MODULES AND INFINITARY LOGICS
SH:977

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Dedicated to Rüdiger Göbel for this 70th birthday

Abstract. We deal with Abelian groups and R-modules. We consider theories in infinitary logic of the form $\mathbb{L}_{\lambda,\theta}$ of such structures M and prove they have elimination of quantifiers up to positive existential formulas, (so ones defining subgroups of some power of M). However, we demand that we expand by enough individual constants. Hence those theories are stable in the appropriate sense and understood to some extent.

In 2026, John Baldwin pointed out a mistake in the end of the proof of the main claim of Section 4, which is used in the theorem in Section 2, and made further requests for clarifications. Here this is corrected, in addition to other improvements. The error is corrected in three ways — in Section 2 we can use a weaker version of Section 4, and in section 4 we also get the original result with more assumptions on the cardinal. Lastly, we provide a shorter and self-contained proof of the main theorem in §2. We can use the older version of Section 4, but then we use somewhat larger cardinals.

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§ 0. Introduction

Much is known on classes of R-modules and first order logic. Szmielew [Szm49] proved the decidability of the theory of Abelian groups. In [Szm55], she proved an elimination of quantifiers in the theory of Abelian groups up to Boolean combinations of p.e. (*positive existential*) formulas.

Eklof [Ekl71] proved the existence of universal homogeneous R-models in λ if $\lambda = \lambda^{<\gamma}$, where γ depends only on R. Fisher improved this to saturated models of elementary classes (see his review of [Ekl71]); this implies stability by a general theorem from [She71, §0] (or [She90, Ch. III]).

Baur [Bau76] proved that for the class of R-modules, any first-order formula is equivalent to a Boolean combination of positive existential formulas, and also proved the stability of $\mathrm{Th}(M)$ for M an R-module.

We like to know for a given ring R how complicated the class of R-modules which are models of a sentence ψ in an infinitary logic.

Question 0.1. Given a ring R, for the class Mod_R of left R-modules:

1) Does it have (for the logic $\mathbb{L}_{\lambda,\mu}$) a kind of elimination of quantifiers (say, up to some depth)?

2) Is it stable? (Say, no formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\infty,\infty}(\tau_R)$ linearly ordering an arbitrarily long sequence of tuples in some models of ψ?)

3) Can we define something like non-forking?

Question 0.2. Do we have a parallel of the main gap — i.e. proving that either every $M \in \mathrm{Mod}_\psi$ can be characterized by some suitable cardinal invariants or that there are many complicated $M \in \mathrm{Mod}_\psi$?

Here we first show that for any R-module, in $\mathbb{L}_{\lambda,\theta}(\tau_R)$ (or better, $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$ — see 1.2(3)), we have a version of eliminating quantifiers up to positive existential formulas. However, we add parameters. Second, by this we can prove some versions and consequences of stability. More specifically:

- After expanding by enough individual constants, every formula in $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$ (see 1.2(3)) is equivalent to a Boolean combination of such positive existential formulas.
- The number of added individual constants is reasonable: $\leq \beth_\gamma\left(|\tau|^{<\theta}\right)$.
- We have stability: i.e. no long sequences of linearly ordered $(<\theta)$-tuples.
- $(\Lambda^{\mathrm{pe}}_{\varepsilon,\alpha}, 2)$-indiscernible implies $\Lambda^{\mathrm{pe}}_{\varepsilon,\alpha}$-indiscernible.
- Convergence follows (see Definition 3.4).

In 2025, this work was continued in a paper with Asgharzadeh and Golshani [AGS25].

We may use models with several *sorts* — that is, multiple distinct structures defined on them. E.g., when constructing an R-module we need a set of objects which are the elements of the module, and a set of elements of the ring R, each with their own operations of addition and (scalar) multiplication. Hence when we need to disambiguate them, we will write something like $x +_{\mathbf{s}} y$, $x -_{\mathbf{s}} v$ for each sort separately. It makes no difference (see 5.2).

§ 1. Preliminaries

Notation 1.1. Let θ^- be σ if $\theta = \sigma^+$ and θ if θ is a limit cardinal.

Definition 1.2. 1) A vocabulary τ consists of function symbols (e.g. individual constants) and predicates (relation symbols). In addition the vocabulary generally assign to each of them its arity (number of places) $\text{arity}_\tau(-)$; here it can be an infinite ordinal. An individual constant is a 0-place function.

2) For a vocabulary τ, we say M is a τ-structure when it consists of:

(a) $|M|$, the *universe* of M; this is a non-empty set of the so-called elements of M. However, we may write $a \in M$, $\bar{a} \in {}^\varepsilon M$, $A \subseteq M$, instead of $a \in |M|$, $\bar{a} \in {}^\varepsilon(|M|)$, etc.

(b) F^M, a function from ${}^\varepsilon M$ to M (possibly partial), for each function symbol F from τ. Here ε is the ordinal $\text{arity}_\tau(F)$.

(c) $P^M \subseteq {}^\varepsilon M$ (where ε is the ordinal $\text{arity}_\tau(P)$), for P a predicate from τ.

We may write $\tau_M = \tau(M) := \tau$.

3) $\mathbb{L}_{\mu,\theta,\alpha}(\tau)$ is the set of formulas $\varphi(\bar{x}) \in \mathbb{L}_{\mu,\theta}$ (so $\ell g(\bar{x}) < \theta$) of quantifier depth $< 1 + \alpha$.

Definition 1.3. 1) We say τ is a θ-*additive* (or a θ-Abelian) vocabulary when τ has the two-place function symbols $x + y, x - y$, the individual constant 0, and the other predicates and function symbols have arity $< \theta$.

2) M is a θ-*additive structure* (or model) when:

(a) τ_M, (the vocabulary of M) is a θ-additive vocabulary.

(b) $G_M := (|M|, +^M, -^M, 0^M)$ is an Abelian group.

(c) If $P \in \tau_M$ is an ε-place predicate then P^M is a subgroup of $(G_M)^\varepsilon$.

(d) If $F \in \tau_M \setminus \{+, -, 0\}$ is an ε-place function symbol then F^M is a partial ε-place function from M to M and
$$\text{graph}\big(F^M\big) := \big\{\bar{a}\hat{\ }\big\langle F^M(\bar{a})\big\rangle : \bar{a} \in \text{dom}(F^M)\big\}$$
is a subgroup of $(G_M)^{\varepsilon+1}$.

3) If $\varphi(\bar{x}, \bar{y})$ is a formula in the vocabulary τ, M is a τ-model, and $\bar{b} \in {}^{\ell g(\bar{y})}M$, then we write $\varphi(M, \bar{b}) = \varphi({}^{\ell g(\bar{x})}M, \bar{b})$ to mean the set
$$\big\{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}, \bar{b}]\big\}.$$

4) We say M is a θ-*affine structure* (or model) when:

(a),(b) As in part (2).

(c) If $P \in \tau_M$ is an ε-place predicate then P^M is an affine subset. (That is, $\bar{a}, \bar{b}, \bar{c} \in P^M \Rightarrow \bar{a} - \bar{b} + \bar{c} \in P^M$.)

(d) As in part (2), but $\text{graph}\big(F^M\big)$ is an affine subset.

Remark 1.4. 1) Definition 1.3(4) is not used in our theorem.

2) In 1.3(4), we may replace $\{+, -, 0\}$ by a three-place function symbol F_{af}, and change clauses (a)-(d) as follows.

(a) $F_{\text{af}} \in \tau_M$

(b) For any $b \in M$, the function $(x, y) \mapsto F^M_{\rm af}(x, b, y)$ defines an additive group $G_{M,b}$ in which b is the zero.

(c) As in 1.3(4), but here 'affine subset' means

$$\bar a, \bar b, \bar c \in P^M \Rightarrow G^M(\bar a, \bar b, \bar c) := \langle F^M_{\rm af}(a_\zeta, b_\zeta, c_\zeta) : \zeta < \varepsilon\rangle \in P^M.$$

(d) If $H \in \tau$ is an ε-place function symbol then $\text{graph}\big(H^M\big)$ is an affine subset.

3) We may also fix a sequence of predicates $\langle P_i, F_i : i < i_*\rangle$ in τ, where

(b)′
- $\langle P^M_i : i < i_*\rangle$ is a partition of M (so P_i is a unary predicate).
- F^M_i is an affine operation on P^M_i (so F_i is a three-place function symbol).

4) In the affine framework above, 1.3(4) can be treated in a perhaps more transparent way: just add more individual constants to τ_M. That is:

⊞ For $\mathcal{I} \subseteq M$, consider replacing τ_M by $\tau_{M,\mathcal{I}} := \tau_M \cup \{c_a : a \in \mathcal{I}\}$, where the c_a-s are individual constants. $M[\mathcal{I}]$ is M expanded to a $\tau_{M,\mathcal{I}}$-model by $c^{M[\mathcal{I}]}_a = a$.

Now the atomic formula '$x = c_a$' is not good for the "additive" version (but is for the affine one).

In this framework, $\Lambda^{\rm pe}_{\alpha,\varepsilon}$ is as in 2.2, but in Claim 2.3:

- If $\varphi(\bar x) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$ for $\mathbb{L}(\tau_{M,\mathcal{I}})$, then $\varphi(M)$ is an affine subset of ${}^\varepsilon M$ — i.e. closed under $\bar a - \bar b + \bar c$. (Or closed under $F^M_{\rm af}$ as in 1.4(2).)

Remark 1.5. 1) Fisher [Fis77] defines and deals with "Abelian structures" in other directions.

2) We use parentheses for formulas $\varphi(\bar x)$, but write (e.g.) $M \models \varphi[\bar a]$ for $\bar a \in {}^{\ell g(\bar x)}M$.

Definition 1.6. 1) We consider an R-module M as a $\tau(R)$-structure, where $\tau_R = \tau(R)$ is the vocabulary of R-modules. I.e. there are binary functions $x+y, x-y$, an individual constant 0, and unary function symbols F_a (interpreted as multiplication by a from the left) for every $a \in R$.

2) If $\bar x$ and $\bar y$ have length ε, <u>then</u> we let $\bar x + \bar y = \langle x_\zeta + y_\zeta : \zeta < \varepsilon\rangle$ and $\bar x - \bar y = \langle x_\zeta - y_\zeta : \zeta < \varepsilon\rangle$; similarly for $a\bar x$ with $a \in R$, and when we replace $\bar x$ and/or $\bar y$ by a member of ${}^\varepsilon M$.

Observation 1.7. 1) *For any ring R, an R-module is an $\aleph_0$-additive structure in the vocabulary τ_R.*

2) *For a τ-additive model M, for every function symbol or τ-term $F(\bar x)$, we have*

(*a*) $M \models$ "$F(\bar a \pm \bar b) = F(\bar a) \pm F(\bar b)$"

(When F^M is partial, this means that if two of the terms are well-defined then so is the third, and the equality holds.)

(*b*) *For P a predicate from τ_M, $M \models P(\bar a \pm \bar b)$ whenever $M \models P(\bar a) \wedge P(\bar b)$.*

§ 2. Eliminating quantifiers

Context 2.1. 1) R is a fixed ring and $\tau = \tau_R$ (see 1.6(1)), or τ is just a θ-additive vocabulary (see 1.3(1), 1.7(1)).

2) $\mathbf{K}$ is the class of R-modules or of τ-additive models.

3) M, N will denote R-modules or just τ-additive models.

4) $\theta = \mathrm{cf}(\theta)$.

Definition 2.2. For $\varepsilon < \theta$ and ordinal α (and τ as in 2.1(1)), we shall define

$$\Lambda^{\mathrm{pe}}_{\alpha,\varepsilon} = \Lambda^{\mathrm{pe},\theta}_{\alpha,\varepsilon} = \Lambda^{\mathrm{pe},\theta}_{\alpha,\varepsilon}(\tau)$$

as a set of formulas $\varphi(\bar{x})$ in $\mathbb{L}_{\infty,\theta}(\tau)$ — in fact, in $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$ — with $\ell g(\bar{x}) = \varepsilon < \theta$. The construction will proceed by induction on the ordinal α.

