in \mathbf{K}_\alpha$ where

- $\boxplus_2$ $(\bar{A},\bar{f}) \in \mathbf{K}_\alpha$ <u>iff</u> $\bar{A} = \langle A_i : i \le \alpha\rangle$ and $\bar{f} = \langle f^i_{c,d} : c,d \in A_i, \mathrm{tp}(c,\emptyset) = \mathrm{tp}(d,\emptyset)$ and $i \le \alpha\rangle$ satisfies:
  - (A) $\bar{A} = \langle A_i : i \le \alpha\rangle$ is increasing continuous: $|A_i| = \lambda, A_i \subseteq \mathfrak{C}$
  - (B) $f^i_{c,d}$ is an elementary mapping, $f^i_{c,d}(c) = d, f^i_{d,c} = (f^i_{c,d})^{-1}$, and for $c,d \in A_j$ the sequence $\langle f^i_{c,d} \colon i \in [j,\alpha]\rangle$ is increasing continuous, and: if $c,d \in A_i$, but $\bigwedge\limits_{j<i}[\{c,d\} \nsubseteq A_j]$ then $\mathrm{Dom}(f^i_{c,d}) = \{c\}$
  - (C) for each i: either

(i) $A_{i+1} = A_i \cup \{a_i\}, \mathrm{tp}(a_i, A_i)$ does not fork over some finite subset B_i of A_i

or

(ii) for some $\mathbf{c}(i), \mathbf{d}(i) \in A_i$, (such that $\mathrm{tp}(\mathbf{c}(i), \emptyset) = \mathrm{tp}(\mathbf{d}(i), \emptyset)$) we have:

$$A_{i+1} = A_i \cup f^{i+1}_{\mathbf{c}(i),\mathbf{d}(i)}(A_i)$$

and

$$(\exists j < i)[\mathrm{Rang}(f^i_{\mathbf{c}(i),\mathbf{d}(i)}) = A_j] \vee [\mathrm{Dom}(f^i_{\mathbf{c}(i),\mathbf{d}(i)}) = \{\mathbf{c}(i)\}].$$

(D) for every $c, d \in A_{i+1}$ such that $\mathrm{tp}(c, \emptyset) = \mathrm{tp}(d, \emptyset)$:

(i) if $\{c, d\}$ is not a subset of A_i, then $\mathrm{Dom}(f^{i+1}_{c,d}) = \{c\}$

(ii) if $c, d \in A_i$, case (i) of (C) holds or case (ii) of (C) holds but $\langle c, d\rangle \notin \{\langle \mathbf{d}(i), \mathbf{d}(i)\rangle, \langle \mathbf{d}(i), \mathbf{c}(i)\rangle\}$, <u>then</u> $f^{i+1}_{c,d} = f^i_{c,d}$

(iii) if $c = \mathbf{c}(i), d = \mathbf{d}(i)$ and case (ii) of (C) holds, then $\mathrm{tp}(f^{i+1}_{\mathbf{c}(i),\mathbf{d}(i)}(A_i), A_i)$ does not fork over $\mathrm{Rang}(f^i_{\mathbf{c}(i),\mathbf{d}(i)})$ and $\mathrm{Dom}(f^{i+1}_{\mathbf{c}(i),\mathbf{d}(i)}) = A_i$ and recall $f^{i+1}_{d,c} = (f^{i+1}_{c,d})^{-1}$

(E) $A_0 = \cup\{\bar{a}_\eta : \eta \in I\}$.

Note that we can prove by induction on α that for any such construction $(\bar{A}, \bar{f}) \in \mathbf{K}$:

$(*)$ If $\mathrm{Dom}(f^i_{c,d}) \neq \{c\}$, then

(i) $(\exists \delta \leq i)[\mathrm{Dom}(f^i_{c,d}) = A_\delta = \mathrm{Rang}(f^i_{c,d})]$ so δ is a limit ordinal or

(ii) $(\exists \epsilon < \zeta \leq i)[\mathrm{Dom}(f^i_{c,d}) = A_\zeta \,\&\, \mathrm{Rang}(f^i_{c,d}) = A_\epsilon \cup (A_{\zeta+1} \setminus A_\zeta)]$ or

(iii) $(\exists \epsilon < \zeta \leq i)[\mathrm{Rang}(f^i_{c,d}) = A_\zeta \,\&\, \mathrm{Dom}(f^i_{c,d}) = A_\epsilon \cup (A_{\zeta+1} \setminus A_\zeta)]$.

We can clearly find $\alpha < \lambda^+$ and $(\bar{A}, \bar{f}) \in \mathbf{K}_\alpha$, i.e. A_i, (for $i \leq \alpha$) $f^i_{c,d}$ (for $i < \alpha$) satisfying (A) - (E) such that:

$(**)$ (i) for every finite $B \subseteq A_\alpha$ and $b \in \mathfrak{C}$,

- if $\lambda \geq \lambda(T)$ then $\mathrm{stp}(b, B)$ is realized by some $a \in A_\alpha$, moreover for λ ordinals $i < \alpha$ clause (i) of (C) holds, $B = B_i \subseteq A_i$ and a_i realizes $\mathrm{stp}(b, B)$,
- if $|T| \leq \lambda < \lambda(T)$ if $\bar{a}$ list B and $\models \varphi[b, \bar{a}]$ then for λ ordinals $i < \alpha, \models \varphi[a_i, \bar{a}]$ and $B_i = B$

(ii) for every $c, d \in A_\alpha, \mathrm{Dom}(f^\alpha_{c,d}) = A_\alpha = \mathrm{Rang}(f^\alpha_{c,d})$.

This is easy by reasonable bookkeeping and clause (C) above. Hence A_α is the universe of a strongly $\aleph_\epsilon$-saturated model if $\lambda \geq \lambda(T)$, and strongly $\aleph_0$-homogeneous (in both cases of model cardinality λ), if $\lambda < \lambda(T)$ (remember we work in $\mathfrak{C}^{\mathrm{eq}}$). We call it M_I (and should have written $\alpha_I < \lambda^+, A^I_i$, etc).

This is close to [She90, Sh.IV,5.13, pg. 213 + §3]. Well, we have constructed the models, but we still need to show the non-embeddability. This is proved just before 3.25, i.e., the end of the sub-section, which deals with the context 3.6 and use 3.7, 3.24, and in particular 3.23.

In 3.6 we can restrict ourselves to Pos. 1. So till we finish the proof of 3.3 we adapt the context 3.6, and for notational simplicity only assume $\lambda \geq \lambda(T)$, (otherwise Claim 3.16 has to be revised). $\square_{3.3}$

Context 3.6. T is a stable (first-order) theory, A, B, C, D denote subsets of the monster $\mathfrak{C} = \mathfrak{C}_T$ of cardinality $< \kappa$, $\mathfrak{C}$ is κ-saturated.
<u>Pos. 1</u>, $\mathbf{F} = \mathbf{F}^f_{\aleph_0}, \kappa = \aleph_0, \mathscr{P} = \mathscr{P}_I = \{D_I\}$ where $D_I = \{a_\eta : \eta \in I\}$ for some I, $\langle a_\eta : \eta \in I\rangle$ as

$\boxplus_2$ In the proof of 3.1 above, $\lambda \geq |D_I| + \lambda(T)$.

<u>Pos. 2</u>: T is a stable theory, $\mathbf{F} = \mathbf{F}^f_\kappa$, see [She90, IV, 3.14] and $\kappa = \kappa_r(T)$, so a regular cardinal, $\mathscr{P}$ a family of sets ($\subseteq \mathfrak{C} = \mathfrak{C}_T$) and $\lambda = \lambda^{<\kappa} + \lambda(T) \geq \sup\limits_{D \in \mathscr{P}} |D|$, and we shall assume

- if $|B| \leq \lambda$, then
$$\lambda \geq |\{\mathrm{tp}(\bar{d}, B) : \lg(\bar{d}) < \kappa \text{ and } \mathrm{Rang}(\bar{d}) \cup B \text{ is } \mathbf{F}\text{-atomic over } B\}|$$

(recall we say A' is $\mathbf{F}$-atomic over A if for every finite $\bar{d} \subseteq A'$ we have $\mathrm{tp}(\bar{d}, A) \in \mathbf{F}(B)$ for some $B \subseteq A$ of cardinality $< \kappa$).
<u>Pos 3</u>: T and F are as in *Pos 2* but $\mathscr{P} = \{(B, D)\}$, where $B \subseteq D$.
<u>Pos 4</u>: As in *Pos 3*, but T is just a singleton (no used below).

Now we define the relevant constructions and prove that the demands parallel to non-forking calculus hold.

We can work in Pos 2 because

Observation 3.7. *1) If Pos 1, then Pos 2.*
2) If Pos 2, then Pos 3 when we identify $D \in \mathscr{P}$ with $(\emptyset, D)$.

