

HJ NUMBERS REVISITED

E117

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ABSTRACT. We improve the bounds on the Hales-Jewett numbers to a tower of exponentiations. Earlier it was WaW (that is, iterations of towers which are themselves iterated exponentiations). we improve the inductive step there (induction on the size of the alphabet, $|\Lambda|$) to 2-exponentiations, instead of towers. In the longer work in typing,

- (A) We present this inductive step as a partition theorem in its own right; (but in this preliminary version we make it just serve the bound on HJ numbers).
- (B) we shall deal with the density version of Hales-Jewett with similar bound.

We are also dealing with the Graham-Rothschild Theorem and the Affine Ramsey Theorem and the polynomial case, and give background.

Annotated Content

§0 Introduction

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We improve on the results from [She:329].

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References like (e.g.) [Sh:950, Th.0.2_{=Ly5}] mean that y5 is the internal label of Theorem 0.2 in the TeXfile of [Sh:950]. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number E117 in Saharon Shelah's list.

§ 0. INTRODUCTION

S0

The history of this problem is well-known; considering early parts, see the book [GraRotSpe90]. (For later parts, see [She:679] or Wikipedia.) Recall the Hales–Jewett Theorem [HalJew63].

z0

Theorem 0.1 (Hales–Jewett Theorem). *For all natural numbers n, ℓ there exists an integer $\text{HJ}(n, \ell)$ such that for all $k \geq \text{HJ}(n, \ell)$, Λ of cardinality n , C of cardinality ℓ , and $\mathbf{c} : {}^k\Lambda \rightarrow C$, there exists a combinatorial line on which $\mathbf{c}$ is constant.*

Where

z1

Definition 0.2. 1) A *combinatorial line* in ${}^M\Lambda$ is a set of the form

$$\{\rho \in {}^M\Lambda : \rho \supseteq \eta \text{ and } \rho \upharpoonright u \text{ is constant}\},$$

for some non-empty $u \subseteq M$ and $\eta \in {}^{M \setminus u}\Lambda$.

2) Generalizing this, we say Γ is a *subspace* of ${}^M\Lambda$ of dimension ℓ (or ℓ -subspace) when it is of the form

$$\{\rho \in {}^M\Lambda : \rho \supseteq \eta, \text{ and } m < \ell \Rightarrow \rho \upharpoonright u_m \text{ is constant}\}$$

for some sequence $\langle u_m : m < \ell \rangle$ of pairwise disjoint non-empty subsets of M and $\eta : M \setminus \bigcup_{m < \ell} u_m \rightarrow \Lambda$.

3) Let $\text{subspace}(V, \ell)$ be the set of ℓ -subspaces of $V := {}^M\Lambda$. If $\Gamma \in \text{subspace}(V, \ell)$ and $1 \leq k < \ell$, let

$$\text{subspace}(\Gamma, k) := \{\Gamma' \in \text{subspace}(V, k) : \Gamma' \subseteq \Gamma\}.$$

We thank Shimoni Garti whose comments help to clarify the presentation.

z2

Notation 0.3. 1) Here $i, j, k, \ell, m, n, r, s$ will denote natural numbers; equivalently, finite ordinals. (So n is identified with $\{i : i < n\}$.)

2) C will denote a non-empty finite set of colors.

3) Λ , a finite alphabet.

4) ε, ζ , and $\varkappa$ will be real numbers in the interval $(0, 1]_{\mathbb{R}}$.

5) $\mathbf{c}$ and $\mathbf{d}$ will be colorings (i.e. functions from some set to C).

6) $\mathbf{s}$ and $\mathbf{t}$ will denote pre-frames (see Definitions ^{la5}1.1)

7) $\exp_m(n) = \exp(m, n) := m^n$ (We will use this to avoid deeply nested superscripts.)

8) ξ will denote a member of the set Ξ from Definition ^{la5}1.1(2)(B), never a real number or ordinal.