For $\zeta < \theta$, we write $\Lambda^{\mathrm{pe}}_{\alpha,\varepsilon,\zeta}$ for the set of $\varphi = \varphi(\bar{x}, \bar{y})$ with $\ell g(\bar{x}) = \varepsilon$ and $\ell g(\bar{y}) = \zeta$ with $\varphi \in \Lambda^{\mathrm{pe}}_{\alpha,\varepsilon+\zeta}$. We define $\Lambda^{\mathrm{pe}}_{\alpha} := \bigcup\limits_{\varepsilon<\theta} \Lambda^{\mathrm{pe}}_{\alpha,\varepsilon}$ and $\Lambda^{\mathrm{pe}}_{\alpha,\varepsilon,<\theta} := \bigcup\limits_{\zeta<\theta} \Lambda^{\mathrm{pe}}_{\alpha,\varepsilon,\zeta}$. If $\tau = \tau_R$ we may write $\Lambda^{\mathrm{pe}}_{\alpha,\varepsilon}(R)$.

The definition is as follows:

Case 1: $\alpha = 0$.

For R-modules:

$\Lambda^{\mathrm{pe}}_{0,\varepsilon}$ is the set of formulas $\varphi = \varphi(\bar{x})$ of the form $\sum\limits_{\ell<n} a_\ell x_{\zeta_\ell} = 0$ with $\zeta_\ell < \ell g(\bar{x})$. Equivalently (but perhaps better phrased), they are of the form $\sum\limits_{\zeta<\varepsilon} a_\zeta x_\zeta = 0$, where $a_\zeta \in R$ is 0_R for all but finitely many ζ-s.

For general τ: (so here, this is the τ-additive case).

$\Lambda^{\mathrm{pe}}_{0,\varepsilon}$ is the set of formulas $\varphi(\bar{x})$ of the form $P(\bar{\sigma}(\bar{x}))$, where $\bar{\sigma}$ is a sequence of length $\mathrm{arity}_\tau(P)$ of terms (in the variables $\bar{x}$). P may be equality, or any predicate from τ of arity equal to $\ell g(\bar{\sigma})$.

Case 2: α a limit ordinal.

$\Lambda^{\mathrm{pe}}_{\alpha,\varepsilon} := \bigcup\limits_{\beta<\alpha} \Lambda^{\mathrm{pe}}_{\beta,\varepsilon}$.

Case 3: $\alpha = \beta + 1$.

We define $\Lambda^{\mathrm{pe}}_{\alpha,\varepsilon}$ as the union of $\Lambda^{\mathrm{pe}}_{\beta,\varepsilon}$ together with the set of all formulas $\psi(\bar{x})$ of the form

$$(\exists \bar{y}_\zeta) \bigwedge \left\{ \varphi(\bar{x}\,\hat{}\,\bar{y}_\zeta) : \varphi(\bar{x}, \bar{y}_\zeta) \in \Phi_\zeta \right\},$$

for some $\zeta < \theta$ and $\Phi_\zeta \subseteq \Lambda^{\mathrm{pe}}_{\beta,\varepsilon,\zeta}$. (Again, keep in mind $\Lambda^{\mathrm{pe}}_{\beta,\varepsilon,\zeta}$ is a set of formulas of the form $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$.)

Claim 2.3. 1) *In 2.2, $\Lambda^{\mathrm{pe}}_{\alpha,\varepsilon}$ is $\subseteq$-increasing with α, and is of cardinality $\leq \beth_\alpha(|\tau| + \aleph_0)$ if $\theta = \aleph_0$, and $\beth_\alpha(|\tau|^{<\theta})$ in general.*

2) *For $M \in \mathbf{K}$ and $\varphi(\bar{x}) \in \Lambda^{\mathrm{pe}}_{\alpha,\varepsilon}(\tau)$, the set*

$$\varphi(\overline{M}) := \left\{ \bar{b} \in {}^{\varepsilon}M : M \models \varphi[\bar{b}] \right\}$$

is an Abelian subgroup of ${}^{\varepsilon}M$, and the set $\left\{ \bar{b} \in {}^{\varepsilon}M : M \models \varphi[\bar{b} - \bar{a}] \right\}$ is affine (that is, closed under $\bar{x} - \bar{y} + \bar{z}$) for any $\bar{a} \in {}^{\varepsilon}M$.

Proof. Easy. $\square_{2.3}$

Theorem 2.4. *For every α and every $M \in \mathbf{K}$, there is a subset $\mathbf{I} = \mathbf{I}_\alpha$ of ${}^{\theta>}M$ of cardinality $\leq \kappa_\alpha := \beth_\alpha(|\tau|^{<\theta})$ such that in M, we have*

$\boxdot_\alpha$ *Every formula $\psi(\bar{x})$ from $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$ (so $\ell g(\bar{x}) < \theta$) is equivalent in M to a Boolean combination of formulas (possibly infinitely many) of the form $\varphi(\bar{x} - \bar{a})$ with $\varphi(\bar{x}) \in \Lambda^{\rm pe}_{\alpha,\ell g(\bar{x})}(\tau)$ and $\bar{a} \in \mathbf{I} \cap {}^{\ell g(\bar{x})}M$.*

Before proving 2.4, we shall note that it implies

Conclusion 2.5. *For every $M \in \mathbf{K}$, $\varepsilon < \theta$, $\mathbf{I}_\alpha$ as in Theorem 2.4 for α a limit ordinal, and $\bar{a} \in {}^\varepsilon M$, for some $i_*, j_* \leq \kappa_\alpha$ and $\varphi_i(\bar{x}), \psi_j(\bar{x}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$ for $i < i_*$, $j < j_*$ we have that*

$$\big\{\bar{a}' \in {}^\varepsilon M : {\rm tp}_{\mathbb{L}^{\rm pe}_{\infty,\theta,\alpha}}(\bar{a}', \varnothing, M) = {\rm tp}_{\mathbb{L}^{\rm pe}_{\infty,\theta,\alpha}}(\bar{a}, \varnothing, M)\big\}$$

is equal to

$$\big\{\bar{a}' \in {}^\varepsilon M : M \models \bigwedge_{i<i_*} \varphi_i(\bar{a}' - \bar{a}) \wedge \bigwedge\{\neg\psi_j(\bar{a}' - \bar{a}'') : j < j_* \text{ and } \bar{a}'' \in \mathbf{I}_\alpha \cap {}^\varepsilon M\}\big\}.$$

Definition 2.6. 1) We say $\bar{b}_1, \bar{b}_2 \in {}^\varepsilon M$ are *α-equivalent over* $\mathbf{I} \subseteq {}^{\theta>}M$ <u>when</u>

$$\varphi(\bar{x}_{[\varepsilon]}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}(R) \wedge \bar{a} \in \mathbf{I} \cap {}^\varepsilon M \;\Rightarrow\; M \models \text{“}\varphi[\bar{b}_1 - \bar{a}] \Leftrightarrow \varphi[\bar{b}_2 - \bar{a}]\text{”}.$$

2) If we write $A \subseteq M$ instead of $\mathbf{I}$, we mean $\mathbf{I} = {}^{\theta>}A$.

We shall use freely

Observation 2.7. *The sequences $\bar{b}_1, \bar{b}_2 \in {}^\varepsilon M$ are α-equivalent over $\mathbf{I} \subseteq {}^\varepsilon M$ <u>iff</u> for any $\varphi(\bar{x}) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$ we have $(a) \vee (b)$, where:*

(*a*) *For some $\bar{a} \in \mathbf{I} \cap {}^\varepsilon M$ we have $M \models \varphi[\bar{b}_1 - \bar{a}] \wedge \varphi[\bar{b}_2 - \bar{a}]$.*
(*b*) *For every $\bar{a} \in \mathbf{I} \cap {}^\varepsilon M$ we have $M \models \neg\varphi[\bar{b}_1 - \bar{a}] \wedge \neg\varphi[\bar{b}_2 - \bar{a}]$.*

Proof. Straightforward, recalling basic properties of cosets of Abelian groups. $\square_{2.7}$

Remark 2.8. Note that the old proof of Theorem 2.4 relied on results in §4, but more is proved there than is necessary here. Specifically, we just need Definition 4.2(1) and clauses (a)+(b) of Claim 4.4.

However, the proof presented below does not rely on §4 at all.

Proof of Theorem 2.4. We choose $\mathbf{I}_\alpha$ by induction on α such that

$\circledast_\alpha$ (a) $\mathbf{I}_\alpha \subseteq {}^{\theta>}M$ is of cardinality $\leq \kappa_\alpha$.
(b) Clause $\boxdot_\alpha$ from the statement of the theorem holds.
(c) ${}^{\theta>}\{0\} \subseteq \mathbf{I}_\alpha$
(d) $\langle \mathbf{I}_\beta : \beta \leq \alpha \rangle$ is $\subseteq$-increasing continuous.

For $\alpha = 0$ choose $\mathbf{I}_0 := {}^{\theta>}\{0^M\}$, and for α a limit ordinal we obviously want $\mathbf{I}_\alpha := \bigcup_{\beta<\alpha} \mathbf{I}_\beta$.

So assume $\alpha = \beta + 1$ and $\mathbf{I}_\beta$ is given, and we shall choose $\mathbf{I}_\alpha$ such that

$\boxplus_{\beta+1}$ (a) $\mathbf{I}_\alpha$ is a subset of ${}^{\theta>}M$.
(b) $|\mathbf{I}_\alpha| \leq 2^{\kappa_\beta}$ (recalling $\kappa_\beta := \beth_\beta(|\tau|^{<\theta})$).
(c) $\mathbf{I}_\beta \subseteq \mathbf{I}_\alpha$

(d) If $\zeta < \theta$ and $\bar{\varphi} = \langle \varphi_i(\bar{x}) : i < i_* \leq \kappa_\beta \rangle$ and $\bar{\varphi}^\bullet = \langle \varphi_i^\bullet(\bar{x}) : i < i_\bullet \rangle$ are sequences in $\Lambda^{\rm pe}_{\beta,\zeta}$, <u>then</u> for some $\gamma_* \leq \kappa_\beta^+$ there exists $\langle \bar{d}_\gamma : \gamma < \gamma_* \rangle \subseteq \mathbf{I}_\alpha$ with $\ell g(\bar{d}_\gamma) := \zeta$ such that

- $\bullet_1$ $\bar{d}_\gamma \in \bigcap\limits_{i<i_*} \varphi_i(M)$ for all $\gamma < \gamma_*$.
- $\bullet_2$ $\bar{d}_{\gamma_2} - \bar{d}_{\gamma_1} \notin \bigcup\limits_{i<i_\bullet} \varphi_i^\bullet(M)$ for all[1] $\gamma_1 < \gamma_2 < \gamma_*$.
- $\bullet_3$ If $\gamma_* < \kappa_\beta^+$ and $\bar{d} \in \bigcap\limits_{i<i_*} \varphi_i(M)$ <u>then</u> there exist $\gamma < \gamma_*$ and $i < i_\bullet$ such that $\bar{d} - \bar{d}_\gamma \in \varphi_i^\bullet(M)$.

Why can we choose $\mathbf{I}_\alpha$? Let Ξ be the set of triples $\mathbf{f} = (\zeta, \bar{\varphi}, \bar{\varphi}^\bullet)$ satisfying the assumptions of clause (d). Clearly Ξ has cardinality $\leq \left|\mathcal{P}(\Lambda^{\rm pe}_{\beta,\varepsilon+\varepsilon}) \times \mathbf{I}_\alpha\right| \leq 2^{\kappa_\beta} + |\mathbf{I}_\alpha| = 2^{\kappa_\beta}$.

