LIMIT LAWS FOR COMPONENT-PRUNED SPARSE RANDOM GRAPHS AND PERCOLATED TORI

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Abstract. We prove an MSO_2 zero–one law for a very sparse Erdős–Rényi graph after pruning by component order. Let $p_n = c_n/n$, where $c_n \to 0$, and delete every component of order less than $f(n)$, where $f(n) \to \infty$. If

$$f(n)\big(\log f(n) + \log(1/c_n)\big) = o(\log n),$$

then the resulting graph satisfies a zero–one law for MSO_2, with quantification over sets of vertices and sets of edges. The proof combines uniform component counts, an MSO Feferman–Vaught decomposition for disjoint unions, and semilinearity of the order spectra of MSO-definable classes of finite trees. We also show that the term $f(n)\log f(n)$ cannot simply be omitted: star components can occur at first-order-visible Poisson thresholds.

We further establish first-order limit laws for bond percolation on the discrete torus T_L^d. In the two-sided subpolynomial regime, pruning below a sufficiently slow threshold yields a zero–one law. For the unpruned model in either one-sided polynomial regime, the reciprocal exponents $\alpha = 1/k$ are precisely the critical scales. At such a scale, an extended limit of Np_N^k or Nq_N^k equal to 0 or ∞ gives a zero–one law; a positive finite limit gives a convergence law but not a zero–one law; and the absence of an extended limit gives failure of convergence. Finally, MSO_1 already detects the parity of the torus side length through bipartiteness, producing a natural obstruction to monadic convergence in a near-deterministic regime.

1. Introduction

The Erdős–Rényi model is a basic meeting point of probability, combinatorics, and logic [7]. A classical starting point is the zero–one law for the dense random graph, proved independently by Glebskii, Kogan, Liogon'kii, and Talanov and by Fagin [12, 8]: for fixed edge probability, every first-order graph property has limiting probability 0 or 1. Sparse random graphs are subtler because local configurations appear at different scales. The work of

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Shelah–Spencer and Lynch gives a detailed model-theoretic analysis in important sparse regimes [23, 18]. Locality is central to many such results: once the quantifier rank is fixed, bounded neighborhoods and their multiplicities control the relevant first-order information. We use the finite bounded-degree form of Hanf locality for this purpose; see [13, 9, 11, 17].

This paper studies logical limit laws in two related sparse settings. The first is obtained by pruning according to component order. For a finite graph G and $m \geq 1$, define

$$\mathrm{Prune}_m(G) = G \restriction \{v \in V(G) : |C_G(v)| \geq m\},$$

where $C_G(v)$ is the connected component containing v. Thus $\mathrm{Prune}_m(G)$ is obtained by deleting every component of order less than m. This operation is unrelated to the usual graph-theoretic k-core, which iteratively deletes vertices of small degree. For a sequence (G_n) and an integer-valued function f, we abbreviate $\mathrm{Prune}_{f(n)}(G_n)$ to $\mathrm{Prune}_f(G_n)$.

We first apply this operation to $G(n, c_n/n)$ with $c_n \to 0$. Before pruning, the graph is a forest with probability tending to one, and its tree components can be counted by standard first- and second-moment estimates [1, 15]. After pruning with $f(n) \to \infty$, no fixed finite component remains. At each fixed monadic quantifier rank, every component type of bounded order disappears, whereas every type having arbitrarily large tree representatives occurs many times. An MSO Feferman–Vaught composition theorem then turns this deterministic component profile into an MSO_2 zero–one law.

The term $f(n) \log f(n)$ in our hypothesis is essential for this argument. First-order logic can detect a star component by asking for a vertex all of whose neighbors are leaves. If $t = t(n) \to \infty$ and $t = o(n^{1/2})$, the expected number of star components on t vertices in $G(n, c_n/n)$ is

$$(1 + o(1))\, n \frac{c_n^{t-1}}{(t-1)!} e^{-c_n t}.$$

Consequently, the weaker condition $f(n) \log(1/c_n) = o(\log n)$ still permits a first-order-visible Poisson threshold at the pruning boundary. The stronger assumption

$$f(n)\big(\log f(n) + \log(1/c_n)\big) = o(\log n)$$

excludes this obstruction, and we give an explicit counterexample under the weaker condition.

Our second setting is bond percolation on the discrete torus

$$T_L^d = (\mathbb{Z}/L\mathbb{Z})^d,$$

where adjacent vertices differ by ± 1 in exactly one coordinate. Write $N = L^d$. In the random subgraph $X_L^d(q_N)$, every torus edge is retained with probability q_N and omitted with probability $p_N = 1 - q_N$, independently. Because the degree is bounded, Hanf locality applies directly. In the two-sided subpolynomial regime, a slowly growing pruning threshold removes all bounded components while leaving many copies of every local type that can occur in an infinite subgraph of $\mathbb{Z}^d$. This yields a first-order zero–one law.

We also classify the unpruned model in the one-sided polynomial regimes. Suppose, for example, that $p_N \to 0$ and $q_N \to 1$. Each bounded rooted neighborhood type has a defect number: the least number of omitted edges needed to produce it from the full lattice neighborhood. A type of defect b has expected count of order Np_N^b. Hence the reciprocal exponents $\alpha = 1/k$ are the critical scales when $p_N = N^{-\alpha+o(1)}$. Away from those exponents, every local type is either absent or saturated. At a reciprocal exponent, finite-range defect clusters have Poisson limits, while rooted type counts are bounded cluster functions of those Poisson variables. This gives convergence at a finite critical parameter, zero–one laws at the two extended endpoints, and failure of convergence when the critical parameter itself does not converge.

The monadic behavior on tori is different. Although component composition controls MSO_2 on forests, monadic logic can express global properties beyond first-order locality on tori. In fact, MSO_1 already expresses bipartiteness and therefore detects the parity of long coordinate cycles. This yields a direct obstruction to monadic convergence when the side length ranges over both even and odd integers.

Relation to previous work. The local-pattern analysis on tori belongs to an established line of work on logical laws for random lattice fields. Coupier, Desolneux, and Ycart proved a first-order zero–one law for independent random images by combining Gaifman locality with threshold estimates for finite pixel patterns [2]. Their work on statistical area thresholding also studies the removal of small connected components as an image-denoising operation, using Poisson approximation to calibrate the area cutoff [3]. Coupier, Doukhan, and Ycart extended the logical local-pattern method to dependent binary fields on lattice tori under mixing assumptions [4]. Our first-order torus arguments use this same locality-and-counting mechanism. The features specific to our torus theorems are the bond-percolation graph language, pruning of percolation components, and the explicit organization of the one-sided critical regimes by the defect quantities Np_N^k and Nq_N^k. Thus the unpruned polynomial classification is an edge-percolation refinement of this framework rather than a new general locality principle.

There is also a substantial literature on monadic limit laws for random tree-like graphs. Woods obtained MSO limiting probabilities for random rooted trees through finite coloring rules [25]; McColm proved MSO zero–one laws for random labeled acyclic graphs [20]. More broadly, Heinig, Müller, Noy, and Taraz established an MSO zero–one law for connected random graphs and an MSO convergence law for unrestricted random graphs from addable minor-closed classes [14]. On the negative side, Kaufmann and Shelah showed that the MSO zero–one law can fail very strongly for uniform random finite relational structures [16]. For sparse Erdős–Rényi graphs, Ostrovsky and Zhukovskii studied bounded-quantifier-depth MSO laws in the very sparse power-law range [21]. The present result has a different source: a deterministic component-size filter is applied to $G(n, c_n/n)$ with $c_n \to 0$, and the conclusion is a zero–one law for every MSO_2 sentence, including quantification over edge sets. Its proof uses the MSO disjoint-union composition theorem and regular-tree size spectra, rather than a new local normal form for MSO.

Main results. For the Erdős–Rényi part, write $G(n, p)$ for the random graph on $[n]$ in which edges are present independently with probability p. The principal theorem of the paper is the following; its proof is completed in Section 5 after the required MSO composition tools are developed.

Theorem 1.1. *Let $p_n = c_n/n$, where $c_n > 0$ and $c_n \to 0$. Let $f : \mathbb{N} \to \mathbb{N}$ satisfy $f(n) \to \infty$ and*

$$f(n)\big(\log f(n) + \log(1/c_n)\big) = o(\log n).$$

Put

$$H_n = \mathrm{Prune}_{f(n)}(G(n, p_n)).$$

Then (H_n) has an MSO_2 zero–one law. In particular, it has an MSO_1 zero–one law and a first-order zero–one law.