9) AB is the set of functions from A to B .

z5

Convention 0.4. If we define some variable as a tuple of diverse objects (e.g.

$$\mathbf{s} := (\bar{m}, \Lambda, \alpha, \beta, C) = (\bar{m}_{\mathbf{s}}, \Lambda_{\mathbf{s}}, \alpha_{\mathbf{s}}, \beta_{\mathbf{s}}, C_{\mathbf{s}}),$$

to pick a relevant example) then we may include or omit the subscript depending on whether the referent is clear from context or not.

§ 1. ON HJ(−)

S1

The proof of Hales-Jewett in [HalJew63] uses double induction: first on n the size of the alphabet denoted here by $(\Lambda, \text{in our notation})$, and then on ℓ (the number of colors). Eventually they get an upper bound in terms of the Ackermann function, so they do not get a primitive recursive bound.

The base idea of [She:329] is that we fix the number of colors $|C|$ and perform our induction only on $n := |\Lambda|$. Let me try to describe the proof of the inductive step on n in [She:329]. That is, from n letters to $n + 1$ letters.

So we let $r := \text{HJ}(n, \ell)$ and fix Λ of cardinality $n + 1$, C of cardinality ℓ , and $\alpha \neq \beta \in \Lambda$. We look for a finite linear order M which is large enough for our purposes and consider a coloring $\mathbf{c} : {}^M\Lambda \rightarrow C$.

Let $\bar{M} = \langle M_i : i < r \rangle$ be a partition of M into intervals; the intention is to use each M_i as a place for one letter. That is, we will try to find $\xi = \langle (k_i, \ell_i, M_i) : i < r \rangle$ with $k_i < \ell_i$ both ¹ from M_i , and restricting ourselves to functions $\eta \in {}^M\Lambda$ of the form

$$i < r \wedge j \in M_i \Rightarrow \eta(j) := \begin{cases} \alpha & \text{if } j < k_i \\ \gamma_i & \text{if } j \in [k_i, \ell_i) \\ \beta & \text{if } j \geq \ell_i. \end{cases}$$

This will be done so that the colour does not change if we replace α by β . The choice of $r = \text{HJ}(n, \ell)$ gives the desired conclusion.

So k_i and ℓ_i will be chosen by downward induction on $i < r$, and we require $|M_i| > |C|^{\sum_{j < i} |M_j|}$, so we get an upper bound.

Here (compared with the original proof on the one hand and [She:329] on the other hand), we intend to improve the bound in the inductive step on n . Here we choose the middle road — not fixing the number of colours. (later we shall treat it as a partition theorem in its own right.)

In the try to find ξ for given M , that is in choosing the pair (k_i, ℓ_i) by downward induction on $i < r$. In the inductive step on r , the number of colors increase, but mildly

For this we chose sets of colors C_i and the size of M_i by downward induction on $i \leq r$ and let M_i be of cardinality $|C_i|$.

a5

Definition 1.1. 1) We say the tuple

$$\mathbf{s} = (\bar{m}, \Lambda, \alpha, \beta, C) = (\bar{m}_{\mathbf{s}}, \Lambda_{\mathbf{s}}, \alpha_{\mathbf{s}}, \beta_{\mathbf{s}}, C_{\mathbf{s}}),$$

is a *pre-frame* (or *0-frame*)² when:

- (A) $\bar{m} = \langle m_i : i < r \rangle$
- (B) Λ and C are non-empty finite sets.
- (C) $\alpha \neq \beta \in \Lambda$
- (D) $r \geq 1$, and each $m_i \geq 2$.
- (E) $m_{r-1} \geq |C|$

Assume $|C| \geq 2$, if not stated otherwise.

1A) We say $\mathbf{s}$ is a 1-frame or just a frame if in addition

- (F) If $i < r - 1$ then ³ $m_i \geq (m_{i+1})^{|\Lambda|+1}$.

¹Pedantically, $M_i \cup \{\infty\}$

²In §2 we will need to keep track of more information, and so in the appropriate definition we will stuff some more variables into this tuple.

³We round upward the bound for transparency

2) Let us define the following sets and relations.

