

CONSISTENCY OF SQUARE BRACKET PARTITION RELATION

SAHARON SHELAH

ABSTRACT. Characteristic earlier results were of the form $\text{CON}(2^{\aleph_0} \rightarrow [\lambda]_{n,2}^2)$, with 2^{\aleph_0} an ex-large cardinal, in the best case the first weakly Mahlo cardinal.

Characteristic new results are $\text{CON}((2^{\aleph_0} = \aleph_m) + \aleph_\ell \rightarrow [\aleph_k]_{n,2}^2)$, for suitable $k < \ell < m$. So we improve in three respects: the continuum may be small (e.g. not a weakly Mahlo), we use no large cardinal, and the cardinals λ involved are $< 2^{\aleph_0}$ after the forcing.

§ 0. INTRODUCTION

In their seminal list of problems [EH71], Erdős and Hajnal posed the question (15(a)): does $2^{\aleph_0} \not\rightarrow [\aleph_1]_3^2$? Recently, Komjáth [Kom25] provided a comprehensive update on this topic.

We continue here works which start with the problem above: [She88, §2], [She92], [She89], [She95] [She96], [She00] and the work with Rabus [RS00], but we try to be self-contained.

The simplest case of our result is (recall 0.3 below):

Theorem 0.1. *Assume GCH for transparency. Then for some ccc forcing notion of cardinality \aleph_6 in the universe $\mathbf{V}^{\mathbb{P}}$, we have $2^{\aleph_0} = \aleph_6$ and for any $n \geq 3$, $\aleph_5 \rightarrow [\aleph_2]_{n,2}^2$.*

Proof. Choose $(\mu, \theta, \partial, \lambda)$ as $(\aleph_6, \aleph_5, \aleph_2, \aleph_0)$ and apply Theorem 0.2 and Fact 1.12 with $\partial_0 = \aleph_1$. □_{0.1}

For Hypothesis 1.1, the main case is:

Theorem 0.2. *Assume $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$, $\partial = \partial^\lambda$ and $2^{\partial^\ell} = \partial^{\ell+1}$ for $\ell = 0, 1, 2$ and $\partial^{+4} \leq \theta$. Then for some λ^+ -cc, $(< \lambda)$ -complete forcing notion \mathbb{P} of cardinality μ (so the forcing does not collapse any cardinal and preserves cardinal arithmetic outside $[\lambda, \mu)$), in the universe $\mathbf{V}^{\mathbb{P}}$ we have, $2^\lambda = \mu$ and for every $\sigma < \lambda$, $\theta \rightarrow [\partial]_{\sigma,2}^2$.*

Proof. All this paper is dedicated to proving this theorem. Pedantically, choose $\partial = \kappa^+$, notice that Hypothesis 1.1 holds (by Fact 1.12) so we can apply Conclusion 1.11. □_{1.11}

We may weaken $\mu = \mu^\theta$ to $\mu = \mu^\partial$ and replace $\partial = \kappa$ by ∂ being a suitable limit cardinal.

Recall,

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Definition 0.3. For possibly finite cardinals θ, ∂, σ and κ , let $\theta \rightarrow [\partial]_{\sigma, \kappa}^2$ mean:

- if \mathbf{c} is a function from $[\theta]^2 := \{u \subseteq \theta : |u| = 2\}$ into σ , then there exists some subset \mathcal{U} of θ of cardinality ∂ such that $\{\mathbf{c}(u) : u \in [\mathcal{U}]^2\}$ has at most κ -many members.

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§ 0(A). **Preliminaries.**

Notation 0.4.

- 1) $\text{cof}(\delta)$ is the class of ordinals of cofinality $\text{cf}(\delta)$.
- 2) For a set x , let $\text{trcl}(x)$ be the transitive closure of x , that is, the minimal set Y such that $x \in Y$ and $(\forall y)(y \in Y \Rightarrow y \subseteq Y)$.
- 3) Let $\mathcal{H}(\lambda) := \{x : |\text{trcl}(x)| < \lambda\}$.
- 4) Let $\text{trcl}_{\text{Ord}}(x)$ be defined similarly to $\text{trcl}(x)$ considering ordinals as atoms (= elements), equivalently, the minimal set Y such that $x \in Y$ and $(\forall y)[y \in Y \wedge (\text{if } y \text{ is not an ordinal, then } y \subseteq Y)]$.
- 5) Let $\mathcal{H}_{<\kappa}(x) = \{x : \text{trcl}_{\text{Ord}}(x) \subseteq \mathcal{H}(x) \text{ but has cardinality } < \kappa\}$.

Notation 0.5.

- (1) \mathbb{P}, \mathbb{Q} and \mathbb{R} are forcing notions.
- (2) p, q, r called *conditions* are members of a forcing notion.
- (3) \mathbf{q} is as in Definition 1.3, some kind of $(< \lambda)$ -support iterated forcing with extra information.

Notation 0.6. We may write e.g. $N[\mathbf{q}, \beta, u]$ instead $N_{\mathbf{q}, \beta, u}$ to help with sub-scripts (or super-script).

Definition 0.7. Let θ, ∂, κ and λ be infinite cardinals. We say that $\theta \rightarrow_{\text{sq}} (\partial)_{\kappa}^{\lambda, 2}$ when $\theta > \partial \geq \kappa \geq \lambda$ and:

- ⊞ If (a) then (b), where:
 - (a) \mathcal{B} is an expansion of $(\mathcal{H}(\chi), \in, <_*)$, where $<_*$ is a well-ordering of $\mathcal{H}(\chi)$, $\chi > \theta$, and its vocabulary $\tau_{\mathcal{B}}$ has cardinality $\leq \lambda$.
 - (b) There is a tuple $\mathbf{s} = (\mathcal{U}, \bar{N}, \bar{\pi})$ solving $\mathbf{p} = (\mu, \theta, \partial, \kappa, \lambda, \mathcal{B})$, which means:
 - ⊞ $_{\mathbf{p}, \mathbf{s}}$ for $u, v \in [\mathcal{U}]^{\leq 2}$,
 - ₁ $\bar{N} = \langle N_u : u \in [\mathcal{U}]^{\leq 2} \rangle$,
 - ₂ $\mathcal{U} \subseteq \theta$ is such that $\text{otp}(\mathcal{U}) = \partial$,
 - ₃ $N_u \prec \mathcal{B}$, $[N_u]^{<\lambda} \subseteq N_u$,
 - ₄ $\varepsilon[\mathbf{s}] := \min(\mathcal{U})$,
 - ₅ $N_u \cap \mathcal{U} = u$,
 - ₆ $\|N_u\| = \kappa$ and $\kappa + 1 \subseteq N_u$,
 - ₇ $N_u \cap N_v \prec N_{u \cap v}$,
 - ₈ $\bar{\pi} = \langle \pi_{u,v} : u, v \in [\mathcal{U}]^{\leq 2} \text{ and } |u| = |v| \rangle$ such that if $|u| = |v|$, then $\pi_{u,v}$ is an isomorphism from N_v onto N_u mapping v onto u ,
 - ₉ if $u_1 \subseteq u_2$ and $v_1 \subseteq v_2$ all from $[\mathcal{U}]^{\leq 2}$ and $|u_2| = |v_2|$, $\pi''_{u_2, v_2}(v_1) = u_1$ then $\pi_{u_1, v_1}, \pi_{u_2, v_2}$ are compatible functions¹,
 - ₁₀ for $\ell = 1, 2$, the sets $N_u \cap \partial$ for $u \in [\mathcal{U}]^{\ell}$ are pairwise equal² and included in N_{\emptyset} .

¹So e.g. it follows that: if $\zeta_1, \zeta_2 \in \mathcal{U}$ then $\pi_{\{\zeta_1\}, \{\zeta_2\}} \upharpoonright (N_{\emptyset} \cap N_{\{\zeta_2\}})$ is the identity map.

²Note that ∂ has two distinct roles: the size of \mathcal{U} and the restriction on $N_u \cap \partial$. We may separate.

Observation 0.8. *If $\bar{N} = \langle N_u : u \in [\mathcal{U}]^{\leq 2} \rangle$ satisfies 0.7(b) $\bullet_1 + \bullet_7$, then:*

- (*) *For every $x \in \cup\{N_u : u \in [\mathcal{U}]^{\leq 2}\}$ the set $\{u \in [\mathcal{U}]^{\leq 2} : x \in N_u\}$ has one of the following forms:*
- (a) $\{u\}$ for some $u \in [\mathcal{U}]^2$,
 - (b) $\{\zeta\}$ for some $\zeta \in \mathcal{U}$,
 - (c) $\{\{\zeta\}\} \cup \{\{\varepsilon, \zeta\} : \varepsilon \in \mathcal{U} \cap \zeta\}$ for some $\zeta \in \mathcal{U}$,
 - (d) $\{\{\zeta\}\} \cup \{\{\zeta, \xi\} : \xi \in \mathcal{U} \setminus (\zeta + 1)\}$ for some $\zeta \in \mathcal{U}$,
 - (e) $\{\emptyset\}$,
 - (f) $\{\emptyset\} \cup \{\{\zeta\} : \zeta \in \mathcal{U}\}$,
 - (g) $\{\emptyset\} \cup \{\{\zeta\} : \zeta \in \mathcal{U}\} \cup \{\{\varepsilon, \zeta\} : \varepsilon < \zeta \text{ are from } \mathcal{U}\}$.

§ 1. THE FORCING

Our aim here is to prove the consistency of the following configuration:

$$2 < \sigma < \lambda = \lambda^{<\lambda} < \partial = \partial^{<\lambda} < \theta < \mu = \mu^\theta = 2^\lambda,$$

and having $\theta \rightarrow [\partial]_{\sigma, 2}^2$.

A continuation is in preparation [S⁺], aiming to further develop the directions explored here, particularly for the case of superscript $\mathbf{n} > 2$, as dealt within [She92]. We also show there that we can weaken the requirements on the cardinals and have more pairs.

