MODULES AND INFINITARY LOGICS
SH:977

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Dedicated to Rüdiger Göbel for this 70th birthday

ABSTRACT. We deal with Abelian groups and R-modules. We consider theories in infinitary logic of the form $\mathbb{L}_{\lambda,\theta}$ of such structures M and prove they have elimination of quantifiers up to positive existential formulas, (so ones defining subgroups of some power of M). However, we demand that we expand by enough individual constants. Hence those theories are stable in the appropriate sense and understood to some extent.

In 2026, John Baldwin 1 pointed out a mistake in the end of the proof of the main claim of Section 4, which is used in the theorem in Section 2, and other comment. Here this is corrected and more improvements. The eroor is corrected in to ways- in section 2 we can use a weaker version of section 4 and in section 4 we get the original result by assuming more on the cardinal.

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§ 0. Introduction

Much is known on classes of R-modules and first order logic. Szmielew [Szm49] proved the decidability of the theory of Abelian groups. In [Szm55], she proved an elimination of quantifiers in the theory of Abelian groups up to Boolean combinations of p.e. (*positive existential*) formulas.

Eklof [Ekl71] proved the existence of universal homogeneous R-models in λ if $\lambda = \lambda^{<\gamma}$, where γ depends only on R. Fisher improved this to saturated models of elementary classes (see his review of [Ekl71]); this implies stability by a general theorem from [She71, §0] (or [She90, Ch. III]).

Baur [Bau76] proved that for the class of R-modules, any first-order formula is equivalent to a Boolean combination of positive existential formulas, and also proved the stability of $\mathrm{Th}(M)$ for M an R-module.

We like to know for a given ring R how complicated the class of R-modules which are models of a sentence ψ in an infinitary logic.

Question 0.1. Given a ring R, for the class Mod_R of left R-modules:

1) Does it have (for the logic $\mathbb{L}_{\lambda,\mu}$) a kind of elimination of quantifiers (say, up to some depth)?

2) Is it stable? (Say, no formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\infty,\infty}(\tau_R)$ linearly ordering an arbitrarily long sequence of tuples in some models of ψ?)

3) Can we define something like non-forking?

Question 0.2. Do we have a parallel of the main gap — i.e. proving that either every $M \in \mathrm{Mod}_\psi$ can be characterized by some suitable cardinal invariants <u>or</u> that there are many complicated $M \in \mathrm{Mod}_\psi$?

Here we first show that for any R-module, in $\mathbb{L}_{\lambda,\theta}(\tau_R)$ (or better, $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$), we have a version of eliminating quantifiers up to positive existential formulas. <u>However</u>, we add parameters. Second, by this we can prove some versions and consequences of stability. More specifically:

- After expanding by enough individual constants, every formula in $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$ is equivalent to a Boolean combination of such positive existential formulas.
- The number of added individual constants is reasonable: $\leq \beth_\gamma\big(|\tau|^{<\theta}\big)$.
- We have stability: i.e. no long sequences of linearly ordered $(<\theta)$-tuples.
- $(\Lambda^{\mathrm{pe}}_{\varepsilon,\alpha}, 2)$-indiscernible implies $\Lambda^{\mathrm{pe}}_{\varepsilon,\alpha}$-indiscernible.
- Convergence follows (see Definition 3.4).

In 2025, this work was continued in a paper with Asgharzadeh and Golshani [AGS25].

We may use models with several *sorts* — that is, multiple distinct structures defined on them. E.g., when constructing an R-module we need a set of objects which are the elements of the module, and a set of elements of the ring R, each with their own operations of addition and (scalar) multiplication. Hence when we need to disambiguate them, we will write something like $x +_{\mathbf{s}} y$, $x -_{\mathbf{s}} v$ for each sort separately. It makes no difference (see 5.2).

§ 1. Preliminaries

Notation 1.1. Let θ^- be σ if $\theta = \sigma^+$ and θ if θ is a limit cardinal.

Definition 1.2. 1) A vocabulary τ consists of function symbols (e.g. individual constants) and predicates (relation symbols). In addition the vocabulary generally assign to each of them its arity (number of places) $\mathrm{arity}_\tau(-)$; here it can be an infinite ordinal. An individual constant is a 0-place function.

2) For a vocabulary τ we say M is a τ-structure when it consists of:

(a) $|M|$, the *universe* of M; this is a non-empty set of the so-called elements of M. However, we may write $a \in M$, $\bar{a} \in {}^\varepsilon M$, $A \subseteq M$, instead of $a \in |M|$, $\bar{a} \in {}^\varepsilon(|M|)$, etc.

(b) F^M, a function from ${}^\varepsilon M$ to M (possibly partial), for each function symbol F from τ. Here ε is the ordinal $\mathrm{arity}_\tau(F)$.

(c) $P^M \subseteq {}^\varepsilon M$ (where ε is the ordinal $\mathrm{arity}_\tau(P)$), for P a predicate from τ.

We may write $\tau_M = \tau(M) := \tau$.

Definition 1.3. 1) We say τ is a θ-*additive* (or a θ-Abelian) vocabulary when τ has the two-place function symbols $x + y, x - y$, the individual constant 0, and the other predicates and function symbols have arity $< \theta$.

2) M is a θ-*additive structure* (or model) when:

(a) τ_M, (the vocabulary of M) is a θ-additive vocabulary.

(b) $G_M := (|M|, +^M, -^M, 0^M)$ is an Abelian group.

(c) If $P \in \tau_M$ is an ε-place predicate then P^M is a subgroup of $(G_M)^\varepsilon$.

(d) If $F \in \tau_M \setminus \{+, -, 0\}$ is an ε-place function symbol then F^M is a partial ε-place function from M to M and

$$\mathrm{graph}\big(F^M\big) := \big\{\bar{a}\hat{\ }\big\langle F^M(\bar{a})\big\rangle : \bar{a} \in \mathrm{dom}(F^M)\big\}$$

is a subgroup of $(G_M)^{\varepsilon+1}$.

3) If $\varphi(\bar{x}, \bar{y})$ is a formula in the vocabulary τ, M is a τ-model, and $\bar{b} \in {}^{\ell g(\bar{y})}M$, then we write $\varphi(M, \bar{b}) = \varphi({}^{\ell g(\bar{x})}M, \bar{b})$ to mean the set

$$\big\{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}, \bar{b}]\big\}.$$

Remark 1.4. Fisher [Fis77] defines and deals with "Abelian structures" in other directions.

Definition 1.5. 1) We consider an R-module M as a $\tau(R)$-structure, where $\tau_R = \tau(R)$ is the vocabulary of R-modules. I.e. there are binary functions $x + y, x - y$, an individual constant 0, and unary function symbols F_a (interpreted as multiplication by a from the left) for every $a \in R$.

2) If $\bar{x}$ and $\bar{y}$ have length ε, then we let $\bar{x} + \bar{y} = \langle x_\zeta + y_\zeta : \zeta < \varepsilon\rangle$ and $\bar{x} - \bar{y} = \langle x_\zeta - y_\zeta : \zeta < \varepsilon\rangle$; similarly for $a\bar{x}$ with $a \in R$, and when we replace $\bar{x}$ and/or $\bar{y}$ by a member of ${}^\varepsilon M$.

Observation 1.6. 1) *For any ring R, an R-module is an $\aleph_0$-additive structure in the vocabulary τ_R.*

2) *For a τ-additive model M, for every function symbol or τ-term $F(\bar{x})$, we have*

(a) $M \models$ "$F(\bar{a} \pm \bar{b}) = F(\bar{a}) \pm F(\bar{b})$"
(When F^M is partial, this means that if two of the terms are well-defined then so is the third, and the equality holds.)

(b) *For P a predicate from τ_M, $M \models P(\bar{a} \pm \bar{b})$ whenever $M \models P(\bar{a}) \wedge P(\bar{b})$.*

§ 2. Eliminating quantifiers

Context 2.1. 1) R is a fixed ring and $\tau = \tau_R$ (see 1.5(1)), <u>or</u> τ is just a θ-additive vocabulary (see 1.3(1), 1.6(1)).

2) $\mathbf{K}$ is the class of R-modules <u>or</u> of τ-additive models.

3) M, N will denote R-modules <u>or</u> just τ-additive models.

4) $\theta = \mathrm{cf}(\theta)$.