- (A) $M = M_{\mathbf{s}} = M_{\bar{m}} = \Sigma_{i < r} M_i$ where $M_i := \{i\} \times m_i$ naturally ordered; but we may write m instead (i, m) and we may write $\eta(i, m)$ instead of $\eta((i, m))$,
- (B) $\Xi_{\mathbf{s}} = \Xi_{\bar{m}} := \{\langle (k_i, \ell_i, m_i) : i < r \rangle : k_i < \ell_i \leq m_i\}$
- (C) For $\xi = \langle (k_i, \ell_i, m_i) : i < r \rangle \in \Xi_{\mathbf{s}}$, we define $\Gamma_{\xi} = \Gamma_{\mathbf{s}, \xi}$ as

$$\left\{ \eta \in {}^{(M_{\mathbf{s}})}\Lambda : (\forall i < r) (\exists \gamma_i \in \Lambda) (\forall n < m_i) \left[\eta(i, n) = \begin{cases} \alpha & \text{if } n < k_i \\ \gamma_i & \text{if } n \in [k_i, \ell_i) \\ \beta & \text{if } n \in [\ell_i, m_i) \end{cases} \right] \right\}.$$

Let $\Gamma_{\mathbf{s}} := \bigcup_{\xi \in \Xi_{\mathbf{s}}} \Gamma_{\xi}$.

- (D) For $\xi \in \Xi_{\mathbf{s}}$, let $E = E_{\xi} = E_{\mathbf{s}, \xi} :=$

$$\{(\eta, \nu) \in \Gamma_{\xi} \times \Gamma_{\xi} : (\forall i < r) [\eta(k_{\xi, i}) = \nu(k_{\xi, i}) \vee \{\gamma_{\xi, \eta, i}, \gamma_{\xi, \nu, i}\} \subseteq \{\alpha, \beta\}]\}$$

a8 **Definition 1.2.** 1) For $\mathbf{s}$ a frame or a pre-frame and $j \leq r_{\mathbf{s}}$, let

$$\mathbf{s} \upharpoonright j := (\bar{m}_{\mathbf{s}} \upharpoonright j, \Lambda_{\mathbf{s}}, \alpha_{\mathbf{s}}, \beta_{\mathbf{s}}, \{i : i < m_{j-1}\}).$$

2) When we say ‘ $\mathbf{s}$ -problem,’ we mean a function $\mathbf{c} : \Gamma_{\mathbf{s}} \rightarrow C_{\mathbf{s}}$.

3) We say that the $\mathbf{s}$ -problem $\mathbf{c}$ is *solvable* (or that the pair $(\mathbf{s}, \mathbf{c})$ is solvable) when there exists a $\xi \in \Xi_{\mathbf{s}}$ such that the function $\mathbf{c} \upharpoonright \Gamma_{\xi}$ respects E_{ξ} .

$$(\text{i.e. } (\eta, \nu) \in E_{\xi} \Rightarrow \mathbf{c}(\eta) = \mathbf{c}(\nu).)$$

We may say that this ξ *solves* $\mathbf{c}$.

4) We say $\mathbf{s}$ is solvable when every $\mathbf{s}$ -problem is solvable.

a10 **Observation 1.3.** 1) If $\mathbf{s}$ is a frame then for all $j \leq r_{\mathbf{s}}$:

- (A) $\mathbf{s} \upharpoonright j$ is a frame
- (B) $\eta E_{\mathbf{s}, \xi} \nu$ implies $\eta \upharpoonright j E_{\mathbf{s} \upharpoonright j, \xi \upharpoonright j} \nu \upharpoonright j$

1A) Similarly for pre-frames.

2) **[Monotonicity:]** If $\mathbf{s}$ is a solvable frame, then so is any $\mathbf{t}$ satisfying the following.

- (a) $r_{\mathbf{t}} \leq r_{\mathbf{s}}$
- (b) $\bar{m}_{\mathbf{t}} \geq \bar{m}_{\mathbf{s}}$ (That is, $(\forall i < r_{\mathbf{s}}) [m_i^{\mathbf{t}} \geq m_i^{\mathbf{s}}]$.)
- (c) $|\Lambda_{\mathbf{t}}| \leq |\Lambda_{\mathbf{s}}|$
- (d) $|C_{\mathbf{t}}| \leq |C_{\mathbf{s}}|$

Proof. Easy.