Rephrasing clause $\boxplus_{\beta+1}$(d) in this notation, we mean:

> For every $\mathbf{f} \in \Xi$ there are $\gamma_{\mathbf{f}} \leq \kappa_\beta^+$ and $\mathfrak{d}_{\mathbf{f}} = \langle \bar{d}_{\mathbf{f},\gamma} : \gamma < \gamma_{\mathbf{f}} \rangle \subseteq \mathbf{I}_\alpha$ such that the conclusions of clause (d) hold.

Let us fix $\mathbf{f} = (\zeta_{\mathbf{f}}, \bar{\varphi}_{\mathbf{f}}, \bar{\varphi}_{\mathbf{f}}^\bullet)$ and <u>try</u> to choose $\bar{d}_{\mathbf{f},\gamma}$ by induction on $\gamma < \kappa_\beta^+$ to satisfy demands $\bullet_1$ and $\bullet_2$ of $\boxplus_{\beta+1}$(d).

Let $\gamma_{\mathbf{f}} \leq \kappa_\beta^+$ be the first ordinal where we fail (or just where we stop). That is,

$\boxplus^1_{\mathbf{f}}$ $\bar{d}_{\mathbf{f},\gamma}$ is defined <u>iff</u> $\gamma < \gamma_{\mathbf{f}}$.

(Note that with these bounds, we will never try to define $\gamma_{\mathbf{f},\kappa_\beta^+}$.)

Let $\mathfrak{d}_{\mathbf{f}}$ denote the sequence $\langle \bar{d}_{\mathbf{f},\gamma} : \gamma < \gamma_{\mathbf{f}} \rangle$. Note that

$\boxplus^2_{\mathbf{f}}$ $\gamma_{\mathbf{f}} \geq 1$ <u>iff</u> $\bigcap\limits_{i<i_*} \varphi_i(M) \neq \varnothing$.

[Why? Because bullet (d) $\bullet_2$ is vacuous when choosing $\bar{d}_{\mathbf{f},0}$, and $\bullet_1$ would be impossible to satisfy were the intersection empty.]

So for proving that $\mathfrak{d}_{\mathbf{f}}$ is as required in $\boxplus_{\beta+1}$(d), we have to verify $\bullet_3$.

Case 1: $\gamma_{\mathbf{f}} < \kappa_\beta^+$.

Here we know $\mathfrak{d}_{\mathbf{f}}$ must also satisfy clause $\boxplus_{\beta+1}$(d)$\bullet_3$, as we cannot choose $\bar{d}_{\mathbf{f},\gamma_{\mathbf{f}}}$.

In more detail: given some $\bar{d} \in \bigcap\limits_{i<i_*} \varphi_i(M)$, we have to prove that there exist $\gamma < \gamma_{\mathbf{f}}$ and $i < i_\bullet$ such that $\bar{d} - \bar{d}_\gamma \in \varphi_i^\bullet(M)$. Assume for the sake of a contradiction that there are no such γ or i: in other words,

$$\bar{d} - \bar{d}_{\gamma_1} \notin \bigcup_{i<i_\bullet} \varphi_i^\bullet(M)$$

for any $\gamma_1 < \gamma_{\mathbf{f}}$, which is precisely (d) $\bullet_2$. $\bar{d}$ also satisfies bullet (d) $\bullet_1$ by assumption, and so $\mathfrak{d}_{\mathbf{f}}{}^\frown\langle \bar{d} \rangle$ satisfies clause (d), contradicting $\boxplus^1_{\mathbf{f}}$ (the maximality of $\mathfrak{d}_{\mathbf{f}}$).

Case 2: $\gamma_{\mathbf{f}} = \kappa_\beta^+$.

In this case we are done, as demand $\bullet_3$ is vacuous.

So we have succeed in:

$\boxtimes_3$
- Choosing $\mathfrak{d}_{\mathbf{f}} = \langle \bar{d}_{\mathbf{f},\gamma} : \gamma < \gamma_{\mathbf{f}} \rangle$ for every $\mathbf{f} \in \Xi$.
- Proving that (d)$\bullet_1$-$\bullet_3$ holds for every $\mathbf{f} \in \Xi$.

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[1] **Note:** $\bullet_2$ does not exactly say that $\varphi_i^\bullet(M) \cap \bigcap\limits_{j<i_*} \varphi_j(M)$ has many cosets inside $\bigcap\limits_{i<i_*} \varphi_i(M)$. If $i_\bullet := 1$ and $\gamma_* := \kappa^+$ then yes, but otherwise it is more complicated.

Now let $\mathbf{I}_\alpha := \mathbf{I}_\beta \cup \bigcup_{\mathbf{f} \in \Xi} \{\bar{d}_{\mathbf{f},\gamma} : \gamma < \gamma_{\mathbf{f}}\}$; clearly it is as required in $\boxplus_{\beta+1}$.

* * *

At this point all we have done is prove that $\mathbf{I}_\alpha$ (for $\alpha = \beta + 1$) can be chosen as required in $\boxplus_{\beta+1}$. Our job now is to prove that this $\mathbf{I}_\alpha$ satisfies the demands of $\circledast_\alpha$.

Clauses $\circledast_\alpha$(a) and (d) are obvious, and clause (c) is satisfied by our construction, so we are left with $\circledast_\alpha$(b) — equivalently, 2.4$\boxdot_\alpha$. (Proving that we actually fulfill what was written in the theorem.)

To this end, clearly it suffices to prove the following.

$\boxtimes_4$ Assume $\varepsilon, \xi < \theta$, $\bar{b}_1, \bar{b}_2 \in {}^\varepsilon M$ are α-equivalent over $\mathbf{I}_\alpha$, and $\bar{c}_1 \in {}^\xi M$. Then for some $\bar{c}_2 \in {}^\xi M$ the sequences $\bar{b}_1 \hat{\ } \bar{c}_1$ and $\bar{b}_2 \hat{\ } \bar{c}_2 \in {}^{\varepsilon+\xi} M$ are β-equivalent over $\mathbf{I}_\beta$.

Let $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}) : i < i_* \rangle$ list the formulas in the set

$$\Phi_1 := \big\{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm pe}_{\beta,\varepsilon,\xi} : \text{there exists } \bar{b} \hat{\ } \bar{c} \in \mathbf{I}_\alpha \text{ such that } \ell g(\bar{b}) = \varepsilon, \\ \ell g(\bar{c}) = \xi, \text{ and } M \models \varphi(\bar{b}_1 - \bar{b}, \bar{c}_1 - \bar{c})\big\}.$$

(Note that if $M \models \varphi[\bar{b}_1, \bar{c}_1]$ then $\varphi(\bar{x}, \bar{y}) \in \Phi_1$ as we can choose $\bar{b} := \bar{0}_\varepsilon$ and $\bar{c} := \bar{0}_\zeta$.)

Similarly, let $\bar{\varphi}^\bullet = \langle \varphi^\bullet_i(\bar{x}, \bar{y}) : i < i_\bullet \rangle$ list the formulas in $\Phi_2 := \Lambda^{\rm pe}_{\beta,\varepsilon,\xi} \setminus \Phi_1$. Clearly, without loss of generality i_* and $i_\bullet$ are ordinals $\le \kappa_\beta$, as $\big|\Lambda^{\rm pe}_{\beta,\varepsilon,\xi}\big| = \kappa_\beta$.

Let $\zeta := \varepsilon + \xi$, and let $\langle \bar{d}_\gamma : \gamma < \gamma_* \rangle$ be as guaranteed to exist by clause $\boxplus_{\beta+1}$(d), for $\mathbf{f} = (\zeta, \bar{\varphi}, \bar{\varphi}^\bullet) \in \Xi$ as above.

Let $\bar{b}^\bullet_\gamma := \bar{d}_\gamma \restriction \varepsilon$ and $\bar{c}^\bullet_\gamma := \bar{d}_\gamma \restriction [\varepsilon, \varepsilon + \xi)$.

Case 1: $\gamma_* < \kappa^+_\beta$.

Let $\bar{d} := \bar{b}_1 \hat{\ } \bar{c}_1$, so clearly $\bar{d} \in \bigcap_{i < i_*} \varphi_i(M)$. By clause $\boxplus_{\beta+1}$(d)$\bullet_3$ we have $\bar{d} - \bar{d}_\gamma \in \varphi^\bullet_i(M)$ for some $\gamma < \gamma_*$ and $i < i_\bullet$. As $\bar{d}_\gamma \in \mathbf{I}_\alpha$, this implies that $\varphi^\bullet_i \in \Phi_1$. But we know $\varphi^\bullet_i \in \Phi_2$ by our definition of $\bar{\varphi}^\bullet$, a contradiction.

Case 2: $\gamma_* = \kappa^+_\beta$.

As $\bar{b}_1$ and $\bar{b}_2$ are α-equivalent (see 2.6, 2.7) there exists $\bar{c}'_2 \in {}^\xi M$ such that $M \models \bigwedge_{i < i_*} \varphi_i(\bar{b}_2, \bar{c}'_2)$. By our choice of the $\bar{d}_\gamma$-s, for every $\gamma < \gamma_*$ we have $M \models \bigwedge_{i < i_*} \varphi_i(\bar{b}^\bullet_\gamma, \bar{c}^\bullet_\gamma)$, hence

$$M \models \bigwedge_{i < i_*} \varphi_i[\bar{b}_2 - \bar{b}^\bullet_\gamma + \bar{b}_1, \bar{c}'_2 - \bar{c}^\bullet_\gamma + \bar{c}_1].$$

For each $i < i_\bullet$ the set

$$\mathcal{S}_i := \big\{\gamma < \gamma_* : M \models \varphi^\bullet_i[\bar{b}_2 - \bar{b}^\bullet_\gamma + \bar{b}_1, \bar{c}'_2 - \bar{c}^\bullet_\gamma + \bar{c}_1]\big\}$$

has at most one member (by $\boxplus_{\beta+1}$(d)$\bullet_2$ plus some algebra).

[In more detail: towards contradiction, assume $\gamma_1 \ne \gamma_2$ are in $\mathcal{S}_i$, so

$$M \models \varphi^\bullet_i[\bar{b}_2 - \bar{b}^\bullet_{\gamma_\ell} + \bar{b}_1, \bar{c}'_2 - \bar{c}^\bullet_{\gamma_\ell} + \bar{c}_1]$$

for $\ell = 1, 2$.

But $\varphi^\bullet_i(M)$ is closed under subtraction, and

$$(\bar{c}'_2 - \bar{c}^\bullet_{\gamma_1} + \bar{c}_1) - (\bar{c}'_2 - \bar{c}^\bullet_{\gamma_2} + \bar{c}_1) = \bar{c}^\bullet_{\gamma_2} - \bar{c}^\bullet_{\gamma_1}$$

and

$$(\bar{b}_2 - \bar{b}^\bullet_{\gamma_1} + \bar{b}_1) - (\bar{b}_2 - \bar{b}^\bullet_{\gamma_2} + \bar{b}_1) = \bar{b}^\bullet_{\gamma_2} - \bar{b}^\bullet_{\gamma_1}.$$

Hence $M \models \varphi^\bullet_i[\bar{b}^\bullet_{\gamma_2} - \bar{b}^\bullet_{\gamma_1}, \bar{c}^\bullet_{\gamma_2} - \bar{c}^\bullet_{\gamma_1}]$, contradicting $\boxplus_{\beta+1}$(d)$\bullet_2$.]