5) $\mathbb{L}_{\mu,\theta,\alpha}(\tau)$ is the set of formulas $\varphi(\bar{x}) \in \mathbb{L}_{\mu,\theta}$ (so $\ell g(\bar{x}) < \theta$) of quantifier depth $< 1 + \alpha$.

Definition 2.2. For $\varepsilon < \theta$ and ordinal α (and τ as in 2.1(1)), we shall define

$$\Lambda^{\rm pe}_{\alpha,\varepsilon} = \Lambda^{{\rm pe},\theta}_{\alpha,\varepsilon} = \Lambda^{{\rm pe},\theta}_{\alpha,\varepsilon}(\tau)$$

as a set of formulas $\varphi(\bar{x})$ in $\mathbb{L}_{\infty,\theta}(\tau)$ (in fact, in $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$) with $\ell g(\bar{x}) = \varepsilon < \theta$. (So $\bar{x} = \langle x_\xi : \xi < \varepsilon \rangle$ if not said otherwise.) The construction will proceed by induction on the ordinal α.

For $\zeta < \theta$, we write $\Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta}$ for the set of $\varphi = \varphi(\bar{x}, \bar{y})$ with $\ell g(\bar{x}) = \varepsilon$ and $\ell g(\bar{y}) = \zeta$ (so $\bar{y} = \langle y_\xi : \xi < \zeta \rangle$, if not said otherwise) with $\varphi \in \Lambda^{\rm pe}_{\alpha,\varepsilon+\zeta}$. We define $\Lambda^{\rm pe}_{\alpha} := \bigcup\limits_{\varepsilon<\theta} \Lambda^{\rm pe}_{\alpha,\varepsilon}$ and $\Lambda^{\rm pe}_{\alpha,\varepsilon,<\theta} := \bigcup\limits_{\zeta<\theta} \Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta}$. If $\tau = \tau_R$ we may write $\Lambda^{\rm pe}_{\alpha,\varepsilon}(R)$.

The definition is as follows:

Case 1: $\alpha = 0$.

<u>For R-modules:</u>

It is the set of $\varphi = \varphi(\bar{x})$ of the form $\sum\limits_{\ell<n} a_\ell x_{\zeta_\ell} = 0$ with $\zeta_\ell < \ell g(\bar{x})$. Equivalently (but perhaps better phrased), they are of the form $\sum\limits_{\zeta<\varepsilon} a_\zeta x_\zeta = 0$, where $a_\zeta \in R$ is 0_R for all but finitely many ζ-s.

<u>For general τ:</u> (so here, this is the τ-additive case).

It is the set of $\varphi(\bar{x})$ of the form $P(\bar{\sigma}(\bar{x}))$, where $\bar{\sigma}$ is a sequence of length $\mathrm{arity}_\tau(P)$ of terms (in the variables $\bar{x}$). P may be equality, or any predicate from τ of arity equal to $\ell g(\bar{\sigma})$.

Case 2: α a limit ordinal.

$\Lambda^{\rm pe}_{\alpha,\varepsilon} := \bigcup\limits_{\beta<\alpha} \Lambda^{\rm pe}_{\beta,\varepsilon}$.

Case 3: $\alpha = \beta + 1$.

We define $\Lambda^{\rm pe}_{\alpha,\varepsilon}$ as the union of $\Lambda^{\rm pe}_{\beta,\varepsilon}$ together with the set of all formulas $\psi(\bar{x})$ of the form

$$(\exists \bar{y}_\zeta) \bigwedge \left\{ \varphi(\bar{x}\hat{\ }\bar{y}_\zeta) : \varphi(\bar{x}, \bar{y}_\zeta) \in \Phi_\zeta \right\},$$

for some $\zeta < \theta$ and $\Phi_\zeta \subseteq \Lambda^{\rm pe}_{\beta,\varepsilon,\zeta}$.

Claim 2.3. 1) *In 2.2, $\Lambda^{\rm pe}_{\alpha,\varepsilon}$ is $\subseteq$-increasing with α, and is of cardinality $\leq \beth_\varepsilon(|\tau| + \aleph_0)$ if $\theta = \aleph_0$, and $\beth_\varepsilon(|\tau|^{<\theta})$ in general.*

2) *For $M \in \mathbf{K}$ and $\varphi(\bar{x}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}(\tau)$, the set*

$$\varphi(\overline{M}) := \left\{ \bar{b} \in {}^\varepsilon M : M \models \varphi[\bar{b}] \right\}$$

is an Abelian subgroup of ${}^\varepsilon M$, and the set $\left\{ \bar{b} \in {}^\varepsilon M : M \models \varphi[\bar{b} - \bar{a}] \right\}$ is affine (that is, closed under $\bar{x} - \bar{y} + \bar{z}$) for any $\bar{a} \in {}^\varepsilon M$.

Proof. Easy. $\square_{2.3}$

Theorem 2.4. *For every α and every $M \in \mathbf{K}$, there is a subset $\mathbf{I} = \mathbf{I}_\alpha$ of ${}^{\theta>}M$ of cardinality $\le \kappa_\alpha := \beth_\alpha(|\tau|^{<\theta})$ such that in M, every formula $\psi(\bar{x})$ from $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$ (so $\ell g(\bar{x}) < \theta$) is equivalent in M to a Boolean combination of formulas of the form $\varphi(\bar{x} - \bar{a})$ with $\varphi(\bar{x}) \in \Lambda^{\rm pe}_{\alpha,\ell g(\bar{x})}(\tau)$ and $\bar{a} \in \mathbf{I} \cap {}^{\ell g(\bar{x})}M$.*

Before this, we shall note that

Conclusion 2.5. *For every $M \in \mathbf{K}$, $\varepsilon < \theta$, $\mathbf{I}_\alpha$ as in Theorem 2.4 for α a limit ordinal, and $\bar{a} \in {}^\varepsilon M$, for some $i_*, j_* \le \kappa_\alpha$ and $\varphi_i(\bar{x}_\varepsilon), \psi_j(\bar{x}_\varepsilon) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$ for $i < i_*$, $j < j_*$ we have that*

$$\left\{\bar{a}' \in {}^\varepsilon M : \mathrm{tp}_{\mathbb{L}^{\rm pe}_{\infty,\theta,\alpha}}(\bar{a}', \varnothing, M) = \mathrm{tp}_{\mathbb{L}^{\rm pe}_{\infty,\theta,\alpha}}(\bar{a}, \varnothing, M)\right\}$$

is equal to

$$\left\{\bar{a}' \in {}^\varepsilon M : M \models \bigwedge_{i<i_*} \varphi_i(\bar{a}' - \bar{a}) \wedge \bigwedge\{\neg\psi_j(\bar{a}' - \bar{a}'') : j < j_* \text{ and } \bar{a}'' \in \mathbf{I}_\alpha \cap {}^\varepsilon M\}\right\}.$$

Definition 2.6. 1) We say $\bar{b}_1, \bar{b}_2 \in {}^\varepsilon M$ are *α-equivalent over* $\mathbf{I} \subseteq {}^{\theta>}M$ <u>when</u>

$$\varphi(\bar{x}_{[\varepsilon]}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}(R) \wedge \bar{a} \in \mathbf{I} \cap {}^\varepsilon M \;\Rightarrow\; M \models \text{“}\varphi[\bar{b}_1 - \bar{a}] \Leftrightarrow \varphi[\bar{b}_2 - \bar{a}]\text{”}.$$

2) If we write $A \subseteq M$ instead of $\mathbf{I}$, we mean $\mathbf{I} = {}^{\theta>}A$.

We shall use freely

Observation 2.7. *The sequences $\bar{b}_1, \bar{b}_2 \in {}^\varepsilon M$ are α-equivalent over $\mathbf{I} \subseteq {}^\varepsilon M$ <u>iff</u> for any $\varphi(\bar{x}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$ we have $(a) \vee (b)$, where:*

- (a) *For some $\bar{a} \in \mathbf{I} \cap {}^\varepsilon M$ we have $M \models \varphi[\bar{b}_1 - \bar{a}] \wedge \varphi[\bar{b}_2 - \bar{a}]$.*
- (b) *For every $\bar{a} \in \mathbf{I} \cap {}^\varepsilon M$ we have $M \models \neg\varphi[\bar{b}_1 - \bar{a}] \wedge \neg\varphi[\bar{b}_2 - \bar{a}]$.*

Proof. Straightforward. $\square_{2.7}$

Proof of Theorem 2.4. We choose $\mathbf{I}_\alpha$ by induction on α so that it satisfies the demands of the theorem, <u>and</u>

$\circledast_\alpha$ $\big(\forall\varepsilon < \theta\big)\big(\forall\varphi(\bar{x}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}(\tau)\big)\big(\forall\bar{a} \in {}^\varepsilon M\big)\big(\exists\bar{b} \in \mathbf{I}_\alpha \cap {}^\varepsilon M\big)\big[M \models \varphi[\bar{b} - \bar{a}]\big]$.