The displayed hypothesis already implies $\log(1/c_n) = o(\log n)$, so no separate subpolynomial assumption on c_n is needed.

We first isolate the first-order component profile, both as a warm-up and for later comparison with the bounded-degree torus argument.

Theorem 1.2. *Let $p_n = c_n/n$, where $c_n > 0$ and $c_n \to 0$. Let $f : \mathbb{N} \to \mathbb{N}$ satisfy $f(n) \to \infty$ and*

$$f(n)\big(\log f(n) + \log(1/c_n)\big) = o(\log n).$$

Put

$$H_n = \mathrm{Prune}_{f(n)}(G(n, p_n)).$$

Then (H_n) has a first-order zero–one law. More precisely, for every quantifier rank s and every fixed K, with probability tending to one H_n is a forest, has no component of any bounded s-component type, and has at least K components of every unbounded s-component type. Consequently, for every s there is a deterministic s-theory of forests which H_n has with probability tending to one. This is the universal large-component forest profile described in Proposition 2.6.

For the torus, fix $d \geq 1$ and let

$$T_L^d = (\mathbb{Z}/L\mathbb{Z})^d$$

with an edge between two vertices if they differ by 1 or -1 modulo L in exactly one coordinate and agree in all other coordinates. The number of vertices is $N = L^d$. In bond percolation on T_L^d, each edge is retained with probability q_N and omitted with probability $p_N = 1 - q_N$, independently. The resulting random subgraph is denoted $X_L^d(q_N)$.

Theorem 1.3. *Fix $d \geq 1$. Let $N = L^d$ and let $X_L = X_L^d(q_N)$. Put*

$$\eta_N = \min\{p_N, q_N\}.$$

Let $f : \mathbb{N} \to \mathbb{N}$ be integer-valued. Assume $\eta_N > 0$ eventually, $f(N) \to \infty$, and

$$f(N)\log(1/\eta_N) = o(\log N).$$

Then

$$Y_L = \mathrm{Prune}_{f(N)}(X_L)$$

has a first-order zero–one law. Its almost-sure first-order theory is determined by the following axiom schemes:

(i) *the maximum degree is at most $2d$, and, for every $r \geq 0$ and every rooted r-ball type τ that is not a d-lattice type, no vertex has rooted r-ball type τ;*
(ii) *for each $m \geq 1$, there is no connected component of size exactly m;*
(iii) *for every $r \geq 0$, every rooted r-ball type τ realized at a vertex of an infinite component of a subgraph of $\mathbb{Z}^d$, and every $h \geq 1$, there are at least h distinct vertices with rooted r-ball type τ.*

Each item is a genuine first-order scheme: bounded degree, a fixed finite rooted ball type, the existence of a component of a fixed finite size, and the existence of h distinct realizations of a fixed type are all first-order expressible.

Because $\eta_N \leq 1/2$, the hypotheses imply $f(N) = o(\log N) = o(L)$ and $\log(1/\eta_N) = o(\log N)$.

The following theorem gives the unpruned one-sided polynomial answer in dimensions at least two. The one-dimensional case is different because deleting an edge from a cycle is a cut; see Remark 4.8.

Theorem 1.4. *Fix $d \geq 2$, let $N = L^d$, and let $X_L = X_L^d(q_N)$.*

(a) *Suppose $p_N \to 0$, $q_N \to 1$, and $p_N = N^{-\alpha+o(1)}$ for some $\alpha > 0$.*
  (i) *If $\alpha \notin \{1/k : k \in \mathbb{N}\}$, then (X_L) has a first-order zero–one law.*
  (ii) *If $\alpha = 1/k$ and $\lambda_N = Np_N^k \to 0$ or $\lambda_N \to \infty$, then (X_L) has a first-order zero–one law.*
  (iii) *If $\alpha = 1/k$ and $\lambda_N = Np_N^k \to \lambda \in (0,\infty)$, then (X_L) has a first-order convergence law but not a zero–one law.*
  (iv) *If $\alpha = 1/k$ and $\lambda_N = Np_N^k$ has no limit in the extended interval $[0,\infty]$, then (X_L) does not have a first-order convergence law.*
(b) *The same four alternatives hold with p_N and q_N interchanged. Thus, if $q_N \to 0$, $p_N \to 1$, and $q_N = N^{-\alpha+o(1)}$, the critical quantity at $\alpha = 1/k$ is Nq_N^k.*

The proof of the monadic theorem and the contrasting torus obstructions are given in Section 5. Proposition 5.4 and Corollary 5.5 show that the torus statements have no direct monadic analogue, even in MSO_1, without additional global restrictions.

The paper is organized as follows. Section 2 collects the locality and finite-type tools used throughout the paper: Hanf locality for bounded-degree graphs, a finite-rank replacement for forests of unbounded degree, and the elementary counting lemmas for local witnesses and rare defect patterns. Section 3 proves the pruning theorem for very sparse Erdős–Rényi graphs and the star obstruction showing that the smallness assumption on f cannot be weakened in the most naive way. Section 4 proves the torus results: first the pruned middle-regime theorem, and then the classification of the unpruned one-sided polynomial regimes. Section 5 proves the main MSO_2 theorem by component composition and tree automata, and then gives the MSO_1 bipartiteness obstruction for tori.

Notation and conventions. All graphs are finite unless explicitly stated otherwise. The first-order language of graphs has one binary relation symbol E, interpreted as a symmetric irreflexive adjacency relation. A sequence of random graphs (X_n) has a first-order convergence law if $\mathbb{P}(X_n \models \varphi)$ converges for every first-order sentence φ, and it has a first-order zero–one law if each of these limits is either 0 or 1.

For monadic second-order logic we use MSO_2 in the usual graph-theoretic sense: one may quantify over sets of vertices and over sets of edges, equivalently over monadic predicates in the incidence structure of the graph. The fragment in which second-order variables range only over subsets of the vertex set is denoted MSO_1. Monadic convergence laws and monadic zero–one laws are defined analogously, with φ ranging over sentences of the corresponding monadic logic.

For a graph G and a vertex v, $C_G(v)$ denotes the connected component of v. If $r \geq 0$, then $B_G(v, r)$ denotes the induced rooted graph on the vertices at graph distance at most r from v, rooted at v. If τ is a rooted r-ball type, then $N_\tau(G)$ denotes the number of vertices of G whose rooted r-ball has type τ.

We write $a_n = o(b_n)$ and $a_n = O(b_n)$ in the usual asymptotic sense. The phrase "with high probability" means with probability tending to one. All pruning thresholds are positive integers, and all threshold functions f are positive integer-valued. For the torus, L is the side length, d is fixed, and $N = L^d$ is the number of vertices. In bond percolation on T_L^d, the retained-edge probability is q_N and the omitted-edge probability is $p_N = 1 - q_N$.

2. Locality and finite-type tools

We first record the bounded-degree locality theorem and the finite-rank composition facts used later.

2.1. Hanf locality in bounded degree.

Definition 2.1. Let $\Delta, r, M \in \mathbb{N}$. Two graphs G and H of maximum degree at most Δ are (r, M)-*Hanf-equivalent* if, for every rooted r-ball type τ of degree at most Δ, either

$$N_\tau(G) = N_\tau(H) < M, \qquad \text{or} \qquad N_\tau(G), N_\tau(H) \geq M.$$

Theorem 2.2 (Bounded-degree Hanf locality). *For every Δ and every first-order quantifier rank s, there are $r = r(\Delta, s)$ and $M = M(\Delta, s)$ such that any two finite or infinite graphs of maximum degree at most Δ that are (r, M)-Hanf-equivalent satisfy the same first-order sentences of quantifier rank at most s.*

The general bounded-degree form goes back to Hanf [13]; the finite form used for the random finite graphs below is due to Fagin, Stockmeyer, and Vardi [9]. See also [17]. In the torus arguments the maximum degree is $2d$.

2.2. **Large-component locality for forests.** The pruned Erdős–Rényi graph is a forest with high probability, but its degree is not bounded. We therefore use a componentwise finite-rank replacement for Hanf locality.

For finite graphs A and B, write $A \equiv_s B$ if they satisfy the same first-order sentences of quantifier rank at most s. There are only finitely many $\equiv_s$-classes of finite trees; we call each such class an *s-component type*. A type is *unbounded* if it contains trees of arbitrarily large order, and *bounded* otherwise.