Hypothesis 1.1. The parameter $\mathbf{p} = (\mu, \theta, \partial, \lambda, \lambda, \mathcal{B})$ consists of the following:

- (a) $\lambda = \lambda^{<\lambda} < \partial < \theta < \mu = \mu^\theta$,
- (b) $\theta \rightarrow_{\text{sq}} (\partial)_{\lambda}^{\lambda, 2}$ (see Definition 0.7, a variant of [She89, 2.1]); in our case using λ twice in intentional.
- (c) σ will vary on the cardinal numbers from $(2, \lambda)$ and the “nice” μ -s are such that $\gamma < \mu \Rightarrow |\gamma|^\theta < \mu$.
- (d)
 - χ is e.g. $\beth_2(\mu)^+$,
 - let \mathcal{B} be an expansion of $(\mathcal{H}(\chi), \in, <_\chi^*)$ with vocabulary of cardinality λ such that for any finite set $u \subseteq \mathcal{H}(\chi)$, the Skolem hull of u $N_u := \text{Sk}(u, \mathfrak{C}_*)$ is of cardinality λ and $|N_u|^{<\lambda} \subseteq N$.

We intend to use $(<\lambda)$ -support iterated forcing of quite a special kind but first, we define the iterand.

Definition 1.2.

(1) Let \mathbf{A} be the set of objects \mathbf{a} consisting of:

- (a)
 - $\gamma < \mu$ and $\sigma \in (2, \lambda)$,
 - \mathbb{P} is a forcing notion such that:
 - $p \in \mathbb{P} \Rightarrow \text{dom}(p) \in [\gamma]^{<\lambda} \wedge (\forall \alpha \in \text{dom}(p))(p(\alpha) \in [\lambda \cup \gamma]^{<\lambda})$,
 - \mathbb{P} is λ^+ -cc and $(<\lambda)$ -complete,
 - the order $\leq_{\mathbb{P}}$ is: $p \leq_{\mathbb{P}} q$ iff:

$$\text{dom}(p) \subseteq \text{dom}(q) \wedge (\forall \alpha \in \text{dom}(p))[p(\alpha) \subseteq q(\alpha)],$$

- (b)
 - \mathfrak{c} is a \mathbb{P} -name of a function from $[\theta]^2$ to σ , (we may write $\mathfrak{c}(\alpha, \beta)$ instead $\mathfrak{c}(\{\alpha, \beta\})$ for $\alpha \neq \beta < \theta$).
- (c) We have $(\mathcal{U}, \bar{N}, \bar{\pi})$ solving $\mathbf{p} = (\mu, \theta, \partial, \lambda, \lambda, \mathcal{B})$, (with \mathcal{B} as in Definition 0.7(b) and 1.1) such that $\mathbb{P}, \mathfrak{c} \in N_u$ for every $u \in [\mathcal{U}]^{\leq 2}$.

(1A) In the context of Definition 1.2(1), $\mathbf{a} = (\gamma, \mathbb{P}, \mathfrak{c}, \mathcal{U}, \bar{N}, \bar{\pi}) = (\gamma_{\mathbf{a}}, \dots)$, so e.g. $N_{\mathbf{a}, u} = N_u$.

(2) We say that the pair (p, \bar{v}) is a *solution* of $\mathbf{a} \in \mathbf{A}$, and write $(\mathbf{a}, p, \bar{v}) \in \mathbf{A}^+$, when:

- (a) $\bar{v} = (\iota_1, \iota_2) \in \sigma \times \sigma$,

- (b) $p \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$, recalling $\varepsilon(\mathbf{a}) = \min(\mathcal{U})$,
- (c) if $p \leq q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ and $\zeta_1 < \zeta_2$ are from \mathcal{U} then there are q_1, q_2, r_1, r_2 such that for $\ell = 1, 2$, we have:
 - ₀ $q \leq_{\mathbb{P}_{\mathbf{a}}} q_\ell$,
 - ₁ $q_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\varepsilon[\mathbf{a}]\}}$ and $q_1 \upharpoonright (N_{\mathbf{a},\emptyset} \cap \gamma_{\mathbf{a}}) = q_2 \upharpoonright (N_{\mathbf{a},\emptyset} \cap \gamma_{\mathbf{a}})$,
 - ₂ $r_\ell \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a},\{\zeta_1, \zeta_2\}}$,
 - ₃ $r_\ell \Vdash \text{“}\mathfrak{c}(\zeta_1, \zeta_2) = \iota_{\mathbf{a},\ell}\text{”}$,
 - ₄ $r_\ell \upharpoonright N_{\mathbf{a},\{\zeta_1\}}$ is $\leq_{\mathbb{P}_{\mathbf{a}}}$ -below $\pi_{\{\zeta_1\},\{\varepsilon[\mathbf{a}]\}}^{\mathbf{a}}(q_\ell)$,
 - ₅ $r_\ell \upharpoonright N_{\mathbf{a},\{\zeta_2\}}$ is $\leq_{\mathbb{P}_{\mathbf{a}}}$ -below $\pi_{\{\zeta_2\},\{\varepsilon[\mathbf{a}]\}}^{\mathbf{a}}(q_{3-\ell})$.

(3) If $\mathbf{b} = (\mathbf{a}, p, \bar{t}) \in \mathbf{A}^+$ then let $\mathbb{Q}_{\mathbf{b}}$ be the \mathbb{P} -name of the following forcing notion:

- (*) For $\mathbf{G} \subseteq \mathbb{P}$ generic over \mathbf{V} ,
- (a) the set of elements of $\mathbb{Q}_{\mathbf{b}} = \mathbb{Q}_{\mathbf{b}}[\mathbf{G}]$ is:

$\left\{ u \in [\mathcal{U}]^{<\lambda} : \text{if } \zeta_1 < \zeta_2 \text{ in } \mathcal{U}, \text{ then } \mathfrak{c}\{\zeta_1, \zeta_2\}[\mathbf{G}] \in \{\iota_1, \iota_2\}, \text{ moreover} \right.$

for some q_1, q_2, r_1, r_2 as in Definition 1.2(1)(c)(•₁)-(•₅), we have $r_1 \in \mathbf{G}$ or $r_2 \in \mathbf{G}$ $\left. \right\}$,

- (b) the order of $\mathbb{Q}_{\mathbf{b}}[\mathbf{G}]$ is inclusion,
- (c) the generic is $\mathcal{Y}_{\mathbf{b}} = \bigcup \mathbf{G}_{\mathbb{Q}_{\mathbf{b}}}$.

Definition 1.3.

(1) Let $\mathbf{Q} := \mathbf{Q}_{\mathbf{p}}$ be the class of \mathbf{q} which consist of (below, $\alpha \leq \text{lg}(\mathbf{q})$ and $\beta < \text{lg}(\mathbf{q})$ and e.g. $\mathbb{P}_{\alpha} = \mathbb{P}_{\mathbf{q},\alpha}$):

- (a) $\text{lg}(\mathbf{q})$ is an ordinal $\leq \mu$,
- (b) $\langle \mathbb{P}_{\alpha}, \mathbb{Q}_{\beta} : \alpha \leq \text{lg}(\mathbf{q}), \beta < \text{lg}(\mathbf{q}) \rangle$ is a $(< \lambda)$ -support iteration,
- (c) \mathbb{P}_{β} satisfies the λ^+ -cc,
- (d) \mathbb{Q}_{β} is $\mathbb{Q}_{\mathbf{b}_{\beta}}$, where:
 - ₁ $\mathbf{b}_{\beta} := (\mathbf{a}_{\beta}, p_{\beta}^*, \bar{t}_{\beta}^*) \in \mathbf{A}^+$,
 - ₂ $\mathbf{a}_{\beta} := (\gamma_{\beta}, \mathbb{P}_{\beta}^{\bullet}, \mathfrak{c}_{\beta}, \mathcal{U}_{\beta}, \bar{N}_{\beta}, \bar{\pi}_{\beta}) \in \mathbf{A}$,
 - ₃ $\mathbb{P}_{\beta}^{\bullet}$ is equal to $\mathbb{P}'_{\xi(\beta)}$ for some $\xi(\beta) = \xi_{\mathbf{q}}(\beta) \leq \beta$ (on \mathbb{P}'_{β} , see below),
 - ₄ The sequence $\langle (\mathbb{P}'_{\gamma}, \mathbb{P}'_{\gamma}, \mathbf{a}_{\gamma}, \mathbf{b}_{\gamma}, \xi(\gamma)) : \gamma < \beta \rangle$ belongs to $N_{\beta,u}$ for every $u \in [\mathcal{U}_{\beta}]^{\leq 2}$.
 - ₅ Let $\mathcal{W}_{\beta} = \bigcup \{ N_{\beta,u} \cap \beta : u \in [\mathcal{U}_{\beta}]^{\leq 2} \}$,
 - ₆ we ³ have: for every $\gamma \in \mathcal{W}_{\beta}$ the set $\mathcal{W}_{\beta} \cap \mathcal{W}_{\gamma}$ has cardinality $\leq \lambda$,
 - ₇ For every $\gamma \in \mathcal{W}_{\beta}$, there is $u = u_{\beta,\gamma} \in [\mathcal{U}_{\beta}]^{\leq 2}$ such that $\mathcal{W}_{\beta} \cap \mathcal{W}_{\gamma} \subseteq N_{\beta,u}$ and without loss of generality u is minimal with this property.
- (e) \mathbb{P}'_{α} is a dense subset of \mathbb{P}_{α} , where,
 - \mathbb{P}'_{α} is \mathbb{P}_{α} restricted to the set of conditions $p \in \mathbb{P}_{\alpha}$ such that:
 - if $\beta \in \text{dom}(p)$ then $p(\beta)$ is a member of \mathbf{V} (not just a \mathbb{P}_{α} -name)
 - and if $\zeta_1 < \zeta_2$ are in $p(\beta) \subseteq \mathcal{U}_{\beta}$, then there are q_1, q_2, r_1, r_2 as in Definition 1.2(2)(c)(•₁)-(•₅) with $\mathbf{a}_{\beta}, \mathbf{b}_{\beta}$ here standing for \mathbf{a}, \mathbf{b} there and

$$\bigvee_{\ell=1}^2 (\forall \gamma \in \text{dom}(r_{\ell})) [\gamma \in \text{dom}(p) \wedge r_{\ell}(\gamma) \subseteq p(\gamma)].$$

- (f) $\gamma_{\mathbf{q}} := \gamma(\mathbf{q}) := \sup\{\gamma_{\mathbf{q},\beta} : \beta < \text{lg}(\mathbf{q})\}$, so $\mathbb{P}'_{\gamma(\mathbf{q})} \subseteq \mathcal{H}_{<\lambda}(\gamma_{\mathbf{q}})$; let $\mathbb{P}_{\mathbf{q}} := \mathbb{P}_{\text{lg}(\mathbf{q})}$ and $\mathbb{P}'_{\mathbf{q}} := \mathbb{P}'_{\text{lg}(\mathbf{q})}$.