Definition 3.8. 1) We say $\mathscr{A} = \langle (A_\alpha, D_\alpha, B_\alpha) : \alpha < \alpha_*\rangle$ is an $(\mathbf{F}, \mathscr{P})$-construction (we may omit $(\mathbf{F}, \mathscr{P})$ when clear from the context) <u>when</u>:

(*a*) $\langle A_\alpha \colon \alpha < \alpha_*\rangle$ is increasing continuous (and we stipulate $A_{\alpha_\star} = \bigcup\limits_{\alpha < \alpha_*} (A_\alpha \cup D_\alpha)$)
(*b*) $A_{\alpha+1} = A_\alpha \cup D_\alpha$
(*c*) $B_\alpha \subseteq A_\alpha \cap D_\alpha$
(*d*) for every finite $\bar{d} \subseteq D_\alpha$ (or just $\bar{d} \subseteq D_\alpha \setminus B_\alpha$) we have $\mathrm{tp}(\bar{d}, A_\alpha) \in \mathbf{F}(B_\alpha)$
(*e*) moreover, for some $B = B_\alpha[\bar{d}, \mathscr{A}] \subseteq B_\alpha$, we have $\mathrm{tp}(\bar{d}, A_\alpha) \in \mathbf{F}(B)$ and $|B_\alpha| < \kappa \Rightarrow B = B_\alpha$.
(*f*) for each α one of the following occurs:
   $\bullet_1$ D_α has cardinality $< \kappa$,
   $\bullet_2$ For *Pos 2*: for some $D'_\alpha \in \mathscr{P}, D_\alpha \cong D'_\alpha$ which means that there is an elementary mapping h_α from D'_α onto D_α (where elementary mapping means in the sense of $\mathfrak{C}$ of course),
   For Pos 4: the pair (D_α, B_α) is isomorphic to some pair from $\mathscr{P}$.
   $\bullet_3$ $D_\alpha \cong D'_\alpha$ for some $D'_\alpha \subseteq A_\alpha$ and h_α is an elementary mapping from D'_α onto D_α.

2) For a construction $\mathscr{A}$ as above we let $\alpha_* = \mathrm{lng}(\mathscr{A}), A_\alpha[\mathscr{A}] = A_\alpha$ for $\alpha \leq \alpha_*, D_\alpha = D_\alpha[\mathscr{A}], B_\alpha = B_\alpha[\mathscr{A}]$ and $A[\mathscr{A}] = A_{\alpha_*}$.
2A) We say that $\mathscr{A}'$ is a *successor* of $\mathscr{A}$, <u>when</u> (both are $(\mathscr{F}, \mathscr{P})$-constructions) and:

(a) $\mathrm{lng}(\mathscr{A}') = \mathrm{lng}(\mathscr{A})$,
(b) $\mathscr{A} \trianglelefteq \mathscr{A}'$.

3) We can replace α^* by any well ordering. We may replace $D_\alpha[\mathscr{A}]$ by (or add to it) $D'_\alpha[\mathscr{A}] \in \mathscr{P}$ and $h_\alpha[\mathscr{A}]$ from clause (e) if $|D_\alpha| \geq \kappa$.
4) For $\alpha < \mathrm{lng}(\mathscr{A})$ we let $w_\alpha[\mathscr{A}] = (\{\beta < \alpha\colon B_\alpha[\mathscr{A}] \cap (A_{\beta+1}[\mathscr{A}] \setminus A_\beta[\mathscr{A}]) \neq \emptyset\})$ so $w_\alpha[\mathscr{A}]$ has cardinality $< \kappa$ by clause (c) of Part (1) so $w_0 = \emptyset$.
5) We call $\mathscr{A}$ *standard* <u>when</u>:

- (a) $\beta \in w_\alpha[\mathscr{A}] \Rightarrow w_\beta[\mathscr{A}] \subseteq w_\alpha[\mathscr{A}] \,\&\, B_\beta[\mathscr{A}] \subseteq B_\alpha[\bar{\mathscr{A}}]$,
- (b) if $\beta < \alpha_*$, $B_\beta[\mathscr{A}]$ is of cardinality $< \kappa$ and $\beta \in w_\alpha[\mathscr{A}]$, <u>then</u> $B_\beta[\mathscr{A}] \subseteq B_\alpha[\mathscr{A}]$,
- (c) if $\beta \in w_\alpha[\mathscr{A}]$ and $\bar{b} \in {}^{\omega>}(D_\beta[\mathscr{A}] \cap B_\alpha[\mathscr{A}])$, <u>then</u> $\mathrm{tp}(\bar{b}, A_\beta)$ do not fork over $(B[\mathscr{A}] \cap A_\beta)$. (Equivalently, for every $\alpha < \mathrm{lng}(\mathscr{A})$, the pair $(w_\alpha, B_\alpha[\mathscr{A}])$ is $\mathscr{A}$-closed, see below.)

6) We say that the pair w, B is $\mathscr{A}$-closed, <u>when</u>:

- (a) $w \subseteq \mathrm{lng}(\mathscr{A})$ and $\beta \in w \Rightarrow w_\beta[\mathscr{A}] \subseteq w$,
- (b) $B \subseteq A[\mathscr{A}]$,
- (c) if $\beta < \mathrm{lng}(\mathscr{A})$ and $B \cap (A_{\beta+1}[\mathscr{A}] \setminus A_\beta[\mathscr{A}]) \neq \emptyset$, then $\beta \in w$,
- (d) if $\beta < \alpha_*$, $B_\beta[\mathscr{A}]$ is of cardinality $< \kappa$ and $\beta \in w$, <u>then</u> $B_\beta[\mathscr{A}] \subseteq B$,
- (e) if $\beta \in w$ and $\bar{b} \in {}^{\omega>}(D_\beta[\mathscr{A}] \cap B_\alpha[\mathscr{A}])$, <u>then</u> $\mathrm{tp}(\bar{b}, A_\beta)$ do not fork over $(B_\alpha[\mathscr{A}] \cap B_\alpha[\mathscr{A}])$, <u>then</u> $\mathrm{tp}(\bar{b}, A_\beta)$.

6A) We say that (w, B) is $(\mathscr{A}, \kappa)$-closed if in addition B is of cardinality $< \kappa$.
7) For $\beta \leq \mathrm{lng}(\mathscr{A})$ let $\mathscr{A} \restriction \beta$ be defined naturally such that $\mathrm{lng}(\mathscr{A}(\beta)) = \beta$.
8) For $b \in A[\mathscr{A}] = A_{\alpha_*}[\mathscr{A}]$, let $\alpha(b) = \alpha(b, \mathscr{A})$ be the β such that $b \in A_{\beta+1}[\mathscr{A}] \setminus A_\beta[\mathscr{A}]$ but for $b \in A_0[\mathscr{A}]$ we stipulate $\alpha(b) = -1$.
9) For $b \in A[\mathscr{A}]$ let $w_b[\mathscr{A}] = w_{\alpha(b)}[\mathscr{A}]$ (where we stipulate $w_{-1}[\mathscr{A}] = \emptyset$, and for a sequence $\bar{b} = \langle b_i : i < \ell g(\bar{b})\rangle$ we let $w_{\bar{b}}[\mathscr{A}] = \bigcup\{w_{b_i}[\mathscr{A}] : i < \ell g(\bar{b})\}$ and $B_{\bar{b}}[\mathscr{A}] = \cup\{\{b_\ell\} \cup B_{\alpha(b_\ell)}[\mathscr{A}] : \ell < \mathrm{lng}(\bar{b})\}$.
10) We may omit $\mathscr{A}$ when clear from the context.

Fact 3.9. 1) For any $w \subseteq \mathrm{lng}(\mathscr{A})$ and $B \subseteq A[\mathscr{A}]$, there is an $\mathscr{A}$-closed pair (w', B') such that $w \subseteq w'$, $B \subseteq B'$ and w', B' are of cardinality $< \kappa + |w|^+ + |B|^+$.
2) For any $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}$ there is a standard $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}'$ such that:

- (a) $\mathrm{lng}(\mathscr{A}') = \mathrm{lng}(\mathscr{A})$
- (b) $A_\alpha[\mathscr{A}'] = A_\alpha[\mathscr{A}]$
- (c) $D_\alpha[\mathscr{A}'] = D_\alpha[\mathscr{A}]$ (and $D'_\alpha[\mathscr{A}'] = D'_\alpha[\mathscr{A}], h_\alpha[\mathscr{A}'] = h_\alpha[\mathscr{A}]$)
- (d) $B_\alpha[\mathscr{A}'] \supseteq B_\alpha[\mathscr{A}]$
- (e) $w_\alpha[\mathscr{A}'] \supseteq w_\alpha[\mathscr{A}]$.