For $\alpha = 0$ choose $\mathbf{I}_0 := {}^{\theta>}\{0^M\}$, and for α a limit ordinal we obviously want $\mathbf{I}_\alpha := \bigcup_{\beta<\alpha} \mathbf{I}_\beta$.

So assume $\alpha = \beta + 1$ and $\mathbf{I}_\beta$ is given, and we shall choose $\mathbf{I}_\alpha$ such that

$\boxplus_\alpha$
- (a) $\mathbf{I}_\alpha$ is a subset of ${}^{\theta>}M$.
- (b) $|\mathbf{I}_\alpha| \le 2^{\kappa_\beta}$, where $\kappa_\beta := \beth_\beta(|\tau|^{<\theta})$.
- (c) $\mathbf{I}_\beta \subseteq \mathbf{I}_\alpha$
- (d) If $\varepsilon < \theta$, $\varphi_i(\bar{x}) \in \Lambda^{\rm pe}_{\beta,\varepsilon}$, $\bar{a}_i \in \mathbf{I}_\beta \cap {}^\varepsilon M$ for $i < i_* \le \kappa_\beta$, and there is $\bar{d} \in {}^\varepsilon M$ such that $M \models \bigwedge_{i<i_*} \varphi_i[\bar{d} - \bar{a}_i]$, <u>then</u> there is such $\bar{d} \in \mathbf{I}_\alpha$.

(e) Assume $\varepsilon < \theta$, $\ell g(\bar{x}) = \varepsilon$, $\psi(\bar{x})$ is a conjunction of formulas from $\Lambda^{\rm pe}_{\beta,\varepsilon}$ and $\varphi_i(\bar{x}) \in \Lambda^{\rm pe}_{\beta,\varepsilon}$ for $i < \kappa_\beta$. Letting $\mu := 2^{\kappa_\beta}$, we apply 4.3 with $\lambda = \mu^+$, κ_β, $\psi({}^\varepsilon M)$, and $\langle \psi({}^\varepsilon M) \cap \varphi_i({}^\varepsilon M) : i < \kappa_\beta \rangle$ here standing in for $\lambda, S, G, \langle G_s : s \in S\rangle$ (i.e. the subgroups of $({}^\varepsilon|M|, +^M)$ with universes as above) there. Then the conclusion of 4.3 gives us a certain family of sets $I \subseteq \mathcal{P}(\kappa_\beta)$ not necessarily an ideal. Further assume that $\kappa_\beta \notin I$. Then

- $\bullet_1$ There are $\bar{d}_\iota \in \mathbf{I}_\alpha \cap \varphi_i({}^\varepsilon M)$ for $\iota < \iota_* \leq \mu$ such that for every $\bar{a} \in \psi({}^\varepsilon M)$ there exists $\iota < \iota_*$ such that
$$\big\{ j < \kappa_\beta : \bar{a} - \bar{d}_\iota \notin \varphi_j({}^\varepsilon M) \big\} \in I.$$
- $\bullet_2$ For any $u \in I$ there is a set u_* with $u \subseteq u_* \in I$ and a sequence
$$\langle \bar{d}_\iota : \iota < \mu \rangle \subseteq \bigcap \{ \varphi_i({}^\varepsilon M) : i \in \kappa_\beta \setminus u_* \} \cap \psi({}^\varepsilon M) \cap \mathbf{I}_\alpha$$
such that
$$\big(\forall i < \kappa_\beta\big)\big(\forall \iota_1 \leq \iota_2 < \mu\big)\big[\bar{d}_{\iota_1} - \bar{d}_{\iota_2} \notin \varphi_i({}^\varepsilon M) \Leftrightarrow i \in u_*\big].$$

(f) if $\varepsilon < \theta$ and $\bar{d}_1, \bar{d}_2 \in \mathbf{I}_\alpha \cap {}^\varepsilon M$ then $\bar{d}_1 + \bar{d}_2 \in \mathbf{I}_\alpha$, $\bar{d}_1 - \bar{d}_2 \in \mathbf{I}_\alpha$, and $\xi < \theta \Rightarrow \bar{0}_\xi {}^\frown \bar{d}_1 \in \mathbf{I}_\alpha$.

This is possible for clause (d) because $\Lambda{\rm pe}|_{\beta,\varepsilon}$ has cardinality κ_β and $\mu = 2^{\kappa_\beta}$. This is possible for (e)$\bullet_1$ by clause (a) of Claim 4.3 and for (e)$\bullet_2$ by 4.3(b).
To prove the induction statement for α, clearly it suffices to prove the following.

$\boxdot$ Assume $\varepsilon, \xi < \theta$, $\bar{b}_1, \bar{b}_2 \in {}^\varepsilon M$ are α-equivalent over $\mathbf{I}_\alpha$, and $\bar{c}_1 \in {}^\xi M$. Then for some $\bar{c}_2 \in {}^\xi M$ the sequences $\bar{b}_1 {}^\frown \bar{c}_1$ and $\bar{b}_2 {}^\frown \bar{c}_2 \in {}^{\varepsilon+\xi} M$ are β-equivalent over $\mathbf{I}_\beta$.

Why does $\boxdot$ hold? Let $\bar{x}$ be of length ε and $\bar{y}$ of length ξ. Let
$$\Phi_1 := \big\{ \varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm pe}_{\beta,\varepsilon+\xi} : \text{for some } \bar{a} \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi} M \text{ we have } M \models \varphi[\bar{b}_1 {}^\frown \bar{c}_1 - \bar{a}] \big\}.$$
For $\varphi(\bar{x}, \bar{y}) \in \Phi_1$, by $\circledast_\alpha$ we can choose $\bar{a}_{\varphi(\bar{x},\bar{y})} \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi} M$ such that $M \models \varphi[\bar{b}_1 {}^\frown \bar{c}_1 - \bar{a}_{\varphi(\bar{x},\bar{y})}]$. Let $\Phi_2 := \Lambda^{\rm pe}_{\beta,\varepsilon+\xi} \setminus \Phi_1$.

So by $\boxplus_\alpha$(d) there is a sequence $\bar{b}^* {}^\frown \bar{c}^* \in \mathbf{I}_\alpha$ such that $\ell g(\bar{b}^*) = \ell g(\bar{b}_1)$, $\ell g(\bar{c}^*) = \ell g(\bar{c}_1)$, and $\varphi(\bar{x}, \bar{y}) \in \Phi_1 \Rightarrow M \models \varphi[\bar{b}^* {}^\frown \bar{c}^* - \bar{a}_{\varphi(\bar{x},\bar{y})}]$. For transparency, note that if $\Phi_2 = \varnothing$ then (as the formula $(\exists \bar{y}) \bigwedge\limits_{\varphi \in \Phi_1} \varphi(\bar{x}, \bar{y})$ is a member of $\Lambda^{\rm pe}_{\alpha,\varepsilon+\xi}$) clearly by the assumption of $\boxdot$ there is $\bar{c}_2 \in {}^\xi M$ such that
$$\varphi(\bar{x}, \bar{y}) \in \Phi_1 \Rightarrow M \models \varphi(\bar{b}_2 {}^\frown \bar{c}_2 - \bar{a}_{\varphi(\bar{x},\bar{y})}).$$
So $\bar{c}_2$ is as required, hence we are done.

So without loss of generality $\Phi_2 \neq \varnothing$. Clearly $|\Phi_2| \leq \kappa_\beta$, and let
$$\Phi'_\ell := \{ \varphi(\bar{0}_\varepsilon, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \Phi_\ell \}$$
for $\ell = 1, 2$.

Let $\{ \neg\varphi_i(\bar{x} {}^\frown \bar{y} - \bar{a}_i) : i < \kappa_\beta \}$ list (possibly with repetitions) the set of formulas $\neg\varphi(\bar{x} {}^\frown \bar{y} - \bar{a})$ satisfied by $\bar{c}_1 {}^\frown \bar{b}_1$ with $\bar{a} \in \mathbf{I}_\beta$ and $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm pe}_{\beta,\varepsilon,\zeta}$ (equivalently, $\varphi(\bar{x}, \bar{y}) \in \Phi_2$). Let $\varphi'_i(\bar{y}) := \varphi_i(0_\varepsilon, \bar{y})$ for $i < \kappa_\beta$, and $\psi'(\bar{y}) := \bigwedge\limits_{\varphi \in \Phi'_1} \varphi(\bar{y})$.

