Discontinuous homomorphisms without Hamel bases

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Abstract

We produce a model of $\mathsf{ZF} + \mathsf{DC}$ in which there exists a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself but no Hamel basis for $\mathbb{R}$, and prove a generalization of this result in terms of internal direct sums.

1 Introduction

We say that a subset of a vector space S is a *Hamel basis* if it is a maximal linearly independent set over the scalar field $\mathbb{Q}$. A permutation of a Hamel basis for $\mathbb{R}$ naturally extends to an automorphism of the group $(\mathbb{R}, +)$; if the permutation moves some elements and fixes others the induced homomorphism is discontinuous. In this paper we show (Theorem 3.3) that the existence of a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself does not (in $\mathsf{ZF} + \mathsf{DC}$) imply the existence of a Hamel basis for any vector space over $\mathbb{Q}$ containing a copy of $(\mathbb{R}, +)$.

Our interest in this problem was inspired by two related results regarding selectors for the Vitali equivalence relation $\mathbb{R}/\mathbb{Q}$. In [6] it was shown that the existence of a discontinuous homomorphism from $\mathbb{R}$ to itself induces (in ZF) a selector for this equivalence relation (this fact is used in Remark 4.5

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below). In [7] it was shown that the existence of a selector for $\mathbb{R}/\mathbb{Q}$ does not imply the existence of a Hamel bases (assuming the consistency of a strongly inaccessible cardinal, which can be removed by combining the proof in [7] with the approached used here). These results naturally lead to the question answered here.

In the second part of the paper we consider ways of writing $\mathbb{R}$ as an internal direct sum of subspaces. Instead of working directly with internal direct sums, however, we consider the associated decompositions of $\mathbb{R}$ into subspaces. The existence of a Hamel basis is equivalent to the existence of a decomposition of $\mathbb{R}$ into one-dimensional subspaces (see Remark 4.5), and a nontrivial decomposition gives rise to a discontinuous homomorphism. This leaves a range of questions regarding when the existence of one type of decomposition implies the existence of one of another type. In Theorem 5.3 we answer one such question by showing that the existence of a nontrivial decomposition does not imply the existence of one into subspaces of countable dimension.

While the main result of the second part of the paper subsumes the original form of the question answered in the first part of the paper, we include both arguments, in part because the argument from the first part can be modified to produce discontinuous homomorphisms which are not induced by decompositions into subspaces. The nonexistence result proved in the first part is generalized in a different way than the corresponding part of the second result. Many similar generalizations are possible and we leave these to the interested reader.

The arguments here are similar to earlier ones to due Horowitz and Shelah [4, 9]. They can also be easily adapted to the methods of [7].

2 Amalgamations

In this section we prove the key lemma of the paper (Lemma 2.5), which will be applied in Sections 3 and 4. We first review some standard facts about amalgamating homomorphisms.

If $(G, +)$ and $(H, +)$ are abelian groups, and $f\colon G \to H$ is a partial function, then we day that f is *additive* (or a *partial homomorphism*) if the equation $f(x + y) = f(x) + f(y)$ holds whenever x, y and $x + y$ are in the domain of f.

A group $(G, +)$ is said to be *divisible* if for each $x \in G$ and $n \in \mathbb{N}$ there is a $y \in G$ such that $ny = x$ (where ny represents y added to itself n times). The additive group of any vector space over a field containing $\mathbb{Q}$ is clearly divisible. We repeatedly and implicitly use the following standard fact.

Lemma 2.1. *If $(G, +)$ is a divisible abelian group, and $h\colon G' \to G$ is a homomorphism from a subgroup G' of G to G, then h extends to a homomorphism from G to G.*

Suppose that $\pi_1, \ldots, \pi_n$ are partial homomorphisms from one abelian group $(G, +)$ to another abelian group $(H, +)$, such that the domain of each π_i is a subgroup of G. We say that $\pi_1, \ldots, \pi_n$ can be *amalgamated* if there is a partial homomorphism $\pi\colon (G, +) \to (H, +)$ containing each π_i whose domain is also a subgroup. It is a standard fact (and easy to see) that $\pi_1, \ldots, \pi_n$ can be amalgamated if and only if, for all $a_i, b_i \in \operatorname{dom}(\pi_i)$ $(1 \leq i \leq n)$ if $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ then $\pi_1(a_1) + \cdots + \pi_n(a_n) = \pi_1(b_1) + \cdots + \pi_n(b_n)$. Letting $c_i = a_i - b_i$, we get the following equivalent condition: whenever $c_i \in \operatorname{dom}(\pi_i)$ $(1 \leq i \leq n)$ are such that $c_1 + \cdots + c_n = 0$, $\pi_1(c_1) + \cdots + \pi_n(c_n) = 0$. We state two specific versions of the fact, both of which will be used in the proof of Lemma 2.5.

Lemma 2.2. *Suppose that h_1 and h_2 are partial homomorphisms from an abelian group $(G, +)$ to another abelian group $(H, +)$, and that the domain of each h_i is a subgroup of G. Then h_1 and h_2 can be amalgamated if and only if they agree on the intersection of their domains.*

Lemma 2.3. *Suppose that h_1, h_2 and h_3 are partial homomorphisms from an abelian group $(G, +)$ to another abelian group $(H, +)$, and that the domain of each h_i is a subgroup of G. Then h_1, h_2 and h_3 can be amalgamated if and only if the equation $h_1(a_1) + h_2(a_2) = h_3(a_3)$ holds whenever a_i $(1 \leq i \leq 3)$ are such that $a_1 + a_2 = a_3$ and each a_i is in the domain of the corresponding h_i.*

It is a classical fact that any Borel (or even Baire-measurable) homomorphism between Polish groups is continuous (see [3, 8]). We will use the following variation of this fact.

Theorem 2.4. *If $\epsilon > 0$ and $f\colon (-\epsilon, \epsilon) \to \mathbb{R}$ is Borel and additive, then there is a real number a such that $f(x) = ax$ for all $x \in (-\epsilon, \epsilon)$.*

Proof. For each real number x, define $g(x)$ to be $nf(x/n)$, for any positive integer n such that $|x/n| < \epsilon/2$. The assumptions imply that g is well-defined, additive and Borel, and that it extends f. □

We will let P denote Cohen forcing, in the form $B(\mathbb{R})/I$, where $B(\mathbb{R})$ denotes the set of Borel subsets of $\mathbb{R}$ and I the ideal of meager sets. It follows then that P is a c.c.c. forcing adding a real number and having continuous reading of names (so every real number in the extension is the result of applying a ground-model Borel function to the generic real), and that, for any Borel set $B \subseteq \mathbb{R}^{n+1}$, the set of $\bar{x} \in \mathbb{R}^n$ for which $(\bar{x}, y) \in B$ for an I-positive set of y is also Borel (see Theorem 16.1 of [5]). Since I is shift-invariant, it follows that if x is a V-generic real added by P, and y is real number in the ground model, then $x + y$ is also V-generic for P. We fix the terms P and I for the rest of the paper, and refer the reader to [1] for more information about Cohen forcing and its products.