(1A) We may write either $\mathbb{P}_{\mathbf{q},\alpha}$ or \mathbb{P}_{α} whenever \mathbf{q} is clear and $(\iota_{\mathbf{q},\beta,1}, \iota_{\mathbf{q},\beta,2})$ is $\bar{t}_{\mathbf{b}_{\beta}}$.

³ Why? By 0.7(b)•₁₀.

(2) Let $\leq_{\mathbf{p}}$ be the following two-place relation on $\mathbf{Q}_{\mathbf{p}}$:

$$\mathbf{q}_1 \leq_{\mathbf{p}} \mathbf{q}_2 \text{ iff } \mathbf{q}_1 = \mathbf{q}_2 \upharpoonright \lg(\mathbf{q}_1), \text{ see below.}$$

(3) For $\mathbf{q}_2 \in \mathbf{Q}_{\mathbf{p}}$ and $\alpha_* \leq \lg(\mathbf{q}_2)$, we define $\mathbf{q}_1 := \mathbf{q}_2 \upharpoonright \alpha_*$ by:

- (a) $\lg(\mathbf{q}_1) = \alpha_*$,
- (b) $(\mathbb{P}_{\mathbf{q}_1, \alpha}, \mathbb{P}'_{\mathbf{q}_1, \alpha}) = (\mathbb{P}_{\mathbf{q}_2, \alpha}, \mathbb{P}'_{\mathbf{q}_2, \alpha})$ for $\alpha \leq \alpha_*$,
- (c) $(\mathbb{Q}_{\mathbf{q}_1, \beta}, \mathbf{b}_{\mathbf{q}_1, \beta}, \xi_{\mathbf{q}_1}(\beta)) = (\mathbb{Q}_{\mathbf{q}_2, \beta}, \mathbf{b}_{\mathbf{q}_2, \beta}, \xi_{\mathbf{q}_2}(\beta))$ for $\beta < \alpha_*$.

(4) We say that two conditions $p, q \in \mathbb{P}'_{\alpha}$ are *isomorphic*, when:

- (a) $\text{otp}(\text{dom}(p)) = \text{otp}(\text{dom}(q))$, and
- (b) if $\beta \in \text{dom}(p) \cap \text{dom}(q)$ then:
 - ₁ $\text{otp}(p(\beta)) = \text{otp}(q(\beta))$,
 - ₂ if $\varepsilon \in p(\beta) \cap q(\beta)$ then $\text{otp}(\varepsilon \cap p(\beta)) = \text{otp}(\varepsilon \cap q(\beta))$,
 - ₃ if $\varepsilon \in p(\beta), \zeta \in q(\beta)$ and $\text{otp}(\varepsilon \cap p(\beta)) = \text{otp}(\zeta \cap q(\beta))$ then:

$$\pi_{\beta, \{\zeta\}, \{\varepsilon\}}(p \upharpoonright N_{\beta, \{\varepsilon\}}) = q \upharpoonright N_{\beta, \{\zeta\}}.$$

- ₄ if $\varepsilon < \varepsilon_1$ belong to $p(\beta), \zeta < \zeta_1$ belong to $q(\beta)$, $\text{otp}(\varepsilon \cap p(\beta)) = \text{otp}(\zeta \cap q(\beta))$ and $\text{otp}(\varepsilon_1 \cap p(\beta)) = \text{otp}(\zeta_1 \cap q(\beta))$ then:

$$\pi_{\beta, \{\zeta, \zeta_1\}, \{\varepsilon, \varepsilon_1\}}(p \upharpoonright N_{\beta, \{\varepsilon, \varepsilon_1\}}) = q \upharpoonright N_{\beta, \{\zeta, \zeta_1\}}.$$

Remark 1.4. If we prefer in clause (d) (•₃) of Definition 1.3 (1) to have $\xi(\beta) = \beta$, i.e., $\mathbb{P}'_{\beta} = \mathbb{P}_{\beta}$, we need to add, e.g. “ μ is regular and e.g. use a preliminary forcing $(\{\mathbf{q} \in \mathbf{Q}_{\mathbf{p}} : \lg(\mathbf{q}) < \mu\}, \triangleleft)$ ”.

Claim 1.5.

(0) For $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, we have: $\mathbb{P}'_{\mathbf{q}} \models “p \leq q”$ iff $\{p, q\} \subseteq \mathbb{P}'_{\mathbf{q}}$, $\text{dom}(p) \subseteq \text{dom}(q)$, and $\beta \in \text{dom}(p) \Rightarrow p(\beta) \subseteq q(\beta)$.

(1) For $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, any increasing sequence of members of length $< \lambda$ of $\mathbb{P}'_{\mathbf{q}}$ has a lub, in fact, if $\delta < \lambda$, $\bar{p} = \langle p_i : i < \delta \rangle \in {}^{\delta}(\mathbb{P}'_{\mathbf{q}})$ is increasing, then the following $p \in \mathbb{P}'_{\mathbf{q}}$ is a lub of \bar{p} ; defined by: $\text{dom}(p) = \bigcup \{\text{dom}(p_i) : i < \delta\}$, and if $\beta \in \text{dom}(p)$ then

$$p(\beta) = \bigcup \{p_i(\beta) : i < \delta \text{ and } \beta \in \text{dom}(p_i)\}.$$

We denote this p by $\lim(\bar{p})$.

(2) For $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$, we have:

- $p \in \mathbb{P}'_{\mathbf{q}}$ iff:
 - (a) p is a function with domain $\in [\lg(\mathbf{q})]^{< \lambda}$,
 - (b) if $\beta \in \text{dom}(p)$ then $p(\beta)$ belongs to $[\mathcal{A}_{\beta}]^{< \lambda}$.
 - (c) If $\beta \in \text{dom}(p)$ and $(\iota_1, \iota_2) = (\iota_{\mathbf{q}, \beta, 1}, \iota_{\mathbf{q}, \beta, 2})$ then for every $\zeta_1 < \zeta_2$ from $p(\beta)$, $(p \upharpoonright \beta) \upharpoonright N_{\mathbf{q}, \beta, \{\zeta_1, \zeta_2\}} \Vdash_{\mathbb{P}_{\mathbf{q}, \beta}} “\mathfrak{c}\{\zeta_1, \zeta_2\} \in \{\iota_1, \iota_2\}”$. Moreover, there are q_1, q_2, r_1, r_2 as in Definition 1.2(2)(c)(•₁)-(•₅) and

$$\bigvee_{\ell=1}^2 (\forall \gamma \in \text{dom}(r_{\ell})) [\gamma \in \text{dom}(p) \cap \beta \wedge r_{\ell}(\gamma) \subseteq p(\gamma)].$$

(3) If $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$ and $\alpha \leq \lg(\mathbf{q})$ then $\mathbf{q} \upharpoonright \alpha \in \mathbf{Q}_{\mathbf{p}}$.

(4) $\leq_{\mathbf{p}}$ is a partial order on $\mathbf{Q}_{\mathbf{p}}$.

(5) If $\bar{\mathbf{q}} = \langle \mathbf{q}_j : j < \delta \rangle$ is $\leq_{\mathbf{p}}$ -increasing then it has a $\leq_{\mathbf{p}}$ -lub, $\lim(\bar{\mathbf{q}})$, of length $\cup \{\lg(\mathbf{q}_j) : j < \delta\}$.

(6) If $\beta < \lg(\mathbf{q})$, $\mathbf{a} = \mathbf{a}_{\mathbf{q}, \beta}$, $u \in [\mathcal{A}_{\mathbf{a}, \beta}]^{\leq 2}$ and $N_u = N_{\mathbf{a}, u}$, then:

- (*) if $p \in \mathbb{P}'_{\mathbf{q}}$ then $q = p \upharpoonright N_{\mathbf{q}, \beta, u}$ satisfies $q \in N_u$ and $q \leq_{\mathbb{P}_{\mathbf{q}}} p$ where q is defined by:

- ₁ $\text{dom}(q) = \text{dom}(p) \cap N_u \cap \beta$
- ₂ If $\gamma \in \text{dom}(q)$ then $q(\gamma) = p(\gamma) \cap N_u$.

(7) If (A) then (B), where:

- (A) (a) $i_* < \lambda$,
 (b) $p_i \in \mathbb{P}'_{\mathbf{q}}$ for $i < i_*$,
 (c) if $i < j < i_*$, then p_i and p_j are essentially comparable, i.e.:
 • if $\beta \in \text{dom}(p_i) \cap \text{dom}(p_j)$ then $p_i(\beta) \subseteq p_j(\beta)$ or $p_j(\beta) \subseteq p_i(\beta)$.
 (d) $\bar{p} = \langle p_i : i < i_* \rangle$.
 (B) \bar{p} has a lub p called $\text{lim}(\bar{p})$ or $\text{lim}(\{p_i : i < i_*\})$ defined by:
 • $\text{dom}(p) = \bigcup \{\text{dom}(p_i) : i < i_*\}$,
 • if $\beta \in \text{dom}(p)$, then

$$p(\beta) = \bigcup \{p_i(\beta) : i < i_* \text{ satisfying } \beta \in \text{dom}(p_i)\}.$$

Proof. Part (2) is crucial but easy to verify. Parts (0), (1), (3), and (4) are also easy.

(5) For this, define $\mathbf{q} := \text{lim}(\bar{\mathbf{q}})$ naturally, but we elaborate.