As in $\boxplus_\alpha$ let $I = I_\lambda \subseteq \mathcal{P}(\kappa_\beta)$ be defined as in Definition 4.1, with $G := \psi'({}^\xi M)$, $G_i := G \cap \varphi'_i({}^\xi M)$ for $i \in S := \kappa_\beta$, and $\lambda := \mu^+ = (2^{\kappa_\beta})^+$.

Case 1: $\kappa_\beta \in I$.
So clearly $M \models \varphi[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^*]$ for every $\varphi(\bar{x}, \bar{y}) \in \Phi_1$.

Let $\psi_*(\bar{x},\bar{y}) = \bigwedge\{\varphi(\bar{x},\bar{y}) : \varphi(\bar{x},\bar{y}) \in \Phi_1\}$; clearly it is a member of $\Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta}$ and $M \models \psi_*[\bar{b}_1,\bar{c}_1]$. Hence by the choice of $(\bar{b}^*,\bar{c}^*)$ we also have $M \models \psi_*[\bar{b}^*,\bar{c}^*]$. As ψ_* is positive existential, clearly $M \models \psi_*[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^*]$, hence

$$M \models (\exists \bar{y})\psi_*[\bar{b}_1 - \bar{b}^*, \bar{y}].$$

But $(\exists \bar{y})\psi(\bar{x},\bar{y}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$, so by the assumption on $\bar{b}_1$ and $\bar{b}_2$ we have

$$M \models (\exists \bar{y})\psi_*[\bar{b}_2 - \bar{b}^*, \bar{y}],$$

hence for some $\bar{c}_2'$ we have $M \models \psi_*[\bar{b}_2 - \bar{b}^*, \bar{c}_2']$. Let $\bar{c}_2'' := \bar{c}_2' + \bar{c}^*$, so

$$M \models \psi_*[\bar{b}_2 - \bar{b}^*, \bar{c}_2'' - \bar{c}^*].$$

By $\boxplus_\alpha(\mathrm{e})\bullet_2$ and our assumption that $\kappa_\beta \in I$, there is a sequence $\langle \bar{e}_\iota : \iota < \mu\rangle$ of members of G (i.e. of $\{\bar{a} \in {}^\xi M : M \models \psi_*(\bar{0}_\varepsilon, \bar{a})\}$), recalling that $\psi'(\bar{y} = \psi(0_\varepsilon,\bar{y})$, such that

$$i < \kappa_\beta \wedge (\iota_1 < \iota_2 < \mu) \ \Rightarrow\ \bar{e}_{\iota_2} - \bar{e}_{\iota_1} \notin G_i.$$

So for every $\iota < \mu$, the sequence $(\bar{b}_2 - \bar{b}^*)\hat{\ }(\bar{c}_2'' - \bar{c}^* + \bar{e}_\iota)$ belongs to $\psi_*({}^{\varepsilon+\xi}M)$ and for each $i < \kappa_\beta$ the set $\{\iota < \mu : (\bar{b}_2 - \bar{b}^*)\hat{\ }(\bar{c}_2'' - \bar{c}^* + \bar{e}_\iota)$ belongs to $(\bar{a}_i - \bar{b}^*\hat{\ }\bar{c}^*) + G_i\}$ has at most one member. As $\kappa_\beta < \mu$, we have

$$(\bar{b}_2 - \bar{b}^*)\hat{\ }(\bar{c}_2'' - \bar{c}^* + \bar{e}_\iota) \notin \bigcup\big\{(\bar{a}_i - \bar{b}^*)\hat{\ }(\bar{c}^* + G_i) : i < \kappa_\beta\big\}$$

for[1] some $\iota < \mu$.

So $\bar{c}_2 := \bar{c}_2'' + \bar{e}_\iota$ is as required.

Case 2: $\kappa_\beta \notin I$.

So there is a sequence $\langle \bar{d}_\iota : \iota < \iota_*\rangle$ of members of $\mathbf{I}_\alpha$ as in $\boxplus_\alpha(\mathrm{e})\bullet_1$ for ξ, G, $\langle G_i : i < \kappa_\beta\rangle$ as above (i.e. with $\bar{\psi}'(\bar{y})$, $\langle \varphi_i'(\bar{y}) : i < \kappa_\beta\rangle$ here standing in for $\psi(\bar{x}), \langle \varphi_i(\bar{x}) : i < \kappa_\beta\rangle$ there). So $\iota_* < (2^{\kappa_\beta})^+ = \mu^+$ and $\iota < \iota_* \Rightarrow \bar{d}_\iota \in \mathbf{I}_\alpha \cap {}^\xi M$. As clearly $\bar{c}_1 - \bar{c}^* \in G$, necessarily for some $\iota < \iota_*$ the set

$$u := \big\{i < \kappa_\beta : (\bar{c}_1 - \bar{c}^* - \bar{d}_\iota) \notin G_i\big\}$$

belongs to I (and of course, $\bar{b}^*\hat{\ }(\bar{c}^* + \bar{d}_\iota) \in \mathbf{I}_\alpha \cap {}^{\varepsilon+\xi}M$) and we have:

$(*)_1$ $M \models \varphi[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^* - d_\iota]$ for $\varphi \in \Phi_1$.

$(*)_2$ If $i \in \kappa_\beta \setminus u$ then $M \models \varphi_i[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^* - \bar{d}_\iota]$.

As in Case 1, there is $\bar{c}_2'' \in {}^\xi M$ such that

$(*)_3$ $M \models \varphi[\bar{b}_2 - \bar{b}^*, \bar{c}_2'' - \bar{c}^* - \bar{d}_\iota]$ for $\varphi \in \Phi_1$.

Hence

$(*)_4$ If $i \in \kappa_\beta \setminus u$ then $M \models \varphi_i[\bar{b}_2 - \bar{b}^*, \bar{c}_2'' - \bar{c}^* - \bar{d}_\iota]$.

As $u \in I$ by $\boxplus_\alpha(\mathrm{e})\bullet_2$ (that is, by 4.3) there are $\bar{\mathbf{e}} = \langle \bar{e}_j : j < \mu\rangle$ and u_* with $u \subseteq u_* \in I$ such that:

$(*)_5$ $\{\bar{e}_j : j < \mu\} \subseteq \bigcap\limits_{i \in \kappa_\beta \setminus u_*} G_i$.

$(*)_6$ $\bar{e}_{j_2} - \bar{e}_{j_1} \notin G_i$ for all $j_1 < j_2 < \mu$ and $i \in u_*$.

So

$(*)_7$ If $j < \mu$ then $(\bar{b}_2 - \bar{b}^*)\hat{\ }(\bar{c}_2'' - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ belongs to $\bigcap\limits_{\varphi \in \Phi_1} \varphi({}^{\varepsilon+\xi}M)$.

[1] Recall that $\bar{b} + G_i$ just means $\{\bar{b} + \bar{a} : \bar{a} \in G_i\}$.

$(*)_8$ If $i \in \kappa_\beta \setminus u_*$ then $i \in \kappa_\beta \setminus u$ as well, so by $(*)_4$+$(*)_5$ the sequence

$$(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j)$$

satisfies $\varphi_i(\bar x\hat{}\bar y - \bar a_i)$ in M, hence $\bar b_2\hat{}(\bar c_2'' - \bar e_j)$ satisfies the formula $\neg\varphi_i(\bar x\hat{}\bar y - \bar a_i)$ in M.

Lastly, by $(*)_6$,

$(*)_9$ For each $i \in u_*$ there is $j_i < \mu$ such that for every $j \in \mu \setminus \{j_i\}$, the sequence $(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j)$ satisfies $\neg\varphi_i(\bar x\hat{}\bar y - \bar a_i)$.

$(*)_{10}$ Moreover,
- (a) The set $\mu \setminus \{j_i : i \in u_*\}$ is non-empty.
- (b) For some (equivalently, 'for every') j in this set we have

$$(\forall i \in u_*)\neg\varphi_i\big[(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j) - \bar a_i\big].$$

[Why? Clause (a) is true simply because $\mu := 2^{\kappa_\beta} > \kappa_\beta \geq |u_*|$, and clause (b) follows from $(*)_9$.]