Real numbers x_1 and x_2 are *mutually P-generic* over M if they are the two P-generic reals over M produced by forcing over M with the product forcing $P \times P$.

The following is the key lemma in the proofs of our main theorems.

Lemma 2.5. *Suppose that x_1 and x_2 are mutually P-generic reals over V, and let $x_3 = x_1 - x_2$.*

1. *If y is an element of $(\mathbb{R}^{V[x_1]} + \mathbb{R}^{V[x_2]}) \cap \mathbb{R}^{V[x_3]}$, then there exist $a, b \in \mathbb{R}^V$ such that $y = ax_3 + b$.*

2. *Let*

    - *h_0 be a homomorphism from $(\mathbb{R}, +)$ to itself in V,*
    - *c be a real number in V, and,*
    - *for each $i \in \{1, 2, 3\}$, h_i be, in $V[x_i]$, a homomorphism from $(\mathbb{R}, +)^{V[x_i]}$ to itself extending h_0, with $h_i(dx_i) = cdx_i$ for all $d \in \mathbb{R}^V$.*

    Then h_1, h_2 and h_3 can be amalgamated in $V[x_1, x_2]$.

Proof. For any x_1, x_2, x_3 and y as given there exist Borel functions r, s and t in V such that $y = r(x_1) + s(x_2) = t(x_3)$. Then there exists a condition $(B_1, B_2) \in P \times P$ (with $x_1 \in B_1$ and $x_2 \in B_2$) such that, letting $\dot{g}_1$ and $\dot{g}_2$ be the standard names for the reals produced by each copy of P, (B_1, B_2) forces that $r(\dot{g}_1) + s(\dot{g}_2) = t(\dot{g}_1 - \dot{g}_2)$. We will show that for any such B_1, B_2, r, s and t, there exists a condition $(B_1', B_2') \leq (B_1, B_2)$ forcing that $t(\dot{g}_1 - \dot{g}_2)$ is equal to $a(\dot{g}_1 - \dot{g}_2) + b$, for some $a, b \in V$.

Shrinking B_1 and B_2 if necessary, we may assume that there exist real numbers c_1, c_2 and $\epsilon > 0$ such that (for each $i \in \{1, 2\}$) B_i is subset of the interval $(c_i - \epsilon, c_i + \epsilon)$ whose complement in this interval is in I. For each $i \in \{1, 2\}$ and $j \in \omega$, let $B_i^j = B_i \cap (c_i - \epsilon/2^j, c_i + \epsilon/2^j)$.

For each $d \in (-\epsilon/2, \epsilon/2) \cap V$, (B_1^1, B_2^1) forces that $r(\dot{g}_1 + d) + s(\dot{g}_2 + d)$ is equal to $t((\dot{g}_1 + d) - (\dot{g}_2 + d)) = t(\dot{g}_1 - \dot{g}_2)$. This implies that (B_1^1, B_2^1) forces that $r(\dot{g}_1 + d) + s(\dot{g}_2 + d)$ is the same for all values of $d \in (-\delta, \delta) \cap V$. It follows then that (B_1^1, B_2^1) forces that $r(\dot{g}_1 + d) - r(\dot{g}_1) = s(\dot{g}_2) - s(\dot{g}_2 + d)$ for all such values of d. Since the realizations of $\dot{g}_0$ and $\dot{g}_1$ will be mutually generic, this value must be in V, and must depend only on d. Let $u(d)$ be the corresponding value as forced by B_1' (for r) and B_2') (for s). Then u is also a Borel function, since $u(d)$ is the unique value taken by the Borel function $r(x + d) - r(x)$ on an I-large subset of $(c_1 - \epsilon/2, c_1 + \epsilon/2)$.

We claim that $u(d + e) = u(d) + u(e)$ for all $d, e \in (-\epsilon/4, \epsilon/4)$. To see this, note that B_1^1 forces that $r(\dot{g}_1 + (d+e)) = r(\dot{g}_1) + u(d+e)$, and B_1^2 forces that

$$r(\dot{g}_1 + (d + e)) = r((\dot{g}_1 + d) + e) = r(\dot{g}_1 + d) + u(e) = r(\dot{g}_1) + u(d) + u(e).$$

Since the restriction of u to $(-\epsilon/4, \epsilon/4)$ is an additive Borel function, it follows from Lemma 2.4 that it is multiplication by some real number a. It follows that there exists a real number b_1 such that B_1^3 forces that $r(\dot{g}_1) - a\dot{g}_1 = b_1$. To see this, suppose that two conditions $C, D \leq B_1^3$ decide the value of $r(\dot{g}_1) - a\dot{g}_1$ to be b_C and b_D respectively. We may assume that $D = C + d$, for some $d \in (-\epsilon/4, \epsilon/4)$. Then C forces that

$$b_D = r(\dot{g}_1 + d) - a(\dot{g}_1 + d) = r(\dot{g}_1) + ad - a\dot{g}_1 - ad = b_C.$$

An analogous argument shows that B_2^3 forces that $s(\dot{g}_2) + a\dot{g}_2 = b_2$, for some $b_2 \in \mathbb{R}$. It follows that (B_1^3, B_2^3) forces that

$$t(\dot{g}_1 - \dot{g}_2) = r(\dot{g}_1) + s(\dot{g}_2) = a(\dot{g}_1 - \dot{g}_2) + (b_1 - b_2).$$

For the second part, by Lemma 2.3 above, it suffices to show that

$$h_1(y_1) + h_2(y_2) = h_3(y_3)$$

holds whenever $y_i \in \mathbb{R}^{V[x_i]}$ $(1 \leq i \leq 3)$ are such that $y_1 + y_2 = y_3$. Fixing such y_1, y_2 and y_3, we have from the first part of the lemma that $y_3 = ax_3 + b$, for some $a, b \in \mathbb{R}^V$. Then

$$\begin{aligned} h_3(y_3) &= h_3(ax_3 + b) \\ &= h_3(ax_3) + h_3(b) \\ &= acx_3 + h_0(b) \\ &= ac(x_1 - x_2) + h_0(b) \\ &= acx_1 + h_1(b) - acx_2 \\ &= h_1(ax_1 + b) - h_2(ax_2). \end{aligned}$$

