

## MAXIMAL FAILURES OF SEQUENCE LOCALITY IN AEC SH932

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*Dedicated to the memory of Andrzej Mostowski*

ABSTRACT. We are interested in examples of AECs  $\mathfrak{k}$  having some (extreme) behavior concerning types, preferably with amalgamation. Note, we deal with  $\mathfrak{k}$  being sequence-local, i.e. local for increasing chains of length a regular cardinal (for types, equality of all restrictions implies equality, some call it *tame*). We construct an AEC  $\mathfrak{k}$  and  $\text{LST}(\mathfrak{k}) = \theta$ ,  $|\tau_{\mathfrak{k}}| = \theta$  such that the following class is maximal:  $\{\kappa : \kappa \text{ is a regular cardinal and } \mathfrak{k} \text{ is not } (2^\kappa, \kappa)\text{-sequence-local}\}$ . In fact, we have a direct characterization of this class of cardinals: the regular  $\kappa$  such that there is no uniform  $\kappa^+$ -complete ultrafilter (on any  $\lambda > \kappa$ ). We also prove a similar result to “ $(2^\kappa, \kappa)$ -compact for types” for  $\mathfrak{k}$ .

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## 0. INTRODUCTION

Recall AECs (abstract elementary classes); were introduced in [She87a]; and their (orbital) types defined in [She87b], see on them [She09b], [Bal09]. It has seemed to us obvious that even with  $\mathfrak{k}$  having amalgamation, those types in general lack some good properties of the classical types in model theory. E.g. “ $(\lambda, \kappa)$ -sequence-locality where,

**Definition 0.1.** 1) We say that an AEC  $\mathfrak{k}$  is a  $(\lambda, \kappa)$ -sequence-local (for types) when  $\kappa$  is regular and for every  $\leq_{\mathfrak{k}}$ -increasing continuous sequence  $\langle M_i : i \leq \kappa \rangle$  of models of cardinality  $\lambda$  and  $p, q \in \mathcal{S}(M_\kappa)$  we have  $(\forall i < \kappa)(p \upharpoonright M_i = q \upharpoonright M_i) \Rightarrow p = q$ . We omit  $\lambda$  when we omit “ $\|M_i\| = \lambda$ ”.

2) We say an AEC  $\mathfrak{k}$  is  $(\lambda, \kappa)$ -local when:  $\kappa \geq \text{LST}(\mathfrak{k})$  and if  $M \in \mathfrak{k}_\lambda$  and  $p_1, p_2 \in \mathcal{S}(M)$  and,

(\*) for every  $N$  we have,  $N \leq_{\mathfrak{k}} M \wedge \|N\| \leq \kappa \Rightarrow p_1 \upharpoonright N = p_2 \upharpoonright N$  then  $p_1 = p_2$ .

3) We may replace  $\lambda$  by  $\leq \lambda, < \lambda, [\mu, \lambda]$  with the obvious meaning (and allow  $\lambda$  to be infinity).

Of course, being sure is not a substitute for a proof; some examples of failures of being local were provided by Baldwin–Shelah [BS08, §2]. Also note our using: “Abelian groups without zero” is similar to e.g., the work [HS90]. Now [BS08] gives an example of the failure of  $(\lambda, \kappa)$ -sequence-locality for  $\mathfrak{k}$ -types in ZFC for some  $\lambda, \kappa$ , actually  $\kappa = \aleph_0$ . This was done by translating our problems to Abelian group problems; using Abelian groups which are not Whitehead. While those problems seem reasonable by themselves, they may hide our real problem.

Here in §1 we get  $\mathfrak{k}$ , an AEC with the class  $\{\kappa : (< \infty, \kappa)\text{-sequence-localness fail for } \mathfrak{k}\}$  being maximal, with amalgamation and the JEP. see Theorem 1.4; here the cardinality of its vocabulary as well as its LST number is  $\theta$  for transparency, for any given  $\theta = \theta^{\aleph_0}$ ; we shall deal with the other  $\theta$  later. Also we deal with “compactness of types”, getting classes with amalgamation; in [BS08], in some cases this was done there only in some universes of set theory; see §2.

We rely on a criterion from [BS08] to prove that  $\mathfrak{k}$  has the JEP and amalgamation.

**Question 0.2.** Can  $\{\kappa : \mathfrak{k} \text{ is } (< \kappa^+, \kappa)\text{-local}\}$  be “wild”? E.g. can it be all odd regular alephs? etc?

Similarly for  $(< \infty, \kappa)$ - sequentially local.

In §2 we deal with sequence-compactness of types.

Mostowski [Mos57] initiated the quest to find strengthenings of first-order logic that still have a “good model theory”. Usually, one may add generalized quantifiers (e.g.,  $(\exists^{\geq \aleph_1} x)$ ) and/or allow certain infinitary operations (e.g.,  $\bigwedge_{\alpha < \lambda} \psi_\alpha$ ). There is much to be said on this topic; see the collection [Bar85] and, later, Väänänen’s book [V11].

In particular, Lindström proved that one cannot expect too much: either the downward Löwenheim–Skolem property to  $\aleph_0$  fails, or  $\aleph_0$ -compactness fails.

Now, *abstract elementary classes* (AECs) continue this, trying to deal directly at the model theory. E.g. concerning “a theory  $T$  in logic  $\mathbb{L}(\exists^{\geq \aleph_1})$ ”, we define the AEC  $\mathfrak{k} = \mathfrak{k}_T$ , by:

- $K_{\mathfrak{k}}$  is the model of  $T$ ,

- $M \leq_{\mathfrak{k}} N$  iff in addition to  $M \prec_{\mathbb{L}(\exists^{\aleph_1})} N$ , which is naturally defined, we demand that, if  $M \models (\exists^{\leq \aleph_0} x)\psi(x, \bar{a})$  then not only  $N \models (\exists^{\leq \aleph_0})\varphi(x, \bar{a})$  but  $N \models \varphi[b, a] \Rightarrow b \in M$ .

Similarly, for e.g. the logic  $\mathbb{L}_{\lambda^+, \aleph_0}$ .

This work is part of the attempt to sort out which properties of first-order logic hold for AECs, particularly when  $\mathfrak{k}$  is an AEC with amalgamation, in §2 the amalgamation property was added lately. This work was submitted to Jouko Väänänen in October 2009 for a volume in honor of Andrzej Mostowski, and deposited in the arXiv. Later and independently, Boney [Bon14] investigated such things mainly for compact cardinals, in particular, has results close to 1.8 (and 2.8).

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It is my pleasure to dedicate this to the memory of Andrzej Mostowski, who contributed so much to mathematical logic and particularly to starting other logics and generalized quantifiers in [Mos57].

## 1. AN AEC WITH MAXIMAL FAILURE OF BEING LOCAL

**Claim 1.1.** *Assume*

- ⊗<sub>1</sub> (a)  $\kappa = \text{cf}(\kappa) > \sigma \geq \aleph_0$  and  $\sigma = \sigma^{\aleph_0}$ ,
- (b) *there is no uniform  $\sigma^+$ -complete ultra-filter  $D$  on  $\kappa$*
- (c)  $\tau_\theta$  *is the vocabulary*  
 $\{E_n, E'_n : n \in [2, \omega)\} \cup \{F_c : c \in [\sigma]^{< \aleph_0}\} \cup \{R_e : e \in {}^\sigma \sigma\} \cup \{R_0, R_1\}$   
*where each  $R_e$  is two-place predicate, each  $F_c$  is an unary function symbol,  $R_0, R_1$  are two-place and unary predicates respectively and  $E'_n, E_n$  are  $(2n)$ -place predicates for  $n \geq 2$ ,*
- (d)  $\tau_\theta^*$  *is  $\tau_\theta \setminus \{R_1\}$ .*

Then

- ⊠ *there are  $I_\alpha, \pi_\alpha, M_{\ell, \alpha}$  (for  $\ell = 1, 2$  and  $\alpha \leq \kappa$ ), and  $g_\alpha$  (for  $\alpha < \kappa$ ) satisfying:*
  - (a)  $I_\alpha$ , *a set of cardinality  $\sigma^\kappa$ , is  $\subseteq$ -increasing continuous with  $\alpha$ ,*
  - (b)  $M_{\ell, \alpha}$ , *a  $\tau_\theta$ -model of cardinality  $\leq \theta^\kappa$ , is increasing continuous with  $\alpha$ ,*
  - (c)  $\pi_\alpha$  *is a function from  $M_{\ell, \alpha}$  onto  $I_\alpha$ , increasing continuous with  $\alpha$ ,*
  - (d)  $|\pi_\alpha^{-1}\{t\}| \leq \sigma$  *for  $t \in I_\alpha, \alpha \leq \kappa$  and  $\ell = 1, 2$ ,*
  - (e) *if  $t \in I_{\alpha+1} \setminus I_\alpha$  then  $\pi_\alpha^{-1}\{t\} \subseteq M_{\ell, \alpha+1} \setminus M_{\ell, \alpha}$ ,*
  - (f) *for  $\alpha < \kappa, g_\alpha$  is an isomorphism from  $M_{1, \alpha}$  onto  $M_{2, \alpha}$  respecting  $\pi_\alpha$  which means  $a \in M_{1, \alpha} \Rightarrow \pi_\alpha(a) = \pi_\alpha(g_\alpha(a))$ ,*
  - (g) *for  $\alpha = \kappa$  there is no isomorphism from  $M_{1, \alpha}$  onto  $M_{2, \alpha}$  respecting  $\pi_\alpha$ .*

*Proof.* Follows from 1.2 which is just a fuller version explicating the unary function  $F_c$  for  $c \in G$ ; anyhow we shall use only 1.2. □<sub>1.1</sub>

**Claim 1.2.** *Assuming ⊗<sub>1</sub> of 1.1 we have:*

- ⊞ there are  $I_\alpha, A_\alpha, \pi_\alpha, M_{\ell, \alpha}$  (for  $\ell = 1, 2, \alpha \leq \kappa$ ) and  $g_\alpha$  (for  $\alpha < \kappa$ ) and  $G$  such that:
- (a)  $G$  is an additive (so Abelian) group of cardinality  $\sigma$ ,
  - (b)  $I_\alpha$  is a set, increasing continuous with  $\alpha$ ,  $|I_\alpha| = \sigma^\kappa$ ,
  - (c)  $M_{\ell, \alpha}$  is a  $\tau_\theta$ -model, increasing continuous with  $\alpha$ , of cardinality  $\sigma^\kappa$  with universe  $A_\alpha$ ,
  - (d)  $\pi_\alpha$  is a function from  $M_{\ell, \alpha}$  onto  $I_\alpha$ , increasing continuous with  $\alpha$ ,
  - (e)  $F_c^{M_{\ell, \alpha}} (c \in G)$  is a permutation of  $M_{\ell, \alpha}$ , increasing continuous with  $\alpha$ ,
  - (f)  $\pi_\alpha(a) = \pi_\alpha(F_c^{M_{\ell, \alpha}}(a))$  for  $a \in M_{\ell, \alpha}$ ,
  - (g)  $F_{c_1}^{M_{\ell, \alpha}}(F_{c_2}^{M_{\ell, \alpha}}(a)) = F_{c_1+c_2}^{M_{\ell, \alpha}}(a)$  for any  $c_1, c_2 \in G$ ,
  - (h)  $\pi_\alpha(a) = \pi_\alpha(b) \Leftrightarrow \bigvee_{c \in G} F_c^{M_{\ell, \alpha}}(a) = b$ ,
  - (i) for  $\alpha < \kappa$ ,  $g_\alpha$  is an isomorphism from  $M_{1, \alpha}$  onto  $M_{2, \alpha}$  which respects  $\pi_\alpha$  which means  $a \in M_{1, \alpha} \Rightarrow \pi_\alpha(a) = \pi_\alpha(g_\alpha(a))$ ,
  - (j) there is no isomorphism from  $M_{1, \kappa} \upharpoonright \tau_\theta$  onto  $M_{2, \kappa} \upharpoonright \tau_\theta$  respecting  $\pi_\kappa$ ,
  - (k)  $M_{1, \alpha} \upharpoonright \tau_\theta^* = M_{2, \alpha} \upharpoonright \tau_\theta^*$ , for  $\alpha \leq \kappa$ .

**Discussion 1.3.** We shall try to shed some light on how we intend to use 1.2 in the proof of 1.4, see Discussion 1.9. Note that, the models  $M_{1, \alpha}, M_{2, \alpha}$  are almost the same.

*Proof.* Let

- (\*)<sub>1</sub> (a) let  $G = ([\sigma]^{< \aleph_0}, \Delta)$ , i.e., the family of finite subsets of  $\sigma$  with the operation of symmetric difference. This is an Abelian group satisfying  $\forall x(x+x=0)$ , but we may identify  $\varepsilon < \sigma$  with  $\{\varepsilon\}$ , so treating ordinals as atoms,
- (b) let  $\langle a_{f, \alpha, u} : f \in {}^\kappa \sigma, \alpha < \kappa, u \in G \rangle$  be a sequence without repetitions,
- (c) for  $\beta \leq \kappa$  let  $A_\beta = \{a_{f, \alpha, u} : f \in {}^\kappa \sigma, \alpha < 1 + \beta \text{ and } u \in G\}$ ,
- (d) for  $\beta \leq \kappa$  let  $I_\beta = ({}^\kappa \sigma) \times (1 + \beta)$ ,
- (e)  $\pi_\beta$  be the function with domain  $A_\beta$  such that  $\pi_\beta(a_{f, \alpha, u}) = (f, \alpha)$  when  $\alpha < 1 + \beta \leq \kappa$ ,
- (f) for each  $\beta < \kappa$  we define a permutation  $g_\beta$  (of order 2) of  $A_\beta$  by  $g_\beta(a_{f, \alpha, u}) = a_{f, \alpha, u+G\{f(\beta)\}}$  hence  $a \in A_\beta \Rightarrow \pi_\beta(g_\beta(a)) = \pi_\beta(a)$ .