- (*) (a) $\text{lg}(\mathbf{q}) = \bigcup \{\text{lg}(\mathbf{q}_i) : i < \delta\}$,
 (b) if $i < \delta$ and $\alpha \leq \text{lg}(\mathbf{q}_i)$, then $(\mathbb{P}_{\mathbf{q},\alpha}, \mathbb{P}'_{\mathbf{q},\alpha}) = (\mathbb{P}_{\mathbf{q}_i,\alpha}, \mathbb{P}'_{\mathbf{q}_i,\alpha})$,
 (c) if $i < \delta$ and $\beta < \text{lg}(\mathbf{q}_i)$, then $(\mathbb{Q}_{\mathbf{q},\beta}, \mathbf{a}_{\mathbf{q},\beta}, \mathbf{b}_{\mathbf{q},\beta}) = (\mathbb{Q}_{\mathbf{q}_i,\beta}, \mathbf{a}_{\mathbf{q}_i,\beta}, \mathbf{b}_{\mathbf{q}_i,\beta})$,
 (d) $(\mathbb{P}_{\mathbf{q},\text{lg}(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\text{lg}(\mathbf{q})})$ is $(\bigcup \{\mathbb{P}_{\mathbf{q}_i} : i < \delta\}, \bigcup \{\mathbb{P}'_{\mathbf{q}_i} : i < \delta\})$ when $\text{cf}(\delta) \geq \lambda$,
 (e) if $\text{cf}(\delta) < \lambda$, then $(\mathbb{P}_{\mathbf{q},\text{lg}(\mathbf{q})}, \mathbb{P}'_{\mathbf{q},\text{lg}(\mathbf{q})})$ are defined as inverse limit. Then,
 • $\mathbb{P}'_{\mathbf{q}} := \mathbb{P}'_{\mathbf{q},\text{lg}(\mathbf{q})}$ is dense in $\mathbb{P}_{\mathbf{q}}$ because by Definition 1.2(3), for each $\beta < \text{lg}(\mathbf{q}_j)$ with $j < \delta$, $\mathbb{Q}_{\mathbf{b}[\beta, \mathbf{q}_j]}$ is closed under increasing unions of length $< \lambda$.

Recalling that in Definition 1.3(1)(c), we use β and not α , “ $\mathbb{P}_{\mathbf{q}}$ satisfies the λ^+ -cc” is not required for proving 1.5 (5), only “if $\beta < \text{lg}(\mathbf{q})$ then $\mathbb{P}_{\mathbf{q},\beta}$ satisfies the λ^+ -cc”, which is clear. Note that even though we formally do not need it here, the chain condition of $\mathbb{P}_{\mathbf{q}}$ will be proved in claim 1.6.

(6) Note that:

- (a) If $\gamma \in \text{dom}(q)$ then $\gamma \in N_u$ and $q(\gamma) \subseteq N_u$,
 (b) As $\text{dom}(q)$ and $q(\gamma)$ for $\gamma \in \text{dom}(q)$ has cardinality $< \lambda$ and $[N_u]^{<\lambda} \subseteq N_u$ so recalling clause (a) obviously $q \in N_u$.
 (c) To prove q is in $\mathbb{P}'_{\mathbf{q}}$ we need, for $\gamma \in \text{dom}(q)$ and $\zeta_1 < \zeta_2$ from $q(\gamma) \subseteq \mathcal{U}_\gamma$ to verify the condition in 1.5(2)(c).
 (d) But as $\gamma \in N_u$ hence $\mathbf{q} \upharpoonright (\gamma + 1)$ and ζ_1, ζ_2 belong to N_u , also $N_{\mathbf{q},\gamma, \{\zeta_1\}}$, $N_{\mathbf{q},\gamma, \{\zeta_2\}}$, $N_{\mathbf{q},\gamma, \{\zeta_1, \zeta_2\}}$ belong to N_u hence are included in it so we can finish easily.

(7) Follows by our definitions. □_{1.5}

We now arrive to the

Crucial Claim 1.6. *If $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$ then $\mathbb{P}_{\mathbf{q}}$ satisfies λ^+ -cc. Moreover $\mathbb{P}_{\mathbf{q}}$ is λ^+ -Knaster.*

Proof. It suffices, by 1.3(1)(e), to prove that $\mathbb{P}'_{\mathbf{q}} = \mathbb{P}'_{\mathbf{q},\text{lg}(\mathbf{q})}$ satisfies the λ^+ -cc, so assume:

- (*)₁ (a) Let $\bar{p} = \langle p_\xi : \xi < \lambda^+ \rangle$, where $p_\xi \in \mathbb{P}'_{\mathbf{q}}$,
 (b) it suffices to prove that for some $\zeta < \xi < \lambda^+$, p_ζ and p_ξ are compatible.

[Why? By the definitions.]

(*)₂ For some stationary set $S \subseteq \text{cof}(\lambda) \cap \lambda^+$, we have:

- ₁ $\langle \text{dom}(p_\xi) : \xi \in S \rangle$ is a Δ -system with heart $w_* \in [\text{lg}(\mathbf{q})]^{<\lambda}$, and
- ₂ if $\beta \in w_*$ then $\langle p_\xi(\beta) : \xi \in S \rangle$ is a Δ -system.

[Why? By the Delta system lemma, the proof using Fodor’s lemma recalling $\lambda = \lambda^{<\lambda}$.]

(*)₃ Without loss of generality, $\langle p_\xi : \xi \in S \rangle$ are pairwise isomorphic (see Definition 1.3(4)).

[Why? Easy because for every \mathbf{a}, u the model $N_{\mathbf{a}, u}$ has cardinality λ .]

(*)₄ For $\gamma < \beta$ from w_* , we have:

- ₁ Let $\mathcal{W}_\beta, u_{\beta, \gamma}$ be as in 1.3(1)(d)•₅.
- ₂ Without loss of generality, $u_{\beta, \gamma}$ is disjoint to $N_{\mathbf{q}, \beta, \{\zeta\}} \setminus N_{\mathbf{q}, \beta, \emptyset} \cap \mu$ for every $\zeta \in \mathcal{U}_\beta$ and is disjoint to $N_{\mathbf{q}, \beta, \{\varepsilon, \zeta\}} \setminus N_{\mathbf{q}, \beta, \emptyset} \cap \mu$ for every $\varepsilon < \zeta$ from \mathcal{U}_β .

[Why? As for any $\gamma < \beta$ from w_* we have to omit from \mathcal{U}_β at most two elements and w_* has cardinality $< \lambda$.]

(*)₅ We fix $\xi(1) \neq \xi(2)$ from S and we shall prove that $p_{\xi(1)}$ and $p_{\xi(2)}$ have a common upper bound; this suffices for proving the Crucial Claim 1.6.

(*)₆ For $\beta \in w_*$:

- (a) for $\ell \in \{1, 2\}$, consider the sequence $\langle \alpha_{\xi(\ell), \varepsilon}^\beta : \varepsilon < \varepsilon_\beta \rangle$ listing the set $p_{\xi(\ell)}(\beta)$ in increasing order
- (b) Why ε_β and not $\varepsilon_{\beta, \ell}$? as the two sequences have the same length because $p_{\xi(1)}, p_{\xi(2)}$ are isomorphic, see Definition 1.3(4) •₁.
- (c) Let $\mathcal{S}_\beta := \{\varepsilon < \varepsilon_\beta : \alpha_{\xi(1), \varepsilon}^\beta \neq \alpha_{\xi(2), \varepsilon}^\beta\}$,
- (d) so by Definition 1.3 (4) •₂ the sets $\{\alpha_{\xi(1), \varepsilon}^\beta : \varepsilon \in \mathcal{S}_\beta\}$, $\{\alpha_{\xi(2), \varepsilon}^\beta : \varepsilon \in \mathcal{S}_\beta\}$ are disjoint and disjoint to $\{\alpha_{\xi(1), \varepsilon}^\beta : \varepsilon \in \varepsilon_\beta \setminus \mathcal{S}_\beta\} = \{\alpha_{\xi(2), \varepsilon}^\beta : \varepsilon \in \varepsilon_\beta \setminus \mathcal{S}_\beta\}$.

Let $\bar{\beta} = \langle \beta_i : i \leq i_* \rangle$ list the closure of $\{\alpha, \alpha+1 : \alpha \in w_*\} \cup \{0, \lg(\mathbf{q})\}$ in increasing order, so necessarily $i_* < \lambda$ and clearly it suffices:

(*)₇ To choose $q_i \in \mathbb{P}'_{\mathbf{q}, \beta_i}$ a common upper bound of $\{p_{\xi(1)} \upharpoonright \beta_i, p_{\xi(2)} \upharpoonright \beta_i\}$ increasing with $i \leq i_*$ by induction on $i \leq i_*$ such that:

(*) If $\beta \in w_* \setminus \{\beta_j : j < i\}$ and $\zeta(1), \zeta(2)$ are from \mathcal{S}_β then:

- ₁ $\text{dom}(q_j) \cap N_{\beta, \{\alpha_{\xi(1), \zeta(1)}, \alpha_{\xi(2), \zeta(2)}\}}$ is a subset of

$$N_{\beta, \{\alpha_{\xi(1), \zeta(1)}\}} \cup N_{\beta, \{\alpha_{\xi(2), \zeta(2)}\}} \cup N_{\beta, \emptyset},$$

- ₂ if $\ell = 1, 2$ and $\gamma \in \text{dom}(q_j) \cap N_{\beta, \{\alpha_{\xi(\ell), \zeta(\ell)}\}}$ then $q_i(\gamma) = p_{\xi(\ell)}(\gamma)$ or $\gamma \in N_{\beta, \emptyset}$

Let us carry the induction.

Case 1: $i = 0$. Clearly, this case is trivial, letting $q_0 = \emptyset$.

Case 2: i is a limit ordinal.

In this case, let $q_i := \lim \langle q_j : j < i \rangle$, so by Claim 1.5(1), q_i is well-defined and is as required by the definition of the order and satisfies (*)₇.