We also have that $y_1+y_2 = y_3 = ax_3+b = a(x_1-x_2)+b = (ax_1+b)-ax_2$. Since h_1 and h_2 are extensions of h_0 existing in mutually generic extensions of V (so the intersection of their domains is $\mathbb{R}^V$) they can be amalgamated, which implies that

$$h_1(y_1) + h_2(y_2) = h_1(ax + b) - h_2(ax).$$

□

3 The first model

Let κ be a cardinal of uncountable cofinality such that $\kappa^{<\kappa} = \kappa$. For each $d \subseteq \kappa$ let P_d be the finite support product of our fixed partial order P, indexed by the elements of d. So a condition p in P_d has the form

$$\{B^p_\alpha : \alpha \in \mathrm{supp}(p)\},$$

where $\mathrm{supp}(p)$ is a finite subset of d and each B^p_α is a Borel I-positive subset of $\mathbb{R}$. Then $p_2 \leq p_1$ in P_d if and only if $\mathrm{supp}(p_1) \subseteq \mathrm{supp}(p_2)$ and $B^{p_2}_\alpha \setminus B^{p_1}_\alpha \in I$ for each $\alpha \in \mathrm{supp}(p_1)$. We write G_d for a generic filter for P_d, and $\dot{g}_\alpha$ for the natural $P_{\{\alpha\}}$-name for the generic real added by $P_{\{\alpha\}}$.

Let Q be the following partial order. Conditions are pairs $(d, \dot{h})$ such that d is a countable subset of κ and $\dot{h}$ is a P_d-name for a homomorphism from $(\mathbb{R}, +)$ to itself. We allow $d = \emptyset$, in which case $\dot{h}$ is a check-name for a homomorphism in the ground model. The order is: $(d_1, \dot{h}_1) \leq (d_0, \dot{h}_0)$ if $d_0 \subseteq d_1$ and $1_{P_{d_1}}$ forces that $\dot{h}_{0,G_{d_0}} \subseteq \dot{h}_{1,G_{d_1}}$. Given $q \in Q$ we write d_q and $\dot{h}_q$ for the first and second coordinates of q, respectively.

We adopt the convention that undertilded names (like $\underset{\sim}{B}$) are Q-names and and dotted names (like $\dot{h}$) are P_d-names (for some $d \subseteq \kappa$), often existing in the ground model V. Combining the two notations, we get things like $\dot{\underset{\sim}{H}}$, the natural Q-name for the P_κ-name $\bigcup\{\dot{h} : (d, \dot{h}) \in K\}$, where K denotes the Q-generic filter.

The two following lemmas show that $\dot{\underset{\sim}{H}}_K$ is, the in the Q-extension $V[K]$, a P_κ-name for a homomorphism from $(\mathbb{R}^{V[K,G_\kappa]}, +)$ to itself. Both lemmas follow from the remarks in the previous section, specifically Lemma 2.1 for $\mathbb{R}$, which says that any partial homomorphism from a subgroup of $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ can be extended to a total homomorphism (in the ground model and in any forcing extension).

Lemma 3.1. *For each $\alpha \in \kappa$, the set of $q \in Q$ with $\alpha \in d_q$ is dense.*

Lemma 3.2. *Every descending ω-sequence in Q has a lower bound.*

We note several consequences of Lemmas 3.1 and 3.2.

- The partial order P_κ is the same in V and in any Q-extension $V[K]$, from which it follows that the forcing iteration $Q * P_\kappa$ and the product forcing $Q \times P_\kappa$ are forcing-equivalent.
- Densely many conditions $(q, p) \in Q \times P_\kappa$ have the property that $\mathrm{supp}(p) \subseteq d_q$. We will call such conditions *normal.*
- Letting (K, G_κ) denote a V-generic filter for $Q * P_\kappa$, every every element of $\mathbb{R}^{V[K,G_\kappa]}$ is an element of $\mathbb{R}^{V[G_d]}$, for some countable $d \subseteq \kappa$. In particular, $\mathbb{R}^{V[K,G_\kappa]} = \mathbb{R}^{V[G_\kappa]}$.

Our first theorem shows that forcing with $Q * P_\kappa$ produces a model with a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$ and no Hamel basis for $\mathbb{R}$. Theorem 3.3 below strengthens the nonexistence part of the theorem by replacing $\mathbb{R}$ with an arbitrary Polish vector space containing a copy of $\mathbb{R}$.

The rest of this section is a proof of the following theorem.

Theorem 3.3. *Suppose that (K, G_κ) is a V-generic filter for $Q * P_\kappa$, and let W be the model*

$$\mathrm{HOD}_{V,\mathbb{R}^{V[G_\kappa]},\dot{\underset{\sim}{H}}_{K,G_\kappa}}.$$

Then the function $\dot{\underset{\sim}{H}}_{K,G_\kappa}$ is a discontinuous homomorphism from $(\mathbb{R}, +)^{V[K,G_\kappa]}$ to itself, and DC holds in W. Furthermore, if, in V, $(S, +_S)$ is a Polish vector space over $\mathbb{Q}$ in for which there exists a continuous injective homomorphism in V from $(\mathbb{R}, +)$ to $(S, +_S)$, then W does not contain a Hamel basis for $(S, +_S)$.

That W satisfies DC follows from standard arguments, using the fact that W is an inner model of $V[K, G_\kappa]$ (a model of Choice), that it contains the reals of this model, and is closed under ordinal definability.[1]

The following amalgamation lemma follows from Lemma 2.2 and the fact that mutually generic extensions have no new real numbers in common.

Lemma 3.4. *Suppose that (q_1, p_1) and (q_2, p_2) are conditions in $Q \times P_\kappa$. Suppose that $q \in Q$ is weaker than both q_1 and q_2, with $d_q = d_{q_1} \cap d_{q_2}$, and that $B^{p_1}_\alpha \cap B^{p_2}_\alpha \in I^+$ for all $\alpha \in \mathrm{supp}(p_1) \cap \mathrm{supp}(p_2)$. Then (q_1, p_1) and (q_2, p_2) are compatible.*

We note that the indices of the conditions in P_κ play no role, and that any partial injection from κ to κ maps conditions to isomorphic conditions.