Hence

$$\Big|\bigcup_{i<i_\bullet}\mathcal{S}_i\Big| \le |i_\bullet| \le i_\bullet \le \kappa_\beta < \gamma_* := \kappa_\beta^+,$$

so there exists $\gamma \in \gamma_* \setminus \bigcup_{i<i_\bullet}\mathcal{S}_i$. Now $\bar{c}_2' - \bar{c}_\gamma^\bullet + \bar{c}_1$ is as required in clause $\boxtimes_4$. $\square_{2.4}$

* * *

Earlier Proof of 2.4: We choose $\mathbf{I}_\alpha$ by induction on α such that

$\circledast_\alpha$ (a) $\mathbf{I}_\alpha \subseteq {}^{\theta>}M$ is of cardinality $\le \kappa_\alpha$.

(b) $(\forall\varepsilon<\theta)\big(\forall\varphi(\bar{x})\in\Lambda^{\rm pe}_{\alpha,\varepsilon}(\tau)\big)\big(\forall\bar{a}\in{}^\varepsilon M\big)\big(\exists\bar{b}\in\mathbf{I}_\alpha\cap{}^\varepsilon M\big)\big[M\models\varphi[\bar{b}-\bar{a}]\big]$

(c) $\langle\mathbf{I}_\beta:\beta\le\alpha\rangle$ is $\subseteq$-increasing continuous.

(d) $\mathbf{I}_\alpha$ satisfies the demands of the theorem (i.e. $\boxdot_\alpha$ holds).

For $\alpha=0$ choose $\mathbf{I}_0 := {}^{\theta>}\{0^M\}$, and for α a limit ordinal we obviously want $\mathbf{I}_\alpha := \bigcup_{\beta<\alpha}\mathbf{I}_\beta$.

So assume $\alpha=\beta+1$ and $\mathbf{I}_\beta$ is given, and we shall choose $\mathbf{I}_\alpha$ such that

$\boxplus_{\beta+1}$ (a) $\mathbf{I}_\alpha$ is a subset of ${}^{\theta>}M$.

(b) $|\mathbf{I}_\alpha|\le 2^{\kappa_\beta}$, where $\kappa_\beta := \beth_\beta(|\tau|^{<\theta})$.

(c) $\mathbf{I}_\beta\subseteq\mathbf{I}_\alpha$

(d) If $\varepsilon<\theta$, $\varphi_i(\bar{x})\in\Lambda^{\rm pe}_{\beta,\varepsilon}$, $\bar{a}_i\in\mathbf{I}_\beta\cap{}^\varepsilon M$ for $i<i_*\le\kappa_\beta$, and there is $\bar{d}\in{}^\varepsilon M$ such that $M\models\bigwedge_{i<i_*}\varphi_i[\bar{d}-\bar{a}_i]$, <u>then</u> there is such $\bar{d}\in\mathbf{I}_\alpha$.

(e) Assume $\varepsilon<\theta$, $\ell g(\bar{x})=\varepsilon$, $\psi(\bar{x})$ is a conjunction of formulas from $\Lambda^{\rm pe}_{\beta,\varepsilon}$ and $\varphi_i(\bar{x})\in\Lambda^{\rm pe}_{\beta,\varepsilon}$ for $i<\kappa_\beta$.
Let G be the Abelian group with set of elements

$$\psi({}^\varepsilon M) = \big\{\bar{a}\in{}^\varepsilon M : M\models\psi[\bar{a}]\big\}.$$

and addition of two such sequences defined coordinatewise. Let $\bar{\varphi} = \langle\varphi_i : i<\kappa_\beta\rangle$ and $G_i := \varphi_i(M)\cap\psi(M)$ (so it is a subgroup of G). Letting $\mu := 2^{\kappa_\beta}$, we apply 4.4 with $\lambda := \mu^+$, $S := \kappa_\beta$, and $\overline{G} := \langle G_i : i\in S\rangle$ (although for this proof we will only need clauses (a) and (b) of the conclusion).
This gives us a certain family of sets $I\subseteq\mathcal{P}(S)$ – not necessarily an ideal. Further assume that $S := \kappa_\beta\notin I$. <u>Then</u>

- $\bullet_1$ There are $\bar{d}_\iota\in\mathbf{I}_\alpha\cap\varphi_i({}^\varepsilon M)$ for $\iota<\iota_*\le\mu$ such that for every $\bar{a}\in G := \psi({}^\varepsilon M)$ there exists $\iota<\iota_*$ such that

$$\big\{j\in S : \bar{a}-\bar{d}_\iota\notin\varphi_j({}^\varepsilon M)\big\}\in I.$$

- $\bullet_2$ For any $u\in I$ there is a set u_* with $u\subseteq u_*\in I$ and a sequence

$$\langle\bar{d}_\iota:\iota<\mu\rangle\subseteq\bigcap\{\varphi_i({}^\varepsilon M) : i\in S\setminus u_*\}\cap\psi({}^\varepsilon M)\cap\mathbf{I}_\alpha$$

such that

$$\big(\forall i<\kappa_\beta\big)\big(\forall\iota_1\le\iota_2<\mu\big)\big[\bar{d}_{\iota_1}-\bar{d}_{\iota_2}\notin\varphi_i({}^\varepsilon M)\Leftrightarrow i\in u_*\big].$$

(f) If $\varepsilon<\theta$ and $\bar{d}_1,\bar{d}_2\in\mathbf{I}_\alpha\cap{}^\varepsilon M$ <u>then</u> $\bar{d}_1+\bar{d}_2\in\mathbf{I}_\alpha$, $\bar{d}_1-\bar{d}_2\in\mathbf{I}_\alpha$, and $\xi<\theta\Rightarrow\bar{0}_\xi{}^\frown\bar{d}_1\in\mathbf{I}_\alpha$.

Why can we choose such $\mathbf{I}_\alpha$? It suffices to show that each of the demands above hold for a club's worth of sets in $[{}^\varepsilon M]^\mu$.

For reference, recall

- For a set X and cardinal Υ, E is a club of $[X]^{\leq\Upsilon}$ iff there exist functions $F_{\xi,n} : X \to X$ for $\xi < \Upsilon$ and $n < \omega$ such that
$$E = \{u \in [X]^{\leq\Upsilon} : u \text{ is closed under the } F_{\xi,n}\text{-s}\}.$$
- If E_i is a club of $[X]^{\leq\Upsilon}$ for $i < i_* \leq \Upsilon$ then so is $\bigcap_{i<i_*} E_i$.

Now $\boxplus_{\beta+1}$(a),(b) are obvious.

This holds for clause (c) by the induction hypothesis, as we will choose $\mathbf{I}_\alpha$ by adding elements to $\mathbf{I}_\beta$.

This holds for clause (d) because $\Lambda^{\rm pe}_{\beta,\varepsilon}$ has cardinality κ_β and $\mu = 2^{\kappa_\beta}$.

This holds for (e)$\bullet_1$ by clause (a) of Claim 4.4and for (e)$\bullet_2$ by 4.4(b).

To prove the induction statement for α, clearly it suffices to prove the following.

$\boxdot$ Assume $\varepsilon, \xi < \theta$, $\bar b_1, \bar b_2 \in {}^\varepsilon M$ are α-equivalent over $\mathbf{I}_\alpha$, and $\bar c_1 \in {}^\xi M$. Then for some $\bar c_2 \in {}^\xi M$ the sequences $\bar b_1{}^\frown \bar c_1$ and $\bar b_2{}^\frown \bar c_2 \in {}^{\varepsilon+\xi}M$ are β-equivalent over $\mathbf{I}_\beta$. (Here we rely on 2.7.)

Why does $\boxdot$ hold? Let $\bar x$ be of length ε and $\bar y$ of length ξ. Let
$$\Phi_1 := \big\{\varphi(\bar x, \bar y) \in \Lambda^{\rm pe}_{\beta,\varepsilon+\xi} : \text{for some } \bar a \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi}M \text{ we have } M \models \varphi[\bar b_1{}^\frown \bar c_1 - \bar a]\big\}.$$
For $\varphi(\bar x, \bar y) \in \Phi_1$, we know (by the induction hypothesis) that $\circledast_\beta$ holds; hence we can choose $\bar a_{\varphi(\bar x,\bar y)} \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi}M$ such that $M \models \varphi[\bar b_1{}^\frown \bar c_1 - \bar a_{\varphi(\bar x,\bar y)}]$. Let $\Phi_2 := \Lambda^{\rm pe}_{\beta,\varepsilon+\xi} \setminus \Phi_1$.

So by $\boxplus_{\beta+1}$(d) there is a sequence $\bar b^*{}^\frown \bar c^* \in \mathbf{I}_\alpha$ such that $\ell g(\bar b^*) = \ell g(\bar b_1)$, $\ell g(\bar c^*) = \ell g(\bar c_1)$, and $\varphi(\bar x, \bar y) \in \Phi_1 \Rightarrow M \models \varphi[\bar b^*{}^\frown \bar c^* - \bar a_{\varphi(\bar x,\bar y)}]$. For transparency, note that if $\Phi_2 = \varnothing$ then (as the formula $(\exists \bar y) \bigwedge_{\varphi\in\Phi_1} \varphi(\bar x, \bar y)$ is a member of $\Lambda^{\rm pe}_{\alpha,\varepsilon+\xi}$) clearly by the assumption of $\boxdot$ there is $\bar c_2 \in {}^\xi M$ such that
$$\varphi(\bar x, \bar y) \in \Phi_1 \Rightarrow M \models \varphi(\bar b_2{}^\frown \bar c_2 - \bar a_{\varphi(\bar x,\bar y)}).$$
So $\bar c_2$ is as required, hence we are done.

So without loss of generality $\Phi_2 \neq \varnothing$. Clearly $|\Phi_2| \leq \kappa_\beta$, and let
$$\Phi'_\ell := \{\varphi(\bar 0_\varepsilon, \bar y) : \varphi(\bar x, \bar y) \in \Phi_\ell\}$$
for $\ell = 1, 2$.

Let $\{\neg\varphi_i(\bar x{}^\frown \bar y - \bar a_i) : i < \kappa_\beta\}$ list (possibly with repetitions) the set of formulas $\neg\varphi(\bar x{}^\frown \bar y - \bar a)$ satisfied by $\bar c_1{}^\frown \bar b_1$ with $\bar a \in \mathbf{I}_\beta$ and $\varphi(\bar x, \bar y) \in \Lambda^{\rm pe}_{\beta,\varepsilon,\zeta}$ (equivalently, $\varphi(\bar x, \bar y) \in \Phi_2$). Let $\varphi'_i(\bar y) := \varphi_i(0_\varepsilon, \bar y)$ for $i < \kappa_\beta$, and $\psi'(\bar y) := \bigwedge_{\varphi\in\Phi'_1} \varphi(\bar y)$.

As in $\boxplus_{\beta+1}$, let $S := \kappa_\beta$ and $I = I_\lambda \subseteq \mathcal{P}(S)$ be defined as in Definition 4.2, with $G := \psi'({}^\xi M)$, $G_i := G \cap \varphi'_i({}^\xi M)$ for $i \in S$, and $\lambda := \mu^+ = (2^{\kappa_\beta})^+$.

Case 1: $S := \kappa_\beta \in I$.

So clearly $M \models \varphi[\bar b_1 - \bar b^*, \bar c_1 - \bar c^*]$ for every $\varphi(\bar x, \bar y) \in \Phi_1$.