Lemma 2.3 (Semilinear tree spectra). *Let $\mathcal{C}$ be an* MSO_2*-definable class of finite (graph-theoretic) trees. Then its order spectrum*

$$S(\mathcal{C}) = \{|V(T)| : T \in \mathcal{C}\}$$

is a semilinear subset of $\mathbb{N}$*.*

Consequently, if $\mathcal{C}_1, \dots, \mathcal{C}_\nu$ are finitely many such classes and each has members of arbitrarily large order, then there is a constant $C \geq 1$ such that, for every $i \leq \nu$ and every $m \geq 1$, some $T \in \mathcal{C}_i$ satisfies

$$m \leq |V(T)| \leq Cm.$$

Proof. View a graph-theoretic tree through its two-sorted incidence structure, with unary labels distinguishing vertex-elements from edge-elements. The incidence structure is itself a finite labeled tree. Choose a root, order the children of each node, and apply the standard first-child/next-sibling binary encoding. The class of valid encodings whose decoded incidence tree belongs to $\mathcal{C}$ is MSO-definable on finite labeled binary trees: validity of the encoding, the two sorts, the incidence relation, and the interpretation of the defining sentence for $\mathcal{C}$ are all MSO-definable. By the MSO–tree-automaton theorem, this class is a regular finite-tree language [24, 6, 5].

Let $\mathcal{A}$ be a finite bottom-up automaton recognizing this language. Convert $\mathcal{A}$ into a context-free grammar as follows. The automaton states are nonterminals. A transition with parent state q, child states $q_1, \dots, q_t$ $(0 \leq t \leq 2)$, and a vertex label gives a production

$$q \longrightarrow x q_1 \cdots q_t,$$

whereas a transition with an edge label gives the same production with y in place of x. A new start symbol S has a production $S \to q$ for each accepting state q. The Parikh vectors of the generated words are exactly the pairs

$(|V(T)|, |E(T)|)$ arising from trees in $\mathcal{C}$. Parikh's theorem therefore makes this set semilinear, and projection onto the first coordinate shows that $S(\mathcal{C})$ is semilinear [22].

An infinite semilinear subset of $\mathbb{N}$ contains an infinite arithmetic progression $u + jh$ with $h > 0$. The least term of this progression that is at least m is at most $m + u + h$, and hence at most $(1 + u + h)m$. Taking the maximum of these constants over the finitely many classes proves the final assertion. □

Lemma 2.4. *For every s there is a constant C_s such that every unbounded s-component type γ has the following property: for every $m \geq 1$ there is a finite tree $T \in \gamma$ with*

$$m \leq |T| \leq C_s m.$$

Proof. Each s-component type is defined, on finite trees, by a characteristic first-order sentence and is therefore MSO_2-definable. Apply the second assertion of Lemma 2.3 to the finite family of unbounded s-component types. □

Lemma 2.5. *For every s there is M_s such that the following holds. If, for each s-component type γ, two forests F, F' either have the same number of components of type γ, this number being less than M_s, or both have at least M_s components of type γ, then $F \equiv_s F'$.*

Proof. Take $M_s = s + 1$. In the s-move Ehrenfeucht–Fraïsse game on F and F', whenever Spoiler first enters a component, Duplicator chooses a fresh component of the same s-component type. At most s components are entered, so the multiplicity hypothesis always provides such a component. On every matched pair, Duplicator follows a winning s-move strategy for the two component trees. Since distinct components have no edges between them, the union of the componentwise partial isomorphisms is a partial isomorphism of the two forests. □

Proposition 2.6. *Fix s. There are constants B_s, M_s and a finite set Γ_s^∞ of unbounded s-component types such that the following holds. If a finite forest F has no component of order at most B_s and has at least M_s components of each type in Γ_s^∞, then the s-theory of F is fixed, independent of F.*

Proof. Let Γ_s^∞ be the set of unbounded s-component types, and let M_s be given by Lemma 2.5. Since there are only finitely many s-component types, the bounded ones have a common size bound B_s; if there are no bounded types, take $B_s = 0$. If F has no component of size at most B_s, then no bounded type occurs in F. The unbounded types all occur at least M_s times by hypothesis. Lemma 2.5 now implies that any two such forests are $\equiv_s$. □

2.3. **Counting local witnesses.**

Lemma 2.7. *Let $(A_i)_{i \in I_N}$ be events, each determined by finitely many independent coin flips. Suppose that $J_N \subseteq I_N$ is a subfamily for which the* *events $(A_i)_{i\in J_N}$ are mutually independent. If*

$$|J_N|\min_{i\in J_N}\mathbb{P}(A_i)\longrightarrow\infty,$$

then, for every fixed m, at least m of the events A_i occur with probability tending to one.

Proof. The number of occurrences in the independent subfamily stochastically dominates a binomial random variable whose mean tends to infinity. A Chernoff bound gives the conclusion. □

Lemma 2.8. *Let $\mathcal{G}$ be a fixed connected locally finite graph, and let a group $\Gamma\le\operatorname{Aut}(\mathcal{G})$ act transitively on its vertices. Let $\pi_N:\mathcal{G}\to\mathcal{G}_N$ be finite covering quotients with $|V(\mathcal{G}_N)|=N\to\infty$ and injectivity radius tending to infinity; that is, for every fixed R, the quotient map restricts to a graph isomorphism on every R-ball for all sufficiently large N. On the edges of $\mathcal{G}_N$, independently put a rare state with probability ρ_N and the common state otherwise, where $\rho_N\to 0$; write $\mathcal{X}_N$ for the resulting random edge-state structure (or for the graph determined by those states).*

Fix $k\ge 1$ and finitely many Γ-orbits $D\in\mathcal{D}$ of bounded marked configurations in $\mathcal{G}$. Choose a common bounded protective window containing all their decision windows. A representative of D consists of a specified k-element set of rare edges in this window; its canonical occurrence requires those edges to be rare and every other edge in the protective window to be common. Descriptions with the same underlying rare-edge set are grouped into a single orbit before counting, so distinct members of $\mathcal{D}$ have distinct orbits of underlying rare-edge sets. Assume that the number of candidate occurrences of D in $\mathcal{G}_N$ is

$$a_DN+o(N)$$

for a constant $a_D>0$. Let $W_{D,N}$ count these underlying occurrences, whose prescribed rare/common states are realized, once per rare-edge set rather than once per possible rooted observation.

If

$$N\rho_N^k\longrightarrow\lambda\in(0,\infty),$$

then $(W_{D,N})_{D\in\mathcal{D}}$ converges jointly to a vector of independent Poisson random variables with respective means $a_D\lambda$.

Suppose, in addition, that $\tau_1,\dots,\tau_m$ are rooted local types such that every realization of any τ_i either uses exactly k rare edges and is represented in $\mathcal{D}$, or uses at least $k+1$ rare edges in one of finitely many bounded decision windows. Then, for every fixed truncation level M, the vector

$$\left(\min\{N_{\tau_i}(\mathcal{X}_N),M\}\right)_{1\le i\le m}$$

has a limiting distribution. The untruncated rooted-type counts need not be Poisson: one underlying defect configuration may produce a bounded cluster of rooted observations, so their limits are in general compound Poisson.

Proof. We use mixed factorial moments. Fix nonnegative integers $(j_D)_{D\in\mathcal{D}}$ and put $j = \sum_D j_D$. Ordered candidate tuples whose decision windows are pairwise disjoint number

$$(1 + o(1)) \prod_{D\in\mathcal{D}} (a_D N)^{j_D}.$$

Their joint occurrence probability is $(1 + o(1))\rho_N^{kj}$, since only a fixed number of common-edge requirements is involved. Their contribution to the mixed factorial moment therefore tends to

$$\prod_{D\in\mathcal{D}} (a_D \lambda)^{j_D}.$$

It remains to discard overlapping tuples. Join two proposed occurrences when their decision windows intersect. Because the local model and all windows are fixed, a connected overlap component has only $O(N)$ possible placements. A non-singleton component contains two distinct underlying k-edge sets and hence requires at least $k + 1$ rare edges. Its contribution is therefore $O(N\rho_N^{k+1}) = o(1)$; the other overlap components contribute at most bounded factors because $N\rho_N^k = O(1)$. Thus every mixed factorial moment has the limit displayed above, proving the joint independent Poisson limit.