Remark 3.5. *If a is a countable subset of κ, and $f \colon a \to \kappa$ is injective, then f induces a function f_Q on the set of $q \in Q$ with $d_q \subseteq a$ and an isomorphism $f_P \colon P_a \to P_{f[a]}$ such that for each $q \in \mathrm{dom}(f_Q)$ and $p \in P_a$,*

- $\mathrm{supp}(f_P(p)) = f[\mathrm{supp}(p)]$;
- $B^{f_P(p)}_{f(\alpha)} = B^p_\alpha$ *for all* $\alpha \in \mathrm{supp}(p)$;
- $d_{f_Q(q)} = f[d_q]$;
- *for any V-generic filter $G \subseteq P_{q_d}$, $f_P[G]$ is a V-generic filter for $P_{f[d_q]}$, and $\dot{h}_{q,G} = \dot{h}_{f_Q(q), f_P[G]}$.*

Suppose toward a contradiction that $\dot{\underset{\sim}{B}}$ is a Q-name for a P_κ-name for a Hamel basis for $(S, +_S)$ in $\mathrm{HOD}_{V, \mathbb{R}^{V[G_\kappa]}, \dot{\underset{\sim}{H}}_{K, G_\kappa}}$, as forced by some $Q \times P_\kappa$-condition (q_0, p_0). Replacing (q_0, p_0) with a stronger condition if neccesary, we may assume that it is normal, and that there exist a $v \in V$, a formula φ, a cardinal λ and $P_{d_{q_0}}$-names $\dot{r}_1, \ldots, \dot{r}_k$ for elements of $\mathbb{R}$ such that

$$(q_0, p_0) \Vdash_{Q \times P_\kappa} \dot{\underset{\sim}{B}} = \{y \in S : V_\lambda[K, G_\kappa] \models \varphi(y, v, \dot{\underset{\sim}{H}}_{K, G_\kappa}, \dot{r}_{1, G_{d_{q_0}}}, \ldots, \dot{r}_{k, G_{d_{q_0}}})\}.$$

Below we write "$\tau \in \dot{\underset{\sim}{B}}$" as an abbreviation for the statement

$$V_\lambda[K, G_\kappa] \models \varphi(y, v, \dot{\underset{\sim}{H}}_{K, G_\kappa}, \dot{r}_{1, G_{d_{q_0}}}, \ldots, \dot{r}_{k, G_{d_{q_0}}}).$$

[1] One is gifted a suitable sequence of real numbers for which the minimal ordinal parameters work.

The first key point is that, for each $d \subseteq \kappa$, the restriction of $\dot{\underset{\sim}{B}}_{K,G_\kappa}$ is decided by the restrictions of K and G_κ to d.

Lemma 3.6. *For each normal $(q,p) \leq (q_0,p_0)$, for each P_{d_q}-name τ for an element of S, the statement $\tau \in \dot{\underset{\sim}{B}}$ is decided by (q,p') for densely many $p' \in P_{d_q}$ below p.*

Proof. If the lemma failed for some (q,p) and τ we could find conditions $(q_1,p_1),(q_2,p_2) \leq (q,p)$ forcing opposite truth values for the statement $\tau \in \dot{\underset{\sim}{B}}$, with $p_1 \restriction q_d = p_2 \restriction q_d$. Applying Remark 3.5, we may assume in addition that $d_{q_1} \cap d_{q_2} = d_q$. This is impossible since by Lemma 3.4, (q_1,p_1) and (q_2,p_2) would also be compatible. $\square$

We can then let, for each $q \leq q_0$, $\dot{B}^q$ be the P_{d_q}-name for the value of $\dot{\underset{\sim}{B}}_{K,G_\kappa} \cap S^{V[G_{d_q}]}$, for any $Q \times P_\kappa$-generic filter (K,G_κ) containing (q_0,p_0). Then p forces in P_{d_q} that the realization of $\dot{B}^q$ will be a maximal linearly independent subset of the $\mathbb{R}$ of the P_{d_q}-extension.

Rephrasing, we have the following fact, which gives us one half of our desired contradiction.

Lemma 3.7. *Let d be a countable subset of κ, and suppose that q_1 and q_2 are conditions below q_0 with $d_{q_1} = d_{q_2} = d$. If G^1_d and G^2_d are V-generic filters for P_d such that*

- $p_0 \in G^1_d \cap G^2_d$,
- $V[G^1_d] = V[G^2_d]$ *and*
- $\dot{h}_{q_1,G^1_d} = \dot{h}_{q_2,G^2_d}$,

then $\dot{B}^{q_1}_{G^1_d} = \dot{B}^{q_2}_{G^2_d}$.

Proof. Let G^* be $V[G^1_d]$-generic for $P_{\kappa\setminus d}$. Then (G^1_d,G^*) and (G^2_d,G^*) naturally induce V-generic G_1 and G_2 for P_κ, and $V[G_1]=V[G_2]$. It suffices to see that there exist $V[G_1]$-generic filters K_1 and K_2 for Q^V, with $q_1 \in K_1$ and $q_2 \in K_2$ such that the models W induced by (K_1,G_1) and (K_2,G_2) are the same. To this end, say that a pair $(q'_1,q'_2) \in Q^V \times Q^V$ is *good* if $q'_1 \leq q_1$, $q'_2 \leq q_2$, $d_{q'_1} = d_{q'_2}$ and $\dot{h}_{q'_1,G_1 \restriction d_{q'_1}} = \dot{h}_{q'_2,G_2 \restriction d_{q'_2}}$. Then (q_1,q_2) is good, and it suffices to show that whenever (q'_1,q'_2) is good and $q''_1 \leq q'_1$, there exists a

$q_2'' \leq q_2'$ such that (q_1'', q_2'') is good (since the corresponding fact with 1 and 2 reversed will follow by symmetry).

Given such (q_1', q_2') and q_1'', let $d_{q_2''}$ be $d_{q_1''}$. Since $V[G_d^1] = V[G_d^2]$, $V[G_1 \restriction d_{q_1''}] = V[G_2 \restriction d_{q_1''}]$, so there exist a condition $p_2 \in G_2 restrict d_{q_1''}$ and a $P_{d_{q_1''}}$-name σ such that $\sigma_{G_2 \restriction d_{q_1''}} = G_1 \restriction d_{q_1''}$ and p_2 forces σ to be a V-generic filter for $P_{d_{q_2''}}$ (giving rise to the entire $P_{d_{q_1''}}$-extension) such that the realization of $\dot{h}_{q_1''}$ by this filter will extend the realization of $\dot{h}_{q_2'}$. Then we can let $\dot{h}_{q_2''}$ be any name for an extension of the realization of $\dot{h}_{q_2'}$ to a homomorphism of $(\mathbb{R}, +)$ to itself such that p_2 forces the realization of $\dot{h}_{q_2''}$ to be the same as the realization of $\dot{h}_{q_1''}$ by the realization of σ. $\square$