Case 3: $i = j + 1$ and $\beta_j \notin w_*$.

In this case, $\text{dom}(p_{\xi(1)}) \cap \text{dom}(p_{\xi(2)}) \cap \beta_i \subseteq \beta_j$, hence the condition

$$q_i := q_j \cup (p_{\xi(1)} \upharpoonright [\beta_j, \beta_i]) \cup (p_{\xi(2)} \upharpoonright [\beta_j, \beta_i])$$

is as promised.

Case 4: $i = j + 1$ and $\beta_j \in w_*$.

By the choice of $\bar{\beta}$, clearly $\beta_i = \beta_j + 1$ and let $\mathcal{S} = \mathcal{S}_{\beta_j}$.

Recalling 1.3(1)(d) and 0.7(b)(•₈), we have:

(*)₈ $\mathbf{a}_{\beta_j} = \mathbf{a}_{\mathbf{q}, \beta_j}$ determine:

- (a) $\bar{\pi}_{\beta_j} = \langle \pi_{u, v} : u, v \in [\mathcal{U}_{\beta_j}]^{\leq 2} \text{ and } |u| = |v| \rangle$,
- (b) $\bar{N}_{\beta_j} = \langle N_u : u \in [\mathcal{U}_{\beta_j}]^{\leq 2} \rangle$,
- (c) for $\varepsilon(1), \varepsilon(2) \in \mathcal{S}$, let:
 - $v[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1), \varepsilon(1)}, \alpha_{\xi(1), \varepsilon(2)}\}$, and
 - $u[\varepsilon(1), \varepsilon(2)] = \{\alpha_{\xi(1), \varepsilon(1)}, \alpha_{\xi(2), \varepsilon(2)}\}$.

- (d) for $\varepsilon \in \mathcal{S}$, let $v[\varepsilon] = \{\alpha_{\xi(1),\varepsilon}\}$ and $u[\varepsilon] = \{\alpha_{\xi(2),\varepsilon}\}$,
- (e) $\bar{v} = \bar{v}_{\beta_j}^*$, see 1.3 (1) (d) \bullet_1 .
- (f) $\gamma_j = \xi_{\mathbf{q}}(\beta_j)$; see 1.3(1)(d) \bullet_3 .

We shall now define $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathcal{S}$ such that:

- (*)₉ (a) $p_{\varepsilon(1),\varepsilon(2)} \in \mathbb{P}_{\gamma_j} \cap N_{u[\varepsilon(1),\varepsilon(2)]}$, hence $\text{dom}(p_{\varepsilon(1),\varepsilon(2)}) \subseteq \gamma_j \cap N_{u[\varepsilon(1),\varepsilon(2)]}$,
- (b) if $\varepsilon(1) = \varepsilon(2)$, then $p_{\varepsilon(1),\varepsilon(2)} \upharpoonright (\gamma_j \cap N_{v[\varepsilon(1)]})$, $p_{\xi(1)} \upharpoonright N_{v[\varepsilon(1)]}$ are essentially comparable; see 1.5(7)(A)(c), moreover the first is $\leq_{\mathbb{P}_{\mathbf{q}}}$ -above the second,
- (c) if $\varepsilon(1) = \varepsilon(2)$, then $p_{\varepsilon(1),\varepsilon(2)} \upharpoonright (\gamma_j \cap N_{u[\varepsilon(2)]})$, $p_{\xi(2)} \upharpoonright N_{u[\varepsilon(2)]}$ are essentially comparable, moreover the first is $\leq_{\mathbb{P}_{\mathbf{q}}}$ -above the second,
- (d) $p_{\varepsilon(1),\varepsilon(2)}$ satisfies 1.3(1)(e) \bullet with $(\gamma_j, \varepsilon(1), \varepsilon(2))$ here standing for $(\beta, \zeta_1, \zeta_2)$ there,
- (e) $\{q_j \upharpoonright N_{\emptyset}\} \cup \{p_{\varepsilon(1),\varepsilon(2)} \upharpoonright N_{\emptyset} : \varepsilon(1), \varepsilon(2) \in \mathcal{S}\}$ are pairwise essentially comparable,
- (f) if $\varepsilon(1) \neq \varepsilon(2)$ then $p_{\varepsilon(1),\varepsilon(2)} \upharpoonright N_{\{\alpha_{\varepsilon(\ell)}\}} \leq p_{\xi(\ell)} \upharpoonright N_{\{\alpha_{\varepsilon(\ell)}\}}$ for $\ell = 1, 2$.
- (g) if $\mathcal{S}_* \subseteq \mathcal{S} \times \mathcal{S}$ then the lub $q_{\mathcal{S}_*}$ of $\{q_j \upharpoonright N_{u[\varepsilon(1),\varepsilon(2)]} : \varepsilon(1), \varepsilon(2) \in \mathcal{S}_*\}$ satisfies the condition in (*)₇.

We have to show two things: \boxplus_1 and \boxplus_2 . The first says we can choose them (the $p_{\varepsilon(1),\varepsilon(2)}$ -s), the second that this is enough.

\boxplus_1 we can choose $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathcal{S}$ as required in (*)₇.

We consider two possible cases:

Case 4.1: $\varepsilon(1) \neq \varepsilon(2)$.

Let $p_{\varepsilon(1),\varepsilon(2)} = \pi(p_{\xi(1)} \upharpoonright N_{v[\varepsilon(1),\varepsilon(2)]})$, where $\pi = \pi_{u[\varepsilon(1),\varepsilon(2)], v[\varepsilon(1),\varepsilon(2)]}$.

Why is (*)₉ preserved? Most clauses are obvious, but (*)₉(g) deserve elaboration, recalling that we have to satisfy (*)₇.

So let $\beta \in \mathcal{W}_* \setminus \{\beta_\iota : \iota < i\}$, hence for some $j(*) < i_*$, we have $\beta = \beta_{j(*)}$, hence we have $\beta_{j(*)} \geq \beta_i$ hence $\beta_{j(*)} > \beta_j$ and we have $\mathcal{S}_* \subseteq \mathcal{S} \times \mathcal{S}$ and deal with $q_{\mathcal{S}_*}$.

For this, it is enough to consider the cases:

- \oplus_1 $\mathcal{S}_* = \{\zeta(1), \zeta(2)\}$, where $\zeta(1) = \varepsilon(1)$ and $\zeta(2) = \varepsilon(2)$ hence from \mathcal{S} , so $\zeta(1) \neq \zeta(2)$,
- \oplus_2 $\mathcal{S}_* = \{\zeta(1), \zeta(2)\}$ where $\zeta(1) \neq \zeta(2)$ are from \mathcal{S} but $(\zeta(1), \zeta(2)) \neq (\varepsilon(1), \varepsilon(2))$.

Easy to check.

Case 4.2: $\varepsilon(1) = \varepsilon(2)$.

In this case, we pick some sequence $\langle p_{\varepsilon,\varepsilon} : \varepsilon \in \mathcal{S} \rangle$ by choosing $p_{\varepsilon,\varepsilon}$ by induction on $\varepsilon \in \mathcal{S}$. Now, $p_{\varepsilon,\varepsilon} \in \mathbb{P}'_{\beta_j} \cap N_{u[\varepsilon(1),\varepsilon(2)]}$ is such that:

- (*) (a) $p_{\varepsilon,\varepsilon}$ is $\leq_{\mathbb{P}'_{\mathbf{q},\beta_j}}$ -above $p_{\xi(1)} \upharpoonright N_{v[\varepsilon]}$ and above the restriction $p_{\xi(2)} \upharpoonright N_{u[\varepsilon]}$,
- (b) $\langle p_{\zeta,\zeta} \upharpoonright N_{\emptyset} : \zeta \in (\varepsilon + 1) \cap \mathcal{S} \rangle$ is $\leq_{\mathbb{P}_{\beta_j}}$ -increasing, and
- (c) there are q_1, q_2, r_1, r_2 as in Definition 1.3(2)(c) (\bullet_1)-(\bullet_5) with $\mathbf{b}_{\mathbf{q},\beta_j}$ standing here for (\mathbf{a}, p, \bar{v}) there such that:

$$\bigvee_{\ell=1}^2 (\forall \gamma \in \text{dom}(r_\ell)) [\gamma \in \text{dom}(p_{\varepsilon,\varepsilon}) \wedge r_\ell(\gamma) \subseteq p_{\varepsilon,\varepsilon}(\gamma)].$$

We can choose $p_{\varepsilon,\varepsilon}$ by the properties of \mathbf{b}_{β_j}

Having defined all the $p_{\varepsilon(1),\varepsilon(2)}$ -s we can proceed.

\boxplus_2 The following set of members of \mathbb{P}_{β_i} has a common upper bound q_* :

- \bullet $p_{\xi(1)}, p_{\xi(2)}$, and
- \bullet $p_{\varepsilon(1),\varepsilon(2)}$ for $\varepsilon(1), \varepsilon(2) \in \mathcal{S}$.

[Why? Recall Claim 1.5(2) and 1.2(1)(c)(\bullet_1) by 1.5(7), clause (A) there holds, in particular sub-clause (A)(c). The main point is that:

(*) $\langle N_{v[\varepsilon(1), \varepsilon(2)]} \cap \gamma_j \setminus (N_{v[\varepsilon(1)]} \cup N_{u[\varepsilon(1)]}) : \varepsilon(1), \varepsilon(2) \in \mathcal{S} \rangle$ is a sequence of pairwise disjoint sets.

Why? As “ $N_u \cap N_v \subseteq N_{u \cap v}$ for $u, v \in [\mathcal{U}_{\beta_j}]^{<2}$ by 0.7•7.