Note that:

- (\*)<sub>2</sub> (a)  $|G| = \sigma$ ,
- (b)  $\langle A_\beta : \beta \leq \kappa \rangle$  is a  $\subseteq$ -increasing continuous, each  $A_\beta$  a set of cardinality  $\sigma^\kappa$ ,
- (c)  $\langle I_\beta : \beta \leq \kappa \rangle$  is  $\subseteq$ -increasing continuous, each  $I_\beta$  of cardinality  $\sigma^\kappa$ ,
- (d)  $\pi_\beta$  is a mapping from  $A_\beta$  onto  $I_\beta$ ,
- (e) if  $t \in I_\alpha \subseteq I_\beta$  then  $\pi_\beta^{-1}\{t\} = \pi_\alpha^{-1}\{t\}$  has cardinality  $|G| = \sigma$ ,
- (f) if  $t \in I_{\alpha+1} \setminus I_\alpha$  then  $\pi_{\alpha+1}^{-1}\{t\} \subseteq A_{\alpha+1} \setminus A_\alpha$ ,
- (g) if  $\alpha \leq \beta \leq \kappa$  then  $g_\beta$  maps  $A_\alpha$  onto itself and  $g_\beta \circ g_\beta$  is the identity.

For each  $n \in [2, \omega)$  and  $\beta \leq \kappa$  we define equivalence relations  $E'_{n,\beta}, E_{n,\beta}$  on  ${}^n(A_\beta)$ :

- (\*)<sub>3</sub>  $\bar{a}E'_{n,\beta}\bar{b}$  iff  $\pi_\beta(\bar{a}) = \pi_\beta(\bar{b})$  where of course  $\pi_\beta(\langle a_\ell : \ell < n \rangle) = \langle \pi_\beta(a_\ell) : \ell < n \rangle$ .
- (\*)<sub>4</sub>  $\bar{a}E_{n,\beta}\bar{b}$  iff  $\bar{a}E'_{n,\beta}\bar{b}$  and there are  $k < \omega$  and  $\bar{a}_0, \dots, \bar{a}_k$  such that:
  - (i)  $\bar{a}_\ell \in {}^n(A_\beta)$ ,
  - (ii)  $\bar{a} = \bar{a}_0$ ,
  - (iii)  $\bar{b} = \bar{a}_k$ ,
  - (iv) for each  $\ell < k$  for some  $\alpha_1, \alpha_2 < \kappa$  we have  $g_{\alpha_2}^{-1}(g_{\alpha_1}(\bar{a}_\ell))$  is well defined and equal to  $\bar{a}_{\ell+1}$  or  $g_{\alpha_2}(g_{\alpha_1}^{-1}(\bar{a}_\ell))$  is well defined and equal to  $\bar{a}_{\ell+1}$ .

Note:

- (\*)<sub>4.1</sub> (a) the two possibilities in (\*)<sub>4</sub>(iv) are one as  $g_\alpha^{-1} = g_\alpha$  so the first one is equal to the second,
- (b)  $g_\alpha$  does not preserve  $\bar{a}/E_{n,\beta}$ !, in fact,  $a, g_\alpha(a)$  are never  $E_{n,\beta}$  equivalent,
- (c) clearly in (\*)<sub>4</sub>(iv) for  $\ell < k$ , the terms are well defined iff  $\bar{a}_\ell \in {}^n(A_{\min\{\alpha_1, \alpha_2\}})$  because if  $\alpha \leq \beta$  then  $g_\beta$  maps  $A_\alpha$  onto itself,
- (d) if  $\alpha \leq \beta$ ,  $a \in A_\alpha$ , then  $g_\beta$  maps  $a/E_{n,\beta}$  onto itself
- (e) if  $\alpha, \beta \leq \kappa$ , then  $g_\alpha, g_\beta$  commute (on the intersection of their domains,  $A_{\min\{\alpha, \beta\}}$ ).

[Why? Easy, e.g.

Clause b: Why? Let  $a = a_{f_1, \gamma_1, u_1}$ ,  $b = a_{f_2, \gamma_2, u_2}$ . Now, on the one hand, if  $g_\alpha(a) = b$  then  $f_1 = f_2$ ,  $\gamma_1 = \gamma_2$  and  $u_1 +_G u_2 = u_1 \Delta u_2$  has cardinality 1, in fact is equal to  $\{f_\alpha(\gamma)\}$ . On the other hand, if  $\langle a \rangle E_{n,\beta} \langle b \rangle$  then (by induction on  $k$  in the definition), we can prove  $f_1 = f_2$ ,  $\gamma_1 = \gamma_2$  and  $u_1 +_G u_2 = u_1 \Delta u_2$  is a set of even cardinality.

Clause (e): Just recalling that  $G$  is a commutative group.]

Note,

- (\*)<sub>5</sub> For  $n \in [2, \omega)$ , we have:
  - (a)  $E'_{n,\beta}, E_{n,\beta}$  are indeed equivalence relations on  ${}^n(A_\beta)$ ,
  - (b)  $E_{n,\beta}$  refine  $E'_{n,\beta}$ ,
  - (c) if  $\bar{a} \in {}^n(A_\beta)$  then  $\bar{a}/E'_{n,\beta}$  has exactly  $\sigma$  members,
  - (d) if  $\alpha < \beta \leq \kappa$  then  $E'_{n,\beta} \upharpoonright {}^n(A_\alpha) = E'_{n,\alpha}$  and  $E_{n,\beta} \upharpoonright {}^n(A_\alpha) = E_{n,\alpha}$  (read (\*)<sub>4</sub>(iv) carefully!),
  - (e) if  $\alpha < \beta \leq \kappa$ ,  $\bar{a} \in {}^n(A_\alpha)$  and  $\bar{b} \in \bar{a}/E'_{n,\beta}$  then  $\bar{b} \in {}^n(A_\alpha)$ ,
  - (f) if  $g_\alpha(\bar{a}_\ell) = \bar{b}_\ell$  for  $\ell = 1, 2$  then:  $\bar{a}_1 E'_{n,\beta} \bar{a}_2$  iff  $\bar{b}_1 E'_{n,\beta} \bar{b}_2$ .

Now we recall the vocabulary  $\tau_\theta$  of cardinality  $2^\sigma$  from 1.1⊗(d) and for  $\alpha \leq \kappa$  we choose a  $\tau_\theta$ -model  $M_{1,\alpha}$  such that:

- (\*)<sub>6</sub> (a)  $M_{1,\alpha}$  increasing with  $\alpha$  and has universe  $A_\alpha$ ,
- (b)  $\bullet_1$  let  $R_0^{M_{1,\alpha}} = \{\bar{a} \in {}^2(A_\alpha) : \text{if } \bar{a} = \langle a_{f_\ell, \alpha_\ell, u_\ell} : \ell < 2 \rangle, \text{ then } u_1 = u_2\}$ , and

- <sub>2</sub> let  $R_1^{M_{1,\alpha}} = \{\bar{a} \in {}^1(A_\alpha) : \text{if } \bar{a} = \langle a_{f,\alpha,u} \rangle, \text{ then } u \text{ has even number of elements}\}$ ,
  - (c)  $\tau_\theta = \tau(M_{1,\alpha})$  is defined in  $\otimes_1(c)$  from 1.1,
  - (d) for every function  $e \in \sigma$ 

$$R_e^{M_{1,\alpha}} = \{(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in A_\alpha \times A_\alpha : f_1 = e \circ f_2 \text{ and} \\ \text{if } i < \sigma \text{ then } i \in u_1 \\ \text{iff } (|\{j \in u_2 : e(j) = i\}| \text{ is odd})\}$$
- recalling  $f_\ell \in {}^\kappa \sigma$ ,
- (e)
    - <sub>1</sub>  $E_n^{M_{1,\alpha}} = E_{n,\alpha}$  for  $n < \omega$ ,
    - <sub>2</sub>  $(E'_{n_1})^{M_{1,\alpha}} = E'_{n,\alpha}$  for  $n < \omega$
    - <sub>3</sub>  $F_c^{M_{1,\alpha}} = F_c$  is defined by  $F_c : A_\alpha \rightarrow A_\alpha$  satisfies  $F_c(a_{f,\alpha,u}) := a_{f,\alpha,u+Gc}$  for  $c \in G$ .
  - (f) if  $\alpha \leq \beta_\ell < \kappa$  for  $\ell = 1, 2$  then  $g_{\beta_2}^{-1} g_{\beta_1} \upharpoonright A_\alpha$  is an automorphism of  $M_{1,\alpha}$
  - (g)  $g_\alpha$  is almost an automorphism of  $M_{1,\alpha}$ , it miss preserving  $R_1$ .

Why is this possible? The universe of  $M_{1,\alpha}$  is defined in clause (a), its vocabulary in clause (c), the interpretation of the predicates in clauses (b), (d), (e)•<sub>1</sub>, •<sub>2</sub> and the interpretations of the functions symbols in clause (e)•<sub>3</sub>. So we are done with clauses (a)-(e), except concerning  $M_{1,\alpha}$  being increasing with  $\alpha$ , see clause (a). Also, we have to prove clauses (f) and (g). Of course, every  $g_\alpha$  (hence  $g_{\beta_2}^{-1} \circ g_{\beta_1}$ ) is a permutation of  $A_\alpha$ , the universe of  $M_{1,\alpha}$ .

Now, to finish the proof of  $(*)_5(a)$ , notice that there is no problem in proving the  $M_{1,\alpha}$ 's are increasing, e.g. by  $(*)_5(d)$ , just check that.

We now prove clause (g) of  $(*)_6$ . Toward this, first, we shall show that for each  $\alpha < \kappa$ ,  $g_\alpha$  maps  $R_e^{M_{1,\alpha}}$  onto itself.

Assume we are given a pair  $(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2})$  from  $A_\alpha \times A_\alpha$  so  $\beta_1, \beta_2 < 1 + \alpha$ . Clearly,  $f_1 \neq e \circ f_2$  implies that

$$(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \notin R_e^{M_{1,\alpha}} \text{ and } (g_\alpha(a_{f_1,\beta_1,u_1}), g_\alpha(a_{f_2,\beta_2,u_2})) \notin R_e^{M_{1,\alpha}}.$$

Hence without loss of generality  $f_1 = e \circ f_2$  so:

$$(*)_{6.1} \quad (a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in R_e^{M_{1,\alpha}} \text{ iff} \\ u_1 = \{e(j) : j \in u_2 \text{ and } (\exists^{\text{odd}} \iota \in u_2)(e(\iota) = e(j))\}.$$

[Why? Read  $(*)_6(d)$  carefully, in particular note that if  $i \notin \{e(j) : j \in u_2\}$  then  $i \notin u_1$ .]

$$(*)_{6.2} \quad (g_\alpha(a_{f_1,\beta_1,u_1}), g_\alpha(a_{f_2,\beta_2,u_2})) \in R_e^{M_{1,\alpha}} \text{ iff} \\ (a_{f_1,\beta_1,u_1+\{f_1(\alpha)\}}, a_{f_2,\beta_2,u_2+G\{f_2(\alpha)\}}) \in R_e^{M_{1,\alpha}} \text{ iff } u_1 +_G \{f_1(\alpha)\} \text{ is equal to} \\ \text{the following set:} \\ \{e(j) : j \in u_2 +_G \{f_2(\alpha)\} \text{ and } [\exists^{\text{odd}} \iota \in (u_2 +_G \{f_2(\alpha)\})][e(\iota) = e(j)]\}.$$

[Why? Inside  $(*)_{6.2}$  the first ‘‘iff’’ holds by the definition of  $g_\alpha$ , the second ‘‘iff’’ holds as in  $(*)_{6.1}$ .]

But  $f_1 = e \circ f_2$  hence

$$(*)_{6.3} \quad f_1(\alpha) = e(f_2(\alpha)).$$

Next

- (\*)<sub>6.4</sub> letting  $x = f_2(\alpha) < \sigma$  the following statements are equivalent:
- (a)  $u_1 = \{e(j) : j \in u_2 \text{ and } (\exists^{\text{odd}} \iota \in u_2)(e(\iota) = e(j))\}$
  - (b)  $u_1 +_G \{e(x)\} = \{e(j) : j \in u_2 +_G \{x\} \text{ and } \exists^{\text{odd}} \iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}$ .

[Why? Straightforward but we shall elaborate, We shall prove it for any  $u_1, u_2 \in G$  and  $x < \sigma$ . By “ $G$  is of order two”, it suffices to prove the “only if” so assume the equality in (\*)<sub>6.4(a)</sub> and we have to prove the equality in (\*)<sub>6.4(b)</sub>.

If  $e(x) \notin u_1$  and  $x \notin u_2$  then the equality is clear: we just add  $e(x)$  to both sides of (\*)<sub>6.4(a)</sub> to get (\*)<sub>6.4(b)</sub>.