Putting together $(*)_7$–$(*)_{10}$, clearly $(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j)$ is as required in $\boxdot$, so we are done. $\square_{2.4}$

Definition 2.8. Let θ be a regular cardinal and γ an ordinal.

1) For an R-module M, we say $\bar{\mathbf{I}}$ is a (θ,γ)-*witness for* M <u>when</u> $\bar{\mathbf{I}} = \langle \mathbf{I}_\beta : \beta \leq \gamma\rangle$ and for each $\alpha \leq \gamma$, $\mathbf{I}_\alpha$ satisfies the conclusion of 2.4.

2) For $\bar\lambda = \langle \lambda_\beta : \beta \leq \gamma\rangle$, we say $\bar{\mathbf{I}}$ is a $(\bar\lambda,\theta,\gamma)$-witness <u>when</u> in addition,

$$\beta \leq \gamma \Rightarrow \lambda_\beta > |\mathbf{I}_\beta|.$$

§ 3. Stability

Context 3.1. 1) R is a fixed ring with $\tau = \tau_R$, <u>or</u> τ is a θ-additive vocabulary; $\mathbf{K}$ is the class of τ-additive models.

2) $M \in \mathbf{K}$ is a fixed R-module.

3) $\theta = \mathrm{cf}(\theta)$ and γ^* is an ordinal — limit, for simplicity.

4) $\bar\lambda = \langle \lambda_\alpha : \alpha \le \gamma^* \rangle$, where $\lambda_\alpha > \kappa_\alpha := \beth_\alpha\big(|R| + \theta^-\big)$.

5) $\bar{\mathbf{I}}^*$ is a $(\bar\lambda, \theta, \gamma^*)$-witness (see 2.8).

6) $A_* := \bigcup\{\bar a : \bar a \in \mathbf{I}_{\gamma^*}\}$.

7) $\Lambda_\varepsilon := \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon}$ for $\varepsilon < \theta$, and $\Lambda := \bigcup\limits_{\varepsilon<\theta} \Lambda_\varepsilon$.

8) $M_* = M_{A_*} := (M, a)_{a \in A_*}$.

Definition 3.2. Assume $\varepsilon < \theta$, $\Lambda \subseteq \Lambda^{\mathrm{pe}}_{\theta,\gamma^*}$, $A_* \subseteq A \subseteq M \in \mathbf{K}$, and $\bar a \in {}^\varepsilon M$.

1) For $\bar a \in {}^\varepsilon M$, let

$$\mathrm{tp}_\Lambda(\bar a, A, M) := \big\{\varphi(\bar x \char`\^ \bar b - \bar c) : \bar b \in {}^\xi A,\ \bar c \in {}^{\varepsilon+\xi} M,\ M \models \varphi[\bar a_1 \char`\^ \bar b - \bar c],\\ \text{and } \varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon+\xi} \cap \Lambda\big\}.$$

2) $\mathbf{S}^\varepsilon_\Lambda(A, M) := \{\mathrm{tp}_\Lambda(\bar a, A, M) : \bar a \in {}^\varepsilon M\}$.

Theorem 3.3 (The Stability Theorem). *Assume $\Lambda \subseteq \Lambda^{\mathrm{pe}}_{\gamma^*}$ and $A \subseteq M \in \mathbf{K}$.*

1) *The set $\mathbf{S}^\varepsilon_\Lambda(A, M)$ has cardinality $\le \big(|A|^{<\theta}\big)^{|\Lambda|}$.*

2) *For any $\kappa \ge 4$ (yes,* four*!) there are no $\langle \bar a_\alpha : \alpha < \kappa \rangle \subseteq {}^\varepsilon M$, $\langle \bar b_\alpha : \alpha < \kappa \rangle \subseteq {}^\zeta M$ and*[2] *$\varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi}$ such that for $\alpha < \beta < \kappa$ we have*

$$M \models \text{“}\varphi[\bar a_\alpha, \bar b_\beta] \wedge \neg\varphi[\bar a_\beta, \bar b_\alpha]\text{”}.$$

3) *If the formula $\varphi(\bar x, \bar y)$ is from $\mathbb{L}_{\infty,\theta,\gamma^*}$ (or is just a Boolean combination of such formulas) and $\kappa \ge \beth_{\gamma^*+2}(|\tau|^{<\theta})^+$, <u>then</u> there are no $M \in \mathbf{K}$, $\langle \bar a_\alpha : \alpha < \kappa \rangle \subseteq {}^\varepsilon M$, and $\langle \bar b_\alpha : \alpha < \kappa \rangle \subseteq {}^\zeta M$ such that*

$$M \models \varphi[\bar a_\alpha, \bar b_\beta] \wedge \neg\varphi[\bar a_\beta, \bar b_\alpha]$$

whenever $\alpha < \beta < \kappa$.
(Actually, $\kappa \ge \beth_{\gamma^*+1}(|\tau|^{<\theta})^+$ will suffice.)

4) *If $p \in \mathbf{S}^\varepsilon_\Lambda(A, M)$, $\varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi}$, and $p \cap \{\varphi(\bar x, \bar b) : \bar b \in {}^\xi A\} \ne \varnothing$, <u>then</u> for some $\bar a_\varphi \in {}^\varepsilon A$ and $\bar b \in {}^\xi A$ we have $\varphi(\bar x - \bar a_\varphi, \bar b) \vdash p \restriction \{\pm\varphi\}$ and $\varphi(\bar x - \bar a_\varphi, \bar b) \in p$.*

Proof. 1) Consider the statement:

⊛ If $\varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi} \cap \Lambda$,

$$p_\ell(\bar x) := \mathrm{tp}_{\{\varphi(\bar x,\bar y)\}}(\bar a_\ell, A, M) \in \mathbf{S}^\varepsilon_{\{\varphi(\bar x,\bar y)\}}(A, M)$$

for $\ell = 1, 2$, $\bar b \in {}^\xi A$ and $\bar c \in {}^{\varepsilon+\xi} A$, and $\varphi(\bar x \char`\^ \bar b - \bar c) \in p_1(\bar x) \cap p_2(\bar x)$, <u>then</u> $p_1(\bar x) = p_2(\bar x)$.

[2] This also holds for $\neg\varphi(\bar x, \bar y)$, but for κ finite we can invert the order.

Why is $\circledast$ true? Assume $\varphi(\bar{x}\hat{\ }\bar{b}'-\bar{c}') \in p_1(\bar{x})$, so $\bar{a}_1\hat{\ }\bar{b}'-\bar{c}' \in \varphi(\overline{M})$. But we are assuming $\varphi(\bar{x}\hat{\ }\bar{b}-\bar{c}) \in p_\ell(\bar{x}) = \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{a}_\ell, A, M)$, hence $\bar{a}_\ell\hat{\ }\bar{b}-\bar{c} \in \varphi(M)$ for $\ell = 1,2$. Together,

$$\bar{a}_2\hat{\ }\bar{b}'-\bar{c}' = (\bar{a}_2\hat{\ }\bar{b}-\bar{c}) - (\bar{a}_1\hat{\ }\bar{b}-\bar{c}) + (\bar{a}_1\hat{\ }\bar{b}'-\bar{c}') \in \varphi(M),$$

hence $\varphi(x\hat{\ }\bar{b}'-c') \in p_2(x)$. So $\varphi(\bar{x}\hat{\ }\bar{b}'-\bar{c}') \in p_1 \Rightarrow \varphi(\bar{x}\hat{\ }\bar{b}'-\bar{c}') \in p_2$, and by symmetry we have '$\Leftrightarrow$,' hence $p_1(\bar{x}) = p_2(\bar{x})$. I.e. we have proved $\circledast$.

Why is $\circledast$ sufficient? For every $\xi < \theta, \varphi(\bar{x},\bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi} \cap \Lambda$ and $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A,M)$ choose $(\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}, \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})})$ such that

$\oplus_1$ (a) $\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^\varepsilon A$ and $\bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^{\varepsilon+\xi}A$
(b) If possible, $\varphi(\bar{x}\hat{\ }\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) \in p(\bar{x})$.

