PARTITION THEOREMS FOR EXPANDED TREES
1176

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Abstract. We look for partition theorems for large subtrees for suitable uncountable trees and colourings parallely to the statement $\lambda \to (\mu)^n_\kappa$ such that possibly $\lambda > \mu$.
We concentrate on sub-trees of ${}^{\kappa \geq}2$ expanded by a well-ordering of each level. However, in the embedding the equality of levels is preserved. The gain is that we get consistency results without large cardinals.
An intention is to apply the results to model theoretic problems.

Annotated Content

[We consider here partition theorems for trees. This is used in §1B.
In 1.1 define **T**.
Definition 1.5 states the partition relation.]

[We work on a partition theorem for trees, for the main case here, we get consistency without the large cardinal. Naturally the price is having to vary the size of the cardinals (parallel to the Erdös-Rado theorem).]

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§ 0. Introduction

§ 0(A). Background and Results.

We continue two lines of research. One is set theoretic: pure partition relations on trees and the other is model theoretic: Hanf numbers and non-deniability of well ordering, in particular related to ω_1. This is related to the existence of GEM (generalized Eherenfuecht-Mostowski) for suitable templates (see [Shec]), and applications to descriptive set theory.

Halpern-Levy [HL71] had proved a milestone theorem on independence of versions of the axiom of choice: in ZF, AC is strictly stronger than the maximal prime ideal theorem (i.e. every Boolean algebra has a maximal ideal).

This work isolated a partition theorem[1] on the tree ${}^{\omega>}2$, necessary for the proof. This partition theorem was subsequently proved by Halpern-Lauchli [HL66] and was a major and early theorem in Ramsey theory, (so the proof above relies on it).

See more in Laver [Lav71], [Lav73] and [She78b, AP,§2] and Milliken [Mil79], [Mil81].

The [HL66] proof uses induction, later Harrington found a different proof using forcing: adding many Cohen reals and a name of a (non-principal) ultrafilter on $\mathbb{N}$. Earlier, (on adding many reals and a partition theorem) see Silver's proof of Π^1_1-equivalence relations, in [Sil80].

Now [She92, §4] turns to uncountable trees, i.e. for some $\kappa > \aleph_0$, we consider trees $\mathscr{T}$ which are sub-trees of $({}^{\kappa>}2, \triangleleft)$, such that (as in [HL66]) for every level $\varepsilon < \kappa$, either $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\eta\hat{\ }\langle 0\rangle, \eta\hat{\ }\langle 1\rangle \in \mathscr{T})$ or $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\exists! \iota < 2)[\eta\hat{\ }\langle \iota\rangle \in \mathscr{T}]$; (and of course the first occurs unboundedly often). But a new point is that we have to use a well ordering of $\mathscr{T} \cap {}^{\varepsilon}2$ for $\varepsilon < \kappa$.

Naturally we add "is closed enough (that is under unions of increasing sequences of length $< \kappa$)". Also colouring with infinite number of colours, the proof uses "measurable κ which remains so when we add λ many κ-Cohens for appropriate λ"; it generalizes Harrington's proof. This was continued in several works, see Dobrinen-Hathaway [DH17] and references there.

We are here mainly interested in a weaker version which is enough for the model theoretic applications we have in mind, we start with a large tree and get one of smaller cardinality, in a sense this is solving the "equations" X /(Erdös-Rado theorem) = [She92]/ (the partition relation of a weakly compact cardinal) = [HL66] / (Ramsey theorem). On other consistent partition relation see Boney-Shelah [S$^+$a], in preparation.

Turning to model theory see [She75], [She76] and Dzamonja-Shelah [DS04] where such indiscernibility is considered in a model theoretic context.

A central direction in model theory in the sixties were two cardinal theorems. For infinite cardinals $\mu > \lambda$, let $K_{\mu,\lambda}$ be the class of models M such that M is of cardinality μ and P^M of cardinality λ. The main problems were transfer, compactness and completeness. For connection to partition theorems, Morley's proof of [Vau65], the Vaught far apart two cardinal theorem used Erdös-Rado theorem; generally see [She71b], [She71a], [She78a] and the survey [DS79]. Jensen's celebrated gap n two

[1] Using not splitting to 2 but other finite splitting makes a minor difference; similarly here.

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cardinal theorem solves those problems for e.g $(\aleph_n, \aleph_0)$ when $\mathbf{V} = \mathbf{L}$. But can we get a nice picture in different universes?

Note that by [She89], [Shec], consistently we have GEM (generalized Eherenfuecht-Mostowski) models for ordered graphs as index models, even omitting types.

On a different direction D.Ulrich has asked me on $(*)_n$ below (and told me it has descriptive set theoretic consequences, see [SU19]). We intend to prove (in the sequel [S⁺b]) that for $n < \omega$:

- $(*)_n$ consistently
  - (a) if $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ has a model M of cardinality $\beth_{n+1}$ with $(P^M, <^M)$ having order type ω_1 then ψ has a model N of cardinality $\beth_{n+1}$ and $(P^N, <^N)$ is not well ordered,
  - (b) moreover, it is enough that M will have cardinality $\aleph_\delta, \delta \geq \beth_n^{++}$,
  - (c) of course, preferably not using large cardinals.

This requires consistency of many cases of partition relations on trees and more complicated structures, analysing GEM models. Much earlier we have intended (mentioned in [She00, 1.15]) to prove the parallel for first order logic; and $(\beth_n, \aleph_0)$, using (many Cohen indestructible) measurables $\kappa_1 < \cdots < \kappa_n$ as in [She92, §4] and forcing by blowing $2^{\aleph_0}$ to κ_1, 2^{κ_1} to κ_2 etc relying on [She92]; but have not carried out that.

In preparation are also solutions to the two cardinal problems above and

- $(*)$
  - (a) $\alpha_\bullet < \omega_1, \beth_{\alpha+1} = (\beth_\alpha)^{+\omega_1+1}$ for $\alpha < \alpha_\bullet$ and well ordering of ω_1 is not definable in $\{\mathrm{EC}_\psi(\beth_{\alpha_\bullet}) \colon \psi \in \mathbb{L}_{\aleph_1,\aleph_0}\}$ or at least,
  - (b) as above but for $\beth_{\alpha+1} = \aleph_{\beth_\alpha^{++}}$,
  - (c) parallel results replacing $\aleph_0$ by μ.

Contrary to the a priori expectation no large cardinal is used.

In a sequel [S⁺c] we intend also to deal with other partition relations and with weakly compact cardinals.

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§ 0(B). Preliminaries.

Definition 0.1. If $\mu = \mu^{<\kappa}$ then "*for a* (μ,κ)*-club of* $u \subseteq X$ *we have* $\varphi(u)$" means that: for some χ such that $\mu, X \in \mathscr{H}(\chi)$ and e.g. $\beth_3(\mu + |X|) < \chi$ and some $x \in \mathscr{H}(\chi)$, if $x \in \mathscr{B} \prec (\mathscr{H}(\chi), \in)$, $\|\mathscr{B}\| = \mu$, $[\mathscr{B}]^{<\kappa} \subseteq \mathscr{B}$ and $\mu + 1 \subseteq \mathscr{B}$, then the set $u = \mathscr{B} \cap X$ satisfies $\varphi(u)$; there are other variants.

Definition 0.2. For κ regular (usually $\kappa = \kappa^{<\kappa}$) and an ordinal γ, the forcing $\mathbb{P} = \mathrm{Cohen}(\kappa,\gamma)$ of adding γ many κ-Cohen reals is defined as follows:

- (A) $p \in \mathbb{P}$ iff:
  - (a) p is a function with domain from $[\gamma]^{<\kappa}$,
  - (b) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \in {}^{\kappa>}2$,
- (B) $\mathbb{P} \models p \leq q$ iff:
  - (a) $p, q \in \mathbb{P}$,
  - (b) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
  - (c) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \trianglelefteq q(\alpha)$.
- (C) for $\alpha < \gamma$ let $\underset{\sim}{\eta}_\alpha = \bigcup\{p(\alpha) \colon p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ satisfies $\alpha \in \mathrm{dom}(p)\}$, so $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}_\alpha \in {}^\kappa 2$",

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(D) for $u \subseteq \gamma$ let $\mathbb{P}_u := \{p \in \mathbb{P}\colon \mathrm{dom}(p) \subseteq u\}$, so $\mathbb{P}_u \lessdot \mathbb{P}$ and $\bar{\eta}_u = \langle \underset{\sim}{\eta}_\alpha : \alpha \in u\rangle$ is generic for $\mathbb{P}_u$.

Notation 0.3. 1) We denote infinite cardinals by $\kappa, \lambda, \mu, \chi, \theta, \partial$, and σ denotes a possibly finite cardinal.

2) We denote ordinals by $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi$ and sometimes i, j.

3) We denote natural numbers by k, ℓ, m, n and sometimes i, j.

4) Instead of e.g. a_i we may write $a[i]$, particularly in sub-script; also $\kappa(+)$ means κ^+.

5) Let $h[u]; = \{h(x) : x \in u\}$.

Notation 0.4. Concerning Definition 1.1:

1) Let $\mathscr{T}$ denote members of $\mathbf{T}_{\mathrm{fl}}$ or of $\mathbf{T}_{\mathrm{wk}}$, writing $\mathbf{T}$ mean it can be either.

2) Similarly about $\mathscr{T}_1 \subseteq_{\mathrm{wk}} \mathscr{T}_2$ or $\mathscr{T}_1 \subseteq_{\mathrm{fl}} \mathscr{T}_2$ (or embedability).

3) Also, in Definition 1.5, we have either $\rightarrow_{\mathrm{fl}}$ or $\rightarrow_{\mathrm{wk}}$.

§ 1. Partition Theorems

§ 1(A). **The definitions.**

Here, we consider partitions on trees. For uncountable trees, we find the need to consider a well-ordering of each level, still preserving equality of level. We may consider embeddings where equality of levels is not preserved, see Dzamonja-Shelah [DS04] (in the web version). This will suffice for the intended model theory application. Also, we may waive the completeness of the tree, but usually still like to have many branches.

We intend to deal with an intermediate one (and with the weakly compact cardinal) in a sequel [S⁺c].

Definition 1.1. 1) Let $\mathbf{T}_{\mathrm{fl}}$ be the class of structures $\mathscr{T}$ such that:

(a) $\mathscr{T} = (u, <_*, E, <, \cap, S, R_0, R_1) = (u_{\mathscr{T}}, <^*_{\mathscr{T}}, E_{\mathscr{T}}, <_{\mathscr{T}}, \cap_{\mathscr{T}}, S_{\mathscr{T}}, R^0_{\mathscr{T}}, R^1_{\mathscr{T}})$ but we may write $s \in \mathscr{T}$ instead of $s \in u$,

(b) $(u, <_*)$ is a well ordering, so linear, u non-empty,

(c) $<_{\mathscr{T}}$ is a partial order included in $<_*$,

(d) $(u, <_{\mathscr{T}})$ is a tree, i.e. if $t \in \mathscr{T}$ <u>then</u> $\{s : s <_{\mathscr{T}} t\}$ is well ordered by $<_{\mathscr{T}}$; the level of t is the order type of this set; the tree is with $\mathrm{ht}(\mathscr{T})$ levels,

Also the tree is normal; that is: if $t_1, t_2 \in \mathscr{T}$ and $\{s : s \leq_{\mathscr{T}} t_1\} = \{s : s \leq_{\mathscr{T}} t_2\}$ and this set has no $<_{\mathscr{T}}$-last element then $t_1 = t_2$.