Now assume  $e(x) \notin u_1$  and  $x \in u_2$  then the left side in clause (\*)<sub>6.4(b)</sub> is the disjoint union of the left side of (\*)<sub>6.4(a)</sub> (which is  $u_1$ ) and  $\{e(x)\}$ . As we are assuming  $e(x) \notin u_1$ , necessarily  $(\exists^{\text{even}} \iota \in u_2)(e(\iota) = e(x))$ , hence the right side of (\*)<sub>6.4(b)</sub> is the disjoint union of the right side of (\*)<sub>6.4(a)</sub> and  $\{e(x)\}$ . As we are assuming the equality in (\*)<sub>6.4(a)</sub>, the last two sentences implies the equality in (\*)<sub>6.4(b)</sub>.

Next assume  $e(x) \in u_1$  and  $x \notin u_2$ . Then the left side in clause (\*)<sub>6.4(b)</sub> is the result of subtracting  $\{e(x)\}$  from the left side of (\*)<sub>6.4(a)</sub> (which is  $u_1$ ). As we are assuming  $x \in u_1$  necessarily  $(\exists^{\text{odd}} \iota \in u_2)(e(\iota) = e(x))$ , hence the right side of (\*)<sub>6.4(b)</sub> is the result of subtracting  $\{e(x)\}$  from the right side of (\*)<sub>6.4(a)</sub>. As we are assuming the equality in (\*)<sub>6.4(a)</sub>, the last two sentences implies the equality in (\*)<sub>6.4(b)</sub>.

Lastly assume  $e(x) \in u_1$  and  $x \in u_2$ . Then the left side in clause (\*)<sub>6.4(b)</sub> is the result of subtracting  $\{e(x)\}$  from the left side of (\*)<sub>6.4(a)</sub> (which is  $u_1$ ). As we are assuming  $e(x) \in u_1$ , necessarily  $(\exists^{\text{odd}} \iota \in u_2)(e(\iota) = e(x))$ , hence the right side of (\*)<sub>6.4(b)</sub> is the result of subtracting  $\{e(x)\}$  from the right side of (\*)<sub>6.4(a)</sub> and  $\{e(x)\}$ . As we are assuming the equality in (\*)<sub>6.4(a)</sub>, the last two sentences implies the equality in (\*)<sub>6.4(b)</sub>.

Hence (\*)<sub>6.4</sub> holds indeed.]

So together by (\*)<sub>6.2</sub> + (\*)<sub>6.4</sub> on the one hand , and (\*)<sub>2</sub> on the other hand we get the equivalence which prove that  $g_\alpha$  maps  $R_e^{M_1, \alpha}$  onto itself.

Hence (toward proving (\*)<sub>6(f)</sub>) the “first” holds. Second, we prove that  $g_\alpha$  preserves “ $\bar{a}, \bar{b}$  are  $E_{n, \alpha}$ -equivalent”, “ $\bar{a}, \bar{b}$  are  $E'_{n, \alpha}$ -equivalent” and their negations.

The  $E'_{n, \alpha}$  case is obvious by (\*)<sub>3</sub> an the  $R_{n, \alpha}$  case holds because  $\bar{a}, g_\alpha(\bar{a})$  are not  $E_{n, \alpha}$ -equivalent, but as  $(\forall \beta)(g_\beta = g_\beta^{-1})$ ,  $\bar{a}, \bar{b}$  being  $E_{n, \alpha}$ -equivalent means that there is an even length pass from  $\bar{a}$  to  $\bar{b}$ , in the graph  $\{(\bar{c}, g_\beta(\bar{c})) : \beta \in [\gamma, \kappa) \text{ and } \bar{c} \in {}^n(A_\gamma)\}$  where  $\gamma = \min\{\gamma : \bar{a}, \bar{b} \in {}^n(A_\gamma)\}$ . This proves another part of clause (g) of (\*)<sub>6</sub>.

Third, ,  $g_\alpha$  commutes with  $F_c^{M_1, \alpha}$  for  $c \in G$  because  $G$  is an Abelian group; thus completing the proof of (\*)<sub>6(g)</sub>.

Fourth, it maps  $R_0^{M_1, \alpha}$  to itself by the definition of  $g_\alpha$  (in (\*)<sub>1(f)</sub>) and of  $R_0^{M_1, \alpha}$  (in (\*)<sub>6(b) \bullet 2</sub>) and of  $G$ .

Lastly it does not preserve  $R_1^{M_1, \alpha}$  onto itself by the definition of  $R_1^{M_1, \alpha}$  in (\*)<sub>6(b) \bullet 2</sub> and of  $g_\alpha$  in (\*)<sub>6(f)</sub>.

This complete the proof of clause (g) of (\*)<sub>6</sub>.

Next, we should check clause (\*)<sub>6(f)</sub>. Now  $g_{\beta_2}^{-1} g_{\beta_1} \upharpoonright A_\alpha = (g_{\beta_2} \upharpoonright A_\alpha)(g_{\beta_1} \upharpoonright A_\alpha)$  by (\*)<sub>2(g)</sub> and it has order 2 because  $G$  is of order 2.

By clause (g) of  $(*)_2$  we have to prove that  $g_{\beta_2}^{-1}g_{\beta_1} \upharpoonright A_\alpha$  maps  $R_0^{M_1, \alpha}$  onto itself, which can be verified by definition of  $M_{2, \alpha}$ .

So we are done proving  $(*)_6$ .

$(*)_7$  for  $\alpha < \kappa$  let  $M_{2, \alpha}$  be the  $\tau_\theta$ -model with universe  $A_\alpha$  such that  $g_\alpha$  is an isomorphism from  $M_{1, \alpha}$  onto  $M_{2, \alpha}$ .

Now we note

$(*)_8$  for  $\alpha < \beta < \kappa$ ,  $M_{2, \alpha} \subseteq M_{2, \beta}$ , that is,  $M_{2, \alpha}$  is a submodel of  $M_{2, \beta}$ .

[Why? By  $(*)_6(g) + (*)_7$  we have just to check that  $R_1^{M_2, \alpha} = R_1^{M_2, \beta} \upharpoonright M_{2, \alpha}$ , which holds because for  $a \in A_\alpha$  we have  $a \in R_1^{M_2, \alpha}$  iff  $g_\alpha^{-1}(a) \notin R_1^{M_1, \alpha}$  iff  $g_\alpha^{-1}(a) \notin R_1^{M_1, \beta}$  iff  $g_\beta(g_\alpha^{-1}(a)) \in R_1^{M_1, \beta}$ . This clearly suffices.]

$(*)_9$  let  $M_{2, \kappa} := \cup\{M_{2, \alpha} : \alpha < \kappa\}$ ,

$(*)_{10}$   $M_{2, \kappa}$  well defined and  $M_{2, \alpha} \subseteq M_{2, \kappa}$  for  $\alpha < \kappa$  by  $(*)_8$ ,

$(*)_{11}$   $\pi_\alpha$  is well defined by  $(*)_1(e)$ ,

$(*)_{12}$  except clause (j) the demands in the conclusion of  $\boxplus$  of 1.2 were proved.

[Why? Just check.]

Note

$(*)_{13}$  if  $(a_{f, \alpha, u_1}, a_{f, \alpha, u_2})$  is  $E_{2, \alpha}$ -equivalent to  $(a_{f, \alpha, v_1}, a_{f, \alpha, v_2})$  then  $G \models "u_1 - u_2 = v_1 - v_2"$ .

[Why? By induction on the  $k$  from  $(*)_4$ .]

So, to finish, we assume toward contradiction:

$\boxtimes$   $h$  is an isomorphism from  $M_{1, \kappa}$  onto  $M_{2, \kappa}$  which respects  $\pi_\kappa$  hence  $\pi_\alpha$  for  $\alpha < \kappa$ , i.e.  $h \upharpoonright M_{1, \alpha}$  respect  $\pi_\alpha$ , see clause  $\boxplus(i)$  of Claim 1.2.

So trivially

$\otimes_1$  if  $\alpha < \kappa$ , then  $h(a_{f, \gamma, u}) \in \{a_{f, \gamma, v} : v \in G\}$  for  $\gamma < 1 + \alpha$ ,  $u \in G$ , and  $\bar{a} \in {}^n(A_\alpha) \Rightarrow h(\bar{a}) \in \bar{a}/E'_{n, \alpha}$ .

[Why? As  $h \upharpoonright M_{1, \alpha}$  respect  $\pi_\alpha$  see  $(*)_1(e)$  and  $(*)_6(g) + (*)_7 + (*)_8 + (*)_9$  clearly  $h(\bar{a}) \in \bar{a}/E'_{n, \alpha}$ .]

Hence,

$\otimes_2$  for  $f \in {}^\kappa\sigma$  and  $\alpha < \kappa$  let  $u_{f, \alpha} \in G$  be the  $u \in G$  such that  $h(a_{f, \alpha, \emptyset}) = a_{f, \alpha, u}$ .

$\otimes_3$  for  $f \in {}^\kappa\sigma$ ,  $\alpha < \kappa$  and  $v \in G$  we have  $h(a_{f, \alpha, v}) = a_{f, \alpha, v +_G u_{f, \alpha}}$ .

[Why? For  $c \in G$ , we know that  $h$  maps  $F_c^{M_1, \alpha}$  onto  $F_c^{M_2, \alpha}$  which is equal to  $F_c^{M_1, \alpha}$ . Apply this to  $c = v$ .]

$\otimes_4$  we define a partial order  $\leq$  on  ${}^\kappa\sigma$  as follows:

$f_1 \leq f_2$  iff there is a function  $e \in {}^\sigma\sigma$  witnessing it; which means  $f_1 = e \circ f_2$

$\otimes_5$  if  $\alpha_1, \alpha_2 < \kappa$  and  $f_1 \leq f_2$  (are from  ${}^\kappa\sigma$ ) then  $|u_{f_1, \alpha_1}| \leq |u_{f_2, \alpha_2}|$ .

[Why? This follows from  $\otimes_6$  below.]

- ⊗<sub>6</sub> if  $e \in {}^\sigma\sigma$ ,  $f_2 \in {}^\kappa\sigma$  and  $f_1 = e \circ f_2 \in {}^\kappa\sigma$  and  $\alpha_1, \alpha_2 < \kappa$  then  
 $u_{f_1, \alpha_1} \subseteq \{e(i) : i \in u_{f_2, \alpha_2}\}$ .

[Why? Choose  $\alpha < \kappa$  such that  $\alpha > \alpha_1, \alpha > \alpha_2$  so  $a_{f_1, \alpha_1, \emptyset}, a_{f_2, \alpha_2, \emptyset} \in M_{\ell, \alpha}$  for  $\ell = 1, 2$ . Recall that  $h$  maps  $R_e^{M_1, \alpha}$  onto  $R_e^{M_2, \alpha}$  by  $\boxtimes$  and  $R_e^{M_2, \alpha} = R_e^{M_1, \alpha}$  because  $g_\alpha$  maps  $R_e^{M_1, \alpha}$  onto itself (by  $(*)_6(g)$  or see the proof of  $(*)_6$  above, the “first” in that proof). Now, let  $x \in \sigma \setminus (u_{f_1, \alpha_1} \cup u_{f_2, \alpha_2})$ , so see  $(*)_6(d)$ , i.e. the definition of  $R_e^{M_1, \alpha}$ , obviously  $(a_{f_1, \alpha_1, \{x\}}, a_{f_2, \alpha_2, \{x\}}) \in R_e^{M_1, \alpha}$  so as  $h$  is an isomorphism from  $M_{1, \kappa}$  onto  $M_{2, \kappa}$  we have  $(h(a_{f_1, \alpha_1, \{x\}}), h(a_{f_2, \alpha_2, \{x\}})) \in R_e^{M_2, \alpha}$  so by the previous sentence and by ⊗<sub>3</sub> and the definitions of  $u_{f_\ell, \alpha_\ell}$  ( $\ell = 1, 2$ ) in ⊗<sub>2</sub> letting  $v_1 = u_{f_1, \alpha_1} \cup \{x\}$  and  $v_2 = u_{f_2, \alpha_2} +_G \{x\}$  we have  $(a_{f_1, \alpha_1, v_1}, a_{f_2, \alpha_2, v_2}) \in R_e^{M_1, \alpha}$  which by the definitions of  $R_e^{M_1, \alpha}$  in  $(*)_6(d)$  implies  $u_{f_1, \alpha_1} \subseteq \{e(i) : i \in u_{f_2, \alpha_2}\} \cup \{x\}$ , which by the choice of  $x$  implies that  $u_{f_1, \alpha_1} \subseteq \{e(i) : i \in u_{f_2, \alpha_2}\}$  as promised.]

- ⊗<sub>7</sub> (a)  $|u_{f, \alpha_1}| = |u_{f, \alpha_2}|$  for  $\alpha_1, \alpha_2 < \kappa, f \in {}^\kappa\sigma$ ,  
 (b)  $\mathbf{n}(f) = |u_{f, \alpha}|$  is well defined for  $\alpha < \kappa$ ,  
 (c) if  $f_1 \leq f_2$  then  $\mathbf{n}(f_1) \leq \mathbf{n}(f_2)$ .

[Why? For clause (a) use ⊗<sub>6</sub> twice for the function  $e = \text{id}_\sigma$  and  $f_1 = f_2 = f$ . Clause (b) follows. Clause (c) holds by ⊗<sub>6</sub> equivalently by ⊗<sub>5</sub>.]