Proof. 1) Straightforward, choose B_α, w_α by introduction on $\alpha \leq \alpha_*$ for $\mathscr{A} \restriction \alpha$, $w \cap \alpha$, $B \cap \mathscr{A}_\alpha[\mathscr{A}]$ recalling that κ is regular by 3.6.
2) Follows. $\square_{3.9}$

Claim 3.10. *Assume that:*

- (a) *$\mathscr{A}$ is a standard $(\mathbf{F}, \mathscr{P})$-construction.*

1) Assume in addition that:

- (b) *π is a one-to-one function from $\alpha = \mathrm{lng}(\mathscr{A})$ onto the ordinal α',*
- (c) *if $\beta \in w_\alpha[\mathscr{A}]$ then $\pi(\beta) < \pi(\alpha)$, and*

(*d*) *If $\beta < \mathrm{lng}(\mathscr{A})$, $|B_\alpha| \ge \kappa$ and $\bar{b} \subseteq D_\beta[\mathscr{A}] \setminus B_\beta[\mathscr{A}]$, then*

$$\bar{b} \subseteq \bigcup\{D_\gamma \colon \gamma < \beta \wedge \pi(\gamma) < \pi(p)\}.$$

Then there is a standard $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}'$ such that:

- (*i*) $\mathrm{lng}(\mathscr{A}') = \alpha'$
- (*ii*) $D_\alpha[\mathscr{A}] = D_{\pi(\alpha)}[\mathscr{A}']$
- (*iii*) $w_{\pi(\alpha)}[\mathscr{A}'] = \{\pi(\beta) : \beta \in w_\alpha[\mathscr{A}]\}$ *and* $B_{\pi(\alpha)}[\mathscr{A}'] = B_\alpha[\mathscr{A}]$
- (*iv*) $A_0[\mathscr{A}'] = A_0[\mathscr{A}]$
- (*v*) $A_{\pi(\alpha)}[\mathscr{A}'] = A_0[\mathscr{A}'] \cup \bigcup\{D_\beta[\mathscr{A}'] \colon \pi(\beta) < \pi(\alpha)\}$.

2) If (w', B') and (w', B'') are $\mathscr{A}$-closed, then $(w' \cap w'', B' \cap B'')$ is $\mathscr{A}$-closed.
3) Assume also that $B \subseteq A_{\lg(\mathscr{A})}[\mathscr{A}]$ and $|B| < \kappa$.

Then there is a $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}'$ satisfying:

- (α) $\mathscr{A}' = \langle A'_\alpha, D'_\alpha, B'_\alpha : \alpha < 1 + \mathrm{lng}(\mathscr{A}')\rangle$, *we use ordinal addition,*
- (β) $A'_0 = A_0[\mathscr{A}]$
- (γ) $A'_1 = A'_0 \cup B$
- (δ) $D'_0 = B$
- (ε) $A'_{1+\alpha} = A'_1 \cup A_\alpha[\mathscr{A}]$
- (ζ) $D'_{1+\alpha} = D_\alpha$
- (η) $B'_{1+\alpha} \supseteq B_\alpha$.

4) In part (3), if for some $\mathscr{A}$-closed $u \subseteq \mathrm{lng}(\mathscr{A})$ we have $\cup\{B_\alpha : \alpha \in u\} \subseteq B \subseteq \cup\{D_\alpha : \alpha \in u\}$ then we can let $B'_{1+\alpha} = B_\alpha \cup B$.

Proof. For 3.10, recall that in Definition 3.8(1)(d) the set $B_\alpha[\bar{b}, \mathscr{A}]$ is part of $\mathscr{A}$ and then see 3.8(6)(d). This helps in particular in 3.10(2). For the others, recall that the proof of [She90, Ch.IV 3.3,3.2,pg.176], (of course, we can strengthen 3.10(1),(3)); [e.g. for part (4) show by induction on $\alpha \le \mathrm{lng}(\mathscr{A})$ then $\bar{d} \subseteq B \Rightarrow \mathrm{tp}(\bar{d}, A_\alpha[\mathscr{A}]) \in \mathbf{F}(B \cap A_\alpha[\mathscr{A}])$; for part (3), just find $B' \supseteq B$ which is as in part (4); part (2) can be proved by induction on $\mathrm{lng}(\mathscr{A})$]. $\square_{3.10}$

Definition 3.11. 1) We say $(\mathscr{A}, \bar{f})$ is a automorphic $(\mathbf{F}, \mathscr{P})$-construction when:

- (*a*) $\mathscr{A}$ is a standard $(\mathbf{F}, \mathscr{P})$-construction
- (*b*) $A_0[\mathscr{A}] = \emptyset$
- (*c*) $\mathrm{lng}(\mathscr{A}) < \lambda^+$
- (*d*) $\bar{f} = \langle f_{i,g} : i \le \mathrm{lng}(\mathscr{A}), g \in \mathscr{G}_{A_i[\mathscr{A}]}\rangle$ where $\mathscr{G}_A$ is a set of elementary mappings from a subset of A into a subset of A such that $g \in \mathscr{G}_{A_i[\mathscr{A}]}$ implies $g^{-1} \in \mathscr{G}_{A_i[\mathscr{A}]}$,
- (*e*) $f_{i,g}$ is an elementary mapping with domain and range $\subseteq A_i[\mathscr{A}]$
- (*f*) $f_{i,g}$ is increasing continuous with $i, f_{i,g^{-1}} = (f_{i,g})^{-1}$
- (*g*) if $g \in \mathscr{G}_{A_i[\mathscr{A}]} \setminus \bigcup\limits_{j<i} \mathscr{G}_{A_j[\mathscr{A}]}$ then $f_{i,g} = g$
- (*h*) Let $\mathscr{G}_{A_{<i}[\mathscr{A}]} = \bigcup_{j<i} \mathscr{G}_{A_j[\mathscr{A}]}$ for $i \le \mathrm{lng}(\mathscr{A})$.

2) The cardinality of $(\mathscr{A}, \bar{f})$ written $\mathrm{card}(\mathscr{A}, \bar{f})$ is the one of $\mathscr{A}$, i.e. $|\mathrm{lng}(\mathscr{A})| + |A[\mathscr{A}]|$.
3) For $\beta \leq \mathrm{lng}(\mathscr{A})$ let $(\mathscr{A}, \bar{f}) \restriction \beta$ be defined naturally.

Fact 3.12. Assume that $(\mathscr{A}, \bar{f})$ is an automorphic $(\mathbf{F}, \mathscr{P})$-construction. Let $\alpha = \mathrm{lng}(\mathscr{A})$, and $B \subseteq D$, $B^* \subseteq A[\mathscr{A}]$, $B_1 \subseteq A[\mathscr{A}]$ and g an elementary mapping from B_1 into $\mathscr{A}[A]$. Then:

1) If $|D^*| < \kappa$, <u>then</u> there is some $\mathscr{A}'$ such that:

- (a) $\mathscr{A}'$ is a successor of $\mathscr{A}$,
- (b) $B_\alpha[\mathscr{A}'] = B$,
- (c) $D_\alpha[\mathscr{A}_\alpha]$ is isomorphic over B to D^*.
- (d) $\mathscr{G}_{A_\alpha[\mathscr{A}']} = \mathscr{G}_{A_{<\alpha}[\mathscr{A}]}$,
- (e) $f_{\alpha,g}[\mathscr{A}'] = f_{\beta,g}[\mathscr{A}]$, when $\beta < \alpha$, $g \in \mathscr{G}_{A_{\beta[\mathscr{A}]}}$.

2) If $D' \in \mathscr{P}$, $D^* \cong D'$ and $|B| < \kappa$, <u>then</u> there is $\mathscr{A}'$ such that (a)-(e) above holds.
3) If $g \notin \mathscr{G}_{A_{<\alpha}[\mathscr{A}]}$, $|D^*| < \kappa$, <u>then</u> there is $\mathscr{A}'$ such that (a), (b), (c), and (e) as in first part holds and

- (d) $\mathscr{G}_{A_\alpha[\mathscr{A}]'} = \mathscr{G}_{A_{<\alpha}[\mathscr{A}]} \cup \{g, g^{-1}\}$.

4) A special case of part (3) is that $u \subseteq \alpha$ has no last member, $g_\beta \in \mathscr{G}_{A_\beta[\mathscr{A}]}$ for $\beta \in u$ is $\subseteq$-increasing with β and $g \coloneqq \{g_\beta \colon \beta \in u\}$.
5) Another special case of part (3) is $\beta < \alpha$, $g_\beta \in \mathscr{G}_{A_\beta[\mathscr{A}]}$ and $g \in \mathscr{G}_{A_\beta[\mathscr{A}]}$ extend g_β and has domain $A_\alpha[\mathscr{A}]$.

Proof. Straightforward (by the existence of non-forking extensions), see more details in 3.14. $\square_{3.12}$

Definition 3.13. For automorphic $(\mathbf{F}, \mathscr{P})$-constructions $(\mathscr{A}^1, \bar{f}^1)$, $(\mathscr{A}^2, \bar{f}^2)$ let $(\mathscr{A}^1, \bar{f}^1) \leq (\mathscr{A}^2, \bar{f}^2)$ means: $\bar{\mathscr{A}}^1 \trianglelefteq \bar{\mathscr{A}}^2$ and $\bar{f}^1 = \langle f^2_{i,g} : i \leq \mathrm{lng}(\mathscr{A}^1), g \in \mathscr{G}_{A_i[\mathscr{A}^1]} \rangle$.