(e) E is an equivalence relation on u, convex under $<_*$,

(f) (α) each E-equivalence class is the set of $t \in \mathscr{T}$ of level ε for some ε, so the set of E-equivalence classes is naturally well ordered,

(β) we denote the ε-th equivalence class by $\mathscr{T}_{[\varepsilon]}$,

(γ) E has no last E-equivalence class if not said otherwise,

(δ) let $\mathrm{lev}_{\mathscr{T}}(s) = \mathrm{lev}(s, \mathscr{T})$ be ε when $s \in \mathscr{T}_{[\varepsilon]}$, equivalently $\{t\colon t <_{\mathscr{T}} s\}$ has order type ε under the order $<_{\mathscr{T}}$,

(ε) so $\mathrm{ht}(\mathscr{T})$ is $\bigcup\{\mathrm{lev}_{\mathscr{T}}(s) + 1\colon s \in \mathscr{T}\}$ and it is a limit ordinal if not said otherwise.

(g) if $s \in u, \mathrm{lev}_{\mathscr{T}}(s) < \zeta < \mathrm{ht}(\mathscr{T})$ then there is $t \in \mathscr{T}_{[\zeta]}$ which is $<_{\mathscr{T}}$-above s,

(h) each $s \in \mathscr{T}$ has exactly two immediate successors by $<_{\mathscr{T}}$,

(i) for $s \in \mathscr{T}$ we let:
   $\bullet_1$ $\mathscr{T}_{\geq s} = \{t \in \mathscr{T} : s \leq_{\mathscr{T}} t\}$,
   $\bullet_2$ $\mathrm{suc}_{\mathscr{T}}(s) = \{t : t \in \mathscr{T}_{[\mathrm{lev}(s)+1]}$ satisfies $s <_{\mathscr{T}} t\}$,

(j) let $s = t{\restriction}\varepsilon$ mean that $\mathrm{lev}_{\mathscr{T}}(s) = \varepsilon \leq \mathrm{lev}_{\mathscr{T}}(t) \wedge (s \leq_{\mathscr{T}} t)$,

(k) for $t_1, t_2 \in \mathscr{T}, t_1 \cap_{\mathscr{T}} t_2$ is the maximal common lower bound of t_1, t_2 (under $\leq_{\mathscr{T}}$) so we demand it always exists, i.e. $(\mathscr{T}, <)$ is normal,

(l) for $\ell = 0, 1$ we have $R_\ell \subseteq \{(s,t) \colon s \in \mathscr{T}$ and $s <_{\mathscr{T}} t\}$, and if $s <_{\mathscr{T}} s_1 <_{\mathscr{T}} s_2$, then $sR_\ell s_1$ iff $sR_\ell s_2$,

(m) if $s \in \mathscr{T}$ then for some $t_0 \neq t_1$ we have $\mathrm{suc}_{\mathscr{T}}(s) = \{t_0, t_1\}$ and $\ell < 2 \Rightarrow (\forall t)(sR_\ell t$ <u>iff</u> $t_\ell \leq_{\mathscr{T}} t)$; so $sR_\ell t$ is the analog to $\eta\hat{\ }\langle \ell \rangle \trianglelefteq \nu$; we may think of $\{t \colon sR_\ell t\}$ as a division to the left side and the right side of the set of the t's above s.

1A) We define $\mathbf{T}_{\mathrm{wk}}$ similarly ("wk" stands for "weak"), but we omit clauses (h), (m), replacing them with:

(h)' is $s \in \mathscr{T}$ and $\mathrm{lev}_{\mathscr{T}}(s) + \omega \leq \mathrm{ht}(\mathscr{T})$, <u>then</u> there is $t \in \mathrm{spl}(\mathscr{T})$ such that $s \leq_{\mathscr{T}} t$, where $\mathrm{spl}(\mathscr{T}) = \{t \in \mathscr{T} : t$ has two immediate successors$\}$, and

(m)' for $s \in \mathscr{T}$ either $\mathrm{suc}_{\mathscr{T}}(s)$ may be as in part (1) or is empty or a singleton.

1B) For $\mathscr{T} \in \mathbf{T}, <_{\mathrm{lex}} := <^{\mathrm{lex}}_{\mathscr{T}}$ is the lexicographic order, i.e.,

$$\eta <_{\mathrm{lex}} \nu \text{ \underline{iff} } (\exists \rho)(\rho\, R_0\, \nu \wedge \rho\, R_1\, \eta) \text{ or } (\eta <_{\mathscr{T}} \nu \wedge \eta R_1 \nu) \text{ or } (\nu <_{\mathscr{T}} \eta \wedge \nu R_0 \eta).$$

2) Let $\mathbf{T}_{\theta,\kappa} = \{\mathscr{T} \in \mathbf{T} \colon$ the tree $\mathscr{T}$ has δ levels, for some ordinal δ of cofinality κ and for every $\varepsilon < \delta$ we have $\theta > |\{s \in \mathscr{T} \colon s$ of level $\leq \varepsilon\}|\}$.

3) Let $\mathscr{T}_1 \subseteq_{\mathrm{fl}} \mathscr{T}_2$ mean:

(a) $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$,
(b) $<_{\mathscr{T}_1} := <_{\mathscr{T}_2} {\restriction} u_{\mathscr{T}_1}$,
(c) if $\mathscr{T}_1 \models$ "$\eta \cap \nu = \rho$" then $\mathscr{T}_2 \models$ "$\eta \cap \nu = \rho$",
(d) $R_{\mathscr{T}_1,\ell} = R_{\mathscr{T}_2,\ell} {\restriction} u_{\mathscr{T}_1}$ for $\ell = 0, 1$,
(e) $<^*_{\mathscr{T}_1} := <^*_{\mathscr{T}_2} {\restriction} u_{\mathscr{T}_1}$;
(f) $E_{\mathscr{T}_1} := E_{\mathscr{T}_2} {\restriction} u_{\mathscr{T}_1}$,

3A) Let $\mathscr{T}_1 \subseteq_{\mathrm{wk}} \mathscr{T}_2$ is defined similarly omitting clause (f).

4) For $s \in \mathscr{T}$ and $\ell \in \{0, 1\}$, let $\mathrm{suc}_{\mathscr{T},\ell}(s)$ be the unique immediate successor of s in $\mathscr{T}$ such that $(s,t) \in R^{\mathscr{T}}_\ell$.

5) We say $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ are *neighbors* <u>when</u> they are equal except that for each $t \in \mathscr{T}_1$ we can change the order $<^*_{\mathscr{T}_1} {\restriction} (t/E_{\mathscr{T}_1})$ to $<^*_{\mathscr{T}_2} (t/E_{\mathscr{T}_2})$.

Definition 1.2. 1) We say f is a $\subseteq$-embedding of $\mathscr{T}_1 \in \mathbf{T}$ into $\mathscr{T}_2 \in \mathbf{T}$ <u>when</u>: f is an isomorphism from $\mathscr{T}_1$ onto $\mathscr{T}'_1$ where $\mathscr{T}'_1 \subseteq \mathscr{T}_2$.

1A) We say that f is a *semi embedding* of $\mathscr{T}_1 \in \mathbf{T}$ into $\mathscr{T}_2 \in \mathbf{T}$, <u>when</u> f is a $\subseteq$-embedding of $\mathscr{T}_1$ into some neighbour $\mathscr{T}'_2$ of $\mathscr{T}_2$.

2) For any ordinal α (limit, if not said otherwise) and sequence $\bar{<} = \langle <_\beta \colon \beta < \alpha \rangle$, with $<_\beta$ a well ordering of ${}^\beta 2$ we define $\mathscr{T} = \mathscr{T}_{\alpha,\bar{<}}$ as follows (omitting $\bar{<}$ means "for some"):

(a) universe ${}^{\alpha >}2$,
(b) $<_{\mathscr{T}}$ is $\triangleleft{\restriction}{}^{\alpha>}2$,
(c) $E_{\mathscr{T}} := \{(\eta,\nu) \colon \eta, \nu \in {}^\beta 2$ for some $\beta < \alpha\}$,

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(d) $<^*_{\mathscr{T}} := \{(\eta,\nu) \colon \eta,\nu \in {}^{\alpha>}2$ and $\ell g(\eta) < \ell g(\nu)$ or $(\exists \beta < \alpha)(\ell g(\eta) = \beta = \ell g(\nu) \wedge \eta <_\beta \nu)\}$,
(e) $R_\ell := \{(\eta,\nu) \colon \eta\hat{\ }\langle \ell \rangle \trianglelefteq \nu \in \mathscr{T}\}$,
(f) $\eta \cap_{\mathscr{T}} \nu := \eta \cap \nu$.

3) For $\mathscr{T} \in \mathbf{T}$ and $\zeta < \mathrm{ht}(\mathscr{T})$, let $<_{\mathscr{T},\zeta}$ be $<^*_{\mathscr{T}} \restriction \mathscr{T}_{[\zeta]}$.

4) For any $\mathscr{T} \in \mathbf{T}$, let

(a) $\mathrm{sub}_1(\mathscr{T}) := \{\mathscr{U} \subseteq \mathscr{T} : \mathscr{U} \neq \emptyset$ and for every $s_1 \in \mathscr{U}, s_2 \in \mathscr{T}$ there is $t \in \mathscr{U}$ such that $s_1 <_{\mathscr{T}} t \wedge \mathrm{lev}_{\mathscr{T}}(s_2) \leq \mathrm{lev}_{\mathscr{T}}(t)\}$,
(b) $\mathrm{sub}_2(\mathscr{T})$ is the set of $\mathscr{U} \in \mathrm{sub}_1(\mathscr{T})$ such that: if $s_1, s_2 \in \mathscr{U}, \mathrm{lev}_{\mathscr{T}}(s_1) = \mathrm{lev}_{\mathscr{T}}(s_2)$ and $|\mathrm{suc}_{\mathscr{T}}(s_1) \cap \mathscr{U}| = 2 = |\mathrm{suc}_{\mathscr{T}}(s_2) \cap \mathscr{U}|$ <u>then</u> $s_1 = s_2$.

5) We say $\mathscr{U} \subseteq \mathscr{T}$ is complete (in $\mathscr{T}$) <u>when</u> if $t \in \mathscr{T}_{[\delta]}, \delta < \mathrm{ht}(\mathscr{T})$ is a limit ordinal and $\{s : s < t\} \subseteq \mathscr{U}$, then $t \in \mathscr{U}$. We say that $\mathscr{T}$ is complete when $u_{\mathscr{T}}$ is complete.

Claim 1.3. *1) If $\theta = \sup\{(2^{|\alpha|})^+ : \alpha < \kappa\}$ and $\bar{<} = \langle <_\beta \colon \beta < \kappa \rangle$ as in 1.2(2) above, <u>then</u> $\mathscr{T}_{\kappa,\bar{<}}$ is well defined and belongs to $\mathbf{T}_{\theta,\kappa}$.*

2) If $\kappa = \kappa^{<\kappa}, \mathscr{T} = \mathscr{T}_{\kappa,\bar{<}}$ are as in part (1), <u>then</u> there is $\mathscr{U} \in \mathrm{sub}_2(\mathscr{T})$ which is complete in $\mathscr{T}$ and for every $s \in \mathscr{U}$ there exists $t \in \mathscr{T}$ such that $s <_{\mathscr{T}} t, \mathrm{lev}_{\mathscr{T}}(t) < \mathrm{lev}_{\mathscr{T}}(s) + 2^{\mathrm{lev}_{\mathscr{T}}(s)}$ and $|\mathrm{suc}_{\mathscr{T}}(t) \cap \mathscr{U}| = 2$.

It follows that $\mathscr{U}$ and $\mathscr{T}$ have the same number of κ-branches. Note however that, $\mathscr{T} \restriction \mathscr{U} \notin \mathbf{T}$ because clause (m) of 1.1(1) may fail still we may demand $\mathscr{T} \restriction \mathscr{U} \in \mathbf{T}_{\mathrm{wk}}$ (see 1.1(1A)).

3) For every $\mathscr{T}_1 \in \mathbf{T}$ such that $\delta = \mathrm{ht}(\mathscr{T}_1)$ satisfying $\alpha < \delta \Rightarrow \alpha\alpha < \delta$, there are $\mathscr{T}_{\mathrm{wk}} \in \mathbf{T}$ and h such that $\mathrm{ht}(\mathscr{T}_1) = \mathrm{ht}(\mathscr{T})$ and $\mathscr{U} \in \mathrm{sub}_2(\mathscr{T})$ and h is an $\subseteq_{\mathrm{wk}}$-embedding of $\mathscr{T}_1$ into $\mathscr{T}$ with range $\mathscr{U}$ and $|\mathscr{U}| = |\mathscr{T}_1| = |\mathscr{T}|$.

Proof. It is clear. $\square_{1.3}$

Definition 1.4. 1) For $\mathscr{T} \in \mathbf{T}$ let $\mathrm{eseq}_n(\mathscr{T})$ be the set of sequences $\bar{a}$ such that:

(a) $\bar{a}$ is an $<^*_{\mathscr{T}}$-increasing sequence of length n of members of $\mathscr{T}$,
(b) $k < \ell < n \Rightarrow a_k \cap a_\ell \in \{a_m : m < n\}$; (in fact, "$\in \{a_m : m \leq k\}$").
(c) $k, \ell < n \wedge \mathrm{lev}(a_k) \leq \mathrm{lev}(a_\ell) \Rightarrow a_\ell \restriction \mathrm{lev}(a_k) \in \{a_m \colon m < n\}$,

1A) For $\mathscr{U} \subseteq \mathscr{T}$ we let $\mathrm{eseq}_n(\mathscr{U}, \mathscr{T})$ be $\mathrm{eseq}_n(\mathscr{T}) \cap ({}^n\mathscr{U})$, similarly for part (2).

2) Let $\mathrm{eseq}(\mathscr{T}) = \mathrm{eseq}_{<\omega}(\mathscr{T}) = \cup\{\mathrm{eseq}_n(\mathscr{T}) \colon n < \omega\}$.

2A) For finite $A \subseteq \mathscr{T}$ we define the sequence $\bar{b} = \mathrm{cl}(A) = \mathrm{cl}_{\mathscr{T}}(A) = \mathrm{cl}(A, \mathscr{T})$ as the unique $\bar{b}$ such that:

(a) $\bar{b} \in \mathrm{eseq}(\mathscr{T})$,
(b) $A \subseteq \mathrm{Rang}(\bar{b})$,
(c) $\mathrm{Rang}(\bar{b})$ is minimal under those restrictions.

Also let $\mathrm{pos}(A) = \mathrm{pos}(A, \mathscr{T})$ be the unique function h from A into $\lg(\bar{b})$ such that for every $a \in A$ we have: $i = h(a)$ <u>iff</u> $b_i = a$.

2B) We may replace above A by a finite sequence $\bar{a}$, and let $c\ell_{\mathscr{T}}(\bar{a})$ be $c\ell_{\mathscr{T}}(\mathrm{rang}(\bar{a}))$ and $\mathrm{pos}(\bar{a})$ be the function mapping $\ell < \lg(\bar{a})$ to k iff $a_\ell = b_k$.

3) We say $\bar{a}, \bar{b} \in \mathrm{eseq}(\mathscr{T})$ are $\mathscr{T}$-similar or $\bar{a} \sim_{\mathscr{T}} \bar{b}$ <u>when</u> for some n we have:

(a) $\bar{a}, \bar{b} \in \mathrm{eseq}_n(\mathscr{T})$,

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(b) for any $k, i, m < n$ we have (notice that, $<^*_{\mathscr{T}}$ is not mentioned):
- $\bullet_1$ $a_k \leq_{\mathscr{T}} a_i$ iff $b_k \leq_{\mathscr{T}} b_i$,
- $\bullet_2$ $(a_k, a_i) \in R^{\mathscr{T}}_\ell$ iff $(b_k, b_i) \in R^{\mathscr{T}}_\ell$ for $\ell = 0, 1$,
- $\bullet_3$ $a_k \cap_{\mathscr{T}} a_\ell = a_m$ iff $b_k \cap_{\mathscr{T}} b_\ell = b_m$, actually follows,
- $\bullet_4$ $a_k = a_\ell \restriction \mathrm{lev}(a_m)$ iff $b_k = b_\ell \restriction \mathrm{lev}(b_m)$,
- $\bullet_5$ $(a_k \cap a_m) R_{\mathscr{T},\ell} a_i$ iff $(b_k \cap b_m) R_{\mathscr{T},\ell} b_i$ for $\ell = 0, 1$; actually follows,
- $\bullet_6$ $\mathrm{lev}_{\mathscr{T}}(a_k) \leq \mathrm{lev}_{\mathscr{T}}(a_\ell)$ iff $\mathrm{lev}_{\mathscr{T}}(b_k) \leq \mathrm{lev}_{\mathscr{T}}(b_\ell)$; actually follows,
- $\bullet_7$ $a_k <^*_{\mathscr{T}} a_i$ iff $b_k <^*_{\mathscr{T}} b_i$.

3A) We say that $\bar a, \bar b \in {}^n\mathscr{T}$ are $\mathscr{T}$*-similar* when $\bar a' = \mathrm{cl}(\bar a), \bar b' = \mathrm{cl}(\bar b)$ are $\mathscr{T}$-similar and $a'_\ell = a_k \Leftrightarrow b'_\ell = b_k$ for any $\ell < \lg(\bar a'), k < n$.

4) For $\bar a \in {}^n\mathscr{T}$, let $\mathrm{Lev}(\bar a)$ be the set $\{\mathrm{lev}_{\mathscr{T}}(a_\ell) \colon \ell < n\}$.

5) We say that $\mathscr{T} \in \mathbf{T}$ is weakly $\aleph_0$-saturated when:

(∗) for every $\varepsilon < \mathrm{ht}(\mathscr{T})$ and $s_0, \dots, s_{n-1}$ from $\mathscr{T}_{[\varepsilon]}$, there are $\zeta \in (\varepsilon, \mathrm{ht}(\mathscr{T}))$ and $t_0 <^*_{\mathscr{T}} \cdots <^*_{\mathscr{T}} t_{n-1}$ from $\mathscr{T}_{[\zeta]}$ satisfying $k < n \Rightarrow s_k <_{\mathscr{T}} t_k$,

6) For $\mathscr{T} \in \mathbf{T}$ let:

(a) $\mathrm{incr}_n(\mathscr{T})$ be the set of $<^*_{\mathscr{T}}$-increasing $\bar a \in {}^n\mathscr{T}$ and let ${}^n\mathscr{T}$ be the set of sequences of length n from $\mathscr{T}$,
(b) $\mathrm{incr}(\mathscr{T}) = \cup\{\mathrm{incr}_n(\mathscr{T}) \colon n < \omega\}$ and $\mathrm{seq}(\mathscr{T}) = \cup\{{}^n\mathscr{T} : n < \omega\}$.

7) For $\mathscr{T} \in \mathbf{T}$:

(a) for $\bar t \in \mathrm{incr}(\mathscr{T})$ or just $\mathrm{seq}(\mathscr{T})$, let $\mathrm{sim}-\mathrm{tp}(\bar t, \mathscr{T})$ be the pair (the similarity type of $\mathrm{cl}(\bar t, \mathscr{T}), \mathrm{pos}(\bar t, \mathscr{T})$), that is all the information from part (3) (of 1.4) and pos,
(b) if in addition, $\mathscr{U} \subseteq \mathscr{T}$ then we let $\mathrm{sim}-\mathrm{tp}(\bar t, \mathscr{U}, \mathscr{T})$ be the function mapping $\bar s \in {}^{\omega>}\mathscr{U}$ to $\mathrm{sim}-\mathrm{tp}(\bar t\hat{\ }\bar s, \mathscr{T})$.

8) Let $\mathbb{S}^n$ be the set of similarity types of sequences of length n in some $\mathscr{T} \in \mathbf{T}$, so the sequences are not necessarily increasing.

9) Naturally $\mathbb{S} = \cup\{\mathbb{S}^n \colon n < \omega\}$.

10) For $\mathscr{T} \in \mathbf{T}$ and $n < \omega$ we define $\mathrm{fseq}_n(\mathscr{T})$ as the set of sequences $\bar s = \langle s_\eta : \eta \in \cup\{{}^m2 : m \leq n\}\rangle$ such that for some ordinals $\alpha(0) < \dots \alpha(n)$ we hav:

- $\bullet_1$ $s_\eta \in \mathscr{T}$ is of level $\alpha(\lg(\eta))$,
- $\bullet_2$ $s_\eta <_{\mathscr{T}} s_\nu$ when η is a proper initial segment of ν,
- $\bullet_3$ for $\iota = 0, 1$ above $s_\eta R^{\mathscr{T}}_\iota s_\nu$ iff $\nu(\lg(\eta))\iota$.

11) The similarity type of $\bar s \in \mathrm{fseq}(\mathscr{T})$ is the following linear order on $\cup\{{}^m2 : m \leq n\}$:

$\eta <_{\bar s} \nu$ iff $\lg(\eta) < \lg(\nu)$ or $\lg(\eta) < \lg(\nu)$ and $\eta < *_{\mathscr{T}}\nu$.

So th enumber of similarity types on members of $\mathrm{fseq}_n(\mathscr{T})$ is $\Pi_{m \leq n}(2^m!)$, and we can express our partition relations using $\mathrm{fseq}_n(\mathscr{T})$
instead $\mathrm{eseq}_n(\mathscr{T})$.

Now comes the main property.

Definition 1.5. 1) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}_{\mathrm{fl}}$ and $n < \omega$ and a cardinal σ let $\mathscr{T}_2 \to_{\mathrm{fl}} (\mathscr{T}_1)^n_\sigma$ mean:

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(∗) if $\mathbf{c}\colon \mathrm{eseq}_n(\mathscr{T}_2) \to \sigma$, <u>then</u> there is a $\subseteq_{\mathrm{fl}}$-embedding g of $\mathscr{T}_1$ into $\mathscr{T}_2$ such that the colouring $\mathbf{c} \circ g$ is homogeneous for $\mathscr{T}_1$, which means:
- if $\bar{a}, \bar{b} \in \mathrm{eseq}_n(\mathscr{T}_1)$ are $\mathscr{T}_2$-similar, then $\mathbf{c}(g(\bar{a})) = \mathbf{c}(g(\bar{b}))$.

2) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$, $k < \omega$ and σ, let $\mathscr{T}_2 \to_{\mathrm{fl}} (\mathscr{T}_1)^{\mathrm{end}(k)}_\sigma$ mean that:

(∗) if $\mathbf{c}\colon \mathrm{eseq}(\mathscr{T}_2) \to \sigma$ <u>then</u> there is an $\subseteq_{\mathrm{fl}}$-embedding g of $\mathscr{T}_1$ into $\mathscr{T}_2$ such that the colouring $\mathbf{c}' = \mathbf{c} \circ g$ (see below) satisfies $\mathbf{c}'(\bar{\eta})$ does not depend on the last k levels, that is:
- $\bullet_1$ the meaning of $\mathbf{c}' = \mathbf{c} \circ g$ is that for every $\bar{s} \in \mathrm{eseq}(\mathscr{T}_1)$ we have $\mathbf{c}'(\bar{s}) = \mathbf{c}(\langle g(\bar{s})\rangle)$,
- $\bullet_2$ if $n < \omega$ and $\bar{a}, \bar{b} \in \mathrm{eseq}_n(\mathscr{T}_2)$ are $\mathscr{T}_2$-similar and
$$\ell < n \wedge (k \leq |\mathrm{Lev}(\bar{a}) \setminus \mathrm{lev}(a_\ell)|) \Rightarrow b_\ell = a_\ell,$$
<u>then</u> $\mathbf{c}'(\bar{a}) = \mathbf{c}'(\bar{b})$.

3) Let $\mathscr{T}_2 \to_{\mathrm{fl}} (\mathscr{T}_1)^{\mathrm{end}(k,m)}_\sigma$ be defined as in part (2), but we restrict in $\bullet_2$ demanding that $n \leq m$, so the length of the relevant sequence $\bar{a} \in \mathrm{eseq}(\mathscr{T}_2)$ is bounded.

4) We define $\mathscr{T}_1 \to'_{\mathrm{fl}} (\mathscr{T}_1)^n_\sigma$ as in part (1), but $\mathbf{c}\colon {}^n(\mathscr{T}_2) \to \sigma$ and $\bar{a}, \bar{b} \in {}^n(\mathscr{T}_2)$.

5) We define similarly $\mathscr{T}_1 \to_{\mathrm{fl}} (\mathscr{T}_2)^{\leq n}_\sigma$ and $\mathscr{T}_1 \to'_{\mathrm{fl}} (\mathscr{T}_2)^{\leq n}_\sigma$.

6) We may replace $\mathbf{T}_{\mathrm{fl}}$, $\subseteq_{\mathrm{fl}}$, $\to_{\mathrm{fl}}$ by $\mathbf{T}_{\mathrm{wk}}$, $\subseteq_{\mathrm{wk}}, \to_{\mathrm{wk}}$ in parts (1)-(5) of Definition 1.5.

Remark 1.6. 1) We may mention some implications among the $\to$,

2) Of course, the equality $\mathbf{c}(g(\bar{a})) = \mathbf{c}(g(\bar{b}))$ is required only if $\bar{a}$ and $\bar{b}$ are $\mathscr{T}_2$-similar since this is the best possible homogeneity, as one can define a coloring according to similarity types.

Claim 1.7. *Let $\mathscr{T} \in \mathbf{T}$.*

1) If $A \subseteq \mathscr{T}$ is finite non-empty with m elements <u>then</u>:

(a) *For some $n \leq (2m-1)m^2$ and $\bar{a} \in \mathrm{eseq}_n(\mathscr{T})$ we have $A \subseteq \mathrm{Rang}(\bar{a})$; moreover $\max\{\mathrm{lev}_{\mathscr{T}}(a)\colon a \in A\} = \max\{\mathrm{lev}_{\mathscr{T}}(a_\ell)\colon \ell < n\}$; in fact $\bar{a} = \mathrm{cl}_{\mathscr{T}}(A)$,*

(b) *If $\mathscr{T} \in \mathbf{T}$ and $A \subseteq \mathscr{T}$ is finite, then $\mathrm{cl}(A, \mathscr{T}), \mathrm{pos}(A, \mathscr{T})$ are well defined.*

2) The number of quantifier free complete n-types realized in some $\mathscr{T} \in \mathbf{T}$ by some $\bar{a} \in \mathrm{eseq}_n(\mathscr{T})$ is, e.g. $\leq 2^{2n^2+n}$ but $\geq n$.

3) If $\mathscr{T} \in \mathbf{T}$ is weakly $\aleph_0$-saturated <u>then</u> $\mathscr{T}$ realizes all possible such types, i.e. each type is realized in some $\mathscr{T}' \in \mathbf{T}$; here "$\mathrm{ht}(\mathscr{T})$ is a limit ordinal" follows.

4) Assume $\mathscr{T} \in \mathbf{T}_{\mathrm{wk}}$ and $\mathscr{U}$ is a subset of $\mathscr{T}$ closed under $<_{\mathscr{T}}$, (that is $s <_{\mathscr{T}} t \in \mathscr{U} \Rightarrow s \in \mathscr{U}$). Let $\mathrm{lev}(\mathscr{U}, \mathscr{T}) = \sup\{\mathrm{lev}(s, \mathscr{T}) + 1\colon s \in \mathscr{U}\}$.

If $\mathrm{lev}(\mathscr{U}, \mathscr{T}) \leq \mathrm{lev}(t, \mathscr{T})$ and $A \subseteq \mathscr{U}$ is finite <u>then</u> $\bar{b} = \mathrm{cl}(A \cup \{t\})$ has the form $\bar{c}\hat{\ }\langle t\rangle$ with $\bar{c} \in \mathrm{eseq}(\mathscr{T}) \cap {}^{\omega >}\mathscr{U}$.

5) If $n < \omega, \mathscr{T}, \mathscr{U}$ are as in 1.3(2) <u>then</u> the number k^*_n of quantifier free complete n-types realized in $\mathscr{T}$ by sequences $\bar{a} \in \mathbf{I}_n$ satisfies the following $k^*_0 = 1 = k^*_1$ and $k^*_{n+1} = nk^*_n(n!)$ for $n \geq 1$, where

$\mathbf{I}_n = \mathbf{I}_n(\mathscr{T}) = \{\bar{a} \in {}^n\mathscr{U}: \bar{a}$ *is without repetitions and* $\langle \mathrm{lev}(a_i) : i < \mathrm{lg}(\bar{a})\rangle$ *is constant*$\}$.

6) If $\mathscr{T} \in \mathbf{T}_{\mathrm{wk}}$, $\mathscr{T} \in \mathrm{sub}_2(\mathscr{T})$ and $\bar{a} \in \mathrm{eseq}_n(\mathscr{T})$, <u>then</u> for every $\varepsilon < \mathrm{ht}(\mathscr{T})$ for at most one $\ell < \mathrm{lg}(\bar{a})$ we have $\bigwedge_{\ell=0}^{1}(\exists w)(aR^{\mathscr{T}}_\ell a_\ell)$ and......

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Proof. Clearly, (3) and (4) hold, and we shall use them freely.

1) Let $B_1 = \{\eta \cap_{\mathscr{T}} \nu \colon \eta, \nu \in A\}$ and note that $\eta \in A \Rightarrow \eta = \eta \cap \eta \in B_1$. Now by induction on $|A|$ easily $|B_1| \le 2m - 1$. Let $B_2 := \{\eta \restriction \mathrm{lev}_{\mathscr{T}}(\nu) \colon \eta \in A, \nu \in B_1$ and $\mathrm{lev}_{\mathscr{T}}(\eta) \ge \mathrm{lev}_{\mathscr{T}}(\nu)\}$.

Easily $B_2 = \mathrm{cl}(A, \mathscr{T})$, also $|B_2| \le m^2|B_1| = m^2(2m-1)$.

We may improve the bound[2] but this does not matter here; similarly below.

2) Considering the class of such pairs $(\bar{a}, \mathscr{T})$, (fixing n); the number of possible $E_{\bar{a}} = \{(k,i) : a_k E_{\mathscr{T}} a_i\}$ is $\le 2^{n^2}$ and the number of $<_{\bar{a}} = \{(k,i) : a_k <_{\mathscr{T}} a_i\}$ is $\le 2^{n^2}$ and the number of $\{(a_k, a_i) : (a_k, a_i) \in R_1^{\mathscr{T}}$ and for no $j, a_k <_{\mathscr{T}} a_j <_{\mathscr{T}} a_i\}$ is $\le 2^n$.

Lastly, from those we can compute $\{(a_k, a_i) : (a_k, a_i) \in R_0^{\mathscr{T}}\}$ as $\{(a_k, a_i) \colon (a_k \cap_{\mathscr{T}} a_i = a_k) \wedge ((a_k, a_i) \notin R_1^{\mathscr{T}}) \wedge a_k \neq a_i\}$, so together the number is $\le 2^{2n^2+n}$.

Clearly, we can get a better bound, e.g. letting $m_n^\bullet(\mathscr{T}) = |\{\mathrm{tp}_{\mathrm{qf}}(\bar{a} \restriction n, \emptyset, \mathscr{T}) : \bar{a} \in \mathrm{eseq}(\mathscr{T})$ has length $\ge n\}|$ then:

$(*)_1$
- $\bullet_1$ $m_n^\bullet(\mathscr{T}) = 1$ for $n = 0, 1$,
- $\bullet_2$ $m_{n+1}^\bullet(\mathscr{T}) \le 4n(m_n^\bullet(\mathscr{T}))$,
- $\bullet_3$ hence $m_n^\bullet(\mathscr{T}) \le 4^{n-1}(n-1)!$.

[Why? e.g. for $\bullet_2$ notice that $\mathrm{tp}_{\mathrm{qf}}(\bar{a} \restriction (n+1), \emptyset, \mathscr{T})$ is determined by $q = \mathrm{tp}_{\mathrm{qf}}(\bar{a} \restriction n, \emptyset, \mathscr{T})$ and the unique triple $(m, \iota, \ell) \in n \times 2 \times 2$ such that:

$(*)_{1.1}$
- (a) $m < n$ is such that $\mathrm{lev}(a_m \cap a_n)$ is maximal, hence $a_m <_{\mathscr{T}} a_n$,
- (b) $a_m R_\iota a_n$,
- (c) $\ell = 0$ iff $\mathrm{Lev}_{\mathscr{T}}(a_n) > \mathrm{Lev}_{\mathscr{T}}(a_{n-1})$.

As there are $\le 4n$ possibilities, we are done.]

It suffices to consider the case $\mathscr{T}$ is weakly $\aleph_0$-saturated (see 1.4(5), 1.7(3)) and then we can get exact values.

Now for $n \ge k \ge 1$ let,

$m_{n,k}^*(\mathscr{T}) := |\{\mathrm{tp}_{\mathrm{qf}}(\bar{a}, \emptyset, \mathscr{T}) \colon \bar{a} \in \mathrm{eseq}_n(\mathscr{T})$ such that $|\{\ell \colon \mathrm{lev}(a_\ell) = \max(\mathrm{Lev}(\bar{a}))\}| = k\}|$.

So,

- $m_{1,1}^*(\mathscr{T}) = 1$, $m_{1,0}^*(\mathscr{T}) = 0$ and stipulate $m_{0,k}^*(\mathscr{T}) = 0$,
- if $n = k \ge 1$, then $m_{n,k}^*(\mathscr{T}) = 1$,

---

[2] in fact the exact bound is:

- $\mathrm{cl}_{\mathscr{T}}(A) = m + (m-1) + \binom{m}{2} + \binom{m-1}{2}$ and it is obtained.

[Why is this bound? For any such A, define the sets $A[0] := A$, $A[1] := \{\eta \cap \nu \colon \eta \neq \nu \in A\}$, $A[2] := \{\eta \restriction \lg(\nu) \colon \eta, \nu \in A$ and $\lg(\nu) < \lg(\eta)\}$ and

$$A[3] := \{\eta \restriction \lg(\nu \cap \rho) \colon \eta, \nu \in A \text{ and } \eta \ntrianglelefteq \nu, \eta \ntrianglelefteq \rho \text{ and } \nu \neq \rho\}.$$

Early $\mathrm{cl}_{\mathscr{T}}(A) = A[0] \cup A[1] \cup A[2] \cup A[3]$; disjoint and we prove the inequality bu induction on m. For $m = 1$, $\mathrm{cl}(A) = A$, so it is clear. If $|A| = m = n+1$, let $a_0 <_I^* \cdots <_I^* a_m$ list A and let $B := A \setminus \{a_m\}$. Easily, $B[i] \subseteq A[i]$ for $i < 4$ and:

- $A[0] \setminus B[0]$ has at most one element,
- $A[1] \setminus B[1]$ has at most one element,
- $A[2] \setminus B[2]$ has at most m elements,
- $A[3] \setminus B[3]$ has at most $m-1$ elements.

Together we are done.]

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- if $2k-1 > n \geq k \geq 1$, then $m^*_{n,k}(\mathscr{T}) = 0$,
- if $n \geq 1$, then $m^*_{n+1,1}(\mathscr{T}) = \Sigma\{2k \cdot m^*_{n,k}(\mathscr{T}) \colon k \in [1,n]\}$,

and more generally,

- if $n > k \geq 1$, then

$$m^*_{n+k,k} = \sum\left\{\ell! \cdot \binom{\ell}{\ell_1} \cdot \binom{\ell-\ell_1}{\ell_2} \cdot 2^\ell \cdot m^*_{n,\ell}(\mathscr{T}) \colon \ell, \ell_0, \ell_1, \ell_2 \in [0,n),\ \ell = \ell_0+\ell_1+\ell_2\right\}.$$

[Why? Considering $p = \mathrm{tp}_{\mathrm{qf}}(\bar{a}{\restriction}(n+k), \emptyset, \mathscr{T})$ we fix $q = \mathrm{tp}(\bar{a}{\restriction}n, \emptyset, \mathscr{T})$, let ℓ be maximal such that $n-\ell \leq i < n \Rightarrow \mathrm{lev}(a_i) = \mathrm{lev}(a_{n-1})$ (equivalently $\mathrm{lev}(a_{n-\ell})) = \mathrm{lev}(a_{n-1})$). For $\iota = 0,1,2$, let $S_\iota = \{m \colon n-\ell \leq m < n$ and $\iota = |\{j < k \colon a_m <_{\mathscr{T}} a_j\}|\}$, so (S_0, S_1, S_2) is a partition of $[n-\ell, n)$. Let $S^\bullet_1 = \{m \in S_1 \colon$ if $j < k$ then $a_m R_1 a_s\}$. Fixing ℓ the number of possibles q's is $m^*_{n,\ell}(\mathscr{T})$ and fixing q (and so ℓ) the freedom left is choosing $\ell_0, \ell_1, \ell_2 \geq 0$ such that $\ell_1 + 2\ell_2 = k$ and then choosing the partition (S_0, S_1, S_2) which have $\binom{\ell}{\ell_1}\binom{\ell-\ell_2}{\ell_2}$ possibilities we have 2^{ℓ_1} possible choices of $S^\bullet_1$ and lastly k possible linear orders of $\{a_i \colon i \in [n, n+k)\}$ clearly we are done.]

3), 4), 5) Clear. $\square_{1.7}$

Claim 1.8. *Let*[3] $\sigma \geq \aleph_0$ *be a cardinal and* $\mathscr{T} \in \mathbf{T}$. *Then:*

1) If $\mathscr{T} \to (\mathscr{T})^{\mathrm{end}(1)}_\sigma$ *<u>then</u>* $\mathscr{T} \to (\mathscr{T})^{\mathrm{end(k)}}_\sigma$ *for every* $k < \omega$.

2) If $\mathscr{T} \to (\mathscr{T})^{\mathrm{end}(1)}_\sigma$ *<u>then</u>* $\mathscr{T} \to (\mathscr{T})^n_\sigma$ *for every* $n < \omega$.

3) If $k \geq 1$ *and* $\mathscr{T}_\ell \in \mathbf{T}$ *for* $\ell = 0, \ldots, k$ *and* $\mathscr{T}_{\ell+1} \to (\mathscr{T}_\ell)^{\mathrm{end(1,m)}}_\sigma$ *for* $\ell < k$, *<u>then</u>* $\mathscr{T}_k \to (\mathscr{T}_0)^{\mathrm{end(k,m)}}_\sigma$, *hence* $\mathscr{T}_k \to (\mathscr{T}_0)^{\leq m}_\sigma$.

Proof. Clear. $\square$

§ 1(B). **Forcing in** ZFC.

Remark 1.9. Concerning the choice of $m(*)$ in 1.13 below (given $\mathbf{m}$), it is minor from the author's point of view, i.e., its value is immaterial for the model theoretic results.

Trivially $m(*) = 2m$ suffices.

Definition 1.10. Let $\mathscr{T} \in \mathbf{T}$ and $\bar{s} \in \mathrm{eseq}(\mathscr{T})$.

(1) Let $\mathrm{last-lev}(\bar{s}) \coloneqq \{\ell < \lg(\bar{s}) \colon$ if $k < \lg(\bar{s})$, then $\mathrm{lev}_{\mathscr{T}}(s_k) \leq \mathrm{lev}_{\mathscr{T}}(s_\ell)\}$ and let $\mathrm{last-ele}(\bar{s}) \coloneqq \{s_\ell \colon \ell \in \mathrm{last-lev}(\bar{s})\}$ and $\mathrm{init-lev}(\bar{s}) = \{\ell < \lg(\bar{s}) \colon \ell \notin \mathrm{last-lev}(\bar{s})\}$.

(2) Let $\mathrm{order}(\bar{s}, \mathscr{T}) \coloneqq \{\sqsubset \colon \sqsubset$ is a linear order on $\mathrm{last-ele}(\bar{s})\}$.

(3) Let $\mathrm{Eseq}(\mathscr{T})$ be the set of pairs $(\bar{s}, \sqsubset)$, where $\bar{s} \in \mathrm{eseq}(\mathscr{T})$ and $\sqsubset \in \mathrm{order}(\bar{s}, \mathscr{T})$.

(4) The similarity type of $(\bar{s}, \sqsubset) \in \mathrm{Eseq}(\mathscr{T})$ is naturally defined: $((\bar{s}_1, \sqsubset_1), (\bar{s}_2, \sqsubset_2) \in \mathrm{Eseq}(\mathscr{T}))$ have the same similarity type <u>iff</u> $\mathrm{sim-tp}(\bar{s}_1, \mathscr{T}_1) = \mathrm{sim-tp}(\bar{s}_2, \mathscr{T}'_2)$ <u>when</u> for $\iota = 1,2$, we let:

- $\bullet_1$ α_ι be such that $\{s_{\iota,\ell} \colon \ell \in \mathrm{last-ter}(\bar{s}_\iota, \mathscr{T}_\iota)\} \subseteq \mathscr{T}_{[\alpha_\iota]}$,

[3] If $\sigma < \aleph_0$ we have parallel results depending on decreasing σ in the conclusion on the bounds from 1.3, that is, for part (2): if $\mathscr{T} \to (\mathscr{T})^{\mathrm{end}(1,m(1))}_{\sigma(1)}$, then $\mathscr{T} \to (\mathscr{T})^{\mathrm{end}(n,m(2))}_{\sigma(2)}$.

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$\bullet_2$ $\mathscr{T}_\iota'$ be like $\mathscr{T}_\iota$ except that we change $<_{\mathscr{T}_\iota,\alpha_\iota}$ for every $\ell, k \in \text{last}-\text{term}(\bar{s}_1)$ we have $s_{\iota,\ell} <_{\mathscr{T}_\iota'} s_{\iota,k} \Leftrightarrow \ell \sqsubset_\iota k$.

(So it does not follows that $\bar{s}_1$, $\bar{s}_2$ have the same similarity type).

(5) For $\mathscr{T}_1$, $\mathscr{T}_2 \in \mathbf{T}$, we have $\mathscr{F}_2 \rightarrow^1 (\mathscr{F}_1)_\sigma^{\text{end}(1,m)}$ when (A) $\Rightarrow$ (B), where:

- (A) $\mathbf{c}\colon \text{Eseq}(\mathscr{T}_2) \rightarrow \sigma$.
- (B) There is a semi embedding h from $\mathscr{T}_1$ into $\mathscr{T}_2$ (see Definition 1.2(1A)) such that:
  - if $(\bar{s}_1, \sqsubset_1), (\bar{s}_2, \sqsubset_2) \in \text{Eseq}_{\leq m}(\mathscr{T}_2)$, $|\text{last}-\text{lev}(\bar{s}_1)| = m = |\text{last}-\text{lev}(\bar{s}_2)|$ and $\bar{s}_2 \restriction \text{init}-\text{lev}(\bar{s}_1) = \bar{s}_2 \restriction \text{init}-\text{lev}(\bar{s}_2)$, then we have $\mathbf{c}(h(\bar{s}_1), \sqsubset_1) = \mathbf{c}(\bar{s}_2, \sqsubset_2)$.

(6) We define $\mathscr{F}_2 \rightarrow^2 (\mathscr{F}_1)_\sigma^{\text{end}(1,m)}$ similarly, but in (B) we demand h to be an embedding.

(7) We define $\mathscr{F}_2 \rightarrow^\ell (\mathscr{F}_1)_\sigma^{\text{end}(k,m)}$ for $\ell = 1, 2$ and $k \in [1, \omega)$ similarly.

Definition 1.11. 1) Fixing $\bar{m} = \langle m_l\colon l \leq l(*)\rangle$, $m_\ell \geq 1$ and $\mathscr{T} \in \mathbf{T}$, let $\text{eseq}_{\bar{m}}(\mathscr{T})$ is the set of $\bar{s} \in \text{eseq}(\mathscr{T})$ such that:

1. the set $\{\lg(s_i)\colon i < \lg(\bar{s})\}$ has $\ell(*) + 1$ members,
2. let $\alpha_0 < \cdots < \alpha_{\ell(*)}$ list it,
3. $|\{\ell\colon \lg(s_\ell) = \alpha_\ell\}| = m_\ell$ for $\ell \leq \ell(*)$.

2) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$, let $\mathscr{T}_2 \rightarrow^\ell (\mathscr{T}_1)_\sigma^{\text{end}(1,\bar{m})}$, mean (A)$\Rightarrow$(B), where:

- (A) $\mathbf{c}\colon \text{Eseq}_{\bar{m}}(\mathscr{T}_2) \rightarrow \sigma$.
- (B) There is a semi embedding h if $\ell = 1$ and an embedding if $\ell = 2$ from $\mathscr{T}_1$ into $\mathscr{T}_2$ such that:
  - if $(\bar{s}_1, \sqsubset_1), (\bar{s}_2, \sqsubset_2) \in \text{Eseq}_{\leq m}(\mathscr{T}_2)$, $|\text{last}-\text{lev}(\bar{s}_1)| = m_{\ell(*)} = |\text{last}-\text{lev}(\bar{s}_2)|$ and $\bar{s}_2 \restriction \text{init}-\text{lev}(\bar{s}_1) = \bar{s}_2 \restriction \text{init}-\text{lev}(\bar{s}_2)$, then we have $\mathbf{c}(h(\bar{s}_1), \sqsubset_1) = \mathbf{c}(\bar{s}_2, \sqsubset_2)$.

3) For $\mathscr{T}_1$, $\mathscr{T}_2 \in \mathbf{T}$, we have $\mathscr{F}_2 \rightarrow^\ell (\mathscr{F}_1)_\sigma^{\bar{m}}$ when (A) $\Rightarrow$ (B), where:

- (A) $\mathbf{c}\colon \text{Eseq}_{\bar{m}}(\mathscr{T}_2) \rightarrow \sigma$.
- (B) There is a semi embedding h if ℓ_1, and an embedding if ℓ_2 from $\mathscr{T}_1$ into $\mathscr{T}_2$ such that:
  - if $\bar{s}_1$, $\bar{s}_2 \in \text{Eseq}_{\bar{m}}(\mathscr{T}_2)$, then we have $\mathbf{c}(h(\bar{s}_1)) = \mathbf{c}(\bar{s}_2)$.

Claim 1.12. *The obvious implications hold. In particular, if $\bar{m} = \langle m_\ell\colon \ell \leq \ell(*)\rangle$ and $\mathscr{T}_{\ell+1} \rightarrow^2 (\mathscr{T}_\ell)_\sigma^{\text{end}(1,\bar{m}\restriction(\ell+1))}$ for $\ell < \ell(*)$, then $\mathscr{T}_{\ell(*)} \rightarrow^2 (\mathscr{T}_0)_\sigma^{\bar{m}}$.*

Major Claim 1.13. *In $\mathbf{V}^{\mathbb{P}}$ we have $\mathscr{T}_2 \rightarrow^2 (\mathscr{T}_1)_\sigma^{\text{end}(1,\mathbf{m})}$ when (see Remark 1.14) for a suitable $m(*)$:*

- (a) *we have:*
  - $(\bullet_1)$ $\kappa = \kappa^{<\kappa}$,
  - $(\bullet_2)$ $\lambda \rightarrow (\kappa^+)_\Upsilon^{m(*)}$, *where $\Upsilon = 2^\kappa$, λ regular, really $\lambda = (\beth_{m(*)-1}(\kappa^+))^+$,*
  - $(\bullet_3)$ $\sigma < \kappa$ *and* $\sigma \geq \aleph_0$.
- (b) $\mathbb{P} = \text{Cohen}(\kappa, \lambda)$,
- (c) $\mathscr{T}_2 \in \mathbf{T}$ *expands* $({}^{\kappa(+)>}2, \vartriangleleft)$ *in* $\mathbf{V}^{\mathbb{P}}$,
- (d) *In $\mathbf{V}^{\mathbb{P}}$, $\mathscr{T}_1 \in \mathbf{T}_{\kappa,\kappa}$ and $\mathscr{T}_1 \subseteq \mathscr{T}_1^+$, where $\mathscr{T}_1^+$ expands $({}^{\kappa>}2, \vartriangleleft)$ and so* $\text{otp}(\mathscr{T}_{1,[\alpha]}, <_{\mathscr{T}_{1,\alpha}}) < \kappa$ *for $\alpha < \kappa$, see 1.2(3),*

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Remark 1.14. 1) As in [She89], $m(*) = 2m$ suffice. More on the value of $m(*)$, see [Shea] (see more in [S$^+$c] and [S$^+$d]).
2) Can we for $\to^2_{\rm wk}$ get a better to $m(*)$? Intend to return to this in [S$^+$c] and [S$^+$d].

Proof. First,

$\boxdot_1$ Without loss of generality, $\mathscr{T}_1 \in \mathbf{V}$ is an object (not just a $\mathbb{P}$-name).

[Why? Let $\underset{\sim}{\mathscr{T}}_1$ be a $\mathbb{P}$-name, then for some $u \in [\lambda]^{\le\kappa}$ we have $\underset{\sim}{\mathscr{T}}_1$ is a $\mathbb{P}_u$-name. We can force by $\mathbb{P}_u$, so as $\mathbb{P}/\mathbb{P}_u = \text{Cohen}(\kappa,\lambda)$, we are done.]

So $\kappa, \mathscr{T}_1, \underset{\sim}{\mathscr{T}}_2$ are well defined ($\underset{\sim}{\mathscr{T}}_2$ a $\mathbb{P}$-name). Let $\underset{\sim}{\mathbf{c}}$ be a $\mathbb{P}$-name, $\underset{\sim}{\mathbf{c}}\colon \text{Eseq}(\underset{\sim}{\mathscr{T}}_2) \to \sigma$, without loss of generality be such that:

$\boxdot_2$ if $(\bar t, \sqsubset) \in \text{Eseq}(\mathscr{T}_2)$, then from $\underset{\sim}{\mathbf{c}}(\bar t, \sqsubset)$ we can compute:
(a) $\underset{\sim}{\mathbf{c}}(\bar t, \sqsubset')$ when $\sqsubset' \in \text{order}(\bar t, \mathscr{T})$,
(b) the similarity type of $\bar t$ in $\mathscr{T}_2$,
(c) the similarity type of $\underset{\sim}{\mathbf{c}}((\bar t, \sqsubset){\restriction}u)$ when $u \subseteq \text{dom}(\bar t)$, $\bar t{\restriction}u \in \text{Eseq}(\underset{\sim}{\mathscr{T}}_2)$.

Let $\underset{\sim}{\bar\eta} = \langle \underset{\sim}{\eta}_\alpha\colon \alpha < \lambda\rangle$ be the generic of $\mathbb{P}$, so $\Vdash$"$\underset{\sim}{\eta}_\alpha \in {}^\kappa 2$" and let $\underset{\sim}{\bar\eta}_u = \langle \underset{\sim}{\eta}_\alpha\colon \alpha \in u\rangle$ for $u \subseteq \lambda$.

Next, in $\mathbf{V}$, we choose:

$(*)_1$ (a) let $\chi > \lambda$ and $<^*_\chi$ a well ordering of $\mathscr{H}(\chi)$,
(b) let $\mathfrak{B} \prec \mathfrak{A}_0 = (\mathscr{H}(\chi), \in, <^*_\chi)$ be of cardinality κ such that $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\lambda, \kappa, \mu, \sigma, \mathscr{T}_1, \underset{\sim}{\mathscr{T}}_2, \underset{\sim}{\mathbf{c}} \in \mathfrak{B}$;
(c) let $u_* = \mathfrak{B} \cap \lambda \in [\lambda]^\kappa$,
(d) let $\mathbf{G}_{u_*} \subseteq \mathbb{P}_{u_*}$ be generic over $\mathbf{V}_0 = \mathbf{V}$, and let $\mathbf{G} \subseteq \mathbb{P}$ be generic over $\mathbf{V}_0$ such that $\mathbf{G}_{u_*} \subseteq \mathbf{G}$,
(e) let $\bar\eta_u = \langle \underset{\sim}{\eta}_\alpha[\mathbf{G}]\colon \alpha \in u\rangle$, for $u \subseteq \lambda$,
(f) let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_{u_*}] = \mathbf{V}_0[\bar\eta_{u_*}]$,
(g) let $\mathbf{V}_2 = \mathbf{V}[G] = \mathbf{V}_0[\bar\eta_\lambda] = \mathbf{V}_1[\bar\eta_{\lambda\setminus u_*}]$.

$(*)_2$ (a) let $\underset{\sim}{\mathscr{T}}_0$ be the $\mathbb{P}$-name of the sub-structure of $\underset{\sim}{\mathscr{T}}_2$ with set of elements $\{\underset{\sim}{\eta}\colon \underset{\sim}{\eta}$ is a canonical $\mathbb{P}$-name of a member of $\underset{\sim}{\mathscr{T}}_2$ and this name belongs to $\mathfrak{B}\}$,
(b) let $\delta_* = \delta(*)$ be $\kappa^+ = \min(\kappa^+ \setminus u_*) = \kappa^+ \cap u_*$, noting δ_* has cofinality κ because $u_* = \mathfrak{B} \cap \kappa^+$, $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\|\mathfrak{B}\| = \kappa$,
(c) let $\langle \delta_\varepsilon\colon \varepsilon < \kappa\rangle$ be increasing continuous with limit δ_* in $\mathbf{V}_0$,
(d) $\mathfrak{B}_2 = \mathfrak{B}[\mathbf{G}_\lambda]$, $\mathfrak{B}_1 = \mathfrak{B}_2{\restriction}\{\underset{\sim}{\tau}[\mathbf{G}_\lambda]\colon \underset{\sim}{\tau}$ is a $\mathbb{P}_u$-name from $\mathfrak{B}$ for some $u \in [\lambda]^{\le\kappa} \cap \mathfrak{B}\}$, $\mathfrak{B}_0 = \mathfrak{B}$, so $\mathfrak{B}_2 \prec \mathfrak{A}_2 = \mathscr{H}(\chi)[\mathbf{G}_\lambda]$ and $\mathfrak{B}_1 \cap {}^{\kappa\ge}\lambda = \mathfrak{B}_2 \cap ({}^{\kappa\ge}\lambda)^{\mathbf{V}_1}$.

Clearly,

$(*)_3$ (a) $\Vdash_{\mathbb{P}_\lambda}$"$\underset{\sim}{\mathscr{T}}_0 \subseteq \underset{\sim}{\mathscr{T}}_2$ is closed under initial segments, is of cardinality κ and has δ_* levels and is closed under unions of increasing chains of length $< \kappa$ and $\nu \in \underset{\sim}{\mathscr{T}}_0 \Rightarrow \nu^\frown\langle 0\rangle, \nu^\frown\langle 1\rangle \in \underset{\sim}{\mathscr{T}}_0$, and $\alpha < \delta_* \Rightarrow (\forall \nu \in \mathscr{T}_0)(\exists\rho)[\nu \triangleleft \rho \in \mathscr{T}_0 \wedge \lg(\rho) \ge \alpha]$, so $\underset{\sim}{\mathscr{T}}_0 \in \mathbf{T}$",
(b) $\underset{\sim}{\mathscr{T}}_0$ is actually a $\mathbb{P}_{u_*}$-name and we can use $\delta_* = \mathfrak{B} \cap \kappa^+$ as its set of levels.

$(*)_4$ (a) let $\mathscr{T}_0 = \underset{\sim}{\mathscr{T}}_0[\mathbf{G}_{u_*}]$, $\mathbf{c}_0 = \mathbf{c}{\restriction}\text{eseq}(\mathscr{T}_0)$ so they are from $\mathbf{V}_1$,

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(b) let $\mathbb{P}_* = \mathbb{P}/\mathbf{G}_{u_*} = \mathbb{P}_{\lambda \setminus u_*}$.

$(*)_5$ (a) for each $\alpha \in \lambda \setminus u_*$, in $\mathbf{V}_1[\eta_\alpha]$ there is $\eta^\bullet_\alpha \in \lim_{\delta_*}(\mathscr{T}_0)^{\mathbf{V}_1[\eta_\alpha]}$ hence $\lg(\eta^\bullet_\alpha) = \delta_*$ such that $\varepsilon < \delta_* \Rightarrow \eta^\bullet_\alpha{\restriction}\varepsilon \in \mathscr{T}_0$ and $\eta^\bullet_\alpha$ is a generic δ_*-branch of $\mathscr{T}_0$ over $\mathbf{V}_1$,

(b) Clearly $\lim_{\delta_*}(\mathscr{T}_0)^{\mathbf{V}_1[\eta_\alpha]} \subseteq \mathscr{T}_2[\mathbf{G}_\lambda]$.

We shall work in $\mathbf{V}_1$.

$(*)_6$ (in $\mathbf{V}_1$) for $\alpha \in \lambda \setminus u_*$, let $\underset{\sim}{\eta}^\bullet_\alpha$ be a $\mathbb{P}_{\{\alpha\}}$-name of $\eta^\bullet_\alpha$, so without loss of generality for some κ-Borel function $\mathbf{B} \colon {}^\kappa 2 \to {}^{\delta(*)}2$, from $\mathbf{V}_1$ we have $\Vdash$"$\underset{\sim}{\eta}^\bullet_\alpha = \mathbf{B}(\underset{\sim}{\eta}_\alpha)$ is as above in $(*)_5$(a)"; note that $\mathbf{B}$ does not depend of α.

[Why? See the proof of $(*)_7$]

$(*)_7$ Without loss of generality,

(a) Recall we had in $(*)_2$(c) an increasing sequence $\langle \delta(\varepsilon) := \delta_\varepsilon \colon \varepsilon < \kappa \rangle$ so that $\delta(*) = \bigcup_{\varepsilon<\kappa} \delta(\varepsilon)$

(b) Let $\langle \mathbf{B}_\varepsilon \colon \varepsilon < \kappa \rangle$ be such that $\mathbf{B}_\varepsilon$ is a κ-Borel function from ${}^\kappa 2$ to ${}^{\delta(\varepsilon)}2$ from $\mathbf{V}_1$ and $\Vdash$"$\underset{\sim}{\eta}^\bullet_\alpha{\restriction}\delta_\varepsilon = \mathbf{B}_\varepsilon(\underset{\sim}{\eta}_\alpha)$" for $\alpha \in \lambda \setminus u_*$,

(c) In $\mathbf{V}_0$, let $\underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{B}}_\varepsilon$ be $\mathbb{P}_{u_*}$-names forced to be as above, can be considered as $\mathbb{P}_{u_*}$-name.

[Why?

For $\eta \in {}^\omega 2$, define $\nu = \nu_\eta \in {}^\kappa\kappa$ as follows: for $\varepsilon < \kappa$, we let $\nu_\eta(\varepsilon)$ be the the unique $\zeta < \chi$ such that $\underset{\sim}{\eta}_\alpha(\zeta) = 1 \wedge \varepsilon = \mathrm{otp}\{\xi < \zeta \colon \underset{\sim}{\eta}_\alpha(\xi) = 1\}$ if there is such ζ and 0 otherwise. Now, $\mathscr{T}_0 \cap {}^{\delta(\varepsilon)}\lambda$ has $\leq \kappa$ members, hence we can interprete $\mathbf{B}_\varepsilon(\eta) = \mathbf{B}_\varepsilon(\eta{\restriction}(\varepsilon+1))$, a member of $\mathscr{T}_0 \cap {}^{\delta(\varepsilon)}\lambda$, which is $\trianglelefteq$-increasing with ε and $\mathbf{B}(\eta) = \bigcup\{\mathbf{B}_\varepsilon(\eta{\restriction}(\varepsilon+1)) \colon \varepsilon < \kappa\}$. That is, we can first choose $\mathbf{B}_\varepsilon$ by induction on $\varepsilon < \kappa$ such that $\mathbf{B}_\varepsilon(\eta)$ depend just on $\nu_\eta{\restriction}(\varepsilon+1)$ and $(*)_8$ below holds and then choose $\mathbf{B}$.]

$(*)_8$ If $\varepsilon < \kappa$, $\nu \in {}^\kappa 2$, and $\mathbf{B}(\nu) = \rho$, then $\mathbf{B}_\varepsilon(\nu{\restriction}\zeta) = \rho \restriction \delta_\varepsilon$.

Recalling we work in $\mathbf{V}_1$ and $\mathbf{V}$, $\mathbf{V}_1$ have the same cardinal arithmetic, there are $\mathscr{U}, \bar{N}$ such that:

$(*)_9$ (a) $\mathscr{U} \subseteq \lambda \setminus u_*$,

(b) $\mathrm{otp}(\mathscr{U})$ is κ^+,

(c) $\bar{N} = \langle N_u \colon u \in [\mathscr{U}]^{\leq \mathbf{m}} \rangle$,

(d) $N_u \cap N_v \subseteq N_{u \cap v}$ when $u, v \in [\mathscr{U}]^{\leq \mathbf{m}}$,

(e) $\kappa, \mathscr{T}_1, \underset{\sim}{\mathscr{T}}_2, \underset{\sim}{\mathbf{c}}, \mathscr{B}, \mathscr{B}_1 \in N_u \prec \mathfrak{A}_0, \|N_u\| = \kappa, [N_u]^{<\kappa} \subseteq N_u$ for $u \in [\mathscr{U}]^{\leq \mathbf{m}}$,

(f) if $u, v \in [\mathscr{U}]^{\leq m}$ and $|u| = |v|$, then there is a unique isomorphism $\mathbf{g}_{u,v}$ from N_v onto N_u and it is the identity on $(\kappa+1) \cup \{\underset{\sim}{\mathbf{c}}, \underset{\sim}{\eta}, \mathbf{B}\}$ hence on $\mathscr{T}_2, \langle \underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}^\bullet_\alpha \colon \alpha < \lambda \setminus u_* \rangle$ and maps v onto u, and if $v_1 \subseteq v, u_1 = \mathbf{g}_{u,v}[v_1] \subseteq u$ then $\mathbf{g}_{u_1,v_1} \restriction N_v \subseteq \mathbf{g}_{u,v}$,

(g) $\langle \underset{\sim}{\eta}^\bullet_\alpha \colon \alpha \in \mathscr{U} \rangle$ is $<^*_{\underset{\sim}{\mathscr{T}}_2}$-increasing.

[Why? By clause (a)($\bullet_3$) of the assumption of 1.13 as in [She89], there is such $\mathbf{m}(*)$ (see 1.14), for clause (g) recall that $<^*_{\underset{\sim}{\mathscr{T}}_2}$ is a (linear) well-ordering.]

$(*)_{10}$ <u>notation</u>:

(a) for finite $u \subseteq \mathscr{T}_{0,[\varepsilon]}$ for some $\varepsilon < \delta_*$, let

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(α) $\mathbf{H}_u := \{h\colon h$ is a one-to-one function from u into $\mathscr{U}\}$,

(β) We let $\mathbf{F}_u$ be the set of f such that for some $(\bar{s}, \sqsubset) = (\bar{s}_f, \sqsubseteq_f)$ and $g = g_f$ we have

($\bullet_1$) $(\bar{s}, \sqsubset) \in \mathrm{Eseq}(\mathscr{T}_1)$,

($\bullet_2$) $\mathrm{last} - \mathrm{ele}[\bar{s}] = v$, and note $\bar{s} = \mathrm{cl}_{\mathscr{T}_1}(\{s_i : i < \lg(\bar{s})\})$ so we let $u = u_f$,

($\bullet_3$) g is an increasing function from $\{\mathrm{lev}_{\mathscr{T}_1}(s_i) : i < \lg(\bar{s})\}$ into $\mathrm{ht}(\mathscr{T}_0) = \delta_*$,

($\bullet_4$) f is a semi embedding of $\mathscr{T}_1 \restriction \{s_i : i < \lg(\bar{s}\}$ into $\mathscr{T}_0$ mapping $\{s_i\colon i \in v\}$ onto u,

($\bullet_5$) so $f(s_i) \in \mathscr{T}_{0,[g(i)]}$ where $\mathrm{level}_{\mathscr{T}_1}(f(s_i)) = g(\mathrm{lev}_{\mathscr{T}_1}(s_i))$.

(b) If in $\mathbf{V}_2$, $(\bar{s}, \sqsubset) \in \mathrm{Eseq}(\mathscr{T}_0)$, $u = \mathrm{last} - \mathrm{ele}[\bar{s}] \in [\mathscr{T}_{0,[\varepsilon]}]^{\leq \mathbf{m}}$ and $h \in \mathbf{H}_u$, then we let $\bar{s}^{[h]}$ be the $(\bar{t}, \sqsubset) \in \mathrm{Eseq}(\mathscr{T}_1)$ such that $\lg(\bar{t}) = \lg(\bar{s})$ and $i \in \lg(\bar{s}) \setminus u \Rightarrow t_i = s_i$ and $i \in \mathrm{last} - \mathrm{lev}(\bar{s}) \Rightarrow t_i = s_i {}^\frown \underset{\sim}{\eta}'_i$ where $\underset{\sim}{\eta}'_i = \eta^\bullet_{h(s_i)} \restriction [\varepsilon(\bar{s}), \delta_*)$.

We define $\mathrm{AP} := \bigcup_{\varepsilon < \kappa} \mathrm{AP}_\varepsilon$, where AP_ε is the set of objects $\mathbf{a}$ which consists of (so $\varepsilon = \varepsilon_{\mathbf{a}}, \bar{\nu} = \bar{\nu}_{\mathbf{a}}$, etc).

$\boxplus^0_{\mathbf{a}}$ (a) $\varepsilon < \kappa$,

(b) $\bullet_1$ $\bar{\nu} = \langle \nu_\rho \colon \rho \in \mathscr{T}_{1,[\varepsilon]}\rangle$ is with no repetitions; (will serve as an approximation to an embedding),

$\bullet_2$ $\mathbf{f}$ is an $\subseteq_{\mathrm{fl}}$-embedding of $\mathscr{T}_{1,[\leq \varepsilon]} = \mathscr{T}_1 \restriction \{\eta \in \mathscr{T}_2 \colon \lg(\eta) \leq \varepsilon\}$ into $\mathscr{T}_0$,

$\bullet_3$ $\mathbf{f}(\rho) = \nu_\rho$ for $\rho \in \mathscr{T}_{1,[\varepsilon]}$.

(c) $\bar{\eta} = \langle \eta_\rho \colon \rho \in \mathscr{T}_{1,[\varepsilon]}\rangle$; ($\nu_\rho$ will serve as condition in $\mathbb{P}_{\{\alpha\}}$ for some α),

(d) $\nu_\rho \in \mathscr{T}_{0,[\zeta]}$ and $\eta_\rho \in {}^{\kappa >}2$ for some $\zeta = \zeta_{\mathbf{a}} < \kappa$ (for all $\rho \in \mathscr{T}_{1,[\varepsilon]}$),

(e) $\mathbf{B}_\zeta(\eta_\rho) = \nu_\rho$,

(f) $\bar{p} = \langle p_{u,h} \colon u \in [\mathscr{T}_{0,[\varepsilon]}]^{\leq \mathbf{m})}, h \in \mathbf{H}_u\rangle$, where $p_{u,h} \in \mathbb{P}_{\lambda \setminus u_*}$,

(g) $p_{u,h} \in N_{h[u]}$ and $[\rho \in u \wedge h(\rho) = \alpha \Rightarrow p_{u,h}(\alpha) = \eta_\rho]$,

(h) if $h_1, h_2 \in \mathbf{H}_u$ and $h_1[u] = h_2[u]$ <u>then</u> $\alpha \in \mathrm{dom}(p_{u,h_1}) \setminus h_1[u] \Rightarrow p_{u,h_1}(\alpha) = p_{u,h_2}(\alpha)$,

(i) if $h_1, h_2 \in \mathbf{H}_u$ and $\rho_1, \rho_2 \in u \Rightarrow [h_1(\rho_1) < h_1(\rho_2) \equiv h_2(\rho_1) < h_2(\rho_2)]$, <u>then</u> $\mathbf{g}_{h_2[u],h_1[u]}$ maps p_{u,h_1} to p_{u,h_2},

(j) if $u_1, u_2 \in [{}^\varepsilon 2]^{\leq \mathbf{m}}$, $h_\ell \in \mathbf{H}_{u_\ell}$ for $\ell = 1, 2$, and h_1, h_2 are compatible, then $(p_{u_1,h_1} \restriction N_{h_1[u_2]})$ and p_{u_2,h_2} are compatible.

Let further

$\boxplus^1_{\mathbf{a}}$ $\mathrm{AP}^+ = \bigcup\{\mathrm{AP}^+_\varepsilon \colon \varepsilon < \kappa\}$, where $\mathrm{AP}^+_\varepsilon$ is the set of $\mathbf{a} \in \mathrm{AP}_\varepsilon$ such that:

(k) $p_{u,h}$ forces a value to $\underset{\sim}{\mathbf{c}}(\bar{s}, \sqsubset)$ <u>when</u> for some $f \in \mathbf{F}_u$ we have $\bar{s} = \bar{s}_f$, $u_f = u$, $\sqsubset = \sqsubset_f = \{(s_\ell, s_k) \colon s_\ell, s_k \in u$ and $\{h(s_\ell) < h(s_k)\}$.

$\boxplus_2$ we define the two-place relation $\leq_{\mathrm{AP}}$ as follows: $\mathbf{a}_1 \leq_{\mathrm{AP}} \mathbf{a}_2$ <u>iff</u>:

(a) $\mathbf{a}_1, \mathbf{a}_2 \in \mathrm{AP}$,

(b) $\varepsilon_1 = \varepsilon_{\mathbf{a}_1} \leq \varepsilon_{\mathbf{a}_2} = \varepsilon_2$,

(c) if $\iota \in \{0, 1\}$, $\rho_1 \in {}^{\varepsilon(1)}2$, $\rho_1^\frown \langle \iota \rangle \trianglelefteq \rho_2 \in {}^{\varepsilon(2)}2$, then $\eta_{\mathbf{a}_1,\rho_1}{}^\frown \langle \iota \rangle \trianglelefteq \eta_{\mathbf{a}_2,\rho_2}$ and $\nu_{\mathbf{a}_1,\rho_1}{}^\frown \langle \iota \rangle \trianglelefteq \nu_{\mathbf{a}_2,\rho_2}$,

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(d) if $n \le m$, $u_1 \in [\mathscr{T}_{1,[\varepsilon(1)]}]^n \subseteq [{}^{\varepsilon(1)}2]^n, u_2 \in [\mathscr{T}_{1,\varepsilon(1)}]^n, u_1 = \{\rho{\restriction}\varepsilon(1) \colon \rho \in u_2\rangle$, and $h_\ell \in \mathbf{H}_{u_\ell}$ for $\ell = 1,2$, then $\rho \in u_2 \Rightarrow h_2(\rho){\restriction}\varepsilon(1) = h_1(\rho{\restriction}\varepsilon(1))$, and $p_{u_1,h_1} \le_{\mathbb{P}_*} p_{u_2,h_2}$.

$\boxplus_3$ $(\mathrm{AP}, <_{\mathrm{AP}})$ is a partial order.

[Why? Read the definitions.]

$\boxplus_4$ for $\varepsilon = 0$ there is $\mathbf{a} \in \mathrm{AP}_\varepsilon$.

[Why? Trivial.]

$\boxplus_5$ (a) If $\varepsilon < \kappa$ is a limit ordinal and $\mathbf{a}_\zeta \in \mathrm{AP}_\zeta$ for $\zeta < \varepsilon$ is $\le_{\mathrm{AP}}$ increasing, then there is $\mathbf{a}_\varepsilon \in \mathrm{AP}_\varepsilon$ such that $\zeta < \varepsilon \Rightarrow \mathbf{a}_\zeta <_{\mathrm{AP}} \mathbf{a}_\varepsilon$.
(b) In clause (a), we can add: if $\mathbf{a}_\zeta \in \mathrm{AP}^+_\zeta$ for $\zeta < \varepsilon$ then $\mathbf{a}_\varepsilon \in \mathrm{AP}^+_\varepsilon$.

[Why? Straightforward recalling that $[N_u]^{<\kappa} \subseteq N_u$.]

$\boxplus_6$ if $\varepsilon < \kappa$ and $\mathbf{a} \in \mathrm{AP}_\varepsilon$ there is $\mathbf{b}$ such that:
(a) $\mathbf{b} \in \mathrm{AP}_\varepsilon$,
(b) $\mathbf{a} \le_{\mathrm{AP}} \mathbf{b}$,
(c) $\varepsilon_{\mathbf{b}} = \varepsilon_{\mathbf{a}} + 1$,

[Why? Straightforward.]

$\boxplus_7$ if $\mathbf{a} \in \mathrm{AP}_\varepsilon$, then there is $\mathbf{b} \in \mathrm{AP}^+_\varepsilon$ such that $\mathbf{a} \le_{\mathrm{AP}} \mathbf{b}$.

Toward this, we first show:

$\boxplus_{7.1}$ if $\mathbf{a} \in \mathrm{AP}_\varepsilon$, $u \in [\mathscr{T}_{1,[\varepsilon]}]^{\le \mathbf{m}}$ and $p, \bar{s}, \sqsubset$ are as in clause (k) of $\boxplus^1_{\mathbf{a}}$ then there is $\mathbf{b} \in \mathrm{AP}_\varepsilon$ such that $\mathbf{a} \le_{\mathrm{AP}} \mathbf{b}$ and $p_{\mathbf{a},u,h}$ forces a value to $\underset{\sim}{\mathbf{c}}(\bar{s}, \sqsubset)$.

[Why? First choose $p \in N_{h[u]} \cap \mathbb{P}$ above $p_{\mathbf{a},u,h}$ forcing a value to $\mathbf{c}(\bar{s}, \sqsubset)$. Then choose $p_{\mathbf{b},u_1,h}$ for relevant pairs by combining $p_{\mathbf{a},u_1,h_1}$ and p (so $p_{\mathbf{b},u,h} = p$) remembering $\boxplus_2$.]

$\boxplus_8$ we can choose $\mathbf{a}_\varepsilon \in \mathrm{AP}_\varepsilon$ by induction on $\varepsilon < \kappa$ such that $\zeta < \varepsilon \Rightarrow \mathbf{a}_\zeta \le_{\mathrm{AP}} \mathbf{a}_\varepsilon$ and $\varepsilon = \zeta + 1 \Rightarrow \mathbf{a}_\varepsilon \in \mathrm{AP}^+_\varepsilon$.

[Why? Use $\boxplus_4$ for $\varepsilon = 0$, use $\boxplus_5$ for ε a limit ordinal and $\boxplus_6 + \boxplus_7$ for $\varepsilon = \zeta + 1$.]
Now define $h \colon \mathscr{T}_1 \to \mathscr{T}_2$ by $\rho \in \mathscr{T}_{1,[\varepsilon]} \Rightarrow h(\rho) = \eta_{\mathbf{a}_\varepsilon,\rho}$ and check.]
Lastly,

$\boxplus_9$ $\bigcup\{\mathbf{f}_{\mathbf{a}_\varepsilon} : \varepsilon < \kappa\}$ is an embedding as is desired.

$\square_{1.13}$

Remark 1.15. 1) Using the end of §1A we get the desired conclusions.
2) In 1.13 we may state and prove the variant with the square bracket. In more details,

(A) We say $\bar{a}, \bar{b} \in \mathrm{eseq}_m(\mathscr{T})$ are weakly $\mathscr{T}$-similar as before but omitting "$a_\ell <^*_{\mathscr{T}} a_k \Leftrightarrow b_\ell <^*_{\mathscr{T}} b_k$"; that is, when $\lg(\bar{a}) = \lg(\bar{b})$ and for some permutation π of $\lg(\bar{a})$ for $k, \ell, m < \lg(\bar{a})$, we have:
(a) $\mathrm{lev}_{\mathscr{T}}(a_\ell) = \mathrm{lev}_{\mathscr{T}}(a_k) \Rightarrow \mathrm{lev}_{\mathscr{T}}(b_{\pi(\ell)}) = \mathrm{lev}_{\mathscr{T}}(b_{\pi(k)})$,
(b) $a_\ell <_{\mathscr{T}} a_k \Leftrightarrow b_{\pi(\ell)} <_{\mathscr{T}} b_{\pi(k)}$.

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(B) We replace "$\mathscr{T}_2 \to (\mathscr{T}_1)^{\text{end(k,m)}}_\sigma$" by "$\mathscr{T}_2 \to [\mathscr{T}_1]^{\text{end(k,m)}}_{\sigma,j}$" for suitable finite j which means that 1.5(1)$(*)\bullet_2$ is replaced by:
- $\bullet_2$ if $n < \omega$ and $\bar{a} \in \text{eseq}_n(\mathscr{T}_2)$, <u>then</u> the following set has at most j elements:
  $\{\mathbf{c}'(\bar{b})\colon \bar{b} \in \text{eseq}_n(\mathscr{T}_2)$ is weakly $\mathscr{T}_1$-similar to $\bar{a}$ and $\ell < n \wedge (k \leq |\text{Lev}(\bar{a})) \setminus \text{lev}(a_\ell)|) \Rightarrow b_\ell = a_\ell\}$.

3) This will be enough for the model theory, and if we use minimal j (well, depends on $\bar{s}{\restriction}v(\bar{s})$), we get back to 1.13.

Conclusion 1.16. *Assume $\kappa = \kappa^{<\kappa}$ and $\kappa < \chi \leq \infty$ is limit and $\theta \in [\kappa, \chi) \Rightarrow 2^\theta = \theta^+$, and if χ is singular then $2^\chi = \chi^+$.*

We can find a forcing notion $\mathbb{P}$ such that:

(a) *$\mathbb{P}$ is a $(<\kappa))$-complete forcing notion of cardinality $\chi^{<\chi}$,*
(b) *$\mathbb{P}$ collapses no cardinal, changes no cofinality,*
(c) *$2^\theta < \theta^{+\omega}$ for $\theta \in [\kappa, \chi)$,*
(d) *in $\mathbf{V}^{\mathbb{P}}$, if $\theta^{+\omega} \leq \chi$, then for every $k, m < \omega$, for some $n < \omega$, for every $\mathscr{T}_1 \in \mathbf{T}$ expanding $({}^{\theta>}2, \lhd)$ there is $\mathscr{T}_2 \in \mathbf{T}$ expanding $({}^{\theta(+n)>}2, \lhd)$ we have $\mathscr{T}_2 \to (\mathscr{T}_1)^{\text{end(k,m)}}_\theta$.*

Discussion 1.17. We may like to replace ${}^{\kappa>}2$ by ${}^{\kappa>}I$ and even use creature tree forcing, see Roslanowski-Shelah [RS99], [RS07], Goldstern-Shelah [GS05]) but (in second thought, for $\kappa = \aleph_0$ maybe see the paper with Zapletal [SZ11]). That is, for $\kappa > \aleph_0$ in each node we have a forcing notion which is quite complete, but of cardinality $< \kappa$ = set of levels.

So we do not have a tree but a sequence of creatures, $\langle \mathfrak{c}_\varepsilon\colon \varepsilon < \text{ht}(\mathscr{T})\rangle$, such that for a colouring we like to find $\mathfrak{d}_\varepsilon \in \Sigma(\mathfrak{c}_\varepsilon)$ for $\varepsilon < \kappa$, which induces a sub-tree in which the colouring is 1-end-homogeneous. Alternatively we have $\langle \mathfrak{c}_\eta\colon \eta \in \mathscr{T}\rangle$ where $\mathfrak{c}_\eta$ is a creature with set of possible values being in $\text{suc}_{\mathscr{T}}(\eta)$, see [RS07].

Clearly the answer is that we can, but it is not clear how interesting it is. We can just,

$\boxdot$ replace 2 by $\Upsilon \in [2, \kappa)$ and ${}^{\kappa>}2$ by ${}^{\kappa>}\Upsilon$; in the Definition 1.1 replace $R_{\mathscr{T},\ell}$ $(\ell < 2)$ by $R_{\mathscr{T},\ell}$ $(\ell < \Upsilon)$ and add: if $s \in \mathscr{T}$, then $\text{suc}_{\mathscr{T}}(s)$ is either a singleton or is $\{s_\ell\colon \ell < \Upsilon\}$, where $s_\ell R_{\mathscr{T},\ell} s$ for $\ell < \Upsilon$.

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