$\square_{\text{1.3}}^{\text{a10}}$

a12 **Lemma 1.4 (Main Lemma).** Every frame is solvable.

Proof. We prove this by induction on r_s for all $\mathbf{s}$ (or just for the set of frames $\mathbf{t}$ with $r_{\mathbf{t}} \leq r_s$, $\bar{m}_{\mathbf{t}} := \bar{m}_{\mathbf{s}} \upharpoonright r_{\mathbf{t}}$ (so $|C_{\mathbf{t}}| \leq m_{\mathbf{s}, r_{\mathbf{t}}}$), and $(\Lambda_{\mathbf{t}}, \alpha_{\mathbf{t}}, \beta_{\mathbf{t}}) = (\Lambda_{\mathbf{s}}, \alpha_{\mathbf{s}}, \beta_{\mathbf{s}})$).

The $r = 0$ case is degenerate.

Case 1: $r = 1$

For $k \leq m_0$, (pedantically, $k \in M_0$), define $\eta_k \in {}^{m_0}\{\alpha, \beta\}$ as follows.

$$\eta_k(n) := \begin{cases} \alpha & \text{if } n < k \\ \beta & \text{if } n \in [k, m_0). \end{cases}$$

So $\langle \mathbf{c}(\eta_k) : k \leq m_0 \rangle$ is a sequence of length $m_0 + 1$ of members of C , and $|C| \leq m_0$. So for some $k < \ell \leq m_0$ we have $\mathbf{c}(\eta_k) = \mathbf{c}(\eta_\ell)$, hence $\xi := \langle (k, \ell, m_0) \rangle$ is as required. Indeed, if $\eta \in \Gamma_\xi$ is neither of the form η_k nor η_ℓ , then the only $\nu \in \Gamma_\xi$ which is E_ξ -equivalent to ν is ν itself.

Case 2: $r = s + 1$ for some $s \geq 1$.

For every $\nu \in (M_{s+1})\Lambda$ we define a function $\mathbf{d}_\nu : (\{s\} \times m_s)\Lambda \rightarrow C$ by

$$\mathbf{d}_\nu(\eta) := \mathbf{c}(\nu \cup \eta).$$

Hence as in Case 1 we can define functions $\eta_{\nu, k} \in (\{s\} \times m_s)\Lambda$ for $k \leq m_s$ by

$$\eta_{\nu, k}(s, n) := \begin{cases} \alpha & \text{if } n < k \\ \beta & \text{if } n \in [k, m_s). \end{cases}$$

So again (as $m_s \geq |C|$ by clause $\text{I.1}^{\text{a5}}(1)(E)$), we can find $k_\nu < \ell_\nu \leq m_s$ such that $\mathbf{d}_\nu(\eta_{\nu, k_\nu}) = \mathbf{d}_\nu(\eta_{\nu, \ell_\nu})$.

Let

$$(*)_0 \quad C_* := \{ \langle k, \ell \rangle^{\wedge} \bar{c} : k < \ell \leq m_s \text{ and } \bar{c} = \langle c_\gamma : \gamma \in \Lambda \setminus \{\beta\} \rangle \subseteq C \}.$$

Now,

$$(*)_1 \quad |C_*| = \binom{m_s+1}{2} \cdot |C|^{| \Lambda | - 1}$$

[Why? Just compute it.]

$$(*)_2 \quad |C_*| \leq m_{s-1}$$

[Why? Recall that $|C| \leq m_s$ by clause $\text{I.1}^{\text{a5}}(1)(E)$. Now

$$|C_*| = \binom{m_s+1}{2} \cdot |C|^{| \Lambda | - 1} \leq \frac{(m_s+1) \cdot m_s}{2} \cdot m_s^{| \Lambda | - 1} = \frac{1}{2}(m_s+1) \cdot m_s^{| \Lambda |}.$$

As

$$m_{s-1} \geq m_s^{| \Lambda | + 1} \geq \frac{1}{2}(m_s^{| \Lambda | + 1} + m_s^{| \Lambda |}) = \frac{1}{2}((m_s+1) \cdot m_s^{| \Lambda |})$$

(the first inequality is $\text{I.1}^{\text{a5}}(1)(F)$ and the second is easily verified) we conclude that $(*)_2$ does indeed hold.]