Claim 3.14. *If $(\mathscr{A}, \bar{f})$ is an automorphic $(\mathbf{F}, \mathscr{P})$-construction, $i \leq \mathrm{lng}(\mathscr{A})$, $g_* \in \mathscr{G}_{A_i}[\mathscr{A}]$ <u>then</u> for some automorphic $(\mathbf{F}, \mathscr{P})$-construction $(\mathscr{A}', \bar{f}')$ we have:*

- (a) $(\mathscr{A}, \bar{f}) \leq (\mathscr{A}', \bar{f}')$
- (b) $\mathrm{card}(\mathscr{A}', \bar{f}') \leq \mathrm{card}(\mathscr{A}, \bar{f}) + \aleph_0$
- (c) $\mathrm{Dom}((f_{j,g_*})[\mathscr{A}']) = A_i[\mathscr{A}]$ *where* $j = \mathrm{lng}(\mathscr{A}')$.

Proof. Let $\mathscr{A}^0 = \mathscr{A} {\restriction} i$, then by 3.10 we can find a standard $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}^1$ and $j_1 \leq \mathrm{lng}(\mathscr{A}^1)$ such that $A_0[\mathscr{A}^1] = A_0[\mathscr{A}^0]$, $A[\mathscr{A}^1] = A[\mathscr{A}^0]$, and $A_{j_1}[\mathscr{A}^1]$ is $\mathrm{Dom}(f_{i,g_*}[\mathscr{A}])$. We can find an elementary mapping h such that: $\mathrm{Dom}(h) = A[\mathscr{A}^0]$, h extends f_{i,g_*}, and for every $\beta \in [j_1, \mathrm{lng}(\mathscr{A}^1)]$, we have

$$\bar{d} \subseteq h(D_\beta[\mathscr{A}^1]) \Rightarrow \mathrm{tp}(\bar{d}, A[\mathscr{A}] \cup h(A_\beta[\mathscr{A}^1])) \in \mathbf{F}(h(B_\beta)).$$

Now we define the automorphic $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}'$: $\mathrm{lng}(\mathscr{A})' = \mathrm{lng}(\mathscr{A}) + (\mathrm{lng}(\mathscr{A}^1) - j_1)$, and $\mathscr{A}' \restriction \mathrm{lng}(\mathscr{A}) = \mathscr{A}$, $D_{\mathrm{lng}(\mathscr{A})+\zeta}[\mathscr{A}'] = h(D_{j_1+\zeta}[\mathscr{A}^1])$, $B_{\mathrm{lng}(\mathscr{A})+\zeta}[\mathscr{A}'] = h(B_{j_1+\zeta}[\mathscr{A}^j])$. Define $\bar{f}' = \langle f'_{\alpha,g} : \alpha \leq \mathrm{lng}(\mathscr{A}'), g \in \mathscr{G}_{A_\alpha[\mathscr{A}']} \rangle$ as follows: for $\alpha \leq \mathrm{lng}(\mathscr{A}'), g \in \mathscr{G}_{A_\alpha[\mathscr{A}']}$ we let:

- (α) if $\alpha \leq \mathrm{lng}(\mathscr{A})$, then $f'_{\alpha,g} = f_{\alpha,g}$
- (β) if $\alpha \geq \mathrm{lng}(\mathscr{A})$ and $g \notin \mathscr{G}_{A_{\mathrm{lng}(\mathscr{A})}}[\mathscr{A}]$ then $f'_{\alpha,g} = g$
- (γ) if $g \in \mathscr{G}_{A_{\mathrm{lng}(\mathscr{A})}}[\mathscr{A}]$, and $g \neq g_*, g_*^{-1}$ then $f'_{\alpha,g} = f_{\mathrm{lng}(\mathscr{A}),g}$

(δ) if $g = g_*$ and $\alpha < \mathrm{lng}(\mathscr{A}')$ then let $f'_{\alpha,g}$ be $f_{\mathrm{lng}(\mathscr{A}),g}$
(ϵ) if $g = g_*$ and $\alpha = \mathrm{lng}(\mathscr{A}')$ then let $f'_{\alpha,g}$ be h.

Now check. $\square_{3.14}$

Claim 3.15. *$\delta < \lambda^+$ is a limit ordinal and if $\langle(\mathscr{A}^\zeta, \bar f^\zeta) : \zeta < \delta\rangle$ is increasing (sequence of automorphic $(\mathbf{F}, \mathscr{P})$-constructions), then it has a* $\mathrm{lub}(\mathscr{A}^\delta, \bar f^\delta)$ *i.e.*

$$\zeta < \delta \Rightarrow (\mathscr{A}^\zeta, \bar f^\zeta) \le (\mathscr{A}^\delta, \bar f^\delta)$$

$$\mathrm{lng}(\mathscr{A}^\delta) = \bigcup_{\zeta<\delta} \mathrm{lng}(\mathscr{A}^\zeta)$$

$$\mathrm{card}(\mathscr{A}^\delta) \le |\delta| + \sup_{\zeta<\delta} \mathrm{card}(\mathscr{A}^\zeta).$$

Proof. Straightforward. $\square_{3.15}$

Claim 3.16. *For every $\theta = \mathrm{cf}(\theta) \in [\kappa, \lambda]$ there is an automorphic $(\mathbf{F}, \mathscr{P})$-construction $\mathscr{A}$ of cardinality λ such that $\mathrm{cf}(\mathrm{lng}(\mathscr{A})) = \theta$ and*

$\otimes_1$ *for $g \in \mathscr{G}_{A[\mathscr{A}]}$, $f_{\mathrm{lng}(\mathscr{A}),g}[\mathscr{A}]$ is an automorphism of $A[\mathscr{A}]$*
$\otimes_2$ *if $B \subseteq A[\mathscr{A}], |B| < \kappa, B \subseteq B'$ and $|B'| < \kappa$ or B' is isomorphic to some $B'' \in \mathscr{P}$ then*

$$\mathrm{lng}(\mathscr{A}) = \mathrm{otp}\{\alpha : \begin{array}{l}\textit{there is an elementary mapping } h \textit{ from } B' \textit{ onto } D_\alpha \\ \textit{which is the identity on } B, \textit{ and } B_\alpha = B\}.\end{array}$$

Proof. By bookkeeping and the assumptions (in 3.6) on λ. $\square_{3.16}$

Claim 3.17. *Suppose:*

(a) *$\mathscr{A}$ is an $(\mathbf{F}, \mathscr{P})$-construction,*
(b) *χ^* large enough and $N_1 \prec N_2 \prec (\mathscr{H}(\chi^*), \in)$*
(c) *$\mathscr{A} \in N_1$ and the monster model $\mathfrak{C}$ belongs to N_1 and $N_1 \cap \kappa$ is an ordinal (possibly κ itself, if $\kappa = \aleph_0$ this is necessarily the case)*
(d) *$\bar b \in {}^{\omega>}(A[\mathscr{A}])$ and $w_{\bar b}[\mathscr{A}] \cap N_2 \subseteq N_1$ (on $w_{\bar b}$ see Definition 3.8(9))*
(e) *if $\alpha \in w_{\bar b} \cap N_1$ then $\mathrm{tp}(\bar b{\restriction}D_\alpha[\mathscr{A}], N_2 \cap A[\mathscr{A}])$ does not fork over $N_1 \cap A[\mathscr{A}]$ where for $\bar b = \langle b_\ell : \ell < n\rangle$ we let $\bar b{\restriction}D_\alpha = \langle b_\ell : \ell < n, b_\ell \in D_\alpha\rangle$.*

Then $\mathrm{tp}(\bar b, A[\mathscr{A}] \cap N_2)$ does not fork over $A[\mathscr{A}] \cap N_1$.

Proof. By 3.10. $\square_{3.17}$

Now to complete the proof of 3.3 we turn back to the model M_I we have constructed before 3.6.

Fact 3.18. For the context 3.6(Pos 1), (so $I \in K^\omega_{\mathrm{tr}}, I \subseteq {}^{\omega\ge}\lambda$ is closed under initial segments of cardinality $\le \lambda$), letting $\kappa = \aleph_0, \mathscr{P} = \{D_I\}$ (see 3.6(Pos 1)) for some $\mathscr{A} = \mathscr{A}^I$ we have

(A) $\mathscr{A}$ is a standard $(\mathbf{F}, \mathscr{P}_I)$-construction $A_1[\mathscr{A}] = D_I$ and $A_0[\mathscr{A}] = \emptyset$
(B) $A[\mathscr{A}^I]$ is strongly $\aleph_\epsilon$-saturated of cardinality λ
(C) $A[\mathscr{A}^I]$ is equal to the model M_I constructed during the beginning of the proof of 3.1.

Remark 3.19. We do not actually use clause (C), as we can just let M_I be the model with universe $A[\mathscr{A}^I]$.

Proof. Straightforward for clause (C) recall Definition 3.11, Claim 3.14 (or just use the model constructed in 3.16). $\square_{3.6}$

Fact 3.20. If χ is regular large enough, $\mathscr{A}^I \in \mathscr{H}(\chi), \mathscr{A}^I \in N_1 \prec N_2 \prec (\mathscr{H}(\chi), \in, <^*_\chi), N_\ell \cap \kappa_r(T) \in \kappa_r(T) + 1, \bar{b} \in M_I$, and $w_{\bar{b}}[\mathscr{A}^I] \cap N_2 \subseteq N_1$ and $\alpha \in w_{\bar{b}}[\mathscr{A}] \Rightarrow \mathrm{tp}(\bar{b} \restriction D_\alpha[\mathscr{A}^I], N_2 \cap M_I)$ does not fork over $N_1 \cap M_I$. Then $\mathrm{tp}(\bar{b}, N_2 \cap M_I)$ does not fork over $N_1 \cap M_I$.