So q_* from \boxplus_2 is a common upper bound of $p_{\xi(1)}, p_{\xi(2)}$, as promised. $\square_{1.6}$

Remark 1.7. 1) No need so far, but we may add in $(*)_4$ of the proof of Crucial Claim 1.6 the following item:

(d) if $\beta \in w_*$ and $\langle \alpha_{\zeta, \beta, i} : i < \iota_{\zeta, \beta} \rangle$ lists in increasing order the members of $p_{\zeta}(\beta)$ for $\zeta \in S$, then:

- $\langle \iota_{\zeta, \beta} : \zeta \in S \rangle$ is constant called i_{β} ,
- for $i < i_{\beta}$, the sequence $\langle \alpha_{\zeta, \beta, i} : \zeta \in S \rangle$ is constant or increasing,
- if $i, j < i_{\beta}$ the sequence of truth values

$\langle \text{Truth value}(\alpha_{\zeta, \beta, i} < \alpha_{\xi, \beta, j}) : \zeta < \xi \text{ are from } S \rangle$

is constant, and

- if $i, j < i_{\beta}$, $\zeta \neq \xi$ are from S and $\alpha_{\zeta, \beta, i} = \alpha_{\xi, \beta, j}$ then $i = j$.

2) We can make our choice of q_1, q_2, r_1, r_2 canonical, that is:

(A) In 1.2(2) we replace (\mathbf{a}, p, \bar{t}) by $(\mathbf{a}, p, \bar{i}, \mathbb{F})$, where:

- ₁ $\mathbb{F}_{\zeta_1, \zeta_2}(q) = (q_1, q_2, r_1, r_2) = \langle \mathbb{F}_{\zeta_1, \zeta_2, \ell}(q) : \ell = 1, 2, 3, 4 \rangle$
- ₂ if also $\zeta_3 < \zeta_4$ are from \mathcal{U} then $\pi_{\zeta_3, \zeta_4, \zeta_1, \zeta_2}^{\mathbf{a}} \mathbb{F}_{\zeta_1, \zeta_2, \ell} = \mathbb{F}_{\zeta_3, \zeta_4, \ell}$, where if $p \leq q \in \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ and $\zeta_1 < \zeta_2$ are from \mathcal{U} , then $\langle \mathbb{F}_{\zeta_1, \zeta_1, \ell}(p, q) : \ell < \mu \rangle$ is the quadruple (q_1, q_2, r_1, r_2) as in 1.2(1)(c)(•₁)-(•₅).

(B) In 1.2(3) similarly and in 1.3(1)(d)

(C) In 1.5(1)(d) use \mathbb{F}_{β} ,

(D) In the proof of 1.6, in $(*)_7\boxplus_1$, case 4.2(*)_{4.2} we use \mathbb{F}_{β_j} ,

(E) Update the proof of 1.8 accordingly.

Claim 1.8. *If (A) then (B), where:*

- (A) (a) $\mathbf{q} \in \mathbf{Q}_{\mathbf{p}}$,
- (b) $2 < \sigma < \lambda$,
- (c) \mathfrak{c} is a $\mathbb{P}_{\mathbf{q}}$ -name of a function from $[\theta]^2$ into σ .
- (d) $p_* \in \mathbb{P}_{\mathbf{q}}$.
- (B) *There is some $\mathbf{b} \in \mathbf{A}^+$ such that $\mathbb{P}_{\mathbf{b}} = \mathbb{P}'_{\mathbf{q}}$ and $\mathfrak{c}_{\mathbf{b}} = \mathfrak{c}$ and $p_* \leq_{\mathbb{P}_{\mathbf{q}}} p_{\mathbf{b}}$.*

Proof. Recalling Hypothesis 1.1(b), on the one hand, it is clear how to choose $\mathbf{a} \in \mathbf{A}$ such that $\mathbb{P}_{\mathbf{a}} = \mathbb{P}'_{\mathbf{q}}$ and $\mathfrak{c}_{\mathbf{a}} = \mathfrak{c}$. On the other hand, the choice of $p_{\mathbf{b}}$ and $\bar{t}_{\mathbf{b}}$ is similar to the proof of [She88, 2.1]. We now elaborate.

First, we can find \mathbf{a} such that:

- (*)_a¹ (a) $\mathbf{a} \in \mathbf{A}$,
- (b) $\mathbb{P}_{\mathbf{a}} = \mathbb{P}'_{\mathbf{q}}$,
- (c) $\gamma = \text{lg}(\mathbf{q})$,
- (d) $\mathfrak{c}_{\mathbf{a}} = \mathfrak{c}$.

Why can we find? Because we have chosen $\mathbb{P}_{\mathbf{a}}$ as in $(*)_{\mathbf{a}}^1$ (b), it is λ^+ -cc by Claim 1.6; also $\gamma, \mathfrak{c}_{\mathbf{a}}$ are as is required in Definition 1.2. Lastly we can choose $(\mathcal{U}_{\mathbf{a}}, \bar{N})$ as required because $\theta \rightarrow_{\text{sq}} (\partial)_{\lambda}^{\lambda, 2}$ holds by Hypothesis 1.1 clause (b) and 0.7 in particular clause $(b)_{\bullet 10}$.

We are left with choosing some appropriate (p, \bar{t}) and then let $\mathbf{b} = (\mathbf{a}, p, \bar{t})$. Let

$$Y := \{(q_1, q_2) : q_1, q_2 \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}} \text{ are above } p_* \text{ and,} \\ q_1 \upharpoonright (N_{\mathbf{a}, \emptyset} \cap \text{lg}(\mathbf{q})) = q_2 \upharpoonright (N_{\mathbf{a}, \emptyset} \cap \text{lg}(\mathbf{q}))\},$$

and let \leq_Y be the following two place relation on Y :

- (*)₂ $(p_1, p_2) \leq_Y (q_1, q_2)$ iff:

- (a) $(p_1, p_2) \in Y$ and $(q_1, q_2) \in Y$,
- (b) $p_1 \leq_{\mathbb{P}'_{\mathbf{q}}} q_1$ and $p_2 \leq_{\mathbb{P}'_{\mathbf{q}}} q_2$.

Clearly,

(*)₃ (Y, \leq_Y) is a $(< \lambda)$ -complete partial order.

[Why? Recalling 1.5(1).]

(*)₄ For $(p_1, p_2) \in Y$, let

(a) $\text{solv}(p_1, p_2)$ be the set of pairs (ι_0, ι_1) such that for any $\zeta_1 < \zeta_2$ from $\mathcal{Q}_{\mathbf{a}}$, there are r_1, r_2 such that for $\ell = 1, 2$ clauses $\bullet_2\text{-}\bullet_5$ of Definition 1.2(2)(c) hold.

(b) $\text{solv}^+(p_1, p_2) := \bigcap \{ \text{solv}(q_1, q_2) : (p_1, p_2) \leq_Y (q_1, q_2) \in Y \}$.

(*)₅ (a) if $(p_1, p_2) \leq_Y (q_1, q_2)$ then:

$$\text{solv}(p_1, p_2) \supseteq \text{solv}(q_1, q_2) \supseteq \text{solv}^+(q_1, q_2) \supseteq \text{solv}^+(p_1, p_2),$$

(b) if $(p_1, p_2) \in Y$ then $\text{solv}(p_1, p_2) \neq \emptyset$.

[Why? The first inclusion in Clause (a) holds because $\leq_{\mathbb{P}'_{\mathbf{q}}}$ is transitive. The other inclusions are clear, and Clause (b) is easy too.]

(*)₆ If $(p_1, p_2) \in Y$ then for some (q_1, q_2) and $\bar{\iota}$, we have:

(a) $(p_1, p_2) \leq_Y (q_1, q_2) \in Y$,

(b) if $(q_1, q_2) \leq_Y (q'_1, q'_2)$ then $\bar{\iota} \in \text{solv}(q'_1, q'_2)$, moreover, $\text{solv}(q_1, q_2) = \text{solv}(q'_1, q'_2) = \text{solv}^+(q'_1, q'_2) = \text{solv}^+(q_1, q_2)$.

[Why? Recalling $\sigma < \lambda$, hence $|\sigma \times \sigma| < \lambda$ and (Y, \leq_Y) is λ -complete by (*)₃.]

(*)₇ For $p \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$, let $\text{solv}(p)$ be the set of $\bar{\iota} \in \sigma \times \sigma$ such that there is (q_1, q_2) such that:

- ₁ $p \leq_{\mathbb{P}'_{\mathbf{q}}} q_1, p \leq_{\mathbb{P}'_{\mathbf{q}}} q_2$ and
- ₂ $(q_1, q_2) \in Y$,
- ₃ $\bar{\iota} \in \text{solv}^+(q_1, q_2)$,
- ₄ $\text{solv}(q_1, q_2) = \text{solv}^+(q_1, q_2)$.

(*)₈ (a) if $p \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then $\text{solv}(p) \neq \emptyset$,

(b) if $p \leq_{\mathbb{P}'_{\mathbf{a}}} q$ are from $\mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then $\text{solv}(p) \supseteq \text{solv}(q)$,

(c) if $p \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ then for some q and $\bar{\iota}$, for every q' , we have $q \leq_{\mathbb{P}'_{\mathbf{q}}} q' \wedge q' \in \mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}} \Rightarrow \bar{\iota} \in \text{solv}(q')$.

[Why? Clause (a) follows by (*)₆, Clause (b) by the definitions, and Clause (c) holds as $\mathbb{P}'_{\mathbf{a}}$ and even $\mathbb{P}'_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ is λ -complete and $|\sigma \times \sigma| < \lambda$.]

Now, applying (*)₈(c) to p_* finish the proof of 1.8. □_{1.8}

Claim 1.9. *If (A) then (B), where:*

- (A) (a) $\mathbf{q} \in \mathbb{Q}_{\mathbf{p}}$ and $\mathbf{q}_0 <_{\mathbf{p}} \mathbf{q}$,
- (b) $\gamma(\mathbf{q}) < \mu$, so $\text{lg}(\mathbf{q}) < \mu$,
- (c) $\mathbf{b} \in \mathbf{A}_{\mathbf{p}}$ and $\mathbb{P}_{\mathbf{b}} = \mathbb{P}_{\mathbf{q}_0}$.

(B) *There exists some \mathbf{q}_1 such that:*

- (a) $\mathbf{q} \leq_{\mathbf{p}} \mathbf{q}_1$,
- (b) $\text{lg}(\mathbf{q}_1) = \text{lg}(\mathbf{q}) + 1$,
- (c) $\mathbf{b}_{\text{lg}(\mathbf{q})}[\mathbf{q}_1] = \mathbf{b}$.