Clearly

$$\mathbf{t} := (\bar{m} \upharpoonright s, \Lambda_{\mathbf{s}}, \alpha_{\mathbf{s}}, \beta_{\mathbf{s}}, C_*)$$

is a frame.

$(*)_3$ We define a coloring $\mathbf{d} : (M_{\mathbf{t}})\Lambda \rightarrow C_*$ as follows.

$$\mathbf{d}(\nu) := \langle (k_\nu, \ell_\nu) \rangle^{\wedge} \langle \mathbf{c}(\eta_{\nu, \gamma}) : \gamma \in \Lambda \setminus \{\beta\} \rangle,$$

where $\text{dom}(\eta_{\nu, \gamma}) := M_{\mathbf{s}}$,

$$\eta_{\nu, \gamma}(s, n) := \begin{cases} \alpha & \text{if } n < k_\nu \\ \gamma & \text{if } n \in [k_\nu, \ell_\nu) \\ \beta & \text{if } n \in [\ell_\nu, m_s), \end{cases}$$

and $\eta_{\nu,\gamma}(i, n) := \nu(i, n)$ for $i < s$ and $n < m_i$.

So applying the induction hypothesis to $\mathbf{t}$ and $\mathbf{d}(-)$, there are $\xi_* \in \Xi_{\mathbf{t}}$ and $c_* \in C_*$ as promised. Write c_* as

$$\langle (k_s, \ell_s) \rangle \wedge \langle c_\gamma : \gamma \in \Lambda \setminus \{\beta\} \rangle$$

and ξ_* as

$$\langle (k_i, \ell_i, m_i) : i < s \rangle,$$

and let $\xi := \xi_* \wedge \langle (k_s, \ell_s, m_s) \rangle$. Now we shall check that ξ is as promised.

Assume that $\nu_1, \nu_2 \in \Gamma_\xi (\subseteq {}^{M_s}\Lambda)$ are E_ξ -equivalent. Clearly $\nu_1 \upharpoonright s$ and $\nu_2 \upharpoonright s$ are E_{ξ_*} -equivalent.

[Why? Just consider the definition of E_{ξ_*} .]

Hence, as ξ_* solves $\mathbf{d}$, necessarily $\mathbf{d}(\nu_1 \upharpoonright s) = \mathbf{d}(\nu_2 \upharpoonright s)$.

Now we can compare the first components of each in the definition of $\mathbf{d}(-)$, so we have $(k_{\nu_1 \upharpoonright s}, \ell_{\nu_1 \upharpoonright s}) = (k_{\nu_2 \upharpoonright s}, \ell_{\nu_2 \upharpoonright s})$. (Call them k_* and ℓ_* .)

Next, if $\nu_1 \upharpoonright (\{s\} \times [k_*, \ell_*])$ is constantly γ (for some $\gamma \in \Lambda \setminus \{\alpha, \beta\}$) then (recalling that ν_1, ν_2 are E_ξ -equivalent) so is $\nu_2 \upharpoonright (\{s\} \times [k_*, \ell_*])$. And by comparing more components of $\mathbf{d}(\nu_1 \upharpoonright s) = \mathbf{d}(\nu_2 \upharpoonright s)$, we conclude $\mathbf{c}(\nu_1) = \mathbf{c}(\nu_2)$.

(More fully, we have $\mathbf{c}(\eta_{\nu_1, \gamma}) = \mathbf{c}(\eta_{\nu_2, \gamma})$, but by $(*)_3$ and our present assumption we have $\eta_{\nu_\iota, \gamma} = \nu_\iota$ for $\iota = 1, 2$.)