Proof. By 3.17. $\square_{3.20}$

For the rest for simplicity assume $\kappa = \aleph_0$.

Fact 3.21. If χ is regular, $I \in K^\omega_{\mathrm{tr}}, \mathscr{A}^I \in \mathscr{H}(\chi), \mathscr{A}^I \in N_1 \prec N_2 \prec (\mathscr{H}(\chi), \in, <^*_\chi), N_\ell \cap \kappa_r(T) \in \kappa_r(T) + 1$ and $\eta \in P^I_\omega, n < \omega, \eta \restriction n \in N_1, \eta(n) \in N_2 \setminus N_1$ and $\mathfrak{C} \in N_1$ then $\mathrm{tp}(\bar{a}_\eta, N_2 \cap M_I)$ fork over $N_1 \cap M_I$.

Proof. Let $\mathscr{A} = \mathscr{A}^I$ be as in 3.18 and let $A^I_i = A_i[\mathscr{A}^I]$, for $i \leq \alpha_I = \mathrm{lng}(\mathscr{A}^I)$, and recall $A^I_1 = \{a_\eta : \eta \in I\}$. For $\bar{c} \subseteq N_\ell \cap M_I$, clearly $\mathrm{tp}(\bar{c}, A^I_1)$ does not fork over $\bigcup\{B_\gamma \cap A^I_1 : \gamma \in w_{\bar{c}}\} \cup (\bar{c} \cap A^I_1) \subseteq N_\ell \cap A^I_1$, hence $\mathrm{tp}(A^I_1, N_\ell \cap M_I)$ does not fork over $N_\ell \cap A^I_1$ recalling $a_\eta \in A^I_1$ we have hence $\mathrm{tp}(a_\eta, N_\ell \cap M_I)$ does not fork over $N_\ell \cap A^I_0$.

But $\mathrm{tp}(\bar{a}_\eta, \{\bar{a}_\nu : \nu \in I, \neg(\eta \restriction n \triangleleft \nu)\})$ does not fork over $\{\bar{a}_\nu : \nu \trianglelefteq \eta \restriction n\}$, (why? as $\langle \bar{a}_\eta : \eta \in I\rangle$ is a non-forking tree). Now the set $\{\bar{a}_\nu : \nu \trianglelefteq \eta \restriction n\}$ is a subset of N_1 hence $\mathrm{tp}(\bar{a}_\eta, \{\bar{a}_\nu : a_\nu \in N_1 \text{ and } \neg(\eta \restriction n \triangleleft \nu)\})$ does not fork over $\{\bar{a}_\nu : \nu \trianglelefteq \eta \restriction n\}$ so by transitivity and previous the sentence, $\mathrm{tp}(\bar{a}_\eta, M_I \cap N_1)$ does not fork over $\{a_{\eta \restriction m} : m \leq n\}$.

On the other hand $\mathrm{tp}(\bar{a}_\eta, M_I \cap N_2)$ forks over it (otherwise $\mathrm{tp}(a_\eta, \{\bar{a}_{\eta \restriction \ell} : \ell \leq n+1\})$ does not fork over $\{a_{\eta \restriction \ell} : \ell \leq n\}$, contradiction), so the conclusion follows. $\square_{3.21}$

Fact 3.22. If I is super unembeddable into J then M_I is not isomorphic to M_J.

Proof. Straightforward by the definition and Facts 3.20, 3.21, but we give some details. Without loss of generality T is countable so $\kappa_r(T) = \aleph_1$, (justified in the proof of 3.23 below).

Let f be an isomorphism from M_I onto M_J and χ be regular large enough. We can find $\langle M_n, N_n : n < \omega\rangle$ an increasing sequence of elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi)$ and η as in Definition 1.1 such that $\mathscr{A}^I, \mathscr{A}^J, f$ belongs to N_0.

By Fact 3.20, $\mathrm{tp}(f(\bar{a}_\eta), M_J \cap N_n)$ does not fork over $M_J \cap M_n$ for every n large enough. By Fact 3.21, $\mathrm{tp}(\bar{a}_\eta, M_I \cap N_n)$ forks over $M_I \cap M_n$. As f maps $M_I \cap N_n$ onto $M_J \cap N_n$ and $M_I \cap M_n$ onto $M_J \cap M_n$ and $\bar{a}_\eta$ to $f(\bar{a}_\eta)$ we finish. $\square_{3.22}$

Fact 3.23. If I is super unembeddable into J then M_I is not elementarily embeddable into M_J.

Proof. Let τ_0 be a countable sub-vocabulary of $\tau(T)$ such that for $\eta \in P^I_\omega, n < \omega$ we have $\mathrm{tp}(a_\eta, \{a_{\eta \restriction \ell} : \ell < n\})$ forks over $\{a_{\eta \restriction \ell} : \ell < n\}$ even in the τ_0-reduct of M_I. Suppose f is an elementary embedding of M_I into M_J (or just of their τ_0-reduct) and we shall get a contradiction. Modulo the proof of 3.22, it suffice to prove:

$(*)_1$ if $\mathrm{tp}(\bar{a}, A)$ does not fork over $B \subseteq A$ in $\mathfrak{C}$ then $\mathrm{tp}_{\mathbb{L}(\tau_0)}(\bar{a}, A)$ does not fork over B in $\mathfrak{C}{\restriction}A$.

[Why? By character by ranks, [She90, Ch.III].]

Fact 3.24. We have:

$(*)_2$ if $\mathscr{A}^I$, $\mathscr{A}^J, f \in N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ <u>then</u> $\mathrm{tp}_{\mathbb{L}(\tau_0)}(N \cap M_J, \mathrm{Rang}(f))$ does not fork over $N \cap \mathrm{Rang}(f)$ in $M_J{\restriction}\tau_0$.

Proof. Why $(*)_2$ holds? As T is stable and τ_0 is countable for every $\bar{c} \in M_J$ there is a countable $B^*_{\bar{c}} \subseteq \mathrm{Rang}(f)$ such that $\mathrm{tp}_{\mathbb{L}(\tau_0)}(\bar{c}, \mathrm{Rang}(f))$ does not fork over $B^*_{\bar{c}}$ in $M_J{\restriction}\tau_0$. As τ_0 is countable, T stable, clearly $\bar{c} \in N \cap M_J \Rightarrow B^*_{\bar{c}} \subseteq N \cap \mathrm{Rang}(f)$. So for $\bar{c} \in N \cap M_J$ the type $\mathrm{tp}_{\mathbb{L}(\tau_0)}(\bar{c}, \mathrm{Rang}(f))$ does not fork over $N \cap \mathrm{Rang}(f)$, as required. $\square_{3.24}$

So we have finished proving 3.23. $\square_{3.23}$

Proof. <u>Continuation of the Proof of 3.3</u>:

Let $\langle I_\alpha : \alpha < 2^\lambda \rangle$ exemplify that K^ω_{tr} has the $(2^\lambda, \lambda, \mu, \aleph_0)$-bigness property and let $M_\alpha = M_{I_\alpha}$. Now apply 3.18 and 3.23. $\square_{3.3}$

Remark 3.25. In 3.22. 3.23 weaker versions of unembeddable suffice.

§ 3(B). **On Generalizations and Abelian p-groups.**

Having finished our digression to stability theory, we look at a strengthening of 2.34, which will be used in 3.28.

Theorem 3.26. *If $\lambda > \mu + \aleph_1$ and $\mu \geq \kappa$ <u>then</u> K^ω_{tr} has the full $(\lambda, \lambda, \mu, \kappa)$-super$^+$-bigness property which means that in the Definition 1.4 we replace super by super$^+$ which means that we define "$I \in K^\omega_{\mathrm{tr}}$ is (μ, κ)-super-unembeddable into $J \in K^\omega_{\mathrm{tr}}$" as in Definition 1.1 but replace $(*)$ there by:*

$(*)^+$ *like $(*)$ of Definition 1.1 adding*

- $(v)^+$ *for each $n, \eta{\restriction}n \in M_n$ and $\eta{\restriction}(n+1) \in N_n \setminus M_n$*
- (vii) *if $\nu \in P^J_\omega$ is in the closure of $M_n \cap I$, (i.e. $\{\nu{\restriction}\ell : \ell < \omega\} \subseteq M_n$) then $\nu \notin N_n \setminus M_n$*
- $(viii)$ *there is $M \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ such that: $\bigcup\limits_{n<\omega} M_n \subseteq M$ and $\eta \notin M$, but for each n we have:*

$$\nu \in P^J_\omega \ \& \ \bigwedge_{\ell<\omega} \nu{\restriction}\ell \in M_n \Rightarrow \nu \in M.$$

Remark 3.27. Compare with 1.21 here + 1.16(2) here.