We now derive the other half of the contradiction. Fix two ordinals, $\alpha_1 < \alpha_2$ in $\kappa \setminus d_{q0}$ and let $d = d_{q0} \cup \{\alpha_1, \alpha_2\}$. We will find generic filters G_d and G_d' for P_d, such that

1. $p_0 \in G_d \restriction P_{d_{q0}} = G_d' \restriction P_{d_{q0}}$,
2. $G_d \restriction P_{\{\alpha_1\}} = G_d' \restriction P_{\{\alpha_1\}}$ and
3. $V[G_d] = V[G_d']$,

and a homomorphism $h \in V[G_d]$ from $(\mathbb{R}^{V[G_d]}, +)$ to itself, extending $\dot{h}_{q0, G_d \restriction P_{q_{d_0}}}$ and having the property that

- $h \restriction V[G_d \restriction P_{d_{q0 \cup \{\alpha_1\}}}] \in V[G_d \restriction P_{d_{q0 \cup \{\alpha_1\}}}]$,
- $h \restriction V[G_d \restriction P_{d_{q0 \cup \{\alpha_2\}}}] \in V[G_d \restriction P_{d_{q0 \cup \{\alpha_2\}}}]$,
- $h \restriction V[G_d' \restriction P_{d_{q0 \cup \{\alpha_2\}}}] \in V[G_d' \restriction P_{d_{q0 \cup \{\alpha_2\}}}]$,

such that $\dot{B}^q_{G_{d_q}} \neq \dot{B}^{q'}_{G_{d_{q'}}}$. This will finish the proof.

Let $G_{d_{q0}}$ be a V-generic filter for $P_{d_{q0}}$ containing p_0. Let x_1 and x_2 be mutually P-generic reals over $V[G_{d_{q0}}]$, and let $x_3 = x_1 - x_2$. Let G_d be the P_d-generic filter extending $G_{d_{q0}}$ giving rise to x_1 in coordinate α_1 and x_2 in coordinate α_2. Let G_d' be the P_d-generic filter extending $G_{d_{q0}}$ giving rise to x_1 in coordinate α_1 and x_3 in coordinate α_2. Then G_d and G_d' satisfy conditions (1)-(3) above.

Fix a real number $c \in V[G_{d_{q0}}]$, and for each $i \in \{1, 2, 3\}$ let h_i in $V[G_{d_{q0}}][x_i]$ be a homomorphism from $(\mathbb{R}, +)^{V[G_{d_{q0}}][x_i]}$ to itself extending $\dot{h}_{q0, G_{d_{q0}}}$, with $h_i(dx_i) = cdx_i$ for all $d \in \mathbb{R}^{V[G_{d_{q_0}}]}$. Applying Lemma 2.5, let h be an amalgamation of h_1, h_2 and h_3 in $V[G_d]$.

The remaining point is that there cannot be a Hamel basis B for $(S, +_S)$ in $V[G_{d_{q_0}}][x_1, x_2]$ whose restriction to each of $V[G_{d_{q_0}}][x_1]$, $V[G_{d_{q_0}}][x_2]$ and $V[G_{d_{q_0}}][x_3]$ is in the corresponding model. To see this, note first that since x_1, x_2 and x_3 are pairwise mutually generic over $V[G_{d_{q_0}}]$, the intersection of any two of these models is $V[G_{d_{q_0}}]$. Now let π be (in V), a continuous injective homomorphism from $(\mathbb{R}, +)$ to $(S, +_S)$. If a B as above did exist, then each of $\pi(x_1)$, $\pi(x_2)$ and $\pi(x_3)$ would be a linear combination of elements of the corresponding set $B \cap V[G_{d_{q_0}}][x_i]$ (with rational coefficients) in a unique way. The equation $\pi(x_3) +_S \pi(x_2) = \pi(x_1)$ however then gives two different linear combinations for $\pi(x_1)$.

Remark 3.8. *The approach above leads to several possible variations, including the following.*

1. *Requiring h to be injective.*
2. *Requiring h to be surjective.*
3. *Requiring h to be bijective.*
4. *Requiring the range of h to be $(\mathbb{Q}, +)$.*
5. *Requiring $h(h(x)) = x$.*
6. *Requiring $\mathbb{R}/\{x : h(x) = x\}$ to have dimension 2.*

Question 3.9 (Zapletal)**.** *Does the existence of a Hamel basis for $\mathbb{R}^2$ imply the existence of a Hamel basis for $\mathbb{R}$?*

We note that if $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is a discontinuous homomorphism, then for some i, j the function $x \mapsto f_i(x\mathbf{e}_j)$ is a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$, where $f_1, \ldots, f_m$ are the component functions of f and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{R}^n$.

4 Internal direct sums

Given a vector space V and subspaces W_i $(i \in I)$, V is the *internal direct sum* $\bigoplus_{i \in I} W_i$ if every vector in V can be written as a finite sum of members of distinct W_i's in a unique way. In this section we generalize the results above by rephrasing them in terms of internal direct sums. We find it convenient to work with the equivalence relations associated to internal direct sums.

Definition 4.1. *A* decomposition of *a vector space* $\mathcal{S}$ into subspaces *is an equivalence relation* E *having the following properties.*

- *The domain of* E *(*$\mathrm{dom}(E)$*) is a set of nonzero vectors from* $\mathcal{S}$*.*
- *Whenever* e *is an equivalence class of* E*,* $e \cup \{0\}$ *is a subspace of* $\mathcal{S}$*.*
- *Every vector in* $\mathcal{S}$ *is a sum of* E*-inequivalent elements of* $\mathrm{dom}(E)$ *in a unique way.*

The last condition in the definition above is easily seen to be equivalent to: the zero vector is not equal to any nonempty linear combination of E-inequivalent elements of the domain of E.

Remark 4.2. *Both Hamel bases and discontinuous homomorphisms are related to interval direct sums, and thus decompositions of vector spaces.*

- *If* B *is a Hamel basis for a vector space* $\mathcal{S}$ *(over* $\mathbb{Q}$*), then the set of pairs* $\{(ax, bx) : x \in B,\ a, b \in \mathbb{Q} \setminus 0\}$ *is a decomposition of* $\mathcal{S}$ *into subspaces, and* $\mathcal{S}$ *is the internal direct sum of the subspaces* $\{ax : a \in Q\}$ *for* $x \in B$*.*
- *If* E *is a decomposition of* $\mathcal{S}$ *into subspaces, and* E *has two equivalence classes which are dense in* $\mathcal{S}$*, then one gets a discontinuous homomorphism from* $\mathcal{S}$ *to* $\mathcal{S}$ *by sending one of these two classes to* 0 *and every other member of the domain of* E *to itself.*

Remark 4.3. *If* E *is decomposition of* $\mathcal{S}$ *into subsapces, and* F *is a coarser equivalence relation on* $\mathrm{dom}(E)$*, then one gets another decomposition of* S *into subspaces by replacing each element of* F *with the span of its union.*

Definition 4.4. *Given decompositions* E *and* F *of a vector space* $\mathcal{S}$*, we say that* F *is* coarser *than* E *if each* F*-class is the span of the union of a set of* E*-classes.*

Internal direct sums and decompositions naturally fall into various subclasses. Given a positive integer n, we say that a decomposition E is $\leq n$-*dimensional* if $e \cup \{0\}$ has dimension at most n, for each E-equivalence class

e (we drop $\leq$ when $n = 1$), and *countable-dimensional* if the dimension of each $e \cup \{0\}$ is countable.