For $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A,M)$, let $\Phi_{p(\bar{x})} := \{\varphi(\bar{x},\bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon,\xi} : \oplus_1\text{(b) does hold}\}$, and let

$$q_{p(\bar{x})} := \big\{\varphi(\bar{x}\hat{\ }\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) : \varphi(\bar{x},\bar{y}) \in \Phi_{p(\bar{x})}\big\}.$$

Now,

$\oplus_2$ If $p_1(\bar{x}), p_2(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A,M)$, $\Phi_{p_1(\bar{x})} = \Phi_{p_2(\bar{x})}$, and $q_{p_1(\bar{x})} = q_{p_2(\bar{x})}$, <u>then</u> $p_1(\bar{x}) = p_2(\bar{x})$.

[Why? Just think about it.]

$\oplus_3$ $\big|\big\{(\Phi_{p(\bar{x})}, q_{p(\bar{x})}) : p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A,M)\big\}\big| \le 2^{|\Lambda|} + (|A|^{<\theta})^{|\Lambda|}$

[Why? Straightforward.]

Clearly we are done.

2) Note that $\varphi(\bar{x},\bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon,\xi}$ implies that

$\boxplus$ $M \models$ "$\varphi[\bar{a},\bar{b}] \wedge \varphi[\bar{a},\bar{b}'] \wedge \varphi[\bar{a}',\bar{b}]$" $\Rightarrow M \models$ "$\varphi[\bar{a}',\bar{b}']$".

[Why? As $\varphi({}^{\varepsilon+\zeta}M)$ is a subgroup of ${}^{\varepsilon+\zeta}M$ and $\bar{a}\hat{\ }\bar{b}$, $\bar{a}'\hat{\ }\bar{b}$ and $\bar{a}\hat{\ }\bar{b}'$ belong to it. Therefore so does $\bar{a}'\hat{\ }\bar{b} + (\bar{a}\hat{\ }\bar{b}') - (\bar{a}\hat{\ }\bar{b})$, but that is equal to $\bar{a}'\hat{\ }\bar{b}'$.]

So we can choose $\bar{a} = \bar{a}_0$, $\bar{a}' = \bar{a}_3$, $\bar{b} = \bar{b}_1$, and $\bar{b}' = \bar{b}_2$, and get a contradiction.

3) Toward contradiction, let $\langle \bar{a}_\alpha : \alpha < \kappa\rangle \subseteq {}^\varepsilon M$ form a counterexample. By the Erdős-Rado Theorem,

$$\beth_{\gamma^*+2}\big(|\tau|^{<\theta}\big)^+ \to (4)^2_{\beth_{\gamma^*+1}(|\tau|^{<\theta})}.$$

Now for $\alpha < \beta < \kappa$, let $p_{\alpha,\beta} := \mathrm{tp}_{\Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\varepsilon}}(\bar{a}_\alpha\hat{\ }\bar{a}_\beta; \varnothing, M)$. So $\{p_{\alpha,\beta} : \alpha < \beta\}$ has cardinality $\le \beth_{\gamma^*+1}(|\tau|^{<\theta})$; hence by the arrow above, for some p and some $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ we have

$$(\forall k < \ell < 4)[p_{\alpha_k,\alpha_\ell} = p].$$

We get a contradiction by part (2).

If κ is just $\ge \beth_{\gamma^*+1}(|\tau|^{<\theta})^+$, use clause $\boxplus$ from the proof of part (2) and repeat a proof of the Erdős-Rado Theorem.

4) Should be clear. $\square_{3.3}$

Recall (from [She09])

Definition 3.4. For $\Phi \subseteq \Lambda$, we say $\mathbf{I} \subseteq {}^\varepsilon M$ is (μ,Φ)-*convergent* <u>when</u> ($|\mathbf{I}| \ge \mu$ and) for every $\xi < \theta$, $\varphi(\bar{x}) \in \Phi_{\varepsilon+\xi}$, and $\bar{b} \in {}^\xi M$ and $\bar{c} \in {}^{\xi+\varepsilon}M$, for all but $< \mu$ of the $\bar{a} \in \mathbf{I}$, the truth value of $\bar{a}\hat{\ }\bar{b} - \bar{c} \in \varphi(M)$ is constant.

Claim 3.5. 1) *The following is a sufficient condition for* $\mathbf{I} = \{\bar a_i : i < \lambda\} \subseteq {}^{\varepsilon}M$ *to be* (μ, Φ)*-convergent:*

$$i < j < \lambda \wedge \varphi(\bar x) \in \Phi \cap \Lambda_\varepsilon \Rightarrow \bar a_j - \bar a_i \in \varphi(M).$$

2) *If* $\varepsilon < \theta$, $\lambda = \mathrm{cf}(\lambda) > \mu \geq \mu_{\gamma^*}$, $(\forall i < \lambda)\big[|i|^{\mu_{\gamma^*}} < \lambda\big]$, *and* $\langle \bar a_i : i < \lambda\rangle \subseteq {}^{\varepsilon}M$ *is without repetition,* then *for some stationary* $S \subseteq \lambda$, $\{\bar a_i : i \in S\}$ *is* (μ^+, Φ)*-convergent.*

Remark 3.6. 1) Note that being $(\mu, \mathbf{I})$-convergent is very close to being $(<\omega)$-indiscernible, and is sometimes the reasonable generalization of indiscernibility.

2) So 3.5(1) says that 2-indiscernible *almost* implies $(<\omega)$-indiscernible.

3) Also, 3.5(2) says that there are $(<\omega)$-indiscernibles.

Proof. Should be clear. $\square_{3.5}$

§ 4. How much does the subgroup exhaust a group?

Definition 4.1. Assume G is a group, $\overline{G} = \langle G_s : s \in S\rangle$ is a sequence of subgroups of G.

1) For $\lambda \geq$ let $I = I_\lambda = I_{G,\overline{G},\lambda}$ be the set of $u \subseteq S$ which are *witnessed* by some sequence $\bar{g} = \langle g_\alpha : \alpha < \lambda\rangle \subseteq G$.

By this we mean

$$s \in u \wedge \alpha < \beta < \lambda \ \Rightarrow\ g_\alpha G_s \neq g_\beta G_s.$$

2) For $\lambda \geq$ let $I_{<\lambda} = I_{G,\overline{G},<\lambda} := \bigcup\limits_{\mu<\lambda} I_\mu$.

3) Let $I^+ := \mathcal{P}(S) \setminus I$.

Observation 4.2. 1) *For any λ, the sequence $\langle I_\mu : \mu < \lambda\rangle$ is $\subseteq$-decreasing.*

2) *If in addition λ has cofinality $> 2^{|S|}$, then the sequence is eventually constant.*

3) *There is $\xi < (2^{|S|})^{+4}$ of cofinality $(2^{|S|})^+$ such that $\lambda := \aleph_\xi$ satisfies all the demands mentioned in* 4.3 *below.*

Also if e.g. $\lambda = (2^{|S|})^+$ satisfies the demands in 4.3(a),(b).

4) *Similarly when $\lambda := \beth_\delta$ with $\delta = (2^{|S|})^+$, or just* $\mathrm{cf}(\delta) > 2^{|S|}$.

Claim 4.3. *Assume $\langle G_s : s \in S\rangle$ is a sequence of subgroups of the group G. The set $I = I_\lambda = I_{G,\overline{G},\lambda}$ satisfies:*

(a) *If $S \notin I$, $\mathrm{cf}(\lambda) > 2^{|S|}$, and $\alpha < \lambda \Rightarrow |\alpha|^{|S|} < \lambda$ (e.g. $(\exists\mu)[\lambda = (\mu^{|S|})^+]$), <u>then</u> there is $A \subseteq G$ of cardinality $< \lambda$ such that*

$$(\forall g \in G)(\exists a \in A)\big[\{s \in S : gG_s \neq aG_s\} \in I\big].$$

(b) *Under the assumptions of clause (a), for every $u \in I$ there exists $\bar{g}$ and $v \in I$ such that*
- $u \subseteq v$
- $\bar{g} = \langle g_\alpha : \alpha < \lambda\rangle$
- $g_\alpha G_s = g_0 G_s$ *for all $\alpha < \lambda$ and $s \in S \setminus v$. Moreover,*

$$\alpha < \lambda \Rightarrow g_\alpha \in \bigcap_{s\in S\setminus v} G_s.$$

- *If $s \in v$ and $0 < \alpha < \beta < \lambda$, then $g_\alpha G_s \neq g_\beta G_s$.*

(c) *$I \subseteq \mathcal{P}(S)$ is closed under subsets.*

(d) *$I = I_\lambda$ is an ideal,* provided that
- $\bullet_1$ *For some $\theta \in \big(|S|, \lambda\big)$ we have $I_\theta = I_\lambda$.*
- $\bullet_2$ *G is Abelian (<u>or</u> just each G_s is a normal subgroup of G).*

(e) *Assuming clause (d)$\bullet_2$ and $\lambda > |S|^+$,*

$$(\forall u_1, u_2 \in I_\lambda)(\forall \mu < \lambda)[u_1 \cup u_2 \in I_\mu].$$

(f) *Assuming clause (d)$\bullet_2$, if $|S| < \aleph_0$ then I is an ideal.*

(g) *The following holds:*
- (a) *if $u \in I_\lambda$ and $\{g_\alpha : \alpha < \alpha_*\}$ is $\subseteq$-maximal such that $\alpha < \beta < \alpha_* \wedge s \in S \Rightarrow g_\alpha G_s \neq g_\beta G_s$ then $\alpha_* \in [\lambda, \lambda^+)$*
- (b) *If λ is a limit cardinal then $I_{<\lambda} = I_\lambda$*
- (c) *If λ is a limit cardinal of cofinality $> 2^{|S|}$ then $(\exists\theta < \lambda)[I_\theta = I_\lambda$*

Proof. Let $I = I_\lambda$ be defined as in 4.1.