- ⊗<sub>8</sub> there are  $f_* \in {}^\kappa\sigma$  and  $\alpha_* < \kappa$  such that:  
 (i) if  $f_* \leq f \in {}^\kappa\sigma$  and  $\alpha < \kappa$  then  $|u_{f_*, \alpha_*}| = |u_{f, \alpha}|$   
 (ii) moreover if  $f_* = e \circ f$  where  $e \in {}^\sigma\sigma$  and  $f \in {}^\kappa\sigma, \alpha < \kappa$  then  $\mathbf{n}(f_*) = \mathbf{n}(f)$   
 (iii) if  $\alpha < \kappa, f_1 = e \circ f_2, f_* = e_1 \circ f_1, f_* = e_2 \circ f_2$  so  $e, e_1, e_2 \in {}^\sigma\sigma$ , then  $e \upharpoonright u_{f_2, \alpha}$  is one-to-one onto  $u_{f_1, \alpha}$ .

[Why? First note that clause (ii), (iii) follows from clause (i)+⊗<sub>6</sub>. Second, if clause then  $e \upharpoonright u_{f, \alpha}$  is one-to-one from  $u_{f, \alpha}$  onto (i) fails, then we can find a sequence  $\langle (f_n, \alpha_n, e_n) : n < \omega \rangle$  such that:

- (α)  $\alpha_n < \kappa, f_n \in {}^\kappa\sigma$  for  $n < \omega$   
 (β)  $f_n \leq f_{n+1}$  say  $f_n = e_n \circ f_{n+1}$  and  $e_n \in {}^\sigma\sigma$   
 (γ)  $(e_n, f_{n+1}, \alpha_{n+1})$  witness that  $(f_n, \alpha_n)$  does not satisfy the demand (i) on  $(f_*, \alpha_*)$  hence  $\mathbf{n}(f_n) < \mathbf{n}(f_{n+1})$ .

Recalling we assume  $\sigma = \sigma^{\aleph_0}$ , there are functions  $\text{pr}_\omega$  and  $e^n \in {}^\kappa\sigma$  for  $n < \omega$  such that  $\text{pr}_\omega : {}^\omega\sigma \rightarrow \sigma$  is 1-to-1, onto and  $\text{pr}_\omega(\bar{\alpha}) = \beta \wedge n < \omega \Rightarrow e^n(\beta) = \alpha_n$ . Now, define  $f \in {}^\kappa\sigma$  by  $f(\beta) = \text{pr}_\omega(\langle f_n(\beta) : n < \omega \rangle)$ , clearly  $f \in {}^\kappa\sigma$  and  $f_n = e^n \circ f$

So  $n < \omega \Rightarrow f_n \leq f$  which by ⊗<sub>7(c)</sub> implies  $\mathbf{n}(f_n) \leq \mathbf{n}(f)$ . As  $\langle \mathbf{n}(f_n) : n < \omega \rangle$  is increasing, easily we get a contradiction.]

- ⊗<sub>9</sub>  $\mathbf{n}(f_*) > 0$ , i.e.  $\alpha < \kappa \Rightarrow u_{f_*, \alpha} \neq \emptyset$ .

[Why? If  $(\forall f \in {}^\kappa\sigma)(\forall \alpha < \kappa)(u_{f, \alpha} = \emptyset)$  then (by ⊗<sub>3</sub>) we deduce  $h$  is the identity, a contradiction because  $R_1^{M_1, \alpha} \neq R_1^{M_2, \alpha}$ . Otherwise assume  $u_{f, \alpha} \neq \emptyset$  hence as in the proof of ⊗<sub>8</sub> there is  $f'$  such that  $f_* \leq f' \wedge f \leq f'$  so by ⊗<sub>5</sub> and ⊗<sub>8</sub> we have  $0 < |u_{f, \alpha}| \leq |u_{f', \alpha}| = |u_{f_*, \alpha_*}|$ .]

- ⊗<sub>10</sub> if  $f \in {}^\kappa\sigma$  and  $\alpha, \beta < \kappa$ , then  $u_{f, \alpha} = u_{f, \beta}$ .

[Why? Recall  $(*)_6(b)\bullet_1$ , hence  $(a_{f,\alpha,\emptyset}, a_{f,\beta,\emptyset}) \in R_0^{M_1,\alpha}$ , so as  $h$  maps  $R_0^{M_1,\alpha}$  onto  $R_0^{M_1,\alpha}$  we have  $(a_{f,u_{\ell,\alpha}}, a_{f,\alpha,u_{\ell,\beta}}) \in R_0^{M_2,\alpha}$ , hence  $u_{f,\alpha} = u_{f,\beta}$  is as promised.]

$\otimes_{11}$  Now fix  $f_*, \alpha_*$  as in  $\otimes_8$  for the rest of the proof, without loss of generality  $f_*$  is onto  $\sigma$ .

[Why? Clearly,  $\text{Range}(f_*)$  is a non-empty subset of  $\sigma$ , and as  $\kappa > \sigma$  is regular, there is  $i < \sigma$  such that  $f_*^{-1}(\{i\})$  has cardinality  $> \sigma$ , (see 1.1,  $\otimes_1(a)$ ). Let  $f \in {}^\kappa\sigma$  be such that  $\alpha < \kappa \wedge f_*(\alpha) \neq i \Rightarrow f(\alpha) = f_*(\alpha)$  and  $f \upharpoonright (f_*^{-1}(\{i\}))$  is onto  $\sigma$ . Let  $e \in {}^\sigma\sigma$  be such that  $j \in \text{Rang}(f_*) \Rightarrow \sigma(j) = j$  and  $j \notin \text{Range}(f_*) \Rightarrow \sigma(j) = i$ . Easily,  $f_* = e \circ f$ , hence  $f_* \leq f$  and  $f$  is onto  $\sigma$ , and so we can replace  $f_*$  by  $f$ , so indeed without loss of generality  $\oplus_{11}$  holds.]

Let  $u_{f_*,\alpha_*} = \{i_\ell^* : \ell < \ell(*)\}$  with  $\langle i_\ell^* : \ell < \ell(*) \rangle$  increasing for simplicity. Now for every  $f \in {}^\kappa\sigma$  such that  $f_* \leq f$  and  $\alpha < \kappa$  by  $\otimes_8(ii), (iii)$  we know that if  $e \in {}^\sigma\sigma \wedge f \in {}^\kappa\sigma \wedge f_* = e \circ f$  then  $e \upharpoonright u_{f,\alpha}$  is a one-to-one mapping from  $u_{f,\alpha}$  onto  $u_{f_*,\alpha_*}$ ; but so  $e \upharpoonright u_{f,\alpha}$  is uniquely determined by  $(f_*, \alpha_*, f, \alpha)$  so let  $i_{f,\alpha,\ell} \in u_{f,\alpha}$  be the unique  $i \in u_{f,\alpha}$  such that  $e(i) = i_\ell^*$  (equivalently  $(\exists \alpha)(f(\alpha) = i \wedge f_*(\alpha) = i_\ell^*)$ ).

Let

$$\mathcal{A} = \{A \subseteq \kappa : \text{for some } f, f_* \leq f \text{ and } \alpha < \kappa \text{ we have } f^{-1}\{i_{f,0}\} \setminus \alpha \subseteq A\}$$

$$\square_1 \mathcal{A} \subseteq \mathcal{P}(\kappa) \setminus [\kappa]^{<\kappa} = [\kappa]^\kappa.$$

[Why? As  $\kappa$  is regular, this means  $A \in \mathcal{A} \Rightarrow A \subseteq \kappa \wedge \sup(A) = \kappa$  which holds by  $\otimes_{10}$ .]

$$\square_2 \kappa \in \mathcal{A}.$$

[Why? By the definition of  $\mathcal{A}$ .]

$$\square_3 \text{ if } A \in \mathcal{A} \text{ and } A \subseteq B \subseteq \kappa \text{ then } B \in \mathcal{A}.$$

[Why? By the definition of  $\mathcal{A}$ .]

$$\square_4 \text{ if } A_1, A_2 \in \mathcal{A} \text{ then } A =: A_1 \cap A_2 \text{ belongs to } \mathcal{A}.$$

[Why? Let  $(f_\ell, e_\ell, \alpha_\ell)$  be such that  $f_* = e_\ell \circ f_\ell$  and  $f_\ell \in {}^\kappa\sigma, \alpha_\ell < \kappa$  and  $f_\ell^{-1}\{i_{f_\ell,0}\} \setminus \alpha_\ell \subseteq A_\ell$  for  $\ell = 1, 2$  and let  $\alpha_0 = \max\{\alpha_1, \alpha_2\}$ . Let  $\text{pr}: \sigma \times \sigma \rightarrow \sigma$  be one-to-one and onto and define  $f \in {}^\kappa\sigma$  by  $f(\alpha) = \text{pr}(f_1(\alpha), f_2(\alpha))$ . Clearly  $f_\ell \leq f$  for  $\ell = 1, 2$  hence  $i_{f,0}$  is well defined and  $i_{f,0} = \text{pr}(i_{f_1,0}, i_{f_2,0})$ . Now for every  $\alpha \in \kappa \setminus \alpha_0$ ,

- <sub>1</sub>  $f(\alpha) = i_{f_2,0} \Rightarrow f_1(\alpha) = i_{f_1,0}$ ,
- <sub>2</sub>  $f_2(\alpha) = i_{f_2,0} \Rightarrow \alpha \in A_1$ ,
- <sub>3</sub>  $\alpha \in A_2 \Rightarrow \alpha \in A_1 \cap A_2 \Rightarrow \alpha \in A$ ; so
- <sub>4</sub>  $f^{-1}\{i_{f,0}\} \subseteq A$ .

Hence  $A \in \mathcal{A}$ .]

$$\square_5 \text{ if } A \subseteq \kappa \text{ then } A \in \mathcal{A} \text{ or } \kappa \setminus A \in \mathcal{A}.$$

[Why? Define  $f \in {}^\kappa\sigma$ :

$$f(\alpha) = \begin{cases} 2f_*(\alpha) & \text{if } \alpha \in A \\ 2f_*(\alpha) + 1 & \text{if } \alpha \in \kappa \setminus A. \end{cases}$$

Let  $e \in {}^\sigma\sigma$  be defined by  $e(2i) = e(2i+1) = i$  so  $f_*(i) = e(f(i))$  for  $i < \sigma$ ; so  $f_* \leq f$ . Let  $i = i_{f,0}$  so by the definition of  $\mathcal{A}$  we have  $f^{-1}\{i\} = f^{-1}\{i_{f,0}\} \in \mathcal{A}$ . But if  $i$  is even then  $f^{-1}\{i\} \subseteq A$  and  $i$  is odd then  $f^{-1}\{i\} \subseteq \kappa \setminus A$  so by  $\square_3$  we are done.]

$\square_6$   $\mathcal{A}$  is a uniform ultrafilter on  $\kappa$ .

[Why? By  $\square_1 - \square_5$ .]

$\square_7$   $\mathcal{A}$  is  $\sigma^+$ -complete.

[Why? Assume  $B_\varepsilon \in \mathcal{A}$  for  $\varepsilon < \sigma$  and let  $B = \bigcap \{B_\varepsilon : \varepsilon < \sigma\}$ .

Define  $A_\varepsilon \subseteq \kappa$  for  $\varepsilon < \sigma$  as follows:

- $A_\varepsilon = \bigcap_{\zeta < \xi} B_\zeta \setminus B_\xi$  if  $\varepsilon = 1 + \xi \geq 2$ ,
- $A_\varepsilon = \kappa \setminus B_0$  if  $\varepsilon = 1$ , and
- $A_\varepsilon = B$  if  $\varepsilon = 0$ .

Clearly  $\langle A_\varepsilon : \varepsilon < \sigma \rangle$  is a partition of  $\kappa$ , let  $f \in {}^\kappa\sigma$  be such that  $f \upharpoonright A_\varepsilon$  is constantly  $\varepsilon$ . Let  $f' \in {}^\kappa\theta$  be such that  $f \leq f' \wedge f_* \leq f'$ . Now  $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$  is included in some  $A_\varepsilon$ . If  $\varepsilon = 0$  this exemplifies  $\bigcap_{\varepsilon < \sigma} B_\varepsilon \in \mathcal{A}$  as required. If

$\varepsilon = 1 + \zeta < \sigma$ , then  $(f')^{-1}\{i_{f',0}\} \subseteq A_\zeta \subseteq \kappa \setminus B_\zeta$ , contradiction to  $\square_6$  because  $B_\varepsilon \in \mathcal{A}$  and  $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$ .]

So by the assumptions of 1.2, that is,  $\otimes_1(b)$  of 1.1 we get a contradiction, coming from the assumption “toward contradiction, clause (j) of  $\boxplus$  of 1.2 fails”, so it holds, and the other clauses were proved so we are done.  $\square_{1.2}$

**Theorem 1.4.** *For every  $\sigma$  there is an  $\mathfrak{k} = \mathfrak{k}_\sigma^*$  such that*

- $\otimes$  (a)  $\mathfrak{k}$  is an AEC with  $\text{LST}(\mathfrak{k}) = \theta$ ,  $|\tau_\mathfrak{k}| = \sigma$
- (b)  $\mathfrak{k}$  admit intersections,
- (c)  $\mathfrak{k}$  has amalgamation,
- (d) if  $\kappa$  is a regular cardinal and there is no uniform  $\sigma^+$ -complete ultrafilter on  $\kappa$ , then:  $\mathfrak{k}$  is not  $(\leq 2^\kappa, \kappa)$ -sequence-local for types, i.e., we can find an  $\leq_\mathfrak{k}$ -increasing continuous sequence  $\langle M_i : i \leq \kappa \rangle$  of models and  $p \neq q \in \mathcal{S}_\mathfrak{k}(M_\kappa)$  such that  $i < \kappa \Rightarrow p \upharpoonright M_i = q \upharpoonright M_i$  and  $M_\kappa$  is of cardinality  $\leq 2^\kappa$ .