Finally, the expected number of bounded windows containing a realization with at least $k + 1$ rare edges is $O(N\rho_N^{k+1}) = o(1)$. The same estimate discards overlaps between distinct critical clusters and additional rare edges in their decision windows. Off this exceptional event, each rooted type count, truncated at M, is a fixed function of the finite vector of underlying cluster counts. The continuous mapping theorem on the discrete state space gives the asserted joint limiting distribution. □

3. Pruning very sparse Erdős–Rényi graphs

Throughout this section $p_n = c_n/n$, where $c_n > 0$ and $c_n \to 0$. Let $f : \mathbb{N} \to \mathbb{N}$ satisfy $f(n) \to \infty$ and

$$f(n)\big(\log f(n) + \log(1/c_n)\big) = o(\log n).$$

In particular, $\log(1/c_n) = o(\log n)$ and $f(n) = o(\log n)$. Write $G_n = G(n, p_n)$ and $H_n = \mathrm{Prune}_{f(n)}(G_n)$.

Lemma 3.1. *With probability tending to one, G_n is acyclic. Consequently, H_n is a forest with probability tending to one.*

Proof. For all sufficiently large n, we have $c_n < 1$. The expected number of cycles in G_n is at most

$$\sum_{\ell\geq 3} \frac{(np_n)^\ell}{2\ell} = \sum_{\ell\geq 3} \frac{c_n^\ell}{2\ell} = o(1).$$

The result follows from Markov's inequality. □

Proposition 3.2. *Let (F_n) be any sequence of finite trees with $t_n = |F_n|$ such that*

$$t_n(\log t_n + \log(1/c_n)) = o(\log n).$$

Let $X_n(F_n)$ be the number of connected components of G_n isomorphic to F_n. Then

$$X_n(F_n) \longrightarrow \infty \quad \text{in probability.}$$

Proof. Since $\log(1/c_n) \to \infty$, the hypothesis implies $t_n = o(\log n)$, and in particular $t_n = o(n^{1/2})$. Let $a_n = |\operatorname{Aut}(F_n)|$. For a fixed t_n-set of vertices, the number of labeled graphs on that set isomorphic to F_n is $t_n!/a_n$. Hence

$$\mathbb{E}X_n(F_n) = \binom{n}{t_n}\frac{t_n!}{a_n}p_n^{t_n-1}(1-p_n)^{t_n(n-t_n)+\binom{t_n}{2}-(t_n-1)}.$$

Put

$$A_n = t_n(n-t_n) + \binom{t_n}{2} - (t_n - 1).$$

Then

$$\log\frac{(n)_{t_n}}{n^{t_n}} = O(t_n^2/n) = o(1), \qquad A_n \log(1-p_n) = -c_n t_n + o(1).$$

Using $a_n \le t_n!$ and $c_n t_n = o(\log n)$ therefore gives

$$\begin{aligned}\log \mathbb{E}X_n(F_n) &\ge \log n - (t_n-1)\log(1/c_n) - \log(t_n!) - O(c_n t_n) - o(1)\\ &\ge \log n - t_n(\log(1/c_n) + \log t_n) - o(\log n)\\ &= (1-o(1))\log n.\end{aligned}$$

Thus $\mathbb{E}X_n(F_n) \to \infty$.

For completeness, write $X_n(F_n) = \sum_S I_S$, where the sum is over t_n-subsets and I_S is the indicator that S induces a component isomorphic to F_n. Two distinct intersecting sets cannot both be connected components. For two disjoint sets, the joint probability differs from the product of the individual probabilities only because the absence of the edges between the two sets is counted twice in the product; the correction factor is

$$(1-p_n)^{-t_n^2} = 1 + o(1).$$

Moreover, the proportion of ordered pairs of t_n-sets which are disjoint is

$$\frac{\binom{n-t_n}{t_n}}{\binom{n}{t_n}} = 1 + o(1),$$

because $t_n^2/n = o(1)$. It follows that

$$\mathbb{E}(X_n(F_n))_2 = (1+o(1))(\mathbb{E}X_n(F_n))^2,$$

and hence

$$\operatorname{Var}(X_n(F_n)) = o((\mathbb{E}X_n(F_n))^2) + O(\mathbb{E}X_n(F_n)),$$

and Chebyshev's inequality gives $X_n(F_n)/\mathbb{E}X_n(F_n) \to 1$ in probability. $\square$

Proof of Theorem 1.2. Fix a first-order sentence φ and let $s = \mathrm{qr}(\varphi)$. Take B_s, M_s, and Γ_s^∞ from Proposition 2.6, and take C_s from Lemma 2.4. For every $\gamma \in \Gamma_s^\infty$, that lemma gives a finite tree $F_{\gamma,n} \in \gamma$ with

$$f(n) \leq |F_{\gamma,n}| \leq C_s f(n).$$

Since $|F_{\gamma,n}| \leq C_s f(n)$, the hypothesis on f implies

$$\begin{aligned}|F_{\gamma,n}|(\log|F_{\gamma,n}| + \log(1/c_n)) &\leq C_s f(n)(\log(C_s f(n)) + \log(1/c_n)) \\ &= o(\log n).\end{aligned}$$

By Proposition 3.2, G_n has at least M_s connected components isomorphic to $F_{\gamma,n}$ with probability tending to one. Since Γ_s^∞ is finite, these events hold simultaneously for all $\gamma \in \Gamma_s^\infty$ with probability tending to one. Since $|F_{\gamma,n}| \geq f(n)$, all of these components survive in H_n.

Every component of H_n has order at least $f(n)$, which exceeds B_s for all sufficiently large n. By Lemma 3.1, H_n is also a forest with probability tending to one. Hence H_n satisfies the hypotheses of Proposition 2.6 with probability tending to one. The truth value of every sentence of rank at most s, and in particular of φ, is then forced. Since φ was arbitrary, (H_n) has a first-order zero–one law.

For the more precise multiplicity assertion, fix K and repeat the same argument with K in place of M_s in the application of Proposition 3.2. The exclusion of bounded types follows from $f(n) > B_s$, and the forest assertion follows from Lemma 3.1. □

Proposition 3.3. *There are sequences $c_n \to 0$ and $f(n) \to \infty$ satisfying*

$$\log(1/c_n) = o(\log n), \qquad f(n)\log(1/c_n) = o(\log n),$$

for which the pruned graphs $\mathrm{Prune}_{f(n)}(G(n, c_n/n))$ *do not have a first-order zero–one law.*

Proof. Fix $\lambda \in (0,\infty)$ and set

$$f(n) = \left\lfloor \frac{\log n}{\log\log n} \right\rfloor.$$

Choose $c_n \in (0,1)$ so that

$$\frac{n c_n^{f(n)-1} e^{-c_n f(n)}}{(f(n)-1)!} = \lambda. \tag{3.1}$$

For all sufficiently large n such a solution exists in the range $c_n \in (0, e^{-1})$. Indeed, the left hand side tends to 0 as $c_n \downarrow 0$, while at $c_n = e^{-1}$ its logarithm is $\log n - f(n)\log f(n) + O(f(n)) \to \infty$. It is unique in this interval because the logarithmic derivative of its c_n-dependent factor is

$$\frac{f(n)-1}{c_n} - f(n) > 0 \qquad (0 < c_n < e^{-1})$$

for all sufficiently large n. Writing $a_n = \log(1/c_n)$, Stirling's formula applied to (3.1) gives

$$a_n = \frac{\log n - \log((f(n)-1)!) - c_n f(n) - \log\lambda}{f(n)-1}$$
$$= \log\log\log n + O(1).$$

Hence $c_n \to 0$, $\log(1/c_n) = o(\log n)$, and $f(n)\log(1/c_n) = o(\log n)$. Notice also that $f(n)\log f(n) = (1+o(1))\log n$, so the stronger hypothesis of Theorem 1.1 is deliberately violated.

Let Z_n be the number of connected components of $G(n, c_n/n)$ that are stars with at least $f(n)$ vertices, and let $\mu_{n,t}$ be the expected number with exactly t vertices. For $t \geq 3$, direct counting gives

$$\mu_{n,t} = \binom{n}{t} t\, p_n^{t-1}(1-p_n)^{t(n-t)+\binom{t}{2}-(t-1)}.$$

Since $f(n) = o(n^{1/2})$, uniformly for $t = O(f(n))$,

$$\mu_{n,t} = (1+o(1))\frac{nc_n^{t-1}e^{-c_n t}}{(t-1)!}.$$

By (3.1), $\mu_{n,f(n)} \to \lambda$. Moreover, for $f(n) \leq t < n$,

$$\frac{\mu_{n,t+1}}{\mu_{n,t}} = \frac{(n-t)p_n}{t}(1-p_n)^{n-t-2} \leq \frac{c_n}{t(1-p_n)} = o(1)$$

uniformly in t. Hence the expected number of star components with more than $f(n)$ vertices is $o(1)$, and $\mathbb{E}Z_n \to \lambda$.