Clearly, a similar argument give the paralel result for ν_2 .

We are left with the case

$$\nu_\iota \upharpoonright (\{s\} \times [k_*, \ell_*]) \text{ is constantly } \alpha \text{ or constantly } \beta$$

for $\iota = 1, 2$.

For this, define ν'_1 to be equal to ν_1 , except (possibly) that

$$n \in [k_*, \ell_*] \Rightarrow \nu'_1(s, n) := \alpha.$$

Hence by the choice of k_* and ℓ_* we have $\mathbf{c}(\nu_1) = \mathbf{c}(\nu'_1)$. Defining ν'_2 similarly, we can show $\mathbf{c}(\nu_2) = \mathbf{c}(\nu'_2)$.

Now the previous argument applies, showing $\mathbf{c}(\nu'_1) = \mathbf{c}(\nu'_2)$. Together, we get $\mathbf{c}(\nu_1) = \mathbf{c}(\nu_2)$, and we are done. $\square_{\text{I.4}}^{\text{a12}}$

Conclusion 1.5. *Given Λ and C of cardinality ≥ 2 and $r \geq 1$, define m_i by downward induction on $i < r$ as follows:*

- $m_{r-1} := |C|$
- For⁴ $i < r-1$, $m_i := (m_{i+1})^{|\Lambda|+1}$.

1) The sequence $\bar{m} = \langle m_i : i < r \rangle$ is as required in Definition $\text{I.1(1)}^{\text{a5}}$.

(That is, $(\bar{m}, \Lambda, \alpha, \beta, C)$ is a frame for any $\alpha \neq \beta \in \Lambda$.)

2) $m_i \leq |C|^{\exp(|\Lambda|+1, r-i-1)} = \exp_{|C|}(\exp_{|\Lambda|+1}(r-i-1))$

(Note that for $i := r-1$ this is indeed equal to $|C|$.)

3) $m_* := \sum_{i < r} m_i \leq |C|^{\exp(|\Lambda|+1, r)}$.

Proof. Obvious ⁵.

(For part (3), recall that we are assuming $|C| \geq 2$; the $|C| = 1$ case is degenerate, and generally $m_0 \leq |C|^{\exp(|\Lambda|+1, r-1)}$.) $\square_{\text{I.5}}^{\text{a15}}$

⁴Of course, $m_i := \frac{1}{2}(m_{i+1})^{|\Lambda|}(m_{i+1} + 1)$ would suffice.

⁵We round upward for transparency

a18 **Claim 1.6.** *If $n \geq 2$ and $\ell \geq 2$ then $HJ(n, \ell) \leq k(n, \ell)$, where*

$$k(2, \ell) := \ell + 1 \quad \text{and} \quad k(n + 1, \ell) := \exp_\ell(\exp_n(k(n, \ell))) \quad \text{for } n \geq 2.$$

(So $n \mapsto k(n, \ell)$ is essentially $\exp_{2n}(\ell + 1)$ – i.e. a power tower of height $2n$.)

Remark 1.7. That is, we define r_j for $j \leq n$ as follows.

- $r_0 := r$
- $r_{i+1} := k^{\exp(n+1, 2r_i)}$

Now $HJ(n, k) \leq r_n$.

Proof. As in [She:329], but replacing [She:329, 1.2-3, p.689] by ^{a5}1.1 and ^{a8}1.2. We prove this by induction on n .

$n = 2$: Here $k(2) = \ell + 1$.

By the definition.

$n = s + 1 > 2$:
Use ^{a5}1.1 and ^{a8}1.2.

□ ^{a18}1.6

In the terminology of [She:329], we have proved, in the MainLemaa ^{a12}1.4

a24 **Claim 1.8.** *The function $f(\ell, c)$ from [She:329, Def.1.2, p.689] has an exponential bound.*

Specifically, if $\mathbf{s}$ is a frame then $f(r_{\mathbf{s}}, |C_{\mathbf{s}}|) \leq m_{\mathbf{s}, 0}$.

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