We shall prove 1.4 below. As in [BS08, 1.2, §4] the aim of the definition of “admit intersections” is to ensure types behave reasonably.

**Definition 1.5.** We say an AEC  $\mathfrak{k}$  admits intersections when there is a function  $cl_\mathfrak{k}$  such that:

- (a)  $cl_\mathfrak{k}(A, M)$  is well defined iff  $M \in K_\mathfrak{k}$  and  $A \subseteq M$
- (b)  $cl_\mathfrak{k}(A, M)$  is preserved under isomorphisms and  $\leq_\mathfrak{k}$ -extensions; that is:
  - $h$  is an isomorphisms from  $M_1 \in K_\mathfrak{k}$  onto  $M_2$  and  $A_1 \subseteq M_1$  then  $cl_\mathfrak{k}(\{h(a) : a \in A_1\}, M_2) = \{h(b) : b \in cl_\mathfrak{k}(A, M_1)\}$ , and
  - if  $A \subseteq M_1 \leq_\mathfrak{k} M_2$  then  $cl_\mathfrak{k}(A, M_1) = cl_\mathfrak{k}(A, M_2)$ .

- (c) for every  $M \in K_{\mathfrak{k}}$  and non-empty  $A \subseteq M$  the set  $B = cl_{\mathfrak{k}}(A, M)$  satisfies:  
 $M \upharpoonright B \in K_{\mathfrak{k}}, M \upharpoonright B \leq_{\mathfrak{k}} M$ ; noting that for every  $M_1, N$  we have  $A \subseteq M_1 \leq_{\mathfrak{k}} N \wedge M \leq_{\mathfrak{k}} N \Rightarrow B \subseteq M_1$ ;
- (d) we may use  $cl_{\mathfrak{k}}(A, M)$  for  $M \upharpoonright cl_{\mathfrak{k}}(A, M)$ .

**Claim 1.6.** *Assume  $\mathfrak{k}$  is an AEC admitting intersections. Then  $ortp_{\mathfrak{k}}(a_1, M, N_1) = ortp_{\mathfrak{k}}(a_2, M, N_2)$  iff letting  $M_{\ell} = N_{\ell} \upharpoonright cl_{\mathfrak{k}}(M \cup \{a_{\ell}\})$ , there is an isomorphism from  $M_1$  onto  $M_2$  over  $M$  mapping  $a_1$  to  $a_2$ .*

*Proof.* It should be clear from the definition. □<sub>1.6</sub>

*Remark 1.7.* In Theorem 1.4 we can many times demand  $\|M_{\kappa}\| = \kappa$ , e.g., if  $(\exists \lambda)(\kappa = 2^{\lambda})$ .

Note we now show that 1.4 is the best possible.

**Claim 1.8.** *1) If  $\mathfrak{k}$  satisfies clause (a) of 1.4, (i.e.  $\mathfrak{k}$  is an AEC with LST-number  $\leq \theta$  and  $|\tau_{\mathfrak{k}}| \leq \theta$ ) and  $\kappa$  is regular and fails the assumption of clause (d) of 1.4, that is,  $\kappa > \theta$  and there is a uniform  $\theta^+$ -complete ultrafilter on  $\kappa$ , then the conclusion of clause (d) of 1.4 fails, that is  $\mathfrak{k}$  is  $\kappa$ -sequence local for types.*

*2) If  $D$  is a  $\theta^+$ -complete ultrafilter on  $\kappa$  and  $\mathfrak{k}$  is an AEC with  $LST(\mathfrak{k}) \leq \theta$  then ultraproducts by  $D$  preserve “ $M \in \mathfrak{k}$ ”, “ $M \leq_{\mathfrak{k}} N$ ”, i.e.*

□ if  $M_i, N_i (i < \kappa)$  are  $\tau(\mathfrak{K})$ -models and  $M = \prod_{i < \kappa} M_i/D$  and  $N = \prod_{i < \kappa} N_i/D$

then:

- (a)  $M \in K$  if  $\{i < \kappa : M_i \in \mathfrak{k}\} \in D$
- (b)  $M \leq_{\mathfrak{k}} N$  if  $\{i : M_i \leq_{\mathfrak{k}} N_i\} \in D$ .

*Proof.* Recall that if  $D$  is  $\theta^+$ -complete, then it is  $\sigma^+$ -complete where  $\sigma = \theta^{\aleph_0}$  (and much more, it is  $\theta'$ -complete for the first measurable  $\theta' > \theta$ ).

1) So assume

- ⊕ (a)  $\langle M_i : i \leq \kappa \rangle$  is  $\leq_{\mathfrak{k}}$ -increasing
- (b)  $M_{\kappa} = N_0 \leq_{\mathfrak{k}} N_{\ell}$  for  $\ell = 1, 2$
- (c)  $p_{\ell} = ortp_{\mathfrak{k}}(a_{\ell}, N_0, N_{\ell})$  for  $\ell = 1, 2$
- (d)  $i < \kappa \Rightarrow p_1 \upharpoonright M_i = p_2 \upharpoonright M_i$ .

We shall show  $p_1 = p_2$ , this is enough.

Without loss of generality

- (\*)<sub>1</sub> (a)  $a_1 = a_2$  call it  $a$ ,
- (b)  $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$ .

By (d) of ⊕ we have:

- (d)<sup>+</sup> for each  $i < \kappa$  there are even  $m_i < \omega$  and  $\langle N_{i,n} : n \leq m_i \rangle$  such that:
  - (α)  $N_{i,0} = N_1$ ,
  - (β)  $N_{i,m_i} = N_2$  or just  $h_i$  is an isomorphism from  $N_{i,m_i}$  onto  $N_2$  such that  $h_i \upharpoonright (M_i \cup \{a\})$  is the identity,
  - (γ)  $a \in N_{i,\ell}$  and  $M_i \leq_{\mathfrak{k}} N_{i,\ell}$ ,
  - (δ) if  $2m + 2 \leq m_i$  then  $N_{i,2m+1} \leq_{\mathfrak{k}} N_{i,2m}, N_{i,2m+2}$ .

As  $\kappa = \text{cf}(\kappa) > \aleph_0$  without loss of generality  $i < \kappa \Rightarrow m_i = n_*$ . Let  $\chi$  be such that  $\langle M_i : i \leq \kappa \rangle, \langle \langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle$  and  $\mathfrak{k}_{\text{LST}(\mathfrak{k})}$  all belongs to  $\mathcal{H}(\chi)$ ; concerning  $\mathfrak{k}_{\text{LST}(\mathfrak{k})}$  this means  $\tau_{\mathfrak{k}}$  and  $\text{LST}(\mathfrak{k})$  belongs to  $\mathcal{H}(\chi)$  and as usual without loss of generality  $\tau_{\mathfrak{k}}$  has cardinality at most  $2^\sigma$  hence  $\{M \in K_{\mathfrak{k}} : M \in \mathcal{H}(\text{LST}_{\mathfrak{k}}^+)\}$  and  $\leq_{\mathfrak{k}} \upharpoonright \mathcal{H}(\text{LST}_{\mathfrak{k}}^+)$  belongs to  $\mathcal{H}(\chi)$ ; those hold by  $(*)_1(b)$ . Let  $\mathfrak{B}$  be the ultrapower  $(\mathcal{H}(\chi), \in)^\kappa / D$  and  $\mathbf{j}_0$  the canonical embedding of  $(\mathcal{H}(\chi), \in)$  into  $\mathfrak{B}$  and let  $\mathbf{j}_1$  be the Mostowski-Collapse of  $\mathfrak{B}$  to a transitive set  $\mathcal{H}$  and let  $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$ . So  $\mathbf{j}$  is an elementary embedding of  $(\mathcal{H}(\chi), \in)$  into  $(\mathcal{H}, \in)$  and even an  $\mathbb{L}_{\theta^+, \theta^+}$ -elementary one. Recall that by  $(*)_1(b)$  we are assuming without loss of generality  $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$  hence  $\mathbf{j}(\tau_{\mathfrak{k}}) = \tau_{\mathfrak{k}}$  hence by part (2),  $\mathbf{j}$  preserves “ $N \in K_{\mathfrak{k}}$ ”, “ $N^1 \leq_{\mathfrak{k}} N^2$ ” and “ $h$  is an isomorphism from  $N'$  onto  $N''$ ”.

So  $\mathbf{j}(\langle M_i : i \leq \kappa \rangle)$  has the form  $\langle M_i^* : i \leq \mathbf{j}(\kappa) \rangle$  but  $\mathbf{j}(\kappa) > \kappa_* := \bigcup_{i < \kappa} \mathbf{j}(i)$  by the uniformity of  $D$  and let  $\mathbf{j}(\langle \langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle) = \langle \langle N_{i,n}^* : n \leq n_* \rangle : i < \mathbf{j}(\kappa) \rangle$  and  $\mathbf{j}(\langle h_i : i < \kappa \rangle) = \langle h_i^* : i < \mathbf{j}(\kappa) \rangle$ .

So

- (a)  $\mathbf{j} \upharpoonright M_\kappa$  is a  $\leq_{\mathfrak{k}}$ -embedding of  $M_\kappa$  into  $M_{\mathbf{j}(\kappa)}^*$  hence even into  $M_{\kappa_*}^*$ ,
- (b)  $M_{\kappa_*}^* \leq_{\mathfrak{k}} N_{i,n}^*$  and  $\mathbf{j}(a) \in N_{i,n}^*$  for  $i < \kappa, n \leq n_*$ ,
- (c)  $N_{i,0}^* = \mathbf{j}(N_1)$ ,
- (d)  $h_{\kappa_*}$  is an isomorphism from  $N_{\kappa_*, n_*}$  onto  $\mathbf{j}(N_2)$ ,
- (e)  $N_{\kappa_*, 2m+1}^* \leq_{\mathfrak{k}} N_{\kappa_*, 2m}^*, N_{\kappa_*, 2m+2}^*$  for  $2m+1 < n_*$ ,
- (f)  $\mathbf{j}(a) \in N_{\kappa_*, m}$  for  $m \leq n_*$ ,
- (g)  $h_{\kappa_*}$  is an isomorphism from  $N_{\kappa_*, n_*}$  into  $\mathbf{j}(N_2)$  over  $M_{\kappa_*}^* \cup \{\mathbf{j}(a)\}$ . Hence,
- (h)  $M_{\kappa_*}^* \leq_{\mathfrak{k}} \mathbf{j}(N_2)$ ,
- (i)  $\text{ortp}(\mathbf{j}(a), M_{\kappa_*}^*, \mathbf{j}(N_1)) = \text{ortp}(\mathbf{j}(a), M_{\kappa_*}^*, \mathbf{j}(N_2))$ ,

Also,

- (j) letting  $M_{\kappa_*}^\bullet = M_{\kappa_*}^* \upharpoonright \{\mathbf{j}(a) : a \in M_\kappa\}$ , we have  $M_{\kappa_*}^\bullet \leq_{\mathfrak{k}} M_{\kappa_*}^*$  and  $\mathbf{j} \upharpoonright M_\kappa$  is an isomorphism from  $M_\kappa$  onto  $M_{\kappa_*}^\bullet$ ,
  - (k) For  $\ell = 1, 2$ , let  $N_\ell^\bullet = \mathbf{j}(N_\ell) \upharpoonright \{\mathbf{j}(a) : a \in N_\ell\}$ , we have  $M_{\kappa_*}^\bullet \leq_{\mathfrak{k}} N_\ell^\bullet \leq_{\mathfrak{k}} \mathbf{j}(N_\ell)$ . Hence (by (i) and monotonicity),
  - (l)  $\text{ortp}(\mathbf{j}(a), M_{\kappa_*}^\bullet, N_1^\bullet) = \text{ortp}(\mathbf{j}(a), M_{\kappa_*}^\bullet, N_2^\bullet)$ .
- By preservation under isomorphisms,
- (m)  $\text{ortp}(a, M_\kappa, N_1) = \text{ortp}(a, M_\kappa, N_2)$ .

Together, we are done.

2) By the representation theorem of AEC, see [She09a, §1]. □<sub>1.8</sub>

**Discussion 1.9.** We try to help the reader by pointing out some things in the proof of Theorem 1.4.

(1) If the reader do not mind having  $\tau_{\mathfrak{k}}$  to be of cardinality  $2^\sigma$ , then we can replace  $R_2$  essentially by the  $R_e$  ( $e \in {}^\sigma \sigma$ ) and omit  $S$  and  $\{d_i : i < \sigma\}$ . This simplifies somewhat, so in the present discussion 1.9 we follow it.

(2) We rely on the conclusion in 1.2. There we have two increasing continuous sequences of models  $\bar{M}_\ell = \langle M_{\ell, \alpha} : \alpha \leq \kappa \rangle$  for  $\ell = 1, 2$ .