Let Y_n count star components with exactly $f(n)$ vertices. Write it as a sum over centered candidates (S, x), where $|S| = f(n)$ and $x \in S$ is the proposed center. For every fixed j, only candidates with pairwise disjoint vertex sets contribute to the jth factorial moment of Y_n. Their joint probability differs from the product of the individual probabilities by the factor

$$(1-p_n)^{-\binom{j}{2}f(n)^2} = 1 + o(1),$$

and the use of disjoint vertex sets changes the corresponding combinatorial factor by $1 + o(1)$ because $f(n) = o(n^{1/2})$. Thus

$$\mathbb{E}(Y_n)_j \longrightarrow \lambda^j.$$

The factorial-moment criterion gives $Y_n \Rightarrow \text{Poisson}(\lambda)$, while $Z_n - Y_n \to 0$ in probability by the preceding tail estimate. Therefore

$$Z_n \Rightarrow \text{Poisson}(\lambda).$$

Let $\psi_\star$ be the first-order sentence

$$\exists x\Big[(\exists y\, E(x,y)) \wedge \forall y(E(x,y) \to \forall z(E(y,z) \to z = x))\Big].$$

With high probability, $G(n, c_n/n)$ is a forest. On this event, the pruned graph satisfies $\psi_\star$ exactly when at least one star component with at least $f(n)$ vertices survives. Therefore

$$\mathbb{P}(\text{Prune}_{f(n)}(G(n, c_n/n)) \models \psi_\star) \longrightarrow 1 - e^{-\lambda},$$

which is strictly between 0 and 1. □

Remark 3.4. The strengthened smallness condition is not cosmetic. Let S_t be the star on t vertices. When $t = t(n) \to \infty$ and $t = o(n^{1/2})$, the expected number of components isomorphic to S_t in $G(n, c_n/n)$ is asymptotic to

$$n \frac{c_n^{t-1}}{(t-1)!} e^{-c_n t}.$$

After pruning, the first-order sentence asserting the existence of a vertex all of whose neighbors are leaves is controlled by star components of order at least $f(n)$. The weaker condition $f(n) \log(1/c_n) = o(\log n)$ permits choices of c_n and $f(n)$ for which the displayed expectation at $t = f(n)$ has a positive finite limit or oscillates. The corresponding sentence then need not have a 0–1 limit. The additional condition $f(n) \log f(n) = o(\log n)$ excludes this automorphism-driven obstruction.

4. Percolated tori

Fix $d \geq 1$ and write $N = L^d$. For every fixed coordinate box, the quotient map $\mathbb{Z}^d \to T_L^d$ is injective on the box and its incident edges once L is sufficiently large.

A rooted r-ball type τ is a *d-lattice type* if $B_X(0, r) \cong \tau$ for some subgraph $X \subseteq \mathbb{Z}^d$. It is *unbounded* if such a realization can be chosen with the root in an infinite component of X. This is stronger than merely requiring the root component in the displayed ball to reach distance r.

4.1. The pruned middle regime.

Throughout this subsection $f : \mathbb{N} \to \mathbb{N}$ is integer-valued.

Let

$$X_L = X_L^d(q_N), \qquad Y_L = \text{Prune}_{f(N)}(X_L).$$

Recall that

$$\eta_N = \min\{q_N, p_N\}, \qquad p_N = 1 - q_N.$$

Lemma 4.1. *Assume $f(N) \to \infty$ and*

$$f(N) \log(1/\eta_N) = o(\log N).$$

Fix $r \geq 0$, an unbounded d-lattice rooted r-ball type τ, and $m \geq 1$. With probability tending to one, Y_L has at least m vertices whose rooted r-ball is τ.

Proof. Choose a subgraph $X \subseteq \mathbb{Z}^d$ such that $B_X(0, r)$ has type τ and 0 lies in an infinite component. That locally finite component contains a self-avoiding ray γ starting at 0. There is a fixed finite set of lattice edges D_τ, contained in a box depending only on r, d, and τ, whose states determine the rooted

r-ball. For example, one may take all lattice edges with both endpoints in $[-r-1, r+1]^d$. Prescribe on D_τ the same states as in X. This includes every boundary decision needed to prevent an additional vertex from entering the r-ball.

Now take an initial segment of γ containing at least $f(N)$ distinct vertices and prescribe all of its edges to be retained. These prescriptions are consistent, since $\gamma \subseteq X$. They force the root component to have at least $f(N)$ vertices while leaving its rooted r-ball equal to τ. The number of queried edges is at most $C_{r,d,\tau}(f(N)+1)$, and their coordinate diameter is $O_{r,d,\tau}(f(N))$. Because $\eta_N \le 1/2$, the hypothesis implies $f(N) = o(\log N)$ and therefore $f(N) = o(L)$. Thus the witness embeds without wrap-around in T_L^d for all sufficiently large L.

A specified translate of this witness occurs with probability at least

$$\eta_N^{C_{r,d,\tau}(f(N)+1)} = \exp\{-C_{r,d,\tau}(f(N)+1)\log(1/\eta_N)\} = N^{-o(1)}.$$

The hypothesis also gives $\log(1/\eta_N) = o(\log N)$, so the additive 1 in the exponent is harmless. Moreover, $f(N) = o(\log N) = N^{o(1)}$. A greedy packing of translates of the witness box gives at least

$$c_{r,d,\tau}\frac{N}{(f(N)+1)^d} = N^{1-o(1)}$$

translates with pairwise disjoint queried edge sets. The product of this quantity and the displayed lower bound tends to infinity, so Lemma 2.7 gives at least m occurrences with probability tending to one. In each occurrence, the root lies in an X_L-component of order at least $f(N)$. Pruning retains that entire component, so the rooted r-ball remains τ in Y_L. □

Lemma 4.2. *For each fixed r, for all sufficiently large L, no vertex of Y_L has a rooted r-ball type that is not unbounded.*

Proof. Fix r. For all sufficiently large L, the coordinate box

$$Q_r(v) = v + [-r, r]^d$$

embeds in T_L^d without wrap-around, and every graph-metric r-ball around v is contained in $Q_r(v)$.

Suppose that the component of v in Y_L contains a vertex outside $Q_r(v)$. Choose a shortest component path from v to the first vertex x outside this box and lift the path to $\mathbb{Z}^d$. One coordinate of $x - v$ has absolute value $r+1$. Starting at x, continue indefinitely in the corresponding outward coordinate direction. Form a subgraph of $\mathbb{Z}^d$ by taking the lifted rooted ball $B_{Y_L}(v, r)$, the part of the chosen path beyond that ball, and this outward ray, while omitting all other edges. Every new vertex is at graph distance greater than r from the root, so the rooted r-ball is unchanged, and the root now lies in an infinite component. Hence the rooted type of v is unbounded.

It follows that a vertex whose rooted r-ball type is not unbounded must have its entire component contained in $Q_r(v)$. Such a component has at most $(2r+1)^d$ vertices. Since $f(N) \to \infty$, this is smaller than $f(N)$ for all sufficiently large L, contradicting the definition of $Y_L = \mathrm{Prune}_{f(N)}(X_L)$. $\square$

Proof of Theorem 1.3. Fix a first-order sentence φ, let $s = \mathrm{qr}(\varphi)$, and take r, M from Theorem 2.2 for maximum degree $2d$ and rank s. For all large L, every rooted r-ball in T_L^d lifts to $\mathbb{Z}^d$ and hence is a d-lattice type; all non-lattice types are therefore deterministically absent. There are only finitely many rooted r-ball types of degree at most $2d$. Lemma 4.1, followed by a union bound over the unbounded ones, shows that each unbounded d-lattice type occurs at least M times in Y_L with probability tending to one. Lemma 4.2 excludes every remaining type. Thus, with probability tending to one, the (r, M)-Hanf profile is the deterministic profile

$$N_\tau(Y_L) \geq M \quad \Longleftrightarrow \quad \tau \text{ is an unbounded } d\text{-lattice type.}$$

Hanf locality makes the truth value of φ constant on this event. This proves the zero–one law, including the case in which the radius supplied by Hanf is $r = 0$.

It remains to justify the asserted axiomatization. The degree clause in scheme (i) holds deterministically, and its remaining clauses hold eventually at every fixed radius by the lifting observation above. Scheme (ii) holds eventually for each fixed m because every component of Y_L has at least $f(N) > m$ vertices. Scheme (iii) is Lemma 4.1, applied with an arbitrary fixed multiplicity h. Hence every displayed axiom belongs to the limiting almost-sure theory. Moreover, every finite subset of the three schemes holds in Y_L with probability tending to one and is therefore satisfiable. By compactness, the full collection of schemes has a model.