Let $\psi_*(\bar x, \bar y) = \bigwedge\{\varphi(\bar x, \bar y) : \varphi(\bar x, \bar y) \in \Phi_1\}$; clearly it is a member of $\Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta}$ and $M \models \psi_*[\bar b_1, \bar c_1]$. Hence by the choice of $(\bar b^*, \bar c^*)$ we also have $M \models \psi_*[\bar b^*, \bar c^*]$. As ψ_* is positive existential, clearly $M \models \psi_*[\bar b_1 - \bar b^*, \bar c_1 - \bar c^*]$, hence
$$M \models (\exists \bar y)\psi_*[\bar b_1 - \bar b^*, \bar y].$$
But $(\exists \bar y)\psi(\bar x, \bar y) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$, so by the assumption on $\bar b_1$ and $\bar b_2$ we have
$$M \models (\exists \bar y)\psi_*[\bar b_2 - \bar b^*, \bar y],$$
hence for some $\bar c'_2$ we have $M \models \psi_*[\bar b_2 - \bar b^*, \bar c'_2]$. Let $\bar c''_2 := \bar c'_2 + \bar c^*$, so
$$M \models \psi_*[\bar b_2 - \bar b^*, \bar c''_2 - \bar c^*].$$

By $\boxplus_{\beta+1}(\mathrm{e})\bullet_2$ and our assumption that $\kappa_\beta \in I$, there is a sequence $\langle \bar e_\iota : \iota < \mu\rangle$ of members of G (i.e. of $\{\bar a \in {}^\xi M : M \models \psi_*(\bar 0_\varepsilon, \bar a)\}$), recalling that $\psi'(\bar y = \psi(0_\varepsilon, \bar y)$, such that

$$i < \kappa_\beta \wedge (\iota_1 < \iota_2 < \mu) \Rightarrow \bar e_{\iota_2} - \bar e_{\iota_1} \notin G_i.$$

So for every $\iota < \mu$, the sequence $(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* + \bar e_\iota)$ belongs to $\psi_*({}^{\varepsilon+\xi}M)$ and for each $i < \kappa_\beta$ the set $\{\iota < \mu : (\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* + \bar e_\iota)$ belongs to $(\bar a_i - \bar b^*\hat{}\bar c^*) + G_i\}$ has at most one member. As $\kappa_\beta < \mu$, we have

$$(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* + \bar e_\iota) \notin \bigcup\{(\bar a_i - \bar b^*)\hat{}(\bar c^* + G_i) : i < \kappa_\beta\}$$

for[2] some $\iota < \mu$.

So $\bar c_2 := \bar c_2'' + \bar e_\iota$ is as required.

Case 2: $S \notin I$.

So there is a sequence $\langle \bar d_\iota : \iota < \iota_*\rangle$ of members of $\mathbf{I}_\alpha$ as in $\boxplus_{\beta+1}(\mathrm{e})\bullet_1$ for ξ, G, $\langle G_i : i < \kappa_\beta\rangle$ as above (i.e. with $\psi'(\bar y)$, $\langle \varphi_i'(\bar y) : i < \kappa_\beta\rangle$ here standing in for $\psi(\bar x), \langle \varphi_i(\bar x) : i < \kappa_\beta\rangle$ there). So $\iota_* < (2^{\kappa_\beta})^+ = \mu^+$ and $\iota < \iota_* \Rightarrow \bar d_\iota \in \mathbf{I}_\alpha \cap {}^\xi M$. As clearly $\bar c_1 - \bar c^* \in G$, necessarily for some $\iota < \iota_*$ the set

$$u := \{i < \kappa_\beta : (\bar c_1 - \bar c^* - \bar d_\iota) \notin G_i\}$$

belongs to I (and of course, $\bar b^*\hat{}(\bar c^* + \bar d_\iota) \in \mathbf{I}_\alpha \cap {}^{\varepsilon+\xi}M$) and we have:

- $(*)_1$ $M \models \varphi[\bar b_1 - \bar b^*, \bar c_1 - \bar c^* - d_\iota]$ for $\varphi \in \Phi_1$.
- $(*)_2$ If $i \in \kappa_\beta \setminus u$ then $M \models \varphi_i[\bar b_1 - \bar b^*, \bar c_1 - \bar c^* - \bar d_\iota]$.

As in Case 1, there is $\bar c_2'' \in {}^\xi M$ such that

- $(*)_3$ $M \models \varphi[\bar b_2 - \bar b^*, \bar c_2'' - \bar c^* - \bar d_\iota]$ for $\varphi \in \Phi_1$.

Hence

- $(*)_4$ If $i \in \kappa_\beta \setminus u$ then $M \models \varphi_i[\bar b_2 - \bar b^*, \bar c_2'' - \bar c^* - \bar d_\iota]$.

As $u \in I$ by $\boxplus_{\beta+1}(\mathrm{e})\bullet_2$ (that is, by 4.4) there are $\bar{\mathbf{e}} = \langle \bar e_j : j < \mu\rangle$ and u_* with $u \subseteq u_* \in I$ such that:

- $(*)_5$ $\{\bar e_j : j < \mu\} \subseteq \bigcap\limits_{i \in \kappa_\beta \setminus u_*} G_i$.
- $(*)_6$ $\bar e_{j_2} - \bar e_{j_1} \notin G_i$ for all $j_1 < j_2 < \mu$ and $i \in u_*$.

So

- $(*)_7$ If $j < \mu$ then $(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j)$ belongs to $\bigcap\limits_{\varphi \in \Phi_1} \varphi({}^{\varepsilon+\xi}M)$.
- $(*)_8$ If $i \in \kappa_\beta \setminus u_*$ then $i \in \kappa_\beta \setminus u$ as well, so by $(*)_4$+$(*)_5$ the sequence
$$(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j)$$
satisfies $\varphi_i(\bar x\hat{}\bar y - \bar a_i)$ in M, hence $\bar b_2\hat{}(\bar c_2'' - \bar e_j)$ satisfies the formula $\neg\varphi_i(\bar x\hat{}\bar y - \bar a_i)$ in M.

Lastly, by $(*)_6$,

- $(*)_9$ For each $i \in u_*$ there is $j_i < \mu$ such that for every $j \in \mu \setminus \{j_i\}$, the sequence $(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j)$ satisfies $\neg\varphi_i(\bar x\hat{}\bar y - \bar a_i)$.
- $(*)_{10}$ Moreover,
  - (a) The set $\mu \setminus \{j_i : i \in u_*\}$ is non-empty.
  - (b) For some (equivalently, 'for every') j in this set we have
  $$(\forall i \in u_*)\neg\varphi_i\big[(\bar b_2 - \bar b^*)\hat{}(\bar c_2'' - \bar c^* - \bar d_\iota - \bar e_j) - \bar a_i\big].$$

[2] Recall that $\bar b + G_i$ just means $\{\bar b + \bar a : \bar a \in G_i\}$.

[Why? Clause (a) is true simply because $\mu := 2^{\kappa_\beta} > \kappa_\beta \geq |u_*|$, and clause (b) follows from $(*)_9$.]

Putting together $(*)_7$–$(*)_{10}$, clearly $(\bar{c}''_2 - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ is as required in $\boxdot$, so we are done. $\square_{2.4OLD}$

Conclusion 2.9. *If λ is a fixed point of the beth sequence (i.e. $\lambda = \beth_\lambda$) then for τ a θ-additive vocabulary for $M \in \mathbf{K}$, every formula $\varphi(\bar{x}) \in \mathbb{L}_{<\lambda,<\lambda}(\tau)$ (i.e. $\bigcup\limits_{\theta<\lambda} \mathbb{L}_{\theta,\theta}(\tau)$) is equivalent to a Boolean combination of positive existential formulas (from $\mathbb{L}_{\theta,\theta}$ for some $\theta = \theta_\varphi < \lambda$).*

§ 3. Stability

Context 3.1. 1) R is a fixed ring with $\tau = \tau_R$, <u>or</u> τ is a θ-additive vocabulary; $\mathbf{K}$ is the class of τ-additive models.

2) M will denote a member of $\mathbf{K}$.

3) $\theta = \mathrm{cf}(\theta)$ and γ^* is an ordinal — limit, for simplicity.

4) $\bar\lambda = \langle \lambda_\alpha : \alpha \le \gamma^* \rangle$, where $\lambda_\alpha > \kappa_\alpha := \beth_\alpha\big(|R| + \theta^-\big)$.

5) $\bar{\mathbf{I}}^*$ is a $(\bar\lambda, \theta, \gamma^*)$-witness (that is, $\mathbf{I}^*_\alpha$ satisfies the conditions of 2.4 and $|\mathbf{I}^*_\alpha| < \lambda_\alpha$ for each $\alpha < \ell g(\bar{\mathbf{I}}^*) := \gamma_*$).

6) $A_* := \bigcup\{\bar a : \bar a \in \mathbf{I}_{\gamma^*}\}$.

7) $\Lambda_\varepsilon := \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon}$ for $\varepsilon < \theta$, and $\Lambda := \bigcup\limits_{\varepsilon<\theta} \Lambda_\varepsilon$.

8) $M_* = M_{A_*} := (M, a)_{a \in A_*}$.

Definition 3.2. Assume $\varepsilon < \theta$, $\Lambda \subseteq \Lambda^{\mathrm{pe}}_{\theta,\gamma^*}$, $A_* \subseteq A \subseteq M \in \mathbf{K}$, and $\bar a \in {}^\varepsilon M$.

1) For $\bar a \in {}^\varepsilon M$, let

$$\mathrm{tp}_\Lambda(\bar a, A, M) := \big\{\varphi(\bar x{}^\frown\bar b - \bar c) : \bar b \in {}^\xi A,\ \bar c \in {}^{\varepsilon+\xi} M,\ M \models \varphi[\bar a_1{}^\frown\bar b - \bar c], \text{ and } \varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon+\xi} \cap \Lambda\big\}.$$

2) $\mathbf{S}^\varepsilon_\Lambda(A, M) := \{\mathrm{tp}_\Lambda(\bar a, A, M) : \bar a \in {}^\varepsilon M\}$.

Theorem 3.3 (The Stability Theorem). *Assume $\Lambda \subseteq \Lambda^{\mathrm{pe}}_{\gamma^*}$ and $A \subseteq M \in \mathbf{K}$.*

1) *The set $\mathbf{S}^\varepsilon_\Lambda(A, M)$ has cardinality $\le \big(|A|^{<\theta}\big)^{|\Lambda|}$.*

2) *For any $\kappa \ge 4$ (yes,* four*!) there are no $\langle \bar a_\alpha : \alpha < \kappa\rangle \subseteq {}^\varepsilon M$, $\langle \bar b_\alpha : \alpha < \kappa \rangle \subseteq {}^\zeta M$ and*[3] *$\varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi}$ such that for $\alpha < \beta < \kappa$ we have*

$$M \models \text{“}\varphi[\bar a_\alpha, \bar b_\beta] \wedge \neg\varphi[\bar a_\beta, \bar b_\alpha]\text{”}.$$

3) *If the formula $\varphi(\bar x, \bar y)$ is from $\mathbb{L}_{\infty,\theta,\gamma^*}$ (or is just a Boolean combination of such formulas) and $\kappa \ge \beth_{\gamma^*+2}(|\tau|^{<\theta})^+$, <u>then</u> there are no $M \in \mathbf{K}$, $\langle \bar a_\alpha : \alpha < \kappa\rangle \subseteq {}^\varepsilon M$, and $\langle \bar b_\alpha : \alpha < \kappa\rangle \subseteq {}^\zeta M$ such that*

$$M \models \varphi[\bar a_\alpha, \bar b_\beta] \wedge \neg\varphi[\bar a_\beta, \bar b_\alpha]$$

whenever $\alpha < \beta < \kappa$.
(Actually, $\kappa \ge \beth_{\gamma^*+1}(|\tau|^{<\theta})^+$ will suffice.)