We write $\Phi_n(S)$ for the assertion that a vector space S has a decomposition into $\leq n$-dimensional subspaces, $\Phi_c(S)$ for the assertion that there exists a decomposition of S into countable-dimensional subspaces.

Remark 4.5. *Strengthening the first part of Remark 4.2 above, we note that the existence of a Hamel basis for $\mathbb{R}$ is equivalent to the existence of a decomposition of $\mathbb{R}$ into one-dimensional subspaces (over $\mathbb{Q}$). The forward direction of this is the first part of the remark. For the reverse direction, the existence of such of decomposition E gives a discontinuous homomorphism from $\mathbb{R}$ to itself, as noted above, and therefore an $\mathbb{E}_0$-selector (by [6]). Since the nonzero rational numbers form an abelian group under multiplication, this gives a selector for the E-classes, by [2], i.e., a Hamel basis.*

We will prove the following.

Theorem 4.6. *If* ZFC *is consistent then so is* ZF $+ \neg\Phi_c(\mathbb{R})$ + *"there is a decomposition of $\mathbb{R}$ into uncountably many subspaces."*

Our proof of Theorem 5.3 uses the corresponding notions of extension and amalgamation for decompositions. Given a vector space $\mathcal{S}$, a subspace $\mathcal{T}$ and a decomposition E of $\mathcal{T}$ into subspaces, a decomposition F of $\mathcal{S}$ into subspaces *extends* E if E is the restriction of F to $\mathcal{T}$.

We make some general observations about decompositions into subspaces.

- Suppose that E is an equivalence relation on a set of nonzero vectors from a vector space $\mathcal{S}$. If every vector v is $\mathcal{S}$ is a member of a subspace $\mathcal{S}'$ for which the restriction of E to $\mathcal{S}'$ is a decomposition of $\mathcal{S}'$ into subspaces, then E is a decomposition of $\mathcal{S}$ into subspaces.

- If $\mathcal{S}'$ is a subspace of a vector space $\mathcal{S}$, E is a decomposition of $\mathcal{S}'$ into subspaces and v is a vector in $\mathcal{S} \setminus \mathcal{S}'$, then one gets a decomposition of the span of $\mathcal{S}' \cup \{v\}$ either by letting $\{av : a \in \mathbb{Q} \setminus \{0\}\}$ be a new equivalence class or by replacing one equivalence class e of E with the nonzero members of the span of $e \cup \{v\}$.

- Using the Axiom of Choice one can show that every decomposition of a subspace of a vector space $\mathcal{S}$ extends to a decomposition of the entire space $\mathcal{S}$.

Definition 4.7. *Given a vector space $\mathcal{S}$, subspaces $\mathcal{S}_1, \ldots, \mathcal{S}_n$ and a decomposition E_i of each $\mathcal{S}_i$ into subspaces, an* amalgamation *of $E_1, \ldots, E_n$ is a decomposition E of $\mathcal{S}$ into subspaces such that E extends each E_i.*

Remark 4.8. *If no member of any S_i is a linear combination of vectors from the other S_j's, then the union of the E_i is a decomposition of the subspace generated by the $\mathcal{S}_i$'s which extends each E_i. Furthermore, any coarsening of this union formed by identifying each class from each E_i with at most one class of each other E_j is also an amalgamation of the E_i's.*

On the other hand, if the zero vector is a linear combination of vectors $\mathbf{v}^i_j$ ($1 \leq i \leq n$, $1 \leq j \leq k_i$), with each $\{\mathbf{v}^i_1, \ldots, \mathbf{v}^i_{k_i}\}$ a set of E_i-inequivalent members of the domain of E_i, then the vectors $\mathbf{v}^i_j$ will all have to be E-equivalent. This is impossible if any k_i is greater than 1.[2]

The following is Lemma 2.5 adapted to decompositions.

Lemma 4.9. *Suppose that x_1 and x_2 are mutually P-generic reals over V, and let $x_3 = x_1 - x_2$.*

1. *Let*
   - *E_0 be, in V, a decomposition of $(\mathbb{R}, +)$ into subspaces, and*
   - *for each $i \in \{1, 2, 3\}$, E_i be, in $V[x_i]$, a decomposition of $(\mathbb{R}, +)^{V[x_i]}$ into subspaces extending E_0, with x_i in the domain of E_i.*

   Then E_1, E_2 and E_3 can be amalgamated in $V[x_1, x_2]$.

Proof. As above, any amalgamation E will have to equate x_1, x_2 and x_3. Let E be the coarsening of the union of E_1, E_2 and E_3 which is formed by identifying these three classes.

We have from the first part of Lemma 2.5 that for each y in $(\mathbb{R}^{V[x_1]} + \mathbb{R}^{V[x_2]}) \cap \mathbb{R}^{V[x_3]}$, there exist $a, b \in \mathbb{R}^V$ such that $y = ax_3 + b$.

Then everything is ok. □

5 The second model

The second model is like the first, except that we add a generic decomposition instead of a generic homomorphism. We let κ and P_κ be exactly as before. For

[2] There should be other ways in which amalgamation should be impossible - find one.

the current proof Q is the following partial order. Conditions are pairs $(d, \dot{e})$ such that d is a countable subset of κ and $\dot{e}$ is a P_d-name for a decomposition of $(\mathbb{R}, +)$ (of the forcing extension) into subspaces. The order is: $(d_1, \dot{e}_1) \leq (d_0, \dot{e}_0)$ if $d_0 \subseteq d_1$ and $1_{P_{d_1}}$ forces that $\dot{1}_{1,G_{d_1}} \cap V[G_{d_0}] = \dot{e}_{0,G_{d_0}}$. Given $q \in Q$ we write d_q and $\dot{e}_q$ for the first and second coordinates of q, respectively.