Proof. Easy. □_{1.9}

Lastly, before arriving at the main conclusion, we have to prove the following.

Claim 1.10.

(1) *Assume $\mathbf{q} \in \mathbb{Q}_{\mathbf{p}}$, $\alpha < \text{lg}(\mathbf{q})$ and $\mathbf{b} = \mathbf{b}_{\mathbf{q}, \alpha} = (\mathbf{a}_{\alpha}, p_{\alpha}, \bar{\iota}_{\alpha}) = (\mathbf{a}, p, \bar{\iota})$, then:*

- $\Vdash_{\mathbb{P}_{\mathbf{q}, \alpha+1}} \text{“} \mathcal{V}_{\mathbb{Q}_{\mathbf{b}}} \in [\mathcal{Q}_{\mathbf{a}_{\alpha}}]^{\partial} \text{ and for every } \alpha \neq \beta \in \mathcal{V}_{\mathbb{Q}_{\mathbf{b}}}, \mathbf{c}_{\mathbf{a}_{\alpha}} \{\alpha, \beta\} \in \{\iota_1, \iota_2\} \text{”}$.

(2) If $\mathbf{b} = (\mathbf{a}, p, \bar{v}) \in \mathbf{A}^+$, $\text{cf}(\partial) > \lambda$, and in $\mathbf{V}^{\mathbb{P}_{\mathbf{a}}}$, $\mathbb{Q}_{\mathbf{b}}$ satisfies the λ^+ -cc, then for some $p \in \mathbb{Q}_{\mathbf{b}} \cap \mathbb{P}_{\mathbf{a}} \cap N_{\mathbf{a}, \{\varepsilon[\mathbf{a}]\}}$ we have⁴ $p \Vdash_{\mathbb{Q}_{\mathbf{b}}} \text{“}\mathcal{V}_{\mathbb{Q}_{\mathbf{b}}} \in [\mathcal{U}_{\mathbf{a}}]^{\partial} \text{ and for every } \alpha \neq \beta \in \mathcal{V}_{\mathbb{Q}_{\mathbf{b}}}, \mathbf{c}_{\mathbf{a}}\{\alpha, \beta\} \in \{\iota_1, \iota_2\}\text{”}$.

Proof. (1) The second phrase in both conclusion holds by the definitions of $\mathbb{Q}_{\mathbf{b}}$.

By the proof of “ $\mathbb{P}_{\mathbf{q}}$ satisfies the λ^+ -cc”, we can show for $\varepsilon < \partial$, the density of the set

$$\mathcal{I}_{\varepsilon} := \{p \in \mathbb{P}'_{\mathbf{q}} : \alpha \in \text{dom}(p) \text{ and there is } \beta \in p(\alpha) \text{ such that } \varepsilon < \text{otp}(\mathcal{U}_{\mathbf{a}_{\alpha}} \cap \beta)\}.$$

(2) Easily, for every $\beta \in \mathcal{U}_{\mathbf{a}}$ we can choose $p_{\beta}^0 = \{\beta\}$, $q_{\beta} = \{(p, p_{\beta}^0)\}$. Clearly, $q_{\beta} \in \mathbb{P}_{\alpha} * \mathbb{Q}_{\mathbf{b}}$ for $\beta \in \mathcal{U}_{\mathbf{a}}$. So by the λ^+ -cc for some $\beta \in \mathcal{U}_{\mathbf{a}}$, $q_{\beta} \Vdash \text{“}\{\varepsilon \in \mathcal{U}_{\mathbf{a}} : q_{\varepsilon} \in \mathbb{Q}_{\mathbf{b}}\} \in [\mathcal{U}_{\mathbf{a}}]^{\partial}$; well assuming $\text{cf}(\partial) > \lambda$. □_{1.10}

Conclusion 1.11. *There exists a forcing notion \mathbb{P} satisfying the following conditions:*

- (a) \mathbb{P} is λ^+ -cc of cardinality μ .
- (b) \mathbb{P} is $(< \lambda)$ -complete; hence, it collapses no cardinals, changes no cofinalities, and preserves cardinal arithmetic outside the interval $[\lambda, \mu]$.
- (c) $\Vdash_{\mathbb{P}} \text{“}2^{\lambda} = \mu\text{”}$.
- (d) $\Vdash_{\mathbb{P}} \text{“}\theta \rightarrow [\partial]_{\sigma, 2}^2\text{”}$ for every $\sigma \in (2, \lambda)$.

Proof. Choose a $\leq_{\mathbf{p}}$ -increasing continuous sequence $\langle \mathbf{q}_{\alpha} : \alpha < \mu \rangle \in {}^{\mu}(\mathbf{Q}_{\mathbf{p}})$ such that $\text{lg}(\mathbf{q}_{\alpha}) = \alpha$, $\mathbb{P}_{\mathbf{q}_{\alpha}}$ has cardinality $\leq (|\alpha| + \lambda)^{< \lambda}$ and,

- if $\alpha < \mu$ and $\Vdash_{\mathbb{P}_{\mathbf{q}_{\alpha}}} \text{“}\mathbf{c} : [\theta]^2 \rightarrow \sigma\text{”}$, then for unboundedly many $\beta \in [\alpha, \mu]$, $\mathbf{c}_{\mathbf{q}_{\beta+1}, \beta} = \mathbf{c}$.

The existence of $\mathbf{b}_{\beta}[\mathbf{q}_{\beta+1}]$ with $\mathbf{c}[\mathbf{b}_{\beta}[\mathbf{q}_{\beta+1}]] = \mathbf{c}$ as required holds by Claim 1.8 and Claim 1.9 below.

Clearly $\bigcup \{\mathbb{P}_{\mathbf{q}_{\beta}} : \beta < \mu\}$ is a forcing notion as is required. □_{1.11}

Conclusion 1.11 is meaningful because:

Fact 1.12. Assume that $\lambda = \lambda^{< \lambda} < \partial < \theta < \mu = \mu^{\theta}$, and $[\alpha < \mu \Rightarrow |\alpha|^{\lambda} < \mu]$, $\theta > \beth_2(\kappa)$ and $\partial = \kappa^+$, $\kappa = \kappa^{\lambda}$. Then the demands in Hypothesis 1.1 hold.

Remark 1.13. To justify the assumption, notice that:

- (A) Omitting $\kappa = \kappa^{\lambda}$ does not help.
- (B) $\theta \rightarrow_{\text{sq}} (\partial)_{\partial}^{2 \leq \lambda}$ implies $\theta \rightarrow (\partial)_{2\partial}^2$, hence necessarily $\theta > 2^{2^{\partial}}$.

With stronger lower bound on θ , see [She89].

The main point is proving $\theta \rightarrow_{\text{sq}} (\partial)_{\partial}^{\leq \lambda, 2}$. For this, see [She89], $\theta = \beth_m(\partial)$ for some small m suffice, on this the bounds in 1.11 depends; we intend to return to this in [S⁺]. Anyhow just $\theta < \partial^{+\omega}$ and GCH in $[\partial, \partial^{+\omega}]$ would suffice for me.

Proof. The point is to prove $\theta \rightarrow_{\text{sq}} (\partial)_{\partial}^{\lambda, 2}$. Let \mathcal{B} be as in 0.7(a), $\partial_1 = 2^{\kappa}$, $\partial_2 = 2^{\partial_1}$, and $\theta > \partial_2$.

Let $\chi > 2^{\mu}$, and \mathfrak{C}_{*} be an expansion of $(\mathcal{H}(\chi), \in, <_{\chi}^{*}, \mathcal{B})$ with vocabulary of cardinality λ such that for any finite set $u \subseteq \mathcal{H}(\chi)$, the Skolem hull of u , $N_u := \text{Sk}(u, \mathfrak{C}_{*})$ is of cardinality λ and $|N_u|^{< \lambda} \subseteq N_u$.

Let $\mathfrak{C}_2 \prec_{\mathbb{L}_{\partial(1)^+, \partial(1)^+}} \mathfrak{C}_{*}$ be of cardinality ∂_2 such that $\partial_2 + 1 \subseteq \mathfrak{C}_2$. Let $\beta_1 := \min(\theta \setminus \mathfrak{C}_2)$. Similarly, choose $\mathfrak{C}_1 \prec_{\mathbb{L}_{\partial, \partial}} \mathfrak{C}_{*}$ of cardinality ∂_1 such that $\partial_1 + 1 \subseteq \mathfrak{C}_1$ and $\{\mathfrak{C}_2, \beta_0\} \subseteq \mathfrak{C}_1$.

Let $\mathfrak{C}_0 = \mathfrak{C}_1 \cap \mathfrak{C}_2$ and choose $\beta_0 \in \beta_1 \cap \mathfrak{C}_2 \subseteq \theta \cap \mathfrak{C}_2$ realizing the $\mathbb{L}_{\partial, \partial}$ -type which β_1 realizes over \mathfrak{C}_0 .

Now,

⁴We may omit p but it does not matter.

(*)₁ choose $\alpha_\varepsilon \in \mathfrak{C}_0 \cap \theta$ by induction on $\varepsilon < \partial$, such that:

- $\alpha_\varepsilon, \beta_1$ realize the same first-order type in \mathfrak{C}_* over the set $\{\beta_2\} \cup (A_\varepsilon \cap \mathfrak{C}_0)$, where:

$$A_\varepsilon = \text{Sk}_{\mathfrak{C}}(\{\alpha_\zeta : \zeta < \varepsilon\} \cup \{\beta_1, \beta_0\}).$$

(*)₂ Let $N_\emptyset^\bullet = N_{\{\beta_0, \beta_1\}} \cap \mathfrak{C}_0$.

Note,

(*)₃ for $\varepsilon < \zeta < \partial$, the following pairs realize the same type over N_\emptyset^* in \mathfrak{C}_* :

- ₁ $(\alpha_\varepsilon, \alpha_\zeta)$,
- ₂ $(\alpha_\varepsilon, \beta_0)$,
- ₃ $(\alpha_\varepsilon, \beta_1)$,
- ₄ (β_0, β_1) .

[Why? For the equality of •₁ and •₂ note the choice of α_ε .

For the equality of •₂ and •₃, note the choice of β_0 .

For the equality of •₃ and •₄ note the choice of α_ε .]

- (*)₄
- ₁ $N_{\{\varepsilon, \zeta\}}^* = N_{\{\alpha_\varepsilon, \alpha_\zeta\}}$, so for $\varepsilon < \zeta < \partial$,
 - ₂ $N_{\{\varepsilon, \zeta\}}^* \prec \mathfrak{C}_0$,
 - ₃ $N_\emptyset^\bullet \prec N_{\{\varepsilon, \zeta\}}^*$.

(*)₅ for $\varepsilon < \zeta < \partial$, let $f_{\{\varepsilon, \zeta\}}$ be the isomorphism from $N_{\{\varepsilon, \zeta\}}^*$ onto $N_{\{\beta_0, \beta_1\}}$.

[Why does it exist? by (*)₃.]

(*)₆ $f_{\{\varepsilon, \zeta\}}$ is the identity on N_\emptyset^\bullet (and $N_\emptyset^\bullet \prec N_{\{\varepsilon, \zeta\}}$).

[Why? By (*)₂.]

(*)₇ if $\varepsilon(0) < \zeta(0) < \partial$, $\varepsilon(1) < \zeta(1) < \partial$ and $\{\varepsilon(0), \zeta(0)\} \cap \{\varepsilon(1), \zeta(1)\} = \emptyset$, then

$$N_{\{\varepsilon(0), \zeta(0)\}}^* \cap N_{\{\varepsilon(1), \zeta(1)\}}^* = N_{\{\beta_0, \beta_1\}} \cap \mathfrak{C}_0 = N_\emptyset^\bullet.$$

[Why? The second equality holds by (*)₂; without loss of generality $\zeta(0) < \zeta(1)$.

Now,

- ₁ $N_{\{\varepsilon(0), \zeta(0)\}}^* \cap N_{\{\varepsilon(1), \zeta(1)\}}^* = N_{\{\varepsilon(0), \zeta(0)\}}^* \cap N_{\{\alpha_{\varepsilon(1)}, \beta_1\}}$ by the choice of $\zeta(1)$.
- ₂ if $\zeta(0) < \varepsilon(1)$ then

$$N_{\{\varepsilon(0), \zeta(0)\}}^* \cap N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} = N_{\{\varepsilon(0), \zeta(0)\}}^* \cap N_{\{\beta_0, \beta_1\}} = N_{\{\varepsilon(0), \zeta(0)\}} \cap N_\emptyset^\bullet = N_\emptyset^\bullet$$
 because the first equality follows by the choice of $\alpha_{\varepsilon(1)}$ second equality by (*)₄•₂ and (*)₂; the third equality by (*)₃.
- ₃ if $\varepsilon(0) < \varepsilon(1) < \zeta(0)$, then:

$$\begin{aligned} N_{\{\varepsilon(0), \zeta(0)\}}^* \cap N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} &= N_{\{\alpha_{\varepsilon(0)}, \beta_0\}} \cap N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} \\ &= \left(N_{\{\alpha_{\varepsilon(0)}, \beta_0\}}^* \cap \mathfrak{C}_0 \right) \cap \left(N_{\{\alpha_{\varepsilon(0)}, \beta_1\}} \cap \mathfrak{C}_0 \right) \\ &= \left(N_{\{\alpha_{\zeta(0)}, \beta_1\}} \cap \mathfrak{C}_0 \right) \cap \left(N_{\{\alpha_{\zeta(0)}, \beta_1\}} \cap \mathfrak{C}_0 \right) \\ &= \left(N_{\{\alpha_{\zeta(0)}, \beta_1\}} \cap \mathfrak{C}_0 \right) \cap \left(N_{\{\beta_0, \beta_1\}} \cap \mathfrak{C}_0 \right) \\ &= \left(N_{\{\alpha_{\varepsilon(0)}, \beta_1\}} \cap \mathfrak{C}_0 \right) \cap N_\emptyset^\bullet = N_\emptyset^\bullet. \end{aligned}$$

[Why? The first equality holds by the choice of β_0 . The second equality as $N_{\{\varepsilon(0), \zeta(0)\}}^* \subseteq \mathfrak{C}_0$ and the first equality. The third equality holds by the choice of β_0 . The fourth equality holds by the choice of $\alpha_{\zeta(0)}$. The fifth equality holds by the choice of N_\emptyset^\bullet i.e., (*)₂. Finally, the sixth equality holds as $N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} \supseteq N_\emptyset^\bullet$ by the choice of $\alpha_{\varepsilon(2)}$.]

- ₄ If $\varepsilon(1) < \varepsilon(0)$, then:

$$\begin{aligned}
N_{\{\varepsilon(0), \zeta(0)\}}^* \cap N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} &= N_{\{\alpha_{\varepsilon(0)}, \beta_0\}} \cap N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} \\
&= \left(N_{\{\alpha_{\varepsilon(0)}, \beta_0\}}^* \cap \mathfrak{C}_0 \right) \cap \left(N_{\{\alpha_{\varepsilon(1)}, \beta_1\}}^* \cap \mathfrak{C}_0 \right) \\
&= \left(N_{\{\alpha_{\varepsilon(0)}, \beta_1\}} \cap \mathfrak{C}_0 \right) \cap \left(N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} \cap \mathfrak{C}_0 \right) \\
&= \left(N_{\{\beta_0, \beta_1\}} \cap \mathfrak{C} \right) \cap \left(N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} \cap \mathfrak{C} \right) \\
&= N_{\emptyset}^* \cap \left(N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} \cap \mathfrak{C}_0 \right) = N_{\emptyset}^*.
\end{aligned}$$

[Why? The first equality holds by the choice of β_0 . The second one holds as $N_{\{\varepsilon(0), \zeta(0)\}} \subseteq \mathfrak{C}_0$ and the first equality. The third equality holds by the choice of β_0 . The fourth equality holds by the choice of $\alpha_{\varepsilon(0)}$. The fifth equality holds by the choice of N_{\emptyset}^* , i.e., by $(*)_2$. Finally, the sixth equality holds as $N_{\{\alpha_{\varepsilon(1)}, \beta_1\}} \supseteq N_{\emptyset}^*$ and by the choice of $\alpha_{\varepsilon(0)}$.

Recalling •₁ and the division to cases in •₂, •₃ and •₄, we are done proving $(*)_6$.]

$(*)_7$ if $\varepsilon < \zeta(1) < \zeta(2) < \partial$, then $N_{\{\varepsilon, \zeta(1)\}} \cap N_{\varepsilon, \zeta(2)} = N_{\{\varepsilon\}}^\uparrow := N_{\{\alpha_\varepsilon, \beta_1\}} \upharpoonright \mathfrak{C}_0$.

[Why? By the choice of $\alpha_{\zeta(2)}$ and $\alpha_{\zeta(1)}$.]

$(*)_8$ if $\zeta_1 < \zeta_2 < \varepsilon < \partial$, then $N_{\{\zeta_1, \varepsilon\}}^* \cap N_{\{\zeta_2, \varepsilon\}}^* = N_{\{\varepsilon\}}^\downarrow$, where $N_{\{\varepsilon\}}^\downarrow := f_{\varepsilon, \varepsilon+1}^{-1}(N_{\{\beta_0, \beta_1\}})$.

[Why? For $\zeta < \partial$, $N_{\{\alpha_\zeta, \beta_0\}} \cap \mathfrak{C}_0 = N_{\{\alpha_{\zeta_3}, \beta_1\}} \cap \mathfrak{C}_0$ by the choice of β_0 , and $\alpha_\varepsilon, \beta_0$ realize the same type of \mathfrak{C}_* over $\{\beta_1\} \cup (A_\varepsilon \cap \mathfrak{C}_0)$.]

$(*)_9$ • Let $N_{\{\varepsilon\}}^*$ be the $\text{Sk}(N_{\{\varepsilon\}}^\uparrow \cup N_{\{\varepsilon\}}^\downarrow, \mathfrak{C}_*)$, and

- let $M_\varepsilon^* = \left(\text{Sk}(\bigcup_{\ell < 5} N_{\{5\varepsilon+\ell\}}^* \cup \{N_{\{5\varepsilon+m, \delta\varepsilon+n\}} : m < n < 5\}^*) \right)_{\ell < 5}$,
- let M_ε^+ be M_ε^* expanded by:
- $c_\ell^{M_\varepsilon^+} = \alpha_{5\varepsilon+\ell}$ for $\ell < 5$,
- $p_\ell^{M_\varepsilon^+} = |N_{\{5\varepsilon+\ell\}}^*|$ for $\ell < 5$,
- $P_{m,n}^{M_\varepsilon^+} = |N_{\{5\varepsilon+m, 5\varepsilon+n\}}^*|$.

$(*)_{10}$ There is some $\mathcal{U}_1 \in [\partial]^\partial$ such that:

- $\langle M_\varepsilon^* : \varepsilon \in \mathcal{U}_1 \rangle$ is a Δ -system with heart N_{\emptyset}^* ,
- the M_ε are pairwise isomorphic.

[Why? Because $\partial = \partial_0$ and $\partial_0 = (\partial_0)^+$ by the Δ -system lemma.]

$(*)_{11}$ $\langle N_u^* : u \in \mathcal{U}_2 \rangle$ is a required when $\mathcal{U}_2 = \{5\varepsilon+2 : \varepsilon \in \mathcal{U}_1\}$ and $N_{\{5\varepsilon+2\}}^* = M_\varepsilon^*$.

Pedantically, $\mathcal{U}_3 = \{\alpha_\zeta : \zeta \in \mathcal{U}_2\}$ and $N_{\{\alpha_\zeta : \zeta \in u\}}^* = N_u^*$ for $u \in [\mathcal{U}_3]^{\leq 2}$. $\square_{1.12}$

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EINSTEIN INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, 9190401,
JERUSALEM, ISRAEL; AND, DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY,
NJ 08854-8019, USA

URL: <https://shelah.logic.at/>