4) *If $p \in \mathbf{S}^\varepsilon_\Lambda(A, M)$, $\varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi}$, and $p \cap \{\varphi(\bar x, \bar b) : \bar b \in {}^\xi A\} \ne \varnothing$, <u>then</u> for some $\bar a_\varphi \in {}^\varepsilon A$ and $\bar b \in {}^\xi A$ we have $\varphi(\bar x - \bar a_\varphi, \bar b) \vdash p \restriction \{\pm\varphi\}$ and $\varphi(\bar x - \bar a_\varphi, \bar b) \in p$.*

Proof. 1) Consider the statement:

⊛ If $\varphi(\bar x, \bar y) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi} \cap \Lambda$,

$$p_\ell(\bar x) := \mathrm{tp}_{\{\varphi(\bar x,\bar y)\}}(\bar a_\ell, A, M) \in \mathbf{S}^\varepsilon_{\{\varphi(\bar x,\bar y)\}}(A, M)$$

for $\ell = 1, 2$, $\bar b \in {}^\xi A$ and $\bar c \in {}^{\varepsilon+\xi} A$, and $\varphi(\bar x{}^\frown\bar b - \bar c) \in p_1(\bar x) \cap p_2(\bar x)$, <u>then</u> $p_1(\bar x) = p_2(\bar x)$.

[3] This also holds for $\neg\varphi(\bar x, \bar y)$, but for κ finite we can invert the order.

Why is ⊛ true? Assume $\varphi(\bar{x}\hat{\ }\bar{b}' - \bar{c}') \in p_1(\bar{x})$, so $\bar{a}_1\hat{\ }\bar{b}' - \bar{c}' \in \varphi(\overline{M})$. But we are assuming $\varphi(\bar{x}\hat{\ }\bar{b} - \bar{c}) \in p_\ell(\bar{x}) = \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{a}_\ell, A, M)$, hence $\bar{a}_\ell\hat{\ }\bar{b} - \bar{c} \in \varphi(M)$ for $\ell = 1, 2$. Together,

$$\bar{a}_2\hat{\ }\bar{b}' - \bar{c}' = (\bar{a}_2\hat{\ }\bar{b} - \bar{c}) - (\bar{a}_1\hat{\ }\bar{b} - \bar{c}) + (\bar{a}_1\hat{\ }\bar{b}' - \bar{c}') \in \varphi(M),$$

hence $\varphi(x\hat{\ }\bar{b}' - c') \in p_2(x)$. So $\varphi(\bar{x}\hat{\ }\bar{b}' - \bar{c}') \in p_1 \Rightarrow \varphi(\bar{x}\hat{\ }b' - \bar{c}') \in p_2$, and by symmetry we have '⇔,' hence $p_1(\bar{x}) = p_2(\bar{x})$. I.e. we have proved ⊛.

Why is ⊛ sufficient? For every $\xi < \theta, \varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\xi} \cap \Lambda$ and $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ choose $(\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}, \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})})$ such that

- $\oplus_1$ (a) $\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^\varepsilon A$ and $\bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^{\varepsilon+\xi} A$
  - (b) If possible, $\varphi(\bar{x}\hat{\ }\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) \in p(\bar{x})$.

For $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$, let $\Phi_{p(\bar{x})} := \{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon,\xi} : \oplus_1\text{(b) does hold}\}$, and let

$$q_{p(\bar{x})} := \big\{\varphi(\bar{x}\hat{\ }\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) : \varphi(\bar{x}, \bar{y}) \in \Phi_{p(\bar{x})}\big\}.$$

Now,

- $\oplus_2$ If $p_1(\bar{x}), p_2(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$, $\Phi_{p_1(\bar{x})} = \Phi_{p_2(\bar{x})}$, and $q_{p_1(\bar{x})} = q_{p_2(\bar{x})}$, <u>then</u> $p_1(\bar{x}) = p_2(\bar{x})$.

[Why? Just think about it.]

- $\oplus_3$ $\big|\big\{(\Phi_{p(\bar{x})}, q_{p(\bar{x})}) : p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)\big\}\big| \le 2^{|\Lambda|} + (|A|^{<\theta})^{|\Lambda|}$

[Why? Straightforward.]

Clearly we are done.

2) Note that $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon,\xi}$ implies that

- ⊞ $M \models$ "$\varphi[\bar{a}, \bar{b}] \wedge \varphi[\bar{a}, \bar{b}'] \wedge \varphi[\bar{a}', \bar{b}]$" $\Rightarrow M \models$ "$\varphi[\bar{a}', \bar{b}']$".

[Why? As $\varphi({}^{\varepsilon+\zeta}M)$ is a subgroup of ${}^{\varepsilon+\zeta}M$ and $\bar{a}\hat{\ }\bar{b}$, $\bar{a}'\hat{\ }\bar{b}$ and $\bar{a}\hat{\ }\bar{b}'$ belong to it. Therefore so does $\bar{a}'\hat{\ }\bar{b} + (\bar{a}\hat{\ }\bar{b}') - (\bar{a}\hat{\ }\bar{b})$, but that is equal to $\bar{a}'\hat{\ }\bar{b}'$.]

So we can choose $\bar{a} = \bar{a}_0$, $\bar{a}' = \bar{a}_3$, $\bar{b} = \bar{b}_1$, and $\bar{b}' = \bar{b}_2$, and get a contradiction.

3) Toward contradiction, let $\langle \bar{a}_\alpha : \alpha < \kappa \rangle \subseteq {}^\varepsilon M$ form a counterexample. By the Erdős-Rado Theorem,

$$\beth_{\gamma^*+2}\big(|\tau|^{<\theta}\big)^+ \to (4)^2_{\beth_{\gamma^*+1}(|\tau|^{<\theta})}.$$

Now for $\alpha < \beta < \kappa$, let $p_{\alpha,\beta} := \mathrm{tp}_{\Lambda^{\mathrm{pe}}_{\gamma^*,\varepsilon,\varepsilon}}(\bar{a}_\alpha\hat{\ }\bar{a}_\beta; \varnothing, M)$. So $\{p_{\alpha,\beta} : \alpha < \beta\}$ has cardinality $\le \beth_{\gamma^*+1}(|\tau|^{<\theta})$; hence by the arrow above, for some p and some $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ we have

$$(\forall k < \ell < 4)[p_{\alpha_k,\alpha_\ell} = p].$$

We get a contradiction by part (2).

If κ is just $\ge \beth_{\gamma^*+1}(|\tau|^{<\theta})^+$, use clause ⊞ from the proof of part (2) and repeat a proof of the Erdős-Rado Theorem.

4) Should be clear. $\square_{3.3}$

Recall (from [She09])

Definition 3.4. For $\Phi \subseteq \Lambda$, we say $\mathbf{I} \subseteq {}^\varepsilon M$ is (μ, Φ)-*convergent* <u>when</u> ($|\mathbf{I}| \ge \mu$ and) for every $\xi < \theta$, $\varphi(\bar{x}) \in \Phi_{\varepsilon+\xi}$, and $\bar{b} \in {}^\xi M$ and $\bar{c} \in {}^{\xi+\varepsilon}M$, for all but $< \mu$ of the $\bar{a} \in \mathbf{I}$, the truth value of $\bar{a}\hat{\ }\bar{b} - \bar{c} \in \varphi(M)$ is constant.

Claim 3.5. 1) *The following is a sufficient condition for* $\mathbf{I} = \{\bar{a}_i : i < \lambda\} \subseteq {}^{\varepsilon}M$ *to be* (μ, Φ)*-convergent:*

$$i < j < \lambda \wedge \varphi(\bar{x}) \in \Phi \cap \Lambda_\varepsilon \Rightarrow \bar{a}_j - \bar{a}_i \in \varphi(M).$$

2) *If* $\varepsilon < \theta$, $\lambda = \mathrm{cf}(\lambda) > \mu \geq \mu_{\gamma^*}$, $(\forall i < \lambda)\big[|i|^{\mu_{\gamma^*}} < \lambda\big]$, *and* $\langle \bar{a}_i : i < \lambda \rangle \subseteq {}^{\varepsilon}M$ *is without repetition,* <u>then</u> *for some stationary* $S \subseteq \lambda$, $\{\bar{a}_i : i \in S\}$ *is* (μ^+, Φ)*-convergent.*

Remark 3.6. 1) Note that being $(\mu, \mathbf{I})$-convergent is very close to being $(<\omega)$-indiscernible, and is sometimes the reasonable generalization of indiscernibility.

2) So 3.5(1) says that 2-indiscernible *almost* implies $(<\omega)$-indiscernible.

3) Also, 3.5(2) says that there are $(<\omega)$-indiscernibles.

Proof. Should be clear. $\square_{3.5}$

§ 4. How much does the subgroup exhaust a group?

Explanation 4.1. 1) The motivation for this section comes from the old proof of Theorem 2.4, but we find it interesting in its own right. It considers the question

> Given a group G and a sequence $\langle G_s : s \in S\rangle$ of its subgroups, under what conditions do we have $G = \bigcup\limits_s G_s$?

2) More specifically, Definition 4.2 is a strong way to say $G \neq \bigcup\limits_s G_s$, and gives a more exact way to measure how different they are.

Question: Why?

Answer: Assume $S \in I_\lambda$, as witnessed by $\langle g_\alpha : \alpha < \lambda\rangle$ (see 4.2) and $G = \bigcup\limits_{s \in S} G_s$, and we shall prove that $\lambda \leq |S|$.

[Why? As $G = \bigcup\limits_s G_s$ and $\{g_\alpha : \alpha < \lambda\} \subseteq G$, clearly there is a function $f : \lambda \to S$ such that $g_\alpha \in G_{f(\alpha)}$ for all $\alpha < \lambda$. Now f must be one-to-one, because

$$\alpha \neq \beta \wedge f(\alpha) = f(\beta) \Rightarrow g_\alpha G_{f(\alpha)} = G_{f(\alpha)} = G_{f(\beta)} = g_\beta G_{f(\beta)} = g_\beta G_{f(\alpha)},$$

contradicting the definition of $\bar{g}$.]

3) Recall that in the old proof of 2.4 we used only 4.2 and 4.4(a)-(b), and from 4.3(3) we use only the sentence starting with 'Moreover.'

Clauses 4.4(c)-(g) are not used elsewhere, but we still find them interesting: in particular clause (d), giving sufficient conditions for I_λ to be an ideal.

4) An obvious point — but still we note that λ is free to vary, so $I_{G,\overline{G},\lambda}$ is defined for all λ, even with $G, \overline{G}$ fixed.

5) In this section, we may even allow the G_s to be affine subsets of G in the Abelian case. (So $G_s = g_s G'_s$ for some $g_s \in G$, where G'_s is a subgroup of G.)

This makes no difference.

Definition 4.2. Assume G is a group and $\overline{G} = \langle G_s : s \in S\rangle$ is a sequence of subgroups of G.

1) For $\lambda \geq 0$, let $I = I_\lambda = I_{G,\overline{G},\lambda}$ be the set of $u \subseteq S$ which are *witnessed* by some sequence $\bar{g} = \langle g_\alpha : \alpha < \lambda\rangle \subseteq G$.

By this we mean

$$s \in u \wedge \alpha < \beta < \lambda \ \Rightarrow\ g_\alpha G_s \neq g_\beta G_s.$$

2) For $\lambda \geq 0$ let $I_{<\lambda} = I_{G,\overline{G},<\lambda} := \bigcap\limits_{\mu<\lambda} I_\mu$.

3) Let $I^+ := \mathcal{P}(S) \setminus I$ for any $I \subseteq \mathcal{P}(S)$.

Claim 4.3. *Let $G, S, \overline{G}$ be as in Definition 4.2; hence $I_\mu = I_{G,\overline{G},\mu}$ is well-defined for any cardinal μ.*

1) *For any λ, the sequence $\langle I_\mu : \mu < \lambda\rangle$ is $\subseteq$-decreasing (that is, $\mu < \lambda \Rightarrow I_\lambda \subseteq I_\mu$).*

2) *If in addition λ has cofinality $> 2^{|S|}$, <u>then</u> the sequence $\langle I_\mu : \mu < \lambda\rangle$ is eventually constant.*

3) *There is $\xi < (2^{|S|})^{+4}$ of cofinality $(2^{|S|})^+$ such that $\lambda := \aleph_\xi$ satisfies all the demands mentioned in 4.4 below.*

Moreover, if (e.g.) $\lambda = (2^{|S|})^+$, <u>then</u> λ satisfies the demands in 4.4(a),(b).