Now  $M_{1, \alpha}, M_{2, \alpha}$  are very similar:

- (\*) For  $\alpha < \kappa$ , they are not just isomorphic but have the same universe, and the difference is only in the interpretation of the predicate  $R_1$ .

Here we define  $M'_{0,\alpha}$  by restricting ourselves to  $\langle I_\alpha, S \rangle$  where  $S$  code  $G$ , and  $I_\alpha$  code  ${}^\kappa\sigma \times \kappa$ .

Now, for  $\ell = 1, 2$  we shall define  $M'_{\ell,\alpha}$  essentially adding  $M_{\ell,\alpha}$  to  $M'_{0,\alpha}$  with both the set of elements  $A_\alpha$  and the relations and function except  $R_1$  and adding a new element  $t_\ell^* \in J^{M_{\ell,\alpha}}$ , which code the  $R_1$ , i.e.

$$R_1^{M_{\ell,\alpha}} = \{\bar{a} \cap \langle t_\ell^* \rangle : \bar{a} \in R_1^{M_{\ell,i}}\}.$$

So this translate “ $M_{1,\alpha} \cong M_{2,\alpha} \iff \alpha < \kappa$ ” to

$$\text{otp}(t_1^*, M'_{0,\alpha}, M'_{1,\alpha}) = \text{otp}(t_2^*, M'_{0,\alpha}, M'_{2,\alpha}) \iff \alpha < \kappa$$

*Proof.* Proof of 1.4

Let  $G = ([\sigma]^{<\aleph_0}, \Delta)$  and let  $\langle c_i : i < \sigma \rangle$  list the members of  $G$ . We define  $\tau = \tau_\mathfrak{k}$ , by:

- ⊕<sub>1</sub>  $\tau = \tau_\sigma^\bullet \cup \{S, S_0, S_1, S_2, J, I, A, \pi, R_1, R_2, R_3\} \cup \{H\} \cup \{F_1, F_2\} \cup \{d_i : i < \sigma\}$ , so of cardinality  $\sigma$ , where:
- (a)  $\tau_\sigma^\bullet = \{E_n, E'_n : n < \omega\} \cup \{R_0\}$ ,
  - (b)  $R_0, R_1, R_3$  are binary predicates,  $E_n, E'_n$  are  $(2n)$ -place predicates for  $n \in [2, \omega)$ ,
  - (c)  $S, S_0, S_1, S_2, J, I, A$  are unary predicates,
  - (d)  $\pi$  is an unary function symbol,
  - (e)  $R_2$  is a three place predicate. respectively,
  - (f)  $H$  is an unary function symbol,
  - (g)  $d_i$  is an individual constant for  $i < \sigma$ .
  - (h)  $H$  is a two place function symbol.
  - (i)  $F_1, +$  are two-place function symbols.

We define  $K$  as a class of  $\tau$ -models by (note that the function symbols are interpreted as partial functions):

⊗<sub>2</sub>  $M \in K$  iff (up to isomorphism):

- (a)  $\langle S_0^M, S_1^M, S_2^M, Q^M, I^M, J^M, A^M \rangle$  is a partition of  $|M|$ , (recall that they are unary),
- (b)  $S^M = S_0^M \cup S_1^M \cup S_2^M$ .
- (c)  $(E'_n)^M \subseteq {}^{2n}(A^M)$  is an equivalence relation on  ${}^n(A^M)$ , and  $E_n^M$  an equivalence relation on  ${}^n(A^M)$  refining it, for  $n \in [2, \omega)$ ,
- (d)  $R_0$  is a binary relation  $\subseteq A^M \times A^M$ ,
- (e)  $R_1^M$  is a binary relation  $\subseteq A^M \times J^M$ ,
- (f)  $\pi^M$  is a function from  $A^M$  onto  $I^M$ ,
- (g)  $R_2^{M_0} \subseteq A^M \times A^M \times S_2^M$  (play the role of the  $R_e$ 's in 1.2),
- (h)  $\{d_i^M : i < \sigma\}$  are pairwise distinct elements of  $S_0^M$ ,
- (i)
  - $+^M$  is a two place function on  $S_1^M$ ,
  - $H^M : S_0^M \rightarrow S_1^M$  is one-to-one,
  - $G = (S_1^M, +^M)$  is an Abelian group satisfying  $(\forall x \in G)(x + x = 0_G)$  and  $\text{rng}(H^M)$  is a basis of  $G$ ,
- (j) (α)  $R_3^M \subseteq A^M \times J$  and for  $t \in J$ , define:
  - $R_3^M[t] = \{a \in A_0 : M \models R_3[a, t]\}$ ,
- (β)
  - $(E'_n)^M \subseteq \bigcup \{B \times B : B \in \mathcal{B}^M\}$ , where  $\mathcal{B}^M = \{R_3^M[t] : t \in J\}$  is a partition of  $A^M$ ,
  - $E_n^M \subseteq (E'_n)^M$ ,

- the domain of  $F_1^M$  is  $A^M \times S_1^M$  and it maps  $B \times S_1^M$  onto  $B$  for  $B \in \mathcal{B}$ ; also if  $a \in A^M$  and  $c_1, c_2 \in S_1^M$  then  $F_1^M(F_1^M(a, c_1), c_2) = F_1^M(a, c_1 + c_2)$ ;  
(it play the role of the  $F_c$ -s in 1.2),
- if  $\bar{a} = \langle a_i : i < n \rangle \in {}^n(A^M)$  then  $\bar{a}/(E'_n)^M$  is equal to  $\{ \langle F_1^M(a_i, c) : c \in S_1 \rangle \}$ ,
- if  $a \in A^M$  then  $\{ \langle b, b \rangle : \langle b, b \rangle (E'_n)^M \langle a, a \rangle \}$  is equal to  $\{ \langle b, b \rangle \in A^M \times A^M : F_2(b) = F_2^M(a) \}$  and to  $\{ \langle b, b \rangle : b = F_1^M(a, c) \text{ for some } c \in S_1^M \}$ .

⊠<sub>3</sub> We define  $\leq_{\mathfrak{k}}$  as being a submodel, in particular  $M \leq_{\mathfrak{k}} N \Rightarrow \pi^N \upharpoonright M = \pi^M$ ,  $F^N \upharpoonright M = F^M$ , etc.

Easily

⊠<sub>4</sub>  $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$  is an AEC where  $\leq_{\mathfrak{k}}$  is the two place relation on  $K$  of being a sub-model.

⊠<sub>5</sub> We define the closure operation  $cl = cl_{\mathfrak{k}}$  with domain  $\{(A, M) : A \subseteq M \in K\}$  such that for every  $(A, M)$  from the domain of  $cl(A, M)$ , is the minimal set  $B$  such that:

- <sub>1</sub>  $B \subseteq M$ ,
- <sub>2</sub>  $A \subseteq B$ ,
- <sub>3</sub>  $B$  is closed under the functions  $\pi^M, F_1^M, +^M$ ,
- <sub>4</sub>  $d_i^M \in B$  for  $i < \sigma$ , hence  $S_1^M \subseteq B$ ,
- <sub>5</sub> If  $\pi(a) = b$  then  $a \in B$  iff  $b \in B$ .

⊠<sub>6</sub> Indeed  $cl$  witnesses  $\mathfrak{k}$  admit intersections.

[Why? Just check.]

⊠<sub>7</sub>  $\mathfrak{k}$  has disjoint amalgamation.

[Why? So assume  $M_0 \subseteq M_\ell$  are from  $K$  and without loss of generality  $M_1 \cap M_2 = M_0$  and we shall find a member  $M$  of  $K$  extending all of them. We define  $M$  as follows:

- <sub>1</sub>  $a \in M$  if  $a \in M_1$  or  $a \in M_2$ ,
- <sub>2</sub> if  $P$  is a predicate from  $\tau$  not equal to  $E'_n$  or  $E_n$  for some  $n \in [2, \omega)$  then  $P^M = P^{M_1} \cup P^{M_2}$ ,
- <sub>3</sub> if  $F$  is a function symbol from  $\mathfrak{k}$  then  $F^M = F^{M_1} \cup F^{M_2}$ ,
- <sub>4</sub> if  $n \in [2, \omega)$  the  $(E'_n)^M$  is the closure of  $(E'_n)^{M_1} \cup (E'_n)^{M_2}$  to an equivalence relation; similarly for  $E_n$ .

Obviously  $M_\ell \subseteq M$  for  $\ell = 0, 1, 2$  so we just have to check that  $M \in K$ . This is straightforward. So ⊠<sub>7</sub> holds indeed.]

Assume  $\kappa$  is as in clause (d) of 1.4, we use the  $M_{\ell, \alpha}$  ( $\ell = 1, 2, \alpha \leq \kappa$ ) as well as  $I_\alpha, \pi_\alpha$  constructed inside the proof of 1.2 (the main relevant properties are stated in 1.2). They are not in the right vocabulary and universe, so let  $M'_{\ell, \alpha}$  be the following  $\tau$ -model:

⊠<sub>8</sub> (a) elements: The universe of the model  $M'_{\ell, \alpha}$  is the disjoint union of the following sets (and we have  $S^{M'_{\ell, \alpha}} = S_0^{M'_{\ell, \alpha}} \cup S_1^{M'_{\ell, \alpha}} \cup S_2^{M'_{\ell, \alpha}}$ ):

- $S_0^{M'_{\ell,\alpha}} = \sigma$
  - $S_1^{M'_{\ell,\alpha}} = [\sigma]^{<\aleph_0}$  ;
  - $S_2^{M'_{\ell,\alpha}} = \sigma\sigma$ ,
  - $I^{M'_{\ell,\alpha}} = I_\alpha$ ,
  - $J^{M'_{\ell,\alpha}} = \{t_\ell^*\}$  where  $t_\ell^*$  is just a new element,
  - $A^{M'_{\ell,\alpha}} = \{\ell\} \times |M_{\ell,\alpha}| = \{\iota\} \times A_\alpha$ ,
- (we assume disjointness)
- (b) for  $\ell = 1, 2$  the mapping  $a \mapsto (\iota, a)$  for  $a \in M_{\ell,\alpha}$  is an isomorphism from  $(M_{\ell,\alpha} \upharpoonright A_\alpha^*) \upharpoonright \tau_\theta^\bullet$  onto  $(M'_{\ell,\alpha} \upharpoonright A_\iota^{M'_{\ell,\alpha}}) \upharpoonright \tau^\bullet$  so  $(E'_n)^{M_{\ell,\alpha}}, E_n^{M_{\ell,\alpha}}, R_1^{M_{\ell,\alpha}}$  are defined.
- (c)  $I^{M'_{\ell,\alpha}} = I_\alpha$ ,
- (d)  $\pi^{M'_{\ell,\alpha}}$  is defined by  $\pi((\ell, x)) = \pi(x)$
- (e) (i)  $R_1^{M'_{\ell,\alpha}}$  is the set of  $\langle (\ell, \alpha_k), t_\ell^* \rangle$  such that  $\langle \alpha_k \rangle \in R_1^{M_{\ell,\alpha}}$ ,
- (ii)  $R_3^{M'_{\ell,\alpha}} = \{(a, t_\ell^*) : a \in A^{M_{\ell,\alpha}}\}$ ,
- (f)  $d_i^{M'_{\ell,\alpha}} = i$  for  $i < \theta$ ,
- (g)  $R_2^{M'_{\ell,\alpha}} = \{((\ell, a), (\ell, b), e) : e \in \sigma\sigma, \text{ and } (a, b) \in R_e^{M_{\ell,\alpha}}\}$ ,
- (h)  $H^M$  is the function mapping  $d_i$  to  $\{d_i\}$ ,
- (i)  $F_1^{M'_{\ell,\alpha}}$  maps  $(a, c) \in A^{M'_{\ell,\alpha}} \times S_1^{M'_{\ell,\alpha}}$  to  $F_c^{M_{\ell,\alpha}}(a)$ .

Let  $M'_{0,\alpha} = M'_{\ell,\alpha} \upharpoonright (S^{M'_{\ell,\alpha}} \cup I^{M'_{\ell,\alpha}} J^{M'_{\ell,\alpha}})$  for  $\ell = 1, 2$  and  $\alpha \leq \kappa$  (we get the same result for  $\ell = 1, 2$ ).

Note easily

$$\boxtimes_9 M'_{0,\alpha} = M'_{1,\alpha} \cap M'_{2,\alpha}$$

$$\boxtimes_{10} \langle M'_{\ell,\alpha} : \alpha \leq \kappa \rangle \text{ is } \leq_{\mathfrak{t}}\text{-increasing and continuous for } \ell = 0, 1, 2,$$

[Why? Easy to check.]

$$\boxtimes_{11} \text{ortp}_{\mathfrak{t}}(t_1^*, M'_{0,\alpha}, M'_{1,\alpha}) = \text{ortp}_{\mathfrak{t}}(t_2^*, M'_{0,\alpha}, M'_{2,\alpha}) \text{ for } \alpha < \kappa.$$

Why? By the isomorphism  $g_\alpha$  from  $M_{1,\alpha}$  onto  $M_{2,\alpha}$  respecting  $\pi_\alpha$  in 1.1. That is we define  $h$  such that it is the isomorphism for  $M'_{1,\alpha}$  onto  $M'_{2,\alpha}$  over  $M'_{0,\alpha}$  mapping  $t_1^*$  to  $t_2^*$  and mapping  $(1, x) \in A^{M'_{1,\alpha}}$  to  $(2, g_\alpha(x))$ .