Conversely, let A be any graph satisfying all three schemes, fix $v \in A$, and fix $r \geq 0$. Scheme (ii) implies that the component of v is infinite. For every $R \geq r$, scheme (i) gives a rooted lattice realization of $B_A(v, R)$. The possible injective root-preserving graph embeddings of this finite ball into $[-R, R]^d$ form a finite set; the image is contained in that box because lattice displacement from the root is at most graph distance. Join an embedding at level $R+1$ to its restriction at level R. The resulting tree is infinite and finitely branching, so Kőnig's lemma supplies a coherent branch. The union of that branch embeds the entire component of v as an infinite subgraph of $\mathbb{Z}^d$. Consequently $B_A(v, r)$ is an unbounded d-lattice type. Scheme (iii) now says that every such type has arbitrarily many realizations. Thus every model of the schemes has, at every radius and cutoff, exactly the deterministic Hanf profile displayed above. Hanf locality shows that the schemes determine a complete first-order theory, namely the limiting theory of (Y_L). $\square$

4.2. **One-sided polynomial regimes without pruning.** Throughout this subsection $d \geq 2$. We prove Theorem 1.4. We give the details when $p_N \to 0$ and $q_N \to 1$; the regime $q_N \to 0$ follows by interchanging retained and omitted edges.

Fix $r \geq 0$ and let W_r be the set of lattice edges with both endpoints in $[-r-1, r+1]^d$. A path of length at most r from the root stays inside $[-r, r]^d$, so, for every vertex v and all sufficiently large L, the edge states in the translate $v + W_r$ determine $B_{X_L}(v, r)$. For a rooted d-lattice r-ball type τ, let $b(\tau)$ be the minimum number of omitted edges among the complete assignments on W_r that produce τ. Equivalently, an assignment attaining the minimum omits $b(\tau)$ edges and retains every other edge in W_r. This definition is independent of enlarging the decision window by finitely many irrelevant edges. In particular, $b(\tau) = 0$ only for the full rooted r-ball of $\mathbb{Z}^d$.

Lemma 4.3. *For every fixed r and every rooted d-lattice r-ball type τ, there is a constant $C_\tau > 0$ such that, uniformly in the root vertex and for all sufficiently large L,*

$$\mathbb{P}(B_{X_L}(v, r) \cong \tau) = C_\tau p_N^{b(\tau)} + O(p_N^{b(\tau)+1}). \tag{4.1}$$

The constant depends only on r, d, and τ.

Proof. The decision window contains only finitely many edges, and the relevant list of assignments is independent of v and L once the window lifts to $\mathbb{Z}^d$. An assignment with a omitted and c retained edges has probability $p_N^a(1-p_N)^c$. Summing the probabilities of the mutually exclusive assignments that realize τ, and grouping them by a, gives a polynomial in p_N. Its least nonzero degree is $b(\tau)$, and its leading coefficient C_τ is the positive number of minimal assignments. This proves the uniform expansion. □

Lemma 4.4. *Assume $d \geq 2$, $p_N \to 0$, $q_N \to 1$, and $p_N = N^{-\alpha+o(1)}$ for some $\alpha > 0$.*

- (i) *If $\alpha \notin \{1/k : k \in \mathbb{N}\}$, then (X_L) has a first-order zero–one law.*
- (ii) *If $\alpha = 1/k$ and $Np_N^k \to 0$ or $Np_N^k \to \infty$, then (X_L) has a first-order zero–one law.*

Proof. Fix a sentence φ and choose r, M from Hanf locality for maximum degree $2d$ and quantifier rank $\mathrm{qr}(\varphi)$. Once L is large enough, every rooted r-ball in X_L is a d-lattice type, so every non-lattice type is deterministically absent. For a d-lattice type τ, equation (4.1) gives

$$\mathbb{E}N_\tau(X_L) = NC_\tau p_N^{b(\tau)}(1 + o(1)).$$

If $b(\tau) = 0$, choose a positive-density set of centers with disjoint decision windows. The corresponding independent events have probabilities tending to $C_\tau > 0$, so the type occurs at least M times with high probability.

For $b(\tau) \geq 1$ and $\alpha \notin \{1/j : j \in \mathbb{N}\}$,

$$Np_N^{b(\tau)} = N^{1-\alpha b(\tau)+o(1)}$$

tends either to 0 or to ∞. If $\alpha = 1/k$, the same conclusion holds for every $b(\tau) \neq k$; for $b(\tau) = k$, it follows from the additional assumption on Np_N^k.

If $Np_N^{b(\tau)} \to 0$, Markov's inequality shows that τ is absent with high probability. If $Np_N^{b(\tau)} \to \infty$, a fixed-window packing provides $c_{r,d}N$ centers with pairwise disjoint decision windows. The corresponding events are independent and have probability asymptotic to $C_\tau p_N^{b(\tau)}$, so Lemma 2.7 shows that τ occurs at least M times with high probability.

There are only finitely many rooted r-ball types of degree at most $2d$. Taking the intersection of the preceding high-probability events gives a deterministic truncated Hanf profile. Hanf locality fixes the truth value of φ with probability tending to one. □

Lemma 4.5. *Assume $d \geq 2$, $p_N \to 0$, $q_N \to 1$, and $Np_N^k \to \lambda \in (0,\infty)$ for some $k \in \mathbb{N}$. Then (X_L) has a first-order convergence law.*

Proof. Fix φ and choose r, M from Hanf locality for maximum degree $2d$ and quantifier rank $\mathrm{qr}(\varphi)$. Non-d-lattice rooted r-ball types are deterministically absent for all large L. For a d-lattice type τ, Lemma 4.3 gives (4.1). If $b(\tau) < k$, then $Np_N^{b(\tau)} \to \infty$, and the disjoint-window argument from Lemma 4.4 shows that τ occurs at least M times with high probability. If $b(\tau) > k$, then $Np_N^{b(\tau)} \to 0$, and Markov's inequality makes τ absent with high probability. Thus only the types with $b(\tau) = k$ have a non-deterministic truncated count.

Take a common fixed decision window for all critical types. List, up to translation in $\mathbb{Z}^d$, all underlying k-edge omitted sets arising in their minimal assignments, merging all rooted descriptions generated by the same set. This is a finite list. Every listed orbit has $a_D N + o(N)$ candidates in T_L^d, with $a_D > 0$, and the quotient maps $\mathbb{Z}^d \to T_L^d$ have injectivity radius tending to infinity. Hence Lemma 2.8, with $\rho_N = p_N$, gives a joint Poisson limit for the underlying clusters. A realization of a critical type not represented by a minimal assignment uses at least $k+1$ omitted edges in a fixed window, and the expected number of all such realizations is

$$O(Np_N^{k+1}) = O((Np_N^k)p_N) = o(1).$$

It follows that the vector of critical rooted-type counts, truncated at M, has a joint limiting distribution. The untruncated counts may be compound Poisson, because one defect set can be observed from several roots.

With probability tending to one all noncritical coordinates of the Hanf profile have their deterministic absent or saturated values. The complete truncated profile takes values in a finite set, and its distribution converges. Hanf locality makes the truth of φ a function of that profile, so $\mathbb{P}(X_L \models \varphi)$ converges. □

Lemma 4.6. *For every $d \geq 2$ and every $k \geq 1$ there are a rooted ball type $\sigma_{d,k}$, of some fixed radius and with minimal omitted-edge defect exactly k, and a constant $a_{d,k} > 0$ with the following property. Let $\theta_{d,k}$ be the first-order sentence asserting that some vertex has rooted type $\sigma_{d,k}$. In the regime $p_N \to 0$, $q_N \to 1$, whenever*

$$Np_N^k \longrightarrow \lambda \in [0,\infty],$$

one has

$$\mathbb{P}(X_L \models \theta_{d,k}) \longrightarrow 1 - e^{-a_{d,k}\lambda},$$

where $e^{-a_{d,k}\infty}$ is interpreted as 0.

Proof. Choose a root at 0 and a set S of k pairwise vertex-disjoint lattice edges in a fixed box, none incident with the root. Take the edges sufficiently separated and parallel to the first coordinate axis. Since $d \geq 2$, choose them so that each has a three-edge detour in the second coordinate direction and these detours avoid S. Consequently the subgraph

$$X^* = \mathbb{Z}^d \setminus S$$

obtained by omitting precisely the edges in S is connected: any lattice path can have each use of an edge in S replaced by its fixed detour.