Proof. The proof is done by cases, so to enlighten the reader, we first list them.

If λ is regular $> \aleph_1$: by case 1

If λ is singular and $(\exists \chi < \lambda)[\chi < \lambda \leq \chi^{\aleph_0}]$: by case 2;

Case 3, see below.

If neither case 1 nor case 2 but $(\exists \chi)[\mu \leq \chi^{\aleph_0} < \lambda \leq 2^\chi]$: by case 4;

So we are left with λ strong limit singular.

If $\lambda = \aleph_{\alpha+\omega}$: by case 6;

If $\mathrm{cf}(\lambda) > 2^{\aleph_0}$: by case 5;

In the remaining case let $\theta = \mu^{+\omega}$, so necessarily $\theta < \lambda$, hence for some increasing sequence $\langle \lambda_n : n < \omega \rangle$ of regular cardinals with limit $\theta < \lambda, \mathrm{cf}(\Pi \lambda_n/\text{finite}) = \theta^+, \lambda_n > \mu$ (exist see [Shed, 3.22=Lpcf.8]), now for $\epsilon < 2^{\aleph_0}, \mathfrak{a}_\epsilon$ be an infinite subset of $\{\lambda_n : n < \omega\}$ such that $\epsilon \neq \zeta \Rightarrow |\mathfrak{a}_\epsilon \cap \mathfrak{a}_\zeta| < \aleph_0$.

So case 3 applies.

<u>Case 1</u>: λ regular $> \aleph_1$. (In fact also the requirements from Def. 1.5(G$^+$) of "super$^{7^+}$" hold.)

Use the proof of 1.16(1) with minor changes:

Choosing $\bar{C}$, by 2.4 we can add the demands:

(c) for any $\zeta < \lambda$, for every club E of λ we have $\{\delta \in S_\zeta : C_\delta \subseteq E\}$ is stationary

(d) $\alpha \in C_\delta \Rightarrow \mathrm{cf}(\alpha) > \aleph_0$.

Choosing $\delta(*) \in E$ we demand also $C_{\delta(*)} \subseteq E$, and let $m_\ell = 2\ell$.

So the condition for super$^{7^+}$ (Def 1.5(G$^+$) (hence from Def.1.1) holds. Clause (v)$^+$ holds by the choice of $\eta_\delta, m_\ell, M_\ell, N_\ell$. Clause (vii) holds by clause (d), i.e. $\mathrm{cf}(\eta_\delta(m)) > \aleph_0, \eta_\delta(m) \in E$.

Lastly, clause (viii) is exemplified by $M = M^*_{\delta(*)}$.

<u>Case 2</u>: There is $\chi, \chi < \lambda \leq \chi^{\aleph_0}$ and λ is singular.

Just Claim 2.1 applies; i.e. the proof of 2.1 but by 2.10(2) we can choose there C_δ such that

$(*)$ $\alpha \in C_\delta \,\&\, \alpha > \sup(C_\delta \cap \alpha) \Rightarrow \mathrm{cf}(\alpha) > \aleph_0$.

The proof gives also $(v)^+, (vii), (viii)$ and even

$(vii)^{++}$ if $\eta \in P^J_\omega, \{\eta{\restriction}\ell : \ell < \omega\} \subseteq M_n$ then $\eta \in M_n$.

[Why? By $(*)$ above or see Case 3's proof; note that if $\eta = \langle i \rangle \underset{\lambda}{\otimes} \nu$ (or $\eta = \langle i \rangle \hat{}\, \nu$) and $\nu \in I_{\rho_i}$ then necessarily $i \in M_n$ hence $I_{\rho_i} \in M_n$.]

$(ii)^+$ $\mu \subseteq M_n$.

<u>Case 3</u>: λ singular, and for some $\theta, \lambda > \theta \geq \mu + \aleph_1, \mathrm{cf}(\theta) = \aleph_0$ and there is a sequence $\langle \mathfrak{a}_\epsilon : \epsilon < \mathrm{cf}(\lambda) \rangle$ as in 2.18.

The proof of 2.18 gives (v)$^+$ trivially. Again (as in the proof of 2.1)

$$[\eta \in P^J_\omega \;\&\; \bigwedge_\ell \eta{\restriction}\ell \in M^*_\alpha \Rightarrow \eta \in M^*_{\alpha+1}]$$

hence

$$[\mathrm{cf}(\alpha) > \aleph_0 \,\&\, \eta \in P^J_\omega \;\&\; \bigwedge_\ell \eta{\restriction}\ell \in M^*_\alpha \Rightarrow \eta \in M^*_\alpha].$$

hence clause (vii) holds.

Lastly, it follows that $M^*_{\delta(*)}$ satisfies the requirement in clause (viii).

<u>Case 4</u>: There is $\chi, \mu \leq \chi < \lambda \leq 2^\chi$ and: λ is singular or at least $(\chi^{\aleph_0})^+ < \lambda$.

Like the proof of 1.16(2).

<u>Case 5</u>: λ is strong limit singular $\mathrm{cf}(\lambda) > 2^{\aleph_0}$.

By the proof of 1.16(3) using models N_η of cardinality $2^{\aleph_0}$, (i.e. choose $\kappa = 2^{\aleph_0}$); and demand $[N_\eta]^{\aleph_0} \subseteq N_\eta$ and using [Shed, 1.16=La45]. Alternatively, in its proof notice that by thinning $\mathscr{T}'$ a bit more we can get: let $h_* \in N_{\langle\rangle}$ be a one to one function from λ onto ${}^{\omega}\lambda$, then:

$$(*)\ \text{ if } k < \lg(\eta)\ \&\ \eta \in \mathscr{T}\ \&\ \alpha \in N_\eta\ \&\ \bigwedge_{\ell<\omega} (h_*(\alpha))(\ell) \in N_{\eta\restriction k} \text{ then } \alpha \in N_{\eta\restriction k}.$$

The point is: without loss of generality $k+1 = \lg(\eta)$ and for each $\nu \in \mathscr{T}$ of length k,

$$|\{\eta : \nu \triangleleft \eta \in \mathscr{T}\ \&\ \ell g(\eta) = k+1 \text{ and } (*) \text{ fails for } \eta, k\}| \le \kappa^{\aleph_0}.$$

For (viii), $\bigcup\limits_{n<\omega} M_n$ is as required by clause (ix) of Subfact 1.20. Note that $(v)^+$ is satisfied by the proof of 1.16.

<u>Case 6</u>: λ strong limit, $\lambda = \aleph_{\alpha+\omega}$.

The proof of 2.25 or even better 2.33 gives this, too (for $(viii)^+$ the suitable "initial segment" of M_A can serve as M).

<u>Case 7</u>: $\lambda = \sum\limits_{i<\text{cf}(\mu)} \mu_i > \mu$, where μ_i is increasing with i, $\text{cf}(\mu_i) = \aleph_0, p(\mu_i) > \mu_i^+$ see [Shed, 3.22=Lpcf.8].

By the proof of 1.21. $\square_{3.26}$

We now turn to separable reduced $\dot{p}$-groups, continuing [Shei, 2.11=Lh5], but the proof apply to more cases.

Claim 3.28. *1) We can define for every $I \in K^\omega_{\text{tr}}$ and prime $\dot{p}$, a separable reduced abelian $\dot{p}$-group $\dot{\mathbb{G}}^a_I$ such that:*

- $(*)_0$ *$\dot{\mathbb{G}}^a_I$ has cardinality $|I| + 2^{\aleph_0}$*
- $(*)_1$ *if $I, J \in K^\omega_{\text{tr}}, I$ is $(2^{\aleph_0}, 2^{\aleph_0})$-super$^+$-unembeddable into J (see 3.26) <u>then</u> $\dot{\mathbb{G}}^a_I$ is not embeddable into $\dot{\mathbb{G}}^a_J$ (i.e. there is no mono-morphism from $\dot{\mathbb{G}}^a_I$ into $\dot{\mathbb{G}}^a_J$; we do not require purity).*

2) For $\lambda > 2^{\aleph_0}$ and prime $\dot{p}$ there is a family of 2^λ separable reduced abelian $\dot{p}$-groups, each of power λ, no one embeddable into another.

Proof. Part (2) follows from part (1) and 3.26.

1) <u>Stage A</u>: On the definition of "super$^+$ unembeddable" see 3.26. We choose a family $\{f_\alpha : \alpha < \alpha^*\}$ with $\alpha^* \le 2^{\aleph_0}$ such that:

⊡ (a) $f_\alpha \in {}^\omega\omega$,
- (b) f_α is (strictly) increasing, $f_\alpha(0) = 0$,
- (c) if h_1 is a function from ${}^{\omega>}\omega$ to ω, <u>then</u> for some α, for infinitely many $n, f_\alpha(n) > h_1(f_\alpha\restriction n)$,
- (d) $\alpha \ne \beta \Rightarrow f_\alpha \ne f_\beta$,
- (e) $\langle f_\alpha(n+1) - f_\alpha(n) : n < \omega\rangle$ goes to infinity (for convenience).