Now, check.

$$\boxtimes_{12} \text{ortp}_{\mathfrak{t}}(t_1^*, M'_{0,\kappa}, M'_{1,\kappa}) \neq \text{ortp}_{\mathfrak{t}}(t_2^*, M'_{0,\kappa}, M'_{2,\kappa}).$$

[Why? By the non-isomorphism in 1.1; extension will not help.]

□<sub>1.4</sub>

## 2. COMPACTNESS OF TYPES IN AEC

Baldwin [Bal09] asks “Can we in ZFC prove that some AEC has amalgamation and JEP but fails compactness of types?”. The background is that in [BS08] we construct one using diamonds.

To me, the question is to show that this class can be very large (in ZFC).

Here we omit amalgamation and accomplish both by direct translations of problems of existence of models for theories in  $\mathbb{L}_{\kappa^+, \kappa^+}$ , first in the propositional logic.

So whereas in [BS08] we have an original group  $G^M$ , here instead we have a set  $P^M$  of propositional “variables” and  $\Gamma^M$ , set of such sentences (and relations and functions explicating this; so really we use coding but are a little sloppy in stating this obvious translation).

In [BS08] we have  $I^M$ , set of indexes, 0 and  $H$ , set of Whitehead cases,  $H_t$  for  $t \in I^M$ , here we have  $I^M$ , each  $t \in I^N$  representing a theory  $P_t^M \subseteq P^M$  and in  $J^M$  we give each  $t \in I^M$  some models  $\mathcal{M}_s^M : P^M \rightarrow \{\text{true}, \text{false}\}$ . This is set up so that amalgamation holds.

*Notation 2.1.* In this section types are denoted by  $\mathbf{p}, \mathbf{q}$  because  $p, q$  are used for propositional variables.

**Definition 2.2.** 1) We say that an AEC  $\mathfrak{k}$  has  $(\leq \lambda, \kappa)$ -sequence-compactness (for types) when: if  $\langle M_i : i \leq \kappa \rangle$  is  $\leq_{\mathfrak{k}}$ -increasing continuous and  $i < \kappa \Rightarrow \|M_i\| \leq \lambda$  and  $\mathbf{p}_i \in \mathcal{S}^n(M_i)$  for  $i < \kappa$  satisfying  $i < j < \kappa \Rightarrow \mathbf{p}_i = \mathbf{p}_j \upharpoonright M_i$  then there is  $\mathbf{p}_\kappa \in \mathcal{S}^n(M_\kappa)$  such that  $i < \kappa \Rightarrow \mathbf{p}_\kappa \upharpoonright M_i = \mathbf{p}_i$ .

2) We define “ $(= \lambda, \kappa)$ -sequence-compactness” similarly. Let  $(\lambda, \theta)$ -sequence-compactness mean  $(\leq \lambda, \kappa)$ -compactness.

**Question 2.3.** Can we find an AEC  $\mathfrak{k}$  with amalgamation and JEP such that  $\{\kappa : \mathfrak{k}$  have  $(\lambda, \kappa)$ -compactness of types for every  $\lambda\}$  is complicated, say:

- (a) not an end segment of the class of cardinals but with “arbitrarily large” members
- (b) any  $\{\kappa : \kappa \text{ satisfies } \psi\}$ ,  $\psi \in \mathbb{L}_{\theta^+, \theta^+}$  (second order) when  $\text{LST}_{\mathfrak{k}} \leq \theta$ .

**Definition 2.4.** Let  $\theta \geq \aleph_0$ , we define  $\mathfrak{k} = \mathfrak{k}_\theta = \mathfrak{k}(\theta)$  as follows:

- (A) the vocabulary  $\tau_{\mathfrak{k}}$  consist of  $F_i (i \leq \theta)$ ,  $R_\ell (\ell = 1, 2)$ ,  $P, \Gamma, I, J, c_i (i < \theta)$ ,  $F_i (i \leq \theta)$ , (pedantically see later),
- (B) the universe of  $M \in K_{\mathfrak{k}}$  is the disjoint union of  $P^M, \Gamma^M, I^M, J^M$  so  $P, \Gamma, I, J$  are unary predicates,
- (C) (a)  $P^M$  a set of propositional variables (i.e. this is how we treat them)
- (b)  $c - i$  an individual constant such that  $c_i^M \in P^M$  are pairwise distinct for  $i < \theta$ ,  $\Gamma^M$  is a set of sentences of one of the forms  $\varphi = (p)$ ,  $\varphi = (r \equiv p \wedge q)$ ,  $\varphi = (q \equiv \neg p)$ ,  $\varphi = (q \equiv \bigwedge_{i < \kappa} p_i)$ , so  $p, q, r, p_i \in P^M$ ,
- (c) the function  $F_i^M : \Gamma^M \rightarrow P^M$  for  $i < \theta$  are such that for every  $i < \theta$  and  $\varphi \in \Gamma^M$  we have (below  $F_0(\varphi)$  tell us how  $\varphi$  is composed,  $F_{i+1}$  give from what):
  - ( $\alpha$ ) if  $\varphi = (p)$  and  $i < \theta$  then  $F_{1+i}(\varphi) = p$ ,  $F_0(\varphi) = c_0$
  - ( $\beta$ ) if  $\varphi = (r \equiv p \wedge q)$  then  $F_i(\varphi)$  is  $c_1$  if  $i = 0$ , is  $p$  if  $i = 1$ , is  $q$  if  $i = 2$ , and is  $r$  if  $i \geq 3$
  - ( $\gamma$ ) if  $\varphi = (q \equiv \neg p)$  then  $F_i(\varphi)$  is  $c_2$  if  $i = 0$ , is  $p$  if  $i = 1$ , and is  $q$  if  $i \geq 2$
  - ( $\delta$ ) if  $\varphi = (q \equiv \bigwedge_{j < \theta} p_j)$  then  $F_i(\varphi)$  is  $c_3$  if  $i = 0$ , is  $q$  if  $i = 1$  and is  $c_{2+j}$  if  $i = 2 + j$
- (d)  $I$  a set of theories, i.e.  $R_1^M \subseteq \Gamma \times I$  and for  $t \in I$  let  $\Gamma_t^M = \{\psi \in \Gamma^M : \psi R_1^M t\} \subseteq \Gamma^M$

- (e)  $J$  is a set of models, i.e.  $R_2^M \subseteq (\Gamma \cup P) \times J$  and for  $s \in J$  we have  $\mathcal{M}_s^M$  is the model, i.e. function giving truth values to (some)  $p \in P^M$ , i.e.
- (α)  $\mathcal{M}_s^M(p)$  is true if  $pR_2^M s$ ; is false if  $\neg(pR_2^M s)$
  - (β)  $(\varphi, s) \in R_2^M$  iff computing the truth value of  $\varphi$  in  $\mathcal{M}_s^M$  we get truth
  - (f)  $F_\theta^M : J^M \rightarrow I^M$  such that  $s \in J^M \Rightarrow \Gamma_{F_\theta^M(s)}^M = \{\varphi \in \Gamma^M : (\varphi, s) \in R_2^M\}$ , i.e. the set of sentences from  $\Gamma^M$  which  $\mathcal{M}_s^M$  satisfies,
  - (g)  $(\forall t \in I^M)(\exists s \in J^M)(F_\theta^M(s) = t)$
- (D)  $M \leq_{\mathfrak{k}} N$  iff  $M \subseteq N$  are  $\tau_{\mathfrak{k}}$ -models from  $K_{\mathfrak{k}}$ .

**Claim 2.5.**  $\mathfrak{k}$  is an AEC and  $LST(\mathfrak{k}) = \kappa$ .

*Proof.* Obvious. □<sub>2.5</sub>

**Claim 2.6.** 1)  $\mathfrak{k}$  has the JEP.

2)  $\mathfrak{k}$  has the amalgamation property.

*Proof.* 1) Just like disjoint unions (also of the relations and functions) except for the individual constants  $c_i$  (for  $i < \theta$ ), or see the proof of part (2).

2) So assume that  $M_0 \subseteq M_\ell$  for  $\ell = 0, 1$  are in  $K_{\mathfrak{k}}$ , without loss of generality  $M_1 \cap M_2 = M_0$  and we shall find  $M \in K_{\mathfrak{k}}$  extending all of them.

- (a) the universe of  $M$  is  $|M_1| \cup |M_2|$ .
- (b) Similarly for the predicates.
- (c) As the functions are all unary, the situation is similar:
  - for  $i < \theta$  and  $\ell \leq 2$ ,  $F_i^{M_\ell} : \Gamma^{M_\ell} \rightarrow \Gamma^{M_\ell}$  and  $F_i^{M_\ell} \upharpoonright \Gamma^{M_0} = F_i^{M_0}$ .
 Clearly  $F_i^M = F_i^{M_1} \cup F_i^{M_2}$  and we have:
  - ⊕  $F_i^M$  is a well-defined function from  $\Gamma^M$  into  $P^M$  extending  $F_i^{M_\ell}$  for  $i < \theta$
- (b) Similarly for  $i = \theta$ .

It is easy to check then that  $\ell = 2 \Rightarrow M_\ell \subseteq M$  and that all clauses of the definition of “ $M \in K_{\mathfrak{k}}$ ” hold. □<sub>2.6</sub>

**Claim 2.7.** Assume  $M_0 \leq_{\mathfrak{k}} M_\ell$  for  $\ell = 0, 1$  and  $|M_0| = P^{M_0} \cup \Gamma^{M_0} = P^{M_\ell} \cup \Gamma^{M_\ell}$  for  $\ell = 1, 2$  and  $a_\ell \in I^{M_\ell}$  for  $\ell = 1, 2$ . Then  $\text{ortp}_{\mathfrak{k}}(a_1, M_0, M_1) = \text{ortp}_{\mathfrak{k}}(a_2, M_0, M_2)$  iff  $\Gamma_{a_1}^{M_1} = \Gamma_{a_2}^{M_2}$ .

*Proof.* The if direction,  $\Leftarrow$ :

Let  $h$  be a one to one mapping with domain  $M_1$  such that  $h \upharpoonright M_0 =$  the identity,  $h(a_1) = a_2$  and  $h(M_1) \cap M_2 = M_0 \cup \{a_2\}$ . Renaming without loss of generality  $h$  is the identity. Now define  $M_3$  as  $M_1 \cup M_2$ , as in 2.6, now  $a_1 = a_2$  does not cause trouble because  $P^{M_0} = P^{M_\ell}$ ,  $\Gamma^{M_0} = \Gamma^{M_\ell}$  for  $\ell = 1, 2$ .

The “only if” direction,  $\Rightarrow$ :

Obvious. □<sub>2.7</sub>

**Claim 2.8.** Assume  $\lambda, \theta, \kappa$  are such that:

- (a)  $\theta \leq \kappa$  are regular  $\leq \lambda$
- (b)  $(\Gamma_i : i \leq \kappa)$  is  $\subseteq$ -increasing continuous sequence of sets of propositional sentences in  $\mathbb{L}_{\theta^+, \aleph_0}$  such that  $[\Gamma_i \text{ has a model} \Leftrightarrow i < \kappa]$

(c)  $|\Gamma_\kappa| \leq \lambda$ .

*Then  $\mathfrak{k}$  fail  $(\lambda, \theta)$ -sequence-compactness (for types).*

*Remark 2.9.* We may wonder but: for  $\theta = \aleph_0$ , compactness holds? Yes.

*Proof.* Without loss of generality  $|\Gamma_\kappa| = \lambda$ . Without loss of generality  $\langle p_\varepsilon^* : \varepsilon < \lambda \rangle$  are pairwise distinct propositions variables appearing in  $\Gamma_\kappa$  (but not necessarily  $\in \Gamma_0$ ) and each  $\psi \in \Gamma_i$  is of the form  $(p)$  or  $r \equiv p \wedge q$  or  $r \equiv \neg p$  or  $r \equiv \bigwedge_{i < \theta} p_i$ .

Let  $P_i$  be the set of propositional variables appearing in  $\Gamma_i$ ; without loss of generality  $|P_i| = \lambda$ .

We choose a  $\tau(\mathfrak{k}_\theta)$ -model  $M_i$  for  $i \leq \kappa$  such that:

- ⊞<sub>1</sub> (a)  $|M_i| = P_i \cup \Gamma_i$ , and  $\tau(M_i) = \tau_{\mathfrak{k}}$ ,
- (b)  $P^{M_i} = P_i$  and  $\Gamma^{M_i} = \Gamma_i$ ,
- (c)  $F_\varepsilon^{M_i}$  (for  $\varepsilon < \kappa$ ) are defined naturally),
- (d)  $I^{M_i} = \emptyset = J^{M_i}$ , hence  $R_1^M = R_2^M = \emptyset = F_\theta^M$ .

Clearly

- ⊞<sub>2</sub> (a)  $M_i \in K_{\mathfrak{k}}$ ,
- (b)  $\langle M_i : i \leq \kappa \rangle$  is  $\leq_{\mathfrak{k}}$ -increasing and continuous.

Let  $\mathcal{M}_i : P_i \rightarrow \{\text{true}, \text{false}\}$  be a model of  $\Gamma_i$ .

We define a model  $N_i \in K_{\mathfrak{k}}$  for  $i < \kappa$  (but not for  $i = \kappa$ !)