Choose R so large that every endpoint of an edge in S, together with all of its lattice neighbors, has X^*-distance strictly less than R from the root. Let

$$\sigma_{d,k} = B_{X^*}(0, R).$$

Finally let P be the complete assignment on the decision window W_R which omits the edges in S and retains every edge in $W_R \setminus S$. The order of these choices is important: W_R is chosen after the observation radius, and it contains every edge whose state can affect the rooted R-ball. Therefore an occurrence of P forces rooted type $\sigma_{d,k}$, regardless of the states outside the translated window.

The displayed realization uses k omitted edges. In $\sigma_{d,k}$ the $2k$ endpoints of S are strictly inside the ball and have degree $2d-1$; here "interior" means graph distance less than R from the root. Every other interior vertex has degree $2d$. In any realization of the same rooted type in a sufficiently large torus, an interior vertex has all of its retained neighbors inside the ball. Thus each of these $2k$ vertices must be incident with an omitted edge. One omitted edge accounts for at most two of the deficits, so every realization requires at least k omitted edges. Hence $b(\sigma_{d,k}) = k$.

Assume first that $Np_N^k \to \lambda \in (0, \infty)$. In the fixed window W_R, list, up to lattice translation, all underlying minimal k-edge sets that realize $\sigma_{d,k}$, forgetting the choice of an observing root and merging descriptions with the same rare-edge set. This list is finite and nonempty. Lemma 2.8, applied to the fixed local model $\mathbb{Z}^d$ and its torus quotients, shows that their underlying cluster counts converge jointly to independent Poisson random variables. Write $a_{d,k} > 0$ for the sum of their orbit densities. The expected number of realizations using at least $k+1$ omitted edges is

$$O(Np_N^{k+1}) = O((Np_N^k)p_N) \longrightarrow 0.$$

Markov's inequality therefore shows that, up to an event of probability $o(1)$, the sentence $\theta_{d,k}$ holds exactly when at least one minimal defect cluster occurs. Hence

$$\mathbb{P}(X_L \models \theta_{d,k}) \longrightarrow 1 - e^{-a_{d,k}\lambda}.$$

If $Np_N^k \to 0$, Lemma 4.3 gives $\mathbb{E}N_{\sigma_{d,k}}(X_L) = O(Np_N^k) = o(1)$, so Markov's inequality gives the same formula with $\lambda = 0$. If $Np_N^k \to \infty$, choose a positive-density family of translates of the particular specification P with pairwise disjoint translates of W_R. Each occurs with probability

$$p_N^k q_N^{|W_R|-k} = (1+o(1))p_N^k,$$

and every occurrence forces rooted type $\sigma_{d,k}$. Lemma 2.7 shows that at least one occurs with probability tending to one. This is the asserted formula for $\lambda = \infty$. □

Lemma 4.7. *Fix $d \geq 2$ and $k \geq 1$, and assume $p_N \to 0$ and $q_N \to 1$.*

(i) *If $Np_N^k \to \lambda \in (0,\infty)$, then the zero–one law fails.*
(ii) *If Np_N^k has no limit in $[0,\infty]$, then the convergence law fails.*

Proof. Use the sentence $\theta_{d,k}$ from Lemma 4.6.

If $Np_N^k \to \lambda \in (0,\infty)$, then

$$\mathbb{P}(X_L \models \theta_{d,k}) \longrightarrow 1 - e^{-a_{d,k}\lambda},$$

which is strictly between 0 and 1. Hence the zero–one law fails.

If Np_N^k has no extended limit, compactness of $[0,\infty]$ gives two distinct subsequential limits λ_1 and λ_2. Along a subsequence on which $Np_N^k \to \lambda_i$, Lemma 4.6 gives

$$\mathbb{P}(X_L \models \theta_{d,k}) \longrightarrow 1 - e^{-a_{d,k}\lambda_i},$$

where $e^{-a_{d,k}\infty}$ is interpreted as 0. The map $\lambda \mapsto 1 - e^{-a_{d,k}\lambda}$ is strictly increasing on the extended half-line, so these two subsequential limits are different. Hence the convergence law fails. □

Proof of Theorem 1.4. Part (a) follows from Lemmas 4.4, 4.5, and 4.7. For part (b), replace "omitted" by "retained" throughout. A rooted type is then measured by the least number $a(\tau)$ of retained edges needed to realize it when all nearby edges are omitted, and the same estimates use $Nq_N^{a(\tau)}$. For the analogue of Lemma 4.6, take a fixed straight lattice path of length k, root it at one endpoint, retain its k edges, and omit every other lattice edge incident with one of its $k+1$ vertices. A fixed decision window then forces the root component to be exactly this path. Observe the component at radius $k+1$, so every path vertex is strictly inside the rooted ball and no retained edge can leave it. Any realization of this rooted type therefore contains a connected graph on $k+1$ vertices and hence at least k retained edges. The prescribed path uses exactly k, so its minimal retained-edge defect is k. The same fixed-local-model Poisson argument gives the critical existence probabilities and completes the proof. □

Remark 4.8. The polynomial theorem was stated for $d \geq 2$ because in dimension one deleted edges are cuts. When $d = 1$ and $L \geq 3$, T_L^1 is a cycle and every component of a proper percolated subgraph is a path. If $p = 1 - q$

and $P_{\ell,L}$ denotes the number of path components with ℓ vertices, then for $1 \le \ell < L$ the exact expectation is

$$\mathbb{E}P_{\ell,L} = Lp^2q^{\ell-1}.$$

For $\ell = L$, a path on all L vertices occurs exactly when one cycle edge is omitted, and hence

$$\mathbb{E}P_{L,L} = Lpq^{L-1}.$$

Thus the first formula is not uniform up to $\ell = L$. These path-component thresholds are one-dimensional phenomena; in dimension $d \ge 2$, components of a percolated torus need not be paths.

5. Monadic second-order logic

We now prove the main monadic theorem. We work with the standard incidence presentation of MSO_2: the structure has vertex and edge sorts, together with the incidence relation, and monadic variables may range over either sort. In MSO_1, monadic variables range only over vertex sets.

The first-order torus arguments rely on locality, which does not extend to monadic logic in this generality because MSO can express global properties such as bipartiteness. The pruned Erdős–Rényi graph, however, is a forest with high probability. On disjoint unions, the Feferman–Vaught composition method applies to MSO [10, 19]; on each tree component, MSO is controlled by finite tree automata [24, 6, 5]. We isolate the finite-rank consequence needed for forests.

Lemma 5.1. *For every s there are a finite set $\mathcal{A}_s$, an MSO_2-definable partition of the finite trees*

$$\tau_s : \{\textit{finite trees}\} \longrightarrow \mathcal{A}_s,$$

and a number M_s with the following property. For a finite forest F and $a \in \mathcal{A}_s$, let

$$n_a(F) = \big|\{C : C \textit{ is a component of } F \textit{ and } \tau_s(C) = a\}\big|.$$

If two finite forests F and F' satisfy, for every $a \in \mathcal{A}_s$,

$$n_a(F) = n_a(F') < M_s \qquad \textit{or} \qquad n_a(F), n_a(F') \ge M_s,$$

then F and F' satisfy the same MSO_2 sentences of quantifier rank at most s. Here "definable partition" means that for each $a \in \mathcal{A}_s$ there is an MSO_2 sentence χ_a such that $\tau_s(T) = a$ if and only if $T \models \chi_a$.

Proof. Represent each graph by its two-sorted incidence structure, with unary predicates for the vertex and edge sorts. The finite-rank MSO Feferman–Vaught theorem for disjoint sums gives, for every s, a finite MSO-definable partition $\mathcal{A}_s$ of the possible summands and an integer r_s such that the rank-s MSO theory of a disjoint sum is determined by the rank-r_s MSO theory of its index structure

$$I_F = \big(\mathrm{Comp}(F); (P_a)_{a \in \mathcal{A}_s}\big),$$

where $P_a(C)$ holds exactly when the component C belongs to the class a [10, 19]. Taking Boolean cells of the finitely many component formulas supplied by the composition theorem gives the asserted partition; each cell is defined by a finite Boolean combination of MSO_2 sentences and hence by a single MSO_2 sentence χ_a.