Obviously, there is such a sequence with $\alpha^* = 2^{\aleph_0}$.

For any $I \in K^\omega_{\text{tr}}$ let $\dot{\mathbb{G}}^a_I$ be the abelian group generated by

$$\{x_{\eta,\rho} : \eta \in P^I_n, \rho \in {}^{n+1}\omega \text{ and } n < \omega\} \cup \{y^n_{\eta,\alpha} : \eta \in P^I_\omega, \alpha < \alpha^* \text{ and } n < \omega\}$$

freely except the equations:

$$\dot{p}^{\rho(n)} x_{\eta,\rho} = 0 \text{ for } \eta \in P^I_n, \rho \in {}^{n+1}\omega, n < \omega$$
$$(\dot{p}^{f_\alpha(n+1)-f_\alpha(n)} y^{n+1}_{\eta,\alpha}) = y^n_{\eta,\alpha} - x_{\eta\restriction n, f_\alpha\restriction(n+1)} \text{ for } \eta \in P^I_\omega, \alpha < \alpha^*, n < \omega$$

so actually

$$y^n_{\eta,\alpha} = \sum\{\dot{p}^{f_\alpha(\ell)-f_\alpha(n)} x_{\eta\restriction\ell, f_\alpha\restriction(\ell+1)} : \ell \in \mathrm{Dom}(f_\alpha), \ell \geq n\}$$

Recall $\dot{\mathbb{G}}^a_I$ is a separable reduced abelian $\dot{p}$-group (see [Fuc73]) and:

⊙ in $\dot{\mathbb{G}}^a_I$, $\|-\|_{\dot{p}}$ is a norm <u>where</u> $\|x\|_{\dot{p}} = \inf\{2^{-n} : x$ is divisible by $\dot{p}^n$ in $\dot{\mathbb{G}}^a_I\}$.
Now,

$(*)_0$ for any $n < \omega, \eta \in P^I_n$, and $\rho \in {}^{n+1}\omega$, there is a projection $\mathbf{h} = \mathbf{h}^I_{\eta,\rho}$ of $\dot{\mathbb{G}}^a_I$ (i.e. an endomorphism of this group which is the identity on its range) defined as follows:

(α) if $m < \omega, \nu \in P^I_m, \varrho \in {}^{m+1}\omega$ then
$$(\nu, \varrho) \neq (\eta, \rho) \Rightarrow \mathbf{h}(x_{\nu,\varrho}) = 0$$
and
$$(\nu, \varrho) = (\eta, \rho) \Rightarrow \mathbf{h}(x_{\nu,\varrho}) = x_{\nu,\varrho}$$

(β) if $\nu \in P^I_\omega, \alpha < \alpha^*, m < \omega$ then:
$$(\nu\restriction n, f_\alpha\restriction(n+1)) \neq (\eta, \rho) \Rightarrow \mathbf{h}(y^m_{\nu,\alpha}) = 0,$$

(γ) $m > n \,\&\, (\nu\restriction n, f_\alpha\restriction(n+1)) = (\eta, \rho) \Rightarrow \mathbf{h}(y^m_{\nu,\alpha}) = 0,$

(δ) $m \leq n \,\&\, (\nu\restriction n, f_\alpha\restriction(n+1)) = (\eta, \rho) \Rightarrow \mathbf{h}(y^m_{\nu,\alpha}) = \dot{p}^{f_\alpha(n)-f_\alpha(m)} x_{\eta,\rho}$.

Also note:

$(*)_1$ if $I \in K^\omega_{\mathrm{tr}}$ for every $z \in \dot{\mathbb{G}}^a_I$ and m there is $z' \in \dot{\mathbb{G}}^a_I$ such that
- (a) $z - z'$ is divisible by $\dot{p}^m$ in $\dot{\mathbb{G}}^a_I$
- (b) $z' \in \sum\{\mathbb{Z}x_{\eta,\rho}$: for some $n < \omega$ we have: $\eta \in P^I_n$ and $\rho \in {}^{n+1}\omega\}$.

<u>Stage B</u>: For proving the claim toward contradiction we assume:

⊞ $I \in K^\omega_{\mathrm{tr}}$ is super$^+$-unembeddable into $J \in K^\omega_{\mathrm{tr}}$, (i.e. as in 3.26) but $\mathbf{g}$ is an embedding of $\dot{\mathbb{G}}^a_I$ into $\dot{\mathbb{G}}^a_J$.

Let χ be large enough and let $\eta \in P^I_\omega, \langle M_n, N_n : n < \omega\rangle$ and M be as guaranteed in 3.26, and the following belongs to M_0:

$\mathbf{g}, I, J, \dot{\mathbb{G}}^a_I, \dot{\mathbb{G}}^a_J$ and the functions $(\eta, \rho) \mapsto x_{\eta,\rho}, (\eta, \alpha, n) \mapsto y^n_{\eta,\alpha}$ and so $(\eta, \rho) \mapsto \mathbf{h}^I_{\eta,\rho}, (\eta, \rho) \mapsto \mathbf{h}^J_{\eta,\rho}$ and $\{\alpha : \alpha < \alpha^*\} \subseteq M_0$.

Remember $\eta\restriction(n+1) \in N_n \setminus M_n$ (by $(v)^+$ there). For $\iota < \omega, \rho \in {}^{\iota+2}\omega$ let

$$k_\rho := \mathbf{n}(\dot{p}^\iota \mathbf{g}(x_{\eta\restriction(\iota+1),\rho}), \dot{\mathbb{G}}^a_J \cap M_\ell)$$

where for $y \in \dot{\mathbb{G}}^a_J$ and $\dot{\mathbb{G}} \subseteq \dot{\mathbb{G}}^a_J$ we let:

$$\mathbf{n}(y, \dot{\mathbb{G}}) = \sup\{k : \text{ for some } z \in \dot{\mathbb{G}}, y - z \text{ is divisible in } \dot{\mathbb{G}}^a_J \text{ by } \dot{p}^k\}.$$

<u>Stage C</u>:

Now,

Fact 3.29. We have:

$\otimes$ $k_\rho < \omega$ <u>when</u> $\rho \in {}^{\ell+2}\omega, \ell < \omega$.

Proof. Otherwise, we can let

$(*)_2$ $\dot{\mathbb{G}}^a_J \vDash \mathbf{g}(x_{\eta\restriction(\ell+1),\rho}) = \sum\limits_{(\nu,\rho)\in u_1} a_{\nu,\rho}x_{\nu,\rho} + \sum\limits_{(\eta,\beta)\in u_2} b_{\eta,\beta}y^{m(\eta,\beta)}_{\eta,\beta}$ with

- (a) $u_1 \subseteq \{(\nu,\rho) : \nu \in P^J_k$ and $\rho \in {}^{k+1}\omega$ for some $k < \omega\}$,
- (b) $u_2 \subseteq \{(\nu,\beta) : \nu \in P^J_\omega$ and $\beta < \alpha^*\}$,
- (c) $a_{\nu,\rho}, b_{\nu,\beta} \in \mathbb{Z}$,
- (d) $m(\nu,\beta) < \omega$,
- (e) $\dot{\mathbb{G}}_J \models$ "$a_\nu, x_{\nu,\rho} \neq 0$, and $b_{\nu,\beta}y^\ell_{\nu,\beta} \neq 0$ (in $\dot{\mathbb{G}}^a_J$)
- (f) u_1, u_2 are finite.

By the way $\dot{\mathbb{G}}^a_J$ was defined we can replace $y^{\mathbf{n}(\nu,\beta)}_{\nu,\beta}$ by $\dot{p}^{f_\beta(\mathbf{n}(\nu,\beta)+1)-f_\beta(\mathbf{n}(\nu,\beta))}y^{\mathbf{n}(\nu,\beta)+1}_{\nu,\beta} + x_{\nu\restriction(\mathbf{n}(\nu,\beta)+1),\beta}$ and repeat this, hence using clause (e) of $\boxdot$, without loss of generality for some $m_0 < m_1 < \omega$ large enough:

$(*)_3$ (a) $(\eta_1,\beta_1) \in u_2 \,\&\, (\eta_2,\beta_2) \in u_2 \,\&\, \eta_1 \neq \eta_2 \Rightarrow \eta_1\restriction m_0 \neq \eta_2\restriction m_0$

(b) $(\eta,\beta_1) \in u_2 \,\&\, (\eta,\beta_2) \in u_2 \,\&\, \beta_1 \neq \beta_2 \Rightarrow f_{\beta_1}\restriction m_0 \neq f_{\beta_2}\restriction m_0$

(c) if $(\eta,\beta) \in u_2$ then

- (α) $\mathbf{n}(\eta,\beta) > m_0$
- (β) if $m_0 \leq m < \mathbf{n}(\eta,\beta)$ then $(\eta\restriction m, f_\beta\restriction(m+1)) \in u_1$ and $\dot{p}^{f_\beta(m(\eta,\beta))-f_\beta(m)}a_{\eta\restriction m,f_\beta\restriction(m+1)} = b_{\eta,\beta}$
- (γ) $|a_{\eta\restriction m_0,f_\beta\restriction(m_0+1)}| < m_1$
- (δ) $b_{\eta,\beta}y^{\mathbf{n}(\eta,\beta)}_{\eta,\beta}$ is divisible by $\dot{p}^{m_1}$ in $\dot{\mathbb{G}}^a_J$.