Now,

(∗) $I \subseteq \mathcal{P}(S)$ is $\subseteq$-downward closed: i.e. $u \in I \wedge v \subseteq u \Rightarrow v \in I$.

[Why? Obvious.]

This covers clause (c).

Toward proving clause (a) of the claim, for each $u \in I^+ := \mathcal{P}(S) \setminus I$, let $\bar{g}_u = \langle g_{u,\alpha} : \alpha < \alpha_u \rangle$ be a maximal sequence of members of G such that

$$\alpha < \beta < \alpha_u \wedge s \in u \ \Rightarrow\ g_{u,\alpha} G_s \neq g_{u,\beta} G_s.$$

As $u \notin I$, necessarily $\alpha_u < \lambda$ (by the definition of I), and as we are assuming $\mathrm{cf}(\lambda) > 2^{|S|}$, clearly $\alpha_* := \sup\{\alpha_u : u \in I^+\} < \lambda$. So

$$B := \{g_{u,\alpha} : u \in I^+,\ \alpha < \alpha_u\}$$

is a subset of G of cardinality $< \lambda$.

Next,

⊛ For every $u \in I$ and $h : S \setminus u \to B$, choose $g_h \in G$ such that

$$\boxplus_{h,g_h} \vee (\forall g \in G)\big[\neg \boxplus_{h,g}\big],$$

where

$\boxplus_{h,g}$
- $g \in G$
- $h : S \setminus u \to B$
- $(\forall s \in S \setminus u)[g G_s = h(s) G_s]$.

Explicitly, if there exists such a g then choose one of them as our g_h; otherwise let $g_h := g_{u,0}$, just so that it is defined.

Now

$$A := \{g_h : h \in {}^{S \setminus u}B,\ u \in I,\ \text{and } \boxplus_{h,g_h}\}$$

is a subset of G of cardinality[3] $\leq |B|^{|S|} < \lambda$.

For showing A is as required in clause (a), fix $g_* \in G$. Let

$$u := \big\{s \in S : (\forall w \in I^+)(\forall \alpha < \alpha_w)[g_* G_s \neq g_{w,\alpha} G_s]\big\}.$$

Now if $u \in I^+$ then $\bar{g}_u = \langle g_{u,\alpha} : \alpha < \alpha_u \rangle$ is well-defined and g_* satisfies $(\forall \alpha < \alpha_u)[g_* G_s \neq g_{u,\alpha} G_s]$, contradicting the maximality of $\bar{g}_u$.

Therefore $u \in I$ by our choices, so by the definition of u we can find a function $h : S \setminus u \to B$ such that

$$s \in S \setminus u \Rightarrow g_* G_s = h(s) G_s.$$

So g_* and h satisfy $\boxplus_{h,g_*}$, hence there is a $g_h \in A$ satisfying $\boxplus_{h,g_h}$, so clause (a) holds with $a := g_h$.

∗ ∗ ∗

We can asume that λ is regular (which anyhow is suffcient for section 2) as by clause (g) we can assume holds. For clause (b), let $u \in I$ be given and let $\langle g_\alpha : \alpha < \lambda \rangle$ witness that $u \in I$. For each $\alpha < \lambda$, let

$$u_\alpha := \big\{s \in S : (\exists \beta < \alpha)[g_\alpha G_s = g_\beta G_s]\big\}.$$

Clearly $u_\alpha \cap u = \varnothing$; let $h_\alpha : u_\alpha \to \alpha$ be such that $s \in u_\alpha \Rightarrow g_\alpha G_s = g_{h_\alpha(s)} G_s$.

As λ is regular and recalling $(\forall \alpha < \lambda)\big[|\alpha|^{|S|} < \lambda\big]$ by the present assumptions, for some $u_* \subseteq S$ and $h : u_* \to \lambda$, the set

$$\mathcal{W} := \{\alpha < \lambda : \mathrm{cf}(\alpha) = |S|^+ + \aleph_0,\ h_\alpha = h,\ u_\alpha = u_*\}$$

[3] Recall that we are assuming $(\forall \alpha < \lambda)\big[|\alpha|^{|S|} < \lambda\big]$.

is a stationary subset of λ. Clearly

$$\alpha, \beta \in \mathcal{W} \wedge s \in u_* \Rightarrow g_\alpha G_s = g_{h(s)} G_s = g_\beta G_s$$

and

$$\alpha \neq \beta \in \mathcal{W} \wedge s \in S \setminus u_* \Rightarrow g_\alpha G_s \neq g_\beta G_s.$$

Letting $\langle \alpha_i : i < \lambda \rangle$ list $\mathcal{W}$ and letting $g'_i := g_{\alpha_i}$ for $\alpha < \lambda$, clearly $v := u_*$ and $\langle g'_i : i < \lambda \rangle$ are as promised in clause (b).

For the 'moreover' bit, recalling clause (b) just says "for some $\bar{g}$ and v," we let $g''_\alpha := (g'_0)^{-1} g_\alpha$ and use the sequence $\langle g''_\alpha : \alpha < \lambda \rangle$. That is, first

$$\sigma < \beta < \lambda \wedge s \in S \Rightarrow g_\alpha G_s \neq g_\beta G_s \Rightarrow (g_0)^{-1}(g_\alpha G_s) \neq (g_0)^{-1}(g_\alpha G_s) \Rightarrow g''_\alpha G_s \neq g''_\beta$$

Second

$$\alpha < \lambda \wedge s \in S \setminus v \Rightarrow g''_\alpha G_s = (g'_0)^{-1}(g_\alpha G_s) = (g'_0)^{-1}(g'_0 G_s) = \big((g'_0)^{-1} g'_0\big) G_s = G_s.$$

Therefore (as $v := S \setminus u_*$) we have

$$\alpha < \lambda \wedge s \in v \;\Rightarrow\; g''_\alpha G_s = G_s \;\Rightarrow\; g''_\alpha \in G_s$$

as promised.

* * *

Lastly, it just remains to prove clause (e), as (d) is an immediate consequence and clause (f) is easier and clause (g) is proved as in clause (g).

Let $u_1, u_2 \in I_\lambda$ be disjoint, and we shall prove that $u := u_1 \cup u_2 \in I_\mu$ when $\mu \in \big(|S|, \lambda\big)$. Let $\langle g_{\ell,\alpha} : \alpha < \lambda \rangle$ witness '$u_\ell \in I_\lambda$' for $\ell = 1, 2$.

We try to choose $g_{3,\varepsilon} \in G$ by induction on $\varepsilon < \mu$ such that

$$\zeta < \varepsilon \wedge s \in u \;\Rightarrow\; g_{3,\varepsilon} G_s \neq g_{3,\zeta} G_s;$$

we shall also demand that $g_{3,\varepsilon} \in \{g_{1,i} g_{2,j} : i, j < \lambda\}$.

Arriving to ε, if for some $i, j < \lambda$ we can choose $g_{3,\varepsilon} := g_{1,i} g_{2,j}$, then we are done.