It remains to analyze the index structures. A pure set with finitely many unary predicates has only finitely many Boolean cells. For this fixed finite unary signature and every quantifier rank r, there is a cutoff $M(r)$ such that two such structures are MSO-equivalent up to rank r whenever, in each cell, the two cardinalities are equal below $M(r)$ or are both at least $M(r)$. This follows directly from the monadic Ehrenfeucht–Fraïssé game. An element move consumes at most one point of one cell. A set move splits every current cell into two; choosing the preceding cutoff at least twice the next cutoff lets Duplicator match every small part exactly and keep every remaining corresponding part above the next cutoff. Induction on the number of remaining moves proves the claim (and simultaneously handles the finitely many pebbled elements).

Apply this cutoff statement with rank r_s and put $M_s = M(r_s)$. The hypothesis of the lemma makes I_F and $I_{F'}$ equivalent up to rank r_s; the composition theorem then makes F and F' equivalent up to rank s in MSO_2. □

Lemma 5.2. *For every s there is a constant C_s such that the following holds. If $a \in \mathcal{A}_s$ has representatives of arbitrarily large order, then for every $m \geq 1$ there is a finite tree T such that*

$$\tau_s(T) = a \qquad \text{and} \qquad m \leq |T| \leq C_s m.$$

Proof. By Lemma 5.1, each class

$$\mathcal{T}_a = \{T : \tau_s(T) = a\}$$

is MSO_2-definable. Apply the second assertion of Lemma 2.3 to the finite family of classes $\mathcal{T}_a$ having representatives of arbitrarily large order. □

Proposition 5.3. *Fix s. There are constants B_s, M_s and a finite set $\mathcal{A}_s^\infty \subseteq \mathcal{A}_s$ such that the following holds. If a finite forest F has no component of order at most B_s and has at least M_s components of each state in $\mathcal{A}_s^\infty$, then the MSO_2 theory of F up to quantifier rank s is fixed, independent of F.*

Proof. Let $\mathcal{A}_s^\infty$ be the set of states $a \in \mathcal{A}_s$ which have finite-tree representatives of arbitrarily large order. Since $\mathcal{A}_s$ is finite, all remaining states have representatives of order bounded by a common constant; call this bound B_s, and take $B_s = 0$ if there are no bounded states. If F has no component of size at most B_s, then no bounded state occurs in F. If, in addition, every state in $\mathcal{A}_s^\infty$ occurs at least M_s times, then Lemma 5.1 shows that the MSO_2 theory of F up to rank s is determined. □

Proof of Theorem 1.1. Fix an MSO_2 sentence φ and let s be its quantifier rank. Take the constants and states from Proposition 5.3. For every $a \in \mathcal{A}_s^\infty$, Lemma 5.2 gives a finite tree $F_{a,n}$ such that

$$\tau_s(F_{a,n}) = a \qquad \text{and} \qquad f(n) \le |F_{a,n}| \le C_s f(n).$$

Since $|F_{a,n}| \le C_s f(n)$, the hypothesis on f gives

$$\begin{aligned}|F_{a,n}|(\log|F_{a,n}| + \log(1/c_n)) &\le C_s f(n)(\log(C_s f(n)) + \log(1/c_n)) \\ &= o(\log n).\end{aligned}$$

By Proposition 3.2, $G(n, p_n)$ has at least M_s connected components isomorphic to $F_{a,n}$ with probability tending to one. There are only finitely many states $a \in \mathcal{A}_s^\infty$, so this holds simultaneously for all of them with probability tending to one. Since $|F_{a,n}| \ge f(n)$, all of these components survive in H_n.

Every component of H_n has order at least $f(n) > B_s$ for all sufficiently large n. By Lemma 3.1, H_n is also a forest with probability tending to one. Hence the hypotheses of Proposition 5.3 hold with probability tending to one, and the truth value of φ is forced on that event. Thus (H_n) has an MSO_2 zero–one law. □

The preceding theorem is special to the pruned Erdős–Rényi forests. There is no corresponding monadic extension of the torus results in the same generality. The obstruction is already present in the weaker MSO_1 fragment: monadic second-order logic can express bipartiteness, and the parity of the side length L is visible through this property.

Proposition 5.4. *Fix $d \ge 1$, let $N = L^d$, and let $X_L = X_L^d(q_N)$ with $q_N = 1 - p_N$. Suppose*

$$Lp_N \longrightarrow 0.$$

If L ranges over all positive integers, then (X_L) does not have an MSO_1 *convergence law, and hence does not have an* MSO_2 *convergence law.*

Proof. In the graph vocabulary, the sentence

$$\beta := \exists U\, \forall x\, \forall y\, \Big(E(x,y) \to ((U(x) \wedge \neg U(y)) \vee (\neg U(x) \wedge U(y)))\Big)$$

says that the graph is bipartite. This is an MSO_1 sentence, and therefore also an MSO_2 sentence. In the two-sorted incidence presentation, x, y are vertex-sort variables, U is a vertex-set variable, and $E(x,y)$ is replaced by

$$x \ne y \;\wedge\; \exists e\, (\mathrm{Inc}(e,x) \wedge \mathrm{Inc}(e,y)),$$

with e of the edge sort. Thus the displayed property has exactly the same meaning in the standard incidence presentation and uses no edge-set quantification.

If L is even, then T_L^d is bipartite, using the parity of the sum of the coordinates. Hence every subgraph of T_L^d is bipartite, and therefore

$$\mathbb{P}(X_L \models \beta) = 1$$

for every even L.

If L is odd, fix the last $d-1$ coordinates and vary the first coordinate. This gives a coordinate cycle of length L in T_L^d. It is an odd cycle. The probability that all its edges are retained is

$$q_N^L = (1-p_N)^L \longrightarrow 1,$$

because

$$0 \le 1 - q_N^L \le Lp_N \longrightarrow 0.$$

Thus, along odd L, the graph X_L contains an odd cycle with probability tending to one. Hence

$$\mathbb{P}(X_L \models \beta) \longrightarrow 0$$

along odd L.

The same MSO_1 sentence has subsequential limiting probabilities 1 and 0. Therefore (X_L) has no MSO_1 convergence law, and a fortiori no MSO_2 convergence law. □

Corollary 5.5. *Under the assumptions of Proposition 5.4, suppose also that* $f : \mathbb{N} \to \mathbb{N}$ *satisfies* $f(N) = o(L)$, *and put*

$$Y_L = \mathrm{Prune}_{f(N)}(X_L).$$

If L ranges over all positive integers, then (Y_L) does not have an MSO_1 *convergence law, and hence does not have an* MSO_2 *convergence law.*

Proof. For even L, the graph T_L^d is bipartite, and therefore every induced subgraph of every subgraph of T_L^d is bipartite. Hence

$$\mathbb{P}(Y_L \models \beta) = 1$$

for every even L, where β is the sentence from the proof of Proposition 5.4.

For odd L, the proof of Proposition 5.4 shows that, with probability tending to one, X_L contains a fully retained coordinate cycle of length L. Since $f(N) = o(L)$, the component containing this cycle has order at least $L > f(N)$ for all sufficiently large L. Hence that component survives in Y_L, and Y_L contains an odd cycle with probability tending to one. Therefore

$$\mathbb{P}(Y_L \models \beta) \longrightarrow 0$$

along odd L.

Thus the probabilities of the same MSO_1 sentence have different subsequential limits, so (Y_L) has neither an MSO_1 nor an MSO_2 convergence law. □

Remark 5.6. For example, Proposition 5.4 applies when $p_N = N^{-\alpha}$ with $\alpha > 1/d$. The obstruction is global rather than local: no single first-order sentence uniformly detects the parity of these unbounded coordinate cycles, whereas monadic second-order logic can express bipartiteness. This does not rule out monadic limit laws for special subsequences, such as even side lengths, or under additional global hypotheses. It only shows that the torus theorems proved above do not have a direct monadic analogue, even in MSO_1, when L ranges over all integers.

Remark 5.7. The arguments follow a common pattern. First, logical questions are reduced to finite local or component data: Hanf locality does this on tori, while the pruned Erdős–Rényi graph is handled by finite component types. The relevant witnesses are then counted. Away from a threshold, each type is absent or saturated; at a threshold, the underlying defect clusters have Poisson limits.

For a fixed first-order torus sentence, only finitely many quantities Np_N^b (or Nq_N^b) arise from the Hanf radius. The polynomial classification in Theorem 1.4 packages this finite collection of thresholds into the usual exponent notation. The monadic obstruction is global and therefore does not rule out monadic convergence on special subsequences, such as even side lengths, or in modified models that suppress the relevant parity phenomenon.

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