(d) if $(\nu,\rho) \in u_1$ then $a_{\nu,\rho}x_{\nu,\rho}$ is not divisible by $\dot{p}^{m_1}$ in $\dot{\mathbb{G}}^a_J$.

So, using $(*)_0 + (*)_1 + (*)_2$ in $\dot{\mathbb{G}}^a_J$ and our assumption toward contradiction that $k_\rho = \omega$, necessarily $u_1 \in M_\ell$, hence $(\nu,\rho) \in u_1 \Rightarrow a_{\nu,\rho}x_{\nu,\rho} \in M_\ell$. Repeating this increasing m_1 (hence the $\mathbf{n}(\eta,\beta)$'s) we get also $(\nu,\beta \in u_2 \Rightarrow \bigwedge\limits_{i<\omega} \nu\restriction i \in M_\ell$, hence by clause (vii) of 3.26 we have $(\nu,\beta) \in u_2 \Rightarrow \nu \in M_\ell \Rightarrow y^m_{\nu,\beta} \in M_\ell \Rightarrow b_{\nu,\beta}y^m_{\nu,\beta} \in M_\ell$. Together by $(*)_2$ in $\otimes$ we have $\mathbf{g}(x_{\eta\restriction(\ell+1),\rho}) \in M_\ell$, but $\mathbf{g} \in M_0$ is one to one, hence $\eta\restriction(\ell+1) \in M_n$, contradiction. So really $k_\rho < \omega$, i.e. $\otimes$ holds. $\square_{3.29}$

<u>Stage D</u>:

By the choice of $\langle f_\alpha : \alpha < \alpha^*\rangle$ for some $\alpha < \alpha^*$, for infinitely many $\ell < \omega$ we have: $f_\alpha(\ell+1) > k_{f_\alpha\restriction(\ell+1)}$.

Now in $\dot{\mathbb{G}}^a_I$ for each $m < \omega$, $y^0_{\eta,\alpha} - \sum\limits_{n<m} \dot{p}^{f_\alpha(n)}x_{\eta\restriction n,f_\alpha\restriction(n+1)}$ is divisible by $\dot{p}^{f_\alpha(m)}$ hence for each $m < \omega$:

$(*)_4$ $\mathbf{g}(y^0_{\eta,\alpha}) - \sum\limits_{n<m} \dot{p}^{f_\alpha(n)}\mathbf{g}(x_{\eta\restriction n,f_\alpha\restriction(m+1)})$ is divisible by $\dot{p}^{f_\alpha(m)}$ in $\dot{\mathbb{G}}^a_J$.

Now $\mathbf{g}(y^0_{\eta,\alpha})$ has, for some $n(0)$, the form

$$\sum\{b_{\eta,\alpha}y^{n(0)}_{\eta,\alpha} : \eta \in \boldsymbol{\Upsilon}_0, \alpha \in u_0\} + \sum\{a_{\eta,\rho}x_{\eta,\rho} : (\eta,\rho) \in \boldsymbol{\Upsilon}_1\}$$

where:

u_0 a finite subset of α^*

$\boldsymbol{\Upsilon}_0$ a finite subset of P^J_ω

$\mathbf{\Upsilon}_1$ a finite subset of $\bigcup_{n<\omega}(P^J_n \times {}^{n+1}\omega)$

$b_{\eta,\alpha}, a_{\eta,\rho} \in \mathbb{Z}$.

Let

$$\mathbf{\Upsilon}'_1 = \{\eta : \text{ for some } \rho \in {}^{\omega>}\omega \text{ we have } (\eta,\rho) \in \mathbf{\Upsilon}_1\}.$$

We can find $n(1) < \omega$ such that:

(α) $n(1) > n(0)$

(β) $\mathbf{\Upsilon}'_1 \cap (\bigcup_{n<\omega} M_n) \subseteq M_{n(1)}$

(γ) $\eta \in \mathbf{\Upsilon}_0 \,\&\, n \geq n(1) \Rightarrow \{\eta\restriction\ell : \ell < \omega\} \cap N_n \subseteq M_n$.

(For (γ) use clause (vi) of ($*$) of 3.26, i.e. of 1.1.)

So by the choice of α we can find ℓ such that:

$$n(1) < \ell < \omega$$

$$f_\alpha(\ell+1) > k_{f_\alpha\restriction(\ell+1)}.$$

Now by $(*)_4$

$$\mathbf{n}(\mathbf{g}(y^0_{\eta,\alpha}), \dot{\mathbb{G}}^a_J \cap N_\ell) \geq f_\alpha(\ell+1)$$

as exemplified by $\sum_{i\leq\ell} \dot{p}^{f_\alpha(i)}\mathbf{g}(x_{\eta\restriction i, f_\alpha\restriction(i+1)})$.

Now if

$$\mathbf{n}(\mathbf{g}(y^0_{\eta,\alpha}), \dot{\mathbb{G}}^a_J \cap M_\ell) \text{ is } \geq f_\alpha(\ell+1)$$

then we get (use again $(*)_4$)

$$\mathbf{n}(\sum_{i\leq\ell} \dot{p}^{f_\alpha(i)}\mathbf{g}(x_{\eta\restriction i, f_\alpha\restriction(i+1)}), \dot{\mathbb{G}}^a_J \cap M_\ell) \text{ is } \geq f_\alpha(\ell+1)$$

but for $i < \ell$

$$\mathbf{g}(x_{\eta\restriction i, f_\alpha\restriction(i+1)}) \in M_i \text{ (as } \eta\restriction i, \mathbf{g} \in M_i)$$

so we get

$$\mathbf{n}(\dot{p}^{f_\alpha(\ell)}\mathbf{g}(x_{\eta\restriction\ell, f_\alpha\restriction(\ell+1)}), \dot{\mathbb{G}}^a_J \cap M_\ell) \text{ is } \geq f_\alpha(\ell+1) > k_{f_\alpha\restriction(\ell+1)}.$$

But this contradicts the definition of $k_{f_\alpha\restriction(\ell+1)}$.

So

$$\mathbf{n}(\mathbf{g}(y^0_{\eta,\alpha}), \dot{\mathbb{G}}^a_J \cap M_\ell) < f_\alpha(\ell+1) \leq \mathbf{n}(\mathbf{g}(y^0_{\eta,\alpha}), \dot{\mathbb{G}}^a_J \cap N_\ell).$$

But this contradicts $\ell > n(1)$. $\square_{3.28}$

Remark 3.30. Really, the proof of 3.28 is a kind of simple black box: we attach to every $\eta \in P^{I_\zeta}_\omega$, a first order theory T_η such that:

if $I = I_\zeta, J = \sum_{\xi\neq\zeta} I_\xi$, and χ^* is regular large enough, $x \in \mathscr{H}(\chi^*)$, then we can find $\eta, \langle M_\eta, N_n : n < \omega\rangle$ as in 1.1, 3.26 and T_η is the first order theory of $(\bigcup_n M_n, M_m, N_m, \eta\restriction m)_{m<\omega}$. We need of course $\kappa \geq 2^{\aleph_0}$.

Remark 3.31. 1) We could have used in the proof only (($*$) of Def.1.1 and) (vii) of 3.26; but as we have used also $(v)^+$ from 3.26 we can add:

(∗) $\alpha^* = \mathfrak{b} = \min\{|\mathscr{F}| : \mathscr{F}$ is a set of functions from ω to ω such that for every $g \in {}^\omega\omega$ for some $f \in \mathscr{F}$ we have $(\exists^\infty n)[g(n) < f(n)]\}$.

Hence we can improve 3.28 in two ways:

(α) we can omit (viii) in 3.26 and add $|\dot{\mathbb{G}}^a_I| = |I| + \mathfrak{b}$

or

(β) we can weaken the "super$^+$" assumption and omit $(v)^+, (viii)$ from 3.26.

Of course (assuming less, getting less)

Conclusion 3.32. *If $\lambda > \aleph_0$ then there are 2^λ separable reduced abelian $\dot{p}$-groups of cardinality λ no one purely embeddable into another.*

Proof. By 2.34. In detail, by 2.34 there is $\langle I_\alpha : \alpha < 2^\lambda\rangle$ such that $\alpha \neq \beta$ implies I_α is $(\aleph_0, \aleph_0)$-super unembeddable into I_β. But $\alpha \neq \beta$ implies I_α is strongly $\varphi_{\rm tr}$-unembeddable into I_α. Now $\dot{\mathbb{G}}_{I_\alpha}$ is defined in [Shei, 2.11=Lh5]. By [Shei, 2.13=h8] we have $\dot{\mathbb{G}}_{I_\alpha}$ is a separable reduced abelian $\dot{p}$-group. Lastly, "$\dot{\mathbb{G}}_{I_\alpha}$ not purely embeddable into $\dot{\mathbb{G}}_{I_\beta}$" by [Shei, 2.14 = Lh11]. $\square_{3.32}$

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