4) *Similarly, when* $\lambda := \beth_\delta$ *with* $\delta = (2^{|S|})^+$, *or just* $\mathrm{cf}(\delta) > 2^{|S|}$.

5) *If* $\lambda > |G|^{<\theta}$ *then* $I_{G,\bar{G},\lambda} = \{\varnothing\}$.

Proof. 1) Obvious.

2) If λ is a successor cardinal this is trivial.

If λ is limit, then $\langle I_\mu : \mu < \lambda\rangle$ is a $\subseteq$-decreasing sequence of subsets of $\mathcal{P}(S)$, which has cardinality $2^{|S|}$. The conclusion is clear.

3) The second sentence is obvious, but for the first sentence we will need to quote results from elsewhere. Recall[4]

$(*)_1$ If κ is regular, $\mu > \mathrm{cf}(\mu) \geq \kappa$, and

$$(\forall \chi < \mu)\big[\mathrm{cf}(\chi) < \kappa \Rightarrow \mathrm{pp}(\chi) < \kappa\big],$$

then $\mathrm{cf}([\mu]^{<\kappa}, \subseteq) = \mu$.

[Why? By [She94a, Ch.II, 5.1,5.4,6.12] (or [She, 5.1,7.1,7.4]).]

$(*)_2$ $\mathrm{cf}(\mu) < \mu \Rightarrow \mathrm{pp}(\mu) \leq \mu^{\mathrm{cf}(\mu)}$.

[Why? Obvious from the definition of $\mathrm{pp}(-)$.]

$(*)_3$ Assume $\kappa = \aleph_{\xi+\zeta}$ is singular, and $\zeta < \aleph_\zeta$. If $|\zeta| = \kappa^{++}$ and $\mathrm{cf}(\zeta) < \kappa = \mathrm{cf}(\kappa)$, then $\mathrm{pp}(\aleph_{\xi+\zeta}) < \aleph_{\xi+|\zeta|^+}$.

[Why? By [She94b, §4].]

$(*)_4$ If $2^\kappa \leq \mu$ and $\mathrm{cf}([\mu]^\kappa, \subseteq) = \mu$ then $\mu^\kappa = \mu$.

[Why? Obvious.]

$(*)_5$ If $\theta \leq \mu = \mu^\theta$ then for some club E of θ^{+4} we have

(a) If $\delta \in E$ with $\mathrm{cf}(\delta) > \theta$ and $\xi := \min(E \setminus \delta + 1)$, then $\mathrm{pp}(\mu^{+\delta}) < \mu^{+\xi}$.

(b) If $\delta \in \mathrm{acc}(E) := \big\{\alpha \in E : \alpha = \sup(E \cap \alpha)\big\}$ and $\mathrm{cf}(\delta) > \theta$, <u>then</u> $\mathrm{cf}\big([\mu^{+\delta}]^{\leq\theta}, \subseteq\big) = \mu^{+\delta}$.

(c) Moreover, in part (b) we have $(\mu^{+\delta})^\theta = \mu^{+\delta}$.

[Why? By the previous statements.]

Now part (3) is a special case of $(*)_5$ as $(2^{|S|})^{|S|} = 2^{|S|}$.

4) Easy as well.

5) If $u \in I$ as witnessed by $\langle g_\alpha : \alpha < \lambda\rangle$ then $\langle g_\alpha G_s : \alpha < \lambda\rangle$ is without repetition, hence $\langle g_\alpha : \alpha < \lambda\rangle$ is without repetition. Hence $|G| \geq \lambda$ — a contradiction. $\square_{4.3}$

Claim 4.4. *Assume $\langle G_s : s \in S\rangle$ is a sequence of subgroups of the group G and λ a cardinal. The set $I = I_\lambda = I_{G,\bar{G},\lambda}$ satisfies:*

(a) *If $S \notin I$, $\mathrm{cf}(\lambda) > 2^{|S|}$, and $\alpha < \lambda \Rightarrow |\alpha|^{|S|} < \lambda$ (e.g. $(\exists\mu)[\lambda = (\mu^{|S|})^+]$), <u>then</u> there is $A \subseteq G$ of cardinality $< \lambda$ such that*

$$(\forall g \in G)(\exists a \in A)\big[\{s \in S : gG_s \neq aG_s\} \in I\big].$$

(b) *If $S \notin I$, $\alpha < \lambda \Rightarrow |\alpha|^{|S|} < \lambda$, and λ is regular, <u>then</u> for every $u \in I$ there exists $\bar{g}$ and $v \in I$ such that*

- $\bullet_1$ *$u \subseteq v$*
- $\bullet_2$ *$\bar{g} = \langle g_\alpha : \alpha < \lambda\rangle$*
- $\bullet_3$ *$g_\alpha G_s = g_0 G_s$ for all $\alpha < \lambda$ and $s \in S \setminus v$. Moreover,*

$$\alpha < \lambda \Rightarrow g_\alpha \in \bigcap_{s \in S \setminus v} G_s.$$

[4] $\mathrm{pp}(\mu)$ is defined for singular cardinals μ in [She94a, Ch.II, 1.1] (or in [She, 6.1]), but this is not used in this section.

$\bullet_4$ *If* $s \in v$ *and* $0 < \alpha < \beta < \lambda$, *then* $g_\alpha G_s \neq g_\beta G_s$.

(c) $I \subseteq \mathcal{P}(S)$ *is closed under subsets.*

(d) $I = I_\lambda$ *is an ideal,* provided that

$\bullet_1$ *For some* $\theta \in \big(|S|, \lambda\big)$ *we have* $I_\theta = I_\lambda$.

$\bullet_2$ G *is Abelian (or just each* G_s *is a normal subgroup of* G*).*

(e) *Assuming clause* (d)$\bullet_2$ *and* $\lambda > |S|^+$,

$$(\forall u_1, u_2 \in I_\lambda)(\forall \mu < \lambda)[u_1 \cup u_2 \in I_\mu].$$

(f) *Assuming clause* (d)$\bullet_2$, *if* $|S| < \aleph_0$ *then* I *is an ideal.*

(g) *The following holds:*

$\bullet_1$ *If* $u \in I_\lambda$ *and* $\{g_\alpha : \alpha < \alpha_*\}$ *is* $\subseteq$*-maximal such that*

$$\alpha < \beta < \alpha_* \wedge s \in S \Rightarrow g_\alpha G_s \neq g_\beta G_s,$$

then $\alpha_* \in [\lambda, \lambda^+)$.

$\bullet_2$ *If* λ *is a limit cardinal of cofinality* $> 2^{|S|}$ *then* $(\exists \theta < \lambda)[I_\theta = I_\lambda]$.

Proof. Let $I = I_\lambda$ be defined as in 4.2.

Now,

$\circledast_0$ $I \subseteq \mathcal{P}(S)$ is $\subseteq$-downward closed: i.e. $u \in I \wedge v \subseteq u \Rightarrow v \in I$.

[Why? Obvious.]

This covers clause (c).

Toward proving clause (a) of the claim, for each $u \in I^+ := \mathcal{P}(S) \setminus I$, let $\bar{g}_u = \langle g_{u,\alpha} : \alpha < \alpha_u \rangle$ be a maximal sequence of members of G such that

$$\alpha < \beta < \alpha_u \wedge s \in u \;\Rightarrow\; g_{u,\alpha} G_s \neq g_{u,\beta} G_s.$$

As $u \in \mathcal{P}(S) \setminus I$, necessarily $\alpha_u < \lambda$ (by the definition of I), and as we are assuming $\mathrm{cf}(\lambda) > 2^{|S|}$, clearly $\alpha_* := \sup\{\alpha_u : u \in I^+\} < \lambda$. So

$$B := \{g_{u,\alpha} : u \in I^+,\ \alpha < \alpha_u\}$$

is a subset of G of cardinality $< \lambda$.

Next,

$\circledast_1$ For every $u \in I$ and $h : S \setminus u \to B$, choose $g_h \in G$ such that

$$\boxplus_{h,g_h} \vee (\forall g \in G)\big[\neg \boxplus_{h,g}\big],$$

where

$\boxplus_{h,g}$
- $g \in G$
- $h : S \setminus u \to B$
- $(\forall s \in S \setminus u)[g G_s = h(s) G_s]$.

Explicitly, if there exists such a g then choose one of them as our g_h; otherwise let $g_h := g_{u,0}$, just so that it is defined.

Now

$$A := \{g_h : h \in {}^{S\setminus u}B,\ u \in I,\ \text{and } \boxplus_{h,g_h}\}$$

is a subset of G of cardinality[5] $\leq |B|^{|S|} < \lambda$.

For showing A is as required in clause (a), fix $g_* \in G$. Let

$$v := \big\{s \in S : (\forall w \in I^+)(\forall \alpha < \alpha_w)[g_* G_s \neq g_{w,\alpha} G_s]\big\}.$$

Now if $v \in I^+$ then $\bar{g}_v = \langle g_{v,\alpha} : \alpha < \alpha_v \rangle$ is well-defined and g_* satisfies $(\forall \alpha < \alpha_v)[g_* G_s \neq g_{v,\alpha} G_s]$, contradicting the maximality of $\bar{g}_v$.

[5] Recall that we are assuming $(\forall \alpha < \lambda)\big[|\alpha|^{|S|} < \lambda\big]$.

Therefore $v \in I$ by our choices (as $I \cup I^+ = \mathcal{P}(S)$ and $v \subseteq S$), so by the definition of v we can find a function $h : S \setminus v \to B$ such that

$$s \in S \setminus v \Rightarrow g_* G_s = h(s) G_s.$$

So g_* and h satisfy $\boxplus_{h,g_*}$, hence by $\circledast_1$ there is a $g_h \in A$ satisfying $\boxplus_{h,g_h}$, so clause (a) holds with $a := g_h$.

* * *

For clause (b), let $u \in I$ be given and let $\langle g_\alpha : \alpha < \lambda \rangle$ witness that $u \in I$. For each $\alpha < \lambda$, let

$$u_\alpha := \big\{ s \in S : (\exists \beta < \alpha)[g_\alpha G_s = g_\beta G_s] \big\}.$$

Clearly $u_\alpha \cap u = \varnothing$; let $h_\alpha : u_\alpha \to \alpha$ be such that $s \in u_\alpha \Rightarrow g_\alpha G_s = g_{h_\alpha(s)} G_s$.

As λ is regular and recalling $(\forall \alpha < \lambda)\big[|\alpha|^{|S|} < \lambda\big]$ by the present assumptions, for some $u_* \subseteq S$ and $h : u_* \to \lambda$, the set

$$\mathcal{W} := \{ \alpha < \lambda : \mathrm{cf}(\alpha) = |S|^+ + \aleph_0,\ h_\alpha = h,\ u_\alpha = u_* \}$$

is a stationary subset of λ. Clearly

$$\alpha, \beta \in \mathcal{W} \wedge s \in u_* \Rightarrow g_\alpha G_s = g_{h(s)} G_s = g_\beta G_s$$

and

$$\alpha, \beta \in \mathcal{W} \wedge \alpha \neq \beta \wedge s \in S \setminus u_* \Rightarrow g_\alpha G_s \neq g_\beta G_s.$$

Letting $\langle \alpha_i : i < \lambda \rangle$ list $\mathcal{W}$ in increasing order and letting $g'_i := g_{\alpha_i}$ for $i < \lambda$, clearly $v := u_*$ and $\langle g'_i : i < \lambda \rangle$ are as promised in clause (b).