- ⊠ (a)  $M_i \leq_{\mathfrak{k}} N_i$ ,
- (b)  $P^{N_i} = P^{M_i}$ ,
- (c)  $\Gamma^{N_i} = \Gamma^{M_i}$ ,
- (d)  $I^M = \{t_j : j < i\}$ ,
- (e)  $J^M = \{s_j : j < i\}$ ,
- (f)  $F_\kappa^{N_i}(s_j) = t_j$ ,
- (g)  $R_1^{N_i} = \bigcup \{\Gamma_j \times \{t_j\} : j < i\}$ ,
- (h)  $R_2^{N_i}$  is chosen such that  $\mathcal{M}_{s_j}^{N_j}$  is  $\mathcal{M}_j$ .

Now

$$(*)_1 \mathbf{p}_i = \text{ortp}_{\mathfrak{k}}(t_i, M_i, N_i) \in \mathcal{S}^1(M_i).$$

[Why? Trivial.]

$$(*)_2 i < j < \kappa \rightarrow \mathbf{p}_i = \mathbf{p}_j \upharpoonright M_j.$$

[Why? Let  $N_{i,j} = N_j \upharpoonright (M_j \cup \{s_j, t_j\})$ .

Easily  $\text{ortp}(t_j, M_i, N_{i,j}) \leq \mathbf{p}_j$  and  $\text{ortp}(t_j, M_i, N_{i,j}) = \mathbf{p}_j$  by the claim 2.7 above.]

$$(*)_3 \text{ there is no } \mathbf{p} \in \mathcal{S}^1(M_\theta) \text{ such that } i < \kappa \Rightarrow \mathbf{p}_j \upharpoonright M_i = \mathbf{p}_i.$$

Why? We prove more:

$$(*)_4 \text{ there is no } (N, t) \text{ such that}$$

- (a)  $M_\kappa \leq_{\mathfrak{k}} N$ ,

- (b)  $t \in I^N$ ,
- (c)  $(\forall \varphi \in \Gamma^{M_\kappa})[\varphi R_1^N t]$ .

[Why? As then  $\Gamma_\kappa = \Gamma^M$  has a model contradiction to an assumption.] □<sub>2.8</sub>

So e.g.

*Conclusion 2.10.* If  $\kappa > \theta$  are regular with no  $\theta^+$ -complete uniform ultrafilter on  $\kappa$  and  $\lambda = 2^\kappa$ , then  $\mathfrak{k} = \mathfrak{k}_\theta$  is not  $(\lambda, \kappa)$ -sequence-compact.

*Remark 2.11.* Recall if  $D$  is an ultrafilter on  $\theta$  then  $\min\{\sigma' : D \text{ is not } \sigma'\text{-complete}\}$  is  $\aleph_0$  or a measurable cardinal.

*Proof.* Not novel but we elaborate.

(\*)<sub>1</sub> Let  $M$  be the model with the following characteristics:

- <sub>1</sub> its universe is  $\mathcal{H}_{\leq \kappa}(\lambda)$ , where  $\mathcal{H}_{\leq \theta}(\lambda)$  considering ordinals as atoms, and for any  $\mu$

$\mathcal{H}_{\leq \theta}(\mu) = \{x : \text{trcl}(x) \text{ and has cardinality } \leq \theta \text{ and every ordinal from it is } < \mu\}$ ,

where  $\text{trcl}(x)$  is the transitive closure of  $x$ ,

- <sub>2</sub>  $P_0^M = \theta$ ,  $c_i^M = i$  for  $i < \theta$  and  $c_\theta = \kappa$ ,
- <sub>3</sub>  $R^M \in \mathcal{H}_{\leq \theta}(\lambda)$ ,
- <sub>4</sub>  $<_*^M$  is a well-ordering of  $\mathcal{H}_{\leq \theta}(\lambda)$ ,
- <sub>5</sub> the vocabulary of  $M$  has cardinality  $\kappa$  and has elimination of quantifiers and Skolem functions.

(\*)<sub>2</sub> Let  $M_\bullet$  be an expansion of  $M$  such that for some  $\chi > 2^\lambda$ , and  $N \prec (\mathcal{H}(\chi), \in)$  of cardinality  $\theta$ ,  $|N| = \theta$ ,  $\theta + 1 \subseteq N$  and  $M \in N$  and  $M_*$  is gotten from  $M$  by:

for every  $\varphi(\bar{c}) \in \mathbb{L}_{\theta^+, \aleph_0}(\tau_M) \cap N$  so  $\text{lg}(\bar{x})$  finite we add  $P_\varphi^M = \{\bar{a} \in \text{lg}(\bar{x})M : M \models \varphi[\bar{a}]\}$  so  $P^M$  in a new  $\text{lg}(\bar{x})$ -place predicate.

(\*)<sub>3</sub> (a) Let  $\tau_* = \tau(M_\bullet) \cup \{c_b : b \in M\} \cup \{c\}$ ,

(b) Let  $M^+$  be the expansion of  $M_\bullet$  to a  $\tau_*$ -model with  $c_b^{M^+} = b$  for  $b \in M$ .

(\*)<sub>4</sub> For  $i \leq \kappa$ , let  $\Gamma_i$  be the union of the following sets:

(a)  $\Gamma_i^1$  is the set of quantifiers free formulas which  $M^+$  satisfies will we consider only those generated from the atomic by  $\neg\varphi$ ,  $\varphi \wedge \psi$ , and

$\varphi \equiv \bigwedge_{i < \theta} \varphi_i$ ,

(b)  $\Gamma_i^2 = \{P_0(c) \wedge b_j R c : j < i\}$ ,

(c)  $\Gamma_i^3 = \{\sigma(\bar{d}) = c_\varepsilon \equiv \neg p_{\sigma(\bar{d})=c_\varepsilon} : \bar{d} \in {}^\omega(|M| \cup \{c\}), \sigma(-, -) \text{ a term for } \tau(M)\}$

(d)  $\Gamma_i^4 = \{q_{\sigma(\bar{d})\varepsilon} P_1(\sigma(\bar{d})) \wedge p_{\sigma(\bar{d}), \varepsilon} : \sigma(\bar{d}), \varepsilon \text{ as above}\}$ ,

(e)  $\Gamma_i^5 = \{q_{\text{sigma}(d)} \equiv \bigwedge_{\varepsilon < \theta} v p_{\sigma(\bar{d}), \varepsilon} : \wedge p_{\sigma(\bar{d}), \varepsilon} : \sigma(\bar{d}) \text{ as above}\}$ ,

(\*)<sub>5</sub> If  $i < \kappa$  then  $\Gamma_i$  has a model.

[Why? Expand  $M^+$  by interpreting  $c$  as  $c_i$ .]

(\*)<sub>6</sub>  $\Gamma_\kappa$  has no model.

[Why? If  $\Gamma_\kappa$  has a model  $N$ , then without loss of generality,  $N \upharpoonright \tau(M^+)$  extend  $M^+$ , and let  $N'$  be the restriction of  $N$  to the closure of  $\{c_b : b \in M^+\} \cup \{c^{N'}\}$  under the Skolem functions. Easily  $N \prec N'$  and  $D = \{b \subseteq \kappa : N' \models "c \in c_b''\}$ . Now check.]

(\*)<sub>7</sub> Now 2.8, applied to  $\langle \Gamma_i : i \leq \kappa \rangle$  gives the desired result.

□<sub>2.10</sub>

*Conclusion 2.12.* In Claim 2.8 if  $\lambda = \lambda^\kappa$  then we can allow  $\langle \Gamma_i : i \leq \kappa \rangle$  to be a sequence of theories in  $\mathbb{L}_{\theta^+, \theta^+}(\tau)$ ,  $\tau$  any vocabulary of cardinality  $\leq \lambda$ .

*Proof.* Without loss of generality, we can add Skolem functions (each with  $\leq \kappa$  places) in particular. So  $\Gamma_i$  becomes universal, and adding propositional variables for each quantifier-free sentence and writing down the obvious sentences, we get a set of propositional sentences, and we get  $\Gamma_i$  as there.  $\square_{2.12}$

Note that:

*Conclusion 2.13.* 1) Let  $\mathbf{C}_\theta = \{\kappa : \kappa = \text{cf}(\kappa) \text{ and for every } \lambda \text{ and AEC } \mathfrak{k} \text{ with } \text{LST}(\mathfrak{k}) \leq \theta, |\tau_{\mathfrak{k}}| = \theta \text{ have } (\lambda, \kappa)\text{-sequence-compactness of types}\}$  is the class  $\{\kappa : \kappa = \text{cf}(\kappa) > \theta \text{ and there is a uniform } \theta^+\text{-complete ultrafilter on } \kappa\}$ .

2) In  $\mathbf{C}_\theta$  we can replace “every  $\lambda$ ” by  $\lambda = 2^\theta + \kappa$ .

*Proof.* Put together 2.10.  $\square_{2.13}$

Of course, a complementary result (showing the main claim is best possible) is:

**Claim 2.14.** *If  $\mathfrak{k}'$  is an AEC,  $\text{LST}(\mathfrak{k}') \leq \theta$  and on  $\kappa$  there is a uniform  $\theta^+$ -complete ultrafilter on  $\kappa$  and  $\theta$  is regular and  $\lambda$  any cardinality then  $\mathfrak{k}'$  has  $(\lambda, \kappa)$ -compactness of types.*

*Proof.* We sha;; write down a set of sentences  $\Gamma_\zeta$  from  $\mathbb{L}_{\theta^+, \theta^+}(\tau_{\mathfrak{k}'})$  for  $\zeta \leq \theta$  expressing the demands.

Let  $\langle M_i : i \leq \kappa \rangle$  be  $\leq_{\mathfrak{k}'}$ -increasing continuous,  $\|M_i\| \leq \lambda$ ,  $\mathbf{p}_i = \text{ortp}_{\mathfrak{k}'}(a_i, M_i, N_i)$  so  $M_i \leq_{\mathfrak{k}'} N_i$  such that  $i < j < \kappa \Rightarrow \mathbf{p}_i = \mathbf{p}_j \upharpoonright M_i$ . Without loss of generality  $\|N_i\| \leq \lambda$ .

Let  $\langle N_{i,j,\ell} : \ell \leq n_{i,j} \rangle, \pi_{i,1}$  witness  $\mathbf{p}_i = \mathbf{p}_j \upharpoonright M_i$  for  $i < j < \kappa$  (i.e.  $M_i \leq_{\mathfrak{k}'} N_{i,j,\ell}$  (without loss of generality  $\|N_{i,j,\ell}\| \leq \lambda$ ),  $N_{i,j,0} = N_i, N_{i,j,n_{i,j}} = N_j, a_i \in N_{i,j,\ell}$ ,  $\bigwedge_{\ell < n_{i,j,\ell}} (N_{i,j,\ell} \leq_{\mathfrak{k}'} N_{i,j,\ell+1} \vee N_{i,j,\ell+1} \leq_{\mathfrak{k}'} N_{i,j,\ell})$  and  $\pi_{i,j}$  be an isomorphism

from  $N_j$  onto  $N_{i,j,n_{i,j}}$  over  $M_i$  mapping  $a_j$  to  $a_i$ ).

Let  $\tau^+ = \tau \cup \{F_{\varepsilon,n} : \varepsilon < \theta, n < \omega\}$  be disjoint union,  $\text{arity}(F_{\varepsilon,n}) = n$ . Let  $\langle M_i^+ : i \leq \kappa \rangle$  be  $\subseteq$ -increasing,  $M_i^+$  a  $\tau^+$ -expansion of  $M_i$  such that  $u \subseteq M_i^+ \Rightarrow M_i \upharpoonright \text{cl}_{M_i^+}(u) \leq_{\mathfrak{k}'} M_i$ . Similarly for  $i < k < \kappa$ , we have  $\langle N_{i,j,\ell}^{+,\varepsilon} : \ell \leq n_{i,j,\ell} \rangle; \varepsilon \in \{1, \ell\}$  such that  $N_{i,j,\ell}^{+,\varepsilon}$  is a  $\tau^+$ -expansion of  $N_{i,j,\ell}$  as above such that  $(\forall \ell < n_{i,j,\ell})(\exists \varepsilon \in \{1, 2\})(N_{i,j,\ell}^{+,\varepsilon} \subset N_{i,j,\ell+1}^{+,\varepsilon} \vee N_{i,j,\ell+1}^{+,\varepsilon} \subset N_{i,j,\ell}^{+,\varepsilon})$ .

Now write down a translation of the question, “is there  $\mathbf{p}$  such that...”

Assume that  $D$  is a uniform  $\theta^+$ -complete ultrafilter on  $\kappa$ .

For each  $i < \kappa$  let  $\mathcal{U}_i \in D$  be such that  $i < j \in \mathcal{U}_i \Rightarrow n_{i,j} = n_i^*$ . Let moreover  $n_*$  and  $\mathcal{W} \in D$  be such that for every  $i \in \mathcal{W}$  we have  $n_i = n_*$ . Let  $N_{i,\kappa,\ell} = \prod_{j \in \mathcal{U}_i} N_{i,j,\ell}/D$ . So  $\langle N_{i,\kappa,\ell} : \ell \leq n_\ell^* \rangle$  are as above. Let  $M = \prod_{i < \kappa} M_i/D, \pi_{i,\theta} = \prod_{j \in \mathcal{U}_i} \pi_{i,j}/D$ , etc.  $\square_{2.14}$

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