We let $\underset{\sim}{\dot{E}}$ be the natural Q-name for the P_κ-name $\bigcup\{\dot{e} : (d, \dot{e}) \in K\}$, where K denotes the Q-generic filter. The two following lemmas show that $\underset{\sim}{\dot{E}}_K$ is, the in the Q-extension $V[K]$, a P_κ-name for a decomposition of $(\mathbb{R}^{V[K,G_\kappa]}, +)$ into subspaces. Both lemmas follow from the remarks in the previous section, specifically[3]

Lemma 5.1. *For each $\alpha \in \kappa$, the set of $q \in Q$ with $\alpha \in d_q$ is dense.*

Lemma 5.2. *Every descending ω-sequence in Q has a lower bound.*

Lemmas 5.1 and 5.2 give the following.

- The partial order P_κ is the same in V and in any Q-extension $V[K]$, from which it follows that the forcing iteration $Q * P_\kappa$ and the product forcing $Q \times P_\kappa$ are forcing-equivalent.
- Densely many conditions $(q, p) \in Q \times P_\kappa$ have the property that $d_q \subseteq \text{supp}(p)$. We will call such conditions *normal*.
- Letting (K, G_κ) denote a V-generic filter for $Q * P_\kappa$, every every element of $\mathbb{R}^{V[K,G_\kappa]}$ is an element of $\mathbb{R}^{V[G_d]}$, for some countable $d \subseteq \kappa$. In particular, $\mathbb{R}^{V[K,G_\kappa]} = \mathbb{R}^{V[G_\kappa]}$.

We will show that forcing with $Q * P_\kappa$ produces a model with a decomposition of $(\mathbb{R}, +)$ into subspaces but no such countable-dimensional decomposition.

The rest of this section is a proof of the following theorem.

Theorem 5.3. *Suppose that (K, G_κ) is a V-generic filter for $Q * P_\kappa$, and let W be the model*

$$\text{HOD}_{V, \mathbb{R}^{V[G_\kappa]}, \underset{\sim}{\dot{E}}_{K,G_\kappa}}.$$

[3]???Lemma 2.1 for $\mathbb{R}$, which says that any partial homomorphism from a subgroup of $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ can be extended to a total homomorphism (in the ground model and in any forcing extension).

Then $\dot{\underset{\sim}{E}}_{K,G_\kappa}$ is a decomposition of $(\mathbb{R},+)^{V[K,G_\kappa]}$ into subspaces, and DC *holds in W. Furthermore, the is no decomposition of $(\mathbb{R},+)^{V[K,G_\kappa]}$ into countable-dimensional subspaces in W.*

The following amalgamation lemma follows from Remark 4.8.

Lemma 5.4. *Suppose that (q_1,p_1) and (q_2,p_2) are conditions in $Q\times P_\kappa$. Suppose that $q\in Q$ is weaker than both q_1 and q_2, with $d_q=d_{q_1}\cap d_{q_2}$, and that $B^{p_1}_\alpha\cap B^{p_2}_\alpha\in I^+$ for all $\alpha\in\operatorname{supp}(p_1)\cap\operatorname{supp}(p_2)$. Then (q_1,p_1) and (q_2,p_2) are compatible.*

The following remark carries over verbatim from our first construction.

Remark 5.5. *If a is a countable subset of κ, and $f\colon a\to\kappa$ is injective, then f induces a function f_Q on the set of $q\in Q$ with $d_q\subseteq a$ and an isomorphism $f_P\colon P_a\to P_{f[a]}$ such that for each $q\in\operatorname{dom}(f_Q)$ and $p\in P_a$,*

- $\operatorname{supp}(f_P(p))=f[\operatorname{supp}(p)]$*;*
- $B^{f_P(p)}_{f(\alpha)}=B^p_\alpha$ *for all* $\alpha\in\operatorname{supp}(p)$*;*
- $d_{f_Q(q)}=f[d_q]$*;*
- *for any V-generic filter $G\subseteq P_{q_d}$, $f_P[G]$ is a V-generic filter for $P_{f[d_q]}$, and $\dot{h}_{q,G}=\dot{h}_{f_Q(q),f_P[G]}$.*

As above, we let W denote the model $\mathrm{HOD}_{V,\mathbb{R}^{V[G_\kappa]},\dot{\underset{\sim}{H}}_{K,G_\kappa}}$. The arguments above for our first model show that W satisfies DC.

Suppose toward a contradiction that $\dot{\underset{\sim}{D}}$ is a Q-name for a P_κ-name for a decomposition of $(\mathbb{R},+)$ into countable-dimensional subspaces in $\mathrm{HOD}_{V,\mathbb{R}^{V[G_\kappa]},\dot{\underset{\sim}{H}}_{K,G_\kappa}}$, as forced by some $Q\times P_\kappa$-condition (q_0,p_0). Replacing (q_0,p_0) with a stronger condition if neccesary, we may assume that it is normal, and that there exist a $v\in V$, a formula φ, a cardinal λ and $P_{d_{q_0}}$-names $\dot{r}_1,\ldots,\dot{r}_k$ for elements of $\mathbb{R}$ such that

$$(q_0,p_0)\Vdash_{Q\times P_\kappa}\dot{\underset{\sim}{D}}=\{y\in S: V_\lambda[K,G_\kappa]\models\varphi(y,v,\dot{\underset{\sim}{H}}_{K,G_\kappa},\dot{r}_{1,G_{d_{q_0}}},\ldots,\dot{r}_{k,G_{d_{q_0}}})\}.$$

We will write "$\tau\in\dot{\underset{\sim}{D}}$" for $V_\lambda[K,G_\kappa]\models\varphi(y,v,\dot{\underset{\sim}{H}}_{K,G_\kappa},\dot{r}_{1,G_{d_{q_0}}},\ldots,\dot{r}_{k,G_{d_{q_0}}})$.

As before, the first key point is that, for each $d\subseteq\kappa$, the restriction of $\dot{\underset{\sim}{D}}_{K,G_\kappa}$ is decided by the restrictions of K and G_κ to d.