Towards contradiction, assume there are no such i and j. Then we have $f : \lambda \times \lambda \to \varepsilon$ and $h : \lambda \times \lambda \to u$ such that for every $(i,j) \in \lambda \times \lambda$ we have

$$g_{1,i} g_{2,j} G_{h(i,j)} = g_{3,f(i,j)} G_{h(i,j)}.$$

For each $i < \lambda$, $\zeta < \varepsilon$, and $s \in u \subseteq S$, let

$$\mathcal{U}^2_{i,\zeta,s} := \{j < \lambda : f(i,j) = \zeta,\ h(i,j) = s\}.$$

Now $j \in \mathcal{U}^2_{i,\zeta,s} \;\Rightarrow\; g_{1,i} g_{2,j} G_s = g_{3,\zeta} G_s \;\Rightarrow\; g_{2,j} G_s = g^{-1}_{1,i} g_{3,\zeta} G_s$; hence if $s \in u_2$ then

$$j \neq k \in \mathcal{U}^2_{i,\zeta,s} \Rightarrow g_{2,j} G_s = (g^{-1}_{1,i} g_{3,\zeta}) G_s = g_{2,k} G_s$$

— this contradicts our choice of $\langle g_{2,j} : j < \lambda \rangle$. Hence $\mathcal{U}^2_{i,\zeta,s}$ has cardinality ≤ 1 for all $i < \lambda$, $\zeta < \varepsilon$, and $s \in u_2$.

For $j < \lambda$, $\zeta < \varepsilon$, and $s \in u$, let

$$\mathcal{U}^1_{j,\zeta,s} := \big\{i < \mu : f(i,j) = \zeta \text{ and } h(i,j) = s\big\}.$$

If G is Abelian, then (as above) we have $\zeta < \varepsilon \wedge j < \lambda \wedge s \in u_1 \Rightarrow |\mathcal{U}^1_{j,\zeta,s}| \leq 1$. If G is non-Abelian and every G_s is a normal subgroup of G, then for any $j < \lambda$,

$\zeta < \mu$, $s \in u_1$ we have

$$\begin{aligned} i \in \mathcal{U}^1_{j,\zeta,s} &\Rightarrow g_{1,i} g_{2,j} G_s = g_{3,\zeta} G_s \\ &\Rightarrow g_{1,i}(G_s g_{2,j}) = g_{1,i}(g_{2,j} G_s) = g_{3,\zeta} G_s \\ &\Rightarrow g_{1,i} G_s = g_{3,\zeta}(G_s g^{-1}_{2,j}). \end{aligned}$$

Hence $i \neq k \in \mathcal{U}^1_{j,\zeta,s} \Rightarrow g_{1,i} G_s = g_{3,\zeta}(G_s g^{-1}_{2,j}) = g_{1,k} G_s$. This is a contradiction, so again $\mathcal{U}^1_{j,\zeta,s}$ has at most one member.

For $\ell \in \{1,2\}$ and $i < \lambda$, let $\mathcal{U}^\ell_i := \bigcup\limits_{\zeta < \varepsilon} \bigcup\limits_{s \in u_\ell} \mathcal{U}^\ell_{i,\zeta,s}$, so as $|u_\ell| \leq |S|$ clearly $|\mathcal{U}^\ell_i| \leq |S| + |\varepsilon|$. Recall that we have $\lambda > \mu > |S| + |\varepsilon|$, so there are $i, j < \lambda$ such that $i \notin \mathcal{U}^1_j \wedge j \notin \mathcal{U}^2_i$; hence the member $g_{1,i} g_{2,j}$ of G satisfies the demand on $g_{3,\varepsilon}$.

So we can carry the induction on $\varepsilon < \lambda$, so we are done proving clause (e). $\square_{4.3}$

Claim 4.4. *In Claim 4.3 there is a $W \subseteq S$ such that*

- (a) *There is a sequence $\bar{s} = \langle s_i : i < i_* \rangle$ listing W such that $\big(\bigcap\limits_{i<j} G_{s_i}, \bigcap G_{s_i}\big)$ is finite for $j < i_*$ (stipulating $\bigcap\limits_{i<0} G_{s_i} := G$).*
- (b) *W is maximal among all subsets of S satisfying clause (a) above.*

Proof. Immediate. $\square_{4.4}$

§ 5. Concluding Remarks

Example 5.1. An example of an additive structure is a ring satisfying $xy = -yx$.

E.g. if $(R, +^R)$ is $\bigoplus\{\mathbb{Z}x_s : s \in I\}$, f is a function from $I \times I$ into R such that $f(x,y) = -f(y,x)$ (so $f(x,x) = 0$), and we have

$$\Big(\sum_{\ell<\ell_*} a_\ell x_{s_\ell}\Big)\Big(\sum_{m<n_*} b_m x_{t_n}\Big) = \sum_{\ell<\ell_*}\sum_{m<m_*} a_\ell b_m x_{f(s_\ell,t_m)}.$$

Remark 5.2. 1) We may use $\tau \supseteq \{+,-,0,1\} \cup \{P_i : i < i_*\}$ with P_i unary, and instead of modules we use τ-models M such that $|M|$ is the disjoint union $\bigcup_{i<i_*} P_i^M$, $+^M$ is a partial two-place function which can be decomposed into

$$+^M := \bigcup\{+^M \restriction P_i^M : i < i_*\},$$

$(P_i^M, +^M)$ an Abelian group, and all relations and functions commute with $+$ (or at least every relation is *affine*).

I.e., let $F_*(x,y,z) := x - y + z$ and demand

$$G(\ldots, F_*(x_i, y_i, z_i), \ldots)_{i<i_*} = F_*\big(G(\bar{x}), G(\bar{y}), G(\bar{z})\big),$$

where $F_*(\bar{a}, \bar{b}, \bar{c}) := \big\langle F_*(a_i, b_i, c_i) : i < \text{arity}(P)\big\rangle \in P^M$ for $\bar{a}, \bar{b}, \bar{c} \in P^M$.

2) However, as we use infinitary logics, if M is the disjoint union of Abelian groups $G_i^M := (P_i^M, +_i^M)$ for $i < i_*$ and we define G_M as the direct sum having predicates for those subgroups, <u>then</u> we have bi-interpretability. When we have only "affine structure," we can expand by choosing an element in each summand to serve as zero.

3) It is natural to extend our logic by cardinality quantifiers which say "the definable group G quotient the definable subgroup H has cardinality $\geq \lambda$."

Remark 5.3. Concerning Theorem 2.4:

1) Note that instead of an R-module M we can use $(M, c_\alpha)_{\alpha<\kappa}$: i.e., expand M by κ-many individual constants. The only difference is that we will use $\beth_\alpha(|R|^{<\theta} + \kappa)$ instead of $\beth_\alpha(|R|^{<\theta})$.

2) Theorem 2.4 has an arbitrary choice — the construction of the $\mathbf{I}_\alpha$-s. Instead of using extra individual constants, in the proof,[4] for any $\psi(\bar{x})$, $\psi(\bar{x}) \wedge \varphi_i(\bar{x})$ for $i < i_* < \kappa_\beta$, I, G, and $\langle G_i : i < i_*\rangle$, we expand M by:

(a) $P^M := \big\{\bar{a} : M \models \psi[\bar{a}]$ and $\{i < \kappa_\beta : \bar{a} \notin G_i\} \in I\big\}$, which is a subgroup.

(b) Predicates for the set $\big\{\bar{a} + P^M : \bar{a} \in \psi(M)\big\}$.

So the proof shows that in M we can eliminate quantifiers to quantifier-free formulas in this expansion.

3) Also, this may give too much information. The result gives elimination of quantifiers, but unlike the first-order case we use more than just the positive existential formulas.

4) We can now define non-forking: hopefully [S$^+$] will deal with this.

[4] See $\boxplus_\alpha$ in the proof of 2.4.

Question 5.4. 1) Are there arbitrarily large Abelian groups G which are not only indecomposable, but even *potentially* so? (I.e. absolutely — even after any forcing G is indecomposable?)

2) Relatives of this — e.g., no *potential* non-trivial automorphism.

Discussion 5.5. We know that up to the minimal cardinal λ satisfying

$$\lambda \to (\omega)^{<\omega}_{\aleph_0},$$

the answer is yes (and more). But if $|G| \geq \lambda$ then absolutely it has non-trivial endomorphisms and even non-trivial embeddings of G into itself (Eklof-Shelah [ES99], Göbel-Shelah [GS07]). We can improve this to "for some $a_1 \neq a_2$ from G;" potentially there are embeddings f_1, f_2 of G into itself such that $f_1(a_1) = a_2$ and $f_2(a_2) = a_1$ — see [S$^+$].

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