For the 'moreover' bit, recalling clause (b) just says "for some $\bar{g}$ and v," we let $g''_\alpha := (g'_0)^{-1} g_\alpha$ and use the sequence $\langle g''_\alpha : \alpha < \lambda \rangle$. That is, first

$$\sigma < \beta < \lambda \wedge s \in S \Rightarrow g_\alpha G_s \neq g_\beta G_s \Rightarrow (g_0)^{-1}(g_\alpha G_s) \neq (g_0)^{-1}(g_\alpha G_s) \Rightarrow g''_\alpha G_s \neq g''_\beta$$

Second

$$\alpha < \lambda \wedge s \in S \setminus v \Rightarrow g''_\alpha G_s = (g'_0)^{-1}(g_\alpha G_s) = (g'_0)^{-1}(g'_0 G_s) = \big((g'_0)^{-1} g'_0\big) G_s = G_s.$$

Therefore (as $v := S \setminus u_*$) we have

$$\alpha < \lambda \wedge s \in v \ \Rightarrow\ g''_\alpha G_s = G_s \ \Rightarrow\ g''_\alpha \in G_s$$

as promised.

* * *

Lastly, it just remains to prove clause (e), as (d) is an immediate consequence and clause (f) is easier and clause (g) is proved as in clause (a).

Let $u_1, u_2 \in I_\lambda$ be disjoint, and we shall prove that $u := u_1 \cup u_2 \in I_\mu$ when $\mu \in \big(|S|, \lambda\big)$. Let $\langle g_{\ell,\alpha} : \alpha < \lambda \rangle$ witness '$u_\ell \in I_\lambda$' for $\ell = 1, 2$.

We try to choose $g_{3,\varepsilon} \in G$ by induction on $\varepsilon < \mu$ such that

$$\zeta < \varepsilon \wedge s \in u \ \Rightarrow\ g_{3,\varepsilon} G_s \neq g_{3,\zeta} G_s;$$

we shall also demand that $g_{3,\varepsilon} \in \{ g_{1,i} g_{2,j} : i, j < \lambda \}$.

Arriving to ε, if for some $i, j < \lambda$ we can choose $g_{3,\varepsilon} := g_{1,i} g_{2,j}$, then we are done.

Towards contradiction, assume there are no such i and j. Then we have $f : \lambda \times \lambda \to \varepsilon$ and $h : \lambda \times \lambda \to u$ such that for every $(i,j) \in \lambda \times \lambda$ we have

$$g_{1,i} g_{2,j} G_{h(i,j)} = g_{3,f(i,j)} G_{h(i,j)}.$$

For each $i < \lambda$, $\zeta < \varepsilon$, and $s \in u \subseteq S$, let

$$\mathcal{U}^2_{i,\zeta,s} := \{j < \lambda : f(i,j) = \zeta,\ h(i,j) = s\}.$$

Now $j \in \mathcal{U}^2_{i,\zeta,s} \ \Rightarrow\ g_{1,i}g_{2,j}G_s = g_{3,\zeta}G_s \ \Rightarrow\ g_{2,j}G_s = g^{-1}_{1,i}g_{3,\zeta}G_s$; hence if $s \in u_2$ then

$$j \neq k \in \mathcal{U}^2_{i,\zeta,s} \Rightarrow g_{2,j}G_s = (g^{-1}_{1,i}g_{3,\zeta})G_s = g_{2,k}G_s$$

— the last statement contradicts our choice of $\langle g_{2,j} : j < \lambda\rangle$. Hence $\mathcal{U}^2_{i,\zeta,s}$ has cardinality ≤ 1 for all $i < \lambda$, $\zeta < \varepsilon$, and $s \in u_2$.

For $j < \lambda$, $\zeta < \varepsilon$, and $s \in u$, let

$$\mathcal{U}^1_{j,\zeta,s} := \big\{i < \mu : f(i,j) = \zeta \text{ and } h(i,j) = s\big\}.$$

If G is Abelian, then (as above) we have $\zeta < \varepsilon \wedge j < \lambda \wedge s \in u_1 \Rightarrow |\mathcal{U}^1_{j,\zeta,s}| \leq 1$. If G is non-Abelian and every G_s is a normal subgroup of G, then for any $j < \lambda$, $\zeta < \mu$, $s \in u_1$ we have

$$\begin{aligned} i \in \mathcal{U}^1_{j,\zeta,s} &\ \Rightarrow\ g_{1,i}g_{2,j}G_s = g_{3,\zeta}G_s \\ &\ \Rightarrow\ g_{1,i}(G_s g_{2,j}) = g_{1,i}(g_{2,j}G_s) = g_{3,\zeta}G_s \\ &\ \Rightarrow\ g_{1,i}G_s = g_{3,\zeta}(G_s g^{-1}_{2,j}). \end{aligned}$$

Hence $i \neq k \in \mathcal{U}^1_{j,\zeta,s} \Rightarrow g_{1,i}G_s = g_{3,\zeta}(G_s g^{-1}_{2,j}) = g_{1,k}G_s$. This is a contradiction, so again $\mathcal{U}^1_{j,\zeta,s}$ has at most one member.

For $\ell \in \{1,2\}$ and $i < \lambda$, let $\mathcal{U}^\ell_i := \bigcup\limits_{\zeta<\varepsilon}\bigcup\limits_{s\in u_\ell}\mathcal{U}^\ell_{i,\zeta,s}$, so as $|u_\ell| \leq |S|$ clearly $|\mathcal{U}^\ell_i| \leq |S| + |\varepsilon|$. Recall that we have $\lambda > \mu > |S| + |\varepsilon|$, so there are $i, j < \lambda$ such that $i \notin \mathcal{U}^1_j \wedge j \notin \mathcal{U}^2_i$; hence the member $g_{1,i}g_{2,j}$ of G satisfies the demand on $g_{3,\varepsilon}$.

So we can carry the induction on $\varepsilon < \mu$, so we are done proving clause (e). $\square_{4.4}$

Claim 4.5. *In Claim 4.4 there is a $W \subseteq S$ such that*

- (a) *There is a sequence $\bar{s} = \langle s_i : i < i_*\rangle$ listing W such that $\big(\bigcap\limits_{i<j} G_{s_i} : \bigcap\limits_{i\leq j} G_{s_i}\big)$ is finite for $j < i_*$ (stipulating $\bigcap\limits_{i<0} G_{s_i} := G$).*
- (b) *W is $\subseteq$-maximal among all subsets of S satisfying clause (a) above.*

Proof. Immediate. $\square_{4.5}$

* * *

We may take a more general perspective.

Definition 4.6. 1) We say $\mathbf{m}$ is a *movement system* when it consists of

- (a) A set S.
- (b) A sequence of sets $\langle \mathcal{S}_s : s \in S\rangle$.
- (c) A set G.
- (d) Mappings $\mathrm{inv}_g \in \prod\limits_s \mathcal{S}_s$ for each $g \in G$.

2) For λ a cardinal, let $I^{\mathbf{m}}_\lambda$ be the family of sets $u \subseteq S$ which are *witnessed* by some $\bar{g} = \langle g_\alpha : \alpha < \lambda\rangle$. By this we mean

- (a) $\{g_\alpha : \alpha < \lambda\} \subseteq G_{\mathbf{m}}$
- (b) For each $s \in u$, the sequence $\langle \mathrm{inv}^{\mathbf{m}}_{g_\alpha}(s) : \alpha < \lambda\rangle$ is without repetition.

Claim 4.7. *The parallel of 4.4 holds for this definition.*

§ 5. Concluding Remarks

Example 5.1. An example of an additive structure is a ring satisfying $xy = -yx$.

E.g. if $(R, +^R)$ is $\bigoplus\{\mathbb{Z}x_s : s \in I\}$, f is a function from $I \times I$ into R such that $f(x,y) = -f(y,x)$ (so $f(x,x) = 0$), and we have

$$\Big(\sum_{\ell<\ell_*} a_\ell x_{s_\ell}\Big)\Big(\sum_{m<n_*} b_m x_{t_n}\Big) = \sum_{\ell<\ell_*}\sum_{m<m_*} a_\ell b_m x_{f(s_\ell,t_m)}.$$

Remark 5.2. 1) We may use $\tau \supseteq \{+,-,0,1\} \cup \{P_i : i < i_*\}$ with P_i unary, and instead of modules we use τ-models M such that $|M|$ is the disjoint union $\bigcup\limits_{i<i_*} P_i^M$, $+^M$ is a partial two-place function which can be decomposed into

$$+^M := \bigcup\{+^M \restriction P_i^M : i < i_*\},$$

$(P_i^M, +^M)$ an Abelian group, and all relations and functions commute with $+$ (or at least every relation is *affine*).

I.e., let $F_*(x,y,z) := x - y + z$ and demand

$$G(\ldots, F_*(x_i,y_i,z_i), \ldots)_{i<i_*} = F_*\big(G(\bar{x}), G(\bar{y}), G(\bar{z})\big),$$

where $F_*(\bar{a},\bar{b},\bar{c}) := \big\langle F_*(a_i,b_i,c_i) : i < \text{arity}(P)\big\rangle \in P^M$ for $\bar{a},\bar{b},\bar{c} \in P^M$.

2) However, as we use infinitary logics, if M is the disjoint union of Abelian groups $G_i^M := (P_i^M, +_i^M)$ for $i < i_*$ and we define G_M as the direct sum having predicates for those subgroups, <u>then</u> we have bi-interpretability. When we have only "affine structure," we can expand by choosing an element in each summand to serve as zero.

3) It is natural to extend our logic by cardinality quantifiers which say "the definable group G quotient the definable subgroup H has cardinality $\geq \lambda$."

Remark 5.3. Concerning Theorem 2.4:

1) Note that instead of an R-module M we can use $(M, c_\alpha)_{\alpha<\kappa}$: i.e., expand M by κ-many individual constants. The only difference is that we will use $\beth_\alpha(|R|^{<\theta} + \kappa)$ instead of $\beth_\alpha(|R|^{<\theta})$.

2) Theorem 2.4 has an arbitrary choice — the construction of the $\mathbf{I}_\alpha$-s. Instead of using extra individual constants, in the proof,[6] for any $\psi(\bar{x})$, $\psi(\bar{x}) \wedge \varphi_i(\bar{x})$ for $i < i_* < \kappa_\beta$, I, G, and $\langle G_i : i < i_*\rangle$, we expand M by:

(a) $P^M := \big\{\bar{a} : M \models \psi[\bar{a}]$ and $\{i < \kappa_\beta : \bar{a} \notin G_i\} \in I\big\}$, which is a subgroup.

(b) Predicates for the set $\big\{\bar{a} + P^M : \bar{a} \in \psi(M)\big\}$.

So the proof shows that in M we can eliminate quantifiers to quantifier-free formulas in this expansion.

3) Also, this may give too much information. The result gives elimination of quantifiers, but unlike the first-order case we use more than just the positive existential formulas.

4) We can now define non-forking: hopefully [S$^+$] will deal with this.

[6] See $\boxplus_{\beta+1}$ in the proof of 2.4.

Question 5.4. 1) Are there arbitrarily large Abelian groups G which are not only indecomposable, but even *potentially* so? (I.e. absolutely — even after any forcing G is indecomposable?)

2) Relatives of this — e.g., no *potential* non-trivial automorphism.

Discussion 5.5. We know that up to the minimal cardinal λ satisfying

$$\lambda \to (\omega)^{<\omega}_{\aleph_0},$$

the answer is yes (and more). But if $|G| \geq \lambda$ <u>then</u> absolutely it has non-trivial endomorphisms and even non-trivial embeddings of G into itself (Eklof-Shelah [ES99], Göbel-Shelah [GS07]). We can improve this to "for some $a_1 \neq a_2$ from G;" potentially there are embeddings f_1, f_2 of G into itself such that $f_1(a_1) = a_2$ and $f_2(a_2) = a_1$ — see [S$^+$].

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