Lemma 5.6. *Each normal $(q,p) \leq (q_0,p_0)$ forces that for each P_{d_q}-name τ for a real number, the $\underset{\sim}{\dot{D}}$-equivalence class of τ will be contained in the P_{d_q}-extension of V.*

Proof. This follows from Lemma 5.4 and Remark 5.5. Suppose that the lemma failed for some (q,p) and τ. Then there exist a condition $(q',p') \leq (p,q)$ and $P_{d_{q'}}$-name σ for a real number which is forces by (q',p') to be $\underset{\sim}{\dot{D}}$-equivalent to τ and not a member of the P_{d_q}-extension of V. Given any $(q'',p'') \leq (q',p')$ and any $\alpha < \omega_1$, letting $f\colon d_{q'} \to \omega_1$ be an injection which is the identity function on d_q and which maps the rest of $d_{q'}$ to elements of $\omega_1 \setminus \alpha \cup d_{q''}$, we get by genericity that there is no countable $d \subseteq \kappa$ such that the $\underset{\sim}{\dot{D}}$-equivalence class of τ will be contained in the P_d-extension. $\square$

We can then let, for each $q \leq q_0$, $\dot{D}^q$ be the P_{d_q}-name for the value of $\underset{\sim}{\dot{D}}_{K,G_\kappa} \restriction \mathbb{R}^{V[G_{d_q}]}$, for any $Q \times P_\kappa$-generic filter (K, G_κ) containing (q_0,p_0). Then p forces in P_{d_q} that the realization of $\dot{D}^q$ will be a decomposition of $\mathbb{R}$ into countable-dimenisonal subspaces in the P_{d_q}-extension.

Rephrasing, we have the following fact, which gives us one half of our desired contradiction.

Lemma 5.7. *If $q,q' \leq q_1$, and G_{d_q} and $G_{d_{q'}}$ are (respectively), V-generic filters for P_{d_q} and $P_{d_{q'}}$ with $p_1 \in G_{d_q} \cap G_{d_{q'}}$ such that*

$$\mathbb{R}^{V[G_{d_q}]} = \mathbb{R}^{V[G_{d_{q'}}]}$$

and $\dot{e}_{q,G_{d_q}} = \dot{e}_{q',G_{d_{q'}}}$, then $\dot{D}^q_{G_{d_q}} = \dot{D}^{q'}_{G_{d_{q'}}}$.

We now derive the other half of the contradiction. Fix two ordinals, $\alpha_1 < \alpha_2$ in $\kappa \setminus d_{q_0}$ and let $d = d_{q_0} \cup \{\alpha_1, \alpha_2\}$. We will find generic filters G_d and G'_d for P_d, such that

1. $p_0 \in G_d \restriction P_{d_{q_0}} = G'_d \restriction P_{d_{q_0}}$,
2. $G_d \restriction P_{\{\alpha_1\}} = G'_d \restriction P_{\{\alpha_1\}}$ and
3. $V[G_d] = V[G'_d]$,

and a decomposition $e \in V[G_d]$ of $\mathbb{R}^{V[G_d]}$ into subspaces, extending $\dot{e}_{q_0,G_d \restriction P_{q_{d_0}}}$ and having the property that

- $e \restriction V[G_d \restriction P_{d_{q_0 \cup \{\alpha_1\}}}] \in V[G_d \restriction P_{d_{q_0 \cup \{\alpha_1\}}}]$,
- $e \restriction V[G_d \restriction P_{d_{q_0 \cup \{\alpha_2\}}}] \in V[G_d \restriction P_{d_{q_0 \cup \{\alpha_2\}}}]$,
- $e \restriction V[G'_d \restriction P_{d_{q_0 \cup \{\alpha_2\}}}] \in V[G'_d \restriction P_{d_{q_0 \cup \{\alpha_2\}}}]$,

such that $\dot{D}^q_{G_{d_q}} \neq \dot{D}^{q'}_{G_{d_{q'}}}$. This will finish the proof.

Let $G_{d_{q_0}}$ be a V-generic filter for $P_{d_{q_0}}$ containing p_0. Let x_1 and x_2 be mutually P-generic reals over $V[G_{d_{q_0}}]$, and let $x_3 = x_1 - x_2$. Let G_d be the P_d-generic filter extending $G_{d_{q_0}}$ giving rise to x_1 in coordinate α_1 and x_2 in coordinate α_2. Let G'_d be the P_d-generic filter extending $G_{d_{q_0}}$ giving rise to x_1 in coordinate α_1 and x_3 in coordinate α_2. Then G_d and G'_d satisfy conditions (1)-(3) above.

For each $i \in \{1, 2, 3\}$ let e_i in $V[G_{d_{q_0}}][x_i]$ be a decomposition of

$$(\mathbb{R}, +)^{V[G_{d_{q_0}}][x_i]}$$

into subspaces, extending $\dot{h}_{q_0, G_{d_{q_0}}}$, with x_i in the domain of e_i. Applying Lemma 4.9, let e be an amalgamation of e_1, e_2 and e_3 in $V[G_d]$.

The remaining point is that there cannot be a decomposition D of $\mathbb{R}^{V[G_d]}$ into subspaces in $V[G_{d_{q_0}}][x_1, x_2]$ whose restriction to each of $V[G_{d_{q_0}}][x_1]$, $V[G_{d_{q_0}}][x_2]$ and $V[G_{d_{q_0}}][x_3]$ is in the corresponding model. To see this, note first that since x_1, x_2 and x_3 are pairwise mutually generic over $V[G_{d_{q_0}}]$, the intersection of any two of these models is $V[G_{d_{q_0}}]$. If a D as above did exist, then each of x_1, x_2 and x_3 would be a sum of D-inequivalent elements in a unique way. The equation $x_3 + x_2 = x_1$ however then gives two different linear combinations for x_1.

6 Questions

The result above induces the following questions (and many other natural variations) which we do not know the answers to.

Question 6.1. *Given a positive integer n, does $\Phi_{n+1}(\mathbb{R})$ imply $\Phi_n(\mathbb{R})$?*

Suppose that E is a decomposition of $\mathbb{R}$ into n-dimensional subspaces. For each E-equivalence class E, let $S(e)$ be the set of linearly independent n-tuples from E. Then each $S(e)$ is an orbit of the action of $\mathrm{GL}_n(\mathbb{Q})$ on the set of linearly independent n-tuples (and the set of all $S(e)$'s is a maximal set of such orbits under the natural corresponding notion of independence). Presumably there is no Borel injection from the set of $\mathrm{GL}_{n+1}(\mathbb{Q})$ orbits into the $\mathrm{GL}_n(\mathbb{Q})$-orbits, but we do not know of a proof.

Let $\Phi_{\mathrm{f}}(S)$ be the assertion that the vector space S has a decomposition into subspaces whose associated subspaces are all finte-dimensional.

Question 6.2. *Does $\Phi_{\mathrm{c}}(\mathbb{R})$ imply $\Phi_{\mathrm{f}}(\mathbb{R})$ or does $\Phi_{\mathrm{f}}(\mathbb{R})$ imply $\Phi_n(\mathbb{R})$ for some or any positive integer n?*

Question 6.3. *Does the existence of a nontrivial homomorphism from $\mathbb{R}$ to itself imply the existence of a decomposition of $\mathbb{R}$ into at least two